RESOLVING SETS AND RESOLVING SEVERAL OBJECTS IN THE FINITE KING GRID

Anni Hakanen

Master's Thesis
August 2017

THE DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF TURKU

UNIVERSITY OF TURKU
The Department of Mathematics and Statistics

HAKANEN, ANNI: Resolving sets and resolving several objects in the finite king grid
Master's thesis, 35 pages
Mathematics
August 2017

An ℓ*-resolving set* of a graph G is a set of vertices that has a unique array of distances to all vertex sets of G with up to ℓ elements. The smallest cardinality of such a set is called the ℓ*-set-metric dimension* of G. In this thesis, we consider mainly the ℓ-set-metric dimension of the finite king grid.

In the first section, we will introduce the problem and its background. We will also give some examples of related topics.

In the second and third sections, we will present the needed definitions and notations regarding the ℓ-resolving sets and the king grid. We will also present examples and some previous results.

In the fourth and fifth sections, we will present our new results concerning the king grid. We will prove exactly what the ℓ-set-metric dimension is for any finite king grid and value of ℓ.

Keywords: Resolving set, metric dimension, set-metric dimension, king grid.

Contents

1 Introduction

Resolving sets are used to locate objects (that is, a subset of vertices) in a graph. The number of objects is restricted, and, of course, cannot be more than the number of vertices of the graph. However, when we are locating the objects, we do not know *exactly* how many there are – we only know that there are at most ℓ objects. If a resolving set can locate uniquely all vertex sets with up to ℓ elements, we call it an ℓ-resolving set of the graph.

A resolving set is a subset of the vertices of the graph. The shortest distance from a vertex of the resolving set to some element of the vertex set we are locating is known for all vertices of the resolving set. An ℓ-resolving set must have a unique set of distances for each subset of vertices with up to ℓ elements.

The research on resolving sets focuses on the smallest possible cardinality of an ℓ-resolving set of a finite graph. This cardinality is called the ℓ-set-metric dimension. The research on 1-resolving sets has been extensive [2, 4, 5, 6, 8, 16, 20]. However, the research on ℓ-resolving sets, where $\ell \geq 2$, is very recent.

The concept of resolving sets was first introduced independently by Slater [21] and Harary and Melter [11]. There are some slightly different variations of ℓ-resolving sets and the ℓ-set-metric dimension. Such are, for example, weighted resolving sets [7], doubly-resolving sets [5], simultaneously resolving sets [19], the strong metric dimension [14], and the k-metric dimension [1]. Resolving sets are also closely related to topics such as identifying codes and locating-dominating sets. Articles on related topics can be found in the list on the web page [17]. Since graphs can be used to model many real-life systems, resolving sets have applications in network discovery and verification [3], chemistry [6], and robot navigation [16] to name a few.

In this thesis, we will find the smallest possible cardinalities of the ℓ-resolving sets of the finite king grid for all values of ℓ. Our approach is somewhat similar as in [15], where the two-dimensional grid graph and the binary hypercube were considered. For research on the king grid in related topics, see for example [9, 13, 18].

The 1-set-metric dimension of the king grid has been previously researched in [2, 20]. The 1-set-metric dimension of a square-shaped king grid was proved to be three in [20]. They also presented a conjecture for the 1-set-metric dimension for rectangular king grids that are not square-shaped. The conjecture was later proved in [2], where they considered strong product graphs. We will present new proofs for the 1-set-metric dimension of the king grid and consider the ℓ-set-metric dimension of the king grid, where $\ell \geq 2$. These results have been published in [10].

2 Resolving sets

In this section, we introduce the ℓ-resolving set and the ℓ-set-metric dimension. We will go through some simple examples and results that clarify the relevant concepts to the reader.

2.1 The ℓ-resolving set of a graph

Consider a connected, finite, and undirected graph G with no loops or multiple edges. Denote by V the set of vertices and by E the set of edges. Let S and X be subsets of V. When we think of S as an ordered set $(s_1,\ s_2,\ \ldots,\ s_{|S|})$, we can form the *distance array*

$$\mathcal{D}_S(X) = (d(s_1, X),\ d(s_2, X),\ \ldots,\ d(s_{|S|}, X)).$$

Here $d(s_i, X) = \min_{x \in X} d(s_i, x)$ is the shortest distance from s_i to some vertex of X. If $X = \{x\}$, we write $\mathcal{D}_S(X) = \mathcal{D}_S(\{x\}) = \mathcal{D}_S(x)$. With the distance array we can try to locate the elements of X.

Definition 2.1. The set S is an *ℓ-resolving set* (or *ℓ-set resolving set*) of $G = (V, E)$, where $\ell \leq |V|$, if for every pair of subsets $X \subseteq V$ and $Y \subseteq V$, with $|X| \leq \ell$ and $|Y| \leq \ell$, we have

$$\mathcal{D}_S(X) \neq \mathcal{D}_S(Y).$$

In other words, an ℓ-resolving set can locate up to ℓ vertices at the same time. Consequently, each ℓ-resolving set must also be a k-resolving set for all $k \in [1, \ell - 1]$ (that is, all integers k such that $1 \leq k \leq \ell - 1$).

Example 2.2. Consider the graph G in Figure 1. The vertices s_1 and s_2 do not form a 1-resolving set of G. We can see this from the distance arrays $\mathcal{D}_{\{s_1,s_2\}}(v)$ that are written next to each vertex $v \in G$ in Figure 1 (a). However, the vertices s_1 and s_3 form a 1-resolving set G, since the distance arrays $\mathcal{D}_{\{s_1,s_3\}}(v)$ in Figure 1 (b) are all unique.

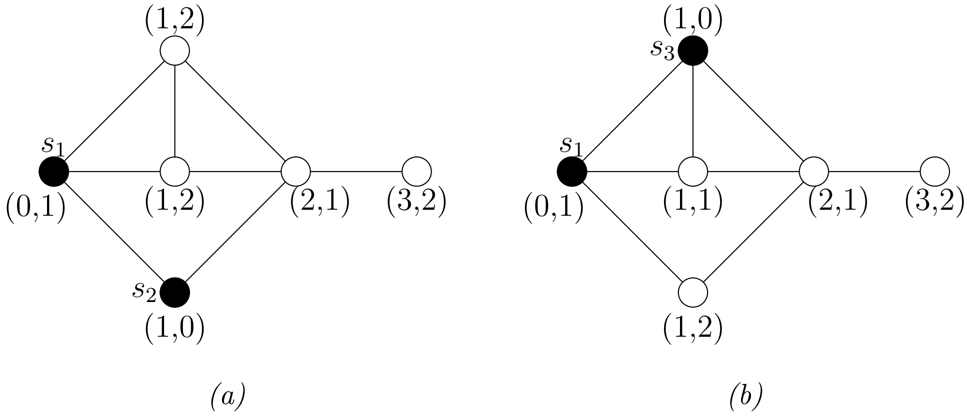


Figure 1: A graph of six vertices with two candidates for a 1-resolving set and the distance arrays.

Example 2.3. Let P_n be a path that has $n \geq 2$ vertices. Denote by u and v the vertices at the ends of P_n. Every vertex of P_n is at a unique distance from u (or v). Therefore both $\{u\}$ and $\{v\}$ are 1-resolving sets of P_n.

It is easy to see that the set $S = \{u, v\}$ is a 2-resolving set of P_n. Let X and Y be two distinct vertex sets of P_n. If both X and Y have only one element, then $\mathcal{D}_S(X) \neq \mathcal{D}_S(Y)$. As we saw earlier, $\{u\}$ is a 1-resolving set, and adding vertices to this set does not change that fact. If $|X| = 1$ and $|Y| = 2$, then $\mathcal{D}_S(X) \neq \mathcal{D}_S(Y)$, since

$$d(u, X) + d(v, X) = d(u, v) > d(u, Y) + d(v, Y).$$

Let $|X| = 2 = |Y|$, and assume that $\mathcal{D}_S(X) = \mathcal{D}_S(Y)$. Since $d(u, X) = d(u, Y)$, the element of X closest to u must be the same as the element of Y closest to u. The same holds for the elements closest to v, and therefore $X = Y$. Thus S is a 2-resolving set of P_n.

2.2 The ℓ-set-metric dimension of a graph

Every graph has an ℓ-resolving set for any $\ell \leq |V|$, since V is always such a set. We can simply check which elements of $\mathcal{D}_V(X)$ are 0 and we have located all elements of X. Therefore the existence of resolving sets is not of interest but the size of them is.

Definition 2.4. The smallest possible cardinality of an ℓ-resolving set of the graph G is called the *ℓ-set-metric dimension* of G. It is denoted by $\beta_\ell(G)$. An ℓ-resolving set that is of cardinality $\beta_\ell(G)$ is called an *ℓ-set-metric-basis* of G.

Example 2.5. The graph in Figure 1 cannot have a 1-resolving set that consists of only one vertex. Indeed, the longest possible distance between two vertices of G is 3. Therefore, we would have only four possible distance arrays but the graph has six vertices.

As we saw in Example 2.2, the graph G has a 1-resolving set that consists of two vertices, and therefore $\beta_1(G) = 2$.

Example 2.6. In Example 2.3 we saw that a path has a 1-resolving set that consists of only one vertex. Thus $\beta_1(P_n) = 1$.

There has been a lot of research on the 1-set-metric dimension of different graphs. However, the research on ℓ-resolving sets and ℓ-set-metric dimensions where $\ell \geq 2$ is relatively new.

The next theorem was proved by contradiction in [16] and by using the diameter of graphs in [6]. We give a more straightforward proof that uses induction.

Theorem 2.7. A graph G has $\beta_1(G) = 1$ if and only if G is a path.

Proof. We already proved in Example 2.6 that the 1-set-metric dimension of a path is 1.

Assume that $G = (V, E)$ is a graph with $\beta_1(G) = 1$, and let $S = \{u_1\}$ be a 1-set-metric basis of G. We will prove that G must be a path.

All vertices v that are neighbours of u_1 have the distance array $\mathcal{D}_S(v) = (1)$. Since S is a 1-resolving set, then all distance arrays must be unique. Therefore u_1 can have only one neighbour, say u_2.

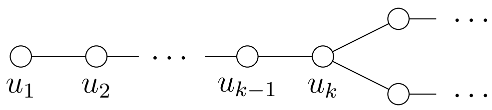


Figure 2: The graph G and the vertex set K in the proof of Theorem 2.7.

