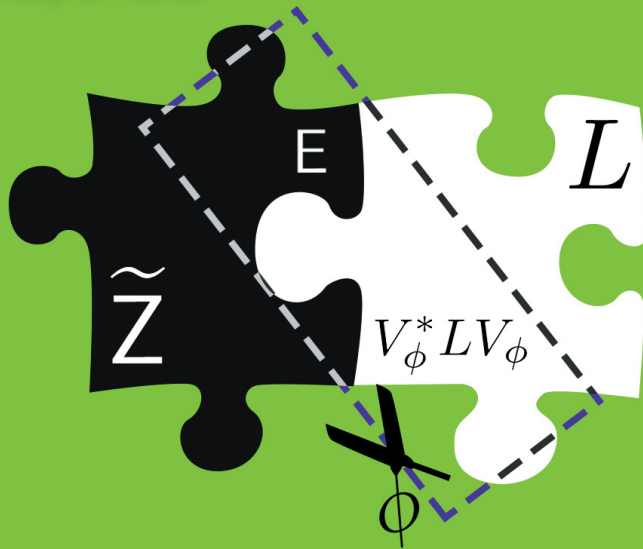




Turun yliopisto
University of Turku



IMPLEMENTABILITY OF OBSERVABLES, CHANNELS AND INSTRUMENTS IN QUANTUM THEORY OF MEASUREMENT

Mikko Tukiainen



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The originality of this thesis has been checked in accordance with the University of Turku quality assurance system using the Turnitin OriginalityCheck service.

ISBN 978-951-29-7009-4 (PRINT)

ISBN 978-951-29-7010-0 (PDF)

ISSN 0082-7002 (PRINT)

ISSN 2343-3175 (PDF)

Painosalama Oy - Turku, Finland 2017

Acknowledgements

I remember, in my childhood, sitting on the back-seat of a car when my family was returning home from my grandparents'. I was leaning back my head and noticing for the first time in my life the spiral arm of the Milky Way galaxy, that arched above us in the otherwise absolute darkness of the midwinterly Northern Karelia. From my standpoint, the stars should not have been much more than just tiny twinkling specks of light on the pitch black sky, but they nevertheless managed to leave a big emotional impact on me. Now that I am finishing my doctoral studies in quantum physics, some 20 years later, I am starting to understand that it is often the smallest things that have the greatest significance. The road that connects these two time periods has been a long one, but luckily I have not had to travel it alone.

My deepest gratitude goes to my parents Lasse and Anneli, who have been extremely supportive throughout my life and my university studies, and have always remained optimistic and cheered me up when the times have been difficult. I also cannot thank enough my lovely wife Naomi and my beautiful daughter Minttu – the two twinkling stars, who are the source of light and positive energy in my life. The completion of my doctoral studies is completely thanks to all of you, my closest family.

I am also thankful to all my friends and to the staff in the Laboratory of theoretical physics in the University of Turku. Special thanks goes to Boris and Ville, who have been responsible for all the craziest shenanigans during my university studies. I want to also thank my closest co-workers and great friends Henri and Erkka, with whom I had the pleasure to work and share countless number of hilarious moments. As an additional note, together with Henri and Erkka and the help of Francesco and Walter, our office earned the controversial nickname “the noisy room”. Thank you Professor Sabrina Maniscalco for all the hard work and dedication you have put into the Laboratory – you bring in the best people and always know how to bring out the best in them.

Finally, I would like to thank my supervisors Juha-Pekka Pellonpää and Teiko Heinosaari, the director of my PhD studies Professor Kalle-Antti Suominen and all the collaborators that I have had the opportunity to work with during my doctoral studies. Thanks to your help, this thesis and its results can now shed some light on the vast mystical quantum world; like the ones preceding and the ones following it; like the stars twinkling on the night sky.

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Abstract

Measurements play an integral part in any scientific theory. Indeed, the observations revealing causal behaviour and symmetries in our world of experience lie at the heart of determining the truth or the falsity of scientific hypotheses. In this regard, quantum theory provides one of the most stringently tested and comprehensive description of the natural world. This thesis is an examination of the mathematical structures of quantum measurements with an emphasis on the measurement implementations of different operational devices: quantum observables, channels and instruments.

The measurability of all the quantum properties a system bears is, in principle, guaranteed by quantum measurement theory. However, as physical processes, measurements are constrained by the laws of physics and it has been long known that certain conservation laws can inhibit the measurements of some of the quantum observables. In this work it is shown that such constraints stem from the fact that the measured observables inherit the symmetries of the measurement devices implementing them.

A quantum measurement set-up can implement a multitude of operational devices as a function of the initial state of the apparatus. In other words, one can realize various maps – even incompatible ones – simply by controlling the apparatus' state. Due its close resemblance with the classical analogue, this operation is often referred as quantum programming. In this thesis the capabilities and restrictions of quantum programming are explored using a measurement-theoretic approach. Moreover, quantitative methods are developed to address the mutual relations between the programming states of an arbitrary pair of observables or channels.

Tiivistelmä

Mittaamisella on keskeinen osa luonnontieteissä. Kartoitamme niin tieteellisten kuin arkipäiväistenkin otaksujen paikkansapitävyyttä tekemällä havaintoja meitä ympäröivistä syy ja seuraus -suhteista sekä erilaisista symmetrioista. Kvanttiteoria tarjoaa tässä suhteessa yhden kattavimmista sekä täsmällisimmistä testatuista luonnonilmiöiden kuvauksista. Tässä väitöskirjatutkielmassa tarkastellaan kvanttimitausten matemaattista rakennetta ja tutkitaan erilaisten kvanttikuvausten, kuten kvanttisuureiden, -kanavien ja -instrumenttien, mitattavuutta.

Kvanttimekaniikan mittausteoria ei periaatteessa yksinään rajoita mitään ominaisuuksia systeemistä voidaan mitata. Erilaiset fysiikan lait voivat kuitenkin rajata kvanttiominaisuuksien mitattavuutta; jo kvanttiteorian alkua ajoista asti on tiedetty, että näin käy erityisesti tiettyjen säilymislakien voimassaollessa. Tässä tutkielmassa osoitetaan, että kyseisen kaltaiset rajoitukset mitattavuudelle voidaan käsittää osana yleisempää ilmiötä, jonka mukaan mittalaitteiden symmetriat periytyvät niillä mitattavissa oleviin kvanttisuureisiin.

Yksittäinenkin mittalaite voi toteuttaa useita kvanttikuvauksia mittalaitteen alkutilan funktiona – jopa sellaisia, jotka ovat kvanttimekaniikan mielessä yhteensopimattomia. Mittalaitteen alkutilan variointia kutsutaan usein kvanttiohjelmoinniksi. Tässä väitöskirjatutkielmassa etsitään vastausta siihen mitä kvanttiohjelmoinnin avulla voi ja ei voi tehdä. Tutkielmassa on keskitytty erityisesti kehittämään matemaattisia menetelmiä, joita apuna käyttäen eri kvanttisuureita tai -kanavia vastaavien ohjelmointitilojen välisiä riippuvuuksia voidaan kvantifioida.

List of papers

This thesis consists of a review of the subject and the following original research articles:

- I Notes on deterministic programming of quantum observables and channels,**
T. Heinosaari, *M. Tukiainen*, Quantum Inf. Process. **14**, 3097 (2015).
- II Thermodynamic power of non-Markovianity,**
B. Bylicka, *M. Tukiainen*, D. Chruściński, J. Piilo, S. Maniscalco, Sci. Rep. **6**, 27989 (2016)
- III Wigner-Araki-Yanase theorem beyond conservation laws,**
M. Tukiainen, Phys. Rev. A **95**, 012127 (2017)
- IV Limitations on post-processing assisted quantum programming,**
T. Heinosaari, T. Miyadera, *M. Tukiainen*, Quantum Inf. Process. **16**, 85 (2017)
- V Fidelity of dynamical maps,**
M. Tukiainen, H. Lyyra, G. Sarbicki, S. Maniscalco, Phys. Rev. A **95**, 052102 (2017)
- VI Quantification of Concurrence via Weak Measurement,**
M. Tukiainen, H. Kobayashi, Y. Shikano, Phys. Rev. A **95**, 052301 (2017)
- VII Minimal normal measurement models of quantum instruments,**
J.-P. Pellonpää, *M. Tukiainen*, Rep. Math. Phys. **79**, 261 (2017)

Chapter 1

Introduction

What is a *theory*? From the philosophical point of view, a theory is defined as a set of self-consistent statements or hypotheses leading to phenomenological predictions of our world of experience whose validity can be objectively tested by doing measurements. Any experimental test of a hypothesis can either corroborate it or, more crucially, falsify it. In fact, under the modern interpretation of philosophy, the falsifiability is often taken as a part of the definition of a scientific theory [1]. In other words, it must be possible to design an experiment, outcome of which can contradict the predictions deduced from the hypothesis. This is reflecting the fact that, although it is in practice never possible from the results of experiments – no matter how numerous – to be fully convinced that a theory is true, even a single measurement has the power to show that it is not. The evolution of science is often guided by these conflicting experiments that pave the way for discoveries of new theoretical structures and eventually new theories that will arise to supersede the previous ones.

In short, measurements allow us to tie theoretical considerations with physical reality. But what is a *measurement*? Every measurement can be split into three stages: the preparation of the system to be measured and the measuring apparatus, their interaction and finally a subsequent reading of the apparatus' pointer scale [2–4]. When such processes are formulated mathematically, the restrictions of the emerging theory of measurements can be studied. Noticeably, besides the concrete experiments, there are remarkably many phenomena that can be seen abstractly as measurement processes.

From the days of its inception, the quantum theory has developed into a flourishing description of Nature, along with an astounding range of applications. In fact, the quantum theory is among the most stringently

tested theories in physics and the measurements of quantum mechanical systems have demonstrated unparalleled accuracy. To give a concrete example, the most accurate measurements of the electron magnetic moment up to date agree within few parts in trillion with the theoretical predictions stated by quantum electrodynamics [5]. As the extremely sensitive and ever-decreasing measuring instruments get more and more immersed into the realm of quantum physics, it becomes important both from foundational and practical points of view to understand the fundamentals of the quantum measurements.

This thesis is an overview of the central conclusions of the research articles I-VII, that have been aimed at studying of the structure of quantum measurements and some of their general constraints. Although solely theoretical, throughout the work there has been an emphasis on the noisy description of measurements in order to preserve the practical significance and realistic testability of the results developed. The appropriate description of physical noise and information leakage is encased in the theory of open quantum systems. In accordance, we begin by recalling some basic facts about states, their transformations and dynamical evolution in Chap. 2.

In Chap. 3 we proceed by formulating the concept of quantum measurement. A special interest is on the measurability of the quantum observables. One of the most remarkable restriction to the quantum measurements is set by the theorem of Wigner, Araki and Yanase [6, 7], which implies constraints on the measurability of observables in the presence of conservation laws. We show that these limitations, rather than arising directly from the conservation laws, can be subsumed under a more general mathematical concept: (a)symmetry transferred from the measuring apparatus to the measured observable. Furthermore, as opposed to the original theory – which only has a very limited range of practical applicability – we prove that such a behaviour is universally present in all quantum measurements.

The last Chap. 4 has been devoted for an analysis of multi-purpose measurements. More precisely, we consider the protocol known as quantum programming [8], where one has the freedom to select and vary the initial state of the measuring apparatus, hence implementing a multitude of different measurements. Such programmable multimeters can transcend above the restrictions set by quantum incompatibility for their single purpose counterparts. However, there is a different universal constraint arising, namely the measurements of certain observables and channels can only be imple-

mented with orthogonal states of the apparatus. A special aim in Chap. 4 is to develop the tools to treat the programmability in a quantitative manner. Furthermore, we study the connections between the joint measurability and the quantum programmability in a hybrid scenario, where, in addition to the state alteration, classical post-processing is considered as a programming resource.

This work is far from a complete survey of the broad list of topics it involves. On the contrary, we have aimed to include only a minimal necessary description for a self-contained treatment and provide the reader with references, in which the subjects involved have been studied more profoundly. Although the thesis is to a great extent based on the research articles I-VII, it also contains some original, previously unpublished results. Such results, when formulated into propositions, have been identified with *-symbol preceding the word "Proposition".

Chapter 2

Open quantum systems

2.1 States and state transformations

Throughout this work we will associate with a quantum system S a complex Hilbert space \mathcal{H} with possibly infinite dimension. In addition, we will always implicitly assume that the investigated Hilbert spaces are separable, i.e., each of them has a countable orthonormal basis. The inner product associated to \mathcal{H} is denoted by $\langle \cdot | \cdot \rangle$. We identify the particular sets of bounded (linear) operators as $\mathcal{L}(\mathcal{H}) = \{A \mid \|A\| \equiv \sup_{\|\varphi\| \leq 1} \|A\varphi\| < \infty\}$, projections as $\mathcal{P}(\mathcal{H}) = \{A \in \mathcal{L}(\mathcal{H}) \mid A^2 = A^* = A\}$ and the trace class operators as $\mathcal{T}(\mathcal{H}) = \{A \in \mathcal{L}(\mathcal{H}) \mid \|A\|_1 \equiv \text{tr}[\sqrt{A^*A}] < \infty\}$. The identity element in $\mathcal{L}(\mathcal{H})$ is denoted by $\mathbb{1}$. The support of an operator A is denoted as $\text{supp}[A]$ and defined as the orthogonal complement of the kernel $\ker[A] \equiv \{\varphi \in \mathcal{H} \mid A\varphi = 0\}$, that is $\text{supp}[A] = \ker[A]^\perp$. Throughout this work, we may add sub- or superscripts for clarification in the previously defined entities.

The *states* of a quantum system S with the associated Hilbert space \mathcal{H} are represented as positive trace-class operators with unit trace and the set of all quantum states is denoted by $\mathcal{S}(\mathcal{H}) = \{\varrho \in \mathcal{T}(\mathcal{H}) \mid \varrho \geq 0, \text{tr}[\varrho] = 1\}$. Clearly $\mathcal{S}(\mathcal{H})$ is a convex set. More generally, $\mathcal{S}(\mathcal{H})$ is a σ -convex set, that is for any sequence $\varrho_i \in \mathcal{S}(\mathcal{H})$ and positive numbers λ_i such that $\sum_i \lambda_i = 1$ the sum $\sum_i \lambda_i \varrho_i$ converges w.r.t $\|\cdot\|_1$ and the limit is in $\mathcal{S}(\mathcal{H})$. Two states ϱ_1 and $\varrho_2 \in \mathcal{S}(\mathcal{H})$ are called orthogonal, in which case we denote $\varrho_1 \perp \varrho_2$, whenever $\text{supp}[\varrho_1]$ and $\text{supp}[\varrho_2]$ are orthogonal subspaces. The extremal elements of $\mathcal{S}(\mathcal{H})$, which will be called *pure* or *vector states*, are the one-dimensional projection operators P_φ , $\varphi \in \mathcal{H}$, $\|\varphi\| = 1$, defined via $P_\varphi\psi = \langle \varphi | \psi \rangle \varphi$ for all $\psi \in \mathcal{H}$. On the contrary, the states that are not pure are called

mixed states. The set of pure states exhausts $\mathcal{S}(\mathcal{H})$ in the sense that every state can be expressed as a σ -convex combination of some pure states. In particular, any $\varrho \in \mathcal{S}(\mathcal{H})$ has a spectral decomposition $\varrho = \sum_i \lambda_i P_{\varphi_i}$, where λ_i form a (possibly infinite) sequence of positive numbers such that $\sum_i \lambda_i = 1$ and φ_i is a sequence of orthonormal vectors in \mathcal{H} .

In the special case $\mathcal{H} = \mathbb{C}^2$ any quantum state $\varrho \in \mathcal{S}(\mathbb{C}^2)$ has a unique real vector $\vec{n} = (n_1, n_2, n_3) \in \mathbb{R}^3$, $\|\vec{n}\| \leq 1$, associated to it, where the correspondence is given by $\varrho \equiv \varrho_{\vec{n}} = \frac{1}{2}(\mathbb{1} + \vec{n} \cdot \vec{\sigma})$. Here $\vec{n} \cdot \vec{\sigma} \equiv \sum_{i=1}^3 n_i \sigma_i$ and the Pauli spin matrices σ_i , $i = 1, 2, 3$, are defined via

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.1)$$

Accordingly, $\mathcal{S}(\mathbb{C}^2)$ can be viewed as a 3-dimensional ball with unit radius. In this case the pure states lie on the surface of this Bloch ball, that is $\varrho_{\vec{n}} \in \mathcal{S}(\mathbb{C}^2)$ is pure if and only if $\|\vec{n}\| = 1$.

The Hilbert space of compound of two systems S_A and S_B is given by the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$ with $\mathcal{H}_{A(B)}$ being associated to $S_{A(B)}$, respectively. The sets $\mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$, $\mathcal{T}(\mathcal{H}_A \otimes \mathcal{H}_B)$, etc. are defined in the usual way. It should be noted, however, that the product operators of the form $A \otimes B$ do not exhaust the respective operator sets. As an important example, in $\mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ there exist pure states $P_\Psi \neq P_{\varphi_A} \otimes P_{\varphi_B}$ for any $P_{\varphi_{A(B)}} \in \mathcal{S}(\mathcal{H}_{A(B)})$; such a state is called *entangled*. More generally, a state $\varrho \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is entangled if it cannot be expressed as a σ -convex combination $\varrho = \sum_i \lambda_i \varrho_i^A \otimes \varrho_i^B$ for any positive numbers λ_i satisfying $\sum_i \lambda_i = 1$ and $\varrho_i^{A(B)} \in \mathcal{S}(\mathcal{H}_{A(B)})$. For a thorough survey on entanglement, we refer to the review articles [9, 10].

For all $T \in \mathcal{T}(\mathcal{H}_A \otimes \mathcal{H}_B)$ we define the mappings

$$\begin{aligned} \text{tr}_{\mathcal{H}_A} : T &\mapsto \sum_{i=1}^{\dim(\mathcal{H}_A)} W_{\varphi_i}^* T W_{\varphi_i} \in \mathcal{T}(\mathcal{H}_B), \\ \text{tr}_{\mathcal{H}_B} : T &\mapsto \sum_{i=1}^{\dim(\mathcal{H}_B)} V_{\phi_i}^* T V_{\phi_i} \in \mathcal{T}(\mathcal{H}_A), \end{aligned} \quad (2.2)$$

where $\{\varphi_i\}_i$ and $\{\phi_i\}_i$ are any orthonormal bases of \mathcal{H}_A and \mathcal{H}_B , respectively, and $W_\varphi(\phi) = \varphi \otimes \phi = V_\phi(\varphi)$, for all $\varphi \in \mathcal{H}_A$ and $\phi \in \mathcal{H}_B$. These

partial traces enable one to assess the local properties of a composite system. Additionally, we will denote by $\text{tr}_\xi : \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B) \rightarrow \mathcal{L}(\mathcal{H}_A)$ the *partial trace with respect to the state* $\xi \in \mathcal{S}(\mathcal{H}_B)$; the mapping uniquely satisfying

$$\text{tr}[\text{tr}_\xi[L] A] = \text{tr}[L A \otimes \xi], \quad (2.3)$$

for all $A \in \mathcal{T}(\mathcal{H}_A)$ and $L \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$. Equivalently, for all $L \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ $\text{tr}_\xi[L] \equiv \sum_i \lambda_i V_{\phi_i}^* L V_{\phi_i}$, where $\{\lambda_i, \phi_i\}_i$ are determined by the spectral decomposition of $\xi = \sum_i \lambda_i P_{\phi_i}$. It is noteworthy that, if $L \mathbb{1} \otimes \xi \in \mathcal{T}(\mathcal{H}_A \otimes \mathcal{H}_B)$, then $\text{tr}_\xi[L] = \text{tr}_{\mathcal{H}_B}[L \mathbb{1} \otimes \xi]$. This holds in particular when $\dim(\mathcal{H}_A) < \infty$.

If a linear mapping $\mathcal{E} : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H}')$ is to be a state transformation, then, for any $T \in \mathcal{T}(\mathcal{H})$, it clearly has to be *positive*, $\mathcal{E}(T) \geq 0$ whenever $T \geq 0$, and *trace preserving*, $\text{tr}[\mathcal{E}(T)] = \text{tr}[T]$. Fulfilling these two requirements is not, however, in general enough to guarantee that $\mathcal{E} \otimes \mathcal{I}_0$ would also act as a state transformation in any composite system $\mathcal{H} \otimes \mathcal{H}_0$, where \mathcal{I}_0 is the identity mapping on $\mathcal{T}(\mathcal{H}_0)$. This broader viewpoint is justified by the fact that the local properties of a subsystem should stay consistent regardless of the compound system it is seen as being part of. Therefore, a linear mapping $\mathcal{E} : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H}')$ is valid to be called a state transformation if, in addition to being trace preserving, it enjoys the property of *complete positivity*: $(\mathcal{E} \otimes \mathcal{I}_0)(T) \geq 0$ for all $T \in \mathcal{T}(\mathcal{H} \otimes \mathcal{H}_0)$, \mathcal{H}_0 being arbitrary. The completely positive and trace preserving (CPTP) linear mappings will also be referred as *quantum channels*.

