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To the memory of my parents

Abstract

The research area of this thesis is Geometric Function Theory, which is a subfield of mathematical analysis. The thesis consists of four published papers. Prepublication versions of these papers are available on the www-pages of the arXiv.org preprint server. The objects of this research are subdomains of the Euclidean n-dimensional space, their geometries, and the function classes defined on these subdomains. Some examples of these function classes are conformal maps, analytic functions, and Möbius transformations in the plane case ($n = 2$), and in the higher-dimensional case ($n \geq 2$) bilipschitz, quasiconformal and quasiregular maps. The main results concern the moduli of continuity of the aforementioned classes of functions with respect to so called hyperbolic type geometries. As a model or ideal we have the classical hyperbolic or non-Euclidean geometry of the unit disk, which was discovered two centuries ago and is invariant under conformal mappings.

During the past 30 years, the research on these questions has been, and continues to be, very active and it has turned out that in dimensions $n \geq 3$ one must give up the full invariance property and replace it with a weaker, quasi-invariance property. The special role of the boundary of the domain is a key feature of hyperbolic type geometries. Each hyperbolic type geometry is based on a specific notion of distance between two points, so called metric. A hyperbolic type metric between two points takes into account, in addition to the position of the points with respect to each other, also the position of the points with respect to the boundary of the domain.

In the first paper, we study the triangular ratio metric and compare it with some other hyperbolic type metrics. Moreover, we prove that quasiregular mappings are Hölder continuous with respect to the triangular ratio metric.

In the second paper, we have a similar aim but this time for the visual angle metric. We show that this metric is comparable to the triangular ratio metric in convex domains and prove that quasiconformal maps are uniformly continuous with respect to the visual angle metric.

In the third paper, we find sufficient conditions on the domains for which two above mentioned metrics are comparable. We also show that bilipschitz maps with respect to the triangular ratio metric are quasiconformal.

In the fourth paper we introduce a new hyperbolic type metric. We show that this metric is comparable to other metrics in so called uniform domains.

The results of this thesis have already found applications also in the works of other researchers. On the basis of these facts, one can say that the results shed new light on Geometric Function Theory. In addition to this, several open problems are formulated in this thesis.

Tiivistelmä

Väitöskirjan tutkimusala on matemaattinen analyysi, tarkemmin sanottuna geometrinen funktioteoria. Tutkimus koostuu neljästä julkaistusta tutkimuksesta, joiden preprint-versiot ovat saatavilla www-sivustoilla arXiv.org. Tutkimuksen kohteena ovat n-ulotteisen euklidisen avaruuden osa-alueiden geometria sekä näissä alueissa määritellyt funktioluokat. Esimerkkeinä funktioluokista ovat tason ($n = 2$) tapauksessa konformikuvaukset, analyyttiset funktiot ja Möbius-kuvaukset sekä avaruudessa ($n \geq 2$) bilipschitz ja kvasikonformikuvaukset. Tutkimuksen päätulokset liittyvät alueiden uudentyyppisten geometrioiden soveltamiseen edellämainittujen funktioluokkien jatkuvuusmoduulien arviointiin. Esikuvana näille geometrioille voidaan pitää konformikuvauksissa invarianttia tason yksikkökiekon epäeuklidista eli hyperbolista geometriaa, joka syntyi kaksi vuosisataa sitten.

Viimeisten 30 vuoden aikana alan kansainvälinen tutkimus on ollut, ja edelleen on, erittäin vilkasta ja on osoittautunut, että tapauksessa $n \geq 3$ invarianssitavoitteesta joudutaan tinkimään ja tyytymään osittaiseen taikka kvasi-invarianssiin. Kunkin hyperbolis-tyyppisen geometrian perustana on geometrian oma kahden pisteen välinen etäisyyskäsite eli metriikka. Tällainen hyperbolis-tyyppinen metriikka huomioi paitsi pisteiden sijainnin toisiinsa nähden myös pisteiden sijainnin alueen reunan suhteen.

Ensimmäisessä julkaisussa tutkimuksen kohteena on kolmiosuhdemetriikka, jota verrataan muihin hyperbolis-tyyppisiin metriikoihin. Lisäksi todistetaan, että kvasisäännölliset kuvaukset ovat tässä metriikassa Hölder-jatkuvia.

Toisessa julkaisussa on samankaltainen tutkimusidea, mutta tarkastellen nyt näkökulma metriikkaa. Osoitetaan, että konvekseissa alueissa näkökulma metriikka ja kolmiosuhdemetriikka ovat vertailtavissa ja kvasikonformikuvaukset ovat tasaisesti jatkuvia näkökulma metriikan suhteen.

Kolmannessa julkaisussa esitetään riittäviä ehtoja alueille, joiden vallitessa kaksi viimeksi mainittua metriikkaa ovat vertailtavissa. Osoitamme myös, että bilipschitz kuvaukset kolmiosuhdemetriikan suhteen ovat kvasikonformisia.

Neljännessä julkaisussa esitellään uusi hyperbolis-tyyppinen metrikka. Osoittautuu, että tämä uusi metriikka on hyvin vertailtavissa edellä mainittuihin metriikoihin ns. uniformeissa alueissa.

Tuloksille on jo löytynyt sovelluksia muiden tutkijoiden töissä. Johtopäätöksenä voidaan todeta, että klassiset geometriset ongelmat tulevat tässä tutkimuksessa hyperbolis-tyyppisen geometrian kautta uuteen valoon. Työssä on esitetty myös aiheita jatkotutkimuksille.

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Turku, Feb 2018
Parisa Hariri

List of original publications

This PhD dissertation consists of a summary and the following four original papers:

[I] J. Chen, P. Hariri, R. Klén, and M. Vuorinen: Lipschitz conditions, triangular ratio metric, and quasiconformal maps, Ann. Acad. Sci. Fenn. 40, 2015, 683–709, doi:10.5186/aasfm.2015.4039, arXiv:1403.6582 [math.CA].

[II] P. Hariri, M. Vuorinen and G. Wang: Some remarks on the visual angle metric, Comput. Methods and Funct. Theory, 16, 2016, 187–201, doi:10.1007/s40315-015-0137-8, arXiv:1410.5943 [math.MG].

[III] P. Hariri, M. Vuorinen, and X. Zhang: Inequalities and bilipschitz conditions for triangular ratio metric, Rocky Mountain J. Math. 47, 4, 2017, 1121-1148, doi:10.1216/RMJ-2017-47-4-1121, arXiv:1411.2747 [math.MG].

[IV] O. Dovgoshey, P. Hariri, and M. Vuorinen: Comparison theorems for hyperbolic type metrics, Complex Var. Elliptic Equ. 61, 11, 2016, 1464–1480, doi:10.1080/17476933.2016.1182517, arXiv:1504.04487 [math.MG].

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1. Introduction

The topic of this PhD thesis is Geometric Function Theory, which is a subfield of Classical Analysis. Some of the basic objects of study are well-known classes of maps defined on subdomains of the Euclidean space $\mathbb{R}^n$. These classes include quasiconformal and quasiregular mappings which are generalizations of conformal mappings and analytic functions, respectively. This research area is accessible to mathematicians familiar with basic real and complex analysis. J. Väisälä's book [Va1, pp.1-40] and his survey [Va2] form an excellent starting point to enter this research.

Our study makes extensive use of metrics. As is well-known, Geometric Function Theory is a field where metrics are recurrent: some examples are Euclidean, chordal and hyperbolic metrics. We introduce in paper [IV] a new metric and find bounds for it in terms of known metrics. Moreover we study several other metrics, too:

- the triangular ratio metric
- the visual angle metric
- the distance ratio metric.

Some of the basic questions are:

- How are these metrics related to other metrics such as hyperbolic or quasi-hyperbolic metric?
- How are these metrics transformed under well-known classes of mappings such as Möbius transformations, quasiconformal maps and Lipschitz maps?

The above questions form the central research themes of this thesis.

