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On Risk Adjusted Valuation: A Certainty Equivalent Characterization of a Class of Stochastic Control Problems

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ABSTRACT

This paper analyzes the certainty equivalent deterministic characterization of a broad class of stochastic control problems arising in the valuation of both single and sequential real investment opportunities and in the rational management of renewable resources. We consider two alternative risk adjustment techniques and study the qualitative properties of these adjustments. We present a set of weak conditions under which the sign of the relationship between increased volatility and the required risk adjustments can be unambiguously characterized. We also present a set of typically satisfied conditions under which the risk of potential liquidation increases the required risk adjustment and, therefore, strengthens the effect of volatility on the adjustment and the required exercise premium.

CONTENTS

1 Introduction

One of the basic lessons of the research considering irreversible but deferrable investment under uncertainty is that the optimal investment threshold and, therefore, the required rate of return is typically an increasing function of the volatility of the underlying state variable even under risk neutrality. However, since the optimal investment threshold is usually a decreasing function of the discount rate and an increasing function of the expected percentage growth rate of the underlying diffusion, we find that the decelerating impact of increased volatility on irreversible investment can be at least partially neutralized by either increasing the discount rate or decreasing the expected percentage growth rate of the underlying state variable. On the other hand, the results of the studies considering optimal exit indicate that the optimal threshold at which irreversible exit is optimal is a decreasing function of volatility and an increasing function of both the discount rate and the expected percentage growth rate of the underlying state variable. Consequently, we observe that the decelerating effect of increased volatility (in the sense that the irreversible decision is postponed) on optimal exit can be at least partially neutralized by either increasing the discount rate or the expected percentage growth rate of the underlying state variable. These observations naturally bring up three important questions. First, given the comparative static properties presented above it is natural to ask whether the impact of increased volatility can always be neutralized by adjusting either the growth rate of the underlying state variable or the discount rate. Second, given that an appropriate risk adjustment exists, it is naturally of importance to characterize the required risk adjustment and especially its monotonicity as a function of the volatility of the underlying state variable thereby providing valuable information on the equilibrium relationship between volatility and the required rate of return. Third, given the significant variety of applied models and exercise payoffs associated with irreversible decision making problems it is of interest to study whether the risk adjustment is state-dependent or not and, therefore, whether the adjustment is sensitive with respect to the current market conditions and the precise nature of the payoff associated with the considered decision making problem.

Motivated by these arguments, it is our purpose in this study to analyze for a broad class of diffusions modelling the underlying state dynamics and potential exercise payoffs the two most familiar forms of risk-adjustment arising in the literature on financial decision making. Namely, the adjustment of the interest rate at which future cash flows are discounted and the adjustment of the infinitesimal growth rate (drift) at which the randomly fluctuating underlying value process is expected to grow. In order to attain this objective, we first analyze a broad class of stochastic Markovian functionals arising typically in both financial and economic applications of stochastic impulse control and optimal stopping of linear diffusions. We state an explicit representation of these functionals by relying on a combination of the classical theory of diffusions and stochastic calculus. We then derive the certainty equivalent deterministic formulations of the considered stochastic valuations by simply adjusting either the discount rate or the growth rate in a way which guarantees that the solutions of the associated boundary value problems representing the values of the considered functionals coincide with each other. It is worth pointing out that even though we consider a certainty equivalent formulation of a class of dynamic programming problems, our approach differs from the classical approaches introduced originally in the seminal studies by Theil (1954, 1957) and Simon (1956) (which were later extended by Malinvaud (1969)). More precisely, instead of considering quadratic programming problems of multidimensional (potentially time dependent) random variables, we analyze directly a broad class of general valuations subject to a one-dimensional but otherwise general diffusion modelling the underlying state variable[1]. Given these certainty equivalent valuations, we state a set of typically satisfied conditions under which increased volatility unambiguously increases the required risk adjustment and, therefore, increases the risk premium associated with the considered class of irreversible decision making problems. More precisely, we prove that the sign of the relationship between increased volatility and the required risk adjustment (and, therefore, the required exercise premium) is positive in most relevant cases where the net appreciation rate of the diffusion modelling the underlying state variable is non-increasing. This observation is of importance, since it emphasizes the role of the net appreciation rate as the principal