Let u_k be a cut vertex of G (that is, if we remove u_k, then the number of connected components of G increases) such that the vertices of $K = \{u_1, u_2, \ldots, u_k\}$ form a path as an induced subgraph of G. Now u_k must be included in all the shortest paths between u_1 and any vertex v of $V \setminus K$. Thus $d(u_1, v) = d(u_1, u_k) + d(u_k, v)$. If u_k has three or more neighbours, then the two neighbours other than u_{k-1} have the same distance array (see Figure 2). Therefore u_k can have only two neighbours: u_{k-1} and u_{k+1}. If u_{k+1} has at least two neighbours, it is a cut vertex of G similar to u_k. If u_{k+1} has only one neighbour, then G is a path of $k + 1$ vertices. □

Theorem 2.8. Let P_n be a path of $n \geq 2$ vertices. Then $\beta_2(P_n) = 2$ and $\beta_\ell(P_n) = n$ for $\ell \geq 3$.

Proof. A path cannot have a 2-resolving set that has only one element, since there would be n possible distance arrays but $\frac{1}{2}n(n+1) > n$ distinct vertex sets to resolve. In Example 2.3 it was shown that a path has a 2-resolving set of two vertices. Thus $\beta_2(P_n) = 2$.

Let u be a vertex at either end of P_n, i.e. u has only one neighbour v (see Figure 3 (a)). Now u must be in any ℓ-resolving set of P_n where $\ell \geq 2$. Indeed, assume to the contrary that S is a 2-resolving set that does not contain u. Since $d(s, u) > d(s, v)$ for all $s \in S$, then $\mathcal{D}_S(\{u, v\}) = \mathcal{D}_S(v)$. Therefore $u \in S$. Since any 3-resolving set must also be a 2-resolving set, the vertices at the ends of the path must be in every 3-resolving set of P_n.

Consider then a vertex u that is not at the end of P_n. Let the two neighbours of u be v and w (see Figure 3 (b)). Let S be a 3-resolving set of P_n that does not contain u. Now for each $s \in S$ either $d(s, v)$ or $d(s, w)$ is smaller than $d(s, u)$. Therefore $\mathcal{D}_S(\{u, v, w\}) = \mathcal{D}_S(\{v, w\})$. Thus all vertices

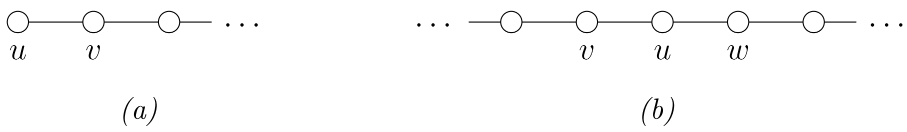


Figure 3

of P_n must be in any 3-resolving set, and $\beta_\ell(P_n) = n$ for all $\ell \geq 3$. □

Example 2.9. Consider C_5, a cycle that consists of the vertices c_i, where $i = 1, \ldots, 5$. From the distance arrays in Figure 4 we can see that the set $\{c_1, c_2\}$ is a 1-resolving set of C_5. In fact, any vertex set of C_5 that has at least two elements is a 1-resolving set of C_5. Now $\beta_1(C_5) = 2$, since according to Theorem 2.7 the 1-set-metric dimension of C_5 cannot be one.

The distances between a vertex $c \in C_5$ and any subset of vertices that has at most two elements are listed in Table 1. Of course, the set $V = \{c_1, \ldots, c_5\}$ is a 2-resolving set of C_5. If we cover one column on the right side of Table 1, we see that all subsets of four vertices are also 2-resolving sets of C_5. However, there are no 2-resolving sets with only three elements. We can see this by considering the sets $S_1 = \{c_1, c_2, c_3\}$ and $S_2 = \{c_1, c_2, c_4\}$ (the other cases are isomorphic to one of these). Since

$$\mathcal{D}_{S_1}(c_2) = \mathcal{D}_{S_1}(\{c_2, c_4\}) = \mathcal{D}_{S_1}(\{c_2, c_5\}) = (1, 0, 1)$$

and

$$\mathcal{D}_{S_2}(\{c_1, c_3\}) = \mathcal{D}_{S_2}(\{c_1, c_5\}) = (0, 1, 1),$$

neither S_1 nor S_2 is a 2-resolving set of C_5. Thus $\beta_2(C_5) = 4$.

If $\ell \geq 3$, then $\beta_\ell(C_5) = 5$. Otherwise there is at least one vertex u that is not in the ℓ-resolving set. As in the proof of Theorem 2.8, the set that consists of the two neighbours of u has the same distance array as the set that consists of u and its neighbours.

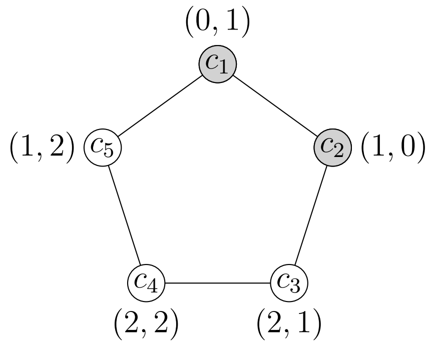


Figure 4: The set $S = \{c_1, c_2\}$ is a 1-resolving set of C_5. The distance array $\mathcal{D}_S(c_i)$ is written next to each vertex $c_i \in C_5$.

$c \in X$					$d(c, X)$				
c_1	c_2	c_3	c_4	c_5	c_1	c_2	c_3	c_4	c_5
x					0	1	2	2	1
x	x				0	0	1	2	1
x		x			0	1	0	1	1
x			x		0	1	1	0	1
x				x	0	1	2	1	0
	x				1	0	1	2	2
	x	x			1	0	0	1	2
	x		x		1	0	1	0	1
	x			x	1	0	1	1	0
		x			2	1	0	1	2
		x	x		2	1	0	0	1
		x		x	1	1	0	1	0
			x		2	2	1	0	1
			x	x	1	2	1	0	0
				x	1	2	2	1	0

Table 1: The distances from a vertex $c \in C_5$ to all subsets of vertices that have at most two elements. The x on the left indicates that the corresponding vertex is in X.

2.3 Some known results

In this section, we will present some previous results on the ℓ-set-metric dimensions of graphs without proofs.

Chartrand *et al.* proved in [6] that a connected graph G on n vertices has 1-set-metric dimension $n-1$ if and only if G is a complete graph on n vertices. They also gave characterisations for all n-vertex graphs with $\beta_1(G) = n-2$.

Let the *diameter* of a graph G be the greatest distance between a pair of vertices. We denote the diameter of G by D_G. Khuller *et al.* [16] and Chartrand *et al.* [6] proved independently that $|G| \leq D_G^{\beta_1(G)} + \beta_1(G)$. However, this bound is achievable only when $D_G \leq 3$ or $\beta_1(G) = 1$. Hernando *et al.* proved a much tighter bound in [12].

There are many formulae for the 1-set-metric dimension of a tree [6, 11, 16, 21]. For instance, Khuller *et al.* [16] proved a formula that counts the legs of vertices. A *leg* of vertex v is a bridge that is a part of a path that begins from v. We denote by ℓ_v the number of legs of v. Let $T = (V, E)$ be a tree that is not a path. Then

$$\beta_1(T) = \sum_{v \in V : \ell_v > 1} (\ell_v - 1).$$

Khuller *et al.* also proved that $\beta_1(G) = d$ for a d-dimensional grid graph G.

The two-dimensional grid graph has recently been studied by Laihonen in [15]. There it was proved that the 2-set-metric dimension of a two-dimensional $m \times n$ grid graph is $\min\{m, n\} + 2$. It was also proved that the ℓ-set-metric dimension of said graph is mn for all $\ell \geq 3$. The ℓ-set-metric dimensions of the d-dimensional binary hypercube were also considered in the same article.

3 The king grid

This thesis focuses on the king grid. In this section we will define it as a strong product of two paths. We will also introduce some notations that are used later.

If two vertices x and y are adjacent, we denote $x \sim y$. Let us denote the *strong product* of two graphs $G = (V, E)$ and $H = (V', E')$ by $G \boxtimes H$. The vertex set of $G \boxtimes H$ is the Cartesian product $V \times V'$. There is an edge between (u_1, v_1) and (u_2, v_2) if one of the three following conditions hold:

1. $u_1 = u_2$ and $v_1 \sim v_2$,

2. $u_1 \sim u_2$ and $v_1 = v_2$,

3. $u_1 \sim u_2$ and $v_1 \sim v_2$.

See Figure 5 for demonstration.

A strong product of two paths $P_m \boxtimes P_n$ is called the $m \times n$ *king grid* (or the *king's graph*). As we can see from Figure 5 (d), the king grid is basically a two-dimensional grid graph with diagonal edges in addition to vertical and horizontal ones. If each vertex represents a square on the chess board, the edges correspond with the legal moves of the king.

The vertices of a king grid can be considered as $\mathbb{N} \times \mathbb{N}$ lattice points. We can give each vertex two coordinates and write the set of vertices of an $m \times n$ king grid as $\{(i, j) \mid i = 1, \ldots, m,\ j = 1, \ldots, n\}$. Now the distance between the vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ is

$$d(u, v) = \max\{|u_1 - v_1|,\ |u_2 - v_2|\}.$$

We denote by $S_r(u) = \{v \in V \mid d(u, v) = r\}$ the set of vertices that are at the distance r from the vertex u (see Figure 6). Note that if $r \neq r'$, then $S_r(u) \cap S_{r'}(u) = \emptyset$.