For any quantum channel $\mathcal{E} : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H}')$ there exists a (possibly infinite) sequence of bounded operators $A_i : \mathcal{H} \rightarrow \mathcal{H}'$ such that

$$\mathcal{E}(T) = \sum_i A_i T A_i^*, \quad \sum_i A_i^* A_i = \mathbb{1}_{\mathcal{H}}. \quad (2.4)$$

Also the opposite holds, that is any sequence $\{A_i : \mathcal{H} \rightarrow \mathcal{H}' \mid \|A_i\| < \infty\}_i$ satisfying $\sum_i A_i^* A_i = \mathbb{1}_{\mathcal{H}}$ defines a quantum channel $\mathcal{E} : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H}')$ via Eq. (2.4) [3]. The decomposition in Eq. (2.4) is called a Kraus representation of \mathcal{E} and the operators A_i Kraus operators. For example, the partial traces defined in Eq. (2.2) are quantum channels, since $\sum_i W_{\varphi_i} W_{\varphi_i}^* = \mathbb{1}$ and $\sum_j V_{\phi_j} V_{\phi_j}^* = \mathbb{1}$. As a consequence $\text{tr}_{\mathcal{H}_{A(B)}}[\varrho_{AB}] \in \mathcal{S}(\mathcal{H}_{B(A)})$ for all $\varrho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$. The decomposition of \mathcal{E} having the smallest number of non-zero Kraus operators is called minimal Kraus representation and the corresponding minimal number of Kraus operators will be referred as the

Kraus rank of \mathcal{E} and denoted by $\text{rank}(\mathcal{E})$.

The channels $\mathcal{E} : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H}')$ form a convex set, extremal elements of which are called the *extremal channels*. There are two particular cases of extremal channels, which will be frequently encountered throughout this work: the *unitary channels* $\mathcal{U} : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H}); T \mapsto UTU^*$, where U is a unitary operator on \mathcal{H} , and the *complete state space contractions* $\mathcal{E}_\varrho : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H}'); T \mapsto \text{tr}[T]\varrho$, where $\varrho \in (\mathcal{H}')$ is some fixed state. Every channel $\mathcal{E} : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H}')$ has a *Stinespring dilation* $\langle \mathcal{K}, W \rangle$, that is, there exist a Hilbert space \mathcal{K} and an isometric linear map $W : \mathcal{H} \rightarrow \mathcal{H}' \otimes \mathcal{K}$ such that

$$\mathcal{E}(T) = \text{tr}_{\mathcal{K}}[WTW^*], \quad T \in \mathcal{T}(\mathcal{H}). \quad (2.5)$$

In particular, every channel $\mathcal{E} : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H})$ can be dilated into a unitary one, i.e., there exists a Stinespring dilation $\langle \mathcal{K}, U, \phi \rangle$, where U is a unitary operator on $\mathcal{H} \otimes \mathcal{K}$ and ϕ is a unit vector in \mathcal{K} , satisfying

$$\mathcal{E}(T) = \text{tr}_{\mathcal{K}}[U(T \otimes P_\phi)U^*], \quad T \in \mathcal{T}(\mathcal{H}). \quad (2.6)$$

The minimal dimension of the auxiliary Hilbert space \mathcal{K} in which a dilation $\langle \mathcal{K}, U, \phi \rangle$ is attainable depends on the channel's Kraus rank, such that $\dim(\mathcal{K})$ equals to either $\text{rank}(\mathcal{E})$ or $\text{rank}(\mathcal{E}) + 1$; see the article VII.

The duality relation

$$\text{tr}[B\mathcal{E}(T)] = \text{tr}[\mathcal{E}^*(B)T], \quad B \in \mathcal{L}(\mathcal{H}'), T \in \mathcal{T}(\mathcal{H}), \quad (2.7)$$

defines a one-to-one connection between a quantum channel $\mathcal{E} : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H}')$ (the Schrödinger picture) and its *dual* – a normal¹, completely positive and unital linear map $\mathcal{E}^* : \mathcal{L}(\mathcal{H}') \rightarrow \mathcal{L}(\mathcal{H})$ (the Heisenberg picture) [12], where the unitality means $\mathcal{E}^*(\mathbb{1}) = \mathbb{1}$. For example, the dual of a unitary channel $\mathcal{U}(T) = UTU^*$, $T \in \mathcal{T}(\mathcal{H})$ is $\mathcal{U}^*(B) = U^*BU$, $B \in \mathcal{L}(\mathcal{H})$. Every unitary channel \mathcal{U} satisfies the homomorphism property $\mathcal{U}^*(AB) = \mathcal{U}^*(A)\mathcal{U}^*(B)$. More generally, we say that a channel $\mathcal{E}^* : \mathcal{L}(\mathcal{H}') \rightarrow \mathcal{L}(\mathcal{H})$ is a $\mathcal{L}(\mathcal{H}') \rightarrow \mathcal{L}(\mathcal{H})$ algebra $*$ -homomorphism, or

¹A positive linear map $\Phi : \mathcal{L}(\mathcal{H}') \rightarrow \mathcal{L}(\mathcal{H})$ is normal if it preserves the limits of increasing and bounded sequences, i.e. $\Phi(A_i) \rightarrow \Phi(A)$ (weakly) in $\mathcal{L}(\mathcal{H})$ for all bounded increasing sequences $\{A_i \in \mathcal{L}(\mathcal{H}')\}_{i \in \mathbb{N}}$ such that $A_i \rightarrow A \in \mathcal{L}(\mathcal{H}')$ (weakly). It is well known that normality of a positive linear map is equivalent to weak*-continuity; see Ref. [11].

shortly a *homomorphic channel*, whenever $\mathcal{E}^*(AB) = \mathcal{E}^*(A)\mathcal{E}^*(B)$, for all $A, B \in \mathcal{L}(\mathcal{H}')$. The following equivalent notions of the homomorphic channels have been proven in the article I.

Lemma 1. *Let $\mathcal{E}^* : \mathcal{L}(\mathcal{H}') \rightarrow \mathcal{L}(\mathcal{H})$ be a channel and $\langle \mathcal{K}, W \rangle$ be its Stinespring dilation. Then the following conditions are equivalent:*

- (i) $\mathcal{E}^*(P) \in \mathcal{P}(\mathcal{H})$, for each $P \in \mathcal{P}(\mathcal{H}')$.
- (ii) $\mathcal{E}^*(AB) = \mathcal{E}^*(A)\mathcal{E}^*(B)$, for all $A, B \in \mathcal{L}(\mathcal{H}')$.
- (iii) $[B \otimes \mathbb{1}_{\mathcal{K}}, WW^*] = 0$, for all $B \in \mathcal{L}(\mathcal{H}')$.

Example 1. It has been proven in the article I that if $\dim \mathcal{H} < \infty$, all homomorphic channels $\mathcal{E}^* : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ are unitary channels. In general this need not be true, if either $\dim \mathcal{H} = \infty$ or the input and the output Hilbert spaces of the channel are allowed to be different. For instance, let $\mathcal{E}^* : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$ be a linear map defined via $\mathcal{E}^*(X) \equiv X \otimes \mathbb{1}_{\mathcal{K}}$. It is immediately verified that \mathcal{E}^* is a homomorphic channel. However, since \mathcal{E}^* is not surjective for any non-trivial \mathcal{K} , it is not generally a unitary channel. This is particularly clear from the corresponding Schrödinger form $\mathcal{E} : \mathcal{T}(\mathcal{H} \otimes \mathcal{K}) \rightarrow \mathcal{T}(\mathcal{H})$, $\mathcal{E}(T) = \text{tr}_{\mathcal{K}}[T]$.

2.2 Dynamical evolutions

A quantum system S is *closed* (or *isolated*) if it is not interacting with and (its state) is not entangled to any other system. The *dynamical evolution*, or *dynamics* in short, of a closed system is described by a strongly continuous unitary group $\{U_t \mid U_{t+s} \equiv U_t U_s, t, s \in \mathbb{R}\}$, so that a state $\varrho \in \mathcal{S}(\mathcal{H})$ evolves according to $\mathcal{U}_t(\varrho) = U_t \varrho U_t^*$, $t \in \mathbb{R}$. Such a unitary group $\{U_t\}_{t \in \mathbb{R}}$ is generated by a unique self-adjoint operator H , so that $U_t = e^{-itH}$ [12]. In particular, the time-evolution of vector states then coincides with the well-known Schrödinger equation

$$i \frac{d}{dt} \psi(t) = H \psi(t). \quad (2.8)$$

A system S is *open* when it is not closed [12, 13]. Let E , with associated Hilbert space \mathcal{K} , denote the composite of all systems either interacting with or entangled to S : we will suggestively call E an *environment* of S . The

dynamical evolution $S + E$, which by construction forms a closed system, is governed by the unitary dynamics $\{U_t\}_{t \in \mathbb{R}}$. We will make an additional assumption that at the time $t = 0$ the system S is fully uncorrelated from E , that is the state of $S + E$ is initially of the product form $\varrho \otimes \xi \in \mathcal{S}(\mathcal{H} \otimes \mathcal{K})$. The reduced dynamical evolution of the open system is then described by the t -parametrized family of channels $\{\mathcal{E}_t\}_{t \geq 0}$ recovered via

$$\mathcal{E}_t(\varrho) = \text{tr}_{\mathcal{K}}[\mathcal{U}_t(\varrho \otimes \xi)], \quad \varrho \in \mathcal{S}(\mathcal{H}). \quad (2.9)$$

In reality it is more often than not the case that quantum systems are open. The dynamical coupling of a system with its environment then unavoidably leads to the degradation of quantum properties, which can be understood as information flowing from the system into the environment. To make this more concrete, let us consider the quantum *mutual information* of a bipartite system $S_A + S_B$ in a state ϱ_{AB} defined via

$$I(A : B) \equiv S(\varrho_A) + S(\varrho_B) - S(\varrho_{AB}), \quad (2.10)$$

where $\varrho_{A(B)} \equiv \text{tr}_{\mathcal{H}_{B(A)}}[\varrho_{AB}]$ and $S(\varrho) \equiv -\text{tr}[\varrho \ln \varrho]$. It can be shown that local CPTP-transformations cannot increase the mutual information $I(A : B)$ [14]. In particular, if the system S_A is subjected to a dynamical evolution $\{\mathcal{E}_t\}_{t \geq 0}$ due to some environment, then the mutual information of S_A and S_B can only decrease in time, that is

$$I(A_t : B) \equiv S(\mathcal{E}_t(\varrho_A)) + S(\varrho_B) - S((\mathcal{E}_t \otimes \mathcal{I})(\varrho_{AB})) \leq I(A : B), \quad (2.11)$$

for all $t \geq 0$ and for any state $\varrho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$.

Interestingly, however, there may exist some intervals in time $0 < t_1 < t_2$ such that

$$I(A_{t_2} : B) > I(A_{t_1} : B), \quad (2.12)$$

for some $\varrho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$. Such a behaviour may be viewed as information returning from the environment back to the system [15]. There exist a variety of figures of merit to quantify the information (back-)flow in the literature; see the review articles [16–18]. These quantifiers enable one to make the following separation: the dynamics $\{\mathcal{E}_t\}_{t \geq 0}$ is *non-Markovian* whenever it shows information back-flow (w.r.t. a given quantifier) and

Markovian whenever such back-flow is inexistent (w.r.t. the quantifier)². Admittedly, this separation vaguely depends on the chosen figure of merit [21–23], but there are also cases where many of the existing quantifiers of (non-)Markovianity agree with each other [24, 25]. Nevertheless, non-Markovianity has gained a special role in the study of open quantum systems, since it can be harnessed in fighting against the inevitable environmental noise by protecting or even reviving quantum properties, that had been previously lost in the course of the dynamic evolution. As an example, the potential usefulness of non-Markovian memory effects in the context of thermodynamic work extraction [26] has been verified in the article II.

²To be precise, some of the quantifiers existing in the literature are capable of resolving both the sufficient and necessary conditions for non-Markovianity, whereas others may only give the sufficient conditions; see Refs. [19, 20].

Chapter 3

Quantum measurements

3.1 Observables

Let Ω be a non-empty set and $\Sigma \subset 2^\Omega$ a σ -algebra. *Quantum observables* are mappings $E : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$, that are positive $E(X) \geq 0$ for all $X \in \Sigma$, σ -additive $E(\cup_i X_i) = \sum_i E(X_i)$ (w.r.t the weak operator topology) whenever $X_i \cap X_j = \emptyset$, for all $i \neq j$, and normalized so that $E(\Omega) = \mathbb{1}$. The operators $E(X)$, $X \in \Sigma$, in the range of an observable E are called effects. In particular, an observable E is called *sharp* whenever all of its effects are projections, that is $E(X)^2 = E(X)$ for every $X \in \Sigma$. Every sharp observable $E : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$ satisfies the norm-1 property $\|E(X)\| = 1$, if $E(X) \neq 0$, however, the converse is not true [27]. We will refer to the observables with the norm-1 property as *approximately sharp observables*.

If $\Omega_N = \{x_1, x_2, \dots\}$ with $N \leq \infty$ elements and $\Sigma_N = 2^{\Omega_N}$ is the corresponding σ -algebra, then $E : \Sigma_N \rightarrow \mathcal{L}(\mathcal{H})$ can be viewed as a collection of positive operators $E(\{x_i\}) \equiv E(i) \in \mathcal{L}(\mathcal{H})$ satisfying $\sum_{i=1}^N E(i) = \mathbb{1}$. By reducing the outcome space, we can (and will) assume that $E(i) \neq 0$, for all $i = 1, \dots, N$. In such a case we say that E is a *discrete N -outcome observable*. A special family of discrete observables is formed by the (unbiased) binary spin-observables $S_{\vec{n}} : \Sigma_2 \rightarrow \mathcal{L}(\mathbb{C}^2)$ defined via $S_{\vec{n}}(\pm) \equiv \frac{1}{2}(\mathbb{1} \pm \vec{n} \cdot \vec{\sigma})$, where $\vec{n} \in \mathbb{R}^3$, $\|\vec{n}\| \leq 1$. It is well known that an observable $S_{\vec{n}}$ is sharp if and only if $\|\vec{n}\| = 1$. Related to the spin-observables, we often use the special notation $\hat{x} \equiv (1, 0, 0)$, $\hat{y} \equiv (0, 1, 0)$ and $\hat{z} \equiv (0, 0, 1)$.

Every observable E has a *Naimark dilation* into a sharp observable, that is there exist a Hilbert space \mathcal{K} , a sharp observable $A : \Sigma \rightarrow \mathcal{L}(\mathcal{K})$ and an

isometry $W : \mathcal{H} \rightarrow \mathcal{K}$ such that

$$\mathbf{E}(X) = W^* \mathbf{A}(X) W, \quad (3.1)$$

for all $X \in \Sigma$ [28]. The following characterization of sharp observables has been noted in Ref. [29].

Lemma 2. *Let \mathbf{E} be an observable and $(\mathcal{K}, \mathbf{A}, W)$ its Naimark dilation. The following are equivalent:*

- (i) $\mathbf{E}(X) \in \mathcal{P}(\mathcal{H})$, for all $X \in \Sigma$.
- (ii) $\mathbf{E}(X \cap Y) = \mathbf{E}(X) \mathbf{E}(Y)$, for all $X, Y \in \Sigma$.
- (iii) $[\mathbf{A}(X), WW^*] = 0$, for all $X \in \Sigma$.

The number $p_\varrho^{\mathbf{E}}(X) \equiv \text{tr}[\mathbf{E}(X)\varrho]$ is interpreted as the probability that a measurement of the observable \mathbf{E} performed on the state $\varrho \in \mathcal{S}(\mathcal{H})$ leads to a result in $X \in \Sigma$. An observable $\mathbf{E} : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$ can be transformed into another one $\mathbf{E}' : \Sigma' \rightarrow \mathcal{L}(\mathcal{H})$ by means of classically processing the obtained measurement statistics – by merging together, relabelling and weighting differently the measurement outcomes in a stochastic sense. In other words, this action of *post-processing* is described as a classical-to-classical channel λ between the probability measures on the measurable spaces (Ω, Σ) and (Ω', Σ') acting via $\lambda(\text{tr}[\mathbf{E}(\cdot)\varrho]) = \text{tr}[\mathbf{E}'(\cdot)\varrho]$, for all $\varrho \in \mathcal{S}(\mathcal{H})$. Throughout this work, every post-processing will without exception be associated with some *Markov kernel*, a map $\lambda : (\Omega, \Sigma') \rightarrow [0, 1]$ such that

- (i) $\lambda(x|\cdot)$ is a probability measure for every $x \in \Omega$,
- (ii) $\lambda(\cdot|Y)$ is measurable for every $Y \in \Sigma'$.

The post-processing of \mathbf{E} into \mathbf{E}' is then described via $\mathbf{E}'(Y) = \int \lambda(x|Y) d\mathbf{E}(x)$ for all $Y \in \Sigma'$: we often shorten this notion by writing $\mathbf{E}' = \lambda \star \mathbf{E}$.

The properties of post-processing have been studied e.g. in Refs. [30–34]. In particular it is known, that the only way to produce an extremal observable from another observable by means of post-processing is by merging together its effects. More specifically, if $\mathbf{E} : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$ is an observable that can be post-processed into an extremal observable $\mathbf{E}' : \Sigma' \rightarrow \mathcal{L}(\mathcal{H})$ with some $\lambda : (\Omega, \Sigma') \rightarrow [0, 1]$, then $\lambda(x|Y) \in \{0, 1\}$ for all $x \in \Omega$ and $Y \in \Sigma'$ [33].

3.2 Measurement models

Every measurement can be split into three stages: (1) preparations of the measured system and the measuring apparatus, (2) the two systems' mutual interaction and (3) subsequent reading of the apparatus' pointer scale. Let us denote by \mathcal{K} the Hilbert space associated to the measuring apparatus. We will impose the intuitive assumption, that the system and the apparatus are initially fully uncorrelated. Hence, in the stage (1) the system-apparatus –composite is prepared in a state $\varrho \otimes \xi$, $\varrho \in \mathcal{S}(\mathcal{H})$ and $\xi \in \mathcal{S}(\mathcal{K})$. In the stage (2) the initial state $\varrho \otimes \xi$ evolves to $\mathcal{V}(\varrho \otimes \xi)$ where $\mathcal{V} : \mathcal{T}(\mathcal{H} \otimes \mathcal{K}) \rightarrow \mathcal{T}(\mathcal{H}' \otimes \mathcal{K})$ is a channel¹ (or possibly dynamics) describing the measurement interaction. After this evolution, in the stage (3) the measurement outcome is read from the pointer scale: to serve this purpose we introduce $Z : \Sigma \rightarrow \mathcal{L}(\mathcal{K})$ as the apparatus' pointer observable. Accordingly, the quantum description of *measurement* is mathematically encased by a quadruple $\langle \mathcal{K}, Z, \mathcal{V}, \xi \rangle$ [2–4].

The probability of obtaining measurement result in $X \in \Sigma$ is given by $p(X) = \text{tr}[Z(X) \text{tr}_{\mathcal{H}'}[\mathcal{V}(\varrho \otimes \xi)]]$. Crucially, there exists a unique POVM $E : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$ satisfying $p(X) = \text{tr}[E(X) \varrho]$ [35]. We will say that E is the *measured observable* in $\langle \mathcal{K}, Z, \mathcal{V}, \xi \rangle$ or, conversely, that $\langle \mathcal{K}, Z, \mathcal{V}, \xi \rangle$ is an *E-measurement*. Since $\text{tr}[Z(X) \text{tr}_{\mathcal{H}'}[\mathcal{V}(\varrho \otimes \xi)]] = \text{tr}[\mathbb{1} \otimes Z(X) \mathcal{V}(\varrho \otimes \xi)]$ for all $\varrho \in \mathcal{S}(\mathcal{H})$, the measured observable is easily solved to be

$$E(X) = \text{tr}_{\xi}[\mathcal{V}^*(\mathbb{1} \otimes Z(X))], \quad X \in \Sigma; \quad (3.2)$$

see Eq.(2.3). Every measurement $\langle \mathcal{K}, Z, \mathcal{V}, \xi \rangle$ realizes some observable $E : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$ via Eq.(3.2). Conversely, every $E : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$ has an infinite number of measurement realizations [36].

If the observable E measured in $\langle \mathcal{K}, Z, \mathcal{V}, \xi \rangle$ is extremal, then one can assume that also Z , \mathcal{V} and ξ are extremal elements (in their respective structural sets); see Ref.[37] and the article I. For example, suppose that ξ is a (mixed) state with decomposition $\xi = \sum_{i=1}^N \lambda_i P_{\phi_i}$, where $0 < \lambda_i \leq 1$ satisfy $\sum_{i=1}^N \lambda_i = 1$. Then by the linearity $E(X) = \sum_{i=1}^N \lambda_i E_i(X)$, where

$$E_i(X) = \text{tr}_{P_{\phi_i}}[\mathcal{V}^*(\mathbb{1} \otimes Z(X))], \quad i = 1, \dots, N. \quad (3.3)$$

¹Here, in principle, we allow the system to change during the interaction, although in practise we often choose $\mathcal{H} = \mathcal{H}'$.

Due to the extremality, it holds that $E = E_i$ and hence also $\langle \mathcal{K}, Z, \mathcal{V}, P_{\phi_i} \rangle$ is an E -measurement for all $i = 1, \dots, N$. More can be said if E is a(n approximately) sharp observable.

