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A Class of Solvable Stochastic Dividend Optimization Problems: On the General Impact of Flexibility on Valuation

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ABSTRACT

We consider the determination of the optimal stochastic lump-sum dividend policy of a corporation facing cash flow uncertainty and present a set of general conditions under which the optimal dividend policy exists and is unique. We also consider a class of associated singular stochastic control and optimal stopping problems and demonstrate that increased flexibility does not only increase the value of a rationally managed corporation, it also increases the rate at which this value grows (i.e. Tobin's marginal q). We also analyze the sensitivity of the optimal dividend policy and its value to changes in the transaction costs and prove that increased transaction costs result into larger but less frequent dividend payments.

CONTENTS

1 INTRODUCTION

In the classical study Miller and Modigliani 1961 established that dividend policy is irrelevant in a perfect and rational market. As Miller and Modigliani 1961 state (on p. 414):

> ". . . there are no "financial illusions" in a rational and perfect economic environment. Values there are determined solely by "real" considerations – in this case the earning power of the firm's assets and its investment policy– and not by how the fruits of the earning power are "packaged" for distribution."

This irrelevance result (and the related findings on the irrelevance of the capital structure on valuation of Modigliani and Miller 1958) were later extended in the general equilibrium framework by Stiglitz 1974. These to some extent controversial findings based on the perfection of the underlying markets have been subsequently challenged in numerous studies by weakening the assumptions and introducing imperfections into the analysis of the determination of the dividend policy (for example, by introducing economics of information (Ross 1977), agency costs (Easterbrook 1984, Jensen 1986), asymmetric information (Miller and Rock 1985), and taxes (Kose and Williams 1985); see also Ross and Westerfield (1988, Chapter 15). Moreover, there is empirical evidence indicating that at least in some industries (for example, in the insurance industry; cf. Akhigbe, Borde and Madura 1993) dividend policy does play a role in the valuation of firms and that dividend policy is actually an important strategic element in the decision making process of these corporations.

Given the arguments stated above, we plan to consider in this study the determination of the optimal dividend policy of a rationally managed corporation in the presence of transaction costs for a broad class of diffusions modelling the stochastically fluctuating dynamics of the underlying cash reserves (retained profits) from which the dividends are paid out. Given the recent interest on stochastic impulse control policies, we model the admissible dividend policy as a stochastic lump-sum impulse policy and, therefore, assume that *the objective of the corporation is to determine both the timing and the size of the optimal dividend policy* (cf. Bar-Ilan, Perry, and Stadje 2004, Cadenillas, Sarkar, and Zapatero 2003 and Peura and Keppo 2003; see also Korn 1999 for an excellent survey on stochastic impulse control applications in finance). Instead of analyzing the stochastic control problem by relying on ordinary dynamic programming techniques we first derive by relying on the classical theory of diffusions an associated mapping measuring the expected cumulative present value of the future dividend flow from the present up to the potentially infinitely distant liquidation date by assuming that the stochastic dividend policy constitutes a stationary impulse control policy characterized by two constant boundaries. Namely, the boundary at which the dividends should be paid out and the boundary at which the underlying stochastic reserve process should be restarted (i.e. the generic initial state). Given this observation and the admissibility of such dividend policy we are able to derive an explicit representation of the value in terms of the exercise payoff accrued every time dividends are paid out and the minimal increasing r-excessive mapping for the controlled diffusion. In this way the original problem is transformed into a simpler two dimensional non-linear programming problem which can be studied by relying on ordinary static optimization techniques. By applying this representation we are able to derive the ordinary first order necessary conditions which necessarily have to be satisfied by the optimal policy within the considered class of admissible dividend policies. We then present a set of general typically satisfied conditions (which are valid, for example, *for most applied mean reverting diffusion models*) under which both the existence and the uniqueness of the optimal dividend policy is always guaranteed and under which the proposed dividend policy satisfying the necessary conditions is indeed optimal. Interestingly, our results unambiguously indicate that *the presence of liquidation risk results into a maximal admissible transaction cost* below which the sequential payment of dividends is optimal. Above this critical cost the sequential payment of dividends is suboptimal and the optimal dividend problem becomes an optimal liquidation problem where the sole objective of the corporation is to determine the threshold at which it should be irreversibly liquidated. Thus, our results unambiguously indicate that the combined effect of the risk of potential liquidation and transaction costs on the nature of the implemented dividend policy may be dramatic depending on the size of the transaction costs.

For the sake of comparison, we also consider two associated stochastic cash flow management problems. Namely, a singular stochastic control problem where the optimal dividend policy is characterized by an exercise threshold at which dividends are paid out in a singular fashion (i.e. the optimal policy typically ranges from periods of intense dividend payments to periods of inactivity; cf. Asmussen and Taksar 1997, Baldursson and Karatzas 1997, Choulli, Taksar, and Zhou 2003, Højgaard and Taksar 1999, 2001, Holt 2003, Jeanblanc-Picqué and Shiryaev 1995, Kobila 1983, Milne and Robertson 1996, Øksendal 2000, and Taksar and Zhou 1998. See also Taksar 2000 for an excellent survey of stochastic dividend optimization models) and an optimal stopping problem where the optimal dividend policy is characterized by an exercise threshold at which all the reserves are paid out as dividends and the corporation is instantaneously liquidated. Somewhat surprisingly, we find that the values of these cash flow management problems are ordered in an exceptionally strong fashion. More precisely, we first demonstrate the intuitively clear finding that the value of the associated singular stochastic dividend control problem dominates the value of the stochastic lump-sum (impulse) dividend control problem which, in turn, dominates the value of the associated optimal liquidation problem. However, we also establish that *the marginal values (and, therefore, the Tobin's marginal q associated with the particular cash flow management problem) are ordered in the same way*. Put formally, we prove that the marginal value of the associated singular stochastic dividend control problem dominates the marginal value of the stochastic lump-sum dividend control problem which, in turn, dominates the marginal value of the associated optimal stopping problem. Therefore, our results clearly support the economically sensible argument that *increased flexibility does not only increase the value of a rationally managed corporation, it also increases the rate at which this value grows*. It is also worth noticing that our results extend previous results establishing a connection between the marginal value of singular stochastic control problems and the value of associated optimal stopping problems (cf. Alvarez 1999, 2001, Baldursson 1987, Benes, Shepp, and Witsenhausen 1980, Boetius and Kohlmann 1998, Hausmann and Suo 1995, Karatzas 1983, Karatzas and Shreve 1984, 1985, Menaldi and Robin 1983, and Menaldi and Rofman 1983) by showing that the values of the considered stochastic control problems are connected through an associated free boundary value problem as well. Similarly, our results extend previous results establishing a connection between the value of the considered stochastic impulse control problem and the value of an associated optimal stopping problem by demonstrating that also the marginal values of these problems are closely connected to each other in the sense that the marginal value of the impulse control problem is greater than or equal to the marginal value of the associated stopping problem.

It is worth observing that our results are of importance in the rational management of renewable resources as well, since all the considered cash flow management problems can be interpreted as *the determination of the admissible harvesting strategy maximizing the expected cumulative present value of future harvesting yields in the presence of stochastic value growth*. Therefore, our results unambiguously indicate that the more flexible the implemented harvesting strategy is, the higher is its value and its marginal value. More specifically, our results demonstrate that typically both the value and the marginal value of the optimal single harvesting strategy are smaller than the value and the marginal value of the optimal ongoing harvesting opportunity, respectively (Alvarez 2003, Sødal 2002, and Willassen 1998 have previously considered the determination of the optimal rotation policy in the presence of stochastic value growth and an exogenously given generic initial state). These values, in turn, are smaller than the value and the marginal value of the singular harvesting opportunity modelling the most flexible harvesting strategy (singular stochastic harvesting strategies have been previously considered, among others, in Alvarez 1998, 2000, Alvarez and Shepp 1998, Lande, Engen, and Sæther 1994, 1995, and Lungu and Øksendal 1996). These observations again imply that the required exercise premia and, consequently, the optimal harvesting thresholds can be ordered accordingly. A natural and economically sensible implication of this observation is that *increased flexibility shortens the expected length of a time interval between two consecutive harvests (i.e. the rotation cycle)*. Put somewhat differently, increased flexibility unambiguously increases the project value by increasing the expected cumulative yield accrued from harvesting and speeds up harvesting by decreasing the optimal harvesting threshold; a finding which is in line with the literature on real options.

The contents of this study are as follows. In section 2 the considered stochastic impulse control problem is presented. In section 3 we then present a set of auxiliary results on linear diffusions and associated stochastic control problems needed later in the analysis. In section 4 we then consider the stochastic impulse control problem and present both a set of necessary conditions from which the optimal policy can be derived and a set of general conditions under which the optimal policy exists and is unique. In section 5 our theoretical results are then explicitly illustrated. Finally, section 6 concludes our study.

2 THE STOCHASTIC IMPULSE CONTROL PROBLEM

Consider a value-maximizing competitive corporation facing cash flow uncertainty. For the sake of simplicity, assume that the reservoir process measuring the retained profits from which dividends are paid out is exogenous and modelled as a general linear diffusion. More precisely, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space satisfying the usual conditions and assume that the dynamics of the controlled cash flow dynamics are described by the process characterized by the generalized Itô-equation

$$\begin{aligned} X_t^\nu = x + \int_0^t \mu(X_s^\nu)ds + \int_0^t \sigma(X_s^\nu)dW(s) \\ - \sum_{\tau_k \leq t} \zeta_k, \quad 0 \leq t \leq \tau_0^\nu, \end{aligned} \tag{2.1}$$

where $\tau_0^\nu = \inf\{t \geq 0 : X_t^\nu \leq 0\}$ denotes the potentially infinite date at which the firm is liquidated and $\mu : \mathbb{R}_+ \mapsto \mathbb{R}$ and $\sigma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ are known sufficiently smooth (at least continuous) mappings guaranteeing the existence of a solution for the stochastic differential equation (2.1) (cf. Borodin and Salminen 2002, pp. 46–47). A stochastic *a lump-sum dividend policy* (i.e. a stochastic impulse control) for the system (2.1) is a possibly finite sequence (cf. Øksendal 1999)

$$\nu = (\tau_1, \tau_2, \dots, \tau_k, \dots; \zeta_1, \zeta_2, \dots, \zeta_k, \dots)_{k\leq N} \quad (N \leq \infty),$$

where $\{\tau_k\}_{k\leq N}$ is an increasing sequence of $\mathcal{F}_t$-stopping times for which $\tau_1 \geq 0$, and $\{\zeta_k\}_{k\leq N}$ denotes a sequence of non-negative dividends (i.e. $\zeta_k \geq 0$ for all k) paid out at the corresponding intervention dates $\{\tau_k\}_{k\leq N}$, respectively. We denote as $\mathcal{V}$ the class of admissible dividend policies ν and assume that $\tau_k \to \tau_0^\nu$ almost surely for all $\nu \in \mathcal{V}$ and $x \in \mathbb{R}_+$. In accordance with most financial and economic applications of stochastic impulse control models, we assume that the upper boundary ∞ is natural (therefore, even though the reserves may be expected to increase, they are never expected to become infinitely high in finite time) and that the lower boundary 0 is either natural, exit, or regular for the controlled diffusion in the absence of interventions. In case it is regular, we assume that it is killing. As usually, we denote as

$$\mathcal{A} = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}$$

the differential operator associated to the controlled diffusion X_t.