[1]It is worth pointing out that our study is related to the analysis of Henry (1974) through the irreversibility of the considered decisions. However, instead of relying on a discrete formulation of the considered class of decision making problems, we rely on a continuous model. Moreover, given the applicability of the considered class of stochastic control problems in the analysis of irreversible investment decisions and the rational management of renewable resources, we also consider sequential decision making problems where an irreversible decision can repeated later in the future.

determinant of the sign of the relationship between volatility and the irreversible decision (for alternative approaches emphasizing the precise form of the exercise payoff see, for example, Bergman, Grundy, and Wiener (1996), El Karoui, Jeanblanc-Picqué, and Shreve (1998), Hobson (1998), and Janson and Tysk (2003)). Since our conclusions are valid both for single (once-and-for-all-type) and sequential irreversible decisions, we find that our results clearly support the findings of studies considering the impact of increased volatility on irreversible investment[2] (see, for example, McDonald and Siegel (1986), Pindyck (1988, 1991), Caballero (1991), Demers (1991), Ingersoll and Ross (1992), and Dixit and Pindyck (1994)). It is also worth emphasizing that since the class of considered valuations arise in the literature on inventory theory (see, for example, Manne (1961)), in stochastic capital theory (see, for example, Brock, Rothschild, and Stiglitz (1988)), in the literature on irreversible investment (see, for example, McDonald and Siegel (1986), Pindyck (1991), and Dixit and Pindyck (1994)), and in the literature on optimal forest rotation (see, for example, Willasen (1998), Sødal (2002), and Alvarez (2004)) our result affect a considerably broad class of problems arising in the literature on irreversible decision making. As intuitively is clear, our findings indicate that the risk adjustment is typically state dependent (cyclical) and, therefore, that the magnitude of the required adjustment may fluctuate considerably depending on the state and on the precise form of the infinitesimal coefficients of the diffusion modelling the dynamics of the underlying state variable. We also present a set of typically satisfied conditions under which the risk of potential liquidation increases the required risk adjustment and, therefore, strengthens the effect of volatility on both the risk-adjusted discount and growth rate (cf. Milne and Robertson (1996)).

The contents of this paper are as follows. In section 2 we characterize both the underlying stochastic dynamics and the associated deterministic processes. In section 3 we then present the considered class of valuations and establish a certainty equivalent formulation for the con-

[2] It is worth pointing out that the sign of the relationship between increased volatility and the required risk adjustment may be locally negative whenever the net appreciation rate is hump shaped. This is the case in many models subject to mean reversion (cf. Alvarez (2001) and Sarkar (2003)). A similar phenomenon also arises in models based on geometric Brownian motion and concave payoffs (cf. Alvarez and Kanniainen (1998) and Sarkar (2000)), in studies considering optimal risk adoption (cf. Alvarez and Stenbacka (2004)), and in studies considering optimal forest rotation in the presence of amenity valuation (cf. Alvarez and Koskela (2003)).

sidered problems. In section 4 we then apply our findings to a broad class of optimal stopping problems arising in real option models of irreversible investment and single rotation problems in forestry economics (single irreversible decision). In section 5 our findings are, in turn, applied to a class of stochastic impulse control problems arising typically in studies considering optimal dividend policies or the optimal ongoing rotation policy in forestry economics (sequential irreversible decisions). Our results are explicitly illustrated in section 6 by relying on geometric Brownian motion and on a mean reverting diffusion. Section 7 finally concludes our study.

2 The Underlying Stochastic Dynamics

The main objective of this study is to consider how a broad class of stochastic control problems arising in the literature on the valuation of real investment opportunities can be solved by relying on deterministic models adjusted to the risk of the underlying value. In order to accomplish this task, assume that the underlying value dynamics evolve in the absence of interventions as an ordinary linear diffusion. More precisely, let $X^\rho = \{X_t^\rho; t \geq 0\}$ be a linear, time-homogeneous, and regular diffusion defined on a complete filtered probability space $(\Omega, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F})$, and evolving on the state-space $\mathcal{I} = (a, b) \subseteq \mathbb{R}$ according to the stochastic dynamics described by the stochastic differential equation

$$dX_t^\rho = \mu(X_t^\rho)dt + \rho\sigma(X_t^\rho)dW_t, \quad X_0^\rho = x, \tag{2.1}$$

where $\rho \in \mathbb{R}_+$ is an exogenously determined scaling factor, and the infinitesimal coefficients $\mu : \mathcal{I} \mapsto \mathbb{R}$ and $\sigma : \mathcal{I} \mapsto \mathbb{R}_+$ (i.e. $\sigma(x) > 0$ on $\mathcal{I}$) are sufficiently smooth (at least continuous) for guaranteeing the existence of a solution for the stochastic differential equation (2.1). In accordance with most economic and financial applications diffusion processes, we also assume that the boundaries a and b of the state-space of the underlying diffusion are either natural, exit, or killing. As usually, we denote as

$$\mathcal{A}_\rho = \frac{1}{2}\rho^2\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}$$

the differential operator associated with the diffusion X_t^ρ. Before proceeding in our analysis, we state the following definition.

