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QUASICONFORMAL MAPPINGS AND INEQUALITIES INVOLVING SPECIAL FUNCTIONS

by

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Abstract

This PhD thesis in Mathematics belongs to the field of Geometric Function Theory. The thesis consists of four original papers. The topic studied deals with quasiconformal mappings and their distortion theory in Euclidean n-dimensional spaces. This theory has its roots in the pioneering papers of F. W. Gehring and J. Väisälä published in the early 1960's and it has been studied by many mathematicians thereafter.

In the first paper we refine the known bounds for the so-called Mori constant and also estimate the distortion in the hyperbolic metric.

The second paper deals with radial functions which are simple examples of quasiconformal mappings. These radial functions lead us to the study of the so-called p-angular distance which has been studied recently e.g. by L. Maligranda and S. Dragomir.

In the third paper we study a class of functions of a real variable studied by P. Lindqvist in an influential paper. This leads one to study parametrized analogues of classical trigonometric and hyperbolic functions which for the parameter value $p = 2$ coincide with the classical functions. Gaussian hypergeometric functions have an important role in the study of these special functions. Several new inequalities and identities involving p-analogues of these functions are also given.

In the fourth paper we study the generalized complete elliptic integrals, modular functions and some related functions. We find the upper and lower bounds of these functions, and those bounds are given in a simple form. This theory has a long history which goes back two centuries and includes names such as A. M. Legendre, C. Jacobi, C. F. Gauss. Modular functions also occur in the study of quasiconformal mappings.

Conformal invariants, such as the modulus of a curve family, are often applied in quasiconformal mapping theory. The invariants can be sometimes expressed in terms of special conformal mappings. This fact explains why special functions often occur in this theory.

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List of original publications

This thesis is based on the following four papers/manuscripts:

I. B. A. Bhayo and M. Vuorinen: *On Mori's theorem for quasiconformal maps in the n-space.*- Trans. Amer. Math. Soc. (to appear) 16pp. arXiv:0906.2853[math.CA].

II. B. A. Bhayo, V. Božin, D. Kalaj, and M. Vuorinen: *Norm inequalities for vector functions.*- J. Math. Anal. Appl. 380 (2011) 768–781, arXiv math 1008.4254 doi: 10.1016/j.jmaa.2011.02.029,14 pp.

III. B. A. Bhayo and M. Vuorinen: *Inequalities for eigenfunctions of the p-Laplacian.*- January 2011, 23 pp. arXiv math.CA 1101.3911.

IV. B. A. Bhayo and M. Vuorinen: *On generalized complete elliptic integrals and modular functions.*- February 2011, 18 pp., arXiv math.CA 1102.1078.

1. Introduction

Classical Analysis is a very wide area of contemporary mathematics and the topics of the papers I-IV may be specified by saying that papers I and II are motivated by geometric function theory whereas papers III and IV deal mainly with mathematical special functions.

We will now make some remarks about the history of these two topics from the point of view of this thesis and list some of the key references. A survey of the topics of geometric function theory discussed below is given in several recent papers, see e.g. F.W. Gehring [19] and M. Vuorinen [32]. The basic references are the monographs of Lehto and Virtanen [26], Väisälä [31] and Vuorinen [33]. The handbook of Kühnau [25] provides a collection surveys of dealing with geometric function theory and quasiconformal mappings in particular. For the theory of special functions our main reference is the monograph of Anderson, Vamanamurthy and Vuorinen [9] and for the most recent results the papers [8], [10].

In the early 1960's, F. W. Gehring and J. Väisälä originated the theory of quasiconformal mappings in the Euclidean n-space. Their work generalized the theory of quasiconformal mappings in the plane due to H. Grötzsch 1928, O. Teichmüller in the period 1938-44, and L. Bers, L. V. Ahlfors from the early 1950's.