We will now briefly review the history and literature of quasiconformal and quasiregular mappings. Quasiconformal and quasiregular mappings in the plane were first studied by H. Grötzsch in 1928 and O. Teichmüller in 1938-44 but the second world war interrupted the progress. In the early 1950's, due to the works of L.V. Ahlfors, L. Bers, I. N. Vekua and their students, fast progress started. The monographs of L.V. Ahlfors [A], K. Astala, T. Iwaniec, G. Martin [AIM], O. Lehto and K. I. Virtanen [LV], and I. N. Vekua [Ve] provide introduction to the theory of quasiconformal and quasiregular mappings in the plane. In the Euclidean space $\mathbb{R}^n$, quasiconformal mappings were studied by F. W. Gehring and J. Väisälä in the early 1960's and quasiregular mappings by Yu. G. Reshetnyak a few years later. See also [C, G2, G3, GMP, R, Ri, Va1, Va2]. There are several collections of surveys on quasiconformal mappings [DHOP, HMMPV, PSV] and the handbook of R. Kühnau [K].

We next outline the roots of this research. F. Klein's Erlangen Program was an important step in the development which led to the present central role of the hyperbolic metric and other conformal invariants in Complex Analysis [B1, KL]. For instance the Schwarz Lemma, a central tool in Complex Analysis, can be expressed as follows: an analytic function of the unit disk into itself is a contraction mapping with respect to the hyperbolic metric. In 1976, F. W. Gehring and B. P. Palka [GP] introduced the quasihyperbolic metric $k_G(x, y)$, $x, y \in G$ which can be defined in every proper subdomain $G \subsetneq \mathbb{R}^n$. For the unit ball $\mathbb{B}^n$ and the half-space $\mathbb{H}^n$ this metric is comparable to the hyperbolic metric. The quasihyperbolic metric has found many applications and is now one of the standard tools of Geometric Function Theory. In 1995, A. F. Beardon [B2] reinvented the Apollonian metric which had been studied by D. Barbilian already in 1934 [B, BS]. Its relation to other metrics is studied in [GH]. During the past twenty years, numerous authors have studied these and some other related metrics, see [HIMPS, Vu4]. In particular, the PhD theses and also later work of P. Seittenranta [SE1], V. Heikkala [He], P. Hästö [H2], Z. Ibragimov [IB1], S. Sahoo [Sa], H. Lindén [L], V. Manojlović [Ma], R. Klén [K1], X. Zhang [Z], G. Wang [W] and M. Mohapatra [MO] dealt with these metrics.

In paper [I], we study the triangular ratio metric s_G, and compare it with some other metrics like the distance ratio metric j_G and hyperbolic metric ρ_G. Furthermore we study the behavior of s_G under quasiregular maps. Paper [II] deals with the visual angle metric v_G, which was introduced and studied recently in [KLVW]. Also the uniform continuity of quasiconformal maps with respect to the visual angle metric and the triangular ratio metric is studied. In paper [III], we continue the study of these metrics, and we introduce two sufficient conditions on domains G under which s_G and v_G are comparable. Moreover we show that bilipschitz maps with respect to the triangular ratio metric are quasiconformal. In paper [IV], we study a new metric called $h_{G,c}$-metric, $c \geq 2$. When the domain is uniform, this metric is comparable to the quasihyperbolic metric, and then it easily follows that $h_{G,c}$- bilipschitz homeomorphisms are quasiconformal.

2. Background and definitions

In Geometric Function Theory, in the study of mappings defined on subdomains of $\mathbb{R}^n$, the Euclidean metric is not adequate. Instead of measuring just distance between points, one should also consider the position of the points with respect to the boundary of the domain. One example of a metric with this feature is the hyperbolic metric of the unit ball in $\mathbb{R}^n$. Many authors have used this idea so as to define metrics of hyperbolic type, and study geometries defined by these metrics in domains more general than the unit ball, see [H1, HIMPS, KL, MVa, RT].

We introduce some terminology and notation, following [Va1]. For $x \in \mathbb{R}^n$ and $r > 0$ let

$$B^n(x,r) = \{\, z \in \mathbb{R}^n : |x-z| < r \,\}, \text{ and}$$
$$S^{n-1}(x,r) = \{\, z \in \mathbb{R}^n : |x-z| = r \,\}$$

denote the ball and the sphere, respectively, centered at x with radius r. The abbreviations $B^n(r) = B^n(0,r)$, $S^{n-1}(r) = S^{n-1}(0,r)$, $\mathbb{B}^n = B^n(1)$, $S^{n-1} = S^{n-1}(1)$ will be used frequently. The dimensions are sometimes omitted: $B(x,r)$, $S(x,r)$. The notation $\measuredangle(x,z,y)$ means the angle in the range $[0,\pi]$ between the segments $[x,z]$ and $[y,z]$. The i'th coordinate unit vector is e_i and hence $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ can be written as $x = \sum_{i=1}^n x_i e_i$.

2.1. **Distance ratio metric.** For a proper open subset $G \subset \mathbb{R}^n$ and for all $x, y \in G$, the distance ratio metric j_G is defined as

$$j_G(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d(x,\partial G), d(y,\partial G)\}}\right).$$

The distance ratio metric was introduced by F.W. Gehring and B.P. Palka [GP], in a slightly different form and in this form by M. Vuorinen in [Vu1]. If confusion seems unlikely, then we write $d(x) = d(x,\partial G)$. For the proof of the triangle inequality for j_G, see [SE1]. We use the abbreviations sh, ch and th for the hyperbolic sine, cosine, and tangent respectively. In addition to j_G we also study the metric

$$j^*_G(x,y) = \operatorname{th}\frac{j_G(x,y)}{2}.$$

Because j_G is a metric, it follows easily, see [AVV, 7.42(1)], that j^*_G is a metric, too.

2.2. **Weighted length.** Let $G \subsetneq \mathbb{R}^n (n \geq 2)$ be a domain and $w : G \to (0,\infty)$ be a continuous function. We define the weighted length of a rectifiable curve $\gamma \subset G$ by

$$\ell_w(\gamma) = \int_\gamma w(z)|dz|$$

and the weighted distance by

$$d_w(x,y) = \inf_\gamma \ell_w(\gamma),$$

where the infimum is taken over all rectifiable curves in G joining x and y. It is easy to see that d_w defines a metric on G and (G, d_w) is a metric space. We say that a curve $\gamma : [0,1] \to G$ is a geodesic joining $\gamma(0)$ and $\gamma(1)$ if for all $t \in (0,1)$, we have

$$d_w(\gamma(0),\gamma(1)) = d_w(\gamma(0),\gamma(t)) + d_w(\gamma(t),\gamma(1)).$$

2.3. **Hyperbolic metric.** The hyperbolic metrics $\rho_{\mathbb{H}^n}$ and $\rho_{\mathbb{B}^n}$ of the upper half space

$$\mathbb{H}^n = \{(x_1,\ldots,x_n) \in \mathbb{R}^n : x_n > 0\}$$

and of the unit ball

$$\mathbb{B}^n = \{z \in \mathbb{R}^n : |z| < 1\}$$

can be defined as weighted metrics with the weight functions $w_{\mathbb{H}^n}(x) = 1/x_n$ and $w_{\mathbb{B}^n}(x) = 2/(1-|x|^2)$, respectively. This definition as such is rather abstract and for applications concrete formulas are needed. By [B1, p.35] we have for $x, y \in \mathbb{H}^n$

$$\operatorname{ch}\rho_{\mathbb{H}^n}(x,y) = 1 + \frac{|x-y|^2}{2x_n y_n}, \tag{2.4}$$

and by [B1, p.40] for $x, y \in \mathbb{B}^n$

$$\operatorname{sh}\frac{\rho_{\mathbb{B}^n}(x,y)}{2} = \frac{|x-y|}{\sqrt{1-|x|^2}\sqrt{1-|y|^2}}. \tag{2.5}$$

From (2.5) we easily obtain

$$\operatorname{th}\frac{\rho_{\mathbb{B}^n}(x,y)}{2} = \frac{|x-y|}{\sqrt{|x-y|^2+(1-|x|^2)(1-|y|^2)}}\,.$$

2.6. **Quasihyperbolic metric.** Let G be a proper subdomain of $\mathbb{R}^n$. For all $x,\, y \in G$, the quasihyperbolic metric k_G is defined as

$$k_G(x,y) = \inf_{\gamma} \int_{\gamma} \frac{1}{d(z,\partial G)}|dz|,$$

where the infimum is taken over all rectifiable arcs γ joining x to y in G [GP]. Hence the quasihyperbolic metric is a weighted metric with the weight function $w(x) = 1/d(x,\partial G)$.