Definition 2.1. *(cf. Borodin and Salminen 2002, pp. 17–20) The Green-function of the diffusion process described in (2.1) reads as*

$$G_r(x,y) = \begin{cases} B^{-1}\psi_\rho(x)\varphi_\rho(y) & x \leq y \\ B^{-1}\psi_\rho(y)\varphi_\rho(x) & x \geq y, \end{cases} \tag{2.2}$$

where $\psi_\rho : \mathcal{I} \mapsto \mathbb{R}_+$ denotes the increasing and $\varphi_\rho : \mathcal{I} \mapsto \mathbb{R}_+$ denotes the decreasing fundamental solutions of the ordinary second order differential equation $(\mathcal{A}_\rho u)(x) = ru(x)$, $B = (\psi_\rho'(x)\varphi_\rho(x) - \varphi_\rho'(x)\psi_\rho(x))/S_\rho'(x) > 0$ denotes the constant Wronskian of these solutions, and

$$S_\rho'(x) = \exp\left(\int \frac{2\mu(x)dx}{\rho^2\sigma^2(x)}\right)$$

denotes the density of the scale function $S_\rho(x)$ *of the diffusion* X_t^ρ. *Especially,*

$$\mathbf{E}_x\left[e^{-r\tau_y^\rho}\right] = \min\left(\frac{\psi_\rho(x)}{\psi_\rho(y)}, \frac{\varphi_\rho(x)}{\varphi_\rho(y)}\right),$$

where $\tau_y^\rho = \inf\{t \geq 0 : X_t^\rho = y\}$ *denotes the first hitting time of process* X_t^ρ *to the state* $y \in \mathcal{I}$.

Given the stochastic dynamics described in (2.1) and the definition of the fundamental solutions $\psi_\rho(x)$ and $\varphi_\rho(x)$, we now define the associated *deterministic processes* $\{\hat{X}_t, t \geq 0\}$ and $\{\tilde{X}_t, t \geq 0\}$ evolving on $\mathcal{I}$ as the processes described by the ordinary time homogenous first order differential equations

$$\hat{X}_t' = \mu(\hat{X}_t), \quad \hat{X}_0 = x \in \mathcal{I}, \tag{2.3}$$

and

$$\tilde{X}_t' = \tilde{\mu}_\rho(\tilde{X}_t), \quad \tilde{X}_0 = x \in \mathcal{I}, \tag{2.4}$$

where $\tilde{\mu}_\rho : \mathcal{I} \mapsto \mathbb{R}$ is a known sufficiently smooth mapping (which will be determined explicitly in the subsequent analysis). Hence, we assume that the underlying processes X_t^ρ, $\hat{X}_t$, and $\tilde{X}_t$ evolve on the same state-space although X_t^ρ is stochastic and $\hat{X}_t$ and $\tilde{X}_t$ are assumed to be deterministic. Especially, it is worth observing that the process $\hat{X}_t$ can be viewed as the process characterizing the dynamics (2.1) in the absence of stochasticity (i.e. as $\rho \downarrow 0$).

3 On Certainty Equivalent Valuation

Having presented the underlying stochastic dynamics and the two associated deterministic processes it is our purpose to now present the considered class of valuations and their certainty equivalent formulations. We are now in position to state the following result characterizing the first type of certainty equivalent valuations subject to a risk adjusted and typically state dependent discount rate.