The study of extremal problems of geometric function theory leads to the study of the special functions that have crucial role in the distortion theory of two-dimensional quasiconformal mappings. Conformal invariants can often be closely associated with particular conformal mappings. This leads to the connection between conformal invariants and special functions, expressed in terms of a conformal mapping of the upper half plane onto a rectangle.

Quasiconformal maps are parametrized by a number $K \geq 1$, the maximal dilatation, which roughly speaking measures how far the maps are from being conformal: conformal maps are the special case $K = 1$. Because quasiconformal maps are differentiable almost everywhere, off a set Z of a measure zero, the local behavior of the mapping at the points of Z is of particular importance. Another problem of particular importance is to study the closeness of quasiconformal maps to conformal maps. For the study of these two topics special functions have an important role as we will see in this thesis, for instance in papers I and II.

2. Mori's theorem

Many authors have proved distortion theorems for quasiconformal and quasiregular mappings in the plane or in the Euclidean n-space, which deal with the estimates for the modulus of continuity and the ways distances between points are changed under these mappings. The Hölder continuity, the counterpart of the Schwarz lemma for quasiconformal mappings and Mori's theorem are some of the important examples. A. Mori [30] gave a result, known as Mori's theorem. He showed that if f is a K-quasiconformal mapping of the unit disk $\mathbb{B}^2$ onto itself with $f(0) = 0$, then

$$|f(x) - f(y)| \leq 16|x - y|^{1/K}$$

for all $x, y \in \mathbb{B}^2$. Some weaker results of the same type had been proved earlier by L. V. Ahlfors and M. A. Lavrentieff. In the case $n = 3$ F. W. Gehring [18, Theorem 14, p.387] proved that quasiconformal mappings are Hölder-continuous. In 1988 this problem was studied by G. D. Anderson and M. K. Vamanamurthy for the higher dimensional case [6].

In the same year, R. Fehlmann and M. Vuorinen [16] studied the least constant $M(n, K)$ such that for every K-quasiconformal mapping $f : \mathbb{B}^n \to \mathbb{B}^n = f(\mathbb{B}^n)$ with $f(0) = 0$ we have for all $x, y \in \mathbb{B}^n$

$$|f(x) - f(y)| \le M(n, K)|x - y|^\alpha, \quad \alpha = K^{1/(1-n)}. \tag{2.1}$$

They also found concrete upper bounds for $M(n, K)$ and showed that $M(n, K) \to 1$ when $K \to 1$ unlike Mori's constant 16 or the constant in [6]. On the other hand as A. Mori pointed out [30], letting $K \to \infty$ we see that the constant 16 cannot be replaced by a smaller constant. P. Hästö [20] proved a counterpart of the Fehlmann-Vuorinen result for the chordal metric.

A domain D in $\overline{\mathbb{R}}^n$ is called a ring domain or, briefly, a ring, if $\overline{\mathbb{R}}^n \setminus D$ consists of two components C_0 and C_1, and it is denoted by $R(C_0, C_1)$. The Grötzsch ring $R_G(s)$, $s > 1$ is defined by

$$R_G(s) = R(\overline{\mathbb{B}}^n, [s\, e_1, \infty]), \quad s > 1\,.$$

The conformal modulus of the Grötzsch ring is denoted by

$$M_n(r) = \mathrm{mod} R_{G,n}(1/r), \quad 0 < r < 1$$

(see [9, (8.35)]). The capacity of the Grötzsch ring is denoted by γ_n [33, (5.52)]. The Grötzsch ring constant λ_n is defined by

$$\log \lambda_n = \lim_{r \to 0_+} (M_n(r) + \log r)$$

and $\lambda_n \in [4, 2e^{n-1})\,, \lambda_2 = 4$, [4], [33, p.89].

