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COVARIANT PHASE OBSERVABLES IN QUANTUM MECHANICS

by

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Contents

Abstract

The aim of this thesis is to present a solution to the quantum phase problem of the single-mode optical field. The solution is based on the use of phase shift covariant normalized positive operator measures. These measures describe realistic direct coherent state phase measurements such as the phase measurement schemes based on eight-port homodyne detection or heterodyne detection.

The structure of covariant operator measures and, more generally, covariant sesquilinear form measures is analyzed in this work. Four different characterizations for phase shift covariant normalized positive operator measures are presented. The canonical covariant operator measure is definded and its properties are studied. Finally, some other suggested phase theories are introduced to investigate their connections to the covariant sesquilinear form measures.

List of papers

This thesis consists of the introductory review part and the following research papers:

I P. J. Lahti and J.-P. Pellonpää,
"Covariant phase observables in quantum mechanics",
J. Math. Phys. **40**, 4688-4698 (1999).

II P. J. Lahti and J.-P. Pellonpää,
"Characterizations of the canonical phase observable",
J. Math. Phys. **41**, 7352-7381 (2000).

III P. Busch, P. J. Lahti, J.-P. Pellonpää, and K. Ylinen,
"Are number and phase complementary observables?"
J. Phys. A: Math. Gen. **34**, 5923-5935 (2001).

IV J.-P. Pellonpää,
"Phase observables, phase operators and operator orderings",
J. Phys. A: Math. Gen. **34**, 7901-7916 (2001).

V G. Cassinelli, E. De Vito, P. Lahti, and J.-P. Pellonpää,
"Covariant localizations in the torus and the phase observables",
J. Math. Phys. **43**, 693-704 (2002).

VI P. J. Lahti and J.-P. Pellonpää,
"The Pegg-Barnett formalism and covariant phase observables",
Phys. Scr. **66**, 66-70 (2002).

VII J.-P. Pellonpää,
"On the structure of covariant phase observables",
J. Math. Phys. **43**, 1299-1308 (2002).

VIII J.-P. Pellonpää,
"The phase representation of covariant phase observables",
`quant-ph/0109026`, submitted to Phys. Scr.

Summary of the research papers

Paper I: Covariant phase observables in quantum mechanics
In this paper we characterize all the phase shift covariant normalised positive operator measures, i.e. phase observables, and we investigate some of their examples. We also characterise those phase observables which arise from the phase space observables as their polar coordinate angle margins.

Paper II: Characterizations of the canonical phase observable
In this paper we investigate various properties of phase observables which could serve to determine the canonical phase observable among the family of all phase observables. We also show that any contractive weighted shift operator defines a unique phase observable, and we characterize phase observables which give the most accurate phase distribution in coherent states in the classical limit.

Paper III: Are number and phase complementary observables
We study various ways of characterizing the quantum optical number and phase as complementary observables.

Paper IV: Phase observables, phase operators and operator orderings
We present quantum phase observables as phase shift covariant normalized positive operator measures. The phase operators are the first moment operators of the phase observables. A phase operator determines the associated phase observable uniquely. We show that the Cahill-Glauber s-ordered phase operators are determined by phase shift covariant generalized operator measures which are ordinary operator measures whenever $\mathrm{Re}\, s < 0$, and phase observables when $s \leq -1$. The Wigner-Weyl quantized phase operator is not determined by any phase observable. We investigate the classical limit of covariant (generalized) operator measures in coherent states.

Paper V: Covariant localizations in the torus and the phase observables
We describe all the localization observables of a quantum particle in a one-dimensional box in terms of sequences of unit vectors in a Hilbert space. An alternative representation in terms of positive semidefinite complex matrices is furnished and the commutative localizations are singled out. As a consequence, we also get a vector sequence characterization of the covariant phase observables.

Paper VI: The Pegg-Barnett formalism and covariant phase observables

We compare the Pegg-Barnett (PB) formalism with the covariant phase observable approach to the problem of quantum phase and show that PB-formalism gives essentially the same results as the canonical (covariant) phase observable. We also show that PB-formalism can be extended to cover all covariant phase observables including the covariant phase observable arising from the angle margin of the Husimi Q-function.

Paper VII: On the structure of covariant phase observables

We study the mathematical structure of covariant phase observables. Such an observable can alternatively be expressed as a phase matrix, as a sequence of unit vectors, as a sequence of phase states, or as an equivalent class of covariant trace-preserving operations. Covariant generalized operator measures are defined by structure matrices which form a W*-algebra with phase matrices as its subset. The properties of the Radon-Nikodým derivatives of phase probability measures are studied.

Paper VIII: The phase representation of covariant phase observables

Covariant phase observables are obtained by defining simple conditions for mappings from the set of phase wave functions (unit vectors of the Hardy space) to the set of phase probability densities. The existence of phase probability density for any phase wave function, the existence of interference effects, and the natural phase shift covariance are those simple conditions. The nonlocalizability of covariant phase observables is proved.

Chapter 1

Introduction

In classical physics the phase of an electromagnetic field is well defined both theoretically and by interference experiments. Diffraction of light, holography, and many other phase dependent phenomena are well understood. It is also easy to describe classically eight-port homodyne detection and other direct measurements of the phase difference of signal and local strong laser beams with fixed phases. A problem arises when, for instance, the signal field in homodyne detection is so weak that one must take into account quantum effects.

The quantum theory of eight-port homodyne detection (see e.g. [69]) shows that when a local laser is strong and its phase is fixed one can measure the so-called Husimi Q-function of a signal state. The Q-function determines a (normalized) positive operator measure defined for the Borel subsets of the interval $[0, 2\pi)$, which is covariant under the shifts generated by the number operator. Using this covariance condition together with the choice of the range for the phase variable one can define a whole family of covariant (normalized) positive operator measures. We call them covariant phase observables. None of these observables is projection valued showing, in particular, that there is no phase shift covariant self-adjoint phase operator. The quantum phase problem is thus a true example of a case where the conventional formulation of quantum mechanics, where observables are self-adjoint operators, cannot be sufficient.

Since there is no phase shift covariant self-adjoint operator the quantum phase problem has remained a thorny question. Alternative descriptions for phase observables have been developed, and over 500 scientific articles and two monographs [97, 33] have been written on the subject since Dirac's famous paper [28] published in the year 1927, see for example the reference lists of [94, 97, 81]. It is beyond the scope of this overview to introduce all different phase theories and cite all the papers written on the field. For instance, the quantum estimation theory of phase shifts originally developed by Helstrom [50] and Holevo [58] is only briefly touched in this work.

The aim of this thesis is to present a simple solution to the quantum phase problem given by the covariant phase observables, to study their properties, and to show how other phase theories are connected with this approach. The order of presentation is logical rather than historical. A reader who is interested in the history of the problem may wish to consult, for example, Lynch's review article [81], Pegg's and Barnett's tutorial review [94], or the special issue T48 of Physica Scripta [102].

The structure of the thesis is as follows: Chapter 2 presents the classical and quantum theories of eight-port homodyne detection to justify the phase shift covariance condition which is used to define phase shift covariant phase observables. Their mathematical structure is studied in Chapter 3, where a more general notion of covariant sesquilinear form valued measures is also investigated. The main results of that chapter are the different characterizations of the phase observables. In Chapter 4 various properties of phase observables and their inter-relations with the number operator are discussed. For instance, complementarity and commutativity of number and phase are studied and the canonical phase observable is characterized in various ways. Some competing phase theories are reviewed in Chapter 5, where their connections to the covariant approach are also investigated. The summary of the thesis collects the main results and open questions.

Chapter 2

Covariant phase observables

In this chapter we introduce the classical and quantum theories of eight-port homodyne detection. We use the coarsest nontrivial approximation for a propagating electromagnetic field, namely, a monochromatic (single-mode) plane wave. This approximation is sufficient for the theory of eight-port homodyne detection in our context and it allows us to present all essential features of ideal laser light. Using the quantum theory of homodyne detection one can justify the phase shift covariance condition and define covariant phase observables. We also recall here Dirac's approach to the problem and we present the related well-known no-go theorem.

2.1 Classical phase

Let us consider a classical electromagnetic field in a cubic cavity. The Maxwell equations with the Coulomb gauge condition lead to the wave equation

$$\nabla^2 \mathbf{A} = \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} \tag{2.1}$$

for the vector potential $\mathbf{A}$ which depends on position $\mathbf{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ and time t, c being the speed of light. The electric and magnetic field vectors are $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}$ and $\mathbf{H} = \mu_0^{-1}\nabla \times \mathbf{A}$ where μ_0 is the magnetic permeability of free space. The monochromatic (single-mode) plane wave is a solution of (2.1):

$$\mathbf{A}(\mathbf{r}, t) = \sqrt{\frac{2\mathcal{E}}{\epsilon_0 \mathcal{V} \omega^2}}\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi)\ \mathbf{u}. \tag{2.2}$$

Here $\mathbf{k}$ is the wavevector, $\omega = c|\mathbf{k}|$, $\mathbf{u}$ is the unit polarization vector ($\mathbf{u} \perp \mathbf{k}$), $\mathcal{E}$ is the cycle-averaged energy content of the plane wave in the volume $\mathcal{V}$, ϵ_0 is the free space permittivity, and ϕ is an *(absolute) phase* of the plane wave.

Assuming the periodic boundary conditions all other solutions of (2.1) can be expressed as linear combinations of these single mode plane waves.

Suppose that the plane wave is moving in the positive direction of the z-axis so that $\mathbf{k} = (\omega/c)\,\hat{k}$ and $\mathbf{u} = \hat{\imath}$. The energy density is then

$$W(\mathbf{r},t) := \frac{\epsilon_0}{2}|\mathbf{E}(\mathbf{r},t)|^2 + \frac{\mu_0}{2}|\mathbf{H}(\mathbf{r},t)|^2 = \frac{2\mathcal{E}}{\mathcal{V}}\sin^2\left((\omega/c)z - \omega t + \phi\right),$$

the Poynting vector

$$\mathbf{E}(\mathbf{r},t) \times \mathbf{H}(\mathbf{r},t) = c\,W(\mathbf{r},t)\,\hat{k},$$

and the cycle-averaged energy content is

$$\int_{\mathcal{V}}\left[\frac{\omega}{2\pi}\int_0^{2\pi/\omega} W(\mathbf{r},t)\mathrm{d}t\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z = \mathcal{E}.$$

Suppose that there is a photodetector at the origin and the active region of the detector is perpendicular to the z-axis. The intensity of light streaming into the detector at time t is

$$c\,W(0,t) = \frac{2c\mathcal{E}}{\mathcal{V}}\sin^2\left(-\omega t + \phi\right).$$

In all experiments measuring light intensity the time of measurement is long compared to the period of a single oscillation. This means that one does not measure $c\,W(0,t)$ but rather its temporal average over a time $2\pi/\omega$. Thus, the optimally measured intensity is

$$\mathcal{I} := \frac{\omega}{2\pi}\int_0^{2\pi/\omega} c\,W(0,t)\mathrm{d}t = \frac{c\mathcal{E}}{\mathcal{V}}.$$

If the efficiency coefficient of the detector is $\alpha \in [0,1]$, then $\alpha\mathcal{I}$ is the actually measured intensity. The measured energy during a time interval $[0,T]$ is

$$\alpha\mathcal{A}\int_0^T c\,W(0,t)\mathrm{d}t = \alpha\mathcal{A}\mathcal{I}\left[T + \frac{\sin(2\phi - 2\omega T) - \sin(2\phi)}{2\omega}\right] \sim \alpha\mathcal{A}T\mathcal{I}$$

when $T \gg 2\pi/\omega$ and $\mathcal{A}$ is the area of the active region of the detector. This justifies the above approximation for intensity when a measurement period is long compared to $2\pi/\omega$. Using $\mathcal{I}$ one can write

$$\mathbf{A}(\mathbf{r},t) = \mathrm{Re}\left(\sqrt{\mathcal{I}}\,e^{i\phi}f_\omega(z,t)\right)\hat{\imath}$$

where

$$f_\omega(z,t) := \sqrt{\frac{2}{c\epsilon_0\omega^2}}e^{i(\omega/c)z - i\omega t}.$$

In photodetection one measures the light intensity $\alpha\mathcal{I}$ which does not depend on the absolute phase ϕ. Therefore, using a single photodetector one cannot measure the absolute phase of an incoming plane wave.

2.2 Classical homodyne detection

In a homodyne detection scheme two light beams (here, plane waves) which differ only by their intensities $\mathcal{I}_1$, $\mathcal{I}_2$ and absolute phases ϕ_1, ϕ_2 are mixed by a 50:50 beam splitter. Suppose that the first resultant wave is propagating along the z-axis and thus is of the form $\mathrm{Re}\left[\left(2^{-1/2}\sqrt{\mathcal{I}_1}\,e^{i\phi_1}+2^{-1/2}i\sqrt{\mathcal{I}_2}\,e^{i\phi_2}\right)f_\omega(z,t)\right]\hat{\imath}$. This wave is then detected using a photodetector with the efficiency α. The measured intensity is $\alpha\mathcal{I}_3$ where

$$\mathcal{I}_3 := \left|2^{-1/2}\sqrt{\mathcal{I}_1}\,e^{i\phi_1}+2^{-1/2}i\sqrt{\mathcal{I}_2}\,e^{i\phi_2}\right|^2=\frac{\mathcal{I}_1}{2}+\frac{\mathcal{I}_2}{2}+\sqrt{\mathcal{I}_1\mathcal{I}_2}\sin(\phi_1-\phi_2).$$

Similarly, the second resultant wave is detected with a photodetector with the same efficiency α, and the measured intensity is $\alpha\mathcal{I}_4$ where

$$\mathcal{I}_4 := \left|2^{-1/2}i\sqrt{\mathcal{I}_1}\,e^{i\phi_1}+2^{-1/2}\sqrt{\mathcal{I}_2}\,e^{i\phi_2}\right|^2=\frac{\mathcal{I}_1}{2}+\frac{\mathcal{I}_2}{2}-\sqrt{\mathcal{I}_1\mathcal{I}_2}\sin(\phi_1-\phi_2).$$

In balanced homodyne detection the difference of intensities $\alpha\mathcal{I}_3$ and $\alpha\mathcal{I}_4$ is registered. This allows us to measure the sine of the *phase difference* $\phi_1-\phi_2$ of the signal fields,

$$\sin(\phi_1-\phi_2)=\frac{\alpha\mathcal{I}_3-\alpha\mathcal{I}_4}{2\sqrt{\alpha\mathcal{I}_1\alpha\mathcal{I}_2}}.$$

If the phase ϕ_1 is shifted by a quarterwave phase shifter before beam splitting then one can measure the cosine of the phase difference since

$$\sin(\phi_1+\pi/2-\phi_2)=\cos(\phi_1-\phi_2).$$

The sine and cosine of the phase difference (and thus $\phi_1-\phi_2$) can be measured simultaneously in the following way: both signal fields are split into two parts using 50:50 beam splitters, and then the sine and cosine of the phase difference are measured using two balanced homodyne detectors. This measurement scheme contains eight input ports and is called eight-port homodyne detection or double homodyne detection.

Figure 2.1: Eight-port homodyne detector with four input ports 1, 2, 10, and 20 and four output ports with detectors D_3, D_4, D_5, and D_6 measuring the photon number differences $n_4 - n_3$ and $n_6 - n_5$. Four identical 50:50 beam splitters are labeled as BS_1, BS_2, BS_3, and BS_4, and $\lambda/4$ is a $\pi/2$-phase shifter (a quarter-wave plate). The picture is from papers of Noh, Fougéres, and Mandel [84, 85, 86, 87].

An eight-port homodyne detection scheme contains four photodetectors D_3, D_4, D_5, and D_6, and their intensities are labeled as $\mathcal{I}_3$, $\mathcal{I}_4$, $\mathcal{I}_5$, and $\mathcal{I}_6$ to get

$$\begin{aligned}
\cos(\phi_2 - \phi_1) &= \frac{\alpha\mathcal{I}_4 - \alpha\mathcal{I}_3}{2\sqrt{\alpha\mathcal{I}_1\alpha\mathcal{I}_2}}, \\
\sin(\phi_2 - \phi_1) &= \frac{\alpha\mathcal{I}_6 - \alpha\mathcal{I}_5}{2\sqrt{\alpha\mathcal{I}_1\alpha\mathcal{I}_2}}.
\end{aligned}$$

As one knows from the theory of photodetection, a photodetector counts the number of photons in some detection time interval $[0, T]$. Hence, in a semiclassical theory the energy of light streaming into a photodetector, $\alpha\mathcal{A}T\mathcal{I}$, is proportional to the measured number of photons n in the detection time interval $[0, T]$. This means that one can write $\mathcal{I} = \beta n/T$ where β is some constant depending on the structure of the photodetector. In particular, in eight-port homodyne detection

$$\cos(\phi_2 - \phi_1) = \frac{n_4 - n_3}{\sqrt{(n_4 - n_3)^2 + (n_6 - n_5)^2}}, \tag{2.3}$$

$$\sin(\phi_2 - \phi_1) = \frac{n_6 - n_5}{\sqrt{(n_4 - n_3)^2 + (n_6 - n_5)^2}} \tag{2.4}$$

$$\tag{2.5}$$

where n_k is the measured number of photons in the kth photodetector. Note that when $n_3 = n_4$ and $n_5 = n_6$ the phase difference $\phi_1 - \phi_2$ is undetermined, and such results are omitted from the measurement data [84].

We continue to study eight-port homodyne detection in Section 2.6 after the quantization of a single mode optical field is introduced. This allows us to present the quantum theory of homodyne detection.

Remark 1 As mentioned earlier, if the physical system under consideration contains only *one* light beam with a fixed phase one cannot measure its phase using a single photodetector. A phase measurement requires at least *two* light beams with fixed phases ϕ_1 and ϕ_2 interfering with each others so that the phase difference can be measured. However, if one measures the phase difference $\phi_1 - \phi_2$ between the unknown signal field with ϕ_1 and the known reference field with the reference phase ϕ_2, which can be chosen to be 0 without restricting generality, then the phase difference is the absolute phase ϕ_1. In this case, the difference between concepts of an absolute phase and a phase difference is more theoretical than practical.

2.3 Quantized electromagnetic field

Let $\mathcal{H}$ be a complex Hilbert space spanned by the number states $\{|n\rangle \mid n \in \mathbb{N}\}$, and let

$$a := \sum_{n=0}^{\infty} \sqrt{n+1}\, |n\rangle \langle n+1| , \qquad a^* := \sum_{n=0}^{\infty} \sqrt{n+1}\, |n+1\rangle \langle n| ,$$

$$N := a^* a = \sum_{n=0}^{\infty} n\, |n\rangle \langle n|$$

be the lowering, raising, and number operators with their respective domains $\mathcal{D}(a) = \mathcal{D}(a^*) := \left\{\psi \in \mathcal{H} \,\middle|\, \sum_{n=0}^{\infty} n|\langle n|\psi\rangle|^2 < \infty\right\}$ and $\mathcal{D}(N) := \left\{\psi \in \mathcal{H} \,\middle|\, \sum_{n=0}^{\infty} n^2|\langle n|\psi\rangle|^2 < \infty\right\}$. Let I be the identity operator of $\mathcal{H}$.

The single-mode electromagnetic vector potential operator [77, Eq. (6.105)] in a cavity is (the closure of) the operator

$$\hat{\mathbf{A}}(\mathbf{r},t) := \sqrt{\frac{\hbar}{2\epsilon_0 \mathcal{V}\omega}}\left(a e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} + a^* e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t}\right)\mathbf{u} \tag{2.6}$$

where $\hbar$ is the Dirac constant. The energy operator of the single-mode system is

$$H := \hbar\omega\left(N + \frac{1}{2}I\right).$$

Since $e^{-i\theta N} a e^{i\theta N} = e^{i\theta} a$, $\theta \in \mathbb{R}$, the energy operator generates time shifts,

$$e^{-iHt'/\hbar}\hat{\mathbf{A}}(\mathbf{r},t)e^{iHt'/\hbar} = \hat{\mathbf{A}}(\mathbf{r},t+t').$$

Coherent states $|z\rangle := e^{-|z|^2/2}\sum_{n=0}^{\infty} z^n/\sqrt{n!}\,|n\rangle$, $z \in \mathbb{C}$, are known to describe ideal laser light in quantum optics [96, p. 232]. We note that $a\,|z\rangle = z\,|z\rangle$ and $\langle z|\,N\,|z\rangle = |z|^2$ for all $z \in \mathbb{C}$. Since laser light is quasi-monochromatic, it suffices to treat a laser as a single-mode optical system in this context. The mean value of the operator $\hat{\mathbf{A}}(\mathbf{r},t)$ in state $|z\rangle$ is

$$\begin{aligned}\left\langle z\left|\hat{\mathbf{A}}(\mathbf{r},t)\right|z\right\rangle &= \sqrt{\frac{\hbar}{2\epsilon_0\mathcal{V}\omega}}\left(z e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} + \overline{z} e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t}\right)\mathbf{u} \\ &= \sqrt{\frac{2\tilde{\mathcal{E}}}{\epsilon_0\mathcal{V}\omega^2}}\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \arg z)\,\mathbf{u} \qquad (2.7)\end{aligned}$$

where

$$\tilde{\mathcal{E}} := \langle z|H|z\rangle - \frac{1}{2}\hbar\omega = \hbar\omega|z|^2$$

is the mean field energy (total energy minus vacuum energy). Comparing Eqs. (2.2) and (2.7) one sees that the classical monochromatic plane wave can be used to describe a quantum laser light in the coarsest nontrivial semi-classical approximation. Also, it is convenient to call $|z|$ and $\arg z$ the *energy parameter* and *phase parameter* of a coherent state $|z\rangle$, respectively. Practically, a laser generates light rays or pulses which are localized either in space or time. Hence, due to the nonlocalizability of a plane wave, the single-mode approximation for a quantum laser is not very accurate.