Theorem 3.1. *(A) Assume that $\mu(x) > 0$ on $\mathcal{I}$ and that*

$$k_\rho(x) = \mu(x)\frac{\psi_\rho'(x)}{\psi_\rho(x)} = r - \frac{1}{2}\rho^2\sigma^2(x)\frac{\psi_\rho''(x)}{\psi_\rho(x)}. \tag{3.1}$$

Then we have for all $g \in C(\mathcal{I})$ and $x, y \in \mathcal{I}$ that

$$\begin{aligned} e^{-\int_0^{t(y)} k_\rho(\hat{X}_s)ds} g(\hat{X}_{t(y)}) &= \mathbf{E}_x\left[e^{-r\tau(y)}g(X^\rho_{\tau(y)})\right] \\ &= \begin{cases} g(x) & x \geq y \\ g(y)\frac{\psi_\rho(x)}{\psi_\rho(y)} & x < y, \end{cases} \end{aligned} \tag{3.2}$$

where $\tau(y) = \inf\{t \geq 0 : X^\rho_t \geq y\}$ denotes the first entrance time of the underlying diffusion to the exercise region $[y, b)$ and $t(y) = \inf\{t \geq 0 : \hat{X}_t \geq y\} = \max(\int_x^y ds/\mu(s), 0)$.
(B) Assume that $\mu(x) < 0$ on $\mathcal{I}$ and that

$$k_\rho(x) = \mu(x)\frac{\varphi_\rho'(x)}{\varphi_\rho(x)} = r - \frac{1}{2}\rho^2\sigma^2(x)\frac{\varphi_\rho''(x)}{\varphi_\rho(x)}. \tag{3.3}$$

Then, we have for all $g \in C(\mathcal{I})$ and $x, y \in \mathcal{I}$ that

$$\begin{aligned} e^{-\int_0^{\tilde{t}(y)} k_\rho(\hat{X}_s)ds} g(\hat{X}_{\tilde{t}(y)}) &= \mathbf{E}_x\left[e^{-r\tilde{\tau}(y)}g(X^\rho_{\tilde{\tau}(y)})\right] \\ &= \begin{cases} g(y)\frac{\varphi_\rho(x)}{\varphi_\rho(y)} & x > y \\ g(x) & x \leq y, \end{cases} \end{aligned} \tag{3.4}$$

where $\tilde{\tau}(y) = \inf\{t \geq 0 : X^\rho_t \leq y\}$ denotes the first entrance time of the underlying diffusion to the exercise region $(a, y]$ and $\tilde{t}(y) = \inf\{t \geq 0 : \hat{X}_t \leq y\} = \max(-\int_y^x ds/\mu(s), 0)$.

Proof. See Appendix A. □

Theorem 3.1 states a set of sufficient conditions under which the expected net present value of an arbitrary continuous exercise payoff accrued at the first time the underlying diffusion enters into an exercise region characterized by a single exercise boundary can be expressed in terms of the associated deterministic process $\hat{X}_t$ and a risk adjusted discount rate. These results are of interest in various financial and economical applications of stochastic analysis since essentially Theorem 3.1 states a set of conditions under which *a random valuation subject to constant discounting can be transformed into an equivalent deterministic valuation subject to a different and typically state dependent discount factor thereby demonstrating that uncertainty can in this way be interpreted as a discount rate effect*. Interestingly, we observe from Theorem 3.1 that given the positivity of the fundamental solutions the required risk adjustment is dependent on the second derivative and, therefore, on the convexity properties of the fundamental solutions.

A second important finding demonstrating how the stochasticity of the underlying diffusion can be alternatively eliminated by adjusting the growth rate of the associated deterministic value dynamics is now summarized in the following.

Theorem 3.2. *(A) Assume that*

$$\tilde{\mu}_\rho(x) = \frac{r\psi_\rho(x)}{\psi'_\rho(x)} = \mu(x) + \frac{1}{2}\rho^2\sigma^2(x)\frac{\psi''_\rho(x)}{\psi'_\rho(x)}. \tag{3.5}$$

Then we have for all $g \in C(\mathcal{I})$ *and* $x, y \in \mathcal{I}$ *that*

$$e^{-r\hat{t}(y)}g(\tilde{X}_{\hat{t}(y)}) = \mathbf{E}_x\left[e^{-r\tau(y)}g(X^\rho_{\tau(y)})\right] = \begin{cases} g(x) & x \geq y \\ g(y)\frac{\psi_\rho(x)}{\psi_\rho(y)} & x < y, \end{cases} \tag{3.6}$$

where $\tau(y) = \inf\{t \geq 0 : X^\rho_t \geq y\}$ *denotes the first entrance time of the underlying diffusion to the exercise region* $[y, b)$ *and* $\hat{t}(y) = \inf\{t \geq 0 : \tilde{X}_t \geq y\} = (\ln\psi_\rho(y) - \ln\psi_\rho(x))^+/r$.
(B) Assume that

$$\tilde{\mu}_\rho(x) = \frac{r\varphi_\rho(x)}{\varphi'_\rho(x)} = \mu(x) + \frac{1}{2}\rho^2\sigma^2(x)\frac{\varphi''_\rho(x)}{\varphi'_\rho(x)}. \tag{3.7}$$