To ease notations, we define the ith *column* for $i \in [1, m]$ as

$$C_i = \{(i, j) \mid j = 1, \ldots, n\}.$$

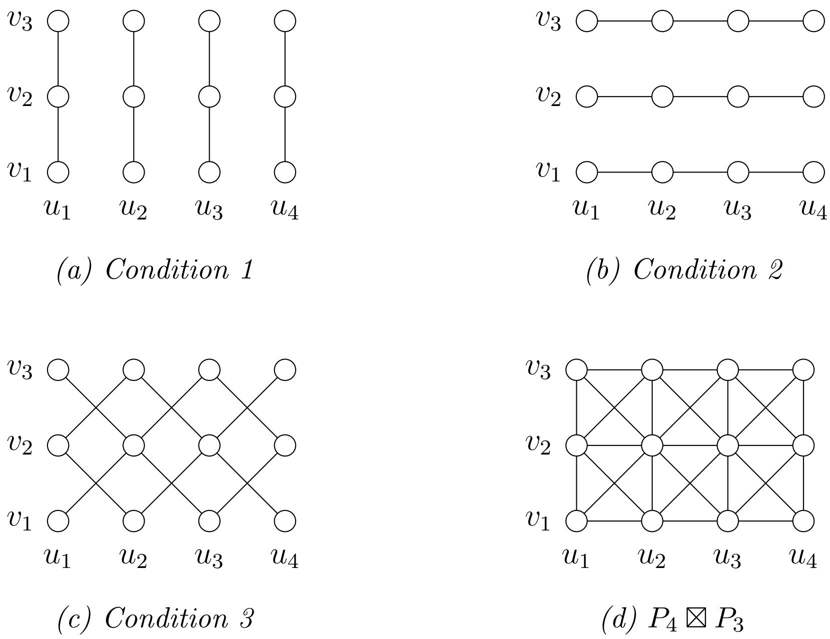


Figure 5: The construction of the strong product of two paths P_4 and P_3.

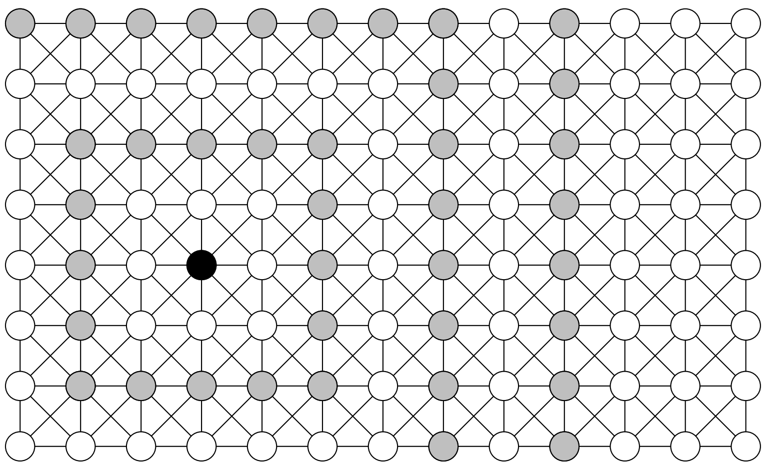


Figure 6: The king grid $P_{13} \boxtimes P_8$ with the vertex $u = (4,4)$ and the sets $S_2(u)$, $S_4(u)$, and $S_6(u)$.

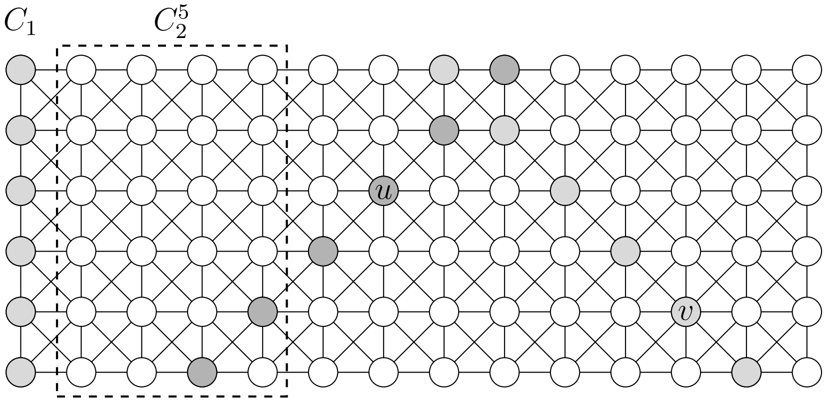


Figure 7: The king grid $P_{14} \boxtimes P_6$ and the column C_1, the four column section C_2^5, vertices $u = (7, 4)$ and $v = (12, 2)$, and the diagonals $L^+(u)$ and $L^-(v)$.

A *section* is the union of consecutive columns:

$$C_i^j = \bigcup_{k=i}^{j} C_k.$$

For illustration, see Figure 7.

The *diagonals* through the vertex $x = (a, b)$ with slope $+1$ and -1 are

$$L^+(x) = \{(a + i, b + i) \mid i \in \mathbb{Z}\}$$

and

$$L^-(x) = \{(a + i, b - i) \mid i \in \mathbb{Z}\}$$

respectively (see Figure 7).

4 Resolving one object in the king grid

The 1-set-metric dimension of the king grid has previously been studied by Rodríguez-Velázquez *et al.* [20] and Barragán-Ramírez and Rodríguez-Velázquez [2]. The 1-set-metric dimension of a square-shaped king grid $P_m \boxtimes P_m$ was proved to be 3 in [20]. They also conjectured that $\beta_1(P_m \boxtimes P_n) = \lceil \frac{m+n-2}{n-1} \rceil$ when $n < m$. This was later proved in [2]. We will present a new and more direct proof for this conjecture in Section 4.2. For completeness, we will first consider the square-shaped king grid.

4.1 Square-shaped king grid

Theorem 4.1. Let $P_m \boxtimes P_m$ be an $m \times m$ king grid with $2 \leq m$. Then

$$\beta_1(P_m \boxtimes P_m) = 3.$$

Proof. The greatest distance between any two vertices is $m - 1$. Therefore each element of $\mathcal{D}_S(X)$ has m possible values. If $|S| = k$, then there are m^k possible distance arrays. Since $\ell = 1$ no distance array can have more than one zero. Since there are only $m^2 - 1$ acceptable distance arrays of length two but m^2 vertices, we have $\beta_1(P_m \boxtimes P_m) \geq 3$.

Let S be a set consisting of three corner vertices of $P_m \boxtimes P_m$. We will show that S is a 1-resolving set.

Without loss of generality, we can assume that

$$S = \{(1,1),(1,m),(m,m)\}.$$

Since $(m,m) \in L^+((1,1))$, there can be at most two vertices that are at the same distance from $(1,1)$ and (m,m) both. In other words, $|S_r((1,1)) \cap S_t((m,m))| \leq 2$ for any r and t. Moreover, if $S_r((1,1)) \cap S_t((m,m)) = \{u, v \mid u \neq v\}$, the vertices u and v are in the same diagonal $L^-(u) = L^-(v)$ (see Figure 8). For any vertex w and integer x the intersection of $L^-(w)$ and $S_x((1,m))$ has at most one element. Therefore $d((1,m),u) \neq d((1,m),v)$ and $\mathcal{D}_S(u) \neq \mathcal{D}_S(v)$.

$\square$

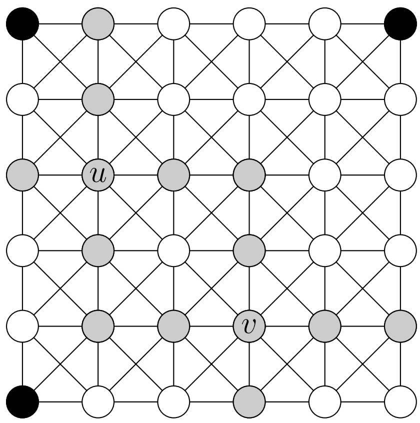


Figure 8: The black vertices form $S = \{(1,1), (1,m), (m,m)\}$ when $m = 6$.

4.2 Non-square rectangular king grid

It is easier to understand what the 1-resolving sets and distance arrays must be like when we consider the king grid in smaller sections. First we will take a look at what a 1-resolving set must be like for it to resolve the vertices at the left or right end of the graph.

Theorem 4.2. Let $P_m \boxtimes P_n$ be an $m \times n$ king grid with $2 \leq n < m$ and let S be a 1-resolving set of $P_m \boxtimes P_n$. Then the sections of n columns C_1^n and $C_{m-(n-1)}^m$ at either end of the graph must contain at least two elements of S.

Proof. Consider the section C_1^n (see Figure 9). Assume that $|C_1^n \cap S| = 0$. Let $v \in C_{n+1}^m$. Now $d(v, u_1) = d(v, u_2)$ for any distinct $u_1,\ u_2 \in C_1$. Therefore $\mathcal{D}_S(u_1) = \mathcal{D}_S(u_2)$ and S cannot be a 1-resolving set.

Assume then that $s \in C_1^n \cap S$. Like we saw in the previous case, the vertices of C_1 cannot be distinguished from each other with the vertices of C_{n+1}^m. In fact, the same also holds for C_2. No matter where s is located in C_1^n there are always at least two vertices in C_1 or C_2 that are at the same distance from s. Therefore if $|C_1^n \cap S| = 1$, then S cannot be a 1-resolving set of $P_m \boxtimes P_n$. $\square$

Corollary 4.3. Let $P_m \boxtimes P_n$ be an $m \times n$ king grid with $2 \leq n < m$ and let S be a 1-resolving set of $P_m \boxtimes P_n$. If $|C_1^n \cap S| = 2$ (or $|C_{m-(n-1)}^m \cap S| = 2$),

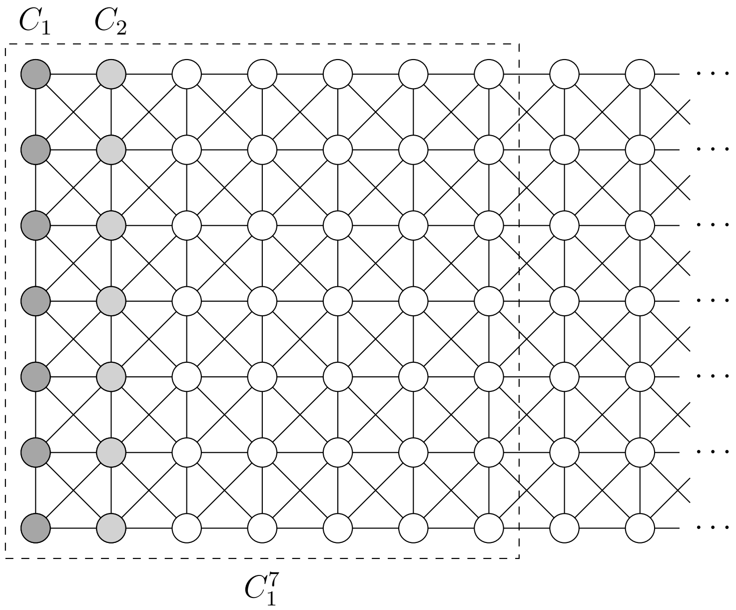


Figure 9: The left end of $P_m \boxtimes P_7$ and the n column section C_1^7.

there must be at least one element of S somewhere else in the graph.