2.7. **Point pair function.** For $x, y \in G \subsetneq \mathbb{R}^n$ we define the point pair function as

$$p_G(x,y) = \frac{|x-y|}{\sqrt{|x-y|^2+4\,d(x)\,d(y)}}.$$

This function was introduced in paper [I] where it turned out to be a very useful function in the study of the triangular ratio metric. However, there are domains G for which p_G is not a metric, for instance when $G = \mathbb{B}^2$ [I, Remark 3.1].

2.8. **Triangular ratio metric.** For a domain $G \subsetneq \mathbb{R}^n$ and $x, y \in G$, the triangular ratio metric s_G is defined as follows:

$$s_G(x,y) = \sup_{z\in\partial G} \frac{|x-y|}{|x-z|+|z-y|} \in [0,1].$$

This metric was introduced by P. Hästö, [H1]. Very recently, the geometry of the balls of s_G for some special domains was studied in [HKLV].

Finding the extremal point $z \in \partial G$ in the above formula is a nontrivial problem. In the case $G = \mathbb{B}^2$ and $x, y \in \mathbb{B}^2$, the extremal point $z \in \partial\mathbb{B}^2$ for the definition of $s_{\mathbb{B}^2}(x,y)$ is the point of contact of the boundary with an ellipse, with foci x, y, contained in $\mathbb{B}^2$. By a basic property of the ellipse with foci x, y, the normal to the ellipse bisects the angle formed by segments joining the foci x, y with the point z. The problem of finding $z \in \partial\mathbb{B}^2$, for which $\measuredangle(x,z,0) = \measuredangle(0,z,y)$, has a long history. It was first formulated by the Greek mathematician Ptolemy (ca. 100-170) in his research of optics. In the three-dimensional case this famous problem reads as follows: "Given a light source and a spherical mirror, find the point on the mirror where the light will be reflected to the eye of an observer." Because of the well-known reflection law of the light, we easily see that this leads, in the two dimensional case, to the question of finding the extremal point of the triangular ratio metric in the unit disk. The solution leads to a polynomial equation of the fourth degree, see [FHMV]. The Arab mathematician Ibn al-Haytham (Alhazen), (ca. 965-1040), also studied Ptolemy's problem and applied it to optics, therefore the problem is sometimes called the Ptolemy-Alhazen problem.

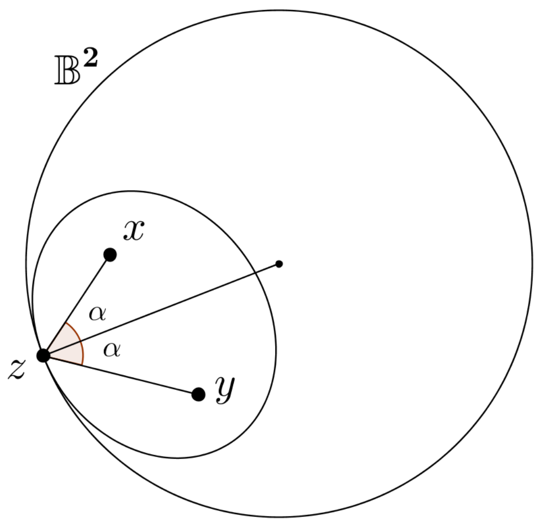

FIGURE 1. The definition of $s_{\mathbb{B}^2}$. The maximal ellipse with foci x and y and contained in $\overline{\mathbb{B}}^2$.

2.9**. Visual angle metric.** The visual angle metric is defined by

$$v_G(x,y) = \sup\{\measuredangle(x,z,y) : z \in \partial G\}, \quad x, y \in G,$$

for domains $G \subsetneq \mathbb{R}^n$, $n \geq 2$, such that ∂G is not a proper subset of a line, see [KLVW, Lemma 2.8].

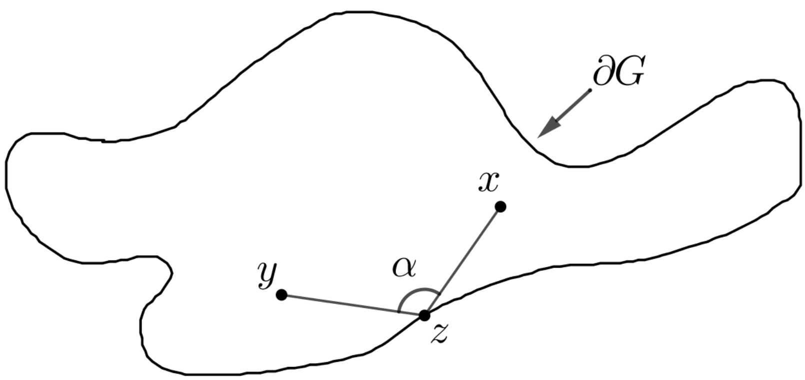

FIGURE 2. The definition of $v_G(x,y)$.

In the case of $G = \mathbb{B}^n$ or $G = \mathbb{H}^n$ the metrics ρ_G, k_G, and j_G can be compared as follows ([AVV, Lemma 7.56], [GP] and [Vu2, Lemma 2.41 (2)]):

$$\frac{1}{2}\rho_{\mathbb{B}^n}(x,y) \le j_{\mathbb{B}^n}(x,y) \le k_{\mathbb{B}^n}(x,y) \le \rho_{\mathbb{B}^n}(x,y)\,, \forall\; x,y \in \mathbb{B}^n\,,$$

$$\frac{1}{2}\rho_{\mathbb{H}^n}(x,y) \le j_{\mathbb{H}^n}(x,y) \le k_{\mathbb{H}^n}(x,y) \equiv \rho_{\mathbb{H}^n}(x,y)\,, \forall\; x,y \in \mathbb{H}^n\,.$$

Because the function th is increasing, by above inequalities, [III, Lemmas 2.3, 2.6, 2.1] and [I, Lemma 3.4(1), (2.4)], we have

$$\operatorname{th}\frac{\rho_{\mathbb{B}^n}(x,y)}{4} \le j^*_{\mathbb{B}^n}(x,y) \le s_{\mathbb{B}^n}(x,y) \le p_{\mathbb{B}^n}(x,y) \le \operatorname{th}\frac{\rho_{\mathbb{B}^n}(x,y)}{2} \le 2j^*_{\mathbb{B}^n}(x,y)\,,$$

for all $x,y \in \mathbb{B}^n$ and

$$\operatorname{th}\frac{\rho_{\mathbb{H}^n}(x,y)}{4} \le j^*_{\mathbb{H}^n}(x,y) \le s_{\mathbb{H}^n}(x,y) \equiv p_{\mathbb{H}^n}(x,y) \equiv \operatorname{th}\frac{\rho_{\mathbb{H}^n}(x,y)}{2} \le 2j^*_{\mathbb{H}^n}(x,y)\,,$$

for all $x,y \in \mathbb{H}^n$.

Because $\rho_{\mathbb{H}^n}$ is a metric we see that $\operatorname{th}(\rho_{\mathbb{H}^n}/2)$ and $p_{\mathbb{H}^n}$ are metrics [AVV, 7.42(1)], too. Recall that by Remark 2.7, $p_{\mathbb{B}^n}$ is not a metric.

2.10. **Remark.** *Let m be one of the above mentioned metrics, $m \in \{j, j^*, k, s, v\}$. Then the following three properties hold:*

(a) *Monotonicity with respect to domain, i.e. if $G_1, G_2 \subsetneq \mathbb{R}^n$ are domains with $G_1 \subset G_2$ and $x,y \in G_1$, then $m_{G_1}(x,y) \ge m_{G_2}(x,y)$.*
(b) *Sensitivity to boundary variation, i.e. if $G \subsetneq \mathbb{R}^n$ is a domain and $x_0 \in G$, then the numerical values of $m_G(x,y)$ and $m_{G\setminus\{x_0\}}(x,y)$ are not comparable if x,y are very close to x_0 .*
(c) *For fixed $x,y \in G \subsetneq \mathbb{R}^n$, one extremal boundary point $z \in \partial G$ determines the numerical value of $m_G(x,y)$.*

For a metric space (G,m) we define the metric ball with the center $x \in G$ and radius $r > 0$ by $B_m(x,r) = \{y \in G : m(x,y) < r\}$.