Then, we have for all $g \in C(\mathcal{I})$ *and* $x, y \in \mathcal{I}$ *that*

$$e^{-r\bar{t}(y)}g(\tilde{X}_{\bar{t}(y)}) = \mathbf{E}_x\left[e^{-r\tilde{\tau}(y)}g(X^\rho_{\tilde{\tau}(y)})\right] = \begin{cases} g(y)\frac{\varphi_\rho(x)}{\varphi_\rho(y)} & x > y \\ g(x) & x \leq y, \end{cases} \tag{3.8}$$

where $\tilde{\tau}(y) = \inf\{t \geq 0 : X^\rho_t \leq y\}$ *denotes the first entrance time of the underlying diffusion to the exercise region* $(a, y]$ *and* $\bar{t}(y) = \inf\{t \geq 0 : \tilde{X}_t \leq y\} = (\ln\varphi_\rho(y) - \ln\varphi_\rho(x))^+/r$.

Proof. See Appendix B. □

Theorem 3.2 states a set of conditions under which the expected net present value of a continuous exercise payoff accrued at the first time the underlying diffusion enters into an exercise region characterized by a single exercise boundary can be expressed in terms of an associated deterministic valuation subject to a risk-adjusted growth rate but to the same discount rate as the original stochastic valuation. Interestingly, we observe that the results of Theorem 3.2 are in accordance with the classical findings on risk premiums arising under risk aversion (cf. Laffont 1989, pp. 19–24) since the required risk adjustment of the growth rate is either of the form $\frac{1}{2}\rho^2\sigma^2(x)(L\psi_\rho)(x)$ or of the form $\frac{1}{2}\rho^2\sigma^2(x)(L\varphi_\rho)(x)$, where $(Lu)(x) = u''(x)/u'(x)$ denotes the percentage growth rate (i.e. logarithmic derivative) of the marginal value $u'(x)$ (in utility theory the factor $-(Lu)(x)$ is known as the *Arrow Pratt measure*). Thus, Theorem 3.2 indicates that the ability to defer an irreversible investment decision results into a risk adjustment resembling the premia arising in studies considering decision making in the presence of uncertainty and risk aversion.

The results of Theorem 3.1 are valid for models subject to a single once-and-for-all-type decision. However, the results of Theorem 3.1 do not show how the discount rate should be adjusted in the sequential case. This task is accomplished in the following theorem extending the results of Theorem 3.1 to the cases where the underlying diffusion is restarted from a given generic initial state whenever it hits an arbitrary threshold on the state space of the underlying diffusion.

Theorem 3.3. *(A) Assume that $\mu(x) > 0$ on $\mathcal{I}$, that $k_\rho(x)$ is defined as in (3.1), and that the mappings $F_\rho : \mathcal{I} \mapsto \mathbb{R}$ and $\tilde{F}_0 : \mathcal{I} \mapsto \mathbb{R}$ satisfy for all $x \in \mathcal{I}$, $x_0 < y$, and $g \in C(\mathcal{I})$ the recursive (running present value) relations*

$$\begin{aligned} F_\rho(x) &= \mathbf{E}_x\left[e^{-r\tau(y)}\left(g(X^\rho_{\tau(y)}) + F_\rho(x_0)\right)\right] \\ \tilde{F}_0(x) &= e^{-\int_0^{t(y)} k_\rho(\hat{X}_s)ds}\left(g(\hat{X}_{t(y)}) + \tilde{F}_0(x_0)\right), \end{aligned}$$

where $\tau(y) = \inf\{t \geq 0 : X^\rho_t \geq y\}$ denotes the first entrance time of the underlying diffusion to the exercise region $[y, b)$ and $t(y) = \inf\{t \geq 0 : \hat{X}_t \geq y\} = \max(\int_x^y ds/\mu(s), 0)$. Then