Given the dynamics in (2.1) and our assumptions on the dynamics of the controlled system, define the expected cumulative present value of the net dividends from the present up to an arbitrarily distant (potentially infinite) future as

$$J_c^{\nu}(x) = \mathbf{E}_x \left[\sum_{k=1}^{N} e^{-r\tau_k}(\zeta_k - c) \right], \tag{2.2}$$

where $r > 0$ denotes the risk free discount rate and $c > 0$ is a known transaction cost incurred each time the irreversible dividend policy ζ is exercised. Given the definition of the expected cumulative dividends $J_c^{\nu}(x)$ we plan to consider in this study the stochastic impulse control problem

$$V_c(x) = \sup_{\nu \in \mathcal{V}} \mathbf{E}_x \left[\sum_{k=1}^{N} e^{-r\tau_k}(\zeta_k - c) \right], \quad x \in \mathbb{R}_+ \tag{2.3}$$

and to determine an admissible lump-sum dividend policy $\nu^* \in \mathcal{V}$ for which $J_c^{\nu^*}(x) = V_c(x)$ for all $x \in \mathbb{R}_+$. Put somewhat differently, we plan to determine an dividend policy $\nu^* \in \mathcal{V}$ maximizing the expected cumulative present value of the paid out dividends from the present up to an arbitrarily distant future. Given our assumptions on the controlled diffusion and the objective function, we now present a verification lemma which is later applied for the verification of optimality of a proposed policy.

Lemma 2.1. *Assume that the mapping $g : \mathbb{R}_+ \mapsto \mathbb{R}$ is r-excessive for the underlying diffusion X_t and that*

$$g(x) \geq \sup_{\zeta \in [0,x]} [\zeta - c + g(x - \zeta)] \tag{2.4}$$

for all $x \in \mathbb{R}_+$. Then, $g(x) \geq V_c(x)$ for all $x \in \mathbb{R}_+$.

Proof. Let $\nu \in \mathcal{V}$ be an admissible stochastic impulse control. Since $\{\tau_j\}_{j \in \mathbb{N}}$ is an increasing sequence of stopping times, we first observe that the assumed r-excessivity of the mapping $g(x)$ implies (see Borodin and Salminen 2002, pp. 32–35 for a precise definition of r-excessive mappings for a diffusion)

$$\mathbf{E}\left[e^{-r\tau_{j+1}} g(X^{\nu}_{\tau_{j+1}-}) | \mathcal{F}_{\tau_j} \right] \leq e^{-r\tau_j} g(X^{\nu}_{\tau_j}).$$

Taking expectations, invoking the tower property of conditional expectations, and reordering terms yields

$$\mathbf{E}_x\left[e^{-r\tau_j}g(X^{\nu}_{\tau_j})\right] - \mathbf{E}_x\left[e^{-r\tau_{j+1}}g(X^{\nu}_{\tau_{j+1}-})\right] \geq 0.$$

Letting $\tau_0 = 0$ and summing terms from $j = 0$ to $j = n$ results in

$$g(x) \geq \mathbf{E}_x\left[e^{-r\tau_{n+1}}g(X^{\nu}_{\tau_{n+1}-})\right] + \mathbf{E}_x\sum_{j=1}^{n} e^{-r\tau_j}\left[g(X^{\nu}_{\tau_{j-}}) - g(X^{\nu}_{\tau_j})\right].$$

Since $X^{\nu}_{\tau_j} = X^{\nu}_{\tau_j-} - \zeta_j$ for any admissible strategy $\nu \in \mathcal{V}$ and the mapping $g(x)$ is non-negative and satisfies the quasi-variational inequality $g(x) \geq \sup_{\zeta\in[0,x]}[\zeta - c + g(x-\zeta)] \geq \zeta - c + g(x-\zeta)$ we find that

$$g(x) \geq \mathbf{E}_x\sum_{j=1}^{n} e^{-r\tau_j}\left[g(X^{\nu}_{\tau_{j-}}) - g(X^{\nu}_{\tau_j-} - \zeta_j)\right] \geq \mathbf{E}_x\sum_{j=1}^{n} e^{-r\tau_j}(\zeta_j - c).$$

Since this inequality is valid for any admissible impulse control $\nu \in \mathcal{V}$, it has to be valid for the optimal as well proving that $g(x) \geq V_c(x)$. □

Lemma 2.1 states a set of sufficient conditions guaranteeing that a mapping dominates the value of the considered stochastic impulse control problem (for related results see, for example, Bensoussan 1982, Brekke and Øksendal 1994, 1996, Harrison, Sellke, and Taylor 1983, Mundaca and Øksendal 1998, Øksendal 1999, and Øksendal 2000). It is worth noticing that the conditions of Lemma 2.1 are considerably weak, since the assumed r-excessivity of the mapping $g(x)$ only guarantees that it is non-negative, continuous and r-superharmonic (cf. Borodin and Salminen 2002, p. 32). An interesting implication of Lemma 2.1 expressing its conditions in a more easily applicable variational form is now summarized in the following.

Corollary 2.2. *Assume that the mapping $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfies the conditions $g \in C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+\backslash\mathcal{D})$, where $\mathcal{D}$ is a set of measure zero and $g''(x\pm) < \infty$ for all $x \in \mathcal{D}$. Assume also that $g(x)$ satisfies the quasi-variational inequality (2.4) for all $x \in \mathbb{R}_+$ and the variational inequality $(\mathcal{A}g)(x) - rg(x) \leq 0$ for all $x \notin \mathcal{D}$. Then, $g(x) \geq V_c(x)$ for all $x \in \mathbb{R}_+$.*

Proof. As was established in Theorem D.1. in Øksendal 1999 (pp. 299-302) the conditions of our corollary guarantee that there a sequence $\{g_n\}_{n=1}^{\infty}$ of mappings $g_n \in C^2(\mathbb{R}_+)$ such that

(i) $g_n \to g$ uniformly on compact subsets of $\mathbb{R}_+$, as $n \to \infty$;

(ii) $(\mathcal{A}g_n) - rg_n \to (\mathcal{A}g) - rg$ uniformly on compact subsets of $\mathbb{R}_+ \backslash \mathcal{D}$, as $n \to \infty$;

(iii) $\{(\mathcal{A}g_n) - rg_n\}_{n=1}^{\infty}$ is locally bounded on $\mathbb{R}_+$.

Applying Itô's theorem to the mapping $(t, x) \mapsto e^{-rt} g_n(x)$, taking expectations, and reordering terms yields

$$e^{-r\tau_j} g_n(X^{\nu}_{\tau_j}) = \mathbf{E}\left[e^{-r\tau_{j+1}} g_n(X^{\nu}_{\tau_{j+1}-}) - \int_{\tau_j}^{\tau_{j+1}-} e^{-rs} G_n(X^{\nu}_s) ds \Big| \mathcal{F}_{\tau_j}\right],$$

where $G_n(x) = (\mathcal{A}g_n)(x) - rg_n(x)$. Letting $n \to \infty$, applying Fatou's theorem, and invoking the variational inequality $(\mathcal{A}g)(x) - rg(x) \leq 0$ yield

$$e^{-r\tau_j} g(X^{\nu}_{\tau_j}) \geq \mathbf{E}\left[e^{-r\tau_{j+1}} g(X^{\nu}_{\tau_{j+1}-}) \Big| \mathcal{F}_{\tau_j}\right].$$

The alleged result now follows from Lemma 2.1. □

3 AUXILIARY RESULTS

Before proceeding in the analysis of the considered stochastic dividend optimization problem, we first derive some auxiliary results needed later in the analysis of the original problem. For the sake of notational simplicity, denote now as X_t the controlled diffusion modelling the stochastic cash flow dynamics in the absence of interventions. As usually, we denote as $\mathcal{L}^1(\mathbb{R}_+)$ the class of measurable mappings $f : \mathbb{R}_+ \mapsto \mathbb{R}$ satisfying the uniform integrability condition

$$\mathbf{E}_x \int_0^{\tau_0} e^{-rs} |f(X_s)| ds < \infty,$$

where $\tau_0 = \inf\{t \geq 0 : X_t \leq 0\}$. That is, $\mathcal{L}^1(\mathbb{R}_+)$ can be interpreted as the collection of cash flows with finite expected cumulative present values. Given the class $\mathcal{L}^1(\mathbb{R}_+)$ we define for $f \in \mathcal{L}^1(\mathbb{R}_+)$ the functional $(R_r f) : \mathbb{R}_+ \mapsto \mathbb{R}$ as

$$(R_r f)(x) = \mathbf{E}_x \int_0^{\tau_0} e^{-rs} f(X_s) ds.$$

As is well-known from the literature on linear diffusions, if $f \in \mathcal{L}^1(\mathbb{R}_+)$ then

$$\begin{aligned}(R_r f)(x) =& B^{-1}\varphi(x)\int_a^x \psi(y)f(y)m'(y)dy+ \\ & B^{-1}\psi(x)\int_x^\infty \varphi(y)f(y)m'(y)dy,\end{aligned}$$

where $\psi(x)$ denotes the increasing and $\varphi(x)$ the decreasing fundamental solution of the ordinary second order differential equation $(\mathcal{A}u)(x) = ru(x)$ (defined on the domain of the operator of the diffusion $\{X_t; t \in [0, \tau_0)\}$; see Borodin and Salminen 2002, pp. 18–20 for a throughout characterization of the fundamental solutions and the Green function of a linear diffusion), $B = (\psi'(x)\varphi(x) - \varphi'(x)\psi(x))/S'(x) > 0$ denotes the constant (with respect to the scale) Wronskian determinant of the fundamental solutions, $S'(x) = \exp\left(-2\int^x (\mu(y)/\sigma^2(y))dy\right)$ denotes the density of the scale function S of X, and $m'(x) = 2/(\sigma^2(x)S'(x))$ denotes the density of the speed measure m of X.

Define the mapping $\rho : \mathbb{R}_+ \mapsto \mathbb{R}$ measuring the net appreciation rate of the reserves X_t as

$$\rho(x) = \mu(x) - rx. \tag{3.1}$$

Given this definition, consider now the associated singular stochastic dividend control problem

$$K(x) = \sup_{Z\in\Lambda} \mathbf{E}_x \int_0^{\tau^Z(0)} e^{-rs}dZ_s, \tag{3.2}$$

where Λ denotes the class of non-negative, non-decreasing, right-continuous, and $\{\mathcal{F}_t\}$-adapted dividend payment processes, $\tau^Z(0) = \inf\{t \geq 0 : X_t^Z \leq 0\}$ denotes the potentially infinite liquidation date, and the underlying reserve process evolves on $\mathbb{R}_+$ according to the dynamics described by the generalized (Itô) stochastic differential equation

$$dX_t^Z = \mu(X_t^Z)dt + \sigma(X_t^Z)dW_t - dZ_t, \quad X_0^Z = x. \tag{3.3}$$

An important result needed later in the analysis of the dividend optimization problem (2.3) (slightly extending the results originally proved in 3 and 5) is now summarized in our next lemma.