The main results of this paper are

2.2. **Theorem.** *(1) For $n \ge 2, K \ge 1$, let $M(n, K)$ be as in (2.1). Then $M(n, K) \le T(n, K)$*

$$T(n, K) = \inf\{h(t) : t \ge 1\}\,, \quad h(t) = (3 + \lambda_n^{\beta-1} t^\beta) t^{-\alpha} \lambda_n^{2(1-\alpha)},\ t \ge 1\,,$$

where $\alpha = K^{1/(1-n)} = 1/\beta$, and $\lambda_n \in [4, 2e^{n-1})\,, \lambda_2 = 4$, is the Grötzsch ring constant [4], [33, p.89].

(2) There exists a number $K_1 > 1$ such that for all $K \in (1, K_1)$ the function h has a minimum at a point t_1 with $t_1 > 1$ and

$$T(n, K) \le h(t_1) = \left[\frac{3^{1-\alpha^2} (\beta - \alpha)^{\alpha^2}}{\alpha^{\alpha^2}} \lambda_n^{\alpha - \alpha^2} + \lambda_n^{\beta - 1} \left(\frac{(3\alpha)^\alpha \lambda_n^{\alpha - 1}}{(\beta - \alpha)^\alpha} \right)^{\beta - \alpha} \right] \lambda_n^{2(1-\alpha)}\,.$$

Moreover, for $\beta \in (1, \min\{2, K_1^{1/(n-1)}\})$ *we have*

$$h(t_1) \le 3^{1-\alpha^2} 2^{5(1-\alpha)} K^5 \left(\frac{3}{2} \sqrt[4]{\beta - \alpha} + \exp(\sqrt{\beta^2 - 1}) \right).$$

In particular, $h(t_1) \to 1$ *when* $K \to 1$.

The hyperbolic metric $\rho(x, y)$, $x, y \in \mathbb{B}^n$, of the unit ball is given by (cf. [24], [33])

$$\mathrm{th}^2 \frac{\rho(x, y)}{2} = \frac{|x - y|^2}{|x - y|^2 + t^2}, \quad t^2 = (1 - |x|^2)(1 - |y|^2).$$

For $n \ge 2$ and $K > 0$, the distortion function $\varphi_{K,n} : [0, 1] \to [0, 1]$ defined by

$$\varphi_{K,n}(r) = \frac{1}{\gamma_n^{-1}(K\gamma_n((1/r)))}, \quad r \in (0, 1),$$

and $\varphi_{K,n}(0) = 0$, $\varphi_{K,n}(1) = 1$ is a homeomorphism. We denote $\varphi_{K,2} = \varphi_K$.

2.3. **Theorem.** *If* $f : \mathbb{B}^2 \to \mathbb{R}^2$ *is a non-constant* K*-quasiregular mapping with* $f\mathbb{B}^2 \subset \mathbb{B}^2$*, and* ρ *is the hyperbolic metric of* $\mathbb{B}^2$ *, then*

$$\rho(f(x), f(y)) \le c(K) \max\{\rho(x, y), \rho(x, y)^{1/K}\}$$

for all $x, y \in \mathbb{B}^2$ *where* $c(K) = 2\mathrm{arth}(\varphi_K(\mathrm{th}\frac{1}{2}))$ *and*

$$K \le u(K - 1) + 1 \le \log(\mathrm{ch}(K\mathrm{arch}(e))) \le c(K) \le v(K - 1) + K$$

with $u = \mathrm{arch}(e)\mathrm{th}(\mathrm{arch}(e)) > 1.5412$ *and* $v = \log(2(1 + \sqrt{1 - 1/e^2})) < 1.3507$*. In particular,* $c(1) = 1$.

The notation ch, th and arch, arth denote the hyperbolic cosine, tangent and their inverse functions, respectively.

Observe that both Theorems 2.2 and 2.3 are asymptotically sharp when $K \to 1$. The proof of sharpness is based on inequalities for special functions.