Next we recall a few basic definitions from the theory of quasiconformal and quasiregular maps. Some of the standard sources in this area are [Va1, R, Ri, Vu2].

Below we assume that D, D' are domains in $\mathbb{R}^n$. The Lebesgue measure of a measurable set $A \subset \mathbb{R}^n$ is denoted by $m_n(A)$. We next introduce the class of ACL-functions for which many properties of smooth functions are valid, except at the points of an exceptional set.

2.11. **Definition.** [Va1, Definitions 26.2, 26.5] *For $j = 1, \ldots, n$, let $R^n_j = \{\,x \in \mathbb{R}^n : x_j = 0\,\}$ and let $T_j : \mathbb{R}^n \to R^n_j$ be the orthogonal projection $T_j x = x - x_j e_j$. Let $D \subset \mathbb{R}^n$ be an open set and $u : D \to \mathbb{R}$ a continuous function. The function u is called absolutely continuous on lines, abbreviated as* ACL, *if for every cube Q with $\overline{Q} \subset D$, the set $A_j \subset T_j D \subset R^n_j$ of all points $z \in T_j Q$ such that the function $t \mapsto u(z + te_j)$, $z + te_j \in Q$, is not absolutely continuous as a function of a single variable, satisfies $m_{n-1}(A_j) = 0$ for all $j = 1, \ldots, n$.*

A vector–valued function is said to be ACL *if and only if each coordinate function is in this class. We say that an* ACL *function* $u : D \to \mathbb{R}^n$ *is* ACL^p, $p \geq 1$, *if* $\partial u(x)/\partial x_j \in L^p(K)$, $j = 1, \ldots, n$, *whenever* $K \subset D$ *is compact.*

One can define the formal derivative of an ACL function as follows. The i'th partial derivative of a mapping $f : G \to \mathbb{R}^n$ is denoted by $\partial_i f$. If all partial derivatives of f exist at a point $x \in G$, the formal derivative of f at x is the linear mapping $f'(x) : \mathbb{R}^n \to \mathbb{R}^n$, defined by $f'(x)e_i = \partial_i f(x)$, $0 \leq i \leq n$.

2.12. **Definition.** *A homeomorphism* $f : D \to D'$ *is* K*-quasiconformal if and only if the following conditions are satisfied:*

(1) f *is* ACL.
(2) f *is differentiable a.e.*
(3) *For almost all* $x \in D$,

$$|f'(x)|^n/K \leq |J(x,f)| \leq K\ell(f'(x))^n,$$

where $\ell(A) = \min_{|h|=1} |Ah|$ *and* $J(x,f)$ *is the Jacobian of f. Here* $f'(x)$ *denotes the formal derivative of* f *at* x.

Definition 2.12 is one of the existing many equivalent definitions of K-quasiconformal mapping. These definitions are studied also in [C]. We follow here Väisälä's definitions [Va1]. In [Va1] also the inner and outer dilatations $K_O(f)$ and $K_I(f)$ are defined and we will occasionally use them here.

2.13. **Definition.** *Let* $G \subset \mathbb{R}^n$ *be a domain. A mapping* $f : G \to \mathbb{R}^n$ *is said to be* K*-quasiregular if* f *is* ACL^n *and if there exists a constant* $K \geq 1$ *such that*

$$|f'(x)|^n \leq KJ(x,f)\ , \quad |f'(x)| = \max_{|h|=1} |f'(x)h|\ , \quad \text{a.e. in } G. \tag{2.14}$$

The classes of quasiconformal and quasiregular mappings generalize the classes of conformal maps and analytic functions and they contain these two classes as particular cases, respectively. The topological properties of quasiregular mappings are similar to those of analytic functions: they map open sets onto open sets and the preimage of every point is a discrete set. Some important analytic properties of quasiregular maps are that they are differentiable a.e. and satisfy Lusin's condition (N), i.e. they map sets of measure zero to sets of measure zero. Moreover, these mappings are locally Hölder continuous. These fundamental properties were proved by Yu. G. Reshetnyak in a series of papers in 1966-70. See his book [R].

2.15. **Definition.** *Let* $A \subset \mathbb{R}^n$ *be open and let* $C \subset A$ *be compact. The pair* $E = (A, C)$ *is called a condenser. Its n-capacity is defined by*

$$\mathrm{cap} E = \inf_u \int_{\mathbb{R}^n} |\nabla u|^n \, dm\ , \tag{2.16}$$

where the infimum is taken over the family of all non–negative ACL^n *functions* u *with compact support in* A *such that* $u(x) \geq 1$ *for* $x \in C$.

The theory of quasiconformal and quasiregular mappings can be based on notions of the capacity of a condenser and the modulus of a curve family and their transformation rules under these mappings [Va1, p. 16], [Ri]. The capacity of a condenser (A, C) is, in fact, equal to the modulus of the family of all curves joining C with ∂A.

2.17. **Definition.** *The Grötzsch condenser $R_{G,n}(s)$ is defined as follows:*

$$R_{G,n}(s) = (\mathbb{R}^n \setminus \{te_1 : t \geq s\}, \overline{\mathbb{B}}^n), \; s \in (1, \infty).$$

We define the functions $\Phi = \Phi_n : (1, \infty) \to (0, \infty)$ and $\gamma_n : (1, \infty) \to (0, \infty)$ by

$$\mathrm{cap} R_{G,n}(s) = \omega_{n-1} \left(\log \Phi(s) \right)^{1-n} = \gamma_n(s)$$

where ω_{n-1} is the $(n-1)$-dimensional surface area of S^{n-1}.

We denote by λ_n the Grötzsch constant which is defined as

$$\log \lambda_n = \lim_{t \to \infty} \left(\log \Phi(t) - \log t \right) . \tag{2.18}$$

Only for $n = 2$ the exact value of the Grötzsch constant is known, $\lambda_2 = 4$ [AVV, p. 169]. Various estimates for λ_n, $n \geq 3$, are given in [G1, p. 518], [C, pp. 239–241], [AN], and [AVV, Chapter 12]. For instance it is known that $\lambda_n \in [4, 2e^{n-1})$.

The Grötzsch condenser and its capacity have a special role because they occur frequently in the study of distortion of quasiconformal mappings. In a typical case of application, geometric arguments lead to inequalities involving the special function γ_n and one needs to apply suitable information about special functions to reach the desired goal. For instance, upper or lower bounds for γ_n are often needed, see [Va1, Vu2]. It should be observed that for $n = 2$ the situation about these special functions is much simpler. In fact, $\gamma_2(s)$ can be expressed in terms of well-known special functions, whereas for $n \geq 3$ no such formulas are known.

2.19. **Elliptic integrals and $\gamma_2(s)$.** The plane Grötzsch ring can be mapped onto an annulus $\{x \in \mathbb{R}^2 : r < |x| < 1\}$, $r = \exp(-\mu(1/s))$, by an elliptic function and

$$\gamma_2(s) = \frac{2\pi}{\mu(1/s)}$$

for $s > 1$ where

$$\mu(r) = \frac{\pi}{2} \frac{\mathcal{K}(\sqrt{1 - r^2}\,)}{\mathcal{K}(r)} \, , \; \mathcal{K}(r) = \int_0^1 [(1 - x^2)(1 - r^2 x^2)]^{-1/2} dx$$

for $0 < r < 1$. The function $\mathcal{K}(r)$ is called a complete elliptic integral of the first kind.