Lemma 3.1. *Assume that* $\rho \in \mathcal{L}^1(\mathbb{R}_+)$ *and that* $\lim_{x\to\infty} \rho(x) < 0$. *Assume also that*

(i) if 0 *is unattainable for* X_t *then there is a unique threshold* $x^* \in (0, \infty)$ *such that* $\rho(x)$ *is increasing on* $(0, x^*)$ *and decreasing on* (x^*, ∞) *and* $\lim_{x\downarrow 0} \mu(x) \geq 0$*;*

(ii) if 0 *is attainable for* X_t *then there is a unique threshold* $x^* \in [0, \infty)$ *such that* $\rho(x)$ *is increasing on* $(0, x^*)$ *and decreasing on* (x^*, ∞) *and* $\lim_{x\downarrow 0} \mu(x) > 0$.

Then, the value of the optimal dividend payment policy reads as

$$K(x) = \begin{cases} x + \frac{\rho(\hat{x})}{r} & x \geq \hat{x} \\ \frac{\psi(x)}{\psi'(\hat{x})} & x < \hat{x}, \end{cases} \tag{3.4}$$

where $\hat{x}$ *is the unique optimal exercise threshold* $\hat{x} = \operatorname{argmin}\{\psi'(x)\}$ *satisfying the ordinary first order condition* $\psi''(\hat{x}) = 0$. *The value of the optimal dividend policy is twice continuously differentiable, monotonically increasing, and concave. Moreover, the marginal value (i.e. Tobin's q) of the optimal dividend policy can be expressed as*

$$K'(x) = \psi'(x) \sup_{y \geq x} \left[\frac{1}{\psi'(y)} \right] = \begin{cases} 1 & x \geq \hat{x} \\ \frac{\psi'(x)}{\psi'(\hat{x})} & x < \hat{x}. \end{cases} \tag{3.5}$$

Proof. Applying Dynkin's theorem to the identity mapping $x \mapsto x$ yields

$$\mathbf{E}_x \left[e^{-r\tau^*} X_{\tau^*} \right] = x + \mathbf{E}_x \int_0^{\tau^*} e^{-rs} \rho(X_s) ds, \tag{3.6}$$

where $\tau^* = \inf\{t \geq 0 : X_t \notin (a, b)\}$ denotes the first exit date of X_t from the open set $(a, b) \subset (0, \infty)$. The r-harmonicity and continuity of the left-hand side of (3.6) implies that

$$\mathbf{E}_x \left[e^{-r\tau^*} X_{\tau^*} \right] = a \frac{\hat{\varphi}(x)}{\hat{\varphi}(a)} + b \frac{\hat{\psi}(x)}{\hat{\psi}(b)},$$

where $\hat{\varphi}(x) = \varphi(x) - \varphi(b)\psi(x)/\psi(b)$ and $\hat{\psi}(x) = \psi(x) - \psi(a)\varphi(x)/\varphi(a)$ denote the decreasing and the increasing solutions of the ordinary differential equation $(\mathcal{A}u)(x) = ru(x)$ subject to the boundary conditions $\hat{\psi}(a) = 0$ and $\hat{\varphi}(b) = 0$, respectively. On the other hand, since the integral expression on the right-hand side of (3.6) satisfies the ordinary differential equation $(\mathcal{A}v)(x) - rv(x) + \rho(x) = 0$ subject to the boundary conditions $v(a) = v(b) = 0$ we observe that

$$\mathbf{E}_x \int_0^{\tau^*} e^{-rs}\rho(X_s)ds = \hat{B}^{-1}\hat{\varphi}(x)\int_a^x \hat{\psi}(y)\rho(y)m'(y)dy + \hat{B}^{-1}\hat{\psi}(x)\int_x^b \hat{\varphi}(y)\rho(y)m'(y)dy,$$

where $\hat{B} = (1 - \psi(a)\varphi(b)/(\psi(b)\varphi(a)))B$ denotes the constant Wronskian of the solutions $\hat{\varphi}(x)$ and $\hat{\psi}(x)$. Combining these expressions now imply that (3.6) can be re-expressed as

$$x = a\frac{\hat{\varphi}(x)}{\hat{\varphi}(a)} + b\frac{\hat{\psi}(x)}{\hat{\psi}(b)} - \hat{B}^{-1}\hat{\varphi}(x)\int_a^x \hat{\psi}(y)\rho(y)m'(y)dy - \hat{B}^{-1}\hat{\psi}(x)\int_x^b \hat{\varphi}(y)\rho(y)m'(y)dy. \tag{3.7}$$

Dividing equation (3.7) with $\hat{\psi}(x)$, differentiating, and multiplying the resulting equation with $\hat{\psi}^2(x)$ yields

$$\hat{\psi}(x) - x\hat{\psi}'(x) = S'(x)\int_a^x \hat{\psi}(y)\rho(y)m'(y)dy - \frac{BaS'(x)}{\varphi(a)} \tag{3.8}$$

Letting $a \downarrow 0$ in (3.8), invoking the absolute integrability condition $\rho \in \mathcal{L}^1(\mathbb{R}_+)$, and multiplying the resulting equation with r now implies that

$$r\psi(x) = rx\psi'(x) + S'(x)r\int_0^x \psi(y)\rho(y)m'(y)dy. \tag{3.9}$$

Subtracting the term $\mu(x)\psi'(x)$ from both sides of equation (3.9) and applying the identity $\frac{1}{2}\sigma^2(x)\psi''(x) = r\psi(x) - \mu(x)\psi'(x)$ then finally yields

$$\frac{1}{2}\sigma^2(x)\frac{\psi''(x)}{S'(x)} = r\int_0^x \psi(y)\rho(y)m'(y)dy - \rho(x)\frac{\psi'(x)}{S'(x)}. \tag{3.10}$$

Denote now the right-hand side of equation (3.10) as $I(x)$. It is clear that our assumptions imply that $I(0) \leq 0$ and that

$$I(x) \leq \rho(x)r\int_0^x \psi(y)m'(y)dy - \rho(x)\frac{\psi'(x)}{S'(x)} = -\rho(x)\frac{\psi'(0)}{S'(0)} \leq 0$$

for all $x \in (0, x^*)$. On the other hand, the assumed monotonicity of $\rho(x)$ on (x^*, ∞) and the assumption $\lim_{x\to\infty} \rho(x) < 0$ imply that there is a threshold $x_0 > x^*$ at which $\rho(x_0) = 0$ and, therefore, at which

$$I(x_0) = r\int_0^{x_0} \psi(y)\rho(y)m'(y)dy > 0.$$

Combining this observation with the continuity and monotonicity of $I(x)$ on (x^*, ∞) then finally implies that equation $I(x) = 0$ and, therefore, that equation $\psi''(x) = 0$ has a unique root $\hat{x} \in (x^*, x_0)$ and that $\hat{x} = \operatorname{argmin}\{\psi'(x)\}$. Consequently, we discover that the proposed value function $K(x)$ is monotonically increasing, concave, and satisfies the variational inequalities $\min\{J'(x) - 1, rJ(x) - (\mathcal{A}J)(x)\} = 0$ proving that it dominates the value of the singular stochastic control problem (3.2). However, since the proposed value can be attained by applying a *local time push-type* dividend strategy and the solution of the stochastic differential equation (3.3) subject to reflection at $\hat{x}$ exists and is unique (cf. Freidlin 1985, Section 1.6), we find that the proposed value function is indeed the value of the singular stochastic control problem (3.2). Moreover, since

$$\frac{d}{dx}\left[\frac{1}{\psi'(x)}\right] = -\frac{\psi''(x)}{\psi'^2(x)} \gtreqless 0, \quad x \lesseqgtr \hat{x}$$

we find that $K'(x)$ can be expressed as in (3.5). □

Lemma 3.1 states a set of weak sufficient conditions under which the associated singular stochastic control problem (3.2) is solvable and under which the value of the optimal policy can be expressed in terms of the increasing minimal r-excessive mapping for the underlying diffusion. Lemma 3.1 has two important capital theoretic implications. First of all, since the optimal dividend threshold is attained on the set where net appreciation rate of the underlying reserve is positive, we find that *dividends are paid out on the set where the expected per capita rate at which the reserves are increasing dominate the opportunity cost of investment*. Second, since the optimal dividend threshold is attained on the set where net appreciation rate of the underlying reserve is decreasing, we find that at the optimum the marginal yield accrued from retaining yet another marginal unit of stock undistributed should be smaller than the interest rate r and, therefore, that the optimal dividend policy diverges from the deterministic golden rule of capital accumulation (cf. Merton 1990, pp. 594-595, see also Alvarez 2001).

It is at this point worth emphasizing that the value (3.4) of the optimal singular stochastic dividend policy can be re-expressed as

$$K(x) = \begin{cases} x - \hat{x} + \frac{\psi(\hat{x})}{\psi'(\hat{x})} & x \geq \hat{x} \\ \frac{\psi(x)}{\psi'(\hat{x})} & x < \hat{x}. \end{cases} \tag{3.11}$$

As we will later observe in our subsequent analysis, this expression is closely related to the value of the considered stochastic lump-sum dividend optimization problem (2.3). An important inequality illustrating the importance of the value of the associated singular stochastic dividend policy is now summarized in the following.