2.4 Dirac's approach

Using the semiclassical photon number $\mathcal{N} := \mathcal{E}/(\hbar\omega)$ Eq. (2.2) gets the form

$$\mathbf{A}(\mathbf{r},t) = \sqrt{\frac{\hbar}{2\epsilon_0\mathcal{V}\omega}}\left(e^{i\phi}\sqrt{\mathcal{N}}\,e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} + e^{-i\phi}\sqrt{\mathcal{N}}\,e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t}\right)\mathbf{u}.$$

Comparing this equation to the operator $\hat{\mathbf{A}}(\mathbf{r},t)$ of Eq. (2.6) one sees that in the quantization of the electromagnetic single-mode field one substitutes the lowering operator a for $e^{i\phi}\sqrt{\mathcal{N}}$. Writing the operator a in the form

$$a = V_\psi \sqrt{N},$$

where $V_\psi := \sum_{n=0}^{\infty} |n\rangle \langle n+1| + |\psi\rangle \langle 0|$, $\psi \in \mathcal{H}$, Dirac [28] assumed that the operator V_ψ could be expressed as $e^{i\Phi}$, where Φ is a self-adjoint phase operator. Actually, Dirac used a decomposition $a = \sqrt{N}e^{-i\Phi}$, but our notation does not change Dirac's basic idea. This would mean that V_ψ is a unitary operator, which is not true since $\langle 0| V_\psi^* V_\psi |0\rangle = \|\psi\|^2 = 1$ only when ψ is a unit vector, whereas $\langle k| V_\psi V_\psi^* |k\rangle = 1 + |\langle k|\psi\rangle|^2 = 1$ for all $k \in \mathbb{N}$ only when ψ is the null vector. Hence, there is no quantum phase operator in Dirac's sense and one has to proceed in another way.

The difficulty in finding a self-adjoint phase operator has led to several phase theories. In the next chapters the following six branches of such theories will be analyzed:

1. Phase observables are defined as phase shift covariant operator measures (Sections 2.5 and 2.6).

2. Self-adjoint phase operators acting on $\mathcal{H}$ are defined by some quantization rules (Section 5.1).

3. Phase observables are defined as phase distributions (Section 5.2).

4. Self-adjoint phase operators are defined in some subspaces of $\mathcal{H}$ (Section 5.3).

5. Self-adjoint phase operators are defined in some extensions of $\mathcal{H}$ (Section 5.3).

6. Phase observables are defined as phase difference operators of two modes acting on $\mathcal{H} \otimes \mathcal{H}$ (Section 5.4).

2.5 Positive operator measures and their generalizations

The traditional way to represent observables in quantum mechanics as self-adjoint operators or, equivalently, as normalized projection measures has turned out to be insufficient in various applications of quantum mechanics. A more suitable way to characterize quantum observables is to represent them as normalized positive operator measures which are not necessarily projection valued (for the theory of positive operator measures and their applications in physics, see e.g. [27, 50, 58, 78, 16]). In this section, we study some general properties of positive operator measures and their generalizations.

Let H be a Hilbert space and $\mathcal{L}(H)$ be a set of bounded operators on H, and let O and I be the null and identity operators of H. Let Ω be a nonempty set and $\mathcal{F}$ a σ-algebra of subsets of Ω. Especially, for any (Borel) subset Ξ of the complex plane, let $\mathcal{B}(\Xi)$ denote the Borel σ-algebra of Ξ.

Definition 1 *The mapping $F:\mathcal{F}\to\mathcal{L}(H)$ is an operator measure if for all $\varphi,\ \psi\in H$ the mapping*

$$\mathcal{F}\ni X\mapsto\langle\varphi|F(X)\psi\rangle\in\mathbb{C}$$

is a complex measure (i.e. a σ-additive set function).

An operator measure F is self-adjoint, positive, or normalized if for all $X\in\mathcal{F}$, $F(X)^*=F(X)$, $F(X)\geq O$, or $F(\Omega)=I$, respectively. A normalized positive operator measure is an observable.

For any observable $F:\mathcal{F}\to\mathcal{L}(H)$ and a state (positive trace-one operator) T the mapping $X\mapsto\mathrm{tr}(TF(X))$ is a probability measure. The number $\mathrm{tr}(TF(X))$ is the probability of getting a measurement outcome from a set $X\in\mathcal{F}$ in the measurement of an observable F when the corresponding physical system is in a state T.

An operator measure F is a projection measure if $F(X)^2=F(X)$, $X\in\mathcal{F}$. Let A be a self-adjoint operator with the domain $D(A)\subseteq H$. The spectral theorem for self-adjoint operators states that there exists a unique normalized projection measure $F_A:\mathcal{B}(\mathbb{R})\to H$ such that $\langle\varphi|A\psi\rangle=\int_{\mathbb{R}}x\,\mathrm{d}\langle\varphi|F_A(x)\psi\rangle$ for all $\varphi\in H$ and $\psi\in D(A)$. Conversely, for any normalized projection measure $F:\mathcal{B}(\mathbb{R})\to H$ the sesquilinear form $H\times D\ni(\varphi,\psi)\mapsto\int_{\mathbb{R}}x\,\mathrm{d}\langle\varphi|F(x)\psi\rangle\in\mathbb{C}$ defines a unique self-adjoint operator with a domain $D:=\left\{\psi\in H\mid\int_{\mathbb{R}}x^2\mathrm{d}\langle\psi|F(x)\psi\rangle<\infty\right\}$. Thus, a self-adjoint operator defines a unique observable defined on $\mathcal{B}(\mathbb{R})$.

The concept of an operator measure can be generalized. As shown in Paper IV the quantization of some classical dynamical variables can be described in a rigorous way using the so-called generalized operator measures or sesquilinear form measures. They are defined as follows: Let K be a linear subspace of a Hilbert space H, and let $\mathcal{SL}(K,K;\mathbb{C})$ be a set of sesquilinear forms from $K\times K$ to $\mathbb{C}$ (conjugate linear in the first argument).

Definition 2 *The mapping $G:\mathcal{F}\to\mathcal{SL}(K,K;\mathbb{C})$ is a sesquilinear form measure if for all $\varphi,\ \psi\in K$ the mapping*

$$\mathcal{F}\ni X\mapsto[G(X)](\varphi,\psi)\in\mathbb{C}$$

is a complex measure.

A sesquilinear form measure G is symmetric, or positive, if $[G(X)](\varphi, \psi) = \overline{[G(X)](\psi, \varphi)}$, or $[G(X)](\psi, \psi) \geq 0$, for all $X \in \mathcal{F}$ and for all φ, $\psi \in K$. If $[G(\Omega)](\varphi, \psi) = \langle \varphi | \psi \rangle$ for all φ, $\psi \in K$ we say that G is normalized.

Any operator measure $F : \mathcal{F} \to \mathcal{L}(H)$ defines the following sesquilinear form measure:

$$\mathcal{F} \ni X \mapsto [(\varphi, \psi) \mapsto \langle \varphi | F(X)\psi \rangle] \in \mathcal{SL}(H, H; \mathbb{C})$$

and, thus, we see that sesquilinear form measures are generalizations of operator measures. Also we see that self-adjoint, positive, or normalized operator measures define symmetric, positive, or normalized sesquilinear form measures, respectively.

Conversely, let $G : \mathcal{F} \to \mathcal{SL}(K, K; \mathbb{C})$ be a sesquilinear form measure, and let K be dense in H. If the sesquilinear form $(\varphi, \psi) \mapsto [G(X)](\varphi, \psi)$ is bounded for all $X \in \mathcal{F}$ then there exists a unique operator measure $F_G : \mathcal{F} \to \mathcal{L}(H)$ such that $[G(X)](\varphi, \psi) = \langle \varphi | F_G(X)\psi \rangle$ for all $X \in \mathcal{F}$ and φ, $\psi \in K$. If G is symmetric, positive, or normalized, then F_G is self-adjoint, positive, or normalized, respectively.

2.6 Coherent state phase measurements

In Section 2.2 classical eight-port homodyne detection was introduced and it was shown that the phase difference of two input fields can be measured by using four photodetectors. The classical model is realistic only when the input fields are strong lasers with fixed phases so that they can be regarded as classical waves. When the input fields are weak then one must consider quantum effects.

Noh, Fougères, and Mandel [84, 85, 86, 87] replaced the numbers n_k in Eqs. (2.3) and (2.4) with number operators associated to the photodetectors and defined sine and cosine phase difference operators S_{M} and C_{M}. These operators satisfy the relations $[S_{\mathrm{M}}, C_{\mathrm{M}}] = O$ and $S_{\mathrm{M}}^2 + C_{\mathrm{M}}^2 = I$, and their spectra are the interval $[-1, 1]$, including the countable dense set of eigenvalues. Theoretical results for S_{M} and C_{M} are in good agreement with the experiments done by Noh, Fougères, and Mandel [84, 85, 86, 87].

As shown in Section 2.3 the expectation value of a vector potential operator in a coherent state is a classical wave. Therefore, it is interesting to study the case when the input fields are in coherent states, say, $|z_1\rangle$ and $|z_2\rangle$. Let $W(n_{43}, n_{65})$ be the joint count probability for the number differences

$n_{43} := n_4 - n_3$ and $n_{65} := n_6 - n_5$. It can be shown [36, 37] that for $|z_2| \gg 0$

$$W\left(n_{43}, n_{65}\right) \approx \frac{1}{\pi |z_2|^2} Q_{|z_1\rangle} \left(\frac{n_{43} - i\, n_{65}}{|z_2|}\, e^{i \arg z_2} \right)$$

where $Q_{|z_1\rangle}(z) := |\langle z|z_1\rangle|^2 = e^{-|z-z_1|^2}$. Moreover, for any (continuous) function $f : [-1,1]\times[-1,1] \to \mathbb{C}$ the expectation value of an operator $f(S_{\mathrm{M}}, C_{\mathrm{M}})$ for large $|z_2|$ is

$$\frac{1}{\pi} \int_0^{2\pi} f(\sin(\arg z_2 - \theta), \cos(\arg z_2 - \theta)) \int_0^\infty Q_{|z_1\rangle}\left(re^{i\theta}\right) r \mathrm{d}r \mathrm{d}\theta.$$

When $|z_1| \to \infty$ then (formally) $\pi^{-1} \int_0^\infty Q_{|z_1\rangle}\left(re^{i\theta}\right) r\mathrm{d}r \to \delta_{2\pi}(\theta - \arg z_1)$, where $\delta_{2\pi}$ is the 2π-periodic Dirac δ-distribution (see Paper II). Hence, the expectation value of an operator $f(S_{\mathrm{M}}, C_{\mathrm{M}})$ tends to

$$f(\sin(\arg z_2 - \arg z_1), \cos(\arg z_2 - \arg z_1))$$

when the energy parameters $|z_1|$ and $|z_2|$ get large. Note that when $|z_1|$, $|z_2| \to \infty$ the expectation values of the field energy and number operators tend to infinity. The situation where the number (and energy) gets large and the phase difference is arbitrarily well defined corresponds to the *classical limit.*

Walker and Carrol [123] made the first eight-port homodyne measurements using coherent state inputs. The second input field was strong ($|z_2| \gg 0$) and they were able to measure the probability density function $Q_{|z_1\rangle}$. Indeed, it can be shown [68, 70, 71, 69] that actually the eight-port homodyne detection scheme with a strong coherent second input beam ($|z_2| \gg 0$) corresponds to a joint measurement of the quadrature components of a signal field. Moreover, one can measure the Husimi Q-function [62]

$$Q_T(z) := \langle z|T|z\rangle, \qquad z \in \mathbb{C},$$

of an arbitrary signal state T or, equivalently, the following phase space observable (positive operator measure) [14, 24]:

$$\mathcal{B}(\mathbb{C}) \ni Z \mapsto A_{|0\rangle}(Z) := \frac{1}{\pi} \int_Z |z\rangle \langle z| \, \mathrm{d}^2 z \in \mathcal{L}(\mathcal{H}).$$

Denoting $r := |z|$ and $\theta := \arg z$ one may define an angle margin observable of $A_{|0\rangle}$,

$$E_{|0\rangle}(X) := \frac{1}{\pi} \int_X \int_0^\infty \left|re^{i\theta}\right\rangle \left\langle re^{i\theta}\right| r\mathrm{d}r\mathrm{d}\theta, \qquad X \in \mathcal{B}([0, 2\pi)),$$

and conclude that $E_{|0\rangle}$ is a phase difference observable between a strong local oscillator and a signal field in eight-port homodyne detection. As noted in Remark 1, choosing the phase parameter $\arg z_2$ of the local oscillator state $|z_2\rangle$ to be 0 one can interpret the above measurement of the Q-function as a measurement of the (absolute) phase of a signal field.

Suppose then that we measure the Q-function $Q_{|z_1\rangle}$ of a signal state $|z_1\rangle$. It is easy to see that $Q_{|z_1\rangle}$ acts covariantly when one shifts the phase parameter $\arg z_1$ of the signal state, that is,

$$Q_{\left|z_1 e^{-i\theta}\right\rangle}(z) = e^{-|z - z_1 e^{-i\theta}|^2} = e^{-|ze^{i\theta} - z_1|^2} = Q_{|z_1\rangle}\left(ze^{i\theta}\right).$$

In terms of $E_{|0\rangle}$ this means that

$$\left\langle z_1 e^{-i\theta}\right| E_{|0\rangle}(X) \left|z_1 e^{-i\theta}\right\rangle = \langle z_1| E_{|0\rangle}(X \dot{+} \theta) |z_1\rangle, \qquad X \in \mathcal{B}([0, 2\pi)),$$

where $\dot{+}$ is the addition modulo 2π, that is,

$$X \dot{+} \theta := \{x \in [0, 2\pi) \mid (x - \theta) \bmod 2\pi \in X\}.$$

The above phase shift covariance of the probability density $Q_{|z_1\rangle}$ is a natural assumption for coherent state phase (parameter) measurements, since when we shift the phase parameter the phase probability density should only move without transforming its shape. Also, since the phase parameter of $|z_1\rangle$ always lies in some phase interval, say $[0, 2\pi)$, it is convenient to generalize the above results and assume that an observable $E : \mathcal{B}([0, 2\pi)) \to \mathcal{L}(\mathcal{H})$ describes a *coherent state phase measurement* if

$$\left\langle ze^{-i\theta}\right| E(X) \left|ze^{-i\theta}\right\rangle = \langle z| E(X \dot{+} \theta) |z\rangle$$

for all $z \in \mathbb{C}$, $\theta \in [0, 2\pi)$, and $X \in \mathcal{B}([0, 2\pi))$.

The number operator N generates shifts of the phase parameter $\arg z$ of a coherent state $|z\rangle$, $z \in \mathbb{C}$, in the following way:

$$e^{i\theta N} |z\rangle = \left|ze^{i\theta}\right\rangle, \qquad \theta \in \mathbb{R}.$$

Thus, one may say that the unitary operator $e^{i\theta N}$, $\theta \in \mathbb{R}$, is a *phase shifter*. In Paper VI the following theorem is proved:

Theorem 1 *Let $E : \mathcal{B}([0, 2\pi)) \to \mathcal{L}(\mathcal{H})$ be an operator measure. Then*

$$\left\langle ze^{-i\theta}\right| E(X) \left|ze^{-i\theta}\right\rangle = \langle z| E(X \dot{+} \theta) |z\rangle$$

for all $z \in \mathbb{C}$, $\theta \in [0, 2\pi)$, and $X \in \mathcal{B}([0, 2\pi))$ if and only if

$$e^{i\theta N} E(X) e^{-i\theta N} = E(X \dot{+} \theta)$$

for all $\theta \in [0, 2\pi)$ and $X \in \mathcal{B}([0, 2\pi))$.

If $e^{i\theta N}E(X)e^{-i\theta N} = E(X\dot{+}\theta)$, $\theta \in [0, 2\pi)$, $X \in \mathcal{B}([0, 2\pi))$, we say that the operator measure E is *phase shift covariant.* On the basis of this theorem we come to the following definition:

Definition 3 *A covariant phase observable is a phase shift covariant normalized positive operator measure* $E : \mathcal{B}([0, 2\pi)) \to \mathcal{L}(\mathcal{H})$.

The theory of covariant phase observables was originally developed by Helstrom [49, 50] and Holevo [56, 57, 58, 59, 60, 61]. Their approach was based on the quantum estimation theory. Later the theory of covariant phase observables has been developed in different contexts by many authors, see for example the following references: [104, 45, 46, 47, 14, 15, 16, 24, 22, 100, 72, 69, 128]. In the next chapter we study mathematical properties of covariant phase observables and more general covariant sesquilinear form measures.

Chapter 3

The structure of covariant sesquilinear form measures

In the previous chapter, the (covariant) phase observables were defined as normalized positive operator measures which are covariant under the shifts generated by the number operator N. The structure of such observables can be determined by using so-called phase matrices. This characterization can be extended to cover all sesquilinear form measures which are covariant under the shifts generated by a generalized number operator. As special cases one gets covariant phase observables of the single mode optical system and position observables of the one dimensional "particle in a box" system which are covariant under the shifts generated by the momentum operator. Next some important definitions and lemmas are introduced.

Fix $J \subseteq \mathbb{Z}$, and let $\mathcal{H}_J$ be a complex Hilbert space with an orthonormal basis consisting of vectors $|n\rangle$ labelled with the index $n \in J$. Then $\mathcal{H}_J$ can be naturally interpreted as a subspace of $\mathcal{H}_\mathbb{Z}$. Let $I_J := \sum_{n\in J} |n\rangle\langle n|$ be the identity operator of $\mathcal{H}_J$ or, equivalently, the projection operator from $\mathcal{H}_\mathbb{Z}$ to $\mathcal{H}_J$. Define a *generalized number operator* $N_J := \sum_{n\in J} n\,|n\rangle\langle n|$ with the domain $\mathcal{D}(N_J) := \left\{\psi \in \mathcal{H}_J \,\middle|\, \sum_{n\in J} n^2 |\langle n|\psi\rangle|^2 < \infty\right\}$. Let $E : \mathcal{B}([0, 2\pi)) \to \mathcal{L}(\mathcal{H}_J)$ be an operator measure. If

$$e^{i\theta N_J} E(X) e^{-i\theta N_J} = E(X \dot{+} \theta)$$

for all $\theta \in [0, 2\pi)$ and $X \in \mathcal{B}([0, 2\pi))$, we say that E is *covariant*. In the case of a single mode optical field $J = \mathbb{N}$, $N_\mathbb{N} = N$, and $\mathcal{H}_\mathbb{N} = \mathcal{H}$, and in the case of a particle in a box $J = \mathbb{Z}$ and $N_\mathbb{Z}$ is the momentum operator.

Let B be a complex Banach space, and let $S : \mathcal{H}_\mathbb{Z} \times \mathcal{H}_\mathbb{Z} \to B$ be a bounded sesquilinear form (conjugate linear in the first argument). Thus, $\|S\| := \sup\left\{\|S(\varphi, \psi)\| \,\middle|\, \|\varphi\| \le 1, \|\psi\| \le 1\right\} < \infty$. As usual, the Banach space of trace-class operators $\mathcal{T}(\mathcal{H}_\mathbb{Z})$ is equipped with the trace norm. Denote

$S_{n,m} := S(|n\rangle, |m\rangle)$ for all $n, m \in \mathbb{Z}$, and $A_{n,m} := \langle n|A|m\rangle$, $n, m \in J$, for any bounded operator A of $\mathcal{H}_J$. The proof of the next lemma is essentially the same as the proof of Proposition 1 of Paper VII.

Lemma 1 *For all $\varphi, \psi \in \mathcal{H}_\mathbb{Z}$,*

$$S(\varphi,\psi) = \lim_{s,t,u,v\to\infty} \sum_{n=-s}^{t} \sum_{m=-u}^{v} S_{n,m}\langle\varphi|n\rangle\langle m|\psi\rangle,$$

and S can be uniquely extended to a continuous linear mapping $\tilde{S} : \mathcal{T}(\mathcal{H}_\mathbb{Z}) \to B$,

$$T \mapsto \tilde{S}(T) := \sum_{n,m\in\mathbb{Z}} S_{n,m}T_{m,n} = \lim_{s,t,u,v\to\infty} \sum_{n=-s}^{t} \sum_{m=-u}^{v} S_{n,m}T_{m,n}.$$

Clearly, $\tilde{S}(|\psi\rangle\langle\varphi|) = S(\varphi,\psi)$ for all $\varphi, \psi \in \mathcal{H}_\mathbb{Z}$.

Remark 2 Any bounded operator A can be written in the form $A =$ w-$\sum_{n,m\in\mathbb{Z}} A_{n,m}|n\rangle\langle m|$ and $\mathrm{tr}(AT) = \sum_{n,m\in\mathbb{Z}} A_{n,m}T_{m,n}$ for any $T \in \mathcal{T}(\mathcal{H}_\mathbb{Z})$. The notation w-$\sum$ means that the sum converges with respect to the weak operator topology; however, for simplicity one sometimes uses a shorter notation $\sum$ for w-$\sum$. Since $\mathcal{H}_J$ is a subspace of $\mathcal{H}_Z$ one may consider a bounded operator A^J of $\mathcal{H}_J$ as a bounded operator $A^\mathbb{Z}$ of $\mathcal{H}_Z$ by defining $A^\mathbb{Z}_{n,m} := A^J_{n,m}$ when $n, m \in J$ and $A^\mathbb{Z}_{n,m} = 0$ otherwise. This extension allows one to write $A^J =$ w-$\sum_{n,m\in J} A^J_{n,m}|n\rangle\langle m| =$ w-$\sum_{n,m\in\mathbb{Z}} A^\mathbb{Z}_{n,m}|n\rangle\langle m|$.