3. Norm inequalities

A geometric generalization of the inner product spaces was given by Fréchet [17], in 1935. It was proved by P. Jordan and J. von Neumann [23] that normed linear spaces satisfying the parallelogram law. There are interesting norm inequalities connected with characterizations of inner product spaces. In 1936, the concept of angular distance

$$\alpha(x, y) = \left| \frac{x}{|x|} - \frac{y}{|y|} \right|$$

between nonzero elements x and y in the normed space was introduced by J. A. Clarkson [13]. In 2006, L. Maligranda considered the p-angular distance

$$\alpha_p(x, y) = \left| \frac{x}{|x|}|x|^p - \frac{y}{|y|}|y|^p \right|, \ p \in \mathbb{R}$$

as a generalization of the concept of angular distance. He proved in [29, Theorem 2] the following theorem in the context of normed spaces.

3.1. **Theorem.**

$$\alpha_p(x,y) \le \begin{cases} (2-p)\dfrac{|x-y|\max\{|x|^p,|y|^p\}}{\max\{|x|,|y|\}} & \textit{if} \quad p\in(-\infty,0) \quad \textit{and} \quad x,y\neq 0;\\[2ex] (2-p)\dfrac{|x-y|}{(\max\{|x|,|y|\})^{1-p}} & \textit{if} \quad p\in[0,1] \quad \textit{and} \quad x,y\neq 0;\\[2ex] p\,(\max\{|x|,|y|\})^{p-1}|x-y| & \text{if} \quad p\in(1,\infty). \end{cases}$$

Thereafter, S. S. Dragomir [14] proved in 2009 the following upper bound for the p-angular distance for nonzero vectors x, y. Numerical tests reported in paper II show that sometimes his bounds are better than those in Theorem 3.1.

3.2. **Theorem.**

$$\alpha_p(x,y) \le \begin{cases} |x-y|(\max\{|x|,|y|\})^{p-1} + ||x|^{p-1}-|y|^{p-1}|\min\{|x|,|y|\} \\ \textit{if} \quad p\in(1,\infty)\,; \\[2ex] \dfrac{|x-y|}{(\min\{|x|,|y|\})^{1-p}} + \left||x|^{1-p}-|y|^{1-p}\right| \min\left\{\dfrac{|x|^p}{|y|^{1-p}},\dfrac{|y|^p}{|x|^{1-p}}\right\} \\ \textit{if} \quad p\in[0,1]\,; \\[2ex] \dfrac{|x-y|}{(\min\{|x|,|y|\})^{1-p}} + \dfrac{||x|^{1-p}-|y|^{1-p}|}{\max\{|x|^{-p}|y|^{1-p},|y|^{-p}|x|^{1-p}\}}, \\ \textit{if} \quad p\in(-\infty,0)\,. \end{cases}$$

Studying sharp constants connected to the p-Laplace operator J. Byström [12, Lemma 3.3] proved in 2005 the following result.

3.3. **Theorem.** *For $p\in(0,1)$ and $x,y\in\mathbb{R}^n$, we have*

$$\alpha_p(x,y)\le 2^{1-p}|x-y|^p$$

with equality for $x=-y$.

We define the function

$$\mathcal{A}_{a,b}(x) = \begin{cases} |x|^{a-1}x & \textit{if } |x|\le 1 \\ |x|^{b-1}x & \textit{if } |x|\ge 1\,, \end{cases}$$

for $a,b>0$, $x\in\mathbb{R}^n$. The following are the main results of the paper II.

3.4. **Theorem.** *Let $0<a\le 1\le b$ and*

$$C(a,b)=\sup_{|x|\le|y|} Q(x,y),$$

where

$$Q(x,y)=\frac{|\mathcal{A}(x)-\mathcal{A}(y)|}{|\mathcal{A}(x)-\mathcal{A}(z)|}\,, \quad x,y\in\mathbb{R}^n\setminus\{0\} \ \textit{with} \ x\neq y\,,$$