***Proposition 1.** *Let $\langle \mathcal{K}, Z, \mathcal{V}, \xi \rangle$ be a measurement of an observable $E : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$ and assume that $\xi = \sum_{i=1}^N \lambda_i P_{\phi_i}$ for some $0 < \lambda_i \leq 1$ such that $\sum_{i=1}^N \lambda_i = 1$. Let us denote by $V_\phi : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{K}$, $\phi \in \mathcal{K}$, $\|\phi\| = 1$, the isometry satisfying $V_\phi(\varphi) = \varphi \otimes \phi$ for all $\varphi \in \mathcal{H}$. Then, for $X \in \Sigma$, $E(X) \in \mathcal{P}(\mathcal{H})$ if and only if $V_{\phi_i} V_{\phi_i}^* \mathcal{V}^*(\mathbb{1} \otimes Z(X)) V_{\phi_i} V_{\phi_i}^* \in \mathcal{P}(\mathcal{H} \otimes \mathcal{K})$ for all $i = 1, \dots, N$.*

Proof. Let $E(X) \in \mathcal{P}(\mathcal{H})$ for some $X \in \Sigma$, that is $E(X)$ is an extremal point of the convex set $\{A \in \mathcal{L}(\mathcal{H}) \mid 0 \leq A \leq \mathbb{1}\}$. Therefore $E(X) = \text{tr}_{P_{\phi_i}}[\mathcal{V}^*(\mathbb{1} \otimes Z(X))]$ for all $i = 1, \dots, N$. Since $V_{\phi_i} V_{\phi_i}^* = \mathbb{1}$, it follows that

$$\begin{aligned} V_{\phi_i} E(X) V_{\phi_i}^* &= V_{\phi_i} V_{\phi_i}^* \mathcal{V}^*(\mathbb{1} \otimes Z(X)) V_{\phi_i} V_{\phi_i}^* \\ &= (V_{\phi_i} V_{\phi_i}^* \mathcal{V}^*(\mathbb{1} \otimes Z(X)) V_{\phi_i} V_{\phi_i}^*)^2, \end{aligned} \quad (3.4)$$

that is $V_{\phi_i} V_{\phi_i}^* \mathcal{V}^*(\mathbb{1} \otimes Z(X)) V_{\phi_i} V_{\phi_i}^* \in \mathcal{P}(\mathcal{H} \otimes \mathcal{K})$ for all $i = 1, \dots, N$. On the contrary, if $V_{\phi_i} V_{\phi_i}^* \mathcal{V}^*(\mathbb{1} \otimes Z(X)) V_{\phi_i} V_{\phi_i}^* \in \mathcal{P}(\mathcal{H} \otimes \mathcal{K})$ then $E(X) = V_{\phi_i}^* V_{\phi_i} V_{\phi_i}^* \mathcal{V}^*(\mathbb{1} \otimes Z(X)) V_{\phi_i} V_{\phi_i}^* V_{\phi_i} = E(X)^2$ for all $i = 1, \dots, N$. \square

It is well known that for projections $P, Q \in \mathcal{P}(\mathcal{H})$ it holds that $PQP \in \mathcal{P}(\mathcal{H})$ if and only if $[P, Q] = 0$ [29]. The following extension of this rule was noticed in Ref. [38].

Lemma 3. *Let $P \in \mathcal{P}(\mathcal{H})$ be a projection and $A \in \mathcal{L}(\mathcal{H})$ such that $0 \leq A \leq \mathbb{1}$. If $PAP \in \mathcal{P}(\mathcal{H})$ then $[P, A] = 0$.*

Proof. Since $0 \leq A \leq \mathbb{1}$, then $A^2 \leq A$. We therefore have

$$0 \leq ((I - P)AP)^* (I - P)AP = PA(I - P)AP = P(A^2 - A)P \leq 0, \quad (3.5)$$

proving that $(I - P)AP = 0$. Hence, $AP = PAP = (PAP)^* = PA$. Clearly the converse is not true, that is from $[P, A] = 0$ it does not in general follow that $PAP \in \mathcal{P}(\mathcal{H})$. \square

***Proposition 2.** *Let $\langle \mathcal{K}, \mathbf{Z}, \mathcal{V}, \xi \rangle$ be a measurement of an approximately sharp observable $\mathbf{E} : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$. Then also $\mathbf{Z} : \Sigma \rightarrow \mathcal{L}(\mathcal{K})$ and $\mathcal{V}^*(\mathbb{1} \otimes \mathbf{Z}) : \Sigma \rightarrow \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$ are approximately sharp. Consequently, if \mathbf{E} is N -outcome, then $\dim(\mathcal{K}) \geq N$.*

Proof. For all $\varphi \in \mathcal{H}$, $\|\varphi\| = 1$, and $X \in \Sigma$ we have

$$\begin{aligned} |\langle \varphi | \mathbf{E}(X) \varphi \rangle| &= |\operatorname{tr} [\mathbb{1} \otimes \mathbf{Z}(X) \mathcal{V}(P_\varphi \otimes \xi)]| \\ &\leq \|\mathbb{1} \otimes \mathbf{Z}(X)\| \|\mathcal{V}(P_\varphi \otimes \xi)\|_1 \\ &= \|\mathbf{Z}(X)\|, \end{aligned} \tag{3.6}$$

and hence $1 = \|\mathbf{E}(X)\| = \sup_{\varphi \in \mathcal{H}, \|\varphi\|=1} |\langle \varphi | \mathbf{E}(X) \varphi \rangle| \leq \|\mathbf{Z}(X)\| \leq 1$, whenever $\mathbf{E}(X) \neq 0$. Therefore, $\|\mathbf{Z}(X)\| = 1$, for all $X \in \Sigma$ for which $\mathbf{Z}(X) \neq 0$. Using $\operatorname{tr} [\mathbb{1} \otimes \mathbf{Z}(X) \mathcal{V}(P_\varphi \otimes \xi)] = \operatorname{tr} [\mathcal{V}^*(\mathbb{1} \otimes \mathbf{Z}(X)) P_\varphi \otimes \xi]$ an analogous calculation shows that $\|\mathcal{V}^*(\mathbb{1} \otimes \mathbf{Z}(X))\| = 1$.

It is well known that if $\mathbf{Z} : \Sigma_N \rightarrow \mathcal{L}(\mathcal{K})$ is an N -outcome sharp observable then $\dim(\mathcal{K}) \geq N$. We will next show that the same implication holds whenever the N -outcome observable \mathbf{Z} is approximately sharp. Firstly, recall that \mathbf{Z} is approximately sharp if and only if for each $\mathbf{Z}(X) \neq 0$ and for any $\varepsilon > 0$ there exists a unit vector $\phi_X \in \mathcal{K}$ such that $\langle \phi_X | \mathbf{Z}(X) \phi_X \rangle \geq 1 - \varepsilon$ [27]. Denoting by X^c the complement $\Omega \setminus X$ one then has $\langle \phi_X | \mathbf{Z}(X^c) \phi_X \rangle \leq \varepsilon$. Since \mathbf{Z} is N -outcome, there exists N disjoint sets $X_i, i < N + 1$, in Σ_N such that $\mathbf{Z}(X_i) \neq 0$. Then for any $i \neq j$ and an arbitrary $\varepsilon > 0$

$$\begin{aligned} &|\langle \phi_{X_i} | \phi_{X_j} \rangle| \\ &= |\langle \phi_{X_i} | (\mathbf{Z}(X_i) + \mathbf{Z}(X_i^c)) \phi_{X_j} \rangle| \\ &\leq |\langle \mathbf{Z}(X_i)^{1/2} \phi_{X_i} | \mathbf{Z}(X_i)^{1/2} \phi_{X_j} \rangle| + |\langle \mathbf{Z}(X_i^c)^{1/2} \phi_{X_i} | \mathbf{Z}(X_i^c)^{1/2} \phi_{X_j} \rangle| \\ &\leq \sqrt{|\langle \phi_{X_j} | \mathbf{Z}(X_i) \phi_{X_j} \rangle|} + \sqrt{|\langle \phi_{X_i} | \mathbf{Z}(X_i^c) \phi_{X_i} \rangle|} \\ &\leq 2\sqrt{\varepsilon}, \end{aligned} \tag{3.7}$$

where the estimates follow from the triangle inequality, the Cauchy-Schwarz inequality and the fact that $\mathbf{Z}(X_i) \leq \mathbf{Z}(X_j^c)$, since $X_i \subset X_j^c$. We also recall that any set of unit vectors $\phi_i \in \mathcal{K}, i = 1, \dots, N$, satisfying $|\langle \phi_i | \phi_j \rangle| \leq \frac{1}{N-1}$ whenever $i \neq j$, are linearly independent in \mathcal{K} [39]. Therefore, choosing $\varepsilon \leq \left(\frac{1}{2(N-1)}\right)^2$ suffices to show that there exists N linearly independent unit vectors in \mathcal{K} , that span an N -dimensional subspace of \mathcal{K} . As a consequence $\dim(\mathcal{K}) \geq N$. \square

We note in the passing that the previous dimensionality bound for the apparatus' Hilbert space $\dim(\mathcal{K}) \geq N$ was derived in the article I in the case where the measured observable \mathbf{E} is N -outcome and sharp. This bound does not hold in general, if \mathbf{E} is not (approximately) sharp. For instance, the trivial observable $X \mapsto p(X)\mathbb{1}$ can be realized with a one-dimensional apparatus $\langle \mathbb{C}, p, 1, 1 \rangle$.

3.2.1 Quantum instruments and compatibility

The properties of a measurement $\langle \mathcal{K}, \mathbf{Z}, \mathcal{V}, \xi \rangle$ that pertain to the measured system are fully captured by the associated *quantum instrument*, a completely positive map-valued mapping $\mathcal{I} : \Sigma \rightarrow \mathcal{L}(\mathcal{T}(\mathcal{H}'))$ defined via

$$\mathcal{I}(X)(T) = \text{tr}_{\mathcal{K}}[\mathbb{1} \otimes \mathbf{Z}(X) \mathcal{V}(T \otimes \xi)], \quad X \in \Sigma, T \in \mathcal{T}(\mathcal{H}). \quad (3.8)$$

For instance, the statistics of the measured observable $\mathbf{E} : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$ are given by $\text{tr}[\mathbf{E}(X)\varrho] = \text{tr}[\mathcal{I}(X)(\varrho)]$, $\varrho \in \mathcal{S}(\mathcal{H})$. The mapping \mathcal{I} can be equivalently recast in the dual form $\mathcal{I}^* : \Sigma \rightarrow \mathcal{L}(\mathcal{L}(\mathcal{H}))$ produced via $\text{tr}[\mathcal{I}^*(X)(B)T] = \text{tr}[B\mathcal{I}(X)(T)]$, $B \in \mathcal{L}(\mathcal{H}'), T \in \mathcal{T}(\mathcal{H})$. Then $\mathbf{E}(X) = \mathcal{I}^*(X)(\mathbb{1})$, for all $X \in \Sigma$.

Physically $\mathcal{I}(X)(\varrho)$ describes the (subnormalized) output state of the system after measurement, given that the system was initially prepared in the state $\varrho \in \mathcal{S}(\mathcal{H})$ and on the condition that a result from $X \in \Sigma$ was obtained. In particular,

$$\mathcal{E}(\varrho) \equiv \mathcal{I}(\Omega)(\varrho) = \text{tr}_{\mathcal{K}}[\mathcal{V}(\varrho \otimes \xi)], \quad \varrho \in \mathcal{S}(\mathcal{H}), \quad (3.9)$$

defines the total state transformation related to the measurement $\langle \mathcal{K}, \mathbf{Z}, \mathcal{V}, \xi \rangle$. Noticeably, the pointer observable $\mathbf{Z} : \Sigma \rightarrow \mathcal{L}(\mathcal{K})$ plays no role in the definition of this (measurement's) *induced channel* $\mathcal{E} : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H}')$ and can thus be omitted altogether. We may interpret this as letting the two systems to evolve, but not reading the apparatus' pointer scale (omitting the stage (3)). In this sense, quantum measurements encase the theory of open quantum system. Indeed, the system and measurement apparatus (and possibly some additional environment) form a closed system and the reduced channel/dynamics $\mathcal{I}(\Omega)(\varrho)$ is of the form of Eq. (2.9).

The dual form of the total state transformation reads

$$\mathcal{E}^*(B) = \text{tr}_{\xi}[\mathcal{V}^*(B \otimes \mathbb{1})], \quad B \in \mathcal{L}(\mathcal{H}'). \quad (3.10)$$

We will be needing the following result, which is similar to what was concluded in Lemma 1. The proof is analogous to that of Prop. 1 and will therefore be omitted.

***Proposition 3.** *Let $\langle \mathcal{K}, \mathcal{Z}, \mathcal{V}, \xi \rangle$ be a measurement and assume that $\xi = \sum_{i=1}^N \lambda_i P_{\phi_i}$ for some $0 < \lambda_i \leq 1$ such that $\sum_{i=1}^N \lambda_i = 1$. Then $\mathcal{E}^*(B) \in \mathcal{P}(\mathcal{H})$ if and only if $V_{\phi_i} V_{\phi_i}^* \mathcal{V}^*(B \otimes \mathbb{1}) V_{\phi_i} V_{\phi_i}^* \in \mathcal{P}(\mathcal{H} \otimes \mathcal{K})$ for all $i = 1, \dots, N$.*

In order to investigate the properties of a quantum system it is intuitive that some information must flow from the system into the measuring apparatus. According to Chap. 2, one would then argue that the system to be measured is necessarily open and some non-unitary state change is to be expected. In fact, it is well known that if $\mathcal{I}(\Omega)(\rho) = G\rho G^*$, for all $\rho \in \mathcal{S}(\mathcal{H})$, where G is a unitary on \mathcal{H} , the measured observable \mathbf{E} is trivial [2–4]; see also Ex. 2 that follows. In other words, no meaningful information can be extracted from the system as long as it stays closed: this result is often referred as *no information without disturbance*. A more general link between the extractable information and disturbance has been solved in Ref. [40]. We will next focus on another viewpoint, which states that the only observables being *compatible* with unitary channels are trivial.

Generally speaking, two quantum devices (observables, channels or instruments) A and B are compatible whenever there exists a fixed measurement that would (in a suitable sense) realize both A and B [41]. On the other hand, the two devices are called *incompatible* if such a joint measurement realization cannot be found. Although this definition would allow to analyse the incompatibility of a rich variety of different devices, in this work we will be concentrating only on the two special instances: incompatibility of pair of observables, and that between observables and channels.

Let us fix two standard Borel spaces $(\Omega_1, \mathcal{B}(\Omega_1))$ and $(\Omega_2, \mathcal{B}(\Omega_2))$ and let $\mathcal{B}(\Omega_1 \times \Omega_2)$ denote the Borel σ -algebra of $\Omega_1 \times \Omega_2$ ². The compatibility of two observables $\mathbf{E}_1 : \mathcal{B}(\Omega_1) \rightarrow \mathcal{L}(\mathcal{H})$ and $\mathbf{E}_2 : \mathcal{B}(\Omega_2) \rightarrow \mathcal{L}(\mathcal{H})$ reduces to finding a third observable $\mathbf{E} : \mathcal{B}(\Omega_1 \times \Omega_2) \rightarrow \mathcal{L}(\mathcal{H})$ having the two as its marginals: $\mathbf{E}_1(X) = \mathbf{E}(X \times \Omega_2)$ and $\mathbf{E}_2(Y) = \mathbf{E}(\Omega_1 \times Y)$. Compatible quantum observables are often aptly called *jointly measurable*. Moreover, the joint measurability of observables \mathbf{E}_1 and \mathbf{E}_2 is traditionally associated with their vanishing commutator $[\mathbf{E}_1(X), \mathbf{E}_2(Y)] = 0$, for all $X \in \mathcal{B}(\Omega_1)$,

²We impose this technical assumption in order to avoid dealing with operator bimeasures [42].

$Y \in \mathcal{B}(\Omega_2)$. This is indeed an equivalent notion of compatibility if either of the two observables is sharp. This fact is encased in the more general relation: if the observables $E_1 : \mathcal{B}(\Omega_1) \rightarrow \mathcal{L}(\mathcal{H})$ and $E_2 : \mathcal{B}(\Omega_2) \rightarrow \mathcal{L}(\mathcal{H})$ satisfy the inequality

$$\| [E_1(X), E_2(Y)] \|^2 > 4 \| E_1(X) - E_1(X)^2 \| \| E_2(Y) - E_2(Y)^2 \|, \quad (3.11)$$

for all $X \in \mathcal{B}(\Omega_1)$, $Y \in \mathcal{B}(\Omega_2)$, then they are incompatible [41, 43]. In particular, this relation envelopes the fact that if one of the observables is sharp, then their compatibility is equivalent to $[E_1(X), E_2(Y)] = 0$. For spin-observables $S_{\vec{m}}$ and $S_{\vec{n}}$ the relation (3.11) simplifies to

$$\| \vec{m} \times \vec{n} \|^2 > (1 - \|\vec{m}\|^2) (1 - \|\vec{n}\|^2), \quad (3.12)$$

using which it is easy to find examples of compatible observables with non-vanishing commutator.

Analogously, a channel $\mathcal{E} : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H}')$ and an observable $E : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$ are compatible, if there exists an instrument $\mathcal{I} : \Sigma \rightarrow \mathcal{L}(\mathcal{T}(\mathcal{H}'))$ such that $\mathcal{I}(\Omega)(\cdot) = \mathcal{E}(\cdot)$ and $\mathcal{I}^*(\cdot)(\mathbb{1}) = E(\cdot)$ [41]. The aforementioned “no information without disturbance” signals the fact that not all channels and observables are compatible. Indeed, if either the induced channel $\mathcal{E}^* : \mathcal{L}(\mathcal{H}') \rightarrow \mathcal{L}(\mathcal{H})$ is a $*$ -homomorphism or the measured observable $E : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$ is sharp, then their compatibility is equivalent the commutativity $[\mathcal{E}^*(B), E(X)] = 0$, $B \in \mathcal{L}(\mathcal{H}')$, $X \in \Sigma$ [36, 44–46]; see also Ex.2 that follows.

3.3 Normal measurement models

In general, a measurement model $\langle \mathcal{K}, Z, \mathcal{V}, \xi \rangle$ describes a situation where there can exist some degree of noise or approximation in the parts of the measuring device. On the contrary, an ideally functioning measuring device would form a closed system together with the system to be measured, ensuring that there is no information leaking outside the system-apparatus-composite. In addition, the detectors of such a device should work with perfect accuracy, and the apparatus should be initially prepared in a state of maximal information. Following this reasoning, one arrives with the definition of a *normal (unitary) measurement* [2] where $Z : \Sigma \rightarrow \mathcal{L}(\mathcal{K})$ is a sharp observable, $\mathcal{V} : \mathcal{T}(\mathcal{H} \otimes \mathcal{K}) \rightarrow \mathcal{T}(\mathcal{H} \otimes \mathcal{K})$ is conjugation with some

unitary operator U on $\mathcal{H} \otimes \mathcal{K}$ and $\xi = P_\phi$ for some unit vector $\phi \in \mathcal{K}$. In the case of normal measurements we will shorten the notation by writing $\langle \mathcal{K}, \mathbf{Z}, U, \phi \rangle$.

The instrument \mathcal{I} associated to a normal measurement model $\langle \mathcal{K}, \mathbf{Z}, U, \phi \rangle$ has a particularly simple form

$$\mathcal{I}^*(X)(B) = V_\phi^* U^*(B \otimes \mathbf{Z}(X)) U V_\phi, \quad X \in \Sigma, B \in \mathcal{L}(\mathcal{H}), \quad (3.13)$$

from which it is immediate to solve the measured observable $\mathbf{E}(X) = V_\phi^* U^*(\mathbb{1} \otimes \mathbf{Z}(X)) U V_\phi$, $X \in \Sigma$ and the induced channel $\mathcal{E}^*(B) = V_\phi^* U^*(B \otimes \mathbb{1}) U V_\phi$, $B \in \mathcal{L}(\mathcal{H})$. It was first proven in Ref. [36] that every (completely positive) instrument \mathcal{I} attains a normal measurement model $\langle \mathcal{K}, \mathbf{Z}, U, \phi \rangle$ so that it has the form of Eq. (3.13).

The minimal dimension of \mathcal{K} in which a normal measurement realization of \mathcal{I} is possible has been analysed in Refs. [44, 47, 48] and in the article VII. Especially, if we only focus on realizing the measured observable or the induced channel the minimality of \mathcal{K} can be concisely stated. In the first case, if \mathbf{E} is an N -outcome discrete observable $\mathbf{E} : \Sigma_N \rightarrow \mathcal{L}(\mathcal{H})$ it holds that $\dim(\mathcal{K}) \geq N$ (regardless of whether \mathbf{E} is approximately sharp or not), and such a minimal normal measurement can always be found; see Refs. [44, 47, 48] and the article VII. In the latter case, the dimension of \mathcal{K} depends on the Kraus rank of the channel, such that always $\dim(\mathcal{K}) \geq \text{rank}(\mathcal{E})$, but this lower bound cannot be met in general. However, for any \mathcal{E} one can find a normal model having $\dim(\mathcal{K}) = \text{rank}(\mathcal{E}) + 1$; see the article VII.