Lemma 3.2. *Define the continuously differentiable mapping* $H : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ *as*

$$H(x, y) = \begin{cases} x - y + \frac{\psi(y)}{\psi'(y)} & x \geq y \\ \frac{\psi(x)}{\psi'(y)} & x < y \end{cases}$$

and assume that the conditions of Lemma 3.1 are satisfied. Then $K(x) = H(x, \hat{x}) > H(x, y)$ *and* $K'(x) = H_x(x, \hat{x}) > H_x(x, y)$ *for all* $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ \setminus \{\hat{x}\}$*. Moreover,* $H_y(x, y) < 0$ *for all* $(x, y) \in \mathbb{R}_+ \times (\hat{x}, \infty)$*.*

Proof. Assume that $y > \hat{x}$. Then

$$H(x, \hat{x}) - H(x, y) = \begin{cases} \theta(\hat{x}) - \theta(y) & \hat{x} < y \leq x \\ x + \theta(\hat{x}) - \frac{\psi(x)}{\psi'(y)} & \hat{x} \leq x < y \\ \frac{(\psi'(y) - \psi'(\hat{x}))\psi(x)}{\psi'(y)\psi'(\hat{x})} & x < \hat{x} < y \end{cases}$$

where the continuously differentiable mapping $\theta : \mathbb{R}_+ \mapsto \mathbb{R}$ is defined as

$$\theta(x) = \frac{\psi(x)}{\psi'(x)} - x.$$

Since

$$\theta'(x) = -\frac{\psi(x)\psi''(x)}{\psi'^2(x)} \gtreqqless 0, \quad x \lesseqqgtr \hat{x},$$

we find that $\hat{x} = \operatorname{argmax}\{\theta(x)\}$ and, therefore, that $\theta(\hat{x}) > \theta(y)$ for all $y \neq \hat{x}$. Moreover, since $\hat{x} = \operatorname{argmin}\{\psi'(x)\}$ we find that $\psi'(y) > \psi'(\hat{x})$

for all $y \neq \hat{x}$ as well. Consequently, it is sufficient to consider the difference $H(x,\hat{x}) - H(x,y)$ on the set $[\hat{x}, y)$. The monotonicity of $\theta(x)$ and $\psi'(x)$ on $[\hat{x}, \infty)$ implies that

$$x + \theta(\hat{x}) - \frac{\psi(x)}{\psi'(y)} > x + \theta(x) - \frac{\psi(x)}{\psi'(y)} = \frac{(\psi'(y) - \psi'(x))\psi(x)}{\psi'(y)\psi'(x)} > 0$$

proving that $H(x,\hat{x}) > H(x,y)$ for all $x \in \mathbb{R}_+$ when $y > \hat{x}$. It remains to consider the case where $y < \hat{x}$. In that case

$$H(x,\hat{x}) - H(x,y) = \begin{cases} \theta(\hat{x}) - \theta(y) & y < \hat{x} \leq x \\ \frac{\psi(x)}{\psi'(\hat{x})} - (x + \theta(y)) & y \leq x < \hat{x} \\ \frac{(\psi'(y) - \psi'(\hat{x}))\psi(x)}{\psi'(y)\psi'(\hat{x})} & x < y < \hat{x}. \end{cases}$$

In light of our observations above, it is sufficient to consider the difference $H(x,\hat{x}) - H(x,y)$ on the set $[y, \hat{x})$. Since

$$\frac{\psi(x)}{\psi'(\hat{x})} - (x + \theta(y)) \geq \frac{\psi(x)}{\psi'(\hat{x})} - (x + \theta(x)) = \frac{(\psi'(x) - \psi'(\hat{x}))\psi(x)}{\psi'(\hat{x})\psi'(x)} > 0$$

we observe that $H(x,\hat{x}) > H(x,y)$ for all $x \in \mathbb{R}_+$ when $y < \hat{x}$ as well and, therefore, that $H(x,\hat{x}) > H(x,y)$ for all $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+ \backslash \{\hat{x}\}$. Establishing that $H_x(x,\hat{x}) > H_x(x,y)$ for all $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+ \backslash \{\hat{x}\}$ is completely analogous. It remains to establish that $H_y(x,y) < 0$ for all $(x,y) \in \mathbb{R}_+ \times (\hat{x}, \infty)$. To observe that this is indeed the case, we find by ordinary differentiation that

$$H_y(x,y) = \begin{cases} \theta'(y) & x \geq y \\ -\frac{\psi(x)}{\psi'^2(y)}\psi''(y) & x < y \end{cases}$$

which is negative since $\theta(y)$ is decreasing and $\psi(y)$ is convex on $(\hat{x}, \infty)$. $\square$

In order to slightly qualify the results of Lemma 3.2 we first observe that the auxiliary mapping $u_y(x) = H(x,y)$ satisfies on $(0,y)$ the absence of arbitrage condition $(\mathcal{A}u_y)(x) = ru_y(x)$, stating that the expected percentage rate of return from the project has to coincide with the risk-free rate of return, subject to the boundary condition $u_y(0) = 0$. On the other hand, since $u_y(x)$ satisfies on (y, ∞) the linear growth condition $u_y'(x) = 1$ we find that Lemma 3.2 essentially demonstrates that given the conditions of Lemma 3.1 the mapping $H(x,\hat{x})$ dominates any other mapping satisfying these variational inequalities.

This dominance is surprisingly strong, since Lemma 3.2 indicates that also $H_x(x, \hat{x})$ dominates the derivative $H_x(x, y)$ for any other chosen boundary y. Consequently, Lemma 3.2 establishes that the variational inequalities stated above *have a unique dominant solution which is not only greater but also grows faster than any other admissible solution*. An interesting implication of our results is now summarized in the following.

Lemma 3.3. *Assume that the conditions of Lemma 3.1 are satisfied. Then* $K(x) > G_0(x)$ *and* $K'(x) > G_0'(x)$*, where*

$$\begin{aligned} G_0(x) &= \sup_{\tau < \tau_0} \mathbf{E}_x \left[e^{-r\tau} X_\tau \right] \\ &= \psi(x) \sup_{y \geq x} \left[\frac{y}{\psi(y)} \right] = \begin{cases} x & x \geq \bar{x}_0 \\ \frac{\psi(x)}{\psi'(\bar{x}_0)} & x < \bar{x}_0, \end{cases} \end{aligned} \tag{3.12}$$

denotes the maximal expected present value of the cash reserves and $\bar{x}_0 \in \rho^{-1}(\mathbb{R}_-)$*, denoting the threshold at which this value is attained, is the unique root of equation* $\psi(\bar{x}_0) = \bar{x}_0 \psi'(\bar{x}_0)$.

Proof. In order to establish (3.12) we first denote as $x_0 > x^*$ the unique interior threshold at which $\rho(x_0) = 0$ and observe that equation (3.9) implies that

$$\frac{\psi(x)}{S'(x)} - x\frac{\psi'(x)}{S'(x)} = \int_0^x \psi(y)\rho(y)m'(y)dy. \tag{3.13}$$

Since

$$\frac{d}{dx}\left[\frac{\psi(x)}{S'(x)} - x\frac{\psi'(x)}{S'(x)}\right] = \psi(x)\rho(x)m'(x) \gtreqqless 0, \quad x \lesseqqgtr x_0,$$

we observe that $(\psi(x) - x\psi'(x))/S'(x) > 0$ for all $x \in (0, x_0)$. Assume now that $x > k > x_0$. The monotonicity of the mapping $\rho(x)$ on (x^*, ∞) and the assumed boundary behavior of the underlying diffusion at ∞ then implies that

$$\begin{aligned} \frac{\psi(x)}{S'(x)} - x\frac{\psi'(x)}{S'(x)} &= \frac{\psi(k)}{S'(k)} - k\frac{\psi'(k)}{S'(k)} + \int_k^x \psi(y)\rho(y)m'(y)dy \\ &\leq \frac{\psi(k)}{S'(k)} - k\frac{\psi'(k)}{S'(k)} + \frac{\rho(k)}{r}\left[\frac{\psi'(x)}{S'(x)} - \frac{\psi'(k)}{S'(k)}\right] \downarrow -\infty, \end{aligned}$$

since $\psi'(x)/S'(x) \to \infty$ as $x \to \infty$ and $\rho(k) < 0$. This demonstrates that there is a unique threshold $\bar{x}_0 \in \rho^{-1}(\mathbb{R}_-)$ at which the ordinary first order necessary condition $\psi(\bar{x}_0) = \bar{x}_0\psi'(\bar{x}_0)$ is satisfied. Moreover, since

$$\frac{\psi^2(x)}{S'(x)}\frac{d}{dx}\left[\frac{x}{\psi(x)}\right] = \frac{\psi(x)}{S'(x)} - x\frac{\psi'(x)}{S'(x)}$$

we also observe that $\bar{x}_0 = \mathrm{argmax}\{x/\psi(x)\}$. Given these findings denote the proposed value function as $\hat{G}_0(x)$. Since

$$\hat{G}_0(x) = \mathbf{E}_x\left[e^{-r\bar{\tau}}X_{\bar{\tau}}\right],$$

where $\bar{\tau} = \inf\{t \geq 0 : X_t \geq \bar{x}_0\}$, we immediately find that $\hat{G}_0(x) \leq G_0(x)$. On the other hand, we observe that the proposed value function is continuously differentiable, twice continuously differentiable on $\mathbb{R}_+\backslash\{\bar{x}_0\}$, and satisfies the variational inequalities

$$\min\{r\hat{G}_0(x) - (\mathcal{A}\hat{G}_0)(x), \hat{G}_0(x) - x\} = 0.$$

Thus, it constitutes a r-excessive majorant of the exercise payoff x for the underlying diffusion X_t. Since $G_0(x)$ is the least of these majorants, we observe that $\hat{G}_0(x) \geq G_0(x)$ proving that $\hat{G}_0(x) = G_0(x)$. In order to prove the inequality $K(x) \geq G_0(x)$ we first observe that the value of the associated singular stochastic control problem satisfies the variational inequality $(\mathcal{A}K)(x) \leq rK(x)$ and the inequality

$$K(x) - x \geq \theta(\min(x, \hat{x})) \geq 0.$$

Thus, $K(x)$ constitutes a r-excessive majorant of the exercise payoff x for the underlying diffusion X_t. Since $G_0(x)$ is the least of these majorants, we observe that $K(x) \geq G_0(x)$. The inequality $K'(x) \geq G_0'(x)$ is now a straightforward consequence of equation (3.5) and Lemma 3.2. □

Lemma 3.3 demonstrates that the value of the optimal singular stochastic dividend policy dominates the maximal expected present value of the future cash reserves. This result is intuitively clear, since the maximal expected present value of the cash reserves can always be attained by choosing the admissible dividend policy $Z_t = X_t 1_{[\bar{x}_0,\infty)}(X_t)$. Since the class of admissible policies is, however, larger than this single dividend payment strategy, we find that *the value of the optimal singular stochastic dividend policy has to dominate the maximal expected present value of the cash reserves*. However, a slightly more surprising result is that also the marginal value of the optimal singular stochastic dividend policy dominates the marginal value of the maximal expected present value of the cash reserves. Therefore, *the yield accrued from retaining a marginal unit of stock undistributed is higher in the case where dividends are paid out sequentially than in the case where dividends are paid out only once* (corresponding to instantaneous liquidation at the optimal threshold). A second important implication of the results of Lemma 3.3 is that $\bar{x}_0 > \hat{x}$. That is, *the required exercise premium is naturally higher in the case where the opportunity to pay out dividends may be exercised only once than in the case where this decision may be subsequently repeated.* Moreover, since $\bar{x}_0$ is attained on the set where the net appreciation rate of the reserves is decreasing we again find that at the optimum the marginal yield accrued from postponing exercise further into the future should be smaller than the opportunity cost of investment measured by the risk free rate r. Hence, the deterministic golden rule of capital accumulation is violated in this case as well.

4 THE OPTIMAL DIVIDEND POLICY

Having presented in the previous section some auxiliary results and an associated singular stochastic dividend optimization problem we now plan to analyze the stochastic lump-sum dividend optimization problem (2.3). In order to present a general detailed treatment of the problem, we first derive a set of necessary conditions which have to be satisfied by a candidate for an optimal policy. We then study the necessary conditions and establish a set of general and typically satisfied conditions under which the necessary conditions admit a unique solution and under which this solution is indeed optimal.