and

$$z = \frac{x}{|x|}(|x| + |x - y|).$$

Then

$$C(a,b) = \frac{2}{3^a - 1} \text{ and } \lim_{a \to 1} C(a,b) = 1.$$

3.5. **Theorem.** *For all $x, y \in \mathbb{R}^n$ and $p \in (0,1)$*

$$\alpha_p(x,y) \le |\mathcal{A}_{p,1/p}(x) - \mathcal{A}_{p,1/p}(y)|,$$

and furthermore, if $|x| \le |y|$, we have also

$$\alpha_p(x,y) \le |\mathcal{A}_{p,1/p}(x) - \mathcal{A}_{p,1/p}(y)| \le \frac{2}{3^p - 1}|\mathcal{A}_{p,1/p}(x) - \mathcal{A}_{p,1/p}(z)| \tag{3.6}$$

where z is as in Theorem 3.4.

Computer tests reported in paper II shows that none of the above bounds in Theorem 3.1-3.3 and 3.5 for $\alpha_p(x,y)$ is better than others. In some cases our bound (3.6) is better than the bounds in Theorems 3.1-3.3.

4. Eigenfunctions

An eigenfunction of a linear operator A, defined on some function space is any nonzero function f in that space which returns from the operator exactly as is, except for a multiplicative scaling factor. A complete set of eigenfunctions is introduced within the Riemann-Hilbert formation for spectral problems associated to some solvable nonlinear evolution equations. The eigenfunctions of one dimensional p-Laplacian operator

$$\begin{cases} -(|u'(x)|^{p-2}u'(x))' = \lambda |u(x)|^{p-2}u(x), \\ u(0) = 0, \quad u(\pi_p) = 0, \quad 0 \le x \le \pi_p \end{cases}$$

are of the form

$$\sin_p(x), \quad \sin_p(2x), \quad \sin_p(3x), \ldots,$$

where $\pi_p = 2\pi/(p\sin(\pi/p))$ and $\sin_p$ is the inverse function of $\arcsin_p$ to be defined below. In a highly cited paper P. Lindqvist [27] studied in 1995 these eigenfunctions and introduced the generalization form of the trigonometric and hyperbolic functions. With J. Peetre [28] he also studied the generalization of Euclidean distance, which is called p-distance(length). Recently P. J. Bushell and D. E. Edmunds studied these p-analogues functions and introduced many relations [11].

Given complex numbers a, b and c with $c \neq 0, -1, -2, \ldots$, the *Gaussian hypergeometric function* is the analytic continuation to the slit place $\mathbb{C} \setminus [1, \infty)$ of the series

$$F(a,b;c;z) = {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{z^n}{n!}, \qquad |z| < 1.$$

Here $(a,0)=1$ for $a\neq 0$, and (a,n) is the *shifted factorial function* or the *Appell symbol*

$$(a,n)=a(a+1)(a+2)\cdots(a+n-1)$$

for $n\in\mathbb{Z}_+$.

We consider the following homeomorphisms

$$\sin_p:(0,a_p)\to I,\ \cos_p:(0,a_p)\to I,\ \tan_p:(0,b_p)\to I,$$
$$\sinh_p:(0,c_p)\to I,\ \tanh_p:(0,\infty)\to I\,,$$

where $I=(0,1)$ and

$$a_p=\frac{\pi_p}{2},\ b_p=\frac{1}{2p}\left(\psi\left(\frac{1+p}{2p}\right)-\psi\left(\frac{1}{2p}\right)\right)=2^{-1/p}F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};\frac{1}{2}\right)\,,$$
$$c_p=\left(\frac{1}{2}\right)^{1/p}F\left(1\,,\frac{1}{p};1+\frac{1}{p}\,;\frac{1}{2}\right)\,.$$