Lemma 2 *Let $q \in \mathbb{Z}$ and let $\nu_q : \mathcal{B}([0,2\pi)) \to \mathbb{C}$ be a σ-additive set function. Then $\nu_q(X \dot{+} \theta) = e^{iq\theta}\nu_q(X)$ for all $X \in \mathcal{B}([0,2\pi))$ and $\theta \in [0,2\pi)$ if and only if $\nu_q(X) = c_q i_q(X)$ for all $X \in \mathcal{B}([0,2\pi))$ where $i_q(X) := (2\pi)^{-1}\int_X e^{iq\theta}\mathrm{d}\theta$ and $c_q \in \mathbb{C}$.*

Proof. Since $[0,2\pi) \dot{+} \theta = [0,2\pi)$ it follows that

$$\nu_q([0,2\pi)) = \nu_q([0,2\pi) \dot{+} \theta) = e^{iq\theta}\nu_q([0,2\pi))$$

and thus $\nu_q([0,2\pi)) = c_0\delta_{0,q}$, where c_0 is a complex constant. The rest of the proof is the same as the proof of Lemma 1 in Paper V (or Phase Theorem 2.2 in Paper I). □

The complex matrix $(c_{n,m})_{n,m\in L}$, $L \subseteq \mathbb{Z}$, is positive semidefinite if $\sum_{n,m\in L} \overline{d_n}c_{n,m}d_m \geq 0$ for all sequences $(d_n)_{n\in L} \subset \mathbb{C}$ for which $d_n \neq 0$ for only finitely many $n \in L$. The positive semidefiniteness of $(c_{n,m})$ implies that $c_{n,n} \geq 0$, $c_{n,m} = \overline{c_{m,n}}$, and $|c_{n,m}| \leq \sqrt{c_{n,n}c_{m,m}}$ for all $n, m \in L$.

3.1 Structure matrices of covariant sesquilinear form measures

Fix $J \subseteq \mathbb{Z}$ and suppose that $\mathcal{K} \subseteq \mathcal{H}_J$ is a linear space. A sesquilinear form measure $G : \mathcal{B}([0,2\pi)) \to \mathcal{SL}(\mathcal{K},\mathcal{K};\mathbb{C})$ is *covariant* if $e^{i\theta N_J}\mathcal{K} \subseteq \mathcal{K}$ and $[G(X)](e^{-i\theta N_J}\varphi, e^{-i\theta N_J}\psi) = [G(X\dot{+}\theta)](\varphi,\psi)$ for all $\theta \in [0,2\pi)$, $X \in \mathcal{B}([0,2\pi))$, and φ, $\psi \in \mathcal{K}$.

Theorem 2 *Let $G : \mathcal{B}([0,2\pi)) \to \mathcal{SL}(\mathcal{K},\mathcal{K};\mathbb{C})$ be a covariant sesquilinear form measure and assume that $|n\rangle$, $|m\rangle \in \mathcal{K}$. Then $[G(X)](|n\rangle, |m\rangle) = c_{n,m}i_{n-m}(X)$ where $c_{n,m} \in \mathbb{C}$. If G is symmetric then $c_{n,m} = \overline{c_{m,n}}$, and if G is normalized and $n = m$ then $c_{n,nh} = 1$. If G is positive and $L \subseteq \mathbb{Z}$ is such that $|n\rangle \in \mathcal{K}$ for all $n \in L$ then the matrix $(c_{n,m})_{n,m\in L}$ is positive semidefinite.*

Proof. Put $q = n - m$ and $\nu_q(X) = [G(X)](|n\rangle, |m\rangle)$ in Lemma 2 to get $[G(X)](|n\rangle, |m\rangle) = c_{n,m}i_{n-m}(X)$. Suppose that G is positive. If $(c_{n,m})_{n,m\in L}$ is not positive semidefinite then there exists a vector $\psi := \sum_{n\in L} d_n |n\rangle \in \mathcal{K}$, $d_n \neq 0$ for finitely many $n \in L$, such that $\sum_{n,m\in L} \overline{d_n}c_{n,m}d_m < 0$. This implies that $[G([0,\epsilon)](\psi,\psi) = (2\pi)^{-1}\int_0^\epsilon \sum_{n,m\in L} \overline{d_n e^{-in\theta}}c_{n,m}d_m e^{-im\theta}\mathrm{d}\theta < 0$ for some $\epsilon > 0$, which contradicts the positivity of G. The rest of the proof is trivial. □

Corollary 1 *Let $E : \mathcal{B}([0,2\pi)) \to \mathcal{L}(\mathcal{H}_J)$ be a covariant operator measure. Then*

$$E(X) = \text{w-}\sum_{n,m\in J} c_{n,m}i_{n-m}(X)\,|n\rangle\langle m|$$

for all $X \in \mathcal{B}([0,2\pi))$ where $c_{n,m} \in \mathbb{C}$ for all n, $m \in J$. If E is normalized then $c_{n,n} = 1$ for all $n \in J$. If E is self-adjoint then $c_{n,m} = \overline{c_{m,n}}$, n, $m \in J$, and if E is positive then $(c_{n,m})_{n,m\in J}$ is positive semidefinite and $\sup\{|c_{n,m}| \mid n,m \in J\} \leq \|E([0,2\pi))\|$.

Proof. This is a direct consequence of Lemma 1, Theorem 2, and the fact that any operator measure defines a unique sesquilinear form measure. If E is positive then for all n, $m \in J$, $|c_{n,m}| \leq \sup\{c_{n,n} \mid n \in J\} = \|E([0,2\pi))\|$. □

Let $G : \mathcal{B}([0,2\pi)) \to \mathcal{SL}(\mathcal{K},\mathcal{K};\mathbb{C})$ be a covariant sesquilinear form measure. Define $\mathcal{M}_J := \mathrm{lin}\{|n\rangle \mid n \in J\}$. If $|n\rangle \in \mathcal{K}$ for all $n \in J$, using sesquilinearity, we get from Theorem 2 that

$$[G(X)](\varphi,\psi) = \sum_{n,m\in J} c_{n,m}i_{n-m}(X)\langle\varphi|n\rangle\langle m|\psi\rangle$$

for all φ, $\psi \in \mathcal{M}_J$ where $(c_{n,m})_{n,m\in J} \in \mathbb{C}^{J\times J}$ is the so-called *structure matrix* of a covariant sesquilinear form measure G. Next we study the converse question of how a complex matrix $(c_{n,m})_{n,m\in J}$ defines a covariant sesquilinear form measure. We have three different cases:

1. Let $(c_{n,m}) \in \mathbb{C}^{J\times J}$ and fix $X \in \mathcal{B}([0,2\pi))$. Define a sesquilinear mapping from $\mathcal{M}_J \times \mathcal{M}_J$ to $\mathbb{C}$,

$$\begin{aligned}(\varphi,\psi) \mapsto [G(X)](\varphi,\psi) &:= \frac{1}{2\pi}\int_X \sum_{n,m\in J} c_{n,m} e^{i(n-m)\theta}\langle\varphi|n\rangle\langle m|\psi\rangle \mathrm{d}\theta \\ &= \sum_{n,m\in J} c_{n,m} i_{n-m}(X)\langle\varphi|n\rangle\langle m|\psi\rangle.\end{aligned}$$

   Now $\mathcal{B}([0,2\pi)) \ni X \mapsto G(X) \in \mathcal{SL}(\mathcal{M}_J,\mathcal{M}_J;\mathbb{C})$ is a covariant sesquilinear form measure.

2. Let $\mathcal{H}_J^1 := \{\psi \in \mathcal{H}_J \mid \sum_{n\in J} |\langle n|\psi\rangle| < \infty\}$, and let $(c_{n,m}) \in \mathbb{C}^{J\times J}$ be bounded, that is, $\|(c_{n,m})\|_\infty := \sup\{|c_{n,m}| \mid n,m \in J\} < \infty$. Then $\theta \mapsto \sum_{n,m\in J} c_{n,m} e^{i(n-m)\theta}\langle\varphi|n\rangle\langle m|\psi\rangle$ is a periodic continuous function for all φ, $\psi \in \mathcal{H}_J^1$ and $[G(X)](\varphi,\psi) = \sum_{n,m\in J} c_{n,m} i_{n-m}(X)\langle\varphi|n\rangle\langle m|\psi\rangle$, $X \in \mathcal{B}([0,2\pi))$, φ, $\psi \in \mathcal{H}_J^1$, defines a covariant sesquilinear form measure $G : \mathcal{B}([0,2\pi)) \to \mathcal{SL}(\mathcal{H}_J^1,\mathcal{H}_J^1;\mathbb{C})$.

3. Let $(c_{n,m}) \in \mathbb{C}^{J\times J}$ be bounded and positive semidefinite. Then, by using the proof of Phase Theorem 2.2 of Paper I, a mapping

$$E : \mathcal{B}([0,2\pi)) \to \mathcal{L}(\mathcal{H}_J),\, X \mapsto E(X) := \text{w-}\sum_{n,m\in J} c_{n,m} i_{n-m}(X)|n\rangle\langle m|$$

   is a covariant positive operator measure. Note that

$$E(X) \le \sum_{n\in J} c_{n,n}\, |n\rangle\, \langle n| \le \|(c_{n,m})\|_\infty\, I_J$$

   for all $X \in \mathcal{B}([0,2\pi))$. If $c_{n,n} = 1$ for all $n \in J$ then E is normalized, that is, an observable.

To conclude, we have the following structure theorem for covariant phase observables:

Theorem 3 *Let $E : \mathcal{B}([0,2\pi)) \to \mathcal{L}(\mathcal{H})$ be a phase observable. Then for any $X \in \mathcal{B}([0,2\pi))$,*

1. $E(X) = \text{w-}\sum_{n,m=0}^{\infty} c_{n,m} i_{n-m}(X)|n\rangle\langle m|$ where

2. $(c_{n,m}) \in \mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ *with* $c_{n,n} = 1$ *for all* $n \in \mathbb{N}$*, and*

3. $(c_{n,m})_{n,m\in\mathbb{N}}$ *is positive semidefinite.*

Conversely, any complex matrix $(c_{n,m})_{n,m\in\mathbb{N}}$ *which has the properties 2 and 3 defines a unique phase observable of the form 1.*

Finally, we define a *phase matrix* as a positive semidefinite complex matrix $(c_{n,m})_{n,m\in\mathbb{N}}$ with $c_{n,n} = 1$ for all $n \in \mathbb{N}$.

Remark 3 Theorem 3 can also be proved by using group theoretical arguments, see Holevo [57, 58, 59, 60]. In Paper V we obtained the same result using a theorem of Cattaneo [19] which generalizes Mackey's imprimitivity theorem for positive operator measures. However, by direct methods, one can obtain results for arbitrary covariant sesquilinear form measures which are not necessarily positive operator measures.

3.2 The canonical covariant observable

Let $e_n(\theta) := e^{-in\theta}$ for all $\theta \in \mathbb{R}$ and $n \in \mathbb{Z}$ so that $\{(2\pi)^{-1/2}e_n \,|\, n \in \mathbb{Z}\}$ is an orthonormal basis of the function space $L^2[0, 2\pi)$. The Hilbert space $\mathcal{H}_\mathbb{Z}$ is isomorphic to $L^2[0, 2\pi)$ via the isomorphism $|n\rangle \mapsto (2\pi)^{-1/2}e_n$. The canonical spectral measure of $L^2[0, 2\pi)$ is the mapping $\tilde{F} : \mathcal{B}([0, 2\pi)) \to \mathcal{L}\left(L^2[0, 2\pi)\right)$ for which $\tilde{F}(X)\psi = \chi_X\psi$ for all $X \in \mathcal{B}([0, 2\pi))$ and $\psi \in L^2[0, 2\pi)$ where χ_X is the characteristic function of X. As is well-known from elementary quantum mechanics, a particle confined to move in a one-dimensional box $[0, 2\pi)$ is described by the Hilbert space $L^2[0, 2\pi)$, and the canonical spectral measure describes the position observable.

Fix $J \subseteq \mathbb{Z}$, and let $F : \mathcal{B}([0, 2\pi)) \to \mathcal{L}(\mathcal{H}_J)$ be a covariant observable with the structure matrix $(c_{n,m}) \in \mathbb{C}^{J\times J}$. Following, for instance, the proof of Proposition 3 in Paper V, one gets the proposition:

Proposition 1 *A covariant observable* F *is projection valued if and only if* $J = \mathbb{Z}$ *and* $c_{n,m} = e^{i(\upsilon_n-\upsilon_m)}$ *for all* n*,* $m \in \mathbb{Z}$ *where* $(\upsilon_n)_{n\in\mathbb{Z}} \subset [0, 2\pi)$*. If* F *is projection valued then* F *is unitarily equivalent, via the isomorphism* $|n\rangle \mapsto (2\pi)^{-1/2}e^{-i\upsilon_n}e_n$*, to the canonical spectral measure* $\tilde{F}$*.*

Proposition 1 implies that *there is no projection valued covariant phase observable.*

Define a unitary representation U_J of the one-dimensional torus $\mathbb{T}$ as $U_J : \mathbb{T} \to \mathcal{L}(\mathcal{H}_J)$, $z \mapsto z^{N_J}$. Let F_1 and F_2 be covariant operator measures with $(c^1_{n,m})_{n,m\in J}$ and $(c^2_{n,m})_{n,m\in J}$, respectively. If W is a unitary operator on $\mathcal{H}_J$ such that $F_1 = WF_2W^*$ and $e^{i\theta N_J} = We^{i\theta N_J}W^*$, $\theta \in [0, 2\pi)$, we say that the $\mathbb{T}$-covariance systems (U_J, F_1) and (U_J, F_2) are equivalent or F_1 is F_2 up to unitary equivalence or, briefly, F_1 is F_2 (u.e.). If F_1 is F_2 (u.e.) then necessarily $W = \sum_{n\in J} e^{iv_n} |n\rangle\langle n|$ where $(v_n)_{n\in\mathbb{Z}} \subset [0, 2\pi)$ and $c^1_{n,m} = e^{i(v_n - v_m)} c^2_{n,m}$ holds for all $n, m \in J$. Thus, the canonical spectral measure is the only covariant projection measure (u.e.).

Definition 4 *The covariant observable $F^J_{\mathrm{can}} : \mathcal{B}([0, 2\pi)) \to \mathcal{L}(\mathcal{H}_J)$, defined by the structure matrix whose elements $c_{n,m} = 1$ for all n, $m \in J$, is the* canonical covariant observable. *Thus,*

$$F^J_{\mathrm{can}}(X) := \text{w-}\sum_{n,m\in J} i_{n-m}(X)|n\rangle\langle m|$$

for all $X \in \mathcal{B}([0, 2\pi))$.

Any canonical covariant observable is the projected canonical spectral measure to the subspace $\mathcal{H}_J$ of $\mathcal{H}_\mathbb{Z}$. Especially, if $J = \mathbb{N}$ we say that $E_{\mathrm{can}} \equiv F^{\mathbb{N}}_{\mathrm{can}}$ is the *canonical phase observable*. From Proposition 1 it follows that the canonical phase observable (u.e.) is the only phase observable which has a covariant projection valued dilation to $\mathcal{H}_\mathbb{Z}$.

3.3 Covariant positive sesquilinear form measures and generalized vectors

Let $G : \mathcal{B}([0, 2\pi)) \to \mathcal{SL}(\mathcal{M}_J, \mathcal{M}_J; \mathbb{C})$ be a sesquilinear form measure defined as

$$[G(X)](\varphi, \psi) := \sum_{n,m\in J} c_{n,m} i_{n-m}(X)\langle\varphi|n\rangle\langle m|\psi\rangle \tag{3.1}$$

for all $X \in \mathcal{B}([0, 2\pi))$ and φ, $\psi \in \mathcal{M}_J$ where $(c_{n,m}) \in \mathbb{C}^{J\times J}$ is positive semidefinite. The equivalence of items 1 and 3 of the following proposition is proved in Paper V (Proposition 2 there). For the proof of the equivalence of 1 and 2, see Page 7353 of Paper II.

Proposition 2 *Let $J \subseteq \mathbb{Z}$ and let $(c_{n,m}) \in \mathbb{C}^{J\times J}$. The following three claims are equivalent:*

1. *$(c_{n,m})$ is positive semidefinite;*
2. *all the principal minors of $(c_{n,m})$ are nonnegative;*
3. *$c_{n,m} = \langle\psi_n|\psi_m\rangle$ for all n, $m \in J$ where $\psi_k \in \mathcal{H}_J$ for all $k \in J$.*

In the context of Proposition 2, defining $f_n^{(k)} := \langle k|\psi_n\rangle$ for all $n,\ k \in J$, it follows that for any positive semidefinite matrix $(c_{n,m}) \in \mathbb{C}^{J\times J}$,

$$\sum_{n,m\in J} \overline{d_n} c_{n,m} b_m = \sum_{k\in J}\sum_{n\in J} \overline{d_n f_n^{(k)}} \sum_{m\in J} b_m f_m^{(k)}$$

for all $(d_n)_{n\in J},\ (b_n)_{n\in J} \subset \mathbb{C}$ with $d_n \neq 0$ and $b_n \neq 0$ for only finitely many $n \in J$. Also $\sum_{k\in J} \left|f_n^{(k)}\right|^2 = c_{n,n} < \infty$ for all $n \in J$. For all $k \in J$ define a sesquilinear form measure $G_k : \mathcal{B}([0,2\pi)) \to \mathcal{SL}(\mathcal{M}_J, \mathcal{M}_J; \mathbb{C})$ as

$$[G_k(X)](\varphi,\psi) := \sum_{n,m\in J} \overline{f_n^{(k)}} f_m^{(k)} i_{n-m}(X)\langle\varphi|n\rangle\langle m|\psi\rangle \tag{3.2}$$

for all $X \in \mathcal{B}([0,2\pi))$ and $\varphi,\ \psi \in \mathcal{M}_J$. Thus, we get the following structure theorem for covariant positive sesquilinear form measures:

Theorem 4 *For a covariant positive sesquilinear form measure G defined in (3.1),*

$$[G(X)](\varphi,\psi) = \sum_{k\in J} [G_k(X)](\varphi,\psi)$$

for all $X \in \mathcal{B}([0,2\pi))$ and $\varphi,\ \psi \in \mathcal{M}_J$ where, for all $k \in J$, G_k is a covariant positive sesquilinear form measure defined in (3.2) and $\sum_{k\in J} \left|f_n^{(k)}\right|^2 < \infty$ for all $n \in J$.

Conversely, if for all $k \in J$, G_k is a covariant positive sesquilinear form measure defined as in (3.2) and $\sum_{k\in J} \left|f_n^{(k)}\right|^2 < \infty$ then $[G(X)](\varphi,\psi) := \sum_{k\in J}[G_k(X)](\varphi,\psi)$ defines a covariant positive sesquilinear form measure $G : \mathcal{B}([0,2\pi)) \to \mathcal{SL}(\mathcal{M}_J, \mathcal{M}_J; \mathbb{C})$ with the positive semidefinite structure matrix $(c_{n,m})$ for which $c_{n,m} := \sum_{k\in J} \overline{f_n^{(k)}} f_m^{(k)}$ for all $n,\ m \in J$.

If $(c_{n,m})$ is positive semidefinite and bounded then G defines a unique covariant positive operator measure

$$X \mapsto F(X) := \text{w-}\sum_{n,m\in J} c_{n,m} i_{n-m}(X)|n\rangle\langle m|. \tag{3.3}$$

Moreover, if $\sup\left\{\left|f_n^{(k)}\right| \;\middle|\; n \in J\right\} < \infty$ then G_k defines a unique covariant positive operator measure

$$X \mapsto F_k(X) := \text{w-}\sum_{n,m\in J} \overline{f_n^{(k)}} f_m^{(k)} i_{n-m}(X)|n\rangle\langle m|. \tag{3.4}$$

Therefore, one gets the following corollary:

Corollary 2 *For a covariant positive operator measure F defined in (3.3),*

$$F(X) = \text{w-}\sum_{k\in J} F_k(X)$$

(summation order is irrelevant) for all $X \in \mathcal{B}([0,2\pi))$ where for all $k \in J$ $F_k : \mathcal{B}([0,2\pi)) \to \mathcal{L}(\mathcal{H}_J)$ is a covariant positive operator measure defined in (3.4) and $\sup\left\{\sum_{k\in J}\left|f_n^{(k)}\right|^2 \;\middle|\; n \in J\right\} < \infty$.