Due to the simpler form of normal measurements, the structural limitations corresponding to those we pointed out in Props. 1 and 3 become slightly more specific.

Proposition 1. *Let $\langle \mathcal{K}, \mathbf{Z}, U, \phi \rangle$ be a normal measurement and let $\mathbf{E} : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$ be the measured observable and $\mathcal{E}^* : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ be the induced channel. For $X \in \Sigma$ and $P \in \mathcal{P}(\mathcal{H})$*

$$\begin{aligned} (i) \quad \mathbf{E}(X) \in \mathcal{P}(\mathcal{H}) &\Leftrightarrow V_\phi V_\phi^* U^*(\mathbb{1} \otimes \mathbf{Z}(X)) U V_\phi V_\phi^* \in \mathcal{P}(\mathcal{H} \otimes \mathcal{K}) \\ &\Leftrightarrow [V_\phi V_\phi^*, U^*(\mathbb{1} \otimes \mathbf{Z}(X)) U] = 0. \end{aligned} \quad (3.14)$$

$$\begin{aligned} (ii) \quad \mathcal{E}^*(P) \in \mathcal{P}(\mathcal{H}) &\Leftrightarrow V_\phi V_\phi^* U^*(P \otimes \mathbb{1}) U V_\phi V_\phi^* \in \mathcal{P}(\mathcal{H} \otimes \mathcal{K}) \\ &\Leftrightarrow [V_\phi V_\phi^*, U^*(P \otimes \mathbb{1}) U] = 0. \end{aligned} \quad (3.15)$$

Proof. The chains of equivalences (i) and (ii) have been originally proven in Ref. [29] and article I, respectively; see also Lemmas 1 and 2. They also follow as special cases from Props. 1 and 3, when recalling that for any projections $P, Q \in \mathcal{P}(\mathcal{H})$ the equivalence $PQP \in \mathcal{P}(\mathcal{H}) \Leftrightarrow [P, Q] = 0$ holds [29]. \square

Example 2. As a corollary of the previous proposition, we can provide a simple proof for the fact that, if $\mathcal{E}(\varrho) = \mathcal{I}(\Omega)(\varrho) = G\varrho G^*$, for all $\varrho \in \mathcal{S}(\mathcal{H})$, where G is unitary on \mathcal{H} , then $\mathbf{E}(X) = \mathcal{I}^*(X)(\mathbb{1}) = p(X)\mathbb{1}$ for some probability measure $p : \Sigma \rightarrow [0, 1]$, that was mentioned earlier. Indeed, recall that any instrument \mathcal{I} can be realized from some normal model $\langle \mathcal{K}, \mathbf{Z}, U, \phi \rangle$. Since now $\mathcal{E}^*(P) = \mathcal{E}^*(P)^2$ for all $P \in \mathcal{P}(\mathcal{H})$, we have

$$\begin{aligned} \mathcal{E}^*(P)\mathbf{E}(X) &= V_\phi^* U^*(P \otimes \mathbb{1}) UV_\phi V_\phi^* U^*(\mathbb{1} \otimes \mathbf{Z}(X)) UV_\phi \\ &= V_\phi^* U^*(P \otimes \mathbf{Z}(X)) UV_\phi = \mathcal{I}^*(X)(P) \\ &= V_\phi^* U^*(\mathbb{1} \otimes \mathbf{Z}(X)) UV_\phi V_\phi^* U^*(P \otimes \mathbb{1}) UV_\phi = \mathbf{E}(X)\mathcal{E}^*(P). \end{aligned} \tag{3.16}$$

Due to the linearity, this commutation extends to $[\mathbf{E}(X), \mathcal{E}^*(B)] = 0$ for all $X \in \Sigma, B \in \mathcal{L}(\mathcal{H})$, which is only possible when $\mathbf{E}(X) = p(X)\mathbb{1}$. As a corollary, we have verified the known result that, if either \mathbf{E} is a sharp observable or \mathcal{E} a homomorphic channel, then $\mathcal{I}^*(X)(B) = \mathbf{E}(X)\mathcal{E}^*(B) = \mathcal{E}^*(B)\mathbf{E}(X)$ and $[\mathbf{E}(X), \mathcal{E}^*(B)] = 0$ for all $X \in \Sigma, B \in \mathcal{L}(\mathcal{H})$ [36, 44–46].

3.3.1 Standard model

Let us assume that one intends to measure a sharp observable $\mathbf{A} : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(\mathcal{H})$ of the system by coupling it to the sharp momentum observable of the apparatus $\mathbf{P} : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(L^2(\mathbb{R}))$ via the unitary coupling

$$U_\lambda \equiv e^{-i\lambda \bar{\mathbf{A}} \otimes \bar{\mathbf{P}}}, \tag{3.17}$$

where $\lambda \in \mathbb{R}$ is a parameter quantifying the strength of the coupling. The notation $\bar{\mathbf{A}}$ stands for the first moment $\bar{\mathbf{A}} = \int_{\sigma(\mathbf{A})} x \mathbf{A}(dx)$, where $\sigma(\mathbf{A})$ denotes the spectrum of $\bar{\mathbf{A}}$, which in this case corresponds to the smallest closed set $S \in \mathcal{B}(\mathbb{R})$ satisfying $\mathbf{A}(S) = \mathbb{1}$. From this we solve the represen-

tation

$$U_\lambda = \int_{\sigma(\mathbf{A})} \mathbf{A}(dx) \otimes e^{-i\lambda x \bar{\mathbf{P}}}, \quad (3.18)$$

which suggests that U_λ will generate measurable shifts in the apparatus' position: $e^{-i\lambda x \bar{\mathbf{P}}} \phi(q) = \phi(q - \lambda x)$ for all $\phi \in L^2(\mathbb{R})$ and $q \in \mathbb{R}$. In order to monitor these shifts it is natural to choose as the pointer the sharp position observable $\mathbf{Q} : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(L^2(\mathbb{R}))$. In addition, to match the measurable spaces of the pointer and the observable measured we introduce a special type of post-processing: a (measurable) pointer function $f : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. The resulting normal measurement $\langle \mathcal{K}, \mathbf{Q}, U_\lambda, \phi, f \rangle$ is typically referred in the literature as the *standard measurement* of \mathbf{A} [2, 4, 49].

Exploiting Eq. (3.18) one finds out that the actual observable measured $\mathbf{E} : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(\mathcal{H})$ in the standard measurement is (usually) not the intended sharp \mathbf{A} but rather a smeared version of it

$$\mathbf{E}(Y) = \int_{\sigma(\mathbf{A})} \langle \phi | \mathbf{Q}(f^{-1}(Y) - \lambda x) \phi \rangle \mathbf{A}(dx) = (\lambda \star \mathbf{A})(Y), \quad Y \in \mathcal{B}(\mathbb{R}), \quad (3.19)$$

with the Markov kernel $\lambda(x|Y) \equiv \langle \phi | \mathbf{Q}(f^{-1}(Y) - \lambda x) \phi \rangle$. In fact, according to Prop. 1 \mathbf{E} is sharp if and only if $[\mathbf{Q}(f^{-1}(Y)), P_\phi] = 0$ for all $Y \in \mathcal{B}(\mathbb{R})$, which on the other hand is equivalent with $\phi \in L^2(\mathbb{R})$ being an eigenvector for all $\mathbf{Q}(f^{-1}(Y))$, $Y \in \mathcal{B}(\mathbb{R})$. Evidently, this can happen only if $\mathbf{A} : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(\mathcal{H})$ is not purely continuous, that is the point spectrum $\sigma_p(\mathbf{A}) \equiv \{a \in \mathbb{R} \mid \mathbf{A}(a) \neq 0\} \subset \sigma(\mathbf{A})$ is non-empty.

Example 3. Let us consider the standard measurement of the sharp spin-observable $\mathbf{S}_{\hat{x}}(\pm) = \frac{1}{2}(\mathbb{1} \pm \sigma_x)$. Obviously, $\bar{\mathbf{S}}_{\hat{x}} = \sigma_x$ and $U_\lambda = e^{-i\lambda \sigma_x \otimes \bar{\mathbf{P}}} = \mathbf{S}_{\hat{x}}(+)\otimes e^{-i\lambda \bar{\mathbf{P}}} + \mathbf{S}_{\hat{x}}(-)\otimes e^{i\lambda \bar{\mathbf{P}}}$. As the pointer function, let us define $f : \mathbb{R} \rightarrow \{-1, +1\}$ such that $f(\mathbb{R}_+) = +1$ and $f(\mathbb{R}_-) = -1$, $\mathbb{R}_+ \equiv [0, \infty)$, $\mathbb{R}_- \equiv (-\infty, 0)$. Then

$$\begin{aligned} \mathbf{E}(+1) &= \langle \phi | \mathbf{Q}([-\lambda, \infty)) \phi \rangle \mathbf{S}_{\hat{x}}(+) + \langle \phi | \mathbf{Q}([\lambda, \infty)) \phi \rangle \mathbf{S}_{\hat{x}}(-) \\ \mathbf{E}(-1) &= \mathbb{1} - \mathbf{E}(+1). \end{aligned} \quad (3.20)$$

From these equations one can gain some intuition of the role of the coupling strength $\lambda \in \mathbb{R}$: for an asymptotically strong coupling $\lambda \rightarrow \infty$, $\mathbf{E}(\pm 1) \rightarrow \mathbf{S}_{\hat{x}}(\pm)$ that is the measurement becomes sharp (regardless of the apparatus

state $\phi \in L^2(\mathbb{R})$). On the other hand, \mathbf{E} becomes the trivial observable $\mathbf{E}(\pm) = \langle \phi | \mathbf{Q}(\mathbb{R}_{\pm}) \phi \rangle \mathbb{1}$ for the vanishing coupling strength $\lambda = 0$. This is to be expected since $U_0 = \mathbb{1}$ implying that $\mathcal{I}_0(\Omega)(\varrho) = \text{tr}_{\mathcal{K}}[U_0(\varrho \otimes \xi)U_0^*] = \varrho$ for all $\varrho \in \mathcal{S}(\mathcal{H})$; see Ex. 2.

3.3.2 Weak measurements and weak values

Let us elaborate on what was found in the previous example. Assume that we would like to have an adjustable parameter $\lambda \in \mathbb{R}$ in a coupling of a normal measurement $\langle \mathcal{K}, \mathbf{Z}, U_\lambda, \phi \rangle$ corresponding to the strength of the measurement. It is natural to assume that $U_{\lambda_1}U_{\lambda_2} = U_{\lambda_1+\lambda_2}$, which turns $\{U_\lambda\}_{\lambda \in \mathbb{R}}$ into a group. We quantify the strength of the measurement in terms of the disturbance $D_{tr}(\mathcal{I}_\lambda(\Omega)(\varrho), \varrho)$, where $\mathcal{I}_\lambda(\Omega)(\varrho) \equiv \text{tr}_{\mathcal{K}}[U_\lambda(\varrho \otimes \xi)U_\lambda^*]$, and in particular we require that for vanishing $\lambda \approx 0$ also $D_{tr}(\mathcal{I}_\lambda(\Omega)(\varrho), \varrho) \approx 0$, or more precisely that for all $\varrho \in \mathcal{S}(\mathcal{H})$ $\lim_{\lambda \rightarrow 0} D_{tr}(\mathcal{I}_\lambda(\Omega)(\varrho), \varrho) = 0$. It follows from the so-called data-processing inequality [14] that, for all $\varrho \in \mathcal{S}(\mathcal{H})$, the relation

$$D_{tr}(\mathcal{I}_\lambda(\Omega)(\varrho), \varrho) \leq D_{tr}(U_\lambda \varrho \otimes P_\phi U_\lambda^*, \varrho \otimes P_\phi) \quad (3.21)$$

holds. Therefore, the constraint on the disturbance is in particular implied, when we assume that $\lim_{\lambda \rightarrow \lambda_0} \|U_\lambda T U_\lambda^* - U_{\lambda_0} T U_{\lambda_0}^*\|_1 = 0$ for all $T \in \mathcal{T}(\mathcal{H} \otimes \mathcal{K})$ and $\lambda_0 \in \mathbb{R}$. Under this condition the group $\{U_\lambda\}_{\lambda \in \mathbb{R}}$ becomes a strongly continuous one-parameter unitary group, which is equivalent to the existence of a fixed self-adjoint operator H such that $U_\lambda = e^{-i\lambda H}$ for all $\lambda \in \mathbb{R}$ [12]. Such a condition holds especially for the previously introduced standard coupling $U_\lambda = e^{-i\lambda \tilde{\mathbf{A}} \otimes \tilde{\mathbf{P}}}$.

Let us consider the 2-dimensional position $\mathbf{Q}(X, Y) \equiv \mathbf{Q}_x(X) \otimes \mathbf{Q}_y(Y)$ and momentum $\mathbf{P}(X, Y) \equiv P_x(X) \otimes P_y(Y)$ observables, $X, Y \in \mathcal{B}(\mathbb{R})$. For small $|\lambda| \approx 0$, according to the above analysis the disturbance caused by the standard measurement $\langle L^2(\mathbb{R}^2), \mathbf{Q}, U_\lambda, \phi \rangle$, $U_\lambda \equiv e^{-i\lambda \tilde{\mathbf{A}} \otimes \tilde{\mathbf{P}}_x} \otimes \mathbb{1}$, is negligible: in this sense it is justified to call $\langle L^2(\mathbb{R}^2), \mathbf{Q}, U_\lambda, \phi \rangle$ a *weak measurement*. Assume that we then make a subsequent measurement of an observable $\mathbf{E} : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$. The joint probability of such an arrangement then reads

$$\begin{aligned} p(X, Y, Z) &\equiv \text{tr}[\mathbf{E}(Z) \mathcal{I}_\lambda(X, Y)(\varrho)] \\ &= \text{tr}[\mathbf{E}(Z) \otimes \mathbf{Q}_x(X) \otimes \mathbf{Q}_y(Y) U_\lambda(\varrho \otimes P_\phi) U_\lambda^*], \end{aligned} \quad (3.22)$$

$X, Y \in \mathcal{B}(\mathbb{R})$ and $Z \in \Sigma$, which allows us to formally³ solve the expectation

$$\bar{p}_x(Z) \equiv \int_{\mathbb{R}} x p(dx, \mathbb{R}, Z) = \text{tr} \left[\mathbf{E}(Z) \otimes \bar{\mathbf{Q}}_x \otimes \mathbb{1} U_\lambda(\varrho \otimes P_\phi) U_\lambda^* \right], \quad (3.23)$$

where $Z \in \Sigma$. We have

$$\frac{\partial}{\partial \lambda} \bar{p}_x(Z) = \text{tr} \left[i [\bar{\mathbf{A}} \otimes \bar{\mathbf{P}}_x, \mathbf{E}(Z) \otimes \bar{\mathbf{Q}}_x] \otimes \mathbb{1} U_\lambda(\varrho \otimes P_\phi) U_\lambda^* \right], \quad (3.24)$$

which for small $|\lambda| \approx 0$ can be approximated as

$$\begin{aligned} \frac{\partial}{\partial \lambda} \bar{p}_x(Z) &\approx \text{tr} \left[i [\bar{\mathbf{A}} \otimes \bar{\mathbf{P}}_x, \mathbf{E}(Z) \otimes \bar{\mathbf{Q}}_x] \otimes \mathbb{1} \varrho \otimes P_\phi \right] \\ &= i \text{tr} \left[\bar{\mathbf{A}} \mathbf{E}(Z) \varrho \right] \text{tr} \left[(\bar{\mathbf{P}}_x \bar{\mathbf{Q}}_x) \otimes \mathbb{1} P_\phi \right] \\ &\quad - i \text{tr} \left[\mathbf{E}(Z) \bar{\mathbf{A}} \varrho \right] \text{tr} \left[(\bar{\mathbf{Q}}_x \bar{\mathbf{P}}_x) \otimes \mathbb{1} P_\phi \right], \end{aligned} \quad (3.25)$$

the right-hand side of which is now constant in λ . Then, by choosing $\phi \in L^2(\mathbb{R}^2)$ as the Laguerre-Gaussian $\phi(x, y) = \pi^{-1/2} (x - iy) e^{-(x^2 + y^2)/2}$, we have $\text{tr} \left[(\bar{\mathbf{Q}}_x \bar{\mathbf{P}}_x) \otimes \mathbb{1} P_\phi \right] = \frac{i}{2} = -\text{tr} \left[(\bar{\mathbf{P}}_x \bar{\mathbf{Q}}_x) \otimes \mathbb{1} P_\phi \right]$ (in the units $\hbar = 1$) implying that $\bar{p}_x(Z) \approx \lambda/2 \text{tr} \left[[\mathbf{E}(Z), \bar{\mathbf{A}}] \varrho \right] = \lambda \Re \left[\text{tr} \left[\mathbf{E}(Z) \bar{\mathbf{A}} \varrho \right] \right]$, \Re denoting the real part of a complex number, $\Re[z] = \frac{1}{2}(z + z^*)$, $z \in \mathbb{C}$. Expressing the same in a slightly different form we have that

$$\bar{p}_x(Z) \approx \lambda \text{tr} \left[\mathbf{E}(Z) \varrho \right] \Re \left[\mathbf{A}^w(\mathbf{E}(Z), \varrho) \right], \quad (3.26)$$

where

$$\mathbf{A}^w(\mathbf{E}(Z), \varrho) = \frac{\text{tr} \left[\mathbf{E}(Z) \bar{\mathbf{A}} \varrho \right]}{\text{tr} \left[\mathbf{E}(Z) \varrho \right]} \quad (3.27)$$

is the so-called *weak value* of \mathbf{A} . In summary, in the sequential measurement scenario described above the pointer position in x -direction gets on average shifted by the amount $\lambda \Re \left[\mathbf{A}^w(\mathbf{E}(Z), \varrho) \right]$ and the pre-factor $\text{tr} \left[\mathbf{E}(Z) \varrho \right] \approx \text{tr} \left[\mathbf{E}(Z) \mathcal{I}_\lambda(\mathbb{R}, \mathbb{R})(\varrho) \right]$ is interpreted as a quantifier of intensity at which the

³In order to preserve the readability, in the treatment presented here we implicitly assume all the necessary regularities. For the mathematical details of the calculations done in this subsection we refer the reader to Ref. [50].

weak value is represented in the experiment⁴.

From the previous setting we could only extract the real part of the complex weak value, which is reflecting the obvious fact that experiments can only produce real numbers. However, in a similar manner, one can show that for small $|\lambda| \approx 0$

$$\begin{aligned} \bar{p}_y(Z) &\equiv \int_{\mathbb{R}} y \operatorname{tr} [\mathbf{E}(Z) \mathcal{I}_\lambda(\mathbb{R}, dy)(\varrho)] \\ &= \operatorname{tr} \left[\mathbf{E}(Z) \otimes \mathbb{1} \otimes \bar{\mathbf{Q}}_y U_\lambda(\varrho \otimes P_\phi) U_\lambda^* \right] \\ &\approx \lambda \operatorname{tr} [\mathbf{E}(Z) \varrho] \Im [\mathbf{A}^w(\mathbf{E}(Z), \varrho)], \quad Z \in \Sigma, \end{aligned} \quad (3.28)$$

with \Im denoting the imaginary part of a complex number, $\Im[z] = \frac{1}{2i}(z - z^*)$, $z \in \mathbb{C}$. The two averages $\bar{p}_x(Z) \propto \Re[\mathbf{A}^w(\mathbf{E}(Z), \varrho)]$ and $\bar{p}_y(Z) \propto \Im[\mathbf{A}^w(\mathbf{E}(Z), \varrho)]$ then enable one to fully extract the complex weak value. It is noteworthy that in the above construction the real and the imaginary parts of the weak value are extracted jointly with a single fixed measurement set-up $\langle L^2(\mathbb{R}^2), \mathbf{Q}, U_\lambda, \phi \rangle$.

Weak measurements and weak values have had a controversial reputation ever since they were first introduced; see Refs. [51–53] and references therein. The complex feature of weak values can be harnessed in many tasks that may seemingly allow one to go beyond the limits of traditional quantum measurements: weak values have been utilized in the direct measurements of quantum state [50, 54–56] and some other conventionally unobservable quantities, such as the geometric phase [57] and entanglement in the article VI, and in “resolving” some of the quantum paradoxes [58], to name a few. Even though it has been successfully demonstrated, both in theory and practice, that the weak measurements and weak values can be beneficial in various information extraction tasks (see e.g. Refs. [59, 60]), there remains disagreement on how the weak values should be physically interpreted [60, 61] – or whether any physical meaning should be ascribed to them at all [59, 62, 63]. This subsection is not aiming to resolve the debate, but to clarify how weak measurements and weak values arise operationally from the theory of (sequential) measurements as first order approximations with respect to the coupling strength λ .

⁴Often in the literature \mathbf{E} is chosen as a rank-1 sharp observable, so that $\mathbf{E}(Z) = |\psi\rangle\langle\psi|$, for some $\psi \in \mathcal{H}$, and its measurement called the *post-selection*. Then, assuming that $\varrho = |\varphi\rangle\langle\varphi|$, for $\varphi \in \mathcal{H}$, weak value takes the more familiar form $\mathbf{A}^w(\psi, \varphi) = \frac{\langle\psi|\mathbf{A}\varphi\rangle}{\langle\psi|\varphi\rangle}$.