4.1 Necessary conditions

Typically, stochastic impulse control problems of the type (2.3) are solved by relying on dynamic programming techniques and, especially, on quasi-variational inequalities. Although such an approach is very general in the sense that it applies in the multidimensional setting as well, it is usually rather difficult to derive expressions independent of the value function with simple and clear economic interpretations. Similarly, marginalistic interpretations providing valuable economic content and general information on the nature of the optimal solution and its value are typically difficult to derive from general approaches based on dynamic programming techniques. Given this argument, we propose in this paper an alternative approach for analyzing and solving the stochastic lump-sum dividend optimization problem (2.3). Instead of considering all admissible dividend policies at once, we follow the approach introduced in 6 and restrict our interest to dividend policies $\nu_{(y,\zeta)} = \{\tau_k^y; \zeta_k^y\}_{k\leq N}$ characterized for all $k \geq 1$ by the sequence of intervention times $\tau_k^y = \inf\{t \geq \tau_{k-1}^y : X_t^\nu \geq y\}$ (with $\tau_0 = 0$) and the sequence of dividend payments $\zeta_k^y = \zeta + (x-y)^+$. That is, we restrict our attention to the sequence of constant-sized dividends (except for the initial impulse which depends on the state) which are exerted every time the underlying diffusion hits a given predetermined constant threshold. Given this subclass of admissible dividend strategies, define the value $F_c : \mathbb{R}_+ \mapsto \mathbb{R}$ accrued from applying the impulse control $\nu_{(y,\zeta)}$ as $F_c(x) = J_c^{\nu_{(y,\zeta)}}(x)$. Since $X^\nu_{\tau_k+} = X^\nu_{\tau_k-} - \zeta$ for all k and the underlying controlled diffusion evolves according to the linear diffusion X_t between two successive intervention times we observe that for all $x < y$ the value of the considered class of dividend policies reads as

$$\begin{aligned} F_c(x) &= \mathbf{E}_x\left[e^{-r\tau_y}(\zeta - c + F_c(y-\zeta))\right] \\ &= (\zeta - c + F_c(y-\zeta))\frac{\psi(x)}{\psi(y)}, \end{aligned} \tag{4.1}$$

where $\tau_y = \inf\{t \geq 0 : X_t = y\}$ denotes the first hitting time of X_t to the state y. Letting $x \to y - \zeta$ in (4.1) then yields that

$$F_c(y-\zeta) = \frac{(\zeta - c)\psi(y-\zeta)}{\psi(y) - \psi(y-\zeta)}$$

implying that the value $F_c(x)$ can be re-expressed on $(0, y)$ as

$$F_c(x) = \frac{(\zeta - c)\psi(x)}{\psi(y) - \psi(y-\zeta)}. \tag{4.2}$$

On the other hand, since the reserves can exceed the threshold y under the proposed impulse policy only at $t = 0$ and $\zeta_1^y = \zeta + (x-y)^+$ we find that on $[y, \infty)$ the value $F_c(x)$ reads as $F_c(x) = x - y + \zeta - c + F_c(y-\zeta)$. Hence, we finally observe that $F_c(x)$ can be re-expressed as

$$F_c(x) = \begin{cases} x - y + \frac{(\zeta-c)\psi(y)}{\psi(y)-\psi(y-\zeta)} & x \geq y \\ \frac{(\zeta-c)\psi(x)}{\psi(y)-\psi(y-\zeta)} & x < y. \end{cases} \tag{4.3}$$

It is worth observing that (4.3) implies the familiar balance identity

$$\zeta + F_c(y-\zeta) = c + F_c(y)$$

stating that *the project value (current revenues + future dividend potential) should be equal to its full costs (transaction costs c + lost option value $F_c(y)$)*. This observation is of interest since it clearly indicates that *the balance identity is an intrinsic property of the considered class of admissible policies and, therefore, is independent of the optimality of the proposed policy*.

Given the definition of the value $F_c(x)$, define now the mapping $h : \mathbb{R}_+^2 \mapsto \mathbb{R}$ as

$$h(\zeta, y) = \frac{(\zeta - c)}{\psi(y) - \psi(y-\zeta)} \tag{4.4}$$

and consider the ordinary inequality constrained non-linear programming problem

$$\sup_{\substack{\zeta\in[0,y],\\ y\in\mathbb{R}_+}} \frac{(\zeta - c)}{\psi(y) - \psi(y-\zeta)}. \tag{4.5}$$

If an admissible pair (ζ_c^*, y_c^*) maximizing the mapping $h(\zeta, y)$ exists, denote the value associated to this pair as $F_c^*(x)$. More precisely, if an admissible pair (ζ_c^*, y_c^*) maximizing the mapping $h(\zeta, y)$ exists, define the mapping $F_c^* : \mathbb{R}_+ \mapsto \mathbb{R}_+$ as

$$F_c^*(x) = \begin{cases} x - y_c^* + h(\zeta_c^*, y_c^*)\psi(y_c^*) & x \geq y_c^* \\ h(\zeta_c^*, y_c^*)\psi(x) & x < y_c^*. \end{cases} \tag{4.6}$$

It is then clear that if an admissible interior pair (ζ_c^*, y_c^*) maximizing the mapping $h(\zeta, y)$ exists, then the ordinary first order necessary conditions

$$\psi(y_c^*) - \psi(y_c^* - \zeta_c^*) = \psi'(y_c^* - \zeta_c^*)(\zeta_c^* - c) \tag{4.7}$$

$$\psi'(y_c^*) = \psi'(y_c^* - \zeta_c^*) \tag{4.8}$$

have to be satisfied. Consequently, we observe that

$$h(\zeta_c^*, y_c^*) = \frac{1}{\psi'(y_c^* - \zeta_c^*)} = \frac{1}{\psi'(y_c^*)}$$

and, therefore, that

$$F_c^*(x) = \begin{cases} x - y_c^* + \frac{\psi(y_c^*)}{\psi'(y_c^*)} & x \geq y_c^* \\ \frac{\psi(x)}{\psi'(y_c^*)} & x < y_c^*. \end{cases} \tag{4.9}$$

It is worth observing that the necessary condition (4.8) implies that if an interior solution of the non-linear programming problem (4.5) exists, then by *Rolle's theorem* there has to be at least one state $\hat{x} \in (y_c^* - \zeta_c^*, y_c^*)$ where the marginal value ${F^*}'_c(x)$ attains an extreme value and, therefore, where $\psi''(\hat{x}) = 0$. Moreover, it is also clear that $F_c^*(x)$ belongs into the class of mappings considered in Lemma 3.2 and, therefore, that $F_c^*(x) \leq K(x)$ and ${F^*}'_c(x) \leq K'(x)$ whenever a unique pair satisfying the necessary conditions (4.7) and (4.8) exists and is unique.

4.2 Existence and sufficiency

Having derived a set of necessary conditions from which the potentially optimal dividend threshold and dividend policy could be derived, we now plan to state a set of general and considerably weak conditions under which these optimal variables exist and are unique, and under which the derived auxiliary mapping indeed constitutes the value of the optimal dividend policy. A set of general conditions under which the necessary conditions (4.7) and (4.8) admit a unique solution is now summarized in the following.

Lemma 4.1. *Assume that $\rho \in \mathcal{L}^1(\mathbb{R}_+)$ and that $\lim_{x\to\infty} \rho(x) = -\infty$. Assume also that either the conditions (i) or conditions (ii) of Lemma 3.1 are satisfied and that $\lim_{x\downarrow 0} \psi'(x) = \infty$. Then there is a unique optimal pair (ζ_c^*, y_c^*) satisfying the necessary conditions (4.7) and (4.8) for all $c \in \mathbb{R}_+$.*

Proof. Consider now the mappings $L_1 : \mathbb{R}_+^2 \mapsto \mathbb{R}$ and $L_2 : \mathbb{R}_+^2 \mapsto \mathbb{R}$ defined as

$$\begin{aligned} L_1(z, y) &= \theta(y) - \theta(z) + c, \\ L_2(z, y) &= \psi'(y) - \psi'(z), \end{aligned}$$

where the continuously differentiable mapping $\theta : \mathbb{R}_+ \mapsto \mathbb{R}$ is defined as in the proof of Lemma 3.2. As was established in Lemma 3.1 and in Lemma 3.2 our assumptions imply that there is a unique threshold $\hat{x} = \operatorname{argmin}\{\psi'(x)\} =$

$\mathrm{argmax}\{\theta(x)\} \in \mathbb{R}_+$ such that $\psi''(x) \lesseqqgtr 0$ and $\theta'(x) \gtreqqless 0$ for $x \lesseqqgtr \hat{x}$. Moreover, since equation (3.9) can be re-expressed as

$$\frac{\psi(x)}{S'(x)} - x\frac{\psi'(x)}{S'(x)} = \int_0^x \psi(y)\rho(y)m'(y)dy$$

we also observe that

$$\theta(x) = \left(\frac{\psi'(x)}{S'(x)}\right)^{-1} \int_0^x \psi(y)\rho(y)m'(y)dy.$$

The assumed boundary behavior of the underlying diffusion at ∞ implies that $\frac{\psi'(x)}{S'(x)} \to \infty$ and $\int_0^x \psi(y)\rho(y)m'(y)dy \downarrow -\infty$ as $x \to \infty$. Hence, L'Hospital's rule implies that $\lim_{x\to\infty} \theta(x) = \lim_{x\to\infty} \frac{\rho(x)}{r} = -\infty$. If 0 is attainable for the underlying diffusion X_t, then we have that $\lim_{x\downarrow 0} \frac{\psi'(x)}{S'(x)} > 0$ implying that $\lim_{x\downarrow 0} \theta(x) = 0$ in that case. On the other hand, if 0 is unattainable for X_t, then $\lim_{x\downarrow 0} \frac{\psi'(x)}{S'(x)} = 0$ and L'Hospital's rule implies that $\lim_{x\downarrow 0} \theta(x) = \lim_{x\downarrow 0} \frac{\mu(x)}{r} \geq 0$. Consequently, we observe that for all $z \leq \hat{x}$ the mapping $L_1(z,y)$ satisfies the conditions $L_1(z,z) = c > 0$, $\lim_{y\to\infty} L_1(z,y) = -\infty$, and

$$\frac{\partial L_1}{\partial y}(z,y) = \theta'(y) \gtreqqless 0 \quad y \lesseqqgtr \hat{x}.$$