Conversely, if for all $k \in J$, $F_k : \mathcal{B}([0,2\pi)) \to \mathcal{L}(\mathcal{H}_J)$ is a covariant positive operator measure defined in (3.4) and $\sup\left\{\sum_{k\in J}\left|f_n^{(k)}\right|^2 \;\middle|\; n \in J\right\} < \infty$ *then $F(X) := \text{w-}\sum_{k\in J} F_k(X)$ exists for all $X \in \mathcal{B}([0,2\pi))$ and defines a covariant positive operator measure $F : \mathcal{B}([0,2\pi)) \to \mathcal{L}(\mathcal{H}_J)$ with the bounded positive semidefinite structure matrix $(c_{n,m})$ for which $c_{n,m} := \sum_{k\in J} \overline{f_n^{(k)}} f_m^{(k)}$ for all n, $m \in J$.*

In the context of Corollary 2, for all $k \in J$, define a generalized vector $|\phi_k) := \sum_{n\in J} f_n^{(k)} |n\rangle$ which belongs to the Banach space

$$\mathcal{H}_J^\infty := \left\{ \sum_{n\in J} d_n |n\rangle \;\middle|\; (d_n)_{n\in J} \subset \mathbb{C},\ \sup_{n\in J}(|d_n|) < \infty \right\}$$

equipped with the sup norm. The space $\mathcal{H}_J^\infty$ is (isomorphic to) the topological dual of the Banach space $\mathcal{H}_J^1$ equipped with the L^1-norm. Denote a bounded operator $\psi \mapsto (\phi_k|\psi\rangle|\phi_k)$ from $\mathcal{H}_J^1$ to $\mathcal{H}_J^\infty$ as $|\phi_k)(\phi_k|$. The operator $|\phi_k)(\phi_k|$, as any bounded operator $\mathcal{H}_J^1 \to \mathcal{H}_J^\infty$, can be interpreted as a sesquilinear form $\mathcal{H}_J^1 \times \mathcal{H}_J^1 \to \mathbb{C}$. Since for all $X \in \mathcal{B}([0,2\pi))$ the sequilinear form

$$(\varphi,\psi) \mapsto \frac{1}{2\pi}\int_X \langle e^{-i\theta N_\mathbb{Z}}\varphi|\phi_k)(\phi_k|e^{-i\theta N_\mathbb{Z}}\psi\rangle \mathrm{d}\theta = \langle\varphi|F_k(X)\psi\rangle$$

is defined on $\mathcal{H}_J^1 \times \mathcal{H}_J^1$ where $\mathcal{H}_J^1$ is a dense subset of $\mathcal{H}_J$ we may formally write

$$F_k(X) = \frac{1}{2\pi}\int_X e^{i\theta N_\mathbb{Z}}|\phi_k)(\phi_k|e^{-i\theta N_\mathbb{Z}}\mathrm{d}\theta$$

and

$$F(X) = \text{w-}\sum_{k\in J}\frac{1}{2\pi}\int_X e^{i\theta N_\mathbb{Z}}|\phi_k)(\phi_k|e^{-i\theta N_\mathbb{Z}}\mathrm{d}\theta$$

for all $X \in \mathcal{B}([0,2\pi))$. If F is normalized and there is only one nonzero term ($|\phi_k) \neq 0$) in the above series then F is the canonical covariant observable (u.e.) and vice versa. This gives the following characterization of covariant phase obsevables:

Corollary 3 *The mapping $E : \mathcal{B}([0,2\pi)) \to \mathcal{L}(\mathcal{H})$ is a phase observable if and only if*

$$E(X) = \text{w-}\sum_{k=0}^{\infty} \frac{1}{2\pi} \int_X e^{i\theta N} |\phi_k)(\phi_k| e^{-i\theta N} \mathrm{d}\theta$$

for all $X \in \mathcal{B}([0,2\pi))$ where, for all $k \in \mathbb{N}$, $|\phi_k) \in \mathcal{H}_{\mathbb{N}}^{\infty}$, and $\sum_{k=0}^{\infty} |\langle n|\phi_k)|^2 = 1$ for all $n \in \mathbb{N}$. The phase obsevable E is defined by a single generalized vector if and only if E is $E_{\rm can}$ (u.e.).

Note that Corollary 3 can be seen as a special case of Theorem 2 of [61]. Moreover, one can write $E_{\rm can}(X) = (2\pi)^{-1} \int_X |\theta)(\theta| \,\mathrm{d}\theta$ where for all $\theta \in \mathbb{R}$, $|\theta) := \sum_{n=0}^{\infty} e^{in\theta} \, |n\rangle \in \mathcal{H}_{\mathbb{N}}^{\infty}$ is the so-called *London phase state* [76, 111].

3.4 Covariant positive operator measures and operations

Fix $J \subseteq \mathbb{Z}$ and let $\mathcal{T}(\mathcal{H}_J)$ and $\mathcal{T}(\mathcal{H}_J)_1^+$ be the sets of trace-class operators and states on $\mathcal{H}_J$, respectively. A linear mapping $\Phi : \mathcal{T}(\mathcal{H}_J) \to \mathcal{T}(\mathcal{H}_J)$ is positive if $\Phi(T)$ is positive, and trace-preserving if $\mathrm{tr}(\Phi(T)) = 1$ for all states T. We say that a positive linear map $\Phi : \mathcal{T}(\mathcal{H}_J) \to \mathcal{T}(\mathcal{H}_J)$ for which $\mathrm{tr}(\Phi(T)) \leq 1$, $T \in \mathcal{T}(\mathcal{H}_J)_1^+$, is an operation (for the theory of operations, see e.g. [27, 65, 16]). A linear mapping Φ is covariant if $e^{iN_{\mathbb{Z}}\theta}\Phi(T)e^{-iN_{\mathbb{Z}}\theta} = \Phi(e^{iN_{\mathbb{Z}}\theta}Te^{-iN_{\mathbb{Z}}\theta})$ for all states T and $\theta \in [0,2\pi)$, and it is pure if $\Phi(|\psi\rangle\langle\psi|)^2 = \Phi(|\psi\rangle\langle\psi|)$ for all unit vectors $\psi \in \mathcal{H}_J$. The next theorem is a generalization of the results of Hall and Fuss [45, 46, 47]. Its proof is almost the same as the proof of Theorem 3 in Paper VII.

Theorem 5 *A mapping $F : \mathcal{B}([0,2\pi)) \to \mathcal{L}(\mathcal{H}_J)$ is a covariant positive operator measure if and only if*

$$\mathrm{tr}(TF(X)) = \mathrm{tr}(\Phi(T)F_{\rm can}^J(X)) \tag{3.5}$$

for all $X \in \mathcal{B}([0,2\pi))$ and $T \in \mathcal{T}(\mathcal{H}_J)$ where $\Phi : \mathcal{T}(\mathcal{H}_J) \to \mathcal{T}(\mathcal{H}_J)$ is a covariant positive linear mapping. Such an operator measure F is normalized if and only if Φ is trace-preserving.

A possible choice for Φ is

$$\Theta(T) := \text{w-}\sum_{n,m \in J} c_{m,n} T_{n,m} \, |n\rangle\langle m| \,, \quad T \in \mathcal{T}(\mathcal{H}_J)$$

where $(c_{n,m}) \in \mathbb{C}^{J\times J}$ is the (bounded positive semidefinite) structure matrix of F. The dual mapping $\Theta^* : \mathcal{L}(\mathcal{H}_J) \to \mathcal{L}(\mathcal{H}_J)$ of Θ defined by the relation

$\mathrm{tr}(T\Theta^*(A)) = \mathrm{tr}(\Theta(T)A)$, $A \in \mathcal{L}(\mathcal{H}_J)$, $T \in \mathcal{T}(\mathcal{H}_J)$, is a positive linear mapping, and

$$\Theta^*(A) = \text{w-}\sum_{n,m\in J} c_{n,m} A_{n,m} |n\rangle \langle m| .$$

From Corollary 2 one gets

$$\Theta(T) = \text{w-}\sum_{k\in J}^{\infty} A_k T A_k^*, \quad T \in \mathcal{T}(\mathcal{H}_J),$$

where $A_k := \sum_{n\in J} f_n^{(k)} |n\rangle \langle n|$ for all $k \in J$ showing that Θ is completely positive [27]. Note that $\sum_{k\in J} A_k A_k^* = \sum_{n\in J} c_{n,n} |n\rangle \langle n|$ and $\Theta^*(A) = \sum_{k\in J} A_k^* A A_k$, $A \in \mathcal{L}(\mathcal{H}_J)$.

Corollary 4 *For any covariant positive operator measure $F : \mathcal{B}([0, 2\pi)) \to \mathcal{L}(\mathcal{H}_J)$,*

$$F(X) = \Theta^* \left(F^J_{\mathrm{can}}(X)\right)$$

for all $X \in \mathcal{B}([0, 2\pi))$.

Let F be normalized and let $\Theta_1^+ : \mathcal{T}(\mathcal{H}_J) \to \mathcal{T}(\mathcal{H}_J)$ be the restriction of the corresponding operation Θ to the set of states. The next theorem characterizes the canonical covariant observable (u.e.) among other covariant observables. Its proof is essentially similar to the proof of Theorem 4 of Paper VII.

Theorem 6
1. *Θ and Θ_1^+ are injections if and only if $c_{n,m} \neq 0$ for all n, $m \in J$;*
2. *Θ_1^+ is a surjection if and only if F is F^J_{can} (u.e.);*
3. *Θ_1^+ is a bijection if and only if F is F^J_{can} (u.e.);*
4. *Θ is pure if and only if F is F^J_{can} (u.e.).*

3.5 Radon-Nikodým derivatives of covariant sesquilinear form measures

Fix $J \subseteq \mathbb{Z}$, and let $(c_{n,m}) \in \mathbb{C}^{J\times J}$. The sesquilinear form measure $\mathcal{B}([0, 2\pi)) \to \mathcal{SL}(\mathcal{M}_J, \mathcal{M}_J; \mathbb{C})$, defined in item 1 of Section 3.1, can be written in the form $[G(X)](\varphi, \psi) = (2\pi)^{-1} \int_X g^G_{|\psi\rangle\langle\varphi|}(\theta) \mathrm{d}\theta$ for all $X \in \mathcal{B}([0, 2\pi))$ and φ, $\psi \in \mathcal{M}_J$, where

$$g^G_{|\psi\rangle\langle\varphi|}(\theta) := \sum_{n,m\in J} c_{n,m} e^{i(n-m)\theta} \langle\varphi|n\rangle\langle m|\psi\rangle, \quad \theta \in \mathbb{R}, \tag{3.6}$$

is the (continuous, bounded, and periodic) Radon-Nikodým derivative of the complex measure $X \mapsto [G(X)](\varphi, \psi)$.

Similarly, if $(c_{n,m})$ is bounded then for the sesquilinear form measure $G : \mathcal{B}([0, 2\pi)) \to \mathcal{SL}(\mathcal{H}^1_J, \mathcal{H}^1_J; \mathbb{C})$, defined in item 2 of Section 3.1, the complex measure $X \mapsto [G(X)](\varphi, \psi)$ has the (continuous, bounded, and periodic) Radon-Nikodým derivative of the form (3.6) for all φ, $\psi \in \mathcal{H}^1_J$.

Suppose then that $(c_{n,m}) \in \mathbb{C}^{J\times J}$ is bounded and positive semidefinite and, adding zeros, represent $(c_{n,m})$ as a (bounded and positive semidefinite) complex matrix defined on the whole set $\mathbb{Z} \times \mathbb{Z}$ (see Remark 2). Hence, when it is notationally convenient (e.g. in Theorem 7), one may assume that $J = \mathbb{Z}$. Define a covariant positive operator measure

$$X \mapsto F(X) := \text{w-}\sum_{n,m\in J} c_{n,m} i_{n-m}(X) \, |n\rangle \, \langle m| \, ,$$

and for any $T \in \mathcal{T}(\mathcal{H}_J)$, define a complex measure $X \mapsto p^F_T(X) := \mathrm{tr}(TF(X))$. Since p^F_T is absolutely continuous with respect to the normalized Lebesgue measure it has a Radon-Nikodým derivative g^F_T such that $p^F_T(X) = (2\pi)^{-1} \int_X g^F_T(\theta) \mathrm{d}\theta$. Defining $e_n(\theta) := e^{-in\theta}$ it follows from Lemma 1 that

$$g^F_T = \lim_{s,t,u,v\to\infty} \sum_{n=-s}^{t} \sum_{m=-u}^{v} T_{m,n} c_{n,m} \overline{e_n} e_m$$

with respect to the L^1-norm (for more details, see Section 5 of Paper VII). This implies the following theorem:

Theorem 7 *There exists an increasing subsequence $\mathbb{N} \ni k \mapsto n_k \in \mathbb{N}$ such that*

$$g^F_T(\theta) = \lim_{k\to\infty} \sum_{n,m=-n_k}^{n_k} T_{m,n} c_{n,m} e^{i(n-m)\theta}$$

for almost all $\theta \in \mathbb{R}$.

Also, one can write (see Section 5 of Paper VII)

$$g^F_T(\theta) = \lim_{\epsilon\to 0+} \sum_{n,m\in J} T_{m,n} c_{n,m} e^{i(n-m)\theta} f^{(j)}_{n-m}(\epsilon), \quad j = 1, 2,$$

for almost all $\theta \in \mathbb{R}$ where for all $k \in \mathbb{Z}$, $f^{(1)}_k(\epsilon) := \left(e^{ik\epsilon} - 1\right)/(ik\epsilon)$, $k \neq 0$, $f^{(1)}_0(\epsilon) = 1$, and $f^{(2)}_k(\epsilon) := (1-\epsilon)^{|k|}$. Thus, for all $k \in \mathbb{Z}$, $\lim_{\epsilon\to 0+} f^{(j)}_k(\epsilon) = 1$ where $j = 1, 2$. Using bounded operators

$$C^{(j)}_\epsilon := \text{w-}\sum_{n,m\in J} c_{n,m} f^{(j)}_{n-m}(\epsilon) |n\rangle\langle m|, \quad j = 1, 2,$$

it follows that

$$g_T^F(\theta) = \lim_{\epsilon \to 0+} \mathrm{tr}\left[TR(\theta)C_\epsilon^{(j)}R(\theta)^*\right], \quad j = 1, 2,$$

for almost all $\theta \in \mathbb{R}$.

To conclude, we may formally write (e.g. in the sense of Theorem 7) that

$$g_T^F(\theta) = \sum_{n,m \in J} T_{m,n} c_{n,m} e^{i(n-m)\theta}.$$

3.6 Moment and cyclic moment forms of covariant sesquilinear form measures

Let H be a Hilbert space, and let K be a linear subspace of H. Let $\mathcal{B}(\Omega)$ be the Borel σ-algebra of a Borel set $\Omega \subseteq \mathbb{C}$, and let $G : \mathcal{B}(\Omega) \to \mathcal{SL}(K, K; \mathbb{C})$ be a sesquilinear form measure. For any Borel function $f : \Omega \to \mathbb{C}$ define

$$K_f^G := \left\{ \psi \in K \,\middle|\, \int_\Omega f(\theta) \mathrm{d}[G(\theta)](\varphi, \psi) \text{ exists for all } \varphi \in K \right\}.$$

Due to the sesquilinearity, K_f^G is a linear space, and we may define the following sesquilinear form

$$K \times K_f^G \ni (\varphi, \psi) \mapsto \int_\Omega f(\theta) \mathrm{d}[G(\theta)](\varphi, \psi) \in \mathbb{C}.$$

We use the symbol $\int_\Omega f(\theta)\mathrm{d}G(\theta)$ for the above form, and say that it is a *generalized operator integral* of f with respect to G.

Remark 4 Any operator measure $F : \mathcal{B}(\Omega) \to \mathcal{L}(H)$ defines an operator integral [66] of a function f as the operator $\int_\Omega f(\theta)\mathrm{d}F(\theta)$ with the domain

$$D_f^F := \left\{ \psi \in H \,\middle|\, \int_\Omega f(\theta) \mathrm{d}\langle \varphi | F(\theta)\psi \rangle \text{ exists for all } \varphi \in H \right\}$$

or, equivalently, as a sesquilinear form

$$H \times D_f^F \ni (\varphi, \psi) \mapsto \int_\Omega f(\theta) \mathrm{d}\langle \varphi | F(\theta)\psi \rangle \in \mathbb{C}. \tag{3.7}$$

Since F can be interpreted as a sesquilinear form measure $\mathcal{B}(\Omega) \to \mathcal{SL}(H, H; \mathbb{C})$ the form (3.7) is the generalized operator integral of f with respect to F.

Fix $J \subseteq \mathbb{Z}$ and let $G : \mathcal{B}([0, 2\pi)) \to \mathcal{SL}(\mathcal{M}_J, \mathcal{M}_J; \mathbb{C})$ be a covariant sesquilinear form measure with the structure matrix $(c_{n,m}) \in \mathbb{C}^{J\times J}$. Define the kth *moment form* of G as the generalized operator integral $\int_0^{2\pi} \theta^k \mathrm{d}G(\theta) \in \mathcal{SL}(\mathcal{M}_J, \mathcal{M}_J; \mathbb{C})$ where $k \in \mathbb{N}$. If G is normalized then the zeroth moment form is the restriction of the inner product to $\mathcal{M}_J \times \mathcal{M}_J$ and, thus, has a unique bounded extension to $\mathcal{H}_J \times \mathcal{H}_J$ which is given by the identity operator. Hence, it is possible that, although the sesquilinear form measure G cannot be interpreted as an operator measure, some of its moment forms can be extended to operators. Some first moment forms of covariant sesquilinear form measures have been studied in the literature (see e.g. Paper IV or s-ordered phase operators in [100, 97]). If $\int_0^{2\pi} \theta^k \mathrm{d}G(\theta)$ is bounded then it can be interpreted as a bounded operator, the kth *moment operator*,

$$\text{w-}\sum_{n,m=0}^{\infty} c_{n,m} \frac{1}{2\pi} \int_0^{2\pi} \theta^k e^{i(n-m)\theta} \mathrm{d}\theta |n\rangle\langle m|.$$

It is interesting to note that G is determined already by its first moment form since

$$c_{n,m} = i(n-m) \int_0^{2\pi} \theta \, \mathrm{d}[G(\theta)](|n\rangle \, , |m\rangle), \quad n \neq m.$$

and

$$c_{n,n} = \frac{1}{\pi} \int_0^{2\pi} \theta \, \mathrm{d}[G(\theta)](|n\rangle \, , |n\rangle).$$

Remark 5 Any bounded operator $A \in \mathcal{L}(\mathcal{H}_J)$ is the first moment operator of a unique covariant sesquilinear form measure $G_A : \mathcal{B}([0, 2\pi)) \to \mathcal{SL}(\mathcal{M}_J, \mathcal{M}_J; \mathbb{C})$ defined by the structure matrix $(a_{n,m}) \in \mathbb{C}^{J\times J}$ for which

$$a_{n,m} = i(n-m)A_{n,m}, \quad n \neq m,$$

and

$$a_{n,n} = \frac{1}{\pi} A_{n,n}.$$

For A to be the first moment operator of a covariant observable requires that $A_{n,n} = \pi$ (or an arbitrary constant, see Paper IV) and the matrix $(a_{n,m})$ is positive semidefinite.

One can also define a kth *cyclic moment form* as the operator integral $\int_0^{2\pi} e^{ik\theta} \mathrm{d}G(\theta) \in \mathcal{SL}(\mathcal{M}_J, \mathcal{M}_J; \mathbb{C})$ where now $k \in \mathbb{Z}$. It always has an (possibly unbounded) extension to $\mathcal{H}_J \times \mathcal{D}_J^k$ which can be interpreted as a kth *cyclic moment operator*

$$\sum_{\substack{n\in J \text{ and} \\ n+k\in J}} c_{n,n+k} \, |n\rangle \, \langle n+k|$$

denoted simply by $\int_0^{2\pi} e^{ik\theta}\mathrm{d}G(\theta)$ and defined on the linear space

$$\mathcal{D}_J^k := \left\{ \psi \in \mathcal{H}_J \;\middle|\; \sum_{\substack{n\in J \text{ and} \\ n+k\in J}} |c_{n,n+k}\langle n+k|\psi\rangle|^2 < \infty \right\}.$$

Any kth cyclic moment operator of G is bounded if and only if

$$\left\| \int_0^{2\pi} e^{ik\theta}\mathrm{d}G(\theta) \right\| = \sup\{|c_{n,n+k}| \,|\, n \in J \text{ and } n+k \in J\} < \infty.$$

If G is symmetric then the kth cyclic moment operators of G associated to the nonnegative integers determines G uniquely.

Chapter 4

Properties of phase observables

In the previous chapter it was shown that one can characterize covariant phase observables in various ways: using phase matrices (Theorem 3), sequences of unit vectors (Proposition 2), sequences of generalized vectors (Corollary 3), or using covariant trace-preserving operations (Theorem 5). It was also shown that there is no projection valued covariant phase observables, and the canonical phase observable was defined.

In this chapter further properties of the canonical phase observable are studied and they are listed in the final section of the chapter. The question of complementarity of number and phase and some important features of phase observables are discussed.