3.4 Limitations of measurability: Wigner-Araki-Yanase theorem

Every quantum observable can be measured in an infinitely many different ways [36]; the theory of quantum measurements *per se* will not impose limitations for the *measurability* of observables. This is an important upside, since in order to comprehend natural phenomena it is crucial to be able to assign values to the physical properties of the system of interest. However, being physical processes, quantum measurements are subjected to the laws of physics; in particular, the conservation laws. In fact, it has been known since the dawn of the quantum theory that certain conservation laws will set limitations to the measurability of quantum observables. More specifically, any (discrete) sharp observable that does not commute with an additive conserved quantity cannot have a perfectly precise repeatable measurement. This limit is known as the WAY-theorem, bearing the initials of its founders Wigner, Araki and Yanase [6, 7].

We will say that a measurement $\langle \mathcal{K}, \mathcal{Z}, \mathcal{V}, \xi \rangle$ of an observable $E : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$ is *repeatable* if any recorded outcome of the measurement does not change upon its immediate repetition. Expressed in terms of the instrument \mathcal{I} , the measurement is repeatable if and only if

$$\mathrm{tr} [E(X) \mathcal{I}(X)(\varrho)] = \mathrm{tr} [E(X) \varrho] , \quad (3.29)$$

for all $X \in \Sigma$ and $\varrho \in \mathcal{S}(\mathcal{H})$. The set of observables that have a repeatable measurement is very limited: only the observables that are discrete and whose every non-zero effect has an eigenvalue 1 can be measured in such a manner [36].

A bounded self-adjoint operator $L \in \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$ is called a *conserved quantity* (with respect to a unitary U) if $\mathrm{tr} [LT] = \mathrm{tr} [LUTU^*]$ for all $T \in \mathcal{T}(\mathcal{H} \otimes \mathcal{K})$. Although the concept of conservation is more transparent from the previous equation, the equivalent notion $[L, U] = 0$ is often more useful. Moreover, if $L = L_1 \otimes \mathbb{1} + \mathbb{1} \otimes L_2$ for some self-adjoint operators $L_1 \in \mathcal{L}(\mathcal{H})$ and $L_2 \in \mathcal{L}(\mathcal{K})$, then L is said to be an *additive conserved quantity*. What the theorem of Wigner, Araki and Yanase states is that, if $L = L_1 \otimes \mathbb{1} + \mathbb{1} \otimes L_2$ is an additive conserved quantity with respect to the interaction unitary U of a repeatable normal measurement $\langle \mathcal{K}, \mathcal{Z}, U, \phi \rangle$ of a sharp discrete observable E , then necessarily $[E, L_1] = 0$.⁵ Since nearly not

⁵For any observable $E : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$ the notation $[E, L] = 0$ is used as a shorthand for

all measurements are repeatable, the WAY-theorem of the this form has only limited applicability. It is, however, known that the same restriction $[E, L_1] = 0$ holds if the assumption of repeatability is replaced with so-called *Yanase condition* $[Z, L_2] = 0$ [2, 64, 65]. We have summarized these findings into the theorem below; for the proof we refer the reader to Ref. [65].

Theorem 1 (WAY-theorem). *Let $\langle \mathcal{K}, Z, U, \phi \rangle$ be a normal measurement of a sharp observable E and let $L = L_1 \otimes \mathbb{1} + \mathbb{1} \otimes L_2 \in \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$, where $L_1 \in \mathcal{L}(\mathcal{H})$ and $L_2 \in \mathcal{L}(\mathcal{K})$ are self-adjoint, be an additive conserved quantity with respect to the measurement interaction U . If $\langle \mathcal{K}, Z, U, \phi \rangle$ is either repeatable or satisfies the Yanase condition $[Z, L_2] = 0$, then $[E, L_1] = 0$.*

For an additive conserved quantity $L = L_1 \otimes \mathbb{1} + \mathbb{1} \otimes L_2$ the Yanase condition $[Z, L_2] = 0$ is equivalent to $[U^*(\mathbb{1} \otimes Z)U, L] = 0$. However, it is possible to find self-adjoint quantities $L \in \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$ – conserved but non-additive, additive but non-conserved or even non-additive and non-conserved ones – such that $[U^*(\mathbb{1} \otimes Z)U, L] = 0$ holds but $[Z, L_2] = 0$ is violated (or ill-defined). In general, we shall refer to the relation $[\mathcal{V}^*(\mathbb{1} \otimes Z), L] = 0$ as the *generalized Yanase condition* associated to the measurement $\langle \mathcal{K}, Z, \mathcal{V}, \xi \rangle$. Using these notations, we get the following generalization of the above WAY-theorem, which was first pointed out for normal measurements in article III.

***Proposition 4.** *Let $\langle \mathcal{K}, Z, \mathcal{V}, \xi \rangle$ be a measurement of a sharp observable $E : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$ and let $L \in \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$ be self-adjoint. If $\langle \mathcal{K}, Z, \mathcal{V}, \xi \rangle$ fulfils the the generalized Yanase condition $[\mathcal{V}^*(\mathbb{1} \otimes Z), L] = 0$, then $[E, \text{tr}_\xi[L]] = 0$.*

Proof. Assume that $\xi = \sum_{i=1}^N \lambda_i P_{\phi_i}$ for some $0 < \lambda_i \leq 1$ satisfying $\sum_{i=1}^N \lambda_i = 1$. We will show that if $[\mathcal{V}^*(\mathbb{1} \otimes Z), L] = 0$, then $[E, V_{\phi_i}^* L V_{\phi_i}] = 0$ for all $i = 1, \dots, N$ from which the claim follows. We first recall from Prop. 1 that $E(X) \in \mathcal{P}(\mathcal{H})$ is equivalent to $V_{\phi_i} V_{\phi_i}^* \mathcal{V}^*(\mathbb{1} \otimes Z(X)) V_{\phi_i} V_{\phi_i}^* \in \mathcal{P}(\mathcal{H} \otimes \mathcal{K})$, for all $i = 1, \dots, N$, where the latter condition further implies that $[\mathcal{V}^*(\mathbb{1} \otimes Z(X)), V_{\phi_i} V_{\phi_i}^*] = 0$; see Lemma 3. Then for all $X \in \Sigma$

$$\begin{aligned}
 E(X) V_{\phi_i}^* L V_{\phi_i} &= V_{\phi_i}^* \mathcal{V}^*(\mathbb{1} \otimes Z(X)) V_{\phi_i} V_{\phi_i}^* L V_{\phi_i} \\
 &= V_{\phi_i}^* L \mathcal{V}^*(\mathbb{1} \otimes Z(X)) V_{\phi_i} \\
 &= V_{\phi_i}^* L V_{\phi_i} V_{\phi_i}^* \mathcal{V}^*(\mathbb{1} \otimes Z(X)) V_{\phi_i} \\
 &= V_{\phi_i}^* L V_{\phi_i} E(X). \tag{3.30}
 \end{aligned}$$

$[E(X), L] = 0$, for all $X \in \Sigma$.

□

Noticeably, Prop. 4 coincides with Thm. 1 in the particular case where the measurement of the sharp observable \mathbf{E} is normal $\langle \mathcal{K}, \mathbf{Z}, U, \phi \rangle$ and $L = L_1 \otimes \mathbb{1} + \mathbb{1} \otimes L_2$ is an additive conserved quantity, namely then $[U^*(\mathbb{1} \otimes \mathbf{Z})U, L] = 0$ implies both $[Z, L_2] = 0$ and $[E, L_1] = 0$. Remarkably, however, in the context of the previous proposition the self-adjoint quantity $L \in \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$ does not have to be additive nor even conserved; see also the article III. In fact Prop. 4 states that the WAY-limitations, rather than arising directly from the conservation laws, attains the following twofold interpretation. Firstly, the commutation $[\mathcal{V}^*(\mathbb{1} \otimes \mathbf{Z}), L] = 0$ signals the fact that the Heisenberg evolved pointer $\mathcal{V}^*(\mathbb{1} \otimes \mathbf{Z})$ has the invariance

$$U_L^* \mathcal{V}^*(\mathbb{1} \otimes \mathbf{Z}(X)) U_L = \mathcal{V}^*(\mathbb{1} \otimes \mathbf{Z}(X)), \quad X \in \Sigma, \quad (3.31)$$

with respect to the unitary $U_L = e^{i\theta L}$. Secondly, the generalized Yanase condition $[\mathcal{V}^*(\mathbb{1} \otimes \mathbf{Z}), L] = 0$ also relates to the compatibility of the Heisenberg evolved pointer with the sharp observable associated to the self-adjoint operator $L \in \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$ [66]; see Ineq. (3.11). Prop. 4 then indicates how this symmetry/compatibility is transferred from the apparatus to the measured sharp observable.

Prop. 4 implies that the WAY-type restrictions are generic in the sense that they are not only confined to normal models. Even so, there seems to be an obvious way to circumvent these restrictions by considering general smeared POVMs as measured observables instead of sharp observables. Nevertheless, the following quantitative version of Prop. 4 holds in the case of smeared observables.

***Proposition 5.** *Let $\langle \mathcal{K}, \mathbf{Z}, \mathcal{V}, \xi \rangle$ be a measurement of an observable $\mathbf{E} : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$ and assume that $\xi = \sum_i \lambda_i P_{\phi_i}$ for some $0 < \lambda_i \leq 1$ satisfying $\sum_i \lambda_i = 1$. Then for all self-adjoint $L \in \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$ and $X \in \Sigma$*

$$\begin{aligned} & \| [\mathbf{E}(X), \text{tr}_\xi[L]] \| \\ & \leq \| [\mathcal{V}^*(\mathbb{1} \otimes \mathbf{Z}(X)), L] \| + 2 \sum_i \lambda_i^2 \| [\mathcal{V}^*(\mathbb{1} \otimes \mathbf{Z}(X)), V_{\phi_i} V_{\phi_i}^*] \| \|L\|. \end{aligned} \quad (3.32)$$

Proof. We recall that for a bounded self-adjoint operator A its operator-norm may be calculated from $\|A\| = \sup_{\|\varrho\|_1=1} |\text{tr}[A\varrho]|$. It is easy to

verify that $i[A, B]$ is a bounded self-adjoint operator whenever $A, B \in \mathcal{L}(\mathcal{H})$ are bounded and self-adjoint. Moreover, $\text{tr}_\xi[L] = \sum_i \lambda_i V_{\phi_i}^* L V_{\phi_i}$ is a bounded self-adjoint operator on \mathcal{H} whenever L is bounded and self-adjoint on $\mathcal{H} \otimes \mathcal{K}$. Hence, by denoting $\tilde{Z}(X) = \mathcal{V}^*(\mathbb{1} \otimes Z(X))$, we have for all $X \in \Sigma$

$$\begin{aligned}
& \| [\mathbf{E}(X), \text{tr}_\xi[L]] \| \\
&= \sup_{\|\varrho\|_1=1} \left| \sum_{i,j} \lambda_i \lambda_j \text{tr} \left[[V_{\phi_i}^* \tilde{Z}(X) V_{\phi_i}, V_{\phi_j}^* L V_{\phi_j}] \varrho \right] \right| \\
&= \sup_{\|\varrho\|_1=1} \left| \sum_{i,j} \lambda_i \lambda_j \text{tr} \left[\left([\tilde{Z}(X), V_{\phi_i} V_{\phi_j}^*] L + [\tilde{Z}(X), L V_{\phi_i} V_{\phi_j}^*] \right) V_{\phi_j} V_{\phi_i}^* \varrho \right] \right| \\
&= \sup_{\|\varrho\|_1=1} \left| \sum_{i,j} \lambda_i \lambda_j \text{tr} \left[\left([\tilde{Z}(X), L] \delta_{ij} + \{ [\tilde{Z}(X), V_{\phi_i} V_{\phi_j}^*], L \} \right) V_{\phi_j} V_{\phi_i}^* \varrho \right] \right| \\
&\leq \| [\tilde{Z}(X), L] \| + \sum_{i,j} \lambda_i \lambda_j \| \{ [\tilde{Z}(X), V_{\phi_i} V_{\phi_j}^*], L \} \| \cdot \| V_{\phi_j} V_{\phi_i}^* \varrho \|_1 \\
&\leq \| [\mathcal{V}^*(\mathbb{1} \otimes Z(X)), L] \| + 2 \sum_i \lambda_i^2 \| [\mathcal{V}^*(\mathbb{1} \otimes Z(X)), V_{\phi_i} V_{\phi_i}^*] \| \| L \|,
\end{aligned} \tag{3.33}$$

where the estimations are due to the triangle inequality, the Cauchy-Schwarz inequality and the fact that $|\text{tr}[AT]| \leq \|A\| \|T\|_1$ for all $A \in \mathcal{L}(\mathcal{H})$, $T \in \mathcal{T}(\mathcal{H})$. \square

It is noteworthy, that on the right-hand side of Ineq. (3.32) the two familiar terms appear: $\| [\mathcal{V}^*(\mathbb{1} \otimes Z(X)), V_{\phi_i} V_{\phi_i}^*] \|$, $i = 1, \dots, N$, related to the “sharpness” of the measured observable and $\| [\mathcal{V}^*(\mathbb{1} \otimes Z(X)), L] \|$ corresponding to the generalized Yanase condition. As opposed to before, from Ineq. (3.11) it can be confirmed that a non-vanishing $\| [\mathcal{V}^*(\mathbb{1} \otimes Z(X)), L] \|$ is not in general directly quantifying incompatibility, although in some special cases it is possible to establish such an operational link [67]. Instead, in order to interpret this term, we recall that for any given $L \in \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$ the quantity $\| [\cdot, L] \|$ is an *asymmetry monotone* [68], so that $\| [\mathcal{V}^*(\mathbb{1} \otimes Z(X)), L] \|$ can be understood as a measure of asymmetry that the Heisenberg evolved pointer has with respect to the symmetry generator $L \in \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$. Under this viewpoint, Prop. 5 states that the Wigner-Araki-Yanase theorem is a consequence of (a)symmetry transferred from the measuring apparatus to the measured observable. Such an interpretation is reinforced by the fact that also earlier attempts to formulate the

WAY-theorem in the context of the resource theory of asymmetry exist in the literature; see Refs. [69, 70]. This formalism is also connected to quantum reference frames and relational observables studied in Refs. [71–73]. Utilizing Prop. 1, it is easy to see that in the case where \mathbf{E} is sharp and $[\mathcal{V}^*(\mathbb{1} \otimes \mathbf{Z}(X)), L] = 0$ Prop. 5 reduces to Prop. 4. Similar calculations to the above reveal that also the following condition holds.

***Proposition 6.** *Let $\langle \mathcal{K}, \mathbf{Z}, \mathcal{V}, \xi \rangle$ be a measurement of an observable $\mathbf{E} : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$. Then for all self-adjoint $L \in \mathcal{L}(\mathcal{H})$ and $X \in \Sigma$*

$$\|[\mathbf{E}(X), L]\| \leq \|[\mathcal{V}^*(\mathbb{1} \otimes \mathbf{Z}(X)), L \otimes \mathbb{1}]\|. \quad (3.34)$$

The above two Props. 5 and 6 share an apparent similarity, but the limitations set by them can be very different; see the article III. This divergence has been demonstrated in the example below.

Example 4. Let us consider the standard measurement $\langle \mathcal{K}, \mathbf{Q}, U_\lambda, \phi, f \rangle$ associated to a sharp observable $\mathbf{A} : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(\mathcal{H})$, where $U_\lambda = e^{-i\lambda \bar{\mathbf{A}} \otimes \bar{\mathbf{P}}}$, and let $\mathbf{E} : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(\mathcal{H})$ denote the observable actually measured in this process. For normal measurements it can be solved from Prop. 6 that the chain of inequalities

$$\|[\mathbf{E}(X), L]\| \leq \| [U^*(\mathbb{1} \otimes \mathbf{Z}(X))U, L \otimes \mathbb{1}] \| \leq 2 \| [U, L \otimes \mathbb{1}] \|, \quad (3.35)$$

holds for all self-adjoint $L \in \mathcal{L}(\mathcal{H})$ – this relation generalizes Prop. 12 in Ref. [73]. In particular, $[U_\lambda, \mathbf{A}(Y) \otimes \mathbb{1}] = 0$ for all $Y \in \mathcal{B}(\mathbb{R})$. It follows that $[\mathbf{E}(X), \mathbf{A}(Y)] = 0$ for all $X, Y \in \mathcal{B}(\mathbb{R})$, which in turn is equivalent to \mathbf{E} and \mathbf{A} being jointly measurable. From Eq. (3.19) one verifies that this is indeed the case. In fact, since $[U_\lambda, \mathbf{A}(Y) \otimes \mathbb{1}] = 0$, this implication holds even regardless of the choice of the pointer observable and the apparatus state. Noticeably, the same cannot be inferred from Prop. 5, since \mathbf{E} is generally unsharp. Accordingly, the “sharpness” term $\| [U_\lambda^*(\mathbb{1} \otimes \mathbf{Z}(X))U_\lambda, V_\phi V_\phi^*] \| \neq 0$; see Prop. 1.

We will return to the implications of Prop. 5 later in Chap. 4.

Chapter 4

Multi-purpose measurements

4.1 Quantum programming

Several protocols in quantum information science rely on the possibility of realizing two or more incompatible devices. However, by definition two devices cannot be realized from a single fixed measurement set-up if they are incompatible. This notion can at first glance lead one to conclude that, in order to realize incompatible devices, physically separate measurements must be set-up. Luckily there exists a more flexible option: quantum programming.

To approach the topic, let us remind that in a measurement $\langle \mathcal{K}, \mathbf{Z}, \mathcal{V}, \xi_i \rangle$ the measured observable $E_i : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$ is given by

$$E_i(X) = \text{tr}_{\xi_i}[\mathcal{V}^*(\mathbb{1} \otimes Z(X))], \quad X \in \Sigma, \quad (4.1)$$

whereas the induced channel $\mathcal{E}_i : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H}')$ reads

$$\mathcal{E}_i(\varrho) = \text{tr}_{\mathcal{K}}[\mathcal{V}(\varrho \otimes \xi_i)], \quad \varrho \in \mathcal{S}(\mathcal{H}). \quad (4.2)$$

Evidently, altering the initial states of the apparatus $\xi_i \in \mathcal{S}(\mathcal{K})$, $i = 1, \dots, N$, while reusing the other parts of the measurement set-up, can result in realizing different devices. This action is called *quantum programming* and the programmable set-up, mathematically described by a triplet $\langle \mathcal{K}, \mathbf{Z}, \mathcal{V} \rangle$, a programmable *quantum multimeter*. If one focuses on programming of channels only, the pointer observable \mathbf{Z} may be omitted; the resulting double $\langle \mathcal{K}, \mathcal{V} \rangle$ is better known in the literature as a programmable

quantum processor or a gate array [8, 37, 39, 74–78].

Example 5. The advantage of having a programmable set-up over a fixed one is in its versatility. As an example, let us elucidate that any given pair of channels $\mathcal{E}_i : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H}')$, $i = 1, 2$, can be realized from a single multimeter. To confirm this, fix any pair of orthogonal unit vectors $\phi_i \in \mathcal{K}$, $i = 1, 2$. Then the pair $\langle \mathcal{K}, \mathcal{V} \rangle$, where $\mathcal{V} : \mathcal{T}(\mathcal{H} \otimes \mathcal{K}) \rightarrow \mathcal{T}(\mathcal{H}' \otimes \mathcal{K})$; $\mathcal{V}(T) = \sum_{i=1}^2 \mathcal{E}_i(V_{\phi_i}^* T V_{\phi_i}) \otimes P_{\phi_i}$ is a CPTP map, defines a quantum multimeter satisfying $\mathcal{E}_i(\varrho) = \text{tr}_{\mathcal{K}}[\mathcal{V}(\varrho \otimes P_{\phi_i})]$, $i = 1, 2$ for all $\varrho \in \mathcal{S}(\mathcal{H})$. As a corollary, any pair of (discrete) observables can also be programmed from a single multimeter. Namely, let $\mathbf{E}_i : \Sigma_N \rightarrow \mathcal{L}(\mathcal{H})$, $i = 1, 2$, be two discrete N -outcome observables¹ and fix an orthonormal basis $\{\varphi_j\}_{j=1}^n$ spanning an N -dimensional Hilbert space \mathcal{H}_N . Then, for $i = 1, 2$, the mappings $\mathcal{E}_i : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H}_N)$; $\mathcal{E}_i(\varrho) = \sum_{j=1}^N \text{tr}[\mathbf{E}_i(j) \varrho] P_{\varphi_j}$ define the quantum-to-classical channels that, as seen above, are programmable from a single multimeter with (orthogonal pure) states. Concatenation of this multimeter with a measurement of the sharp observable $\mathbf{E} : \Sigma_N \rightarrow \mathcal{L}(\mathcal{H}_N)$; $\mathbf{E}(j) = P_{\varphi_j}$ results to new a multimeter that outputs the correct statistics of the observables \mathbf{E}_i , $i = 1, 2$.