2.20. **A special function.** For $0 < r < 1$ and $K > 0$ we define the special function $\varphi_K : [0, 1] \to [0, 1]$ as follows:

$$\varphi_K(r) = \frac{1}{\gamma_n^{-1}(K \gamma_n(1/r))} = \varphi_{K,n}(r) \tag{2.21}$$

and set $\varphi_K(0) = 0$, $\varphi_K(1) = 1$. It is easy to see that $\varphi_K : [0,1] \to [0,1]$ is a homeomorphism. For the proof of Theorem 2.22 we will need the inequality

$$r^\alpha \le \varphi_{K,n}(r) \le \lambda_n^{1-\alpha} r^\alpha,\ \alpha = K^{1/(1-n)}\,,$$

for $K \ge 1$, $r \in (0,1)$, see [Vu2, 7.47] and [AVV, 9.14]. For $n = 2$, $K > 0$, $r \in (0,1)$ we have

$$\varphi_{K,2}(r) = \mu^{-1}(\mu(r)/K)\,.$$

We now state the first result which is a quasiregular counterpart of the Schwarz lemma.

2.22. **Theorem.** [Vu2, Theorem 11.2], [I, Theorem 5.4] *Let G, D be either $\mathbb{B}^n$ or $\mathbb{H}^n$ and $f : G \to fG \subset D$ be a non-constant $K-$quasiregular mapping and let $\alpha = K_I(f)^{1/(1-n)}$. Then for all $x, y \in G$*

$$(1)\ \ \mathrm{th}\left(\tfrac{1}{2}\rho_D(f(x), f(y))\right) \le \varphi_{K,n}\left(\mathrm{th}\left(\tfrac{1}{2}\rho_G(x,y)\right)\right) \le \lambda_n^{1-\alpha}\left(\mathrm{th}\left(\tfrac{1}{2}\rho_G(x,y)\right)\right)^\alpha,$$

$$(2)\ \ \rho_D(f(x), f(y)) \le K_I(f)(\rho_G(x,y) + \log 4)\,.$$

Here $K_I(f)$ is the inner dilatation [Ri].

In the case when $n = 2$ this result can be further refined and simplified as the next theorem shows.

2.23. **Theorem.** [WV, Theorem 3.6] *If $f : \mathbb{B}^2 \to \mathbb{R}^2$ is a K-quasiconformal map with $f\mathbb{B}^2 \subset \mathbb{B}^2$ and ρ is the hyperbolic metric of $\mathbb{B}^2$, then*

$$\rho_{\mathbb{B}^2}(f(x), f(y)) \le c(K)\max\{\rho_{\mathbb{B}^2}(x,y),\ \rho_{\mathbb{B}^2}(x,y)^{1/K}\}$$

for all x, $y \in \mathbb{B}^2$, where $c(K) = 2\mathrm{arth}(\varphi_K(\mathrm{th}\frac{1}{2}))$ and, in particular, $c(1) = 1$.

Many authors have proved results similar to Theorems 2.22 and 2.23, but with less precise constants, see [Vu2, 11.50].

3. Lipschitz conditions, triangular ratio metric, and quasiconformal maps [I]

The triangular ratio metric s_G was introduced by P. Hästö in [H1], where he also considered metrics more general than s_G and proved the triangle inequality in [H1, Lemma 6.1]. In a short note A. Barrlund [BA] [1] had also studied a related notion. Very recently, the geometry of the balls of s_G for some special domains was studied in [HKLV, HKV]. Our goal here is to continue the study of this metric and to explore its behavior under Möbius transformations and quasiregular mappings. We also give upper and lower bounds for this metric in terms of some other metrics, and study the smoothness of s_G disks when G is a triangle or a rectangle in the plane.

The main results of the paper [I] are as follows:

[1] Anders Barrlund 1962-2000 was a Swedish mathematician and [BA] was his last paper.

3.1. **Theorem.** [I, Theorem 1.2]

(1) *Let $f:\mathbb{H}^n\to\mathbb{H}^n$ be a $K-$quasiregular mapping. Then for $x,y\in\mathbb{H}^n$ we have*
$$s_{\mathbb{H}^n}(f(x),f(y))\le\lambda_n^{1-\alpha}(s_{\mathbb{H}^n}(x,y))^\alpha\,.$$

(2) *Let $f:\mathbb{B}^n\to\mathbb{B}^n$ be a $K-$quasiregular mapping. Then for $x,y\in\mathbb{B}^n$ we have*
$$s_{\mathbb{B}^n}(f(x),f(y))\le 2^\alpha\lambda_n^{1-\alpha}(s_{\mathbb{B}^n}(x,y))^\alpha\,.$$

(3) *Let $f:\mathbb{B}^n\to\mathbb{H}^n$ be a $K-$quasiregular mapping. Then for $x,y\in\mathbb{B}^n$ we have*
$$s_{\mathbb{H}^n}(f(x),f(y))\le 2^\alpha\lambda_n^{1-\alpha}(s_{\mathbb{B}^n}(x,y))^\alpha\,.$$

(4) *Let $f:\mathbb{H}^n\to\mathbb{B}^n$ be a $K-$quasiregular mapping. Then for $x,y\in\mathbb{H}^n$ we have*
$$s_{\mathbb{B}^n}(f(x),f(y))\le\lambda_n^{1-\alpha}(s_{\mathbb{H}^n}(x,y))^\alpha.$$

Here $\lambda_n\in[4,2e^{n-1}]$ is the Grötzsch ring constant (2.18), *and $\alpha=K^{1/(1-n)}$.*

3.2. **Remark.** Note that for the dimension $n=2$, Theorem 3.1 can be refined if we apply Theorem 2.23. Of particular interest is the special case $K=1$. The question about the best constant in Theorem 3.1 (2) deserves some attention for the case when $K=1=\alpha$. The constant on the right hand side is then 2.

For a detailed study of this constant we define for $a\in(0,1)$ the class $C(a)$ of all Möbius transformations $h:\mathbb{B}^n\to\mathbb{B}^n$ with $|h(0)|=a$ and the constant

$$L(a)=\sup\{s_{\mathbb{B}^n}(h(x),h(y))/s_{\mathbb{B}^n}(x,y):\ x,y\in\mathbb{B}^n,x\ne y,h\in C(a)\}. \tag{3.3}$$

3.4. **Theorem.** [I, Theorem 1.5] *For $n=2$, $L(a)\ge 1+a$.*

Computer experiments support the conjecture that
$$L(a)=1+a\,.$$
We have been able to prove that $L(a)\le\frac{1+|a|}{1-|a|}$.

The following theorems give inequalities for the function p_G and the metrics j_G,ρ_G,s_G.

3.5. **Theorem.** [I, Theorem 1.7] *Let G be a proper subdomain of $\mathbb{R}^n$. Then for all $x,y\in G$ we have*
$$p_G(x,y)\le\frac{1}{\sqrt{2}}j_G(x,y),$$
and
$$s_G(x,y)\le\frac{1}{\log 3}j_G(x,y),$$
where the constant $\frac{1}{\log 3}\approx 0.91$ is best possible.

For a refined version of the second inequality of Theorem 3.5, see Theorem 5.3 below.

3.6. **Theorem.** [I, Theorem 3.23] *For $x, y \in \mathbb{B}^n$ we have*

$$\operatorname{th}\left(\frac{\rho_{\mathbb{B}^n}(x,y)}{2}\right) \le 2 s_{\mathbb{B}^n}(x,y). \tag{3.7}$$

There is room for improvement in inequality (3.7) as one can see by looking at Lemma 5.2. This observation implies, in particular, that Theorem 3.1 can also be refined because its proof is based on Theorem 3.6, see [III].

3.8. **Theorem.** [I, Theorem 1.8] *(1) Let $t \in (0,1)$ and $m \in \{j,p,s\}$. There exists a constant $c_m = c_m(t) > 1$ such that for all $x, y \in \mathbb{B}^n$ with $|x|, |y| < t$ we have*

$$m_{\mathbb{B}^n}(x,y) \le c_m m_{\mathbb{R}^n \setminus \{e_1\}}(x,y).$$

Moreover, $c_m(t) \to 1$ as $t \to 0$ and $c_m(t) \to \infty$ as $t \to 1$, for all $m \in \{j,p,s\}$.