4.1 Strong phase observables and the semigroup of number shifts

Let $J \subseteq \mathbb{Z}$ and let $F : \mathcal{B}([0, 2\pi)) \to \mathcal{L}(\mathcal{H}_J)$ be a normalized covariant operator measure with the structure matrix $(c_{n,m}) \in \mathbb{C}^{J\times J}$. As before, adding zeros, one may assume that $J = \mathbb{Z}$ when it is notationally more convenient. Following Schreiber [103] we say that F is *strong* if for all $k \in \mathbb{N}$

$$\begin{aligned}
\int_0^{2\pi} e^{ik\theta}\mathrm{d}F(\theta) &= \left[\int_0^{2\pi} e^{i\theta}\mathrm{d}F(\theta)\right]^k, \\
\int_0^{2\pi} e^{-ik\theta}\mathrm{d}F(\theta) &= \left\{\left[\int_0^{2\pi} e^{i\theta}\mathrm{d}F(\theta)\right]^*\right\}^k.
\end{aligned}$$

Note that $A^0 := I_J$ for all $A \in \mathcal{L}(\mathcal{H}_J)$. If F is strong, then F is self-adjoint and

$$c_{n,n+k} = \prod_{l=0}^{k-1} c_{n+l,n+l+1}, \qquad n \in \mathbb{Z},\ k \in \mathbb{Z}^+.$$

In that case matrix elements ..., $c_{-2,-1}$, $c_{-1,0}$, $c_{0,1}$, $c_{1,2}$,... suffice to determine F. If $F : \mathcal{B}([0,2\pi)) \to \mathcal{L}(\mathcal{H}_J)$ is positive, then $|c_{n,n+1}| \leq 1$, $n \in \mathbb{Z}$, and the first cyclic moment operator

$$\int_0^{2\pi} e^{i\theta} \mathrm{d}F(\theta) = \sum_{n\in\mathbb{Z}} c_{n,n+1} \left|n\right\rangle \left\langle n+1\right|$$

of F is contractive. Conversely, for any contractive weighted shift operator

$$V_f := \sum_{\substack{n\in J \text{ and} \\ n+1\in J}} f_n \left|n\right\rangle \left\langle n+1\right|,$$

$f_n \in \mathbb{C}$, $0 < |f_n| \leq 1$, $n \in J$, and $n+1 \in J$ there exists a unique strong covariant observable $F_f : \mathcal{B}([0,2\pi)) \to \mathcal{L}(\mathcal{H}_J)$ such that

$$\int_0^{2\pi} e^{ik\theta} \mathrm{d}F_f(\theta) = V_f^k, \qquad k \in \mathbb{N}$$

(for the proof, see Paper II, Theorem 4.1). The following proposition is easy to prove (for more details, see Remark 2 of Paper V):

Proposition 3 *V_f is unitary if and only if $J = \mathbb{Z}$ and $|f_n| = 1$ for all $n \in \mathbb{Z}$. In this case F_f is $F^{\mathbb{Z}}_{\mathrm{can}}$ (u.e.).*

The canonical spectral measure is associated with the polar decomposition of any weighted shift operator. Indeed, let $J \subseteq \mathbb{Z}$ and $(f_n)_{n\in J} \subset \mathbb{C}\setminus\{0\}$, and define

$$V_f := \sum_{n\in J} f_n \left|n\right\rangle \left\langle n+1\right|.$$

Then $V_f = W_f|V_f|$, $W_f = \int_0^{2\pi} e^{i\theta}\mathrm{d}F_f(\theta)$, and F_f is F^J_{can} (u.e.) (see Section 4.2 of Paper II). In other words, F_f is unitarily equivalent to the projected canonical spectral measure to the subspace $\mathcal{H}_J$ of $\mathcal{H}_{\mathbb{Z}}$.

As was shown in Section 2.4 there is no self-adjoint phase operator acting on $\mathcal{H} = \mathcal{H}_{\mathbb{N}}$ associated with the polar decomposition of a. However,

$$a = \int_0^{2\pi} e^{i\theta} \mathrm{d}F^{\mathbb{Z}}_{\mathrm{can}}(\theta)\, \sqrt{N},$$

where $F_{\rm can}^{\mathbb{Z}}$ is the canonical spectral measure. But $F_{\rm can}^{\mathbb{Z}}$ acts on the larger space $\mathcal{H}_{\mathbb{Z}}$ and it cannot be regarded as a phase observable. Its projection to $\mathcal{H}$ is the canonical phase observable $E_{\rm can}$, and one may write $a = \int_0^{2\pi} e^{i\theta} {\rm d}E_{\rm can}(\theta)\sqrt{N}$.

Strong phase observables constitute an important class of phase observables since they generate the semigroup of number shifts, as we will see in the following.

Suppose first that $J = \mathbb{Z}$. The Hilbert space $\mathcal{H}_{\mathbb{Z}}$ is isomorphic to $L^2[0, 2\pi)$ and describes a "particle in a box" system where $F_{\rm can}^{\mathbb{Z}}$ is the (canonical) position observable and $N_{\mathbb{Z}}$ is the momentum operator of the particle. The covariance condition $e^{i\theta N_{\mathbb{Z}}} F_{\rm can}^{\mathbb{Z}}(X) e^{-i\theta N_{\mathbb{Z}}} = F_{\rm can}^{\mathbb{Z}}(X \dot{+} \theta)$ means that the momentum generates position shifts. Equivalently, this can be expressed in the following (generalized) Weyl form (see [59, Prop. 1] and [99]):

$$e^{i\theta N_{\mathbb{Z}}} \int_0^{2\pi} e^{inx} {\rm d}F_{\rm can}^{\mathbb{Z}}(x)\, e^{-i\theta N_{\mathbb{Z}}} = e^{-in\theta} \int_0^{2\pi} e^{inx} {\rm d}F_{\rm can}^{\mathbb{Z}}(x), \quad \theta \in [0, 2\pi),\ n \in \mathbb{Z}.$$

Any covariant observable $F : \mathcal{B}([0, 2\pi)) \to \mathcal{L}(\mathcal{H}_{\mathbb{Z}})$ satisfies the above covariance condition, or its Weyl form, and can be characterized as a position observable. Conversely, one may now ask which position observable generates momentum shifts. The momentum shifts form an additive group $\mathbb{Z}$ and, when $F : \mathcal{B}([0, 2\pi)) \to \mathcal{L}(\mathcal{H}_{\mathbb{Z}})$ is a strong covariant observable, it has the following representation on $\mathcal{H}_{\mathbb{Z}}$:

$$k \mapsto \int_0^{2\pi} e^{ik\theta} {\rm d}F(\theta).$$

The only strong covariant observable F which generates momentum shifts, that is,

$$\int_0^{2\pi} e^{-ik\theta} {\rm d}F(\theta)\, |n\rangle \langle n| \int_0^{2\pi} e^{ik\theta} {\rm d}F(\theta) = |n+k\rangle \langle n+k|, \quad n, k \in \mathbb{Z},$$

is the canonical covariant observable $F_{\rm can}^{\mathbb{Z}}$ (u.e.). The representation $k \mapsto \int_0^{2\pi} e^{ik\theta} {\rm d}F_{\rm can}^{\mathbb{Z}}(\theta)$ is now unitary. Note that $F_{\rm can}^{\mathbb{Z}}$ and $N_{\mathbb{Z}}$ also form a Heisenberg pair:

$$\left[N_{\mathbb{Z}}, \int_0^{2\pi} \theta\, {\rm d}F_{\rm can}^{\mathbb{Z}}(\theta) \right] = iI_{\mathbb{Z}}$$

in a dense domain which does not contain the eigenstates $|n\rangle$ of $N_{\mathbb{Z}}$.

In the case of phase, that is, when $J = \mathbb{N}$, one can repeat the above calculations. The covariance condition of a phase observable E equals the (generalized) Weyl commutation relation (see [59, Prop. 1] and [121, 122]):

$$e^{i\theta N}\int_0^{2\pi} e^{inx}\mathrm{d}E(x)\, e^{-i\theta N} = e^{-in\theta}\int_0^{2\pi} e^{inx}\mathrm{d}E(x), \quad \theta\in[0,2\pi),\ n\in\mathbb{N},$$

and the pair $(N,\, E_{\mathrm{can}})$ fulfills the Heisenberg commutation relation

$$\left[N, \int_0^{2\pi}\theta\,\mathrm{d}E_{\mathrm{can}}(\theta)\right] = iI$$

in a dense domain which does not contain the eigenstates $|n\rangle$ of N [41, 38, 82]. The number shifts form an additive semigroup $\mathbb{N}$ and, when E is a strong phase observable, it can be represented on $\mathcal{H}$ via $k\mapsto\int_0^{2\pi}e^{ik\theta}\mathrm{d}E(\theta)$. Similarly as in the case of $J=\mathbb{Z}$ *the only phase observable* E *which generates number shifts is the canonical phase observable* E_{can} *(u.e.).*

4.2 Commutativity of covariant operator measures

An operator measure $F:\mathcal{F}\to\mathcal{L}(H)$ is *commutative* if

$$F(X)F(Y) = F(Y)F(X)$$

for all $X,\, Y\in\mathcal{F}$. One gets from Proposition 1 of Paper V the following proposition:

Proposition 4 *Let* $F:\mathcal{B}([0,2\pi))\to\mathcal{L}(\mathcal{H}_{\mathbb{Z}})$ *be a covariant operator measure with the structure matrix* $(c_{n,m})$. F *is commutative if and only if*

$$c_{n,n+k}c_{n+k,m} = c_{n,m-k}c_{m-k,m}$$

for all $n,\ m,\ k\in\mathbb{Z}$.

This implies for $n=m$ the following necessary condition for a commutative F:

$$|c_{n,n+k}| = |c_{n-k,n}|, \quad n,\, k\in\mathbb{Z}.$$

Let $J\subseteq\mathbb{Z}$ be bounded from below or above, and let $F':\mathcal{B}([0,2\pi))\to\mathcal{L}(\mathcal{H}_J)$ be a covariant normalized operator measure. Using the above necessary condition the next proposition follows (see Proposition 2 in Paper III):

Proposition 5 F' *is commutative if and only if* $F'(X) = F^J_{\mathrm{triv}}(X) := i_0(X)\, I_J$ *for all* $X\in\mathcal{B}([0,2\pi))$.

Especially it follows that the only commutative phase observable is the *trivial phase observable* $X \mapsto E_{\mathrm{triv}}(X) := i_0(X)\, I$. It is interesting to note that the canonical phase observable E_{can} (u.e.) is totally noncommutative (see Corollary 5 in Paper III), but its extension $F^{\mathbb{Z}}_{\mathrm{can}}$ (u.e.) is commutative.

4.3 Complementarity of number and phase

Let $F : \mathcal{B}([0, 2\pi)) \to \mathcal{L}(\mathcal{H}_J)$ be a covariant observable. Since the condition $|n\rangle\langle n|\, F(X) = F(X)\, |n\rangle\langle n|$ for all $n \in J$ and $X \in \mathcal{B}([0, 2\pi))$ implies that $F = F^J_{\mathrm{triv}}$, it follows that N_J and any nontrivial $F : \mathcal{B}([0, 2\pi)) \to \mathcal{L}(\mathcal{H}_J)$ are *noncoexistent observables* (for more details, see e.g. Section II.2.2 of [16] and Section 4 of Paper III). The noncoexistence of observables implies that they cannot be measured together and they do not have a joint probability measure in all states. Especially, for a phase observable E for which $c_{n,m} \neq 0$ for all n, $m \in \mathbb{N}$ (e.g. E_{can} and $E_{|0\rangle}$) the commutativity domain of N and E is $\{0\}$ and they do not have a joint probability measure in *any* state (Proposition 7 in Paper III).

Like the concept of coexistence, the notion of complementary observables has both measurement theoretical and probabilistic aspects. For example, complementary observables do not admit any joint measurements [16]. Following [16, Section IV.2.3] we say that any observable $F : \mathcal{B}([0, 2\pi)) \to \mathcal{L}(\mathcal{H}_J)$ and N_J are *complementary* if for any finite set $\{n_1, ..., n_k\} \subseteq J$ and for any $X \in \mathcal{B}([0, 2\pi))$, for which $O \neq F(X) \neq I$, the greatest lower bound

$$\left(\sum_{l=1}^{k} |n\rangle\langle n| \right) \wedge F(X) = O.$$

The canonical spectral measure $F^{\mathbb{Z}}_{\mathrm{can}}$ (u.e.) and $N_{\mathbb{Z}}$ are complementary observables [67], which implies that they are also probabilistically complementary. We say that $F : \mathcal{B}([0, 2\pi)) \to \mathcal{L}(\mathcal{H}_J)$ and N_J are *probabilistically complementary* if

$$\begin{aligned} \sum_{i=1}^{k} |\langle \psi \,|\, n_i \rangle|^2 &= 1 \text{ implies that } 0 < \langle \psi \,|\, F(X)\psi \rangle < 1, \\ \langle \psi \,|\, F(X)\psi \rangle &= 1 \text{ implies that } 0 < \sum_{i=1}^{k} |\langle \psi \,|\, n_i \rangle|^2 < 1, \end{aligned}$$

for all $\psi \in \mathcal{H}_J$, for all nonempty sets $\{n_1, \cdots, n_k\} \subseteq J$, and for any $X \in \mathcal{B}([0, 2\pi))$ such that $O \neq F(X) \neq I$. The following proposition (Theorem 1 in Paper VIII) implies that any phase observable E and N are probabilistically complementary:

Proposition 6 *For any state $T \in \mathcal{L}(\mathcal{H})$ and a phase observable E*

$$\mathrm{tr}(TE(X)) < 1$$

for all $X \in \mathcal{B}([0, 2\pi))$ such that $i_0(X) < 1$.

Open Problem 1 *Are the canonical phase E_{can} (u.e.) (or $E_{|0\rangle}$) and the number N complementary observables?*

Remark 6 Proposition 10 of Paper III states that any phase observable E and N are also *value complementary* observables, that is, for any sequence of unit vectors $(\psi_k)_{k\in\mathbb{N}}$ for which the probability measures $X \mapsto \langle\, \psi_k \,|\, E(X)\psi_k \,\rangle$ tend (with $k \to \infty$) to a Dirac measure δ_θ, $\theta \in [0, 2\pi)$, the number probabilities $|\,\langle\, \psi_k \,|\, n \,\rangle\,|^2$ tend to zero for all $n \in \mathbb{N}$. In this case $\langle\, \psi_k \,|\, N\psi_k \,\rangle \to \infty$. This situation, where the number and, thus, the energy gets large and the phase is arbitrarily well defined, corresponds to the classical limit for a single mode field.

4.4 The phase representation

As we have seen in Section 3.3 the operator measure F^J_{can} can be written in the form

$$F^J_{\mathrm{can}}(X) = \frac{1}{2\pi}\int_X e^{i\theta N_{\mathbb{Z}}}|\phi)(\phi|e^{-i\theta N_{\mathbb{Z}}}\mathrm{d}\theta$$

where $|\phi) := \sum_{n\in J}|n\rangle \in \mathcal{H}^\infty_J$. Using the isomorphism $|n\rangle \mapsto (2\pi)^{-1/2}e_n$ one may identify $\mathcal{H}_J$ with a subspace of $L^2[0, 2\pi)$. Thus,

$$\langle\, \varphi \,|\, F^J_{\mathrm{can}}(X)\psi \,\rangle = \frac{1}{2\pi}\int_X \overline{\varphi(\theta)}\psi(\theta)\mathrm{d}\theta$$

for all φ, $\psi \in \mathcal{H}_J$ where $\psi(\theta) := \sum_{n\in J}\langle\, n \,|\, \psi \,\rangle\, e^{-in\theta}$ for dθ-almost all $\theta \in \mathbb{R}$. All other covariant observables F with $(c_{n,m})$ are of the form:

$$\langle\, \varphi \,|\, F(X)\psi \,\rangle = \sum_{n\in J}\frac{1}{2\pi}\int_X \overline{v_n\varphi(\theta)}v_n\psi(\theta)\mathrm{d}\theta$$

where v_n, $n \in J$, is a contraction (see Paper VIII). Using the kernel function $C_s(x, y) := \sum_{n,m=-s}^{s} e^{-inx}c_{n,m}e^{iny}$ one gets

$$\langle\, \varphi \,|\, F(X)\psi \,\rangle = \lim_{s\to\infty}\frac{1}{2\pi}\int_X\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2\pi}\int_0^{2\pi}\overline{\varphi(x)}C_s(x-\theta, y-\theta)\psi(y)\mathrm{d}x\mathrm{d}y\right]\mathrm{d}\theta.$$

The canonical covariant observable has the "sharpest" kernel in the formal sense that $\lim_{s\to\infty} C_s(x,y) = 2\pi\delta_{2\pi}(x)\,2\pi\delta_{2\pi}(y)$ where $\delta_{2\pi}$ is the 2π-periodic Dirac δ-distribution (see Paper VIII).

Since the momentum operator $N_{\mathbb{Z}}$ acts on $L^2[0,2\pi)$ as the derivative operator $i\partial_\theta$ and the position $F^{\mathbb{Z}}_{\rm can}$ is the canonical spectral measure $X \mapsto \chi_X$ of the multiplicative operator, we say that $\mathcal{H}_{\mathbb{Z}}$ is the momentum representation space and $L^2[0,2\pi)$ is the position representation space of the "particle in a box" system. The operator $e^{i\alpha N_{\mathbb{Z}}}$ operates as $\psi(\theta) \mapsto \psi(\theta-\alpha)$ in $L^2[0,2\pi)$, that is, the momentum generates position shifts. Similarly, restricting ourselves to the subspace $\mathcal{H} \subset \mathcal{H}_{\mathbb{Z}}$, isomorphic to the Hardy space $H^2 \subset L^2[0,2\pi)$, we say that $\mathcal{H}$ is the number representation space and H^2 is the *phase representation space* of the single-mode optical system [111, 41, 94].

4.5 Discrete phase shifts and approximation sequences for phase observables

Fix $s \in \mathbb{N}$ and define $\theta_{s,k} := 2\pi k/(s+1)$ for all $k = 0,1,...,s$. Following Paper VI or [45, 61, 22], define a group of discrete phase shifts

$$D_s := \{\theta_{s,k} \,|\, k = 0,1,...,s\}$$

with the addition modulo 2π. It has a unitary representation $\theta_{s,k} \mapsto e^{i\theta_{s,k}N_J}$ on $\mathcal{H}_J$. Let $\mathcal{P}(D_s)$ denote the power set of D_s and suppose that $\mathcal{K} \subseteq \mathcal{H}_J$ is a linear space. A sesquilinear form measure $G : \mathcal{P}(D_s) \to \mathcal{SL}(\mathcal{K},\mathcal{K};\mathbb{C})$ is *s-covariant* if $e^{i\theta_{s,k}N_J}\mathcal{K} \subseteq \mathcal{K}$ and $[G(\{\theta_{s,k}\})](e^{-i\theta_{s,l}N_J}\varphi, e^{-i\theta_{s,l}N_J}\psi) = [G(\{\theta_{s,k+l}\})](\varphi,\psi)$ for all k, $l \in \{0,1,...,s\}$ and φ, $\psi \in \mathcal{K}$. The next theorem is easy to prove:

Theorem 8 *Let $G : \mathcal{P}(D_s) \to \mathcal{SL}(\mathcal{K},\mathcal{K};\mathbb{C})$ be an s-covariant sesquilinear form measure, and let $|n\rangle$, $|m\rangle \in \mathcal{K}$ for some n, $m \in \mathbb{N}$. Then $[G(\{\theta_{s,k}\})](|n\rangle, |m\rangle) = c_{n,m}e^{i(n-m)\theta_{s,k}}$ for all $k = 0,1,...,s$ where $c_{n,m} \in \mathbb{C}$.*

If G is symmetric, then $c_{n,m} = \overline{c_{m,n}}$, and, if G is normalized and $n = m$, then $c_{n,n} = 1$. If G is positive and $L \subseteq \mathbb{Z}$ is such that $|n\rangle \in \mathcal{K}$ for all $n \in L$, then the matrix $(c_{n,m})_{n,m\in L}$ is positive semidefinite.

Let $G : \mathcal{B}([0,2\pi)) \to \mathcal{SL}(\mathcal{M}_J,\mathcal{M}_J;\mathbb{C})$ be a covariant sesquilinear form measure with the structure matrix $(c_{n,m})_{n,m\in J}$ and let $s \mapsto (c^s_{n,m})_{n,m\in J}$ be the sequence of complex matrices such that $\lim_{s\to\infty} c^s_{n,m} = c_{n,m}$ for all n, $m \in J$. For a fixed $s \in \mathbb{N}$, the matrix $(c^s_{n,m})_{n,m\in J}$ defines an s-covariant sesquilinear form measure $G_s : \mathcal{P}(D_s) \to \mathcal{SL}(\mathcal{M}_J,\mathcal{M}_J;\mathbb{C})$ for which $[G_s(\{\theta_{s,k}\})](|n\rangle, |m\rangle) = c^s_{n,m} e^{i(n-m)\theta_{s,k}}$ for all $k = 0, 1, ..., s$ and n, $m \in J$. Since, by the definition of the Riemann integral,

$$\lim_{s\to\infty} \frac{1}{s+1} \sum_{\theta_{s,k}\in[a,b)} e^{ik\theta_{s,k}} = \frac{1}{2\pi}\int_a^b e^{ik\theta}\mathrm{d}\theta, \quad 0 \le a < b \le 2\pi,$$

it follows that for all φ, $\psi \in \mathcal{M}_J$

$$[G([a,b))](\varphi,\psi) = \lim_{s\to\infty} \sum_{\substack{k\in\{0,\dots,s\}\\ \theta_{s,k}\in[a,b)}} [G_s(\{\theta_{s,k}\})](\varphi,\psi), \quad 0 \le a < b \le 2\pi.$$

In this case, we say that $(G_s)_{s\in\mathbb{N}}$ is an *approximation sequence*, or a *discretization* of G. It satisfies the (generalized) spectral accuracy condition (see [33, p. 495], [52], or Paper VI) in the sense that the "support" of G_s fills the interval $[0, 2\pi]$ when $s \to \infty$.

For any covariant sesquilinear form measure $G : \mathcal{B}([0, 2\pi)) \to \mathcal{SL}(\mathcal{M}_J, \mathcal{M}_J; \mathbb{C})$ with $(c_{n,m}) \subset \mathbb{C}^{J\times J}$ choosing $c^s_{n,m} = c_{n,m}$, $-s \le n, m \le s$, n, $m \in J$, and $c^s_{n,m} = 0$ otherwise, one gets an approximation sequence $(G_s)_{s\in\mathbb{N}}$ where every s-covariant sesquilinear form measure G_s can be interpreted as a bounded operator. Especially for any covariant observable $F : \mathcal{B}([0, 2\pi)) \to \mathcal{L}(\mathcal{H}_J)$ there exists an approximation sequence of s-covariant observables $F_s : \mathcal{P}(D_s) \to \mathcal{L}(\mathcal{H}_J)$ for which

$$F([a,b)) = \operatorname*{w-lim}_{s\to\infty} \sum_{\substack{k\in\{0,\dots,s\}\\ \theta_{s,k}\in[a,b)}} F_s(\{\theta_{s,k}\}).$$

In the case of the canonical phase observable E_{can} (u.e.), there exists an approximation sequence of spectral measures $E^s_{\mathrm{can}} : \mathcal{P}(D_s) \to \mathcal{L}(\mathcal{H}_{\{0,\dots,s\}})$ (Paper VI). This is quite striking since any E^s_{can} is commutative and E_{can} is totally noncommutative. We say that E_{can} is *almost projection valued.* Recall that E_{can} also has a commutative (projection valued) extension $F^{\mathbb{Z}}_{\mathrm{can}}$.