Multimeters can raise above the limitations that are set by incompatibility for their single purpose counterparts. As a consequence, multimeters can be useful in tasks of quantum information processing, such as state discrimination [74, 78, 79]. However, no multimeter is universal in the sense that it could implement all the quantum observables and/or channels, if it is supposed to work in a deterministic manner. The purpose of this chapter is to develop tools that allow one to investigate what can and cannot be done with programmable quantum multimeters.

4.2 Programmability of quantum channels

4.2.1 Normal multimeters

We will say that a multimeter $\langle \mathcal{K}, \mathbf{Z}, \mathcal{V} \rangle$ is *normal* when \mathbf{Z} is sharp and \mathcal{V} is a conjugation with some unitary operator G on $\mathcal{H} \otimes \mathcal{K}$: in such a case we shorten the notion by writing $\langle \mathcal{K}, \mathbf{Z}, G \rangle$. The programmability of

¹If the number of outcomes N_i of the observables \mathbf{E}_i , $i = 1, 2$, are different, one may fix $N := \max(N_1, N_2)$ and expand the smaller one by adding zero effects.

normal multimeters was first studied by Nielsen and Chuang in the late 90's. The main result of their preliminary paper [8] was to show that when such a multimeter is programmed to realize two different unitary channels, the programming (pure) states are necessarily orthogonal. For the later purposes, it is illustrative to review the original proof of the result. To that end, fix any two unitary operators U_i , $i = 1, 2$, on \mathcal{H} satisfying $U_1^*U_2 \neq c\mathbb{1}$, where $c \in \mathbb{C}$, $|c| = 1$. Let \mathcal{K} be a separable Hilbert space, let $\phi_i \in \mathcal{K}$, $i = 1, 2$, be unit vectors and assume that G is a unitary operator on $\mathcal{H} \otimes \mathcal{K}$ such that for any $\varphi \in \mathcal{H}$

$$G(\varphi \otimes \phi_i) = U_i\varphi \otimes \phi'_i, \quad i = 1, 2. \quad (4.3)$$

Let us make an assumption that the final states of the apparatus satisfy $\langle \phi'_1 | \phi'_2 \rangle \neq 0$. Then

$$\langle \varphi | U_1^*U_2\varphi \rangle = \frac{\langle \phi_1 | \phi_2 \rangle}{\langle \phi'_1 | \phi'_2 \rangle}, \quad (4.4)$$

for all $\varphi \in \mathcal{H}$. However, this equation only holds when $U_1^*U_2 = c\mathbb{1}$ for some $c \in \mathbb{C}$, $|c| = 1$, since its right-hand side is independent of φ . This is contradicting the initial assumption and therefore $\langle \phi'_1 | \phi'_2 \rangle = 0$. The proof of the claim then follows from Eq. (4.3), namely

$$\langle \phi_1 | \phi_2 \rangle = \langle G(\varphi \otimes \phi_1) | G(\varphi \otimes \phi_2) \rangle = \langle U_1\varphi | U_2\varphi \rangle \langle \phi'_1 | \phi'_2 \rangle = 0. \quad (4.5)$$

The conclusion has been summarized into the following theorem.

Theorem 2. *Let $\langle \mathcal{K}, G \rangle$ be a normal multimeter and, for $i = 1, 2$, let $P_{\phi_i} \in \mathcal{S}(\mathcal{K})$ be pure states programming $\langle \mathcal{K}, G \rangle$ to realize unitary channels \mathcal{U}_i , respectively. If $\mathcal{U}_1 \neq \mathcal{U}_2$, then $\langle \phi_1 | \phi_2 \rangle = 0$.*

Let us examine the implications of Prop. 2. Firstly, it is important to realize that limiting one's attention only on pure programming states in the above scenario is not a restriction of generality. Indeed, if $\xi = \sum_i \lambda_i P_{\phi_i}$, where $0 < \lambda_i \leq 1$ and $\sum_{i=1}^N \lambda_i = 1$, implements a unitary channel $\mathcal{U}(\cdot) = U(\cdot)U^*$ via

$$\mathcal{U}(\varrho) = \text{tr}_{\mathcal{K}}[G(\varrho \otimes \xi)G^*] \quad \varrho \in \mathcal{S}(\mathcal{H}), \quad (4.6)$$

then it follows from the extremality of the unitary channels that

$$\mathcal{U}(\varrho) = \text{tr}_{\mathcal{K}}[G(\varrho \otimes P_{\phi_i})G^*] \quad (4.7)$$

for all $i = 1, \dots, N$. More generally, any extremal channel that can be realized from a multimeter with a mixed programming state can also be realized with a pure programming state; there is no benefit to use mixed states when programming extremal channels.

Secondly, assume that one wishes to implement N different unitary channels $\mathcal{U}_i(\cdot) = U_i(\cdot)U_i^*$, $i = 1, \dots, N$ using a normal multimeter. Since *any* pair of unequal unitary channels requires mutually orthogonal programming vectors/pure states, the dimension of the multimeter must be greater than or equal to N . Noticeably, this lower bound can always be met, since the channels can be programmed from the unitary interaction $G = \sum_{i=1}^N U_i \otimes P_{\phi_i}$ with the multimeter Hilbert space \mathcal{K} being the one spanned by the orthonormal vectors ϕ_i , $i = 1, \dots, N$. In order to implement *all* the unitary channels would then require an uncountable amount of orthogonal vectors. This, however, is impossible, since the (separable) Hilbert space of the multimeter has at most a countably infinite basis. In other words, there does not exist a universal multimeter capable of implementing all the quantum (unitary) channels: this result is often referred in the literature as the “no-programming”-theorem [14, 80, 81].

Thirdly, the above orthogonality result sheds some light on the question “can quantum resources be advantageous in programming?” Let us recall that two pure quantum states P_{ϕ_1} and P_{ϕ_2} can be perfectly discriminated if and only if they are orthogonal. The fact that not all pure states can be discriminated perfectly lies at the heart of many quantum applications. As an example, the existence of indistinguishable pure quantum states results to the impossibility of perfect cloning, which, on the other hand, enables a quantum benefit, e.g., in cryptography. Importantly, the indistinguishability of pure quantum states does not have a classical analogue: classical states may only be indistinguishable due to our ignorance, i.e., when they are statistically mixed. What Thm. 2 shows is that one cannot have a quantum benefit from indistinguishability when programming different unitary channels.

4.2.2 General multimeters

Analogously to the normal measurements, normal multimeters should be considered as mathematical idealizations: in physically realistic situations there may exist some noise in various entries of the multimeter. An additional downside of the applicability of Thm. 2 is that it only discusses the mutual dependencies of the programming vectors of noiseless unitary channels. For these reasons, Thm. 2 remains mainly a theoretical curiosity with only minor practical relevance. It would be desirable to have the mathematical tools to address the programmability of quantum channels in full generality; from the viewpoint of open quantum systems. This is exactly the task that we will tackle next.

The following observation was first made in case of normal multimeters in the article I as an attempt to generalize Thm. 2. It should be noted that, even though the main implications of the two results are same, the mathematical arguments used to prove them are distinct from each other.

***Proposition 7.** *Let $\langle \mathcal{K}, \mathcal{V} \rangle$ be a multimeter and, for $i = 1, 2$, let $\xi_i \in \mathcal{S}(\mathcal{K})$ be states that program the extremal channels $\mathcal{E}_i^* : \mathcal{L}(\mathcal{H}') \rightarrow \mathcal{L}(\mathcal{H})$, respectively. If either of the channels is a $*$ -homomorphism and $\mathcal{E}_1^* \neq \mathcal{E}_2^*$, then $\xi_1 \perp \xi_2 = 0$.*

Proof. Let us assume that \mathcal{E}_1^* is a $*$ -homomorphism and that $\xi_1 = \sum_{i=1}^N \lambda_i P_{\phi_i}$ for some $0 < \lambda_i \leq 1$ satisfying $\sum_{i=1}^N \lambda_i = 1$. Recall from Prop. 3 and Lemma 3 that then necessarily $[\mathcal{V}^*(B \otimes \mathbb{1}), V_{\phi_i} V_{\phi_i}^*] = 0$, for all $B \in \mathcal{L}(\mathcal{H}')$ and $i = 1, \dots, N$. Similarly, let $\xi_2 = \sum_{j=1}^M \mu_j P_{\eta_j}$ for some $0 < \mu_j \leq 1$ satisfying $\sum_{j=1}^M \mu_j = 1$. Due to the extremality each of the pure states P_{ϕ_i} (resp. P_{η_j}) programs \mathcal{E}_1 (resp. \mathcal{E}_2) via $\langle \mathcal{K}, \mathcal{V} \rangle$. To prove the orthogonality $\xi_1 \perp \xi_2$ is equivalent with proving the condition $\langle \phi_i | \eta_j \rangle = 0$ for all $i = 1, \dots, N, j = 1, \dots, M$. Make a counter-assumption that there exist some ϕ_i and η_j , such that $\langle \phi_i | \eta_j \rangle \neq 0$. Hence $\eta_j = \alpha \phi_i + \beta \zeta$ for some $\alpha, \beta \in \mathbb{C}$, $\alpha \neq 0$, $|\alpha|^2 + |\beta|^2 = 1$ and a unit vector $\zeta \in \mathcal{K}$ satisfying

$\langle \phi_i | \zeta \rangle = 0$. Then for all $B \in \mathcal{L}(\mathcal{H}')$

$$\begin{aligned}
\mathcal{E}_2^*(B) &= V_{\eta_j}^* \mathcal{V}^*(B \otimes \mathbb{1}) V_{\eta_j} \\
&= |\alpha|^2 V_{\phi_i}^* \mathcal{V}^*(B \otimes \mathbb{1}) V_{\phi_i} + \alpha \beta^* V_{\zeta}^* \mathcal{V}^*(B \otimes \mathbb{1}) V_{\phi_i} \\
&\quad + \alpha^* \beta V_{\phi_i}^* \mathcal{V}^*(B \otimes \mathbb{1}) V_{\zeta} + |\beta|^2 V_{\zeta}^* \mathcal{V}^*(B \otimes \mathbb{1}) V_{\zeta} \\
&= |\alpha|^2 V_{\phi_i}^* \mathcal{V}^*(B \otimes \mathbb{1}) V_{\phi_i} + |\beta|^2 V_{\zeta}^* \mathcal{V}^*(B \otimes \mathbb{1}) V_{\zeta} \\
&= |\alpha|^2 \mathcal{E}_1^*(B) + |\beta|^2 \mathcal{E}^*(B), \tag{4.8}
\end{aligned}$$

where $\mathcal{E}^*(B) = V_{\zeta}^* \mathcal{V}^*(B \otimes \mathbb{1}) V_{\zeta}$ defines some quantum channel. The cross-terms in above equation vanish due to the commutativity noted above, for example

$$V_{\zeta}^* \mathcal{V}^*(B \otimes \mathbb{1}) V_{\phi_i} = V_{\zeta}^* V_{\phi_i} V_{\phi_i}^* \mathcal{V}^*(B \otimes \mathbb{1}) V_{\phi_i} = \langle \zeta | \phi_i \rangle \mathcal{E}_1^*(B), \tag{4.9}$$

for all $B \in \mathcal{L}(\mathcal{H}')$. The counter-assumption $\langle \phi_i | \eta_j \rangle \neq 0$ thus leads to contradiction with the extremality of \mathcal{E}_2^* whenever $\mathcal{E}_1^* \neq \mathcal{E}_2^*$, and therefore $\xi_1 \perp \xi_2$. \square

Let us return to the proof of Thm. 2, however, take an alternative step in Eq. (4.3) by considering the transformation of two unit vectors $\varphi_i \in \mathcal{H}$, $i = 1, 2$. As a consequence, we get the relation

$$|\langle \varphi_1 | \varphi_2 \rangle| |\langle \phi_1 | \phi_2 \rangle| = |\langle U_1 \varphi_1 | U_2 \varphi_2 \rangle| |\langle \phi'_1 | \phi'_2 \rangle|. \tag{4.10}$$

Recalling the definition of (Uhlmann) *fidelity of states*

$$F(\varrho_1, \varrho_2) \equiv \text{tr} \left[\sqrt{\sqrt{\varrho_2} \varrho_1 \sqrt{\varrho_2}} \right], \tag{4.11}$$

it can be concluded from Eq. (4.10) that the inequality

$$F(P_{\varphi_1}, P_{\varphi_2}) F(P_{\phi_1}, P_{\phi_2}) \leq F(\mathcal{U}_1(P_{\varphi_1}), \mathcal{U}_2(P_{\varphi_2})) \tag{4.12}$$

holds for arbitrary pure states $P_{\varphi_i} \in \mathcal{S}(\mathcal{H})$, $i = 1, 2$. In fact, such a relation holds more generally; see the article V.

Proposition 2. *Let $\langle \mathcal{K}, \mathcal{V} \rangle$ be a multimeter and, for $i = 1, 2$, let $\xi_i \in \mathcal{S}(\mathcal{K})$ be states that program the channels $\mathcal{E}_i : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H}')$, respectively. Then*

$$F(\varrho_1, \varrho_2) F(\xi_1, \xi_2) \leq F(\mathcal{E}_1(\varrho_1), \mathcal{E}_2(\varrho_2)) \tag{4.13}$$

holds for all states $\varrho_i \in \mathcal{S}(\mathcal{H})$, $i = 1, 2$.

Proof. The claim follows from the basic properties of fidelity of states [14]. Namely, for all $\varrho_i \in \mathcal{S}(\mathcal{H})$, $i = 1, 2$,

$$\begin{aligned}
 F(\varrho_1, \varrho_2) F(\xi_1, \xi_2) &= F(\varrho_1 \otimes \xi_1, \varrho_2 \otimes \xi_2) \\
 &\leq F(\mathcal{V}(\varrho_1 \otimes \xi_1), \mathcal{V}(\varrho_2 \otimes \xi_2)) \\
 &\leq F(\text{tr}_{\mathcal{K}}[\mathcal{V}(\varrho_1 \otimes \xi_1)], \text{tr}_{\mathcal{K}}[\mathcal{V}(\varrho_2 \otimes \xi_2)]) \\
 &= F(\mathcal{E}_1(\varrho_1), \mathcal{E}_2(\varrho_2)).
 \end{aligned} \tag{4.14}$$

□

In the previous proof we repeatedly used the data processing inequality $F(\varrho_1, \varrho_2) \leq F(\mathcal{E}(\varrho_1), \mathcal{E}(\varrho_2))$, which holds for all channels $\mathcal{E} : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H}')$ and states $\varrho_i \in \mathcal{S}(\mathcal{H})$, $i = 1, 2$ [14]. As a matter of fact, Ineq. (4.13) may be understood as a generalization of this relation for the case of unequal state transformations; the data processing inequality follows as a corollary, when choosing $\xi_1 = \xi_2$. This formalism enables us to analyse the mutual dependencies of the programming states of given pair of quantum channels. In order to see this, let us define the quantity

$$\mathcal{F}(\mathcal{E}_1, \mathcal{E}_2) \equiv \inf_{\varrho_1, \varrho_2 \in \mathcal{S}(\mathcal{H})} \frac{F(\mathcal{E}_1(\varrho_1), \mathcal{E}_2(\varrho_2))}{F(\varrho_1, \varrho_2)}. \tag{4.15}$$

It then follows from Ineq. (4.13) that

$$F(\xi_1, \xi_2) \leq \mathcal{F}(\mathcal{E}_1, \mathcal{E}_2). \tag{4.16}$$

The quantifier \mathcal{F} measures how well a pair of quantum channels can preserve the overlap of a pair of input states at worst. It shares many of the properties of fidelity of states, that have been listed below.

Proposition 3. *For all channels $\mathcal{E}_i : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H}')$, $i = 1, 2$, the quantifier \mathcal{F} satisfies the following properties:*

$$(F1) \quad \mathcal{F}(\mathcal{E}_1, \mathcal{E}_2) = \mathcal{F}(\mathcal{E}_2, \mathcal{E}_1),$$

$$(F2) \quad \mathcal{F}(\mathcal{E}_1, \mathcal{E}_2) \in [0, 1],$$

$$(F3) \quad \mathcal{F}(\mathcal{E}_1, \mathcal{E}_2) = 1 \text{ if and only if } \mathcal{E}_1 = \mathcal{E}_2,$$

(F4) $\mathcal{F}(\mathcal{E}_1, \mathcal{E}_2) = \mathcal{F}(\mathcal{U}_1 \circ \mathcal{E}_1 \circ \mathcal{U}_2, \mathcal{U}_1 \circ \mathcal{E}_2 \circ \mathcal{U}_2)$, for all unitary channels \mathcal{U}_1 and \mathcal{U}_2 .

(F5) $\mathcal{F}(\mathcal{E}_1, \mathcal{E}_2) \leq \mathcal{F}(\mathcal{E} \circ \mathcal{E}_1, \mathcal{E} \circ \mathcal{E}_2)$, for all channels $\mathcal{E} : \mathcal{T}(\mathcal{H}') \rightarrow \mathcal{T}(\mathcal{H}'')$,

(F6) $\mathcal{F}(\mathcal{E}_1, \mathcal{E}_2) \leq \mathcal{F}(\mathcal{E}_1 \circ \mathcal{E}, \mathcal{E}_2 \circ \mathcal{E})$, for all channels $\mathcal{E} : \mathcal{T}(\mathcal{H}'') \rightarrow \mathcal{T}(\mathcal{H})$.

Proof. See the article V. □

The orthogonality of states $\xi_1 \perp \xi_2$ is equivalent to $F(\xi_1, \xi_2) = 0$ [14]. A connection of Ineq. (4.16) with the previously presented Thm. 2 then comes from the fact that $\mathcal{F}(\mathcal{U}_1, \mathcal{U}_2) = 0$ for all unequal unitary channels implying that the corresponding programming states are necessarily orthogonal (regardless of the multimeter). To prove this, it is sufficient to notice that for any unitary operator $U \neq c\mathbb{1}$, with $c \in \mathbb{C}$, $|c| = 1$, and any unit vector $\varphi_1 \in \mathcal{H}$, such that $|\langle \varphi_1 | U\varphi_1 \rangle| \neq 1$, the unit vector

$$\varphi_2 \equiv \sqrt{1 - |\langle \varphi_1 | U\varphi_1 \rangle|^2} \varphi_1 - \frac{\langle \varphi_1 | U\varphi_1 \rangle}{\sqrt{1 - |\langle \varphi_1 | U\varphi_1 \rangle|^2}} (U^* \varphi_1 - \langle \varphi_1 | U^* \varphi_1 \rangle \varphi_1) \in \mathcal{H} \quad (4.17)$$

satisfies $\langle \varphi_1 | U\varphi_2 \rangle = 0$ and $\langle \varphi_1 | \varphi_2 \rangle \neq 0$. Namely, choosing $U = U_1^* U_2$ then implies that $\mathcal{F}(\mathcal{U}_1, \mathcal{U}_2) \leq \frac{F(\mathcal{U}_1(P_{\varphi_1}), \mathcal{U}_2(P_{\varphi_2}))}{F(P_{\varphi_1}, P_{\varphi_2})} = 0$.

The relation in Ineq. (4.16) allows to shed light on programmability questions that are beyond the scopes of Thm. 2 and Prop. 7. For instance, let us assume that $d = \dim(\mathcal{H}) < \infty$ and suppose that one is tasked with building a minimal multimeter with respect to $\dim(\mathcal{K})$ implementing the channels $\mathcal{E}_i(\varrho) = (1 - \varepsilon)\mathcal{U}_i(\varrho) + \varepsilon\frac{1}{d}\mathbb{1}$ with programming states $\xi_i \in \mathcal{K}$, $i = 1, \dots, N$, respectively. Here \mathcal{U}_i , $i = 1, \dots, N$ are unequal unitary channels, $\frac{1}{d}\mathbb{1}$ corresponds to state-space contraction to the maximally mixed state in $\mathcal{S}(\mathcal{H})$ and $\varepsilon \in [0, 1]$ describes a fixed error rate. Prop. 7 implies that $\dim(\mathcal{K}) \geq N$ in absence of the noise, $\varepsilon = 0$. It can be shown that $F(\mathcal{E}_i(\varrho_1), \mathcal{E}_j(\varrho_2)) \leq \text{tr}[\mathcal{E}_i(\varrho_1) \mathcal{E}_j(\varrho_2)] + \sqrt{(1 - \text{tr}[\mathcal{E}_i(\varrho_1)^2])(1 - \text{tr}[\mathcal{E}_j(\varrho_2)^2])}$ [82] using which we get the relation

$$\frac{F(\mathcal{E}_i(P_{\varphi_1}), \mathcal{E}_j(P_{\varphi_2}))}{F(P_{\varphi_1}, P_{\varphi_2})} \leq \frac{(1 - \varepsilon)^2 |\langle \varphi_1 | U_i^* U_j \varphi_2 \rangle|^2}{|\langle \varphi_1 | \varphi_2 \rangle|} + \frac{(2 - \varepsilon)\varepsilon}{|\langle \varphi_1 | \varphi_2 \rangle|}, \quad (4.18)$$

for all pure states $P_{\varphi_k} \in \mathcal{S}(\mathcal{H})$, $k = 1, 2$. In particular, if we fix any φ_1 such that $|\langle \varphi_1 | U_i^* U_j \varphi_1 \rangle| \neq 1$ and choose φ_2 as in Eq. (4.17) for $U = U_i^* U_j$, then Ineq. (4.16) implies that

$$F(\xi_i, \xi_j) \leq \inf_{P_{\varphi_1} \in \mathcal{S}(\mathcal{H})} \frac{(2 - \varepsilon)\varepsilon}{\sqrt{1 - |\langle \varphi_1 | U_i^* U_j \varphi_1 \rangle|^2}} \equiv g_{ij}(\varepsilon) \quad (4.19)$$

whenever $i \neq j$.