Proof. The section C_1^n can be considered as an $n \times n$ king grid. According to Theorem 4.1, a 1-resolving set of $P_n \boxtimes P_n$ has at least three elements. $\square$

Theorem 4.4. Let $P_m \boxtimes P_n$ be an $m \times n$ king grid with $2 \leq n < m < 2n$. Then

$$\beta_1(P_m \boxtimes P_n) = 3.$$

Proof. The lower limit $\beta_1(P_m \boxtimes P_n) \geq 3$ follows immediately from Corollary 4.3.

We will show that $S = \{(1,1), (n,n), (m,1)\}$ is a 1-resolving set of $P_m \boxtimes P_n$ (see Figure 10). Assume that there are two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ such that $\mathcal{D}_S(u) = \mathcal{D}_S(v)$.

If $u_1 \leq n$ and $v_1 > n$, then $d((1,1), u) \neq d((1,1), v)$. Therefore both u and v are in either C_1^n or C_{n+1}^m.

If u and v are in C_1^n, then like in the proof of Theorem 4.1 they must be at the same diagonal $L^-(u) = L^-(v)$. Now $v = (u_1 + t, u_2 - t)$ for some

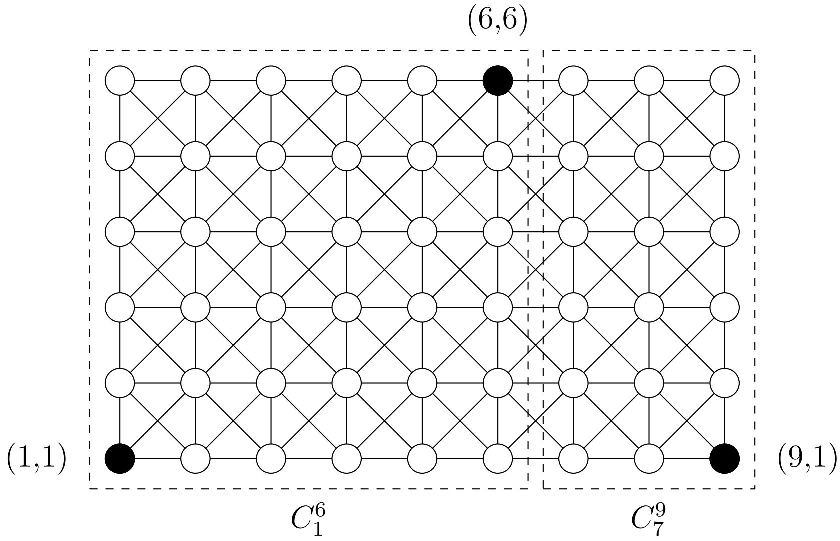


Figure 10: The vertices $(1,1)$*,* $(6,6)$*, and* $(9,1)$ *form a 1-set-metric basis of* $P_9 \boxtimes P_6$*.*

integer $t \neq 0$ and

$$d(v,(m,1)) = \max\{m - u_1 - t, u_2 - t - 1\}.$$

Notice that

$$d(u,(m,1)) = \max\{m - u_1, u_2 - 1\}.$$

If $d(u,(m,1)) = m - u_1$, then $d(v,(m,1)) = m - u_1 - t$. If $d(u,(m,1)) = u_2 - 1$, then $d(v,(m,1)) = u_2 - t - 1$. Since $t \neq 0$, $d(u,(m,1)) \neq d(v,(m,1))$ and therefore $\mathcal{D}_S(u) \neq \mathcal{D}_S(v)$.

Assume then that both u and v are in C_{n+1}^m (C_7^9 in Figure 10). The distance from $(1,1)$ to u and v both is at least n. Therefore $d(u,(1,1)) = d(v,(1,1))$ if and only if u and v are in the same column C_i for some $i \in [n+1,m]$. Without loss of generality we can assume that $u_2 < v_2$. Now

$$d(u,(n,n)) = \max\{i - n,\ n - u_2\},$$
$$d(v,(n,n)) = \max\{i - n,\ n - v_2\},$$
$$d(u,(m,1)) = \max\{m - i,\ u_2 - 1\},$$

and

$$d(v,(m,1)) = \max\{m-i,\ v_2-1\}.$$

Clearly $d(u,(n,n)) = d(v,(n,n))$ if and only if $u_2 \geq 2n-i$ since $u_2 < v_2$. Similarly $d(u,(m,1)) = d(v,(m,1))$ if and only if $v_2 \leq m-i+1$. Therefore if $\mathcal{D}_S(u) = \mathcal{D}_S(v)$, then

$$2n-i+1 \leq u_2+1 \leq v_2 \leq m-i+1 \leq 2n-i,$$

since $m < 2n$.

Since $\mathcal{D}_S(u) \neq \mathcal{D}_S(v)$ for all u and v in $P_m \boxtimes P_n$ such that $u \neq v$, S is a 1-set-metric basis of $P_m \boxtimes P_n$. $\square$

To calculate the lower limit for the 1-set-metric dimension of $P_m \boxtimes P_n$, where $m \geq 2n$, we need to consider the $n-1$ column sections that do not contain C_1 or C_m.

Theorem 4.5. Let $P_m \boxtimes P_n$ be an $m \times n$ king grid with n even and $2 \leq n < m$. If for a set $X \subseteq V$ the intersection $X \cap C_i^{i+n-2}$ is empty for any $i \in [2, m-n+1]$, then X cannot be a 1-resolving set of $P_m \boxtimes P_n$.

Proof. Assume that $|X \cap C_i^{i+n-2}| = 0$. Consider the vertices $u = (i+\frac{n-2}{2}, \frac{n}{2})$ and $v = (i+\frac{n-2}{2}, \frac{n}{2}+1)$ (see Figure 11). Let $x = (a,b) \in X$. Now

$$d(x,u) = \max\{\left|i+\tfrac{n-2}{2}-a\right|,\ \left|\tfrac{n}{2}-b\right|\} \text{ and}$$
$$d(x,v) = \max\{\left|i+\tfrac{n-2}{2}-a\right|,\ \left|\tfrac{n}{2}+1-b\right|\}.$$

The distances $d(x,u)$ and $d(x,v)$ are different if and only if $|i+\frac{n-2}{2}-a| < |\frac{n}{2}-b|$ or $|i+\frac{n-2}{2}-a| < |\frac{n}{2}+1-b|$. Without loss of generality, we can assume that $x \in C_1^{i-1}$. Since $a \in [1, i-1]$,

$$\left|i+\tfrac{n-2}{2}-a\right| \geq \left|i+\tfrac{n-2}{2}-i+1\right| = \tfrac{n}{2}.$$

However, $|\frac{n}{2}-b|$ and $|\frac{n}{2}+1-b|$ are both at most $\frac{n}{2}$. Therefore $d(x,u) = d(x,v)$ for all $x \in X$ and $\mathcal{D}_X(u) = \mathcal{D}_X(v)$. $\square$

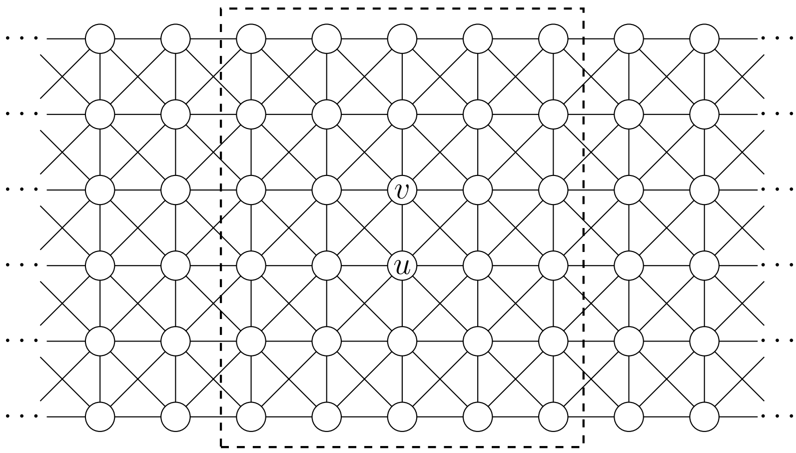


Figure 11: $P_m \boxtimes P_6$ with an $n-1$ column section C_i^{i+4}. The vertices that are not in C_i^{i+4} cannot distinguish between u and v.

Theorem 4.6. Let $P_m \boxtimes P_n$ be an $m \times n$ king grid with n odd and $2 \leq n < m$ and let S be a 1-resolving set of $P_m \boxtimes P_n$. If the intersection $S \cap C_i^{i+n-2}$ is empty for some $i \in [2, m-n+1]$, then

$$|S \cap C_{i-1}^{i+n-1}| \geq 4.$$

Proof. Assume S and i are such that $|S \cap C_i^{i+n-2}| = 0$. Consider the vertices $u = (i + \lfloor \frac{n-2}{2} \rfloor, \frac{n-1}{2})$ and $v = (i + \lfloor \frac{n-2}{2} \rfloor, \frac{n-1}{2} + 1)$ (see Figure 12). Like in the proof of Theorem 4.5 the vertices of the sections C_{i+n}^m and C_1^{i-2} are too far away to distinguish u and v from each other. Consider the vertices of C_{i-1}. The vertex $w = (i-1, n)$ can distinguish u and v since $d(w, u) = \lfloor \frac{n-2}{2} \rfloor + 2$ and $d(w, v) = \lfloor \frac{n-2}{2} \rfloor + 1$. For any other vertex $w' \in C_{i-1}$ other than w, $d(w', u) = d(w', v) = \lfloor \frac{n-2}{2} \rfloor + 1$. Therefore $\mathcal{D}_S(u) \neq \mathcal{D}_S(v)$ only if $w \in S$.

Similarly, if we consider the vertices $u = (i + \lfloor \frac{n-2}{2} \rfloor, \frac{n-1}{2} + 1)$ and $v = (i + \lfloor \frac{n-2}{2} \rfloor, \frac{n-1}{2} + 2)$, we can show that $(i-1, 1) \in S$, since otherwise $\mathcal{D}_S(u) = \mathcal{D}_S(v)$. Therefore the column C_{i-1} must contain at least two elements of S.