Remark 7 The Pegg-Barnett formalism [92, 7, 93, 94] gives the same results as the canonical phase observable [104, 44, 117]. Indeed, the Pegg-Barnett theory gives an approximation sequence for E_{can}, and the results of this section, or Paper VI, are generalizations of the Pegg-Barnett theory for arbitrary covariant sesquilinear form measures. Hall and Fuss [45] defined another discretization sequence for a phase observable dividing the phase interval $[0, 2\pi)$ into s equal bins and getting thus an s-covariant positive operator measure.

4.6 Phase space phase observables

In Section 2.6 we defined the phase space observable

$$A_{|0\rangle}(Z) = \frac{1}{\pi}\int_Z D(z)|0\rangle\langle 0|D(z)^* \mathrm{d}^2 z,$$

$D(z) := e^{za^*-\overline{z}a}$, and its covariant angle margin $E_{|0\rangle}$. These definitions can be generalized as follows (see Section 4 of Paper IV): For any trace-class operator $T \in \mathcal{L}(\mathcal{H})$, the mapping

$$A_T : \mathcal{B}(\mathbb{C}) \to \mathcal{L}(\mathcal{H}),\ Z \mapsto A_T(Z) := \frac{1}{\pi}\int_Z D(z)TD(z)^* \mathrm{d}^2 z$$

is a *phase space operator measure*. A phase space operator measure A_T is an observable (a phase space observable) if and only if T is a state. Let $E_T : \mathcal{B}([0, 2\pi)) \to \mathcal{L}(\mathcal{H})$ be an angle margin of A_T:

$$E_T(X) := \frac{1}{\pi}\int_X \int_0^\infty D\left(re^{i\theta}\right) TD\left(re^{i\theta}\right)^* r\mathrm{d}r\mathrm{d}\theta, \quad X \in \mathcal{B}([0, 2\pi)).$$

The next theorem is proved in Papers I and IV (Theorem 4.1 in both papers).

Theorem 9 *For any trace-class operator $T \in \mathcal{L}(\mathcal{H})$, the angle margin E_T is covariant if and only if*

$$T = \sum_{k=0}^\infty \lambda_k |k\rangle \langle k|$$

where $(\lambda_k)_{k\in\mathbb{N}} \subset \mathbb{C}$ and $\sum_{k=0}^\infty |\lambda_k| < \infty$. The angle margin E_T is a phase observable if and only if $T = \sum_{k=0}^\infty \lambda_k |k\rangle \langle k|$ where $(\lambda_k)_{k\in\mathbb{N}} \subset [0, \infty)$ and $\sum_{k=0}^\infty \lambda_k = 1$.

If E_T is a phase observable we say that it is a *phase space phase observable* (for more details, see e.g. [14, 15]). For any $T = |n\rangle \langle n|$, $n \in \mathbb{N}$, a simpler notation $E_{|n\rangle} := E_{|n\rangle\langle n|}$ is used. In the context of Theorem 9 one gets

$$E_T = \text{w-}\sum_{k=0}^\infty \lambda_k E_{|k\rangle}.$$

By a simple calculation, one can prove that all phase observables cannot be represented as the above convex combination of phase observables $E_{|k\rangle}$ Indeed, take E_{can} and calculate some matrix elements assuming first the above condition to get a contradiction. Choose $\lambda_k = (1-\lambda)\lambda^k$, $\lambda \in \mathbb{C}$, $|\lambda| < 1$

to get a trace-one operator $(1-\lambda)\lambda^N$ and define a covariant normalized operator measure

$$E_\lambda := E_{(1-\lambda)\lambda^N} = (1-\lambda)\sum_{k=0}^{\infty}\lambda^k E_{|k\rangle}$$

These covariant operator measures can be seen as special cases of the Cahill-Glauber s-parametrized [17, 100] normalized covariant sesquilinear form measures (Section 5 of Paper IV)

$$\begin{aligned} &G_\lambda : \mathcal{B}([0,2\pi)) \to \mathcal{SL}(\mathcal{M}_{\mathbb{N}},\mathcal{M}_{\mathbb{N}};\mathbb{C}), \\ &X \mapsto \left[(\varphi,\psi)\mapsto \frac{1-\lambda}{\pi}\int_X\int_0^\infty\sum_{k=0}^{\infty}\lambda^k\langle\varphi|D(re^{i\theta})|k\rangle\langle k|D(re^{i\theta})^*|\psi\rangle r\mathrm{d}r\mathrm{d}\theta\right] \end{aligned}$$

where $\lambda\in\mathbb{C}$ is such that $\mathrm{Re}\,\lambda<1$. The s-parameter is defined as $s := (\lambda+1)/(\lambda-1)$. A covariant sesquilinear form measure G_λ is symmetric if and only if $\lambda\in\mathbb{R}$ (and $\lambda<1$), so we always assume that λ is a real number less than one (Proposition 5.1 in Paper IV). If $0\le\lambda<1$ then G_λ defines a phase observable E_λ (see Proposition 5.2 in Paper IV or [100]).

Finally, note that any $E_{|n\rangle}$ can be seen as a noisy version of E_{can} in the sense that $E_{|n\rangle}(X) = \int_0^\infty T_n(r)E_{\mathrm{can}}(X)T_n(r)^*\mathrm{d}r^2$ (weakly) where $T_n(r)\in\mathcal{L}(\mathcal{H})$ for all $n\in\mathbb{N}$ and $r\in[0,\infty)$ [15].

4.7 Measures of uncertainty of periodic probability densities and coherent state phase measurements

Let $f:\mathbb{R}\to[0,\infty]$ be a 2π-periodic probability density, that is, $f(\theta+2\pi)=f(\theta)$ for (almost) all $\theta\in\mathbb{R}$ and $(2\pi)^{-1}\int_0^{2\pi}f(\theta)\mathrm{d}\theta = 1$. There are many suggestions in the literature [13, 12, 46, 97] for the measure of uncertainty and the mean value of f. Two of these measures are studied here.

Since f can be understood as a probability measure on the unit circle $\mathbb{T}$ via the mapping $\theta\mapsto e^{i\theta}$, a natural choice for the measure of uncertainty is the *cyclic variance* or angle variance [6, 75, 57, 13]

$$\mathrm{Cvar}(f) := \inf_{v\in\mathbb{T}}\int_{\mathbb{T}}|z-v|^2 f(\arg z)\mathrm{d}\lambda(z)$$

where λ is the normalized Lebesgue measure transferred by the map $\theta\mapsto e^{i\theta}$. In addition, there is a simple mechanical analogue for cyclic variance and mean where the cyclic mean value corresponds to the centre of mass of a mass-distribution on a circle [88].

For any phase observable E with $(c_{n,m})$ and a state T one gets a phase probability density g_T^E of $X \mapsto \mathrm{tr}(TE(X))$, and thus the cyclic variance

$$\mathrm{Cvar}(g_T^E) = 1 - \left| \sum_{n=0}^{\infty} T_{n+1,n} c_{n,n+1} \right|^2 .$$

This measure depends only on the second diagonal $(c_{n,n+1})_{n\in\mathbb{N}}$ of the phase matrix, so it does not describe the uncertainty of g_T^E well. Indeed, for any phase observable E which is not strong (e.g. $E_{|0\rangle}$) there exists a strong phase observable which has the same second diagonal, and thus the same cyclic variances, as E. Also, cyclic variance gives the intuitively wrong expectation value (see Ref. 16 of Paper II). However, for any vector state ψ, E_{can} (u.e.) *minimizes the cyclic variance* of $g^E_{|\psi\rangle\langle\psi|}$ (see e.g. [58] or Section 6.2 of Paper II).

One problem of using variance as a measure of the uncertainty of f is how to choose an integration interval. Namely, one can define a set of *α-shifted variances*, which are not equal in nontrivial cases, as follows:

$$\mathrm{Var}_\alpha(f) := \inf_{\beta\in[\alpha-\pi,\alpha+\pi)} \frac{1}{2\pi} \int_{\alpha-\pi}^{\alpha+\pi} (\theta-\beta)^2 f(\theta)\mathrm{d}\theta, \qquad \alpha \in \mathbb{R}.$$

This leads us to define the *minimum variance* or the Lévy measure of f [74, 63, 64, 13] as the infimum of α-shifted variances:

$$\mathrm{VAR}(f) := \inf_{\alpha\in\mathbb{R}} \mathrm{Var}_\alpha(f) = \min_{\alpha\in[0,2\pi)} \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta^2 f(\theta+\alpha)\mathrm{d}\theta.$$

The minimum variance is an intuitively right measure of uncertainty but it is very complicated algebraically when the shape of f is not "simple", e.g., a Gaussian curve. Naturally, numerical results can be obtained. Next, different phase observables are compared using the minimum variance in coherent states. The task is to show that the canonical phase observable gives smaller minimum variance in coherent states than measured phase observales.

Remark 8 Recall that in Section 2.6 it was mentioned that a phase space phase observable $E_{|0\rangle}$ is measured by using an eight-port homodyne detection scheme, or equivalently, a joint measurement of quadrature components. Note that $E_{|0\rangle}$ can also be measured by using a heterodyne detection scheme [131, 105, 45] or by using linear amplification [6, 90, 42, 101]. There are also measurement schemes for other phase space observables $E_{|k\rangle}$, $k \in \mathbb{N}$ [14].

Indeed, Leonhardt and Paul [71] have shown that we do not actually measure $E_{|0\rangle}$ in realistic devices (fully efficient detectors are not involved), but unsharp $E_{|0\rangle}$, $E_\lambda \approx E_{|0\rangle} + \lambda E_{|1\rangle} + ...$, where λ is a small positive constant (see also [21]). Other phase observables which can be measured (via adaptive single-shot phase measurements) are suggested by Wiseman and Killip [126, 127, 128] (see also [25]). Note that there is no realistic measurement for the canonical phase observable, although a proposal to that aim has been made by Pegg and Barnett [10, 95]. However, the canonical phase distribution of the signal state can be *sampled* [118, 109, 11, 107, 108, 69, 20, 91, 34], for example, by using optical homodyne tomography (OHT) since using OHT one can determine the signal state and thus all phase distributions (for an overview of phase measurements, see Section III of [81]).

Open Problem 2 *Is there a realistic measurement for the canonical phase observable?*

Let $g^E_{|z\rangle}$ be the probability density of the probability measure $X \mapsto \langle z|E(X)|z\rangle$ where E is a phase observable and $|z\rangle$, $z \in \mathbb{C}$, a coherent state. The probabilty density of the canonical phase in that state has the highest peak, that is,

$$g^E_{|z\rangle}(\arg z) < g^{E_{\rm can}}_{|z\rangle}(\arg z)$$

for all phase observables $E \neq E_{\rm can}$. As noted earlier, the canonical phase also minimizes the cyclic variance of phase observables in a coherent state. It would be interesting to get a similiar result for the canonical phase observable using minimum variance instead of cyclic variance. This seems to be a hard task due to the algebraic complexity of minimum variance.

Open Problem 3 *Does the canonical phase observable minimize minimum variance in an arbitrary state?*

However, one has some asymptotic results for measured phase observables $E_{|0\rangle}$ and E_λ, and for $E_{\rm can}$: for large $|z|$,

$$\begin{aligned}
\mathrm{VAR}\left(g^{E_{\rm can}}_{|z\rangle}\right) &\sim \frac{1}{4|z|^2}, \\
\mathrm{VAR}\left(g^{E_{|k\rangle}}_{|z\rangle}\right) &\sim \frac{k+1}{2|z|^2} > \mathrm{VAR}\left(g^{E_{\rm can}}_{|z\rangle}\right), \quad k \in \mathbb{N}, \\
\mathrm{VAR}\left(g^{E_\lambda}_{|z\rangle}\right) &\sim \frac{1}{2(1-\lambda)|z|^2} > \mathrm{VAR}\left(g^{E_{|0\rangle}}_{|z\rangle}\right) > \mathrm{VAR}\left(g^{E_{\rm can}}_{|z\rangle}\right), \quad 0 < \lambda < 1.
\end{aligned}$$

(see Papers II and IV). Note that the variance of the number operator in a coherent state $|z\rangle$ is $|z|^2$, so that the above results give the uncertainty relations of some phase observables E and N in $|z\rangle$ of the form

$$\mathrm{VAR}\left(g^E_{|z\rangle}\right)|z|^2 \sim constant.$$

If one gives up the positivity of E and studies Cahill-Glauber covariant sesquilinear form measures in coherent states one gets, for a fixed $z \in \mathbb{C}$, a (periodic) probability distribution of a sesquilinear form measure in $|z\rangle$ such that the corresponding minimum variance is arbitrarily small (this is the case when $\lambda \to -\infty$, see Paper IV).

In the classical limit $|z| \to \infty$, where the mean value of the number tends to infinity, $g^E_{|z\rangle}$ tends to the 2π-periodic Dirac δ-distribution concentrated on $\arg z$ if and only if $\lim_{n\to\infty} c_{n,n+k} = 1$ for all $k \in \mathbb{N}$ (Theorem 7.1 in Paper II). This condition holds, e.g., for E_{can}, $E_{|k\rangle}$, $k \in \mathbb{N}$, and E_λ, $0 \le \lambda < 1$, and we say that they behave well in the classical limit.

We started our covariant approach to the quantum phase problem discussing coherent state phase (parameter) measurements in Section 2.6. Then we defined the canonical phase observable and studied its properties. Finally, in this section, we saw that the canonical phase observable gives the most accurate phase distribution in coherent state phase measurements comparing it to the measured phase observables.

4.8 Properties of the canonical phase observable

The main properties of the canonical phase observable E_{can} are collected in the following:

1. E_{can} (u.e.) is the only phase observable which generates number shifts (Paper II);

2. E_{can} is a unique phase observable associated to the polar decomposition of the lowering operator and other unilateral shift operators (Paper II);

3. the phase matrix of E_{can} is the indentity of the structure matrix W*-algebra and the upper bound of the partially ordered set (of equivalence classes) of phase matrices (Paper VII);

4. the phase matrix of E_{can} (u.e.) is an extreme point of the convex set of phase matrices (Paper III);

5. E_{can} (u.e.) is the only phase observable which is defined by a single generalized vector (Paper VI);

6. all phase observables are of the form $\Theta^*(E_{\rm can})$, where Θ is a covariant trace-preserving operation (Paper VII);
7. the operation Θ_1^+ associated to a phase observable E is a surjection (and a bijection) if and only if $E = E_{\rm can}$ (u.e.) (Paper VII);
8. Θ_1^+ is pure if and only if $E = E_{\rm can}$ (u.e.) (Paper VII);
9. $E_{\rm can}$ (u.e.) is strong (Paper II);
10. $E_{\rm can}$ (u.e.) and N form a Heisenberg pair (Paper I);
11. $E_{\rm can}$ (u.e.) is totally noncommutative (Paper III);
12. $E_{\rm can}$ (u.e.) and N are noncoexistent, probabilistically and value complementary observables (Paper III);
13. $E_{\rm can}$ (u.e.) is the only phase observable which has a projection valued covariant extension to $\mathcal{H}_{\mathbb{Z}}$. The extension is unitarily equivalent to the canonical spectral measure of $L^2[0, 2\pi)$ (Paper V).
14. $E_{\rm can}$ (u.e.) has a projection valued approximation sequence (Paper VI);
15. $E_{\rm can}$ has the "sharpest" kernel (Paper VIII);
16. $E_{\rm can}$ (u.e.) minimizes the cyclic variance in any state (Paper II);
17. $E_{\rm can}$ gives smaller minimum variance in large amplitude coherent states than phase space phase observables (Paper II);
18. the probabilty density of $E_{\rm can}$ in $|z\rangle$ has the highest peak at $\arg z$;
19. the phase probability distribution of $E_{\rm can}$ in coherent states tends to a Dirac delta distribution in the classical limit (Paper II).

Chapter 5

Covariant phase observables and different phase theories

In addition to the covariant approach introduced in the previous chapters, there are several other phase theories considered in the literature (for an overview, see e.g. [81, 33]). The common factor in most of these theories is that they present a phase observable as a self-adjoint operator. None of those theories satisfies all the requirements needed for a satisfactory phase theory. These requirements are (see e.g. Paper VI or [94, 72, 100, 69]):

1. The Hilbert space of a phase observable is $\mathcal{H}$ spanned by $\{|n\rangle \mid n \in \mathbb{N}\}$;
2. The set of values of a phase observable is a phase interval, which can be chosen to be e.g. $[0, 2\pi)$ (see Section 2.1 of Paper IV);
3. The phase distribution in a number state $|n\rangle$ is uniform;
4. The number observable N generates phase shifts;
5. The phase observable generates number shifts.

These requirements force a phase observable to be the canonical (covariant) phase observable (u.e.) which is not a projection measure. Hence, if we want to represent a phase observable as a self-adjoint operator (or a projection measure) we have to give up some of the above requirements. In this chapter, we study some phase theories found in the literature and compare them to the covariant approach.

5.1 The first moment and cyclic moment operators of covariant sesquilinear form measures as phase observables

Let $G:\mathcal{B}([0,2\pi))\to\mathcal{SL}(\mathcal{M}_{\mathbb{N}},\mathcal{M}_{\mathbb{N}};\mathbb{C})$ be a covariant normalized sesquilinear form measure with the structure matrix $(c_{n,m})\in\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ for which $c_{n,n}=1$ for all $n\in\mathbb{N}$. Suppose that the first moment form $\int_0^{2\pi}\theta\mathrm{d}G(\theta)\in\mathcal{SL}(\mathcal{M}_{\mathbb{N}},\mathcal{M}_{\mathbb{N}};\mathbb{C})$ is symmetric and bounded and, thus, defines a bounded self-adjoint operator

$$\Phi_G := \pi\, I + \sum_{n\neq m\in\mathbb{N}} \frac{c_{n,m}}{i(n-m)}\,|n\rangle\,\langle m|\,.$$

In the literature (see e.g. [81, 100, 33]), some of this type of operators Φ_G have been suggested to be phase operators. The Hilbert space of Φ_G is adequate and the spectrum of Φ_G may be the whole phase interval $[\alpha-\pi,\alpha+\pi]$, $\alpha\in\mathbb{R}$, but the spectral measure of Φ_G can *never* be covariant with respect to phase shifts. Moreover, if the phase distribution of Φ_G in any number state $|n\rangle$, $n\in\mathbb{N}$, were uniform, then the variance

$$\mathrm{Var}\,(\Phi_G,|n\rangle)=\left\langle n\left|(\Phi_G)^2\right|n\right\rangle-\left(\langle n\left|\Phi_G\right|n\rangle\right)^2$$

should be $\pi^2/3$. In this case, since

$$\mathrm{Var}\,(\Phi_G,|n\rangle)=\sum_{\substack{m=0\\m\neq n}}^{\infty}\frac{|c_{n,m}|^2}{(n-m)^2},$$

the following equation should hold:

$$\sum_{\substack{m=0\\m\neq n}}^{\infty}\frac{|c_{n,m}|^2}{(n-m)^2}=\frac{\pi^2}{3} \tag{5.1}$$

for all $n\in\mathbb{N}$. If G is positive, that is, a phase observable, then $(c_{n,m})$ is positive semidefinite and thus $|c_{n,m}|\leq 1$. Putting this into Eq. 5.1 one sees that the probability distribution of the first moment operator of a phase observable in a number state can *never* be uniform.

Suppose that G is a phase observable. Then there are two (normalized) positive operator measures, namely G and the spectral measure of Φ_G associated to the first moment operator Φ_G. Both of these measures are determined by Φ_G *uniquely*. Thus, we can say that Φ_G is the *phase operator*, but the phase observable associated to Φ_G is G, not the spectral measure. Note that, according to Remark 5, any bounded operator $A \in \mathcal{L}(\mathcal{H})$, for which $n \mapsto \langle n|A|n\rangle$ is a constant mapping, is the first moment operator of a covariant normalized sesquilinear form measure G_A. Therefore, to call such a bounded operator A a phase operator, it is essential that G_A is positive, that is, a phase observable.

Another way to form self-adjoint operators using a covariant normalized symmetric sesquilinear form measure is to calculate the first cyclic moment form

$$V_G := \sum_{n=0}^{\infty} c_{n,n+1} |n\rangle \langle n+1|$$

and define self-adjoint sine and cosine operators

$$S_G := \frac{1}{2i}\left(V_G - V_G^*\right), \quad C_G := \frac{1}{2}\left(V_G + V_G^*\right)$$

which are bounded if and only if $\sup_{n\in\mathbb{N}}\{|c_{n,n+1}|\} < \infty$. Note that S_G and C_G are extensions of the generalized operator integrals $\int_0^{2\pi} \sin\theta \, \mathrm{d}G(\theta)$ and $\int_0^{2\pi} \cos\theta \, \mathrm{d}G(\theta)$, respectively. Although S_G and C_G would be self-adjoint bounded operators, a simple calculation shows that they never commute with each others (except in the trivial case $S_G = O = C_G$) and, thus, there are states where S_G and C_G do not have a joint probability measure. Also, the spectra of S_G and C_G may not be the whole interval $[-1, 1]$ and the joint probability distribution (if it exists) needs not be uniform in number states. If $n \mapsto c_{n,n+1}$ is an increasing sequence of positive numbers and $\lim_{n\to\infty} c_{n,n+1} = 1$, then the spectrum of S_G and C_G is the whole interval $[-1, 1]$ [73]. But in this case there exists a unique strong phase observable E_G such that $S_G = \int_0^{2\pi} \sin\theta \, \mathrm{d}E_G(\theta)$ and $C_G = \int_0^{2\pi} \cos\theta \, \mathrm{d}E_G(\theta)$ (Theorem 4.1 in Paper II).