For any set of unequal states $\xi_i \in \mathcal{S}(\mathcal{H})$, $i = 1, \dots, K$, one can find a set of unequal unit vectors $\phi_i \in \mathcal{K}$, $i = 1, \dots, K$, such that $F(P_{\phi_i}, P_{\phi_j}) \leq F(\xi_i, \xi_j)$ for all $i, j = 1, \dots, K$. Indeed, fixing the decompositions of any two of the states $\xi_1 = \sum_i \lambda_i P_{\phi_i}$, $\sum_i \lambda_i = 1$, and $\xi_2 = \sum_j \mu_j P_{\eta_j}$, $\sum_j \mu_j = 1$, the strong concavity of fidelity [14] implies that

$$F(\xi_1, \xi_2) \geq \sum_{i,j} \sqrt{\lambda_i \mu_j} F(P_{\phi_i}, P_{\eta_j}). \quad (4.20)$$

If it were for all $F(P_{\phi_i}, P_{\eta_j}) > F(\xi_1, \xi_2)$, then

$$F(\xi_1, \xi_2) \geq \sum_{i,j} \sqrt{\lambda_i \mu_j} F(P_{\phi_i}, P_{\eta_j}) > \sqrt{\sum_{i,j} \lambda_i \mu_j} F(\xi_1, \xi_2) = F(\xi_1, \xi_2). \quad (4.21)$$

This contradiction proves that the claim $F(P_{\phi_i}, P_{\phi_j}) \leq F(\xi_i, \xi_j)$ holds pairwise. The fact that the states ξ_i , $i = 1, \dots, K$, are unequal implies that every ϕ_i , $i = 1, \dots, K$, must also be different, namely $F(P_{\phi_i}, P_{\phi_j}) \leq F(\xi_i, \xi_j) < 1$, for any $i \neq j$. We recall that any unit vectors $\phi_i \in \mathcal{K}$, $i = 1, \dots, K$, satisfying $F(P_{\phi_i}, P_{\phi_j}) \leq \frac{1}{K-1}$ whenever $i \neq j$, are linearly independent in \mathcal{K} [39]. Let us denote by K_ε the largest integer satisfying $K_\varepsilon < \max_{i \neq j \in \{1, \dots, N\}} (1/g_{ij}(\varepsilon) + 1)$. Hence, if $N \leq K_\varepsilon$, then there exists vectors $\phi_i = 1, \dots, N$ satisfying $F(P_{\phi_i}, P_{\phi_j}) \leq F(\xi_i, \xi_j) \leq \frac{1}{K_\varepsilon - 1} \leq \frac{1}{N-1}$ that are therefore linearly independent and $\dim(\mathcal{K}) \geq N$. On the other hand, if $N > K_\varepsilon$, then $\dim(\mathcal{K}) \geq K_\varepsilon$.

Example 6. (Example from article V.) Let us consider the programming of the four qubit channels $\mathcal{E}_i(\varrho) = (1 - \varepsilon)\sigma_i \varrho \sigma_i^* + \varepsilon \frac{1}{2} \mathbb{1}$, $i = 0, \dots, 3$, where $\sigma_0 = \mathbb{1}$ and σ_i , $i = 1, 2, 3$, are the Pauli unitary matrices. For all $i \neq j = 0, \dots, 3$, it can be shown that the function $g_{ij}(\varepsilon)$ defined in Ineq. (4.19) simplifies to $g_{ij} = (2 - \varepsilon)\varepsilon$. Consequently, any multimeter implementing the above

noisy unitary channels is at least four dimensional for $0 \leq \varepsilon < \frac{1}{3}(3 - \sqrt{6})$, at least three dimensional for $\frac{1}{3}(3 - \sqrt{6}) \leq \varepsilon < \frac{1}{2}(2 - \sqrt{2})$ and at least two dimensional for $\frac{1}{2}(2 - \sqrt{2}) \leq \varepsilon < 1$. For comparison, in Ref. [77] has been concluded that the multimeter is at least four dimensional for $0 \leq \varepsilon < \frac{1}{3}(7 - 4\sqrt{3})$, at least three dimensional for $\frac{1}{3}(7 - 4\sqrt{3}) \leq \varepsilon < \frac{1}{2}(5 - 2\sqrt{6})$ and at least two dimensional for $\frac{1}{2}(5 - 2\sqrt{6}) \leq \varepsilon < 1$, which, compared to the bounds derived above, are too loose.

Mimicking the proof of Prop. 2 it is easy to derive various inequalities quantifying the distinguishability of the programming states in terms of the implemented channels, as long as the figure of merit for distinguishability satisfies the data processing inequality and an appropriate notion of additivity/multiplicativity. For instance, in the article V a family of inequalities

$$F_\alpha(\varrho_1, \varrho_2)F_\alpha(\xi_1, \xi_2) \leq F_\alpha(\mathcal{E}_1(\varrho_1), \mathcal{E}_2(\varrho_2)) \quad (4.22)$$

was proven, which holds for all $\varrho_i \in \mathcal{S}(\mathcal{H}), i = 1, 2$, and $\alpha \in [1/2, 1)$. Here, F_α are defined via

$$F_\alpha(\varrho_1, \varrho_2) \equiv \text{tr} \left[\left(\varrho_2^{\frac{1-\alpha}{2\alpha}} \varrho_1 \varrho_2^{\frac{1-\alpha}{2\alpha}} \right)^\alpha \right]. \quad (4.23)$$

In fact, the earlier Ineq. (4.16) follows when choosing $\alpha = 1/2$. Although F_α are all measures of distinguishability [83–87], there is one shortcoming they all share: none of them is an actual *distance* in the state space².

A natural distance measure related to the distinguishability of quantum states is the trace distance $D_{tr}(\varrho_1, \varrho_2) \equiv \frac{1}{2} \|\varrho_1 - \varrho_2\|_1$. Since it satisfies the data processing inequality $D_{tr}(\mathcal{E}(\varrho_1), \mathcal{E}(\varrho_2)) \leq D_{tr}(\varrho_1, \varrho_2)$ for all channels $\mathcal{E} : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H}')$ and the sub-additivity $D_{tr}(\varrho_1 \otimes \xi_1, \varrho_2 \otimes \xi_2) \leq D_{tr}(\varrho_1, \varrho_2) + D_{tr}(\xi_1, \xi_2)$ for all $\varrho_i \in \mathcal{S}(\mathcal{H})$ and $\xi_i \in \mathcal{S}(\mathcal{K}), i = 1, 2$, we can

²Generally $F_\alpha(\varrho_1, \varrho_2) \neq F_\alpha(\varrho_2, \varrho_1)$, that is the quantifiers F_α fail to be symmetric, which is one of axioms stipulated from a metric. In fact, the symmetry condition holds generally only for $\alpha = 1/2$. However, neither the Uhlmann fidelity $F = F_{1/2}$ is a metric, since $F(\varrho_1, \varrho_2) = 1$ if and only if $\varrho_1 = \varrho_2$, which again contradicts one of the metric axioms. It is not even an inverse metric in the sense that $\bar{F}(\varrho_1, \varrho_2) \equiv 1 - F(\varrho_1, \varrho_2)$ would define a distance function, since it is easily seen that \bar{F} fails to satisfy the triangle inequality. However, there exist various fidelity induced metrics in the literature, such as the Sine metric $C(\varrho_1, \varrho_2) \equiv \sqrt{1 - F(\varrho_1, \varrho_2)}$; see for example Ref. [88].

solve that

$$\mathcal{D}(\mathcal{E}_1, \mathcal{E}_2) \equiv \sup_{\varrho_1, \varrho_2 \in \mathcal{S}(\mathcal{H})} (D_{tr}(\mathcal{E}_1(\varrho_1), \mathcal{E}_2(\varrho_2)) - D_{tr}(\varrho_1, \varrho_2)) \leq D_{tr}(\xi_1, \xi_2). \quad (4.24)$$

Analogously to the proof of Prop. 3 it is possible to show that $\mathcal{D}(\mathcal{E}_1, \mathcal{E}_2) \in [0, 1]$, where $\mathcal{D}(\mathcal{E}_1, \mathcal{E}_2) = 0$ holds if and only if $\mathcal{E}_1 = \mathcal{E}_2$. However, from Ineq. (4.24) one cannot in general deduce the orthogonality of the programming states of unequal unitary channels³: in this case Ineq. (4.16) results in stronger limitations. Nevertheless, Ineq. (4.24) can be useful in some situations as has been alluded in the example below.

Example 7. After years of peaceful collaboration, Alice has turned greedy and decided to exploit her colleague Bob financially. The cunning scheme of Alice – who has lately acquired some skills in quantum programming – involves selling copy-protected quantum programs to Bob. The programs are supposed to implement different precious algorithms on a “black box” quantum multimeter, that Alice has suspiciously generously donated to Bob. Alice remembers that, if the algorithms worked precisely unitarily (\mathcal{U}_i , $i = 1, \dots, N$), then the corresponding programs would inevitably be perfectly discriminable. As she fears that Bob would ultimately realize that the perfect discriminability enables perfect copying, Alice decides to go for a noisy implementation of the algorithms (channels $\mathcal{E}_i = (1 - \varepsilon)\mathcal{U}_i + \varepsilon\mathcal{N}$). She reasons that the programming states could then remain indistinguishable and hence copy-protected, and, with a clever choice of fixed noise (channel \mathcal{N}) and error ($\varepsilon \in [0, 1]$), Bob should barely even notice the imperfection on the black box’ function.

Unfortunately for Alice, she soon realizes that her scheme is doomed to fail. Namely, as

$$\mathcal{D}(\mathcal{E}_i, \mathcal{E}_j) \geq \sup_{\varrho \in \mathcal{S}(\mathcal{H})} D_{tr}(\mathcal{E}_i(\varrho), \mathcal{E}_j(\varrho)) = \frac{(1 - \varepsilon)}{2} \|\mathcal{U}_i - \mathcal{U}_j\|_1, \quad (4.25)$$

where $\|\mathcal{E}_1 - \mathcal{E}_2\|_1 \equiv \sup_{\varrho \in \mathcal{S}(\mathcal{H})} \text{tr} [|\mathcal{E}_1(\varrho) - \mathcal{E}_2(\varrho)|]$ defines a distance in the space of quantum channels, Ineq. (4.24) then implies that the programming

³In fact, one can find pairs of unequal unitary channels \mathcal{U}_i , $i = 1, 2$, such that $\mathcal{D}(\mathcal{U}_1, \mathcal{U}_2)$ will attain any value from $(0, 1]$: this can be easily confirmed with suitably chosen qubit unitary channels.

states ξ_i , $i = 1, \dots, N$, will satisfy $(1 - \varepsilon) \|\mathcal{U}_i - \mathcal{U}_j\|_1 \leq \|\xi_i - \xi_j\|_1$. Accordingly, for small errors ($\varepsilon \ll 1$) the only way for the programming states to remain sufficiently indistinguishable from each other ($\|\xi_i - \xi_j\|_1 \approx 0$) is when the programmed unitary transformation process data in practically equivalent manner ($\|\mathcal{U}_i - \mathcal{U}_j\|_1 \approx 0$). Vice versa, for algorithms that would operate significantly differently, a large error must be accepted in order to keep the programs indistinguishable. These two conclusions were first made in Ref. [89]. Alice must find a different way to scam her colleague.

It is worth mentioning, that identical relations to those derived above for time-fixed quantum channels would also hold for the continuous-time dynamical transformations; see the article V. For example, the dynamical versions of Eqs. (4.16) and (4.24) would read

$$\begin{aligned} F(\xi_1, \xi_2) &\leq \mathcal{F}(\mathcal{E}_t^{(1)}, \mathcal{E}_t^{(2)}), \\ \mathcal{D}(\mathcal{E}_t^{(1)}, \mathcal{E}_t^{(2)}) &\leq D_{tr}(\xi_1, \xi_2) \end{aligned} \quad (4.26)$$

for all $t \geq 0$. These inequalities, together with the fact that $F(\xi_1, \xi_2) \leq \sqrt{1 - D_{tr}(\xi_1, \xi_2)^2}$ [14], lead to the relation

$$F(\xi_1, \xi_2) \leq \min \left[\mathcal{F}(\mathcal{E}_t^{(1)}, \mathcal{E}_t^{(2)}), \sqrt{1 - \mathcal{D}(\mathcal{E}_t^{(1)}, \mathcal{E}_t^{(2)})^2} \right]. \quad (4.27)$$

It is possible to harness such a relation to probe and estimate properties of an environment, when it is interpreted as the multimeter apparatus. Interestingly, this probing can be done without any a priori knowledge on the interaction between the probe and the environment. In short, the protocol consist of first preparing the environment into a known calibration state, bringing a probe system and the environment into contact and measuring the dynamics induced on the probe, e.g. via process-tomography. This known dynamics can then be used as a relative baseline against which any other dynamics, induced by an unknown environment state, can be compared in terms of the quantities appearing on the right hand side of Eq. (4.27). Finally, Eq. (4.27) gives a bound for the unknown environment state relative to the calibration state. For more details on this protocol and some of its potential applications we refer the reader to the article V.

4.3 Programmability of quantum observables

Fidelity of states may be recast in several equivalent forms, e.g., $F(\varrho_1, \varrho_2) = \min_{\mathbf{E} \in \text{POVM}} \sum_i \sqrt{p_{\varrho_1}^{\mathbf{E}}(i) p_{\varrho_2}^{\mathbf{E}}(i)}$. This definition is closely connected to so-called *Bhattacharyya coefficient* $B(p, q) \equiv \sum_i \sqrt{p(i) q(i)}$, which is a measure of overlap between two probability distributions $p, q : \Sigma \rightarrow [0, 1]$, $\sum_i p(i) = 1 = \sum_i q(i)$. Using these notations, and the fact that any (discrete) observable $\mathbf{E} : \Sigma_N \rightarrow \mathcal{L}(\mathcal{H})$ may be viewed as quantum-to-classical channel $\mathbf{E} : \varrho \mapsto \sum_{i=1}^N p_{\varrho}^{\mathbf{E}}(i) P_{\varphi_i}$, where $\{\varphi_i\}_{i=1}^N$ is an orthonormal basis, we get the following result as a consequence of Prop. 2; see also the article IV.

Proposition 4. *Let $\langle \mathcal{K}, \mathcal{Z}, \mathcal{V} \rangle$ be a multimeter and, for $i = 1, 2$, let $\xi_i \in \mathcal{S}(\mathcal{K})$ be states that program observables $\mathbf{E}_i : \Sigma_N \rightarrow \mathcal{L}(\mathcal{H})$, respectively. Then*

$$F(\varrho_1, \varrho_2) F(\xi_1, \xi_2) \leq B(p_{\varrho_1}^{\mathbf{E}_1}, p_{\varrho_2}^{\mathbf{E}_2}), \quad (4.28)$$

for all $\varrho_i \in \mathcal{S}(\mathcal{H})$, $i = 1, 2$.

Motivated by the analysis done in the previous section, we define a quantity

$$\mathcal{B}(\mathbf{E}_1, \mathbf{E}_2) \equiv \inf_{\varrho_1, \varrho_2 \in \mathcal{S}(\mathcal{H})} \frac{B(p_{\varrho_1}^{\mathbf{E}_1}, p_{\varrho_2}^{\mathbf{E}_2})}{F(\varrho_1, \varrho_2)}. \quad (4.29)$$

This quantity sets the tightest upper bound attainable via Ineq. (4.28) for the distinguishability of the programming states. It has the following properties (for the proof see the article IV).

Proposition 5. *For any $\mathbf{E}_1, \mathbf{E}_2 : \Sigma_N \rightarrow \mathcal{L}(\mathcal{H})$ the quantifier \mathcal{B} satisfies*

$$(B1) \quad \mathcal{B}(\mathbf{E}_1, \mathbf{E}_2) = \mathcal{B}(\mathbf{E}_2, \mathbf{E}_1),$$

$$(B2) \quad \mathcal{B}(\mathbf{E}_1, \mathbf{E}_2) \in [0, 1],$$

$$(B3) \quad \mathcal{B}(\mathbf{E}_1, \mathbf{E}_2) = 1 \text{ if and only if } \mathbf{E}_1 = \mathbf{E}_2,$$

$$(B4) \quad \mathcal{B}(\mathbf{E}_1, \mathbf{E}_2) = \mathcal{B}(\mathcal{U}^*(\mathbf{E}_1), \mathcal{U}^*(\mathbf{E}_2)), \text{ for all unitary channels } \mathcal{U},$$

$$(B5) \quad \mathcal{B}(\mathbf{E}_1, \mathbf{E}_2) \leq B(\mathcal{E}^*(\mathbf{E}_1), \mathcal{E}^*(\mathbf{E}_2)),$$

$$(B6) \quad \mathcal{B}(\mathbf{E}_1, \mathbf{E}_2) \leq B(\lambda \star \mathbf{E}_1, \lambda \star \mathbf{E}_2),$$

for all channels $\mathcal{E}^* : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H}')$ and post-processings $\lambda : (\Omega_N, \Sigma_M) \rightarrow [0, 1]$.

It can be shown that $\mathcal{B}(\mathbf{E}_1, \mathbf{E}_2) = 0$ for all unequal sharp observables $\mathbf{E}_i : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$ (see the article IV) and, as a consequence, the corresponding programming states are necessarily orthogonal. This fact eliminates the possibility of having a universal multimeter implementing all quantum observables via state programming. Interestingly, sharp observables are exactly the $\Sigma \rightarrow \mathcal{L}(\mathcal{H})$ algebra homomorphisms [29, 45]; see Lemma 2. Keeping in mind that all unitary channels are also $*$ -homomorphisms in $\mathcal{L}(\mathcal{H})$ (see Lemma 1), the two results $\mathcal{B}(\mathbf{E}_1, \mathbf{E}_2) = 0$ and $\mathcal{F}(\mathcal{U}_1, \mathcal{U}_2) = 0$ seem to be related. In fact, let $\mathcal{E}_i^* : \mathcal{L}(\mathcal{H}') \rightarrow \mathcal{L}(\mathcal{H})$, $i = 1, 2$, be unequal channels with the $*$ -homomorphism property $\mathcal{E}_i^*(AB) = \mathcal{E}_i^*(A)\mathcal{E}_i^*(B)$, $A, B \in \mathcal{L}(\mathcal{H}')$. Fix any projection $P \in \mathcal{P}(\mathcal{H}')$ such that $\mathcal{E}_1^*(P) \neq \mathcal{E}_2^*(P)$ and define a discrete 2-outcome sharp observable $\mathbf{A} : \Sigma_2 \rightarrow \mathcal{L}(\mathcal{H}')$ via $\mathbf{A}(1) = P$ and $\mathbf{A}(2) = \mathbb{1} - P$. Then

$$\begin{aligned} F(\mathcal{E}_1(\varrho_1), \mathcal{E}_2(\varrho_2)) &= \min_{\mathbf{E} \in \text{POVM}} \sum_i \sqrt{p_{\mathcal{E}_1(\varrho_1)}^{\mathbf{E}}(i) p_{\mathcal{E}_2(\varrho_2)}^{\mathbf{E}}(i)} \\ &= \min_{\mathbf{E} \in \text{POVM}} \sum_i \sqrt{p_{\varrho_1}^{\mathcal{E}_1^*(\mathbf{E})}(i) p_{\varrho_2}^{\mathcal{E}_2^*(\mathbf{E})}(i)} \\ &\leq B(p_{\varrho_1}^{\mathcal{E}_1^*(\mathbf{A})}, p_{\varrho_2}^{\mathcal{E}_2^*(\mathbf{A})}), \end{aligned} \quad (4.30)$$

for all $\varrho_i \in \mathcal{S}(\mathcal{H})$, $i = 1, 2$. Since $\mathcal{E}_i^*(\mathbf{A}) : \Sigma_2 \rightarrow \mathcal{L}(\mathcal{H})$, $i = 1, 2$ are unequal sharp observables, we conclude that $\mathcal{F}(\mathcal{E}_1, \mathcal{E}_2) \leq \mathcal{B}(\mathcal{E}_1^*(\mathbf{A}), \mathcal{E}_2^*(\mathbf{A})) = 0$. Analogously to Prop.7 one can prove the following result, which further underpins this similarity; the result was first noted in the case normal multimeters in the article I.