The same also holds for C_{i+n-1}. By considering the vertices $(i + \lfloor \frac{n-2}{2} \rfloor + 1, \frac{n-1}{2})$ and $(i + \lfloor \frac{n-2}{2} \rfloor + 1, \frac{n-1}{2} + 1)$ we can show that $(i+n-1, n) \in S$. Similarly $(i+n-1, 1) \in S$, since otherwise the vertices $(i + \lfloor \frac{n-2}{2} \rfloor + 1, \frac{n-1}{2} + 1)$ and

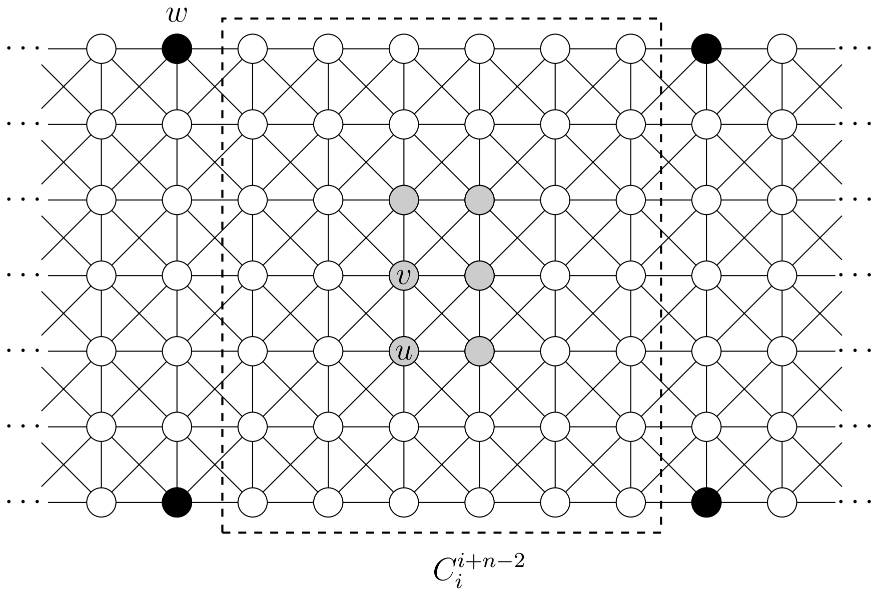


Figure 12: In order to resolve the gray vertices, the black vertices must be included in the 1-resolving set.

$(i + \lfloor \frac{n-2}{2} \rfloor + 1, \frac{n-1}{2} + 2)$ would have the same distance array. □

Next we make an observation of consecutive sections of $n-1$ columns.

If C_i^{i+n-2} contains exactly one element of S, then the $n-1$ column sections at either side of C_i^{i+n-2} must both contain at least one element of S. Otherwise, if n is odd, the section C_i^{i+n-2} would contain more than one element of S according to Theorem 4.6. If n is even, then according to Theorem 4.5 each $n-1$ column section must contain at least one element of S.

Now we have all the necessary tools to obtain the lower bound.

Theorem 4.7. Let $P_m \boxtimes P_n$ be an $m \times n$ king grid with $2 \leq n < m$. Then

$$\beta_1(P_m \boxtimes P_n) \geq \left\lceil \frac{m+n-2}{n-1} \right\rceil .$$

Proof. According to Theorem 4.4, $\beta_1(P_m \boxtimes P_n) = 3$. This proves the claim for $m < 2n$, since

$$\left\lceil \frac{m+n-2}{n-1} \right\rceil \leq \left\lceil \frac{2n-1+n-2}{n-1} \right\rceil = 3$$

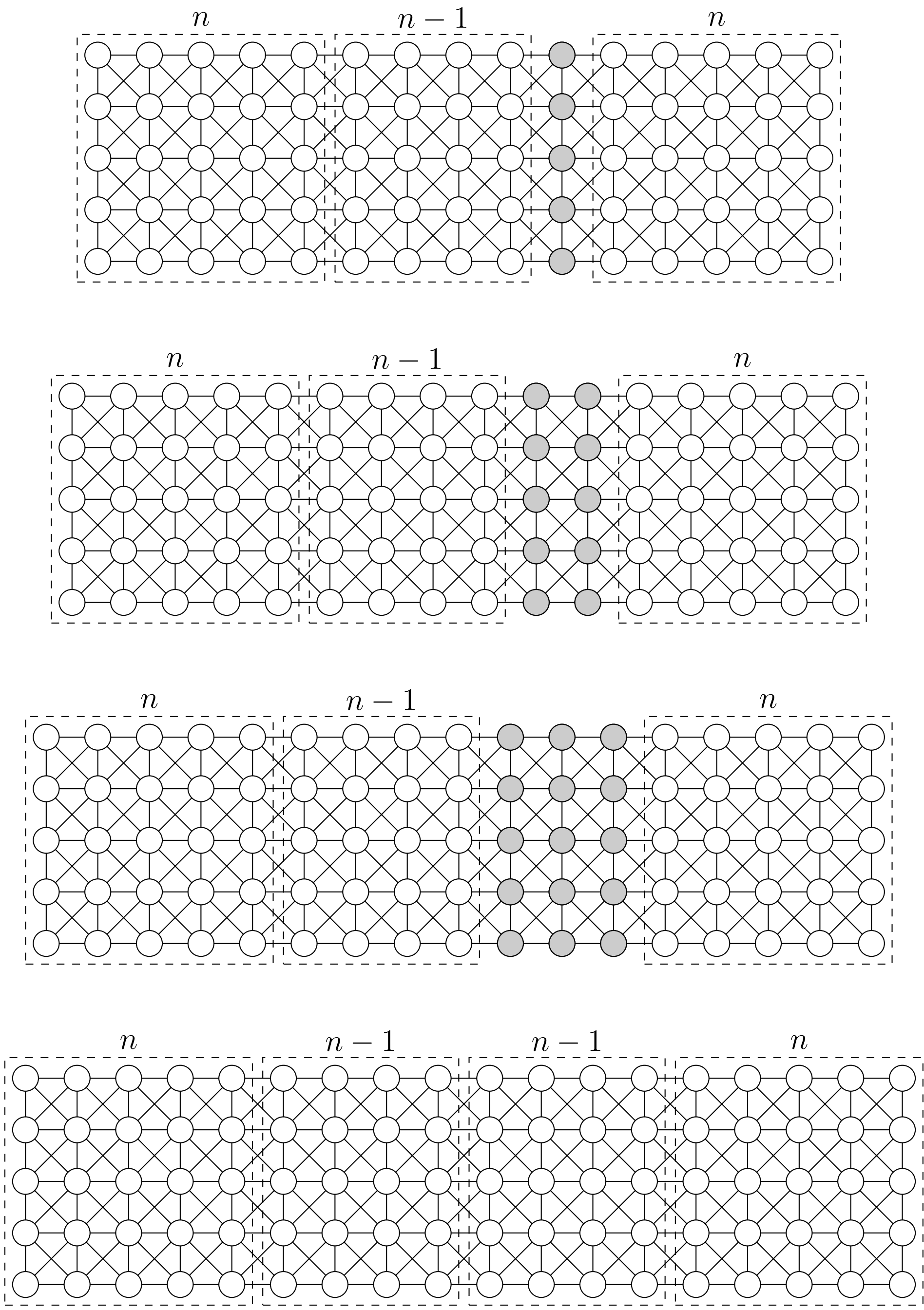


Figure 13: The partitions of $P_m \boxtimes P_5$ into sections of n and $n-1$ columns, when $m = 15, \ldots, 18$.

and
$$\left\lceil \frac{m+n-2}{n-1} \right\rceil \geq \left\lceil \frac{n+1+n-2}{n-1} \right\rceil = \left\lceil \frac{n}{n-1} \right\rceil + 1 = 3.$$

When $m \geq 2n$, we prove the claim by partitioning the graph into sections of n and $n-1$ columns, see Figure 13. Assume S is a 1-resolving set of $P_m \boxtimes P_n$. According to Theorem 4.2, the sections of n columns at either end of the graph must both contain at least two elements of S. Therefore $\beta_1(P_m \boxtimes P_n) \geq 4$.

Consider a section of $n-1$ columns C_i^{i+n-2} with $i \in [n+1, m-2(n-1)]$, i.e. C_i^{i+n-2} does not intersect with the n column sections at the ends of the graph. If n is even, then according to Theorem 4.5 the section C_i^{i+n-2} must contain at least one element of S. If n is odd and the section C_i^{i+n-2} does not have any elements of S, then according to Theorem 4.6 the $n-1$ column sections at either side of C_i^{i+n-2}, denoted by C' and C'', both contain at least two elements of S. If also the $n-1$ column section at the other side of C' (or C'') does not contain an element of S, then C' (or C'') must contain at least four elements of S. Therefore in this case each $n-1$ column section must contain on average at least $\frac{4}{3}$ elements of S. However, if the section C_i^{i+n-2} contains one element of S, we can do with only three vertices in three consecutive sections of $n-1$ columns. To obtain the lower limit we count that each complete section of $n-1$ columns contains at least one element of S.