Therefore, we find that for all $z \leq \hat{x}$ there is a unique $\tilde{y}_c(z) \in (\hat{x}, \infty)$ satisfying the equation $L_1(z, \tilde{y}_c(z)) = 0$. Moreover, we also find that $\tilde{y}_c(0+) < \infty$, $\tilde{y}_c(\hat{x}) > \hat{x}$, and

$$\tilde{y}_c'(z) = \frac{\theta'(z)}{\theta'(\tilde{y}_c(z))} = \frac{\psi(z)\psi'^2(\tilde{y}_c(z))\psi''(z)}{\psi(\tilde{y}_c(z))\psi'^2(z)\psi''(\tilde{y}_c(z))} < 0$$

Consider now, in turn, the mapping $L_2(z,y)$. The strict convexity of $\psi(x)$ on $(\hat{x}, \infty)$ and the mean value theorem imply that $\psi'(x) \to \infty$ as $x \to \infty$. Consequently, we find that for all $z \in (0, \hat{x})$ the mapping $L_2(z,y)$ satisfies the conditions $L_2(z,z) = 0$, $\lim_{y\to\infty} L_2(z,y) = \infty$, and

$$\frac{\partial L_2}{\partial y}(z,y) = \psi''(y) \lesseqqgtr 0 \quad y \lesseqqgtr \hat{x}$$

Therefore, we again find that for all $z \leq \hat{x}$ there is a unique $\hat{y}(z) \in [\hat{x}, \infty)$ satisfying the equation $L_2(z, \hat{y}(z)) = 0$. Moreover, we also find that $\hat{y}(\hat{x}) = \hat{x} < \tilde{y}_c(\hat{x})$, and

$$\hat{y}'(z) = \frac{\psi''(z)}{\psi''(\hat{y}(z))} < 0.$$

Given these findings, we observe that if $\psi'(0+) = \infty$ then $\hat{y}(0+) = \infty > \tilde{y}_c(0+)$ and, therefore, that equation $\hat{y}(z) - \tilde{y}_c(z) = 0$ has at least at one root $z^* \in (0, \hat{x})$. Since

$$\tilde{y}_c'(z^*) = \frac{\psi(z^*)\psi''(z^*)}{\psi(\tilde{y}_c(z^*))\psi''(\tilde{y}_c(z^*))} > \frac{\psi''(z^*)}{\psi''(\tilde{y}_c(z^*))} = \hat{y}'(z^*)$$

we find that z^* is unique. □

Lemma 4.1 presents a set of typically satisfied conditions under which a pair (ζ_c^*, y_c^*) maximizing the mapping $h(\zeta, y)$ and satisfying the necessary conditions (4.7) and (4.8) exist and is unique for all $c \in \mathbb{R}_+$. It is worth pointing out that the conditions of Lemma 4.1 are typically satisfied in the cases where the lower boundary 0 is unattainable for the underlying diffusion X_t. Whenever 0 is attainable for the underlying diffusion X_t we typically have that $\psi'(0) < \infty$ and, therefore, that the conditions of Lemma 4.1 are no longer satisfied. A set of conditions extending the results of Lemma 4.1 to that case as well are presented in the following.

Lemma 4.2. *Assume that $\rho \in \mathcal{L}^1(\mathbb{R}_+)$ and that $\lim_{x\to\infty} \rho(x) = -\infty$. Assume also that the conditions (ii) of Lemma 3.1 are satisfied and that $\lim_{x\downarrow 0} \psi'(x) < \infty$. Then, there is a critical cost $\hat{c}$ such that there is a unique optimal pair (ζ_c^*, y_c^*) satisfying the necessary conditions (4.7) and (4.8) whenever $0 < c < \hat{c}$.*

Proof. To establish the existence of the critical cost $\hat{c}$ we first consider the mappings

$$\begin{aligned} f_1(x) &= \psi(x) - \psi'(x)x \\ f_2(x) &= \psi'(x) - \psi'(0). \end{aligned}$$

It is now clear that our assumptions and the results of Lemma 3.1 imply that $f_1(0) = f_2(0) = 0$ and that $\hat{x} = \operatorname{argmax}\{f_1(x)\} = \operatorname{argmin}\{f_2(x)\}$. Moreover, standard differentiation implies that $f_1'(x) = -xf_2'(x)$. Integrating this equation from 0 to x and applying integration by parts then yields

$$f_1(x) = \int_0^x f_2(y)dy - xf_2(x).$$

As in the proof of Lemma 4.1 denote now as $\hat{y}(0) < \infty$ (since $\psi'(0) < \infty$) the interior root of equation $f_2(x) = 0$. Then

$$f_1(\hat{y}(0)) = \int_0^{\hat{y}(0)} f_2(y)dy < 0$$

implying that $\hat{y}(0) > \tilde{y}_0(0)$, where $\tilde{y}_0(0)$ denotes the root of the interior root of equation $\theta(x) = 0$ (which, by definition, coincides with the interior root of equation $f_1(x) = 0$). Since $\partial\tilde{y}_c(0)/\partial c > 0$ we finally observe that there is a critical $\hat{c} > 0$ such that $\hat{y}(0) > \tilde{y}_c(0)$ for all $c < \hat{c}$. However, since $\hat{y}(\hat{x}) < \tilde{y}_c(\hat{x})$ the existence and uniqueness of the root z^* of equation $\hat{y}(z) - \tilde{y}_c(z) = 0$ follows from the proof of Lemma 4.1. □

As is now clear from Lemma 4.2 in case $\psi'(0) < \infty$ there is a maximal admissible transaction costs $\hat{c}$ under which the necessary conditions (4.7) and (4.8) are satisfied whenever $\psi'(0) < \infty$. Since this condition arises typically in cases where the underlying boundary is attainable for the reserves X_t, we find that *the risk of potential liquidation results into a maximal admissible transaction cost*. As we will later observe, this critical cost can be interpreted as the maximal cost the firm is prepared to incur in order to pay out dividends sequentially in first place. Our first result characterizing the optimal dividend policy and its value is now stated in the following.

Theorem 4.3. *Assume that the conditions of either Lemma 4.1 or Lemma 4.2 are satisfied. Then, the optimal lump sum dividend policy is $\nu^* = \nu_{(y_c^*,\zeta_c^*)}$ and its value reads as $V_c(x) = F_c^*(x)$.*

Proof. It is now sufficient to establish that the proposed value satisfies the sufficient quasi-variational inequalities, since the admissibility of the considered class of impulse controls naturally implies that $V_c(x) \geq F_c^*(x)$. We first observe that $F_c^* \in C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+ \backslash \{y_c^*\})$, $F_c^{*\prime\prime}(y_c^*+) = 0$, and $F_c^{*\prime\prime}(y_c^*-) = h(\zeta_c^*, y_c^*)\psi''(y_c^*) < \infty$. Moreover, since continuous mappings are bounded on compacts and $X_t^\nu \in (0, y_c^*)$ except for a t-set of measure zero we find that $\lim_{t\to\infty} \mathbf{E}_x[e^{-rt}F_c^*(X_t^\nu)] = 0$ for all $x \in \mathbb{R}_+$. Define the mapping $A_1 : \mathbb{R}_+ \backslash \{y_c^*\} \mapsto \mathbb{R}$ as $A_1(x) = (\mathcal{A}F_c^*)(x) - rF_c^*(x)$. It is clear that

$$A_1(x) = \begin{cases} \mu(x) - r\left(x - y_c^* + \frac{\psi(y_c^*)}{\psi'(y_c^*)}\right) & x > y_c^* \\ 0 & x < y_c^* \end{cases}$$

implying that

$$\lim_{x\downarrow y_c^*} A_1(x) = \frac{1}{\psi'(y_c^*)}\left[\mu(y_c^*)\psi'(y_c^*) - r\psi(y_c^*)\right] = -\frac{1}{2}\sigma^2(y_c^*)\frac{\psi''(y_c^*)}{\psi'(y_c^*)} < 0$$

since y_c^* is attained on the set where $\psi(x)$ is convex. However, since $A_1(x) = \rho(x) - r\left(\frac{\psi(y_c^*)}{\psi'(y_c^*)} - y_c^*\right)$ for all $x > y_c^*$ and y_c^* is on the set where the net appreciation rate $\rho(x)$ is decreasing, we find that $A_1(x) \leq 0$ for all $x \in \mathbb{R}_+ \setminus \{y_c^*\}$. It now remains to establish that $F_c^*(x)$ satisfies the quasi-variational inequality $F_c^*(x) \geq \sup_{\zeta\in[0,x]}[\zeta - c + F_c^*(x-\zeta)]$ for all $x \in \mathbb{R}_+$. To accomplish this task, we first observe that this quasi-variational inequality can be rewritten as $F_c^*(x) \geq x - c + \sup_{y\in[0,x]}[F_c^*(y) - y]$. Define now the mapping $A_2 : \mathbb{R}_+ \mapsto \mathbb{R}$ as

$$A_2(x) = F_c^*(x) - (x-c) - \sup_{y\in[0,x]}[F_c^*(y) - y].$$

Since $\psi'(x)/\psi'(y_c^*) < 1$ for all $x \in (y_c^* - \zeta_c^*, y_c^*)$ we first observe that

$$\sup_{y\in[0,x]}[F_c^*(y) - y] = \begin{cases} F_c^*(y_c^* - \zeta_c^*) - (y_c^* - \zeta_c^*) & x > y_c^* - \zeta_c^* \\ F_c^*(x) - x & x \leq y_c^* - \zeta_c^*. \end{cases}$$

Consequently, we find that

$$A_2(x) = \begin{cases} 0 & x \geq y_c^* \\ \frac{\psi(x) - \psi(y_c^*)}{\psi'(y_c^*)} + y_c^* - x & x \in (y_c^* - \zeta_c^*, y_c^*) \\ c & x \leq y_c^* - \zeta_c^*. \end{cases}$$

Since $\lim_{x\to y_c^*-} A_2(x) = 0$ and $A_2'(x) = \frac{\psi'(x)}{\psi'(y_c^*)} - 1 < 0$ on $(y_c^* - \zeta_c^*, y_c^*)$ we find that $A_2(x) > 0$ on $(y_c^* - \zeta_c^*, y_c^*)$ and, therefore, that $A_2(x) \geq 0$ for all $x \in \mathbb{R}_+$. Thus, $F_c^*(x) \geq V_c(x)$ implying that $F_c^*(x) = V_c(x)$ and, therefore, that $\nu^* = \nu_{(y_c^*, \zeta_c^*)}$. □

Theorem 4.3 demonstrates that the conditions of both Lemma 4.1 and Lemma 4.2 are actually sufficient for guaranteeing that the auxiliary mapping $F_c^*(x)$ indeed constitutes the maximal attainable expected cumulative present value of the future dividend payments. As intuitively is clear, the optimal dividend policy is completely characterized by the optimal threshold y_c^* at which a lump-sum dividend ζ_c^* is paid out. Hence, the state $y_c^* - \zeta_c^*$ can be viewed as a generic initial state at which the diffusion process modelling the retained profits is restarted after the dividends have been paid out. An important set of results characterizing the relationship between the associated optimal dividend problems is now presented in the following.