5.1.1 The canonical phase, the Toeplitz phase operator, and sine and cosine operators

Choosing $G = E_{\mathrm{can}}$ we get the Toeplitz phase operator

$$\Phi_{E_{\mathrm{can}}} = \pi I + \sum_{n\neq m\in\mathbb{N}} \frac{1}{i(n-m)} |n\rangle \langle m|$$

suggested by Garrison and Wong [41] (see also [48, 38, 82, 98]), the sine and cosine operators $S_{E_{\mathrm{can}}}$ and $C_{E_{\mathrm{can}}}$ proposed by Susskind and Glogover

[111] (see also [18, 73, 120]), and the operator $V_{E_{\rm can}}$ suggested by Lévy-Leblond [75] (see also [73, 120]). Using the isomorphism $|n\rangle \mapsto (2\pi)^{-1/2}e_n$ the operator $\Phi_{E_{\rm can}}$ can be represented on the Hardy space H^2 as a projected multiplicative operator of $L^2[0,2\pi)$. The spectrum of $\Phi_{E_{\rm can}}$ is $[0,2\pi]$ [41, 38, 82] but, of course, $\Phi_{E_{\rm can}}$ is not phase shift covariant and it does not have uniform distribution in number states (see also [9, 39]). The spectra of $S_{E_{\rm can}}$ and $C_{E_{\rm can}}$ are $[-1,1]$ [111, 18, 73, 120], but their commutator is $(2i)^{-1}|0\rangle\langle 0|$. Moreover, their joint distribution in number states $|n\rangle$, $n \geq 1$, may not be uniform [23]. Finally, the London phase states $|\theta)$ are the generalized eigenvectors of $V_{E_{\rm can}}$ and they determine $E_{\rm can}$ (see [75] and Section 3.3).

5.1.2 The Cahill-Glauber s-quantized phase angle

The first moment operator $\Phi_{E_{|0\rangle}}$ of the phase space phase observable $E_{|0\rangle}$ is proposed to be used as the phase operator by Turski [114] and Paul [90]. This operator is sometimes called the Bargmann-Segal phase operator [33, Definition 10.4]. More generally, one may suggest that any $\Phi_{E_T} = \text{w-}\sum_{n=0}^\infty \lambda_n \Phi_{E_{|n\rangle}}$ is a phase operator where E_T is an arbitrary covariant operator measure generated by a trace-one operator $T = \sum_{n=0}^\infty \lambda_n |n\rangle\langle n|$, $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{R}$, $\sum_{n=0}^\infty |\lambda_n| < \infty$, $\sum_{n=0}^\infty \lambda_n = 1$.

Let G_λ be the Cahill-Glauber s-parametrized sesquilinear form measure (Sec. 4.6). Its first moment form $\int_0^\infty \theta \mathrm{d}G_\lambda(\theta)$ is symmetric when $\lambda \in \mathbb{R}$ and $\lambda < 1$, and it generates a bounded operator Φ_{G_λ} when $-1 \leq \lambda < 1$ (Paper IV). In this case we say that Φ_{G_λ} is the *s-quantized phase angle* or the *s-ordered phase operator* [100, 97, 33]. When $\lambda = 0$ then the s-quantized phase angle is $E_{|0\rangle}$ and we say that it is the *antinormally ordered phase operator*. The normal ordering corresponds to the case $\lambda \to -\infty$, which cannot determine any bounded, or even unbounded, operator at all. The case $\lambda = -1$ determines the *symmetrically ordered phase operator* or the *Wigner-Weyl quantized phase angle* $\Phi_{G_{\lambda=-1}}$ (for the theory of different orderings, see e.g. [17, 1, 55]).

Similarly, like the phase angle θ, the exponential phase angle $e^{i\theta}$ (or $\sin\theta$ and $\cos\theta$) can be s-quantized to get operators V_{G_λ} (or S_{G_λ} and G_{G_λ}). And of course, E_T also determines the cyclic moment operator V_{E_T} and the corresponding sine and cosine operators. All these operators Φ_{E_T}, V_{E_T}, Φ_{G_λ}, and V_{G_λ} have similiar problems (e.g. lack of covariance) as the operators $\Phi_{E_{\rm can}}$ and $V_{E_{\rm can}}$ defined in the previous section. Next we study the Wigner-Weyl quantized phase angle somewhat closer, since it has a connection to a simple laser model [32, 33].

The Wigner-Weyl quantized phase angle[1]

First we recall the properties of the Wigner-Weyl quantized phase angle $\Phi_{G_{\lambda=-1}}$ [106, 30, 52, 51, 31, 32, 33, 100]. It is a bounded self-adjoint operator determined by the generalized operator integral of a phase angle with respect to a covariant normalized symmetric sesquilinear form measure $G_{\lambda=-1}$ which is not positive (see Paper IV or [29]). It is not known if $G_{\lambda=-1}$ is bounded and, thus, if it determines an operator measure. In fact, if X is a finite union of subintervals of $[0, 2\pi)$, then $G_{\lambda=-1}(X)$ is a bounded form and thus can be regarded as a bounded operator (see [124, Prop. 2] or [33, Sec. 9.4.5]); however, if $G_{\lambda=-1}(X)$ were a bounded form (or a bounded operator) for all $X \in \mathcal{B}([0, 2\pi))$ then for all $R > 0$ there would exist such a (Borel) set $X_R \subseteq [0, 2\pi)$ that $\|G_{\lambda=-1}(X_R)\| > R$ [124, Prop. 3]. Numerical calculations [30] suggest that the spectrum of $\Phi_{G_{\lambda=-1}}$ is $[0, 2\pi]$, but this is not rigorously proved [33, p. 304]. In any case it does not satisfy the covariance and the uniform number distribution conditions [33, p. 458].

The operator $\Phi_{G_{\lambda=-1}}$ arises from the theoretical analysis of the Dicke laser model in the following way [32, 33]. The Dicke laser model consists of $\mathbf{N} \in \mathbb{N}$ non-interacting two-state atoms, each interacting with a single mode optical field (see Section 2.3) via the electric dipole coupling. The lowering and raising operators a and a^* of the optical field are scaled as $a/\sqrt{\mathbf{N}}$ and $a^*/\sqrt{\mathbf{N}}$ to get a scaled energy operator, or the energy density operator, $\hbar\omega a^*a/\mathbf{N}$. The scaled state T of the optical field is defined as

$$T^{(\mathbf{N};\alpha)} := D\left(\sqrt{\mathbf{N}}\alpha\right) T D\left(\sqrt{\mathbf{N}}\alpha\right)^*, \quad \alpha \in \mathbb{C}.$$

(Note that Dubin, Hennings, and Smith use an operator $W(z) := D(-i\overline{z})$ instead of $D(z)$ [32, 33].) The expectation value of the scaled displacement operator $D\left(z/\sqrt{\mathbf{N}}\right) = e^{za^*/\sqrt{\mathbf{N}} - \overline{z}a/\sqrt{\mathbf{N}}}$ in the scaled state $T^{(\mathbf{N};\alpha)}$ is

$$\mathrm{tr}\left(T^{(\mathbf{N};\alpha)} D\left(z/\sqrt{\mathbf{N}}\right)\right) = e^{\overline{\alpha}z - \alpha\overline{z}} \mathrm{tr}\left(T D\left(z/\sqrt{\mathbf{N}}\right)\right) \to e^{\overline{\alpha}z - \alpha\overline{z}}, \quad \mathbf{N} \to \infty.$$

In the following calculations, "operators" must be undestood as sesquilinear forms defined on a suitable dense subset of $\mathcal{H}$ such that the integrals exist in the weak sense. Since

$$\Phi_{G_{\lambda=-1}} := \int_0^{2\pi} \theta \mathrm{d}G_{\lambda=-1}(\theta) = \frac{1}{\pi}\int_{\mathbb{C}} \left(\frac{1}{\pi}\int_{\mathbb{C}} \arg\xi \, e^{\xi\overline{z} - \overline{\xi}z} \, \mathrm{d}^2\xi\right) D(z) \mathrm{d}^2 z$$

(see e.g. Paper IV) defining the scaled Wigner-Weyl quantized phase angle

$$\Phi^{(\mathbf{N})}_{G_{\lambda=-1}} := \frac{1}{\pi}\int_{\mathbb{C}} \left(\frac{1}{\pi}\int_{\mathbb{C}} \arg\xi \, e^{\xi\overline{z} - \overline{\xi}z} \, \mathrm{d}^2\xi\right) D\left(z/\sqrt{\mathbf{N}}\right) \mathrm{d}^2 z$$

[1] The material of this subsection is a part of a work in progress.

one gets, when $\mathbf{N} \to \infty$,

$$\operatorname{tr}\left(T^{(\mathbf{N};\alpha)}\Phi^{(\mathbf{N})}_{G_{\lambda=-1}}\right) \to \frac{1}{\pi}\int_{\mathbb{C}}\left(\frac{1}{\pi}\int_{\mathbb{C}} \arg\xi\, e^{\xi\overline{z}-\overline{\xi}z}\,\mathrm{d}^2\xi\right) e^{\overline{\alpha}z-\alpha\overline{z}}\mathrm{d}^2 z = \arg\alpha$$

(see e.g. [32]). Thus, the expectation value of the scaled Wigner-Weyl quantized phase operator in any scaled state tends to a classical phase parameter $\arg\alpha$ in the limit $\mathbf{N} \to \infty$. This observation seems to support $\Phi_{G_{\lambda=-1}}$ to be the "right" phase operator [32, 33]. But there exist other scaled phase operators which give similar results. Namely, all Cahill-Glauber s-quantized phase angles, which can be regarded as bounded self-adjoint operators when $-1 \leq \lambda < 1$, are such operators. This is due to the fact that (see e.g. Paper IV)

$$\Phi_{G_\lambda} := \int_0^{2\pi} \theta \mathrm{d}G_\lambda(\theta) = \frac{1}{\pi}\int_{\mathbb{C}}\left(\frac{1}{\pi}\int_{\mathbb{C}} \arg\xi\, e^{\xi\overline{z}-\overline{\xi}z}\,\mathrm{d}^2\xi\right) D_\lambda(z)\mathrm{d}^2 z$$

where $D_\lambda(z) := e^{(\lambda+1)|z|^2/[2(\lambda-1)]}D(z)$, and defining

$$\Phi^{(\mathbf{N})}_{G_\lambda} := \frac{1}{\pi}\int_{\mathbb{C}}\left(\frac{1}{\pi}\int_{\mathbb{C}} \arg\xi\, e^{\xi\overline{z}-\overline{\xi}z}\,\mathrm{d}^2\xi\right) D_\lambda(z/\sqrt{\mathbf{N}})\mathrm{d}^2 z$$

one gets

$$\lim_{\mathbf{N}\to\infty} \operatorname{tr}\left(T^{(\mathbf{N};\alpha)}\Phi^{(\mathbf{N})}_{G_\lambda}\right) = \arg\alpha.$$

Especially, this holds for the antinormally ordered phase operator $\Phi_{G_{\lambda=0}}$.

To conclude, the Wigner-Weyl quantized phase operator seems to have no special role in the Dicke laser model. The underlining of the Wigner-Weyl quantized phase operator in that model is due to a special scaling. Actually, we must scale the operator $D_\lambda(z)$ in the above way, not only scale $D(z)$ neglecting the factor $e^{(\lambda+1)|z|^2/[2(\lambda-1)]}$ in the definition of $D_\lambda(z)$. If we neglect the factor, then the results in the classical limit are "good" only for the Wigner-Weyl quantized phase operator. A fact which supports our scaling is seen, for example, in the case of $\lambda = 0$ when $D_{\lambda=0}(z) = e^{-|z|^2/2}D(z) = e^{-\overline{z}a}e^{za^*}$ is antinormally ordered: just replace operators a and a^* with $a/\sqrt{\mathbf{N}}$ and $a^*/\sqrt{\mathbf{N}}$ in $e^{-\overline{z}a}e^{za^*}$ to get a scaled operator $D_{\lambda=0}(z/\sqrt{\mathbf{N}})$ which is antinormally ordered. Next we show how the angle margin of the Q-function is related to the Dicke laser model.

If the angle characteristic function $\xi \mapsto \chi_{[0,x)}(\arg\xi)$, $x \in (0, 2\pi]$, is quantized instead of $\xi \mapsto \arg\xi$ one gets, in the case $|\lambda| < 1$, an operator $E_\lambda([0,x))$ of the covariant operator measure E_λ defined in Section 4.6. Since $E_\lambda([0,x))$ is defined by the bounded sesquilinear form

$$G_\lambda([0,x)) = \frac{1}{\pi}\int_{\mathbb{C}}\left(\frac{1}{\pi}\int_{\mathbb{C}} \chi_{[0,x)}(\arg\xi)\, e^{\xi\overline{z}-\overline{\xi}z}\,\mathrm{d}^2\xi\right) D_\lambda(z)\mathrm{d}^2 z$$

defining

$$G_\lambda^{(\mathbf{N})}([0,x)) := \frac{1}{\pi}\int_{\mathbb{C}}\left(\frac{1}{\pi}\int_{\mathbb{C}}\chi_{[0,x)}(\arg\xi)\,e^{\xi\bar{z}-\bar{\xi}z}\,\mathrm{d}^2\xi\right)D_\lambda(z/\sqrt{\mathbf{N}})\mathrm{d}^2 z$$

and its extension to a bounded operator $E_\lambda^{(\mathbf{N})}([0,x))$ one gets

$$\lim_{\mathbf{N}\to\infty}\operatorname{tr}\left(T^{(\mathbf{N};\alpha)}E_\lambda^{(\mathbf{N})}([0,x))\right) = \delta_{\arg\alpha}([0,x))$$

where δ_p is the Dirac measure concentrated on p. Thus, we conclude that the Cahill-Glauber s-parametrized operator measures E_λ are related to the Dicke laser model in a similar manner as their first moments. In particular, the measured phase space phase observable $E_{\lambda=0} = E_{|0\rangle}$ and (its "fuzzy" versions) E_λ, $\lambda\in(0,1)$, $\lambda\approx 0$, arise from the Dicke laser model in the above way. Note that we can also scale the phase space observables $A_\lambda := A_{(1-\lambda)\lambda^N}$, $|\lambda|<1$, (see Section 4.6) in a similar way to get scaled operators $A_\lambda^{(\mathbf{N})}([x_1,x_2]\times[y_1,y_2])$, $x_1<x_2$, $y_1<y_2$, which behave well in the classical limit $\mathbf{N}\to\infty$:

$$\lim_{\mathbf{N}\to\infty}\operatorname{tr}\left(T^{(\mathbf{N};\alpha)}A_\lambda^{(\mathbf{N})}([x_1,x_2]\times[y_1,y_2])\right) = \delta_\alpha([x_1,x_2]\times[y_1,y_2]).$$

5.2 Phase distributions

Due to the difficulty in finding a satisfactory self-adjoint phase operator acting on $\mathcal{H}$, alternative phase theories have been developed. Since one knows that the Q-function of a signal state can be measured, it would be convenient to include this observation in the alternative phase theory. One way to proceed is to consider the Q-function as a mapping from the set of states (or unit vectors) of $\mathcal{H}$ to the set of probability densities. The angle margin function of the Q-function then describes the phase distribution of a state in question. Other angle margins of functions from the set of states to quasi probability densities (they may get negative values) have also been considered (for an overview, see e.g. [112, 81, 100]). We collect here some (quasi) phase distributions suggested in the literature and show how they arise from covariant sesquilinear form measures.

Suppose that one makes a set of phase measurements after preparing the signal state to be a unit vector $\psi\in\mathcal{H}$. Then one measures the phase of the single mode system and gets a 2π-periodic phase probability density $f_\psi:\mathbb{R}\to[0,\infty]$ for which $(2\pi)^{-1}\int_0^{2\pi}f_\psi(\theta)\mathrm{d}\theta=1$. Assuming the existence of interference terms, $\psi\mapsto f_\psi$ can be extended to a sesquilinear mapping (see Paper VIII). Then, adopting the natural covariance condition $f_{R(-\alpha)\psi}(\theta) =$

$f(\theta+\alpha)$, the mapping $\psi \mapsto f_\psi$ can be represented using a unique phase observable E in the form $f_\psi(\theta) = g_\psi^E(\theta)$ (the Radon-Nikodým derivative of the probability measure $X \mapsto \langle\psi|E(X)\psi\rangle$) (Paper VIII).

To descibe some other quasi phase distributions using covariant sesquilinear form measures as above (using positive operator measures for covariant phase distributions) we generalize the mapping $\psi \mapsto f_\psi$ in the following way. Let $\mathcal{K}$ be a linear dense subspace of $\mathcal{H}$ for which $R(\alpha)\mathcal{K} \subseteq \mathcal{K}$ for all $\alpha \in [0, 2\pi)$. Suppose that for all $\varphi, \psi \in \mathcal{K}$ there is an integrable (over $[0, 2\pi)$) 2π-periodic function $g_{\varphi,\psi} : \mathbb{R} \to \mathbb{C}$ such that the mapping $(\varphi, \psi) \mapsto g_{\varphi,\psi}$ is sesquilinear and covariant in the sense that

$$g_{R(-\alpha)\varphi,R(-\alpha)\psi}(\theta) = g_{\varphi,\psi}(\theta + \alpha).$$

We say that $(\varphi, \psi) \mapsto g_{\varphi,\psi}$ is a *generalized phase distribution*. Note that we do not assume the positivity of $g_{\psi,\psi}$. It is easy to show that for any generalized phase distribution for which $\mathcal{M}_\mathbb{N} \subseteq \mathcal{K}$ there exists a unique covariant sesquilinear form measure $G : \mathcal{B}([0, 2\pi)) \to \mathcal{SL}(\mathcal{M}_\mathbb{N}, \mathcal{M}_\mathbb{N}; \mathbb{C})$ with $(c_{n,m}) \in \mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ such that $g_{\varphi,\psi}(\theta) = \sum_{n,m=0}^\infty c_{n,m} e^{i(n-m)\theta}\langle\varphi|n\rangle\langle m|\psi\rangle$ for all φ, $\psi \in \mathcal{M}_\mathbb{N}$ and for $\mathrm{d}\theta$-almost all $\theta \in \mathbb{R}$. Most of the quasi phase distributions arising from the literature satisfy the covariance condition and thus can be represented using covariant sesquilinear form measures or, equivalently, using the structure constants $c_{n,m}$. We have collected some of them in the following list. Suppose next that $\psi \in \mathcal{H}$ is such that the corresponding integral exists.

1. The London phase distribution [76] or the Pegg-Barnett phase distribution [94]
$$\theta \mapsto \left|\sum_{n=0}^\infty \langle n|\psi\rangle e^{-in\theta}\right|^2.$$
The associated positive operator measure is E_{can}.

2. The angle margin of the Cahill-Glauber s-parametrized distribution [17, 112] is associated to the sesquilinear form measure E_λ. The matrix elements $c_{n,m}$ are given in Paper IV and e.g. in [112, 100].

3. The angle margin of the Husimi Q-function [62, 37]
$$\theta \mapsto \int_0^\infty \left|\langle re^{i\theta}|\psi\rangle\right|^2 r\mathrm{d}r.$$
The associated positive operator measure is $G_{\lambda=0} \equiv E_{|0\rangle}$ with $c_{n,m} = \frac{\Gamma((n+m)/2+1)}{\sqrt{n!m!}}$ (see e.g. [90, 24]).

4. The angle margin of the Glauber-Sudarshan P-distribution [43, 110, 81] is associated to the limit of sesquilinear form measures G_λ when $\lambda \to -\infty$ (Paper IV). Note that this limit can be determined only for some particular vectors, e.g., coherent states.

5. The angle margin of the Wigner function [125, 53, 40]

$$\theta \mapsto \int_0^\infty 2 \left\langle \psi \middle| D\left(2re^{i\theta}\right) e^{i\pi N} \psi \right\rangle r \mathrm{d}r.$$

   The associated nonpositive sesquilinear form measure is $G_{\lambda=-1}$. The matrix elements are given in Paper IV and e.g. in [124, 106].

6. The quadrature-based phase distribution [118, 119, 54]

$$\theta \mapsto \frac{1}{\sqrt{2\pi}} \left| \sum_{n=0}^{\infty} \sqrt{\frac{(2n)!}{2^{2n}(n!)^2}} \langle 2n|\psi\rangle e^{-i2n\theta} \right|^2 .$$

   This distribution is positive but not normalized and it is π-periodic. The associated positive nonnormalized covariant operator measure is determined by the structure constants

$$c_{2n,2m} = \sqrt{\frac{(2n)!(2m)!}{2^{2n+2m}(n!m!)^2}}, \qquad n,\ m, \in \mathbb{N},$$

   and other $c_{n,m} = 0$. This distribution can be measured using homodyne techniques [119, 107, 108, 54].