***Proposition 8.** *Let $\langle \mathcal{K}, \mathcal{Z}, \mathcal{V} \rangle$ be a multimeter and, for $i = 1, 2$, let $\xi_i \in \mathcal{S}(\mathcal{K})$ be states that program the extremal observables $\mathbf{E}_i : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$, respectively. If either of the observables is sharp and $\mathbf{E}_1 \neq \mathbf{E}_2$, then $\xi_1 \perp \xi_2$.*

4.3.1 Post-processing assisted quantum programming

It may seem that programming of quantum observables is completely analogous to programming of quantum channels. There is, however, an important distinction between the two scenarios associated with the available resources *viz.* any post-measurement manipulation of the statistics, being a

classical-to-classical transformation, may be considered a free resource. We recall that the appropriate mathematical notion for such a transformation is post-processing; see Chap. 3. In accordance, the programmed observables depend, not only on the programming states $\xi_i \in \mathcal{S}(\mathcal{K})$, but also on the post-processings $\lambda_j : (\Omega, \Sigma') \rightarrow [0, 1]$ so that

$$E_{ij}(Y) = \int_{\Omega} \lambda_j(x|Y) \operatorname{tr}_{\xi_i} [\mathcal{V}^* (\mathbb{1}_{\mathcal{H}} \otimes Z(dx))], \quad Y \in \Sigma'. \quad (4.31)$$

We will call such a scenario *post-processing assisted quantum programming*.

It was first observed in Ref. [76] that with post-processing assisted quantum programming it is possible to defeat the limitations that the two procedures, classical post-processing and quantum programming, possess when they are performed individually. Namely, there exist multimeters that implement, after post-processing, non-commuting sharp observables with non-orthogonal programming states; see also Ex. 8 that follows. Since the post-processings present a loophole around the above orthogonality results, they could, in principle, assist in making quantum programming universal. The fiducial formalism developed above for quantum programming enables us to show that this, however, is not the case. To this end, we define the *fidelity of post-processings* λ_1 and $\lambda_2 : (\Omega_N, \Sigma_M) \rightarrow [0, 1]$ via

$$\begin{aligned} F(\lambda_1, \lambda_2) &= \inf_{p \in \mathbf{Prob}(\Sigma_N)} B(\lambda_1 \star p, \lambda_2 \star p) \\ &= \inf_{p \in \mathbf{Prob}(\Sigma_N)} \sum_{j=1}^M \sqrt{\sum_{i=1}^N \lambda_1(i|j)p(i)} \sqrt{\sum_{i=1}^N \lambda_2(i|j)p(i)}. \end{aligned} \quad (4.32)$$

Using this quantifier of “closeness” for post-processings we get the following generalization of Prop. 5; for the proof we refer the reader to the article IV.

Proposition 6. *Suppose that for $i = 1, 2$ the states ξ_i program the observables $E_i : \Sigma_N \rightarrow \mathcal{L}(\mathcal{H})$, respectively. Then for any post-processings $\lambda_i : (\Omega_N, \Sigma_M) \rightarrow [0, 1]$, $i = 1, 2$,*

$$F(\varrho_1, \varrho_2) F(\xi_1, \xi_2) F(\lambda_1, \lambda_2) \leq B(\lambda_1 \star p_{\varrho_1}^{E_1}, \lambda_2 \star p_{\varrho_2}^{E_2}), \quad (4.33)$$

for all $\varrho_1, \varrho_2 \in \mathcal{S}(\mathcal{H})$.

It follows that $F(\xi_1, \xi_2) F(\lambda_1, \lambda_2) \leq B(\lambda_1 \star E_1, \lambda_2 \star E_2)$. Assume now that the pointer observable Z is discrete and N -outcome ($N < \infty$) and sup-

pose that one aims to implement different M -outcome sharp observables via post-processing assisted programming⁴. Recalling that the quantifier \mathcal{B} vanishes for sharp observables, it then holds that necessarily either $F(\xi_1, \xi_2) = 0$ or $F(\lambda_1, \lambda_2) = 0$. Accordingly, one can use the post-processing assisted multimeter $\langle \mathcal{K}, \mathcal{Z}, \mathcal{V} \rangle$ to implement at most $\dim(\mathcal{K}) \cdot \text{Bin}(N, M)$ M -outcome sharp observables, where $\text{Bin}(N, M) = \frac{N!}{M!(N-M)!}$ characterizes the size of the set $\{\lambda_i : (\Omega_N, \Sigma_M) \rightarrow [0, 1] \mid F(\lambda_i, \lambda_j) = 0, i \neq j\}$. Since the number of (M -outcome) sharp observables is uncountable, the impossibility of universal post-processing assisted multimeter can be concluded from this upper bound.

We finish this subsection by noting that the fidelity of post-processings can be more easily calculated from the relation

$$F(\lambda_1, \lambda_2) = \min_{i \in \{1, \dots, n\}} \sum_j \sqrt{\lambda_1(i|j)} \sqrt{\lambda_2(i|j)} \quad (4.34)$$

(for the proof see the article IV).

4.3.2 Programming of covariant observables

Throughout this subsection we let \mathcal{H} be a finite d -dimensional Hilbert space, G be a finite group with $\#G$ elements and $U : G \rightarrow \mathcal{U}(\mathcal{H})$ its irreducible projective representation. Recall that each $\xi \in \mathcal{S}(\mathcal{H})$ defines a covariant observable $E_\xi : G \rightarrow \mathcal{L}(\mathcal{H})$ via formula $E_\xi(g) = \frac{d}{\#G} U(g) \xi U(g)^*$. Accordingly, the structure of E_ξ is characterized by the state $\xi \in \mathcal{S}(\mathcal{H})$, which we shall call the *seed* of E_ξ . We shorten the notation by writing E_ψ when the seed corresponds to the pure state P_ψ .

It has been noted in Ref. [37] that there exists a multimeter that implements all observables E_ξ , $\xi \in \mathcal{S}(\mathcal{H})$, related to a fixed representation U ; we will recap this result here. Fix an orthonormal basis $\{\varphi_i\}_{i=1}^d$ of \mathcal{H} . For each $g \in G$, we define an observable $Z(g) = \frac{d^2}{\#G} |u(g)\rangle\langle u(g)|$ on $\mathcal{H} \otimes \mathcal{H}$, where $u(g) \in \mathcal{H} \otimes \mathcal{H}$ are the (Choi-)vectors defined via $u(g) = \frac{1}{\sqrt{d}} \sum_i U(g) \varphi_i \otimes \varphi_i$. A direct calculation shows that for all $\varrho \in \mathcal{S}(\mathcal{H})$

$$\text{tr} [Z(g) \varrho \otimes \xi^T] = \frac{d}{\#G} \text{tr} [\varrho U(g) \xi U(g)^*] = \text{tr} [E_\xi(g) \varrho], \quad (4.35)$$

⁴Notice that, then necessarily $M \leq N$, since sharp observables can be post-processed from a “smeared” one only by merging together its effects [33].

where the transpose ξ^T is defined with respect to the basis $\{\varphi_i\}_{i=1}^d$. Let $\mathcal{V} : \mathcal{T}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H})$ be the partial SWAP channel $\mathcal{V}(A \otimes B \otimes C) = B \otimes A \otimes C$ for all $A, B, C \in \mathcal{T}(\mathcal{H})$. Then $\langle \mathcal{H} \otimes \mathcal{H}, \mathcal{Z}, \mathcal{V} \rangle$ is a multimeter that implements the observables \mathbf{E}_ξ with programming states $\eta \otimes \xi^T$ regardless of the choice of the (fixed) auxiliary state $\eta \in \mathcal{S}(\mathcal{H})$.

As an interesting consequence of Prop. 5, whenever two covariant observables \mathbf{E}_{ξ_1} and \mathbf{E}_{ξ_2} are associated to a common irreducible unitary representation and are unequal sharp observables, or can be made into such with a fixed post-processing, then their seeds necessarily satisfy $F(\xi_1, \xi_2) = 0$. However, this needs not be true if the post-processings are allowed to differ. To verify this, let us make an additional assumption that G has a cyclic subgroup $H = \langle h \rangle$ of order $\#G/d$. Fix any eigenvector $\psi \in \mathcal{H}$, $\|\psi\| = 1$, of the unitary operator $U(h)$, which, due to cyclicity, is then also an eigenvector for all $U(h')$, $h' \in H$. Therefore,

$$\sum_{g \in H} \mathbf{E}_\psi(g) = \frac{d}{\#G} \sum_{g \in H} U(g) P_\psi U(g)^* = P_\psi. \quad (4.36)$$

Similarly, for a different left coset $g'H$, $g' \notin H$, we have $\sum_{g \in g'H} \mathbf{E}_\psi(g) = U(g') P_\psi U(g')^*$. To conclude, if one divides G into the left cosets gH and performs the merging-type post-processings of \mathbf{E}_ψ by summing together the outcomes in each of the cosets, then a sharp observable with d outcomes will be obtained.

Assume now that G has several cyclic subgroups $H_1 = \langle h_1 \rangle, \dots, H_n = \langle h_n \rangle$ of order $\#G/d$. In addition, suppose that $[U(h_i), U(h_j)] \neq 0$ when $i \neq j$, so that one can always find non-orthogonal (and non-parallel) eigenvectors ψ_1, \dots, ψ_n of $U(h_1), \dots, U(h_n)$, respectively. Each of these eigenvectors will, after appropriate post-processing specified by the different subgroups, then lead to a different d -outcome sharp observable $\mathbf{E}_{\psi_1}, \dots, \mathbf{E}_{\psi_n}$. As seen above, the non-orthogonal seeds $P_{\psi_1}, \dots, P_{\psi_n}$ can be chosen as the programming states of these observables.

The above construction, along with examples demonstrating its usage, was first introduced in the article IV. It is noteworthy, that the method may succeed even if some of the mentioned assumptions are not fulfilled. This is a useful fact, since it may be difficult find groups whose projective representation would satisfy all the restrictions listed above. The next example elucidates this situation.

Example 8. Consider the group $U = \langle \sigma_1, i\sigma_2 \rangle$. It can be shown that $U = \{\pm \mathbb{1}, \pm \sigma_1, \pm i\sigma_2, \pm \sigma_3\}$ with the group generators satisfying $\sigma_1^2 = \mathbb{1} = (i\sigma_2)^4$ and $\sigma_1(i\sigma_2)\sigma_1 = (i\sigma_2)^{-1}$. In other words, U is a two-dimensional unitary representation of the dihedral group $D_8 = \langle a, b \mid a^2 = 1 = b^4, aba^{-1} = b^{-1} \rangle$. The representation is irreducible, since $\sum_{g \in D_8} \text{tr}[U(g)]^2 = 8 = \#D_8$. Let us fix the seeds

$$P_{\psi_1} = \frac{1}{2}(\mathbb{1} + \sigma_1), \quad P_{\psi_2} = \frac{1}{2}(\mathbb{1} + \sigma_2), \quad P_{\psi_3} = \frac{1}{2}(\mathbb{1} + \sigma_3) \quad (4.37)$$

and define the observables $E_{\psi_i}(g) = \frac{1}{4}U(g)P_{\psi_i}U(g)^*$, $i = 1, 2, 3$, respectively. Noticeably, D_8 has only one cyclic subgroup of order 4, which is generated by the element $a \simeq \sigma_1$. Nevertheless, using three different post-processings we get the sharp spin-observables

$$\begin{aligned} S_{\hat{x}}(+) &= \frac{1}{4}(P_{\psi_1} + (-\mathbb{1})P_{\psi_1}(-\mathbb{1}) + \sigma_1 P_{\psi_1} \sigma_1 + (-\sigma_1)P_{\psi_1}(-\sigma_1)) \\ S_{\hat{y}}(+) &= \frac{1}{4}(P_{\psi_2} + (-\mathbb{1})P_{\psi_2}(-\mathbb{1}) + i\sigma_2 P_{\psi_2} (i\sigma_2)^* + (-i\sigma_2)P_{\psi_2}(-i\sigma_2)^*) \\ S_{\hat{z}}(+) &= \frac{1}{4}(P_{\psi_3} + (-\mathbb{1})P_{\psi_3}(-\mathbb{1}) + \sigma_3 P_{\psi_3} \sigma_3 + (-\sigma_3)P_{\psi_3}(-\sigma_3)), \end{aligned} \quad (4.38)$$

with $S_i(-) = \mathbb{1} - S_i(+)$, $i = \hat{x}, \hat{y}, \hat{z}$. Using Eq.(4.34) it is easy to confirm that the three post-processings are mutually orthogonal with respect to their corresponding fidelity. This is what one would expect, since the seeds/programming states satisfy $F(P_{\psi_i}, P_{\psi_j}) = 1/\sqrt{2}$ for $i \neq j$.

4.3.3 Wigner-Araki-Yanase theorem and programming

Let us continue from where we left with the WAY-theorem in Chap. 3. Consider a programmable multimeter $\langle \mathcal{K}, \mathcal{Z}, \mathcal{V} \rangle$ that realizes a sharp observable $E_1 : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$ with a programming state $\xi_1 \in \mathcal{S}(\mathcal{K})$. Consequently, Prop. 5 then reduces to

$$\| [E_1(X), \text{tr}_{\xi_1}[L]] \| \leq \| [\mathcal{V}^*(\mathbb{1} \otimes Z(X)), L] \|. \quad (4.39)$$

Let $E_2 : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$ be another (not necessarily sharp) observable that is programmable from $\langle \mathcal{K}, \mathcal{Z}, \mathcal{V} \rangle$ with a state $\xi_2 \in \mathcal{K}$ and let \mathcal{E} be any channel

on $\mathcal{T}(\mathcal{K})$ satisfying $\mathcal{E}(\xi_1) = \xi_2$. Then, in particular

$$\begin{aligned} E_2(Y) &= \text{tr}_{\xi_2}[\mathcal{V}^*(\mathbb{1} \otimes Z(Y))] \\ &= \text{tr}_{\xi_1}[(\mathcal{I} \otimes \mathcal{E})^*(\mathcal{V}^*(\mathbb{1} \otimes Z(Y)))] . \end{aligned} \quad (4.40)$$

Finally, by denoting $\mathcal{V}_{\mathcal{E}} \equiv \mathcal{V} \circ (\mathcal{I} \otimes \mathcal{E})$, we may conclude that the following proposition holds true.

***Proposition 9.** *Let $\langle \mathcal{K}, Z, \mathcal{V} \rangle$ be a multimeter realizing the observables $E_i : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$ with programming states ξ_i , $i = 1, 2$, respectively. Assume that E_1 is sharp. Then, for all $X, Y \in \Sigma$,*

$$\| [E_1(X), E_2(Y)] \| \leq \| [\mathcal{V}^*(\mathbb{1} \otimes Z(X)), \mathcal{V}_{\mathcal{E}}^*(\mathbb{1} \otimes Z(Y))] \| \quad (4.41)$$

holds for any channel $\mathcal{E} : \mathcal{T}(\mathcal{K}) \rightarrow \mathcal{T}(\mathcal{K})$ satisfying $\mathcal{E}(\xi_1) = \xi_2$.

We notice that choosing above $\mathcal{E} = \mathcal{E}_{\xi_2}$ as the complete state space contraction $\mathcal{E}_{\xi_2} : T \mapsto \text{tr}[T] \xi_2$ reproduces the same implication

$$\| [E_1(X), E_2(Y)] \| \leq \| [\mathcal{V}^*(\mathbb{1} \otimes Z(X)), E_2(Y) \otimes \mathbb{1}] \|, \quad X, Y \in \Sigma, \quad (4.42)$$

that could be deduced from Prop. 6⁵.

Importantly, Prop. 9 relates, not just the programming states, but all the different entries of the multimeter to the programmed observables and can in this sense directly linked to the optimal multimeter design. If the multimeter is normal $\langle \mathcal{K}, Z, U \rangle$, the result has also a particularly prominent interpretation. Namely since now E_1 and $U^*(\mathbb{1} \otimes Z)U$ are sharp, the commutators appearing in Prop. 9 can be directly viewed as quantifiers of incompatibility; see Ineq. (3.11). In this regard, Prop. 9 states that the programmed observables E_1 and E_2 are necessarily more compatible than the Heisenberg-evolved pointers $U^*(\mathbb{1} \otimes Z)U$ and $\mathcal{I} \otimes \mathcal{E}^*(U^*(\mathbb{1} \otimes Z)U)$ realizing them. As such, the relation could be applied in the field of so-called equivalent multimeters, that have been studied in Ref. [90]. In the general case, $\mathcal{V}^*(\mathbb{1} \otimes Z(X))$ is only approximately sharp (see Prop. 2), so that the incompatibilities of the aforementioned entities are not as straightforwardly linked.

⁵However, in the context of Prop. 6 the observable E_1 does not need to be sharp.

Chapter 5

Summary

In this thesis an overview of the central conclusions of the research articles written by the PhD student Mikko Tukiainen, in collaboration with his supervisors and other colleagues, has been presented. Shortly summarized, the thesis deals with the theory of quantum measurements with the main emphasis on the implementability and programmability of different quantum devices: observables, channels or instruments. Although fully theoretical by nature, the results have been conducted in the mathematical language of open quantum systems, keeping an eye on their practical pertinence.

Entanglement lies in the heart of many quantum applications [9, 10]. One of scenarios where quantum entanglement provides an advantage over the classical case is the thermodynamic work extraction by information erasure [26]. As opposed to the classical case, where information erasure can only consume energy, it is possible to gain positive work by erasing information stored in a quantum system. A potential application of the phenomenon is to provide a solution to the problem of heating in computation. Indeed on a fundamental level, the heat generated in computation ultimately traces back to the cost of erasure [91]. Thanks to the quantum benefit in information erasure, it is actually possible to design the computation in a way that cools down the device – a fact highly contrary to the classical case. The inevitable and inescapable environmental noise may, however, rapidly degrade the entangled state and render it useless for the purposes of the quantum protocol at hand. Nevertheless, if the open quantum system evolution is non-Markovian, memory effects can allow for a (partial) recovery of the entanglement previously lost into the environment. In the article II the non-Markovian benefit in the thermodynamic work extraction has been studied. Moreover, in the article VI it has been

shown how weak measurements and weak values can be harnessed to quantify the amount entanglement in 2-qubit systems with a fixed measurement setup.

From the viewpoint of measurability, the work at hand describes the structure of an apparatus realizing a given quantum device. In particular, a complete characterization of the minimal apparatuses is presented in the case where the model of the measurement realization is assumed to be normal. Such minimal measurement descriptions can be used, for example, in the field of open quantum systems to solve the smallest possible environment – hence with least (potentially harmful) environmental degrees of freedom – in which the system-environment-composite evolves unitarily. It is worth mentioning, that the minimal models are also the effective realizations in the sense that any such model can be isometrically embedded into any other normal measurement of the same device. This topic of minimality has been explored shortly in Chap. 3 and more thoroughly in the article VII.

The restrictions of measurements of quantum observables imposed by the conservation laws were investigated in Chap. 3 and in the research article III. Historically, these limitations were first noticed by Wigner in measurements of spin- $\frac{1}{2}$ -systems already in the early days of quantum theory and were soon after generalized by Araki and Yanase. In this thesis a new formulation of the Wigner-Araki-Yanase theorem was introduced, that, instead of involving the conservation laws *per se*, rephrases the measurability limitations in terms of (a)symmetry transferred from the measuring apparatus to the measured system.

The programmable measurements, quantum multimeters, that implement different observables and channels were examined in Chap. 4 and the articles I, III, IV and V. An example of a programmable arrangement is the common computer; the virtue of programmability and reusability of a fixed (array of) circuit(s), taking in different instructions that describe how the circuit should process the data, is that there is no need to build a different computer for each given task. A quantum multimeter, where the state of the apparatus is considered as the programming instruction, works in a completely analogous way, except that the instructions may come in a form of a quantum state. Such an arrangement can implement measurements of different observables and channels – even incompatible ones.

Curiously, when one aims to implement a pair of unequal sharp observables or so-called homomorphic channels, quantum programming coincides

with the classical case in the sense that the different programs are perfectly distinguishable from each other, regardless of the multimeter design. Besides observing this occurrence qualitatively, in this thesis different figures of merit were developed to quantify the mutual relations between the programming states of arbitrary pairs of observables or channels (or more generally dynamics). Based on some of these quantifiers, in the article V we have proposed and demonstrated a new method for quantum probing fused with quantum programming. When one focuses on programming observables, classical post-processings can be considered as additional programming resources. The usefulness of this kind of post-processing assisted quantum programming was demonstrated in the article IV, especially in the case of covariant observables. Finally, the aforementioned new formalism of the Wigner-Araki-Yanase theorem was applied to multimeters in the article III in order to expose further restrictions in quantum programming.

There remain many questions yet to be solved. Firstly, the Wigner-Araki-Yanase-limitations were only considered in case of quantum observables; Prop. 1 plays a central role in proving these restrictions. Similar constraints could be straightforwardly developed for quantum channels by harnessing Prop. 3 instead. Secondly, only the deterministic quantum programming of two kinds of quantum devices, observables and channels, were studied. As opposed to the deterministic case, in a probabilistic programming scenario a correct device is implemented with a probability less than 1. It may be possible to link the methods developed in this thesis to the probabilistic programming e.g. by studying the programmability of quantum instruments. Indeed, quantum instruments can naturally be interpreted as conditional state transformations i.e. probabilistic channels. There exist also higher levels of hierarchies of programmability to be considered, such as so-called process positive operator valued measures [92] and more general quantum networks [93–96] and their programming with different resources: with states, channels, observables, instruments or ultimately even with other networks. Although some interest has been paid on programmable devices of these levels, e.g. the quantum switch [96], the overall field is still to large extend open for exploration.

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Annales Universitatis Turkuensis



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ISBN 978-951-29-7009-4 (PRINT)
ISBN 978-951-29-7010-0 (PDF)
ISSN 0082-7002 (PRINT) | ISSN 2343-3175 (PDF)