Theorem 4.4. *Assume that the conditions of either Lemma 4.1 or Lemma 4.2 are satisfied. Then,*

$$K(x) \geq V_c(x) \geq G_c(x) \quad \textit{and} \quad K'(x) \geq V_c'(x) \geq G_c'(x),$$

where

$$\begin{aligned} G_c(x) &= \sup_{\tau} \mathbf{E}_x \left[e^{-r\tau}(X_\tau - c) \right] \\ &= \psi(x) \sup_{y \geq x} \left[\frac{y-c}{\psi(y)} \right] = \begin{cases} x - c & x \geq \bar{x}_c \\ \frac{\psi(x)}{\psi'(\bar{x}_c)} & x < \bar{x}_c, \end{cases} \end{aligned} \tag{4.10}$$

denotes the value of an associated optimal stopping problem and $\bar{x}_c > c$ is the unique root of equation $\psi(\bar{x}_c) = (\bar{x}_c - c)\psi'(\bar{x}_c)$. Moreover, $\bar{x}_c > y_c^* > \hat{x}$ for all admissible costs $c > 0$.

Proof. Inequality $K(x) \geq V_c(x)$ follows directly from Lemma 3.2 and the representation (4.9). On the other hand, as was established in the proof of Theorem 4.3, the value function $V_c(x)$ is continuously differentiable on $\mathbb{R}_+$, twice continuously differentiable on $\mathbb{R}_+ \setminus \{y_c^*\}$ and satisfies the variational inequality $(\mathcal{A}V_c)(x) - rV_c(x) \leq 0$ for all $x \in \mathbb{R}_+ \setminus \{y_c^*\}$. Moreover, since

$$V_c(x) \geq \sup_{\zeta \leq x} [\zeta - c + V_c(x - \zeta)] \geq x - c$$

we observe that $V_c(x)$ constitutes a r-excessive majorant of the exercise payoff $x - c$ and, therefore, that $V_c(x) \geq \sup_\tau \mathbf{E}_x \left[e^{-r\tau}(X_\tau - c) \right]$. Establishing equation (4.10) is analogous with the proof of Lemma 3.3. It is now clear from the proof of Lemma 4.1 and Lemma 4.2 that $y_c^* > \hat{x}$. Moreover, for all $x \in (0, \min(y_c^*, \bar{x}_c))$ the inequality

$$V_c(x) - G_c(x) = \frac{(\psi'(\bar{x}_c) - \psi'(y_c^*))\psi(x)}{\psi'(y_c^*)\psi'(\bar{x}_c)} \geq 0$$

implies that $\bar{x}_c > y_c^*$ since both thresholds are attained on the set where $\psi(x)$ is convex. It remains to establish that $K'(x) \geq V_c'(x) \geq G_c'(x)$. The inequality

$K'(x) \geq V_c'(x)$ follows directly from Lemma 3.2. Since $\bar{x}_c > y_c^* > \hat{x}$ we find that

$$V_c'(x) - G_c'(x) = \begin{cases} 0 & y_c^* < \bar{x}_c \leq x \\ \frac{\psi'(\bar{x}_c) - \psi'(x)}{\psi'(\bar{x}_c)} & y_c^* \leq x < \bar{x}_c \\ \frac{(\psi'(\bar{x}_c) - \psi'(y_c^*))\psi'(x)}{\psi'(y_c^*)\psi'(\bar{x}_c)} & x < y_c^* < \bar{x}_c \end{cases}$$

which is non-negative since the thresholds $\bar{x}_c$ and y_c^* are attained on the set where $\psi(x)$ is convex. □

Theorem 4.4 demonstrates that given the conditions of both Lemma 4.1 and Lemma 4.2 both the values and the marginal values of the considered dividend optimization problems are completely ordered. More precisely, Theorem 4.4 proves that the value of the associated singular stochastic control problem dominates the value of the stochastic impulse control problem which, in turn, dominates the value of the associated optimal stopping problem (single dividend payment). An important implication of this finding is that the optimal dividend threshold associated with the single dividend payment dominates the optimal dividend threshold of the sequential lump-sum dividend policy which, in turn, dominates the optimal dividend threshold of the optimal singular dividend policy. Put somewhat differently, Theorem 4.4 shows that the required exercise premium is highest in the single dividend payment case and lowest in the singular dividends case. Somewhat surprisingly, Theorem 4.4 also proves that not only the values of the considered different types of optimal dividend policies are ordered but also the marginal values of these policies are ordered. Since the marginal value of the optimal policy can be interpreted as the (marginal) *Tobin's q* we find that according to the findings of Theorem 4.4 the marginal value of the reserves in the presence of a singular dividend policy is higher than in the presence of a sequential lump-sum dividend policy which, in turn, dominates the marginal value of the reserves in the single dividend payment case. This result is very interesting since it formalizes the intuitively clear argument that *increased dividend payment flexibility does not only increase the value of the optimal policy, it also increases the marginal benefits (and, therefore, Tobin's marginal q) associated to the increased flexibility*. Another important result illustrating the importance of the risk of potential liquidation is now summarized in the following.

Theorem 4.5. *Assume that the conditions of Lemma 4.2 are satisfied and assume that $c \geq \hat{c}$, where the critical cost $\hat{c}$ satisfies the condition $\hat{c} = -\theta(\hat{y}(0))$,* *where $\hat{y}(0) > \hat{x}$ satisfies the equation $\psi'(y) = \psi'(0)$. Then, the optimal policy is $\nu^* = \nu_{(\bar{x}_c, \bar{x}_c)}$ (i.e. instantaneous liquidation at $\bar{x}_c$) and its value reads as $V_c(x) = G_c(x)$.*

Proof. As was established in the proof of Theorem 4.4, the value of the optimal stopping policy reads as

$$G_c(x) = \begin{cases} x - c & x \geq \bar{x}_c \\ \frac{\psi(x)}{\psi'(\bar{x}_c)} & x < \bar{x}_c. \end{cases}$$

Since $\bar{x}_c$ is attained on the set where $\psi(x)$ is convex and $\psi'(0) < \psi'(\bar{x}_c)$ we find that $G_c'(x) \leq 1$ for all $x \in \mathbb{R}_+$. Consequently, we observe that $G_c(x)$ satisfies the quasi-variational inequality $G_c(x) \geq (x-c)+\sup_{y\in[0,x]}[G_c(y)-y] = x-c$. Since $G_c(x)$ satisfies the condition $(\mathcal{A}G_c)(x) \leq rG_c(x)$ for all $x \in \mathbb{R}_+\backslash\{\bar{x}_c\}$ as well, we find that $G_c(x) \geq V_c(x)$ proving that $G_c(x) = V_c(x)$. Finally, since the policy $(\zeta_c^*, y_c^*) = (\bar{x}_c, \bar{x}_c)$ and the stopping time $\tau^* = \inf\{t \geq 0 : X_t \geq \bar{x}_c\}$ are admissible, and $V_c(x)$ is attained by implementing this policy, we find that $\nu_{(\bar{x}_c, \bar{x}_c)}$ is optimal. □

Theorem 4.5 demonstrates that the presence of potential liquidation risk (in the sense that the underlying reserve may vanish in finite time even in the absence of a dividend strategy) results into a maximal admissible cost at which the sequential payment of dividends becomes suboptimal. In that case, the problem can actually be interpreted as an optimal liquidation (or exit) problem where the objective of a rationally managed corporation is only to determine the optimal exercise threshold at which the firm should be liquidated and all the retained profits should be instantaneously paid out as dividends. An interesting special case where liquidation is also the optimal policy is presented in the next corollary.

Corollary 4.6. *Assume that $\lim_{x\downarrow 0}\mu(x) \leq 0$, that the net appreciation rate $\rho(x)$ is non-increasing, and that $\lim_{x\to\infty}\rho(x) < -rc$. Then, the optimal policy is $\nu^* = \nu_{(\bar{x}_c, \bar{x}_c)}$ (i.e. instantaneous liquidation at $\bar{x}_c$) and its value reads as $V_c(x) = G_c(x)$.*

Proof. We first observe that under the assumptions of our Corollary $\rho(x) \leq 0$ for all $x \in \mathbb{R}_+$ since $\rho(0+) = \mu(0+) \leq 0$ and $\rho(x)$ is non-increasing. On the other hand, the assumed monotonicity of the net appreciation rate $\rho(x)$ and equation (3.10) imply that

$$\begin{aligned}\frac{1}{2}\sigma^2(x)\frac{\psi''(x)}{S'(x)} &\geq \rho(x)\left(\frac{\psi'(x)}{S'(x)} - \frac{\psi'(0)}{S'(0)}\right) - \rho(x)\frac{\psi'(x)}{S'(x)} \\ &= -\rho(x)\frac{\psi'(0)}{S'(0)} \geq 0\end{aligned} \tag{4.11}$$

proving that the convexity of the increasing fundamental solution $\psi(x)$. Establishing that the value of the optimal stopping policy reads as

$$G_c(x) = \begin{cases} x - c & x \geq \bar{x}_c \\ \frac{\psi(x)}{\psi'(\bar{x}_c)} & x < \bar{x}_c \end{cases}$$

is now analogous with the proof of Lemma 3.3. Combining this observation with the convexity of the increasing fundamental solution implies that $G_c(x)$ is convex and satisfies the inequality $G_c'(x) \leq 1$ for all $x \in \mathbb{R}_+$. Consequently, we notice that $G_c(x)$ satisfies the quasi-variational inequality $G_c(x) \geq (x - c) + \sup_{y\in[0,x]}[G_c(y) - y] = x - c$. Since $G_c(x)$ satisfies the condition $(\mathcal{A}G_c)(x) \leq rG_c(x)$ for all $x \in \mathbb{R}_+\setminus\{\bar{x}_c\}$ as well, we find that $G_c(x) \geq V_c(x)$ proving that $G_c(x) = V_c(x)$. Finally, since the policy $(\zeta_c^*, y_c^*) = (\bar{x}_c, \bar{x}_c)$ and the stopping time $\tau^* = \inf\{t \geq 0 : X_t \geq \bar{x}_c\}$ are admissible, and $V_c(x)$ is attained by implementing this policy, we find that $\nu_{(\bar{x}_c,\bar{x}_c)}$ is optimal. □

Corollary 4.6 states a set of conditions under which the sequential payment of dividends is suboptimal and, therefore, under which the value of the considered stochastic impulse control problem coincides with the value of the associated optimal stopping problem corresponding to the optimal liquidation of the firm. It is worth observing that this case arises in situations where the net appreciation rate is negative and, therefore, in cases where the optimal singular dividend strategy is to liquidate the corporation immediately and pay out all the reserves instantaneously (the so-called *take the money and run-strategy*).

Our main results on the sensitivity of the optimal policy and its value to changes in the transaction cost c are now summarized in the following.