For $x\in I$, their inverse functions are defined as

$$\begin{aligned}
\operatorname{arcsin}_p x &= \int_0^x (1-t^p)^{-1/p}dt = x\,F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};x^p\right)\\
&= x(1-x^p)^{(p-1)/p}F\left(1,1;1+\frac{1}{p};x^p\right)\,,\\
\operatorname{arctan}_p x &= \int_0^x (1+t^p)^{-1}dt = x\,F\left(1,\frac{1}{p};1+\frac{1}{p};-x^p\right)\,,\\
\operatorname{arsinh}_p x &= \int_0^x (1+t^p)^{-1/p}dt = xF\left(\frac{1}{p}\,,\frac{1}{p};1+\frac{1}{p};-x^p\right)\,,\\
\operatorname{artanh}_p x &= \int_0^x (1-t^p)^{-1}dt = xF\left(1\,,\frac{1}{p};1+\frac{1}{p};x^p\right)\,,
\end{aligned}$$

and by [11, Prop 2.2] $\operatorname{arccos}_p x=\operatorname{arcsin}_p((1-x^p)^{1/p})$. For the particular case $p=2$ one obtains the familiar elementary functions [9, 1.20].

Some of the main results of this paper read as follows

4.1. **Theorem.** *For $p>1$ and $x\in(0,1)$, we have*

(1) $\left(1+\dfrac{x^p}{p(1+p)}\right)x<\operatorname{arcsin}_p x<\dfrac{\pi_p}{2}\,x,$

(2) $\left(1+\dfrac{1-x^p}{p(1+p)}\right)(1-x^p)^{1/p}<\operatorname{arccos}_p x<\dfrac{\pi_p}{2}\,(1-x^p)^{1/p},$

(3) $\dfrac{(p(1+p)(1+x^p)+x^p)x}{p(1+p)(1+x^p)^{1+1/p}}<\operatorname{arctan}_p x<2^{1/p}\,b_p\left(\dfrac{x^p}{1+x^p}\right)^{1/p}.$

4.2. **Theorem.** *For $p>1$ and $x\in(0,1)$, we have*

(1) $z\left(1+\frac{\log(1+x^p)}{1+p}\right)<\operatorname{arsinh}_p x<z\left(1+\frac{1}{p}\log(1+x^p)\right),\ z=\left(\dfrac{x^p}{1+x^p}\right)^{1/p},$

(2) $x\left(1-\frac{1}{1+p}\log(1-x^p)\right) < \operatorname{artanh}_p x < x\left(1-\frac{1}{p}\log(1-x^p)\right)$.

5. Generalized complete elliptic integrals

In 1655, John Wallis first used the term "hypergeometric series". L. Euler studied hypergeometric series, but the first full systematic treatment was given by J. C. F. Gauss in 1813. Gauss hypergeometric function $F(a,b;c;z)$ is a special function represented by the hypergeometric series. The investigation of integral addition theorems introduced the discovery of elliptic functions. An addition theorem for a function f is a formula expressing $f(u+v)$ in terms of $f(u)$ and $f(v)$. A. M. Legendre investigated elliptic integrals, he showed that integrations of the elliptic integral $\int R(t)/\sqrt{P(t)}\,dt$, where $R(t)$ is a rational function of t and $P(t)$ is a polynomial of fourth degree, can be reduced to the integration of the three integrals

$$\int \frac{dx}{\sqrt{1-x^2}\sqrt{1-l^2x^2}},\ \int \frac{x^2\,dx}{\sqrt{1-x^2}\sqrt{1-l^2x^2}},\ \int \frac{dx}{(x-a)\sqrt{1-x^2}\sqrt{1-l^2x^2}},$$

which he called the elliptic integrals of the first, second, and third kinds, respectively.

The study of the elliptic integrals of the first kind introduces several special functions. In [5], [21], these special functions are generalized, and many results are given there. We introduce some notation here. For $0<a\le 1/2$ and $a,r\in(0,1)$, the generalized elliptic integrals are defined by

$$\mathcal{K}_a(r)=\frac{\pi}{2}\,F(a,1-a;1;r^2), \qquad \mathcal{E}_a(r)=\frac{\pi}{2}\,,$$

with $\mathcal{K}_{1/2}=\mathcal{K}$ and $\mathcal{E}_{1/2}=\mathcal{E}$. The decreasing homeomorphism $\mu_a:(0,1)\to(0,\infty)$ is defined by

$$\mu_a(r)=\frac{\pi}{2\,\sin(\pi\,a)}\frac{\mathcal{K}_a(r')}{\mathcal{K}_a(r)} \tag{5.1}$$

for $r\in(0,1)$ and $r'=\sqrt{1-r^2}$.