Now
$$\beta_1(P_m \boxtimes P_n) \geq 4 + \left\lfloor \frac{m-2n}{n-1} \right\rfloor = 3 + \left\lceil \frac{m-2n+1}{n-1} \right\rceil = \left\lceil \frac{m+n-2}{n-1} \right\rceil.$$

□

Next we will construct a 1-set-metric basis of $P_m \boxtimes P_n$ (see also [20]). This was already done for $m < 2n$ in the proof of Theorem 4.4. Therefore, we assume that $m \geq 2n$. Let

$$T = \{(2i(n-1)+1, 1) \mid i = 0, \ldots, \lceil \tfrac{m-1}{2(n-1)} \rceil - 1\},$$
$$U = \{((2j+1)(n-1)+1, 1) \mid j = 0, \ldots, \lceil \tfrac{m-n}{2(n-1)} \rceil - 1\},$$

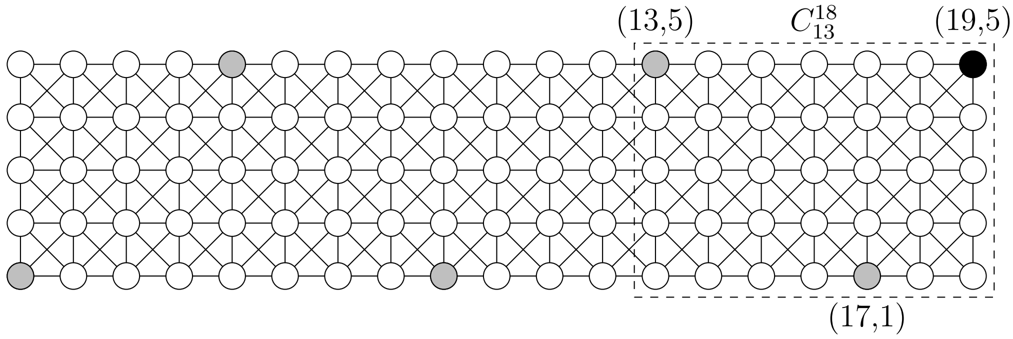


Figure 14: The gray vertices form the set S' for $P_{19} \boxtimes P_5$. The vertices $(13,5)$, $(17,1)$, and $(19,5)$ can resolve the vertices of C_{13}^{18} according to Theorem 4.4. The set $S' \cup \{(19,5)\}$ is therefore a 1-set-metric basis of $P_{19} \boxtimes P_5$.

and

$$S' = T \cup U.$$

For illustration, see Figure 14. The shortest distance between any two vertices of S' is $n-1$ and for any vertex of S' there are at most two such vertices. The elements of S' are of the form $(i(n-1), j)$, where $i = 0, \ldots, \lceil \frac{m-1}{n-1} \rceil - 1$ and $j \in \{1, n\}$. Now $|S'| = \lceil \frac{m-1}{n-1} \rceil$, since all elements of S' are in different columns.

Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be any two vertices of S' such that $d(u,v) = n-1$. Now u and v are in the same diagonal, i.e. either $L^+(u) = L^+(v)$ or $L^-(u) = L^-(v)$. Without loss of generality we can assume that $u_1 < v_1$. Let a and b be two distinct vertices of $C_{u_1}^{v_1}$. We will show that $\mathcal{D}_{S'}(a) \neq \mathcal{D}_{S'}(b)$. If $d(u,a) \neq d(u,b)$ or $d(v,a) \neq d(v,b)$, then we are done. Assume that $d(u,a) = d(u,b)$ and $d(v,a) = d(v,b)$. Now a and b must be in the same diagonal. Because $m \geq 2n$, S' has at least three elements. Let $w \in S' \setminus \{u,v\}$. Since $d(w,u) \geq n-1$ and $d(w,v) \geq n-1$, the distance between w and any vertex of $C_{u_1}^{v_1}$ is also at least $n-1$. Therefore the intersection of $C_{u_1}^{v_1}$ and $S_t(w)$ is either empty or a column of $C_{u_1}^{v_1}$. Since $a \neq b$, they are in different columns and therefore at different distances from w. Now $\mathcal{D}_{S'}(a) \neq \mathcal{D}_{S'}(b)$ and we can resolve all vertices of $C_{u_1}^{v_1}$ with the vertices of S'.

However, the vertices of S' cannot distinguish between all vertices of the at most $n-1$ columns without vertices of S' at the right end of the graph. Let $x=(x_1,x_2)$ and $y=(y_1,y_2)$ be the two rightmost vertices of S' with $x_1<y_1$. The section $C^m_{x_1}$ can be considered as a $P_k \boxtimes P_n$ king grid, where $k=m-x_1+1<2n$. According to the proof of Theorem 4.4 the set $\{x,y,z\}$, where $z=(m,1)$ if $y_2=n$ or $z=(m,n)$ if $y_2=1$, is a 1-resolving set of $P_k \boxtimes P_n$.

Therefore the set $S=S' \cup \{z\}$ is a 1-resolving set of $P_m \boxtimes P_n$. Since

$$|S| = \left\lceil \frac{m-1}{n-1} \right\rceil + 1 = \left\lceil \frac{m+n-2}{n-1} \right\rceil,$$

S is a 1-set-metric basis of $P_m \boxtimes P_n$, and the next result is immediate.

Theorem 4.8. Let $P_m \boxtimes P_n$ be an $m \times n$ king grid with $2 \le n < m$. Then

$$\beta_1(P_m \boxtimes P_n) = \left\lceil \frac{m+n-2}{n-1} \right\rceil.$$

5 Resolving several objects in the king grid

In this section we present our new results concerning the ℓ-set-metric dimension of the king grid when $\ell \geq 2$.

5.1 Two objects

When $\ell = 2$, the vertices at the frame of the king grid can "hide" behind its neighbour closer to the center. Therefore all vertices at the frame of the grid must be in any 2-resolving set, and it turns out that this condition is sufficient.

Theorem 5.1. Let $P_m \boxtimes P_n$ be an $m \times n$ king grid with $2 \leq n \leq m$. Then

$$\beta_2(P_m \boxtimes P_n) = 2m + 2n - 4.$$

Proof. Let $u = (u_1,\ u_2)$ be a vertex at the frame of the graph i.e. $u_1 \in \{1, m\}$ or $u_2 \in \{1, n\}$.

Assume that $u_2 = 1$ and let $u' = (u_1,\ 2)$ (see Figure 15). We will show that there is no vertex closer to u than u' other than u itself. Let $v = (v_1, v_2) \neq u$. Now $d(u, v) = \max\{|u_1 - v_1|,\ |1 - v_2|\}$ and $d(u', v) = \max\{|u_1 - v_1|,\ |2 - v_2|\}$. If $v_2 = 1$, then $|u_1 - v_1| \geq 1$, since $v \neq u$. Now $d(u, v) = d(u', v)$. If $v_2 \geq 2$, then $|2 - v_2| < |1 - v_2|$ and therefore $d(u, v) \geq d(u', v)$.

Let S be a 2-resolving set of $P_m \boxtimes P_n$ and assume that $u \notin S$. Consider two vertex sets $A = \{u'\}$ and $B = \{u, u'\}$. Now $\mathcal{D}_S(A) = \mathcal{D}_S(B)$ since no vertex of S can be closer to u than u' as we saw above. Therefore $u \in S$. The other cases are handled similarly, namely $u_1 = 1$, $u_1 = m$, and $u_2 = n$. This shows that all vertices at the frame of the graph must be included in the resolving set.

Denote by F the vertices at the frame of the graph. We will show how to locate the elements of X when we know $\mathcal{D}_F(X)$.

Let f and f' be two neighbouring vertices of F. Now $|d(f, X) - d(f', X)| \leq 1$, since $d(f, X) \leq d(f', X) + d(f, f') = d(f', X) + 1$ and similarly $d(f', X) \leq$

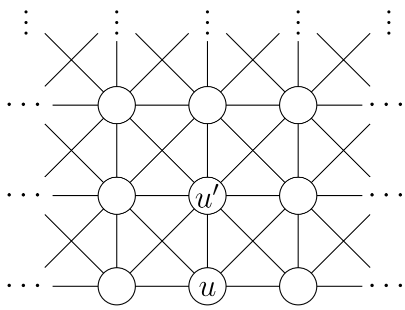


Figure 15: The vertices u and u'. Note that all neighbours of u are also neighbours of u'.

$d(f, X) + 1$.

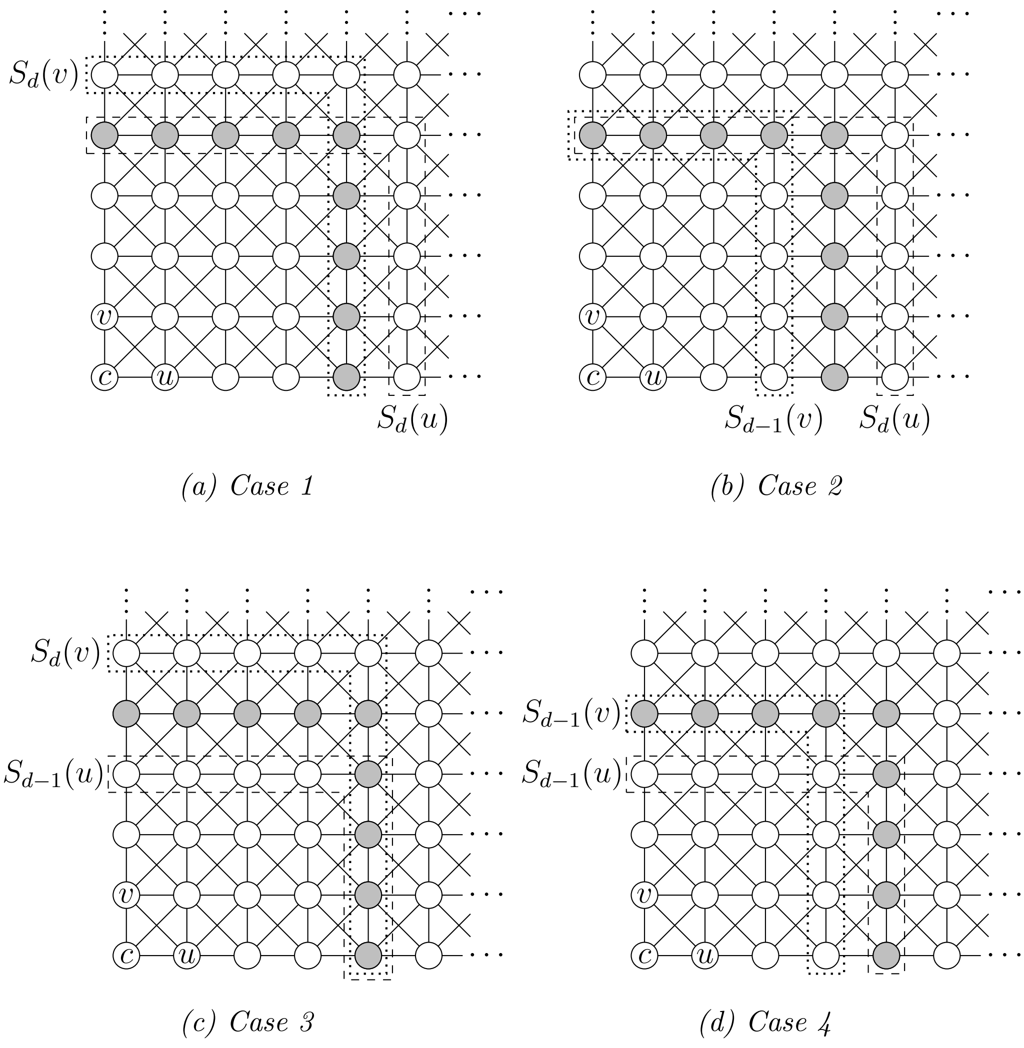
Consider three vertices of F: the corner $c_1 = (1, 1)$ and its neighbours $u = (2, 1)$ and $v = (1, 2)$. If $d(c_1, X) = 0$, we have already located one element of X. Let $d(c_1, X) = d \geq 1$. At least one vertex of $S_d(c_1) = \{(i, d+1),\ (d+1, i) \mid i \in [1, d+1]\}$ is in X. It is clear that $d(u, w)$ and $d(v, w)$ are at most d for all $w \in S_d(c_1)$, since otherwise $d(c_1, X) > d$. Due to the previous observation $d(u, X) \geq d - 1$ and $d(v, X) \geq d - 1$, and therefore $d(u, X),\ d(v, X) \in \{d - 1, d\}$. Let $H = \{(i, d + 1) \mid i = 1, \ldots, d\}$ and $K = \{(d + 1, i) \mid i = 1, \ldots, d\}$. We have four cases (see Figure 16):