(2) Let $G \subset \mathbb{R}^n$, $x \in G$, $t \in (0,1)$ and $m \in \{j,p,s\}$. Then there exists a constant $c_m = c_m(t)$ such that for all $y, z \in G \setminus B^n(x, t d_G(x))$ we have

$$m_{G \setminus \{x\}}(y,z) \le c_m m_G(y,z).$$

Moreover, the constant is best possible as $t \to 1$. This means that $c_j, c_p, c_s \to 2$ as $t \to 1$.

In the next theorem we study the smoothness of s_G-metric disks in the case when G is either a triangle or rectangle in the plane.

Let us denote by $T_{\frac{\pi}{6},2}$ the equilateral triangle with vertices $(0,0)$, $(\sqrt{3},1)$, $(\sqrt{3},-1)$, and by $R_{a,b}$ the rectangle with vertices (a,b), $(a,-b)$, $(-a,b)$, $(-a,-b)$, where $a \ge b > 0$.

3.9. **Theorem.** [I, Theorem 1.9] *(1) Let $G = T_{\frac{\pi}{6},2}$, $x = (x_1, x_2) \in G$, $r > 0$. Then the metric ball $B_{s_G}(x,r)$ is smooth if and only if $r \le r_0$ or $r \le r_1$, where*

$$r_0 = \min\left\{\frac{2|x_2|}{|x|}, \frac{|x_2| - \sqrt{3}x_1 + 2}{\sqrt{(x_1-\sqrt{3})^2 + (1-|x_2|)^2}}\right\}, \text{ and } r_1 = \frac{\sqrt{3}x_1 - 2 - |x_2|}{\sqrt{(x_1-\sqrt{3})^2 + (1-|x_2|)^2}}.$$

(2) Let $G = R_{a,b}$, $x = (x_1, x_2) \in G$, $r > 0$. Then the metric ball $B_{s_G}(x,r)$ is smooth if and only if $r \le r_2$ or $r \le r_3$, where

$$r_2 = \min\left\{\frac{|x_2|}{b}, \frac{(a-|x_1|)-(b-|x_2|)}{\sqrt{(a-|x_1|)^2 + (b-|x_2|)^2}}\right\}, \text{ and } r_3 = \min\left\{\frac{|x_1|}{a}, \frac{(b-|x_2|)-(a-|x_1|)}{\sqrt{(a-|x_1|)^2 + (b-|x_2|)^2}}\right\}.$$

Figures 3 and 4 seem to suggest that in polygonal plane domains G, the s_G-disks have a shape which looks like the Euclidean disk for small values of the radius and which reflects the shape of the domain G for large values of the radius.

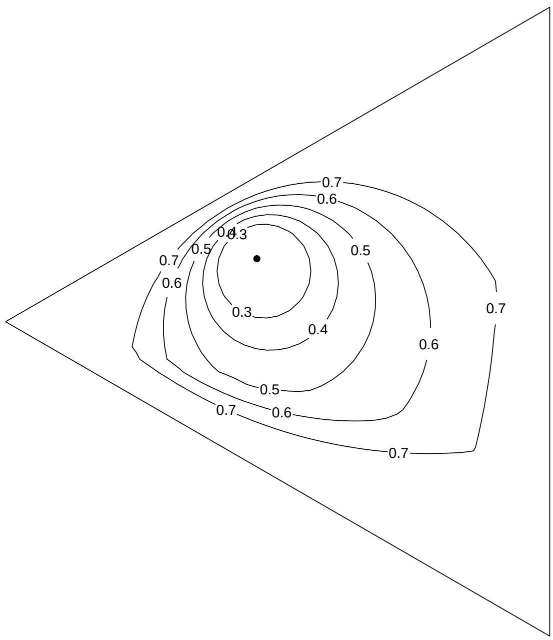

FIGURE 3. Some triangular ratio metric balls $B_{s_G}(x,r)$ in the triangle $T_{\frac{\pi}{6},2}$ [I, Figure 3].

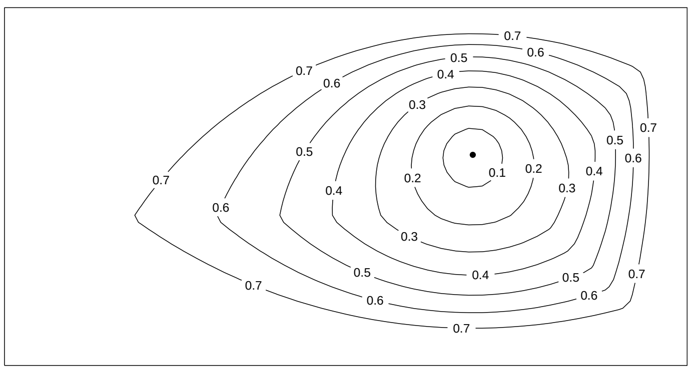

FIGURE 4. Some triangular ratio metric balls $B_{s_G}(x,r)$ in the rectangle $R_{a,b}$ [I, Figure 4].

4. Some remarks on the visual angle metric [II]

The visual angle metric defined in 2.9 was introduced and studied very recently in [KLVW]. It is clear that a point $z \in \partial G$ exists for which the supremum in 2.9 is attained. Such a point z is called an extremal point for $v_G(x,y)$.

The visual angle metric and the triangular ratio metric defined in 2.8, are closely related, for instance, both depend on extremal boundary points, and also both metrics are monotone with respect to domain. On the other hand, we will see that these two metrics are not comparable in some domains, see Remark 4.3. In the case of domains satisfying suitable hypotheses, these metrics can be compared, as we will see below, in Theorems 5.5 and 5.7.

In the paper [II] we begin by comparing the visual angle metric and the triangular ratio metric in convex domains. Our main results are about uniform continuity of quasiconformal maps with respect to the triangular ratio metric and the visual angle metric.

We first recall the following relation between the visual angle metric and the hyperbolic metric: for all $x\,,y\in\mathbb{B}^n$,

$$\arctan\left(\mathrm{sh}\frac{\rho_{\mathbb{B}^n}(x,y)}{2}\right)\le v_{\mathbb{B}^n}(x,y)\le 2\arctan\left(\mathrm{sh}\frac{\rho_{\mathbb{B}^n}(x,y)}{2}\right)\,, \tag{4.1}$$

see [KLVW, Theorem 3.11].

4.2. **Theorem.** [II, Theorem 2.17] *Let $D\subsetneq\mathbb{R}^n$ be a convex domain. Then for all $x,y\in D$ we have*

$$s_D(x,y)\le v_D(x,y)\le \pi s_D(x,y).$$

4.3. **Remark.** [II, Remark 2.18] The visual angle metric and the triangular ratio metric both highly depend on the boundary of the domain. If we replace the convex domain D in Theorem 4.2 with $G=\mathbb{B}^2\setminus\{0\}$, then the visual angle metric and the triangular ratio metric are not comparable in G. To this end, we consider two sequences of points $x_k=(1/k,0)$, $y_k=(1/k^2,0)\,(k=2,3,\dots)$. Then

$$s_G(x_k,y_k)=\frac{k-1}{k+1}\,.$$

By [II, (2.8)], we get

$$v_G(x_k,y_k)=v_{\mathbb{B}^2}(x_k,y_k)=\arctan\left(\frac{k}{(k+1)\sqrt{1+k^2}}\right)<\arctan\left(\frac{1}{k+1}\right).$$

Therefore,

$$\frac{s_G(x_k,y_k)}{v_G(x_k,y_k)}\ge\frac{k-1}{(k+1)\arctan\left(\frac{1}{k+1}\right)}\to\infty\,,\ \text{as } k\to\infty\,.$$

Theorems 2.22 and 3.1 show that quasiregular mappings are Hölder-continuous with respect to the hyperbolic metric $\rho_{\mathbb{B}^n}$ and the triangular ratio metric $s_{\mathbb{B}^n}$, respectively. The next two theorems give similar results for the case when the mapping is defined in a general domain. The Hölder exponent will be the same as earlier.