5.3 Phase operators defined on subspaces or extensions of $\mathcal{H}$

One way to proceed seeking a self-adjoint phase operator is to change the Hilbert space of the system and define a phase operator in that new space. One can define phase operators on finite dimensional subspaces of $\mathcal{H}$ (the theories of Pegg-Barnett [94] and Popov-Yarunin [98]) and then calculate the limit when the dimension of the corresponding subspaces tends to infinity. However, these subspace methods do not give anything new to the phase problem compared to the covariant approach. The theory of Pegg and Barnett gives the same results as the covariant phase observable E_{can} (see Paper VI or [44, 117]). On the other hand, the theory of Popov and Yarunin gives the first moment operator $\int_0^{2\pi} \theta \mathrm{d}E_{\mathrm{can}}(\theta)$ of the canonical phase in the limit.

Another method is to extend $\mathcal{H}$ to find a phase operator on the new Hilbert space, and finally to project it back to $\mathcal{H}$. The first phase theory based on an extension of $\mathcal{H}$ was given by Newton [83]. The new Hilbert space suggested by Newton is isomorphic to $\mathcal{H}_{\mathbb{Z}}$, and his phase operator is $\int_0^{2\pi} \theta \mathrm{d}F^{\mathbb{Z}}_{\mathrm{can}}(\theta)$ where $F^{\mathbb{Z}}_{\mathrm{can}}$ is a spectral measure whose projection to $\mathcal{H}$ is the canonical phase observable E_{can}. Thus, Newton's theory does not give anything new compared to the covariant approach.

Ozawa [89] constructed a quite interesting extension for $\mathcal{H}$ and defined a phase operator using nonstandard analysis and the Pegg-Barnett formalism. The new Hilbert space is "very large" in the sense that it is not separable, that is, it does not have a countable basis. Also, the new Hilbert space contains nonstandard number states which are eigenvectors of an extended number operator. They correspond to the infinite eigenvalues of the extended number operator, and Ozawa describes these states as macroscopic states. However, when we project the spectral measure of Ozawa's phase operator back to $\mathcal{H}$ we get the canonical phase observable E_{can}.

Vaccaro [115] has also constructed a new Hilbert space H_{sym} whose basis contains the number states $|n\rangle$, $n \in \mathbb{N}$, and the new vectors labeled by $|\widetilde{\infty} - n\rangle$, $n \in \mathbb{N}$. It is interesting to note that these new states are formally similiar to Ozawa's infinite number states. When one projects the spectral measure of Vaccaro's phase operator back to $\mathcal{H}$ one gets again E_{can}. Note that Vaccaro and Bonner [116] have suggested a phase theory based on a vector space extension of $\mathcal{H}$ which also gives results similar to E_{can}.

5.4 Two mode theories and phase difference operators[2]

As was mentioned in Sections 2.1 and 2.2, the classical absolute phase of a single mode field cannot be measured using only one photodetector. For a measurement of an absolute phase one needs a (strong) reference field with a fixed phase (which can be chosen to be 0) and a measurement scheme for measuring the phase difference of input fields. Thus, in realistic phase measurements like heterodyne and eight-port homodyne detections, the total system contains *two* input fields.

The quantum model of eight-port homodyne detection by Noh, Fougéres,

[2]The material of this section is a part of a work in progress with T. Heinonen and P. Lahti.

and Mandel (see Section 2.6) is based on the Hilbert space $\mathcal{H}\otimes\mathcal{H}$ of two input modes. When the second input port is in a large amplitude coherent state this theory reduces to a one mode theory based on $\mathcal{H}$, and it allows one to measure the single mode phase observable $E_{|0\rangle}$ whose phase variable describes the difference of the phase of a signal field and the fixed phase parameter of the strong coherent reference field. If neither input ports is strong, then the reduction to a one mode theory cannot be done. Next we consider the most general two mode phase difference theory based on a natural covariance condition of two mode phase difference measurements.

Choose the Hilbert space of the two mode system to be $\mathcal{H}\otimes\mathcal{H}$ and define the phase difference observable as a difference of two phase observables in the following way. Indeed, suppose that $E_1 : \mathcal{B}([0,2\pi)) \to \mathcal{L}(\mathcal{H})$ and $E_2 : \mathcal{B}([0,2\pi)) \to \mathcal{L}(\mathcal{H})$ are phase observables (or E_2 is a fixed-phase observable, see Remark 9). The positive operator bimeasure

$$\mathcal{B}([0,2\pi)) \times \mathcal{B}([0,2\pi)) \ni (X,Y) \mapsto E_1(X) \otimes E_2(Y) \in \mathcal{L}(\mathcal{H}\otimes\mathcal{H})$$

can be extended to the positive operator measure $W : \mathcal{B}([0,2\pi) \times [0,2\pi)) \to \mathcal{L}(\mathcal{H}\otimes\mathcal{H})$ [129, Th. 4.5]. Using the Borel function

$$f : [0,2\pi) \times [0,2\pi) \to [0,2\pi),\ (x,y) \mapsto x-y \ (\mathrm{mod}\ 2\pi)$$

we define the *phase difference observable* of E_1 and E_2 as the observable

$$E^{\mathrm{diff}} : \mathcal{B}([0,2\pi)) \to \mathcal{L}(\mathcal{H}\otimes\mathcal{H}),\ X \mapsto W(f^{-1}(X)).$$

Actually, if $E_1 = E_2 = E_{|0\rangle}$ then E^{diff} can be measured by using two eight-port homodyne detectors which have the same strong fixed-phase laser as their reference field [113, 35, 80].

The phase difference observable E^{diff} of phase observables E_1 and E_2 behaves under rotations in the following way:

$$e^{i\alpha N\otimes I + i\beta I\otimes N} E^{\mathrm{diff}}(X) e^{-i\alpha N\otimes I - i\beta I\otimes N} = E^{\mathrm{diff}}(X \dot{+} (\alpha-\beta))$$

for all $X \in \mathcal{B}([0,2\pi))$ and $\alpha,\ \beta \in \mathbb{R}$. This is a natural covariance condition for a general phase difference observable. Thus, we define:

Definition 5 *A normalized positive operator measure* $E^{\mathrm{diff}} : \mathcal{B}([0,2\pi)) \to \mathcal{L}(\mathcal{H}\otimes\mathcal{H})$ *is a covariant phase difference observable if*

$$e^{i\alpha N\otimes I + i\beta I\otimes N} E^{\mathrm{diff}}(X) e^{-i\alpha N\otimes I - i\beta I\otimes N} = E^{\mathrm{diff}}(X \dot{+} (\alpha-\beta))$$

for all $X \in \mathcal{B}([0,2\pi))$ *and* $\alpha,\ \beta \in \mathbb{R}$.

Especially, E^{diff} is *invariant* with respect to the shifts generated by $\Sigma N := N\otimes I + I\otimes N$ and *covariant* with respect to the shifts generated by $\Delta N := (N\otimes I - I\otimes N)/2$. Indeed, since

$$e^{i\alpha N\otimes I + i\beta I\otimes N} = e^{i\alpha\Sigma N}e^{i2\alpha\Delta N}e^{i\beta\Sigma N}e^{-i2\beta\Delta N},$$

for an operator measure $E:\mathcal{B}([0,2\pi))\to\mathcal{L}(\mathcal{H}\otimes\mathcal{H})$ to be covariant in the sense of the previous definition, it suffices that E is invariant with respect to the shifts generated by ΣN and covariant with respect to the shifts generated by ΔN. The next theorem is the structure theorem for covariant phase difference operators:

Theorem 10 *Let $E^{\text{diff}}:\mathcal{B}([0,2\pi))\to\mathcal{L}(\mathcal{H}\otimes\mathcal{H})$ be a covariant phase difference observable. Then, for all $X\in\mathcal{B}([0,2\pi))$*

$$E^{\text{diff}}(X) = \text{w-}\sum_{n,m,k,l=0}^{\infty}\delta_{n-m,l-k}c_{n,m,k}i_{n-m}(X)\,|n\rangle\,\langle m|\otimes|k\rangle\,\langle l|$$

where $c_{n,m,k}\in\mathbb{C}$ and $c_{n,n,k}=1$ for all $n,\ m,\ k\in\mathbb{N}$. If E^{diff} is the phase difference observable of phase observables E_1 and E_2 with $(c^1_{n,m})$ and $(c^2_{n,m})$, respectively, then $c_{n,m,k}=c^1_{n,m}c^2_{k,k+n-m}$ for all $n,\ m,\ k\in\mathbb{N}$ for which $k\geq m-n$ holds, and $c_{n,m,k}=0$ otherwise.

Proof. Denoting $|n,k\rangle := |n\rangle\otimes|k\rangle$ for all $n,\ k\in\mathbb{N}$ and using the covariance condition one gets

$$\langle n,k|E^{\text{diff}}(X\dot{+}(\alpha-\beta))|m,l\rangle = e^{i\alpha(n-m)+i\beta(k-l)}\langle n,k|E^{\text{diff}}(X)|m,l\rangle$$

for all $n,\ k,\ m,\ l\in\mathbb{N}$, $\alpha,\ \beta\in\mathbb{R}$, and $X\in\mathcal{B}([0,2\pi))$. Choosing $\alpha=\beta$ it follows that $\langle n,k|E^{\text{diff}}(X)|m,l\rangle = 0$ if $n-m\neq l-k$, and choosing $n-m=l-k$ one can use Lemma 2 to complete the first part of the proof. The second part follows immediately. $\square$

Since $\int_0^{2\pi}e^{i\theta}\mathrm{d}E^{\text{diff}}(\theta)\,|0\rangle\otimes|0\rangle = 0$ it follows that $\int_0^{2\pi}e^{i\theta}\mathrm{d}E^{\text{diff}}(\theta)$ is not unitary and, thus, E^{diff} *is never projection valued.* If another mode is in a mixture of number states then the phase difference distribution is uniform. For example, if the first mode is in an arbitrary state T and the second mode in $|n\rangle\,\langle n|$, $n\in\mathbb{N}$, one gets

$$\text{tr}\left(T\otimes|n\rangle\,\langle n|\,E^{\text{diff}}(X)\right) = i_0(X),\quad X\in\mathcal{B}([0,2\pi)).$$

Remark 9 It is known from the theory of homodyne detection [69, 5] that when the reference mode is in a large amplitude coherent state $|z\rangle$, $|z| \gg 0$, the lowering operator a of the reference mode can be replaced with the "classical" observable $z\,I$ in calculations. This means that the energy and phase of the reference field are well known and fixed.

Let then E_1 and E_2 be phase observables with the phase matrices $(c^1_{n,m})$ and $(c^2_{n,m})$, and suppose that $\lim_{n\to\infty} c^2_{n,n+k} = 1$ for all $k \in \mathbb{N}$ (e.g. E_2 is $E_{\rm can}$ or $E_{|k\rangle}$, $k \in \mathbb{N}$). Let $E^{\rm diff}$ be the phase difference observable of E_1 and E_2. It follows from Theorem 7.1 of Paper II that for all states T of $\mathcal{H}$

$$\lim_{|z|\to\infty} \operatorname{tr}\left(T \otimes |z\rangle\langle z|\, E^{\rm diff}(X)\right) = \operatorname{tr}\left(TE^1(X\dot{+} \arg z)\right), \quad X \in \mathcal{B}([0,2\pi)).$$

This means that the theory reduces to the single-mode theory. There is another way to get the same result.

Let $\alpha \in [0,2\pi)$ and define a *fixed-phase observable*

$$F_\alpha : \mathcal{B}([0,2\pi)) \to \mathcal{L}(\mathcal{H}),\ X \mapsto \delta_\alpha(X)\, I$$

where δ_α is the Dirac measure concentrated on α. The fixed-phase observable F_α is the spectral measure of a self-adjoint operator $\alpha\, I$ and, thus, it is not a phase observable. If we choose the phase observable E_2 to be the fixed-phase observable F_α, then $E^{\rm diff}(X) = E_1(X\dot{+}\alpha) \otimes I$, that is, the phase difference $E^{\rm diff}$ and the single-mode phase E_1 are practically the same observables (up to unitary equivalence or the choice of the reference phase α).

Next we define the canonical phase difference observable and show how it is connected with other published works (for an introduction, see [130]).

When $E_1 = E_{\rm can}$ and $E_2 = E_{\rm can}$ we say that $E^{\rm diff}$ of E_1 and E_2 is the *canonical phase difference observable* and denote it by $E^{\rm diff}_{\rm can}$. Its structure constants $c_{n,m,k} = 1$ for all n, m, $k \in \mathbb{N}$. Using the phase representation (Section 4.4) we may write for any φ, $\psi \in \mathcal{H}$ that

$$\left\langle \varphi \otimes \psi \middle| E^{\rm diff}_{\rm can}(X)\varphi \otimes \psi\right\rangle = \frac{1}{2\pi}\int_X \frac{1}{2\pi}\int_0^{2\pi} |\varphi(\theta+\eta)|^2 |\psi(\theta)|^2 {\rm d}\theta{\rm d}\eta.$$

The above phase difference distribution was first suggested by Barnett and Pegg [8, 94].

The first phase difference operators studied in the literature were suggested by Sussking and Glogover [111] (see also [18, 120]). Their operators were the so-called cosine and sine phase difference operators which can be represented as $C_{12} = \int_0^{2\pi} \cos\theta\, {\rm d}E^{\rm diff}_{\rm can}(\theta)$ and $S_{12} = \int_0^{2\pi} \sin\theta\, {\rm d}E^{\rm diff}_{\rm can}(\theta)$, respectively. The operators C_{12} and S_{12} do not commute, and their spectra are the inteval $[-1,1]$, including the countable dense set of eigenvalues [111, 18, 120].

Lévy-Leblond [75] defined the relative exponential phase operator $\int_0^{2\pi} e^{i\theta}\mathrm{d}E_{\mathrm{can}}(\theta) \otimes \int_0^{2\pi} e^{-i\theta}\mathrm{d}E_{\mathrm{can}}(\theta) = \int_0^{2\pi} e^{i\theta}\mathrm{d}E^{\mathrm{diff}}_{\mathrm{can}}(\theta)$ by analogy with the classical expression $e^{i(\theta_1-\theta_2)} = e^{i\theta_1}e^{-i\theta_2}$. The operator $\int_0^{2\pi} e^{i\theta}\mathrm{d}E^{\mathrm{diff}}_{\mathrm{can}}(\theta)$ is not unitary but it is associated with the polar decompositon of $a \otimes a^*$ in the following way: using $|a \otimes a^*| = \sqrt{N \otimes (N+I)}$,

$$a \otimes a^* = \int_0^{2\pi} e^{i\theta}\mathrm{d}E^{\mathrm{diff}}_{\mathrm{can}}(\theta)\,\sqrt{N \otimes (N+I)}.$$

We can add an extra operator $T := \sum_{n=0}^{\infty} |n\rangle\langle 0| \otimes |0\rangle\langle n|$ to $\int_0^{2\pi} e^{i\theta}\mathrm{d}E^{\mathrm{diff}}_{\mathrm{can}}(\theta)$ and it still satisfies the polar decomposition relation of $a \otimes a^*$. When doing this we get a unitary operator $\mathcal{E}_{12} := \int_0^{2\pi} e^{i\theta}\mathrm{d}E^{\mathrm{diff}}_{\mathrm{can}}(\theta) + T$ and, thus, a self-adjoint operator Φ_{12} such that $\mathcal{E}_{12} = e^{i\Phi_{12}}$. Obviously, Φ_{12} is not the first moment operator of a covariant phase difference observable. Luis and Sánchez-Soto have shown [79] that the point spectrum of Φ_{12} consists of eigenvalues $\{2\pi r/(n+1) \,|\, n \in \mathbb{N},\ r = 0,1,...,n\} \subset [0,2\pi)$ and the spectrum of Φ_{12} is its closure, that is, $[0,2\pi]$. When the second mode is in a large amplitude coherent state $|z\rangle$, the spectral measure of Φ_{12} gives essentially the same results as E_{can} (or E^{diff} of E_{can} and $F_{\arg z}$) [79, 26].

There is another two mode theory which gives a phase difference operator, namely, the theory developed by Ban [2, 3, 4]: Let us define the following relative-number states of $\mathcal{H} \otimes \mathcal{H}$ for all $m \in \mathbb{N}$ and $n \in \mathbb{Z}$:

$$|n,m\rangle\rangle := \begin{cases} |m+n\rangle \otimes |m\rangle & \text{when } n \geq 0, \\ |m\rangle \otimes |m-n\rangle & \text{when } n < 0. \end{cases}$$

Thus, $\Sigma N|n,m\rangle\rangle = (2m+|n|)|n,m\rangle\rangle$, $\Delta N|n,m\rangle\rangle = n/2\,|n,m\rangle\rangle$, and $\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} |n,m\rangle\rangle\langle\langle n,m| = I$. It is easy to see that $\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} |n,m\rangle\rangle\langle\langle n+1,m|$ is a unitary operator, and thus it can be represented in the form $\int_0^{2\pi} e^{i\theta}\mathrm{d}E^{\mathrm{diff}}_{\mathrm{Ban}}(\theta)$ where

$$E^{\mathrm{diff}}_{\mathrm{Ban}}(X) := \sum_{n,m=-\infty}^{\infty} \frac{1}{2\pi}\int_X e^{i(n-m)\theta}\mathrm{d}\theta \sum_{k=0}^{\infty} |n,k\rangle\rangle\langle\langle m,k|, \qquad X \in \mathcal{B}([0,2\pi)),$$

is a spectral measure [2] and, thus, not a covariant phase difference observable. When the second mode is in the vacuum state and the first mode is in an arbitrary state T

$$\mathrm{tr}\left(T \otimes |0\rangle\langle 0|\; E^{\mathrm{diff}}_{\mathrm{Ban}}(X)\right) = \mathrm{tr}\left(T E_{\mathrm{can}}(X)\right), \qquad X \in \mathcal{B}([0,2\pi)).$$

To conclude, the canonical (single-mode) phase observable E_{can} arises from various two-mode phase difference theories. Especially, when the reference mode is in a large amplitude coherent state (with the fixed phase) or in the vacuum state (with the uniform phase), the probability measure of the two-mode observable reduces to the probability measure of E_{can} in a signal state.

5.5 Phase theories and covariant phase observables

We have collected various phase theories in the following table. The names of authors and the publication years are presented in the first two columns. The associated covariant phase observable (or the covariant sesquilinear form measure) is given in the next column. If the suggested phase operator is obtained by a (generalized) operator integral of a function, this function is shown in the final column.

Theory	*Year*	*Measure*	*Function*
Dirac [28]	1927	E_{can}	$e^{i\theta}$
Susskind, Glogover [111]	1964	E_{can}	$\sin\theta$, $\cos\theta$
Lerner, Huang, Walters [73]	1970	all phase obs.	$\sin\theta$, $\cos\theta$
Garrison, Wong [41]	1970	E_{can}	θ
Turski, Paul [114, 90]	1972	$E_{\lvert 0\rangle}$	θ
Popov, Yarunin (see Ref. in [98])	1973	E_{can}	θ
Helstrom, Holevo [49, 57]	1974	E_{can}	—
Lévy-Leblond [75]	1976	E_{can}	$e^{i\theta}$
Newton [83]	1980	E_{can}	—
Shapiro, Wagner (heterodyne det.) [105]	1984	$E_{\lvert 0\rangle}$	—
Walker, Carroll (homodyne det.) [123]	1986	$E_{\lvert 0\rangle}$	—
Pegg, Barnett [92]	1988	E_{can}	—
Ban [2]	1991	E_{can}	—
Noh, Fougeres, Mandel [84]	1991	$E_{\lvert 0\rangle}$	—
Vogel, Schleich [119]	1991	see Sec. 5.2	—
Smith, Dubin, Hennings [106]	1992	$E_{\lambda=0}$	θ
Luis, Sanchez-Soto [79]	1993	E_{can}	—
Busch, Grabowski, Lahti [14]	1994	$E_{\lvert k\rangle}$	—
Vaccaro [115]	1995	E_{can}	—
Wiseman [126]	1995	see [128]	—
Royer [100]	1996	E_{λ}	θ
Ozawa [89]	1997	E_{can}	—

Chapter 6

Summary

Covariant phase observables constitute a simple solution to the phase problem of a single-mode optical field: these observables descibe realistic coherent state phase measurements like the phase measurement scheme based on eight-port homodyne detection. Among covariant phase observables, there is no spectral measure, that is, we do not have a covariant self-adjoint phase operator. However, the first moment operator of a phase observable can be understood as a phase operator, since the corresponding phase observable is uniquely determined by its first moment operator. Covariant phase observables can be characterized at least four different ways: by using phase matrices, sequences of unit vectors, sequences of generalized vectors, or by using covariant trace-preserving operations.

The canonical phase observable has many properties which distinguish it from other phase observables. For example, the canonical phase observable generates the number shifts similarly as the number observable generates phase shifts; this suggests that the canonical phase and the number constitute a true canonical pair (as the canonical position and momentum).

Different phase theories existing in the literature seem to be connected to the covariant phase observables. In many cases the suggested single-mode phase operator is a (generalized) operator integral of some function of the phase angle with respect to a covariant phase observable (or a covariant sesquilinear form measure which has a similar structure as the covariant phase observables). Also, two mode theories seem to give covariant phase observables of the first mode when the phase of the second (reference) mode is fixed (or uniform). These notions justify the covariant approach to the quantum phase problem.

There are still some open problems left. For instance, a realistic direct measurement scheme for the canonical phase is lacking. Moreover, it is not known if the canonical phase and the number are complementary observables in the strict sense of the measurement theory. Another interesting open question is whether the canonical phase minimizes the minimum (phase) variance in any state. The characterization of the extremal elements of the convex set of phase matrices [57] is also open.

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