H. Alzer and S.-L. Qiu have given the following bounds for $\mathcal{K}$ in [3, Theorem 18]

$$\frac{\pi}{2}\left(\frac{\operatorname{artanh}(r)}{r}\right)^{3/4} < \mathcal{K}(r) < \frac{\pi}{2}\left(\frac{\operatorname{artanh}(r)}{r}\right) . \tag{5.2}$$

In the following theorem we generalize their result. For the case $a=1/2$ our upper bound is better than their bound.

5.3. **Theorem.** *For $p\ge 2$ and $r\in(0,1)$, we have*

$$\begin{aligned}\frac{\pi}{2}\left(\frac{\operatorname{artanh}_p(r)}{r}\right)^{1/2} &< \frac{\pi}{2}\left(1-\frac{p-1}{p^2}\log(1-r^2)\right)\\ &< \mathcal{K}_a(r) < \frac{\pi}{2}\left(1-\frac{2}{p\,\pi_p}\log(1-r^2)\right) ,\end{aligned}$$

where $a=1/p$ and $\pi_p=2\pi/(p\sin(\pi/p))$, see [27].

5.4. **Theorem.** *The function* $f(x) = 1/\mathcal{K}_a(1/\cosh(x))$ *is increasing and concave from* $(0,\infty)$ *onto* $(0, 2/\pi)$*. In particular,*

$$\frac{\mathcal{K}_a(r)\,\mathcal{K}_a(s)}{\mathcal{K}_a(rs/(1+r's'))} \le \mathcal{K}_a(r) + \mathcal{K}_a(s) \le \frac{2\,\mathcal{K}_a(r)\,\mathcal{K}_a(s)}{\mathcal{K}_a(\sqrt{rs/(1+rs+r's')})} \le \frac{2\,\mathcal{K}_a(r)\,\mathcal{K}_a(s)}{\mathcal{K}_a(rs)}\,,$$

for all $r, s \in (0,1)$*, with equality in the third inequality if and only if* $r = s$*.*

5.5. **Theorem.** *For* $p \ge 2$ *and* $r \in (0,1)$*, let*

$$l_p(r) = \left(\frac{\pi_p}{2}\right)^2 \left(\frac{p^2 - (p-1)\log r^2}{p\,\pi_p - 2\log r'^2}\right) \quad \textit{and} \quad u_p(r) = \left(\frac{p}{2}\right)^2 \left(\frac{p\,\pi_p - 2\log r^2}{p^2 - (p-1)\log r'^2}\right).$$

(1) The following inequalities hold

$$l_p(r) < \mu_a(r) < u_p(r)\,,$$

where $a = 1/p$*.*
(2) For $p = 2$ *we have*

$$u_2(r) < \frac{4}{\pi}\,l_2(r)\,.$$

6. Conclusions and open problems

The study of quasiconformal mappings in paper II and IV shows that conformal invariants together with special functions provide a powerful tool when examining the case when mappings have a small maximal dilatation $K > 1$. It is natural to expect that further progress is possible using this approach. This research has led to several open problems and we list here some of them.

1. What is the sharpest constant for the Theorem 2.3 [I, Theorem 1.10] in the higher dimensional case?
2. Do there exist analogues of addition formulas for the p-functions e.g. in the form of an inequality?
3. Let $l_p(r)$ and $u_p(r)$ be as in Theorem 5.5. Is it is true that $u_p(r) < (4/\pi_p) l_p(r)$? For $p = 2$ see [IV, Theorem 1.9].

Also the publications [5], [9] and IV list a few open problems.

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