1. $d(u, X) = d(v, X) = d$:

   Since $d(u, k) = d - 1$ for all $k \in K$, there cannot be any elements of X in K. Similarly, since $d(v, h) = d - 1$ for all $h \in H$, there cannot be any elements of X in H. However, $S_d(c_1)$ contains at least one element of X. The only element of $S_d(c_1)$ that is in neither K nor H is $(d + 1, d + 1)$, and therefore $(d + 1, d + 1) \in X$.

2. $d(u, X) = d$ and $d(v, X) = d - 1$:

   The distance between c_1 and a vertex of $S_{d-1}(v)$ is either d or $d - 1$. Since $d(c_1, X) = d$, there must be at least one element of X in $S_d(c_1) \cap S_{d-1}(v) = H$. Like in Case 1, we know that $K \cap X = \emptyset$. Now if $S_d(c_1)$ contains one element of X, it must be in H. If $S_d(c_1)$ contains two



(a) Case 1

(b) Case 2

(c) Case 3

(d) Case 4

Figure 16: The four cases for $d = 4$. The gray vertices form $S_d(c)$.

elements of X, they must be in $H \cup \{(d+1, d+1)\}$.

3. $d(u, X) = d - 1$ and $d(v, X) = d$:

   This case is symmetrical to Case 2, and we omit the details.

   If $S_d(c_1)$ contains one element of X, it must be in K. If $S_d(c_1)$ contains two elements of X, they must be in $K \cup \{(d+1, d+1)\}$.

4. $d(u, X) = d(v, X) = d - 1$:

   According to Cases 2 and 3, we know that both H and K must contain at least one element of X. Since $|X| \leq 2$, there cannot be elements of X anywhere else in the graph.

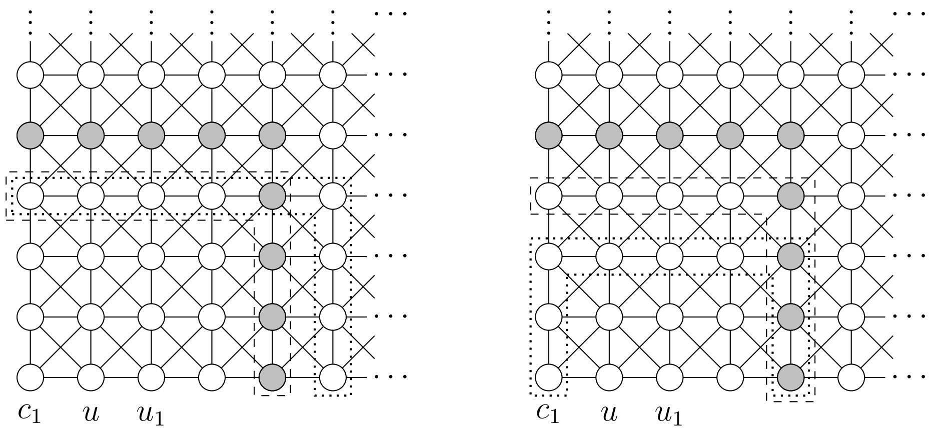
Now we have successfully located one element of X, or we know that H or K contains at least one element of X.

Consider the case where $d(u, X) = d-1$ and $|K \cap X| \geq 1$ (the other case, i.e. $d(v, X) = d - 1$ and $|H \cap X| \geq 1$, is symmetrical and goes similarly). We will show how to locate one element of X with the help of the vertices $(i, 1) \in F$, where $i = 1, \ldots, d + 1$.

Let $u_1 \in F$ be the neighbour of u on the right, i.e. $u_1 = (3, 1)$ (see Figure 17). Now for all $k \in K \setminus \{(d+1, d)\}$, the distance between k and u_1 is $d-2$. If $k = (d + 1, d)$, then $d(k, u_1) = d - 1$. The element of X closest to u_1 is in K and therefore $d(u_1, X) \in \{d - 2, d - 1\}$.

If $d(u_1, X) = d-1 = d(u, X)$, then $(d+1, d) \in X$ and $K_1 = K \setminus \{(d+1, d)\}$ cannot contain any elements of X. However, if $d(u_1, X) = d - 2 \neq d(u, X)$, then K_1 must contain at least one element of X (note that $(d + 1, d)$ may or may not be an element of X). In this case, we move on to consider $u_2 = (4, 1)$, the right neighbour of u_1. Again, if $d(u_2, X) = d(u_1, X)$, then $(d+1, d-1) \in X$. If $d(u_2, X) = d(u_1, X) - 1$, then we know that $|(K_1 \setminus \{(d+1, d-1)\}) \cap X| \geq 1$, and we move on to consider the right neighbour of u_2.

We continue in this fashion until we find two consecutive vertices that have the same distance to X. If there are no such vertices, we will eventually reach the vertex $(d+1, 1)$. Because the distance $d(u_s, X)$ decreases with each



(a) $d(u_1, X) = d - 1$ *(b)* $d(u_1, X) = d - 2$

Figure 17: The dashed line outlines the vertices of $S_{d-1}(u)$ *and the dotted line the vertices of* $S_{d(u_1,X)}(u_1)$*. The gray vertices form* $S_d(c_1)$*.*

step, we now have $d((d+1, 1), X) = 0$. In both cases, we have located one element of X.

If K contains two elements of X, we will locate the one closer to $(d+1, 1)$ with the procedure described above.

Since we can locate one element of $H \cap X$ with a similar procedure, we can now locate one element of X in Cases 2 and 3, and two elements in Case 4.

Assume that we have located only one element of X. Let $X = \{x, y\}$ with $x \neq y$, and let $x = (x_1, x_2)$ be the element we have already located. We will show how to locate $y = (y_1, y_2)$.

Clearly $y_1 \geq x_1$ or $y_2 \geq x_2$, since otherwise $d(c_1, y) < d = d(c_1, X)$.

If $y_1 = x_1$, then $y_2 > x_2$. Otherwise x and y are both in K and y is closer to $(d+1, 1)$ than x is. Since our procedure finds the element of $K \cap X$ that is closest to $(d+1, 1)$, we would have found y first. Similarly, if $y_2 = x_2$, then $y_1 > x_1$. Otherwise $H \cap X = \{x, y\}$ and $d((1, d+1), y) < d((1, d+1), x)$, and we would have found y first. Thus $y_1 > x_1$ or $y_2 > x_2$.

If $y_1 > x_1$ and $y_2 < x_2$, then we can locate y by repeating our procedure to the corner $c_2 = (m, 1)$. Indeed, now $d(y, c_2) < d(x, c_2)$ and y is the only element of X in $S_{d(c_2,X)}(c_2)$. Similarly, if $y_1 > x_1$ and $y_2 > x_2$, or $y_1 < x_1$ and $y_2 > x_2$, then we can locate y by repeating our procedure to $c_3 = (m, n)$ and $c_4 = (1, n)$ respectively.

If $y_1 = x_1$, then we can locate y by considering either c_3 or c_4. Indeed, y is closer to the corner we have chosen than x is, or they are at the same distance from the corner. In the latter case we can locate y since it is closer to the frame vertices at the top of the graph than x is. Similarly, if $y_2 = x_2$, then we can locate y by considering either c_2 or c_3.

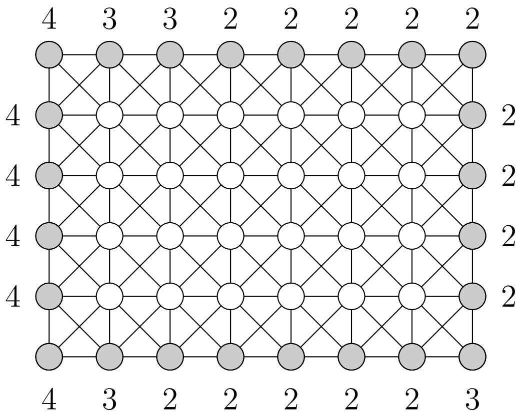
If X contains only one element, our procedure finds only this one vertex no matter from which corner we begin our search. Therefore, if we repeat this procedure to all four corner vertices, we will resolve X whether it has one element or two. $\square$

The next example demonstrates how we can use our procedure in practice and that it pays off to choose the corners we consider carefully.

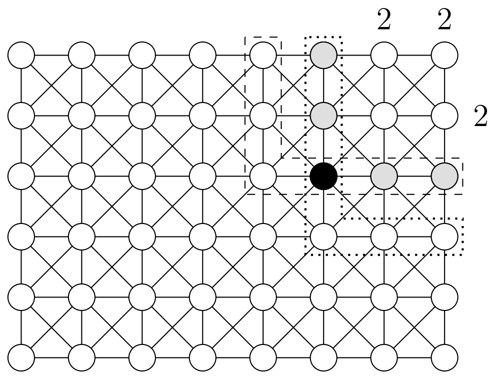
Example 5.2. Consider the 8×6 king grid $P_8 \boxtimes P_6$ in Figure 18 (a). Let X be a vertex set of at most two elements. The distance $d(f, X)$ is written next to each frame vertex $f \in F$ in Figure 18 (a). We will locate the elements of X by using the procedure presented in the proof of Theorem 5.1.