Theorem 4.7. *Assume that the conditions of either Lemma 4.1 or Lemma 4.2 are satisfied. Then,*

$$\begin{aligned}
\frac{dy_c^*}{dc} &= \frac{\psi'(y_c^*)}{\psi''(y_c^*)(\zeta_c^* - c)} > 0 \\
\frac{d\zeta_c^*}{dc} &= \frac{\psi'(y_c^*)(\psi''(y_c^* - \zeta_c^*) - \psi''(y_c^*))}{\psi''(y_c^* - \zeta_c^*)\psi''(y_c^*)(\zeta_c^* - c)} > 0.
\end{aligned}$$

Moreover, $\lim_{c\downarrow 0} y_c^* = \hat{x}$, $\lim_{c\downarrow 0} \zeta_c^* = 0$, $\lim_{c\downarrow 0} dy_c^*/dc = \infty$, *and*

$$\lim_{c\downarrow 0} \frac{\partial V_c}{\partial c}(x) = -\infty \tag{4.12}$$

for all $x \in \mathbb{R}_+$.

Proof. The comparative statics of the optimal variables y_c^* and ζ_c^* can be obtained from the ordinary first order conditions (4.7) and (4.8) by implicit differentiation. The limits $\lim_{c\downarrow 0} y_c^* = \hat{x}$ and $\lim_{c\downarrow 0} \zeta_c^* = 0$ follow directly from the proofs of Lemma 4.1 and Lemma 4.2. The continuity of $\psi'(x)$, $\psi''(x)$, y_c^*, and ζ_c^* then imply that $\lim_{c\downarrow 0} dy_c^*/dc = \infty$ since $\lim_{c\downarrow 0} \psi''(y_c^*) = \psi''(\lim_{c\downarrow 0} y_c^*) = \psi''(\hat{x}) = 0$, $\lim_{c\downarrow 0} \zeta_c^* - c = 0$, and $\lim_{c\downarrow 0} \psi'(y_c^*) = \psi'(\lim_{c\downarrow 0} y_c^*) = \psi'(\hat{x}) > 0$. It remains to establish the limit (4.12). Standard differentiation yields that

$$\frac{\partial V_c}{\partial c}(x) = -\frac{\min(\psi(x), \psi(y_c^*))}{\psi'(y_c^*)(\zeta_c^* - c)}$$

which finally implies (4.12). □

Theorem 4.7 establishes the intuitively clear result that increased transaction costs not only increase the required exercise premium by increasing the optimal threshold at which dividends should be optimally be paid out but it simultaneously increases the size of the optimal dividend policy. An interesting implication of this conclusion is that *increased transaction costs should result into larger but less frequent dividends*. Moreover, we are also able to verify that *the impact of the transaction costs on the value of the optimal policy is dramatic* in the sense that the sensitivity of the value function with respect to changes in the costs becomes unbounded as the transaction costs tend to zero (see, for example, Øksendal 1999 and Øksendal 2000).

5 ILLUSTRATIONS

5.1 Brownian motion with drift

In order to illustrate our results in the case where the lower boundary is regular for the underlying diffusion X_t and, consequently, $\psi'(0+) < \infty$, we now assume that in the absence of interventions the underlying diffusion evolves according to a Brownian motion with drift characterized by the stochastic differential equation

$$dX_t = \mu dt + \sigma dW_t \quad X_0 = x.$$

In this case, $\psi(x) = e^{\kappa x} - e^{\lambda x}$ where

$$\kappa = -\frac{\mu}{\sigma^2} + \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}} > 0$$

and

$$\lambda = -\frac{\mu}{\sigma^2} - \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}} < 0$$

denote the positive and the negative root of the characteristic equation $\sigma^2 b^2 + 2\mu b - 2r = 0$, respectively. In this case the conditions of Lemma 4.2 are satisfied and, therefore, there is a critical cost $\hat{c}$ such that there is a unique optimal pair (ζ_c^*, y_c^*) satisfying the necessary conditions (4.7) and (4.8) whenever $0 < c < \hat{c}$. In this case, the critical threshold at which $\psi''(x)$ vanishes reads as

$$\hat{x} = \frac{1}{(\kappa - \lambda)} \ln\left(\frac{\lambda^2}{\kappa^2}\right).$$

This example is illustrated numerically in Table 1 for various values of the volatility coefficient σ (with $\mu = 0.1$, $r = 0.025$, and $c = 0.1$).

σ	0.1	0.2	0.3	0.4	0.5
$\hat{x}$	0.43	1.13	1.78	2.30	2.68
$\bar{x}_c$	4.15	4.29	4.51	4.77	5.03
y_c^*	1.26	1.93	2.64	3.25	3.74
ζ_c^*	0.99	1.15	1.39	1.64	1.88
$y_c^* - \zeta_c^*$	0.27	0.78	1.25	1.61	1.86

Table 1 The Optimal Thresholds, Intervention Size, and Generic Initial State

Along the lines of previous studies considering the determination of a rational dividend strategy Table 1 clearly indicates that increased volatility increases the required exercise premium in all cases and, therefore, that increased volatility leads to the postponement of dividends. However, it is also worth noticing that our numerical results seem to indicate that increased volatility increases the optimal size of the paid out dividends at a lower rate than it increases the optimal exercise threshold. Thus, our findings show that *increased volatility increases the generic initial state and, therefore, leads to a higher capital requirement in terms of the reserves*. Although this result is intuitively clear, it is of importance since it demonstrates that *the presence of liquidation risk should result into greater capital buffers*. The critical cost $\hat{c}$ at which liquidation becomes optimal is illustrated numerically in Table 2 for various values of the volatility coefficient σ (with $\mu = 0.1$ and $r = 0.025$).

σ	0.1	0.2	0.3	0.4	0.5
$\hat{c}$	13.84	8.94	6.47	4.97	3.96

Table 2 The critical cost $\hat{c}$ as a function of volatility

The results of Table 2 indicate that increased volatility decreases the critical cost $\hat{c}$. Therefore, our numerical results support the intuitively clear result that increased liquidation risk decrease the maximal admissible transaction cost under which the sequential payment of dividends can be sustained.

In order to illustrate the results of Theorem 4.4 as well, we illustrate the values of the optimal dividend policies in Figure 1 and the marginal values of these policies in Figure 2 under the assumption that $\mu = 0.1$, $r = 0.025$, $c = 0.1$, and $\sigma = 0.3$. As was established in Theorem 4.4 we observe from these figures that both $K(x) \geq V_c(x) \geq G_c(x)$ and $K'(x) \geq V_c'(x) \geq G_c'(x)$.

5.2 Logistic diffusion

In order to illustrate our results in the case where the lower boundary is natural (and, therefore, unattainable) for the underlying diffusion X_t we now assume that in the absence of interventions the underlying diffusion evolves according to a logistic diffusion characterized by the stochastic differential equation

$$dX_t = \mu X_t(1 - \gamma X_t)dt + \sigma X_t dW_t \quad X_0 = x.$$

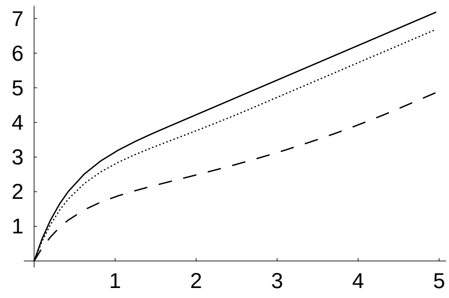

Figure 1: The Value Functions $K(x)$, $V_c(x)$, and $G_c(x)$

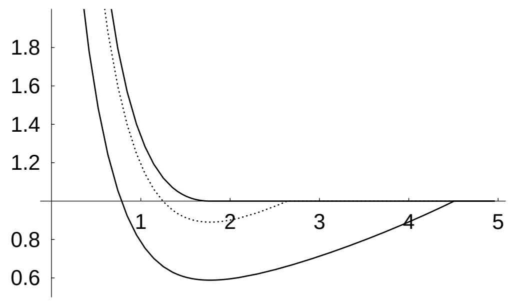

Figure 2: The Marginal Values $K'(x)$, $V_c'(x)$, and $G_c'(x)$

In this case, $\psi(x) = x^\eta M(\eta, 1+\eta-\alpha, 2\mu\gamma x/\sigma^2)$, where M denotes the Kummer confluent hypergeometric function and

$$\eta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 0$$

denotes the positive and

$$\alpha = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} < 0$$

denotes the negative root of the quadratic characteristic equation $\sigma^2 a(a-1) + 2\mu a - 2r = 0$. It is well-known that if $\mu > r$ then $\eta < 1$ and the conditions of our Theorem 4.1 are satisfied. This example is illustrated in Table 3 for the parameter values are $\mu = 0.1$, $r = 0.025$, $c = 1$, and $\gamma = 0.5$.

σ	0.1	0.2	0.3	0.4	0.5
$\hat{x}$	0.8	0.93	1.08	1.22	1.32
$\bar{x}_c$	2.16	2.72	3.31	3.84	4.31
y_c^*	2.05	2.54	3.08	3.62	4.13
ζ_c^*	1.96	2.41	2.90	3.42	3.93
$y_c^* - \zeta_c^*$	0.09	0.13	0.18	0.2	0.2

Table 3 The Optimal Thresholds, Intervention Size, and Generic Initial State

6 CONCLUDING COMMENTS

In this study we considered the determination of the optimal lump-sum stochastic dividend payment policy for a broad class of linear diffusion modelling the random dynamics of the underlying cash reserves. Instead of tackling the stochastic control problem directly via ordinary dynamic programming techniques we first derived an associated mapping depending on both the exercise payoff accrued every time dividends are paid out and on the minimal increasing r-excessive mapping for the diffusion modelling the cash reserves. Having derived this expression, we then presented a set of general conditions under which the existence and uniqueness of an optimal policy is always guaranteed by relying on a combination of stochastic calculus, the classical theory of diffusions, and ordinary nonlinear programming techniques. Interestingly, our results demonstrate that the presence of liquidation risk results into a maximal admissible transaction cost below which the sequential payment of dividends is optimal. Above this critical cost the sequential payment of dividends becomes suboptimal and the optimal dividend problem becomes an optimal liquidation problem where the objective of the corporation is to determine the threshold at which it should be irreversibly liquidated.

We also considered two associated stochastic cash flow management problems and established that these values are ordered in an exceptionally strong way. More precisely, we found that the value of the associated singular stochastic control problem dominates the value of the impulse (lump-sum) control problem which, in turn, dominates the value of the associated optimal stopping problem. However, we also demonstrated that the marginal values (and, therefore, Tobin's q associated with these particular problems) are ordered in the same way. In other words, we found that the marginal value of the associated singular stochastic control problem dominates the marginal value of the impulse (lump-sum) control problem which, in turn, dominates the marginal value of the associated optimal stopping problem. Hence, our results clearly support the economically sensible argument that increased flexibility should increases the value of a rationally managed corporation.

While our results are considerably general, they are based on a model where the cash flow process is exogenous and, therefore, overlooks the capital accumulation dynamics and financing decisions of a corporation. Thus, a natural way to extend our analysis would be to introduce endogenous capital accumulation and financial constraints into the model. Unfortunately, such an extension is out of the scope of the present study and is, therefore, left for future research.

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