4.4. **Theorem.** [II, Theorem 4.6] *Let D,D' be two proper subdomains of $\mathbb{R}^n$ such that ∂D is connected. Let $f:D\to D'=fD$ be a K-quasiconformal mapping. Then for all $x,y\in D$,*

$$s_{D'}(f(x),f(y))\le C_1 s_D(x,y)^{\alpha},\ \alpha=K^{1/(1-n)}\,,$$

where

$$C_1 = \max\left\{2\lambda_n^{2-\alpha}(2+t_0)^\alpha, \left(\frac{2+t_0}{t_0}\right)^\alpha\right\},$$

and $t_0 = (\lambda_n^{\alpha-2}/2)^{1/\alpha}$, *where* $\lambda_n \in [4, 2e^{n-1}]$ *is the Grötzsch ring constant* (2.18).

4.5. **Theorem.** [II, Theorem 4.9] *Let* D, D' *be two proper subdomains of* $\mathbb{R}^n$ *such that* D *is convex. Let* $f : D \to D' = fD$ *be a* K*-quasiconformal mapping. Then for all* $x, y \in D$,

$$v_{D'}(f(x), f(y)) \le C_2 v_D(x,y)^\alpha,\ \alpha = K^{1/(1-n)},$$

where

$$C_2 = \max\left\{2^{3+2\alpha}\lambda_n^{2-\alpha}, \pi\left(\frac{4}{t_1}\right)^\alpha\right\},$$

and $t_1 = (\lambda_n^{\alpha-2}/4)^{1/\alpha}$.

5. Inequalities and bilipschitz conditions for triangular ratio metric [III]

It is easy to see that there exist domains G with isolated boundary points such that the metrics s_G and v_G are not comparable (see also 4.3). Here we introduce two conditions on domains G for which s_G and v_G are comparable. The first condition applies to domains G which satisfy that ∂G is "locally uniformly nonlinear", whereas the second condition applies to domains satisfying the "exterior ball condition". We also show, motivated in part by J. Väisälä's work [Va3], that bilipschitz maps with respect to the triangular ratio metric are quasiconformal.

First we present refined versions of some inequalities in papers [I, II].

5.1. **Lemma.** [III, Lemma 2.1] *Let* G *be a proper subdomain of* $\mathbb{R}^n$. *If* $x, y \in G$, *then*

$$j_G^*(x,y) = \operatorname{th}\frac{j_G(x,y)}{2} = \frac{|x-y|}{|x-y| + 2\min\{d(x), d(y)\}}$$

and

$$j_G^*(x,y) \le s_G(x,y) \le \frac{e^{j_G(x,y)} - 1}{2}.$$

The first inequality is sharp for $G = \mathbb{R}^n \setminus \{0\}$.

The following lemma refines [I, Theorem 3.23].

5.2. **Lemma.** [III, Lemma 2.6] *For* $x, y \in \mathbb{B}^n$ *we have*

$$\operatorname{th}\frac{\rho_{\mathbb{B}^n}(x,y)}{4} \le s_{\mathbb{B}^n}(x,y) \le p_{\mathbb{B}^n}(x,y) \le \operatorname{th}\frac{\rho_{\mathbb{B}^n}(x,y)}{2} \le 2\operatorname{th}\frac{\rho_{\mathbb{B}^n}(x,y)}{4}.$$

5.3. **Theorem.** [III, Theorem 2.9] *For a convex domain* $G \subsetneq \mathbb{R}^n$ *and all* $x, y \in G$ *we have*

(1) $s_G(x,y) \le \sqrt{2}\, j_G^*(x,y)$,
(2) $v_G(x,y) \ge (1/\sqrt{2})\, p_G(x,y)$.

We will next introduce two sufficient conditions under which v_G has a lower bound in terms of s_G.

5.4. **Definition.** [III, Definition 3.2] *Suppose that $G \subset \mathbb{R}^2$ is a domain. We say that ∂G satisfies the nonlinearity condition if there exists $\delta \in (0,1)$, such that for every $z \in \partial G$ and for every $r \in (0, d(G))$ and for every line L with $L \cap B(z,r) \neq \emptyset$, there exists a point*

$$w \in B(z,r) \cap \partial G \setminus \bigcup_{y \in L} B(y, \delta r)\,.$$

One example for such domain is the snowflake domain see [AVV, xxii].

5.5. **Theorem.** [III, Theorem 3.3] *Let $G \subset \mathbb{R}^2$ be a domain such that ∂G satisfies the nonlinearity condition. If $x, y \in G$ and $s_G(x,y) < 1$ then*

$$v_G(x,y) > \arctan\left(\frac{\delta}{6} s_G(x,y)\right)\,.$$

5.6. **Definition.** [III, Definition 3.7] *Let $\delta \in (0, 1/2)$. We say that a domain $G \subset \mathbb{R}^n$ satisfies the condition $H(\delta)$ if for every $z \in \partial G$ and all $r \in (0, d(G)/2)$ there exists $w \in \mathbb{B}^n(z,r) \cap (\mathbb{R}^n \setminus G)$ such that $\mathbb{B}^n(w, \delta r) \subset \mathbb{B}^n(z,r) \cap (\mathbb{R}^n \setminus G)$.*

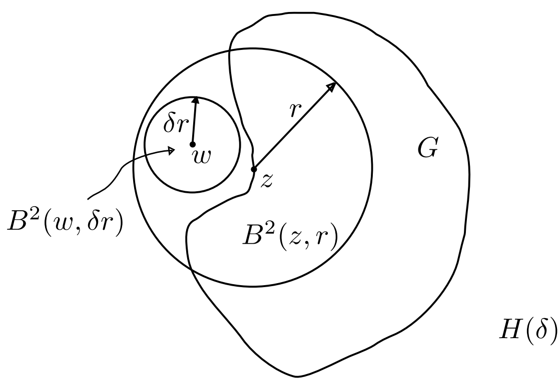

FIGURE 5. Condition $H(\delta)$ [III, Figure 3].

The condition $H(\delta)$ is similar to the so-called porosity condition. For porosity condition see [MVu, VVW, KLV].

5.7. **Theorem.** [III, Theorem 3.8] *Let $G \subset \mathbb{R}^2$ be a domain satisfying the condition $H(\delta)$. Then for all $x, y \in G$ we have*

$$\sin v_G(x,y) \geq \frac{\delta}{2} j^*_G(x,y)\,.$$

P. Hästö [H3] has proved that the Apollonian metric α_G of a domain G has a lower bound in terms of j_G under the hypothesis that ∂G satisfies the thickness condition of [VVW]. Notice the analogy between Hästö's result and Theorem 5.7.

The next theorem shows that L-bilipschitz homeomorphisms $f : (G, s_G) \to (fG, s_{fG})$ are quasiconformal with linear dilatation bounded by L^2.

5.8. **Theorem.** [III, Theorem 4.4] *Let $G \subsetneq \mathbb{R}^n$ be a domain and let $f : G \to fG \subset \mathbb{R}^n$ be a sense-preserving homeomorphism, satisfying the L-bilipschitz condition with respect to the triangular ratio metric, i.e.*

$$s_G(x,y)/L \le s_{fG}(f(x), f(y)) \le L s_G(x,y),$$

holds for all $x, y \in G$. Then f is quasiconformal with the linear dilatation $H(f) \le L^2$.

6. Comparison theorems for hyperbolic type metrics [IV]

In paper [IV], a new metric has been defined, see Theorem 6.1. This metric is comparable to the metrics discussed in earlier sections. In the one-dimensional case of $\mathbb{R}$ this metric was studied by P. Hästö [H1].

6.1. **Theorem.** [IV, Theorem 1.1] *Let D be an open set in a metric space (X, ρ) and let $\partial D \ne \varnothing$. Then the function*

$$h_{D,c}(x,y) = \log\left(1 + c\frac{\rho(x,y)}{\sqrt{d_D(x) d_D(y)}}\right),$$

where $d_D(x) = d(x, \partial D)$, is a metric for every $c \ge 2$. The constant 2 is best possible here.

It is easy to see that the h-metric has the properties listed in Remark 2.10.

This metric has its roots in the study of distance ratio metric j_G, which is recurrent in the study of quasiconformal maps. See also [Vu2, 2.43] for a related quantity.

In the next results we compare $h_{G,c}$ with j_G and ρ_G.