The distances next to the corner vertices indicate that the corners $(8, 1)$ and $(8, 6)$ have Cases 4 and 1 respectively. The distances of the neighbours of $(1, 1)$ and $(1, 6)$ are not equal, and therefore these corners have Cases 2 or 3. We will begin from the top right corner $(8, 6)$ and Case 1 (see Figure 18 (b)), since we can locate one element of X immediately. Indeed, the only element of $S_2((8, 6))$ that can be an element of X is $(6, 4)$. Thus $(6, 4) \in X$.

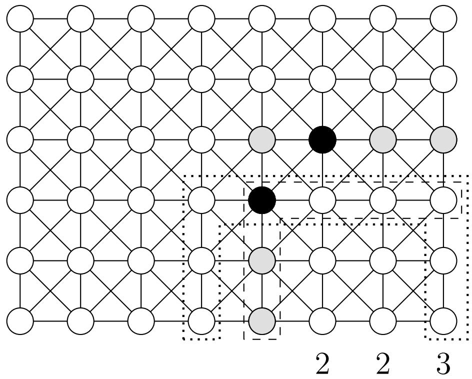
Next we will consider the corner $(8, 1)$. In Case 4 there are two sets, $K = \{(5, i) \mid i = 1, \ldots, 3\}$ and $H = \{(i, 4) \mid i = 6, \ldots, 8\}$, that must both contain one element of X. Therefore X must have two elements. Since we already know that $(6, 4) \in X \cap H$, the other element of X must be in K. Since $d((6, 1), X) = 2$ the only element of K that can be an element of X is $(5, 3)$ (see Figure 18 (c)). Now $X = \{(5, 3), (6, 4)\}$.



(a) The distance $d(f, X)$ for each frame vertex $f \in F$.



(b) With the distances $d((7,6), X)$, $d((8,6), X)$, and $d((8,5), X)$ we can locate one element of X.



(c) The other element of X can be located easily with the distances $d((6,1), X)$, $d((7,1), X)$, and $d((8,1), X)$.

Figure 18: How to locate the objects of X in $P_8 \boxtimes P_6$.

We would have arrived at the same conclusion no matter from which corner we begin our search. However, if we would have begun from either $(1, 1)$ or $(1, 6)$ (Cases 2 or 3), we would have gone through a lot more steps. Generally speaking, the best corner to begin the search from is the one with Case 1 (if such exists), since we can locate one element with the help of only three frame vertices. Another good starting point is a corner with Case 4. In this case, we have two quite small vertex sets that both contain one element of X.

5.2 Three objects or more

When $\ell \geq 3$, we cannot leave any vertex out of the ℓ-resolving set. If we do, we can always find two sets of vertices that have the same distance array.

Theorem 5.3. *Let $P_m \boxtimes P_n$ be an $m \times n$ king grid with $2 \leq n \leq m$. Then*

$$\beta_{\geq 3}(P_m \boxtimes P_n) = mn.$$

Proof. In Theorem 5.1 we saw that the vertices at the frame of the graph must be included in any 2-resolving set. Therefore they must also be in any 3-resolving set. If $n = 2$ or $m = 2$, all vertices are at the frame of the graph and the claim holds.

Let S be a 3-resolving set of $P_m \boxtimes P_n$ where $2 < n \leq m$. Assume that $u = (u_1, u_2) \notin S$ where $u_1 \in [2, m-1]$ and $u_2 \in [2, n-1]$. Let $v = (u_1 - 1, u_2)$ and $w = (u_1 + 1, u_2)$ (see Figure 19).

Assume that there is a vertex $s = (s_1, s_2) \in S$ such that $d(s, u) < d(s, v)$ and $d(s, u) < d(s, w)$.

- If $s_1 = u_1$, then $d(s, u) < d(s, v)$ implies that $|s_2 - u_2| \leq |s_1 - u_1| = 0$ and therefore $s_2 = u_2$. Now $s = u$ but this is a contradiction, since $u \notin S$.

- If $s_1 < u_1$, then $s_1 - u_1 < 0$ and therefore $|s_1 - u_1 + 1| < |s_1 - u_1|$. In fact $|s_1 - u_1 + 1| = |s_1 - u_1| - 1$. Since $d(s, u) < d(s, v)$,

$$\max\{|s_1 - u_1|,\ |s_2 - u_2|\} < \max\{|s_1 - u_1 + 1|,\ |s_2 - u_2|\}.$$

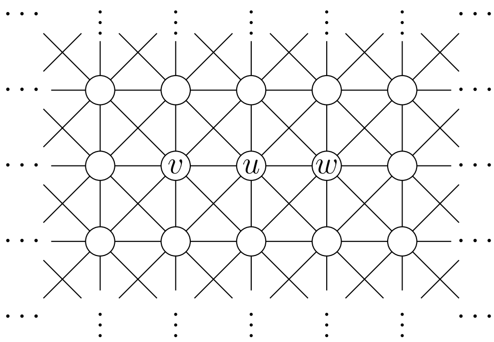


Figure 19

Now $|s_2 - u_2| < |s_1 - u_1|$, because otherwise $d(s,u) = |s_2 - u_2| = d(s,v)$. But now $|s_2 - u_2| \leq |s_1 - u_1| - 1 = |s_1 - u_1 + 1|$, and therefore $d(s,u) = |s_1 - u_1| > |s_1 - u_1 + 1| = d(s,v)$, which is a contradiction.

- If $s_1 > u_1$, we can just replace v and $|s_1 - u_1 + 1|$ with w and $|s_1 - u_1 - 1|$ in the previous case.

Therefore, every $s \in S$ is as close or closer to either v or w than u. But now $\mathcal{D}_S(A) = \mathcal{D}_S(B)$, where $A = \{v, w\}$ and $B = \{u, v, w\}$. Therefore S cannot be a 3-resolving set if it does not include all vertices of the graph. $\square$

References

[1] R. Adar and L. Epstein. *The k-metric dimension.* Journal of Combinatorial Optimization, 34, pp. 1–30, 2017.

[2] G. A. Barragán-Ramírez and J. A. Rodríquez-Velázquez. *The Local Metric Dimension of Strong Product Graphs.* Graphs and Combinatorics, 32, pp. 1263–1278, 2016.

[3] Z. Beerliova, F. Eberhard, T. Erlebach, A. Hall, M. Hoffmann, M. Mihal'ák, and L. S. Ram. *Network discovery and verification.* Graph-theoretic concepts in computer science, Lecture Notes in Comput. Sci., 3787, pp. 127–138, Springer, Berlin, 2005.

[4] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, and M. L. Puertas. *On the metric dimension of infinite graphs.* Discrete Appl. Math., 160, pp. 2618–2626, 2012.

[5] J. Cáceres, C. Hernando, M. Mora, I.M. Pelayo, M. L. Puertas, C. Seara, and D.R. Wood. *On the metric dimension of Cartesian products of graphs.* SIAM J. Discrete Math., 21, pp. 423–441, 2007.

[6] G. Chartrand, L. Eroh, M. A. Johnson, and O. R. Oellermann. *Resolvability in graphs and the metric dimension of a graph.* Discrete Appl. Math., 105, pp. 99–113, 2000.

[7] L. Epstein, A. Levin, and G. J. Woeginger. *The (Weighted) Metric Dimension of Graphs: Hard and Easy Cases.* Algorithmica, 72(4), pp. 1130–1171, 2015.

[8] H. Fernau, P. Heggernes, P. van't Hof, D. Meister, and R. Saei. *Computing the metric dimension for chain graphs.* Information Processing Letters, 115, pp. 671-–676, 2015.

[9] F. Foucaud, T. Laihonen, and A. Parreau. *An Improved Lower Bound for* $(1, \leq 2)$*-Identifying Codes in the King Grid.* Advances in Mathematics of Communications, 8(1), pp. 35–52, 2014.

[10] A. Hakanen and T. Laihonen. *On resolving several objects in the king grid.* Proceedings of the Fourth Russian Finnish Symposium on Discrete Mathematics : Organized in Turku on 16.-19.5.2017, pp. 60–64, 2017

[11] F. Harary and R. Melter. *On the metric dimension of a graph.* Ars Combinatoria, 2, pp. 191–195, 1976.

[12] C. Hernando, M. Mora, I. M. Pelayo, C. Seara, and D. R. Wood. *Extremal Graph Theory for Metric Dimension and Diameter.* The Electronic Journal of Combinatorics, 17, 2010.

[13] I. Honkala and T. Laihonen. *Codes for Identification in the King Lattice.* Graphs and Combinatorics, 19, pp. 505–516, 2003.

[14] D. Kuziak, I. G. Yero, and J. A. Rodríguez-Velázquez. *On the Strong Metric Dimension of Cartesian Sum Graphs.* Fundamenta Informaticae, 141, pp. 57-–69, 2015.

[15] T. Laihonen. *The metric dimension for resolving several objects.* Information Processing Letters, 116, pp. 694–700, 2016.

[16] S. Khuller, B. Raghavachari, and A. Rosenfeld. *Landmarks in graphs.* Discrete Appl. Math., 70, pp. 217–229, 1996.

[17] A. Lobstein. *Watching systems, identifying, locating-dominating and discrminating codes in graphs, a bibliography.* Published electronically at `http://perso.telecom-paristech.fr/~lobstein/debutBIBidetlocdom.pdf`.

[18] M. Pelto. *On Identifying and Locating-Dominating Codes in the Infinite King Grid.* TUCS Dissertations, 155, 2012.

[19] Y. Ramírez-Cruz, O. R. Oellermann, and J. A. Rodríguez-Velázquez *The simultaneous metric dimension of graph families.* Discrete Appl. Math., 198, pp. 241—250, 2016.

[20] J. A. Rodríguez-Velázquez, D. Kuziak, I. G. Yero, and J. M. Sigarreta. *The metric dimension of strong product graphs.* Carpathian J. Math., 31, No. 2, pp. 261–268, 2015.

[21] P. Slater. *Leaves of trees.* Proceedings of the Sixth Southeastern Conference on Combinatorics, Graph Theory, and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1975), pp. 549–559. Congressus Numerantium, No. XIV, Utilitas Math., Winnipeg, Man., 1975.