6.2. **Lemma.** [IV, Lemma 4.4] *Let $G \subsetneq \mathbb{R}^n$ be a domain. Then for $c > 0$ and all $x, y \in G$, we have*

$$\frac{c}{2(1+c)} j_G(x,y) \le \log\left(1 + 2c\,\mathrm{sh}\frac{j_G(x,y)}{2}\right) \le h_{G,c}(x,y) \le c j_G(x,y).$$

6.3. **Theorem.** [IV, Theorem 4.6] *Let $G \in \{\mathbb{B}^n, \mathbb{H}^n\}$, and let ρ_G stand for the respective hyperbolic metric. If $c \ge 2$, then for all $x, y \in G$*

$$\frac{1}{c} h_{G,c}(x,y) \le \rho_G(x,y) \le 2 h_{G,c}(x,y).$$

A function related to $h_{G,c}$ was recently introduced by Z. Ibragimov [IB2] and studied by M. Mohapatra in his PhD thesis [MO]. It is defined as follows,

$$\tilde{\tau}_G(x,y) = \log\left(1 + \sup_{p\in\partial G} \frac{|x-y|}{\sqrt{|x-p||y-p|}}\right).$$

It was proved by Z. Ibragimov [IB2, Theorem 3.3] that for every proper domain $G \subset \mathbb{R}^n$ and for all $x, y \in G$

$$\frac{1}{2} j_G(x,y) \le \tilde{\tau}_G(x,y) \le j_G(x,y)\,. \tag{6.4}$$

Comparing Lemma 6.2 and (6.4) we see that $\tilde{\tau}_G(x,y)$ and $h_{G,c}(x,y)$ are closely related.

The next lemma shows that under Möbius transformations $h_{G,c}$ is increased at most by a factor $m \le 2$.

6.5. **Lemma.** [IV, Lemma 2.6] *Let $g : \mathbb{B}^n \to G$, $G \in \{\mathbb{B}^n, \mathbb{H}^n\}$ be a Möbius transformation with $g(\mathbb{B}^n) = G$. Then for $c > 0$ the inequality*

$$h_{G,c}(g(x), g(y)) \le 2h_{\mathbb{B}^n,c}(x,y)$$

holds for all $x, y \in \mathbb{B}^n$.

6.6. **Definition.** *A domain $G \subsetneq \mathbb{R}^n$ is said to be uniform, if there exists a constant $U \ge 1$ such that for all $x, y \in G$*

$$k_G(x,y) \le U j_G(x,y).$$

In [GO, Theorem 2] a similar characterization of uniform domains was given but with the expression $a j_G(x,y) + b$ on the right hand side. It was pointed out in [Vu1, 2.50 (2)] that the pair of constants (a,b) can be replaced by $(c,0)$ where c is some constant. In other words the above definition of uniform domain is quantitatively equivalent to the definition in [GO]. The class of uniform domains is very wide: for instance quasidisks in $\mathbb{R}^2$ are such domains [GH].

The motivation for the study of the $h_{G,c}$- metric derives from applications to quasiconformal maps. Our main applications are about the quasiconformality of $h_{G,c}$-bilipschitz homeomorphisms. Some other applications are given in [NA, NT]. The next two theorems are similar to earlier results in Sections 3-5, but now expressed in terms of the $h_{G,c}$-metric.

6.7. **Theorem.** [IV, Theorem 4.9] *Let $f : G \to fG$ be a K-quasiconformal map, let $G, fG \subsetneq \mathbb{R}^n$ and let G be a U-uniform domain. Then for all $c > 0$ there exists $e \in (0,1)$ such that*

$$h_{fG,c}(f(x), f(y)) \le \frac{1}{e}\max\{h_{G,c}(x,y), h_{G,c}(x,y)^\alpha\}$$

where $\alpha = K_I(f)^{1/(1-n)}$. Here e depends on K, c, and U.

6.8. **Theorem.** [IV, Theorem 4.10] *Let $c > 0$ and $f : G \to fG$ be a homeomorphism, let $G, fG \subsetneq \mathbb{R}^n$ and suppose that there exists $L \geq 1$ such that for all $x, y \in G$*

$$h_{G,c}(x,y)/L \leq h_{fG,c}(f(x), f(y)) \leq L h_{G,c}(x,y).$$

Then f is quasiconformal with linear dilatation $H(f) \leq L^2$.

7. Concluding remarks

The quasihyperbolic metric introduced by F.W. Gehring in joint papers with his students B. Palka [GP] and B. Osgood [GO] has become a basic tool in Geometric Function Theory in the plane, Euclidean spaces of dimension $n \geq 3$, and also in metric spaces. Väisälä's definition of quasiconformal mappings in Banach spaces [Va3] is based on the quasihyperbolic metric. The quasihyperbolic metric can be defined in every proper subdomain G of the Euclidean space and it enjoys many properties of the hyperbolic metric. Parallel to this development several other metrics have been studied: Ferrand's metric [F1, HJ], the Apollonian metric introduced by D. Barbilian [B] and rediscovered by A. F. Beardon [B2], the Möbius invariant metric of P. Seittenranta [SE2], and the conformally invariant metrics μ_G and λ_G (see [LF] and [Vu2, Chapter 8]). These metrics have been studied in several PhD theses, P. Hästö [H2], V. Heikkala [He], Z. Ibragimov [IB1], V. Manojlović [Ma], H. Lindén [L], R. Klén [K1], X. Zhang [Z], G. Wang [W], S. K. Sahoo [Sa] and M. Mohapatra [MO]. The study of metrics does indeed provide a wide territory of research, which combines function theory and geometry. For an overview of some open problems, see [Vu3, Chapter 8] and [Vu4].

Finally, let us formulate some open problems or areas of further work, motivated by this thesis and [Vu3, Vu4].

1. Let $f : G \to fG$ be a quasiregular mapping, where $G, fG \subsetneq \mathbb{R}^n$ and let m_G be a metric on the domain G. Is $f : (G, m_G) \to (fG, m_{fG})$ uniformly continuous? We know that by the Schwarz lemma 2.22 this is the case if $G = \mathbb{B}^n$, $fG \subset \mathbb{B}^n$ and $m_G = \rho_{\mathbb{B}^n}$. Under which conditions on m_G this uniform continuity holds? Note that this question can be studied for several metrics.

2. For $x, y \in G \subsetneq \mathbb{R}^n$, can we find upper or lower bounds for triangular ratio metric s_G and visual angle metric v_G, in terms of conformally invariant metrics μ_G, and λ_G? In which domains these metrics are comparable?

3. For a metric m_G of G and $x \in G$ denote by $B_{m_G}(x, T) = \{z \in G : m_G(x, z) \leq T\}$, the metric ball with center x and radius T. We expect that balls with small radii are more or less like Euclidean balls whereas balls with large radii are more or less like the domain G. The problem is to study this transition from small to large radii. For instance are the balls of small radii smooth? (Or at least bilipschitz equivalent to Euclidean ball. See for instance Figures 3, 4 and [KRT].) Note that it is easy to give examples of situations when two balls of equal radii $B_{m_G}(x_1, T)$ and $B_{m_G}(x_2, T)$ are not homeomorphic, when $x_1 \neq x_2$.

4. As we saw in Section 6, the class of uniform domains is defined in terms of a comparison inequality of two metrics, the quasihyperbolic metric and the distance ratio metric. Suppose now that we have a collection $\mathcal{M}$ of metrics, each metric defined for a proper subdomain G of $\mathbb{R}^n$. Is it possible to define "generalized uniform domains" by a comparison property of two metrics of the class $\mathcal{M}$? In other words, fix a proper subdomain G and two metrics $m_1, m_2 \in \mathcal{M}$ and require that these metrics are comparable. What can we say about the domain? For instance we can consider the case, when

$$\mathcal{M} = \{s_G, j_G, v_G\}\,.$$

After this PhD thesis my work on this topic has continued in [FHMV, HKV, HKVZ]. Several of the metrics studied here and most recently the metric $h_{D,c}$ are included in [DD]. The metric $h_{D,c}$ has also found recent applications [NA, NT, MO].

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