$$F_\rho(x) = \tilde{F}_0(x) = \begin{cases} g(x) + H^a_\rho(y)\psi_\rho(x_0) & x \in [y, b) \\ H^a_\rho(y)\psi_\rho(x) & x \in (a, y), \end{cases} \tag{3.9}$$

where

$$H_\rho^a(y) = \frac{g(y)}{\psi_\rho(y) - \psi_\rho(x_0)}. \tag{3.10}$$

(B) Assume that $\mu(x) < 0$ on $\mathcal{I}$, that $k_\rho(x)$ is defined as in (3.3), and that the mappings $K_\rho : \mathcal{I} \mapsto \mathbb{R}$ and $\tilde{K}_0 : \mathcal{I} \mapsto \mathbb{R}$ satisfy for all $x \in \mathcal{I}$, $x_0 > y$, and $g \in C(\mathcal{I})$ the recursive (running present value) relations

$$\begin{aligned} K_\rho(x) &= \mathbf{E}_x\left[e^{-r\tilde{\tau}(y)}\left(g(X^\rho_{\tilde{\tau}(y)}) + K_\rho(x_0)\right)\right] \\ \tilde{K}_0(x) &= e^{-\int_0^{\tilde{t}(y)} k_\rho(\hat{X}_s)ds}\left(g(\hat{X}_{\tilde{t}(y)}) + \tilde{K}_0(x_0)\right), \end{aligned}$$

$\tilde{\tau}(y) = \inf\{t \geq 0 : X_t^\rho \leq y\}$ denotes the first entrance time of the underlying diffusion to the exercise region $(a, y]$ and $\tilde{t}(y) = \inf\{t \geq 0 : \hat{X}_t \leq y\} = \max(-\int_y^x ds/\mu(s), 0)$. Then

$$K_\rho(x) = \tilde{K}_0(x) = \begin{cases} H_\rho^b(y)\varphi_\rho(x) & x \in (y, b) \\ g(x) + H_\rho^b(y)\varphi_\rho(x_0) & x \in (a, y] \end{cases} \tag{3.11}$$

where

$$H_\rho^b(y) = \frac{g(y)}{\varphi_\rho(y) - \varphi_\rho(x_0)}. \tag{3.12}$$

Proof. See Appendix C. □

Theorem 3.3 extends the findings of Theorem 3.1 to the case where the exercise payoff can be sequentially accrued every time the underlying diffusion hits an arbitrary fixed exercise threshold in the state-space of the underlying diffusion. More precisely, Theorem 3.3 states a set of conditions under which the expected cumulative net present value of a continuous exercise payoff accrued every time the underlying diffusion enters into an exercise region characterized by a single exercise boundary can be expressed in terms of an associated deterministic valuation subject to a risk-adjusted discount rate. The results of Theorem 3.2 are, in turn, extended to the sequential stopping case in the following.

Theorem 3.4. *(A) Assume that $\tilde{\mu}_\rho(x)$ is defined as in (3.5) and that the mappings $F_\rho : \mathcal{I} \mapsto \mathbb{R}$ and $\hat{F}_0 : \mathcal{I} \mapsto \mathbb{R}$ satisfy for all $x \in \mathcal{I}$, $x_0 < y$, and $g \in C(\mathcal{I})$ the recursive (running present value) relations*

$$\begin{aligned} F_\rho(x) &= \mathbf{E}_x\left[e^{-r\tau(y)}\left(g(X^\rho_{\tau(y)}) + F_\rho(x_0)\right)\right] \\ \hat{F}_0(x) &= e^{-r\hat{t}(y)}\left(g(\tilde{X}_{\hat{t}(y)}) + \hat{F}_0(x_0)\right), \end{aligned}$$

where $\tau(y) = \inf\{t \geq 0 : X_t^\rho \geq y\}$ *denotes the first entrance time of the underlying diffusion to the exercise region* $[y, b)$ *and* $\hat{t}(y) = \inf\{t \geq 0 : \tilde{X}_t \geq y\} = (\ln\psi_\rho(y) - \ln\psi_\rho(x))^+/r$. *Then* $F_\rho(x) = \hat{F}_0(x)$ *and the value can be expressed as in (3.9).*
(B) Assume that $\tilde{\mu}_\rho(x)$ *is defined as in (3.7) and that the mappings* $K_\rho : \mathcal{I} \mapsto \mathbb{R}$ *and* $\hat{K}_0 : \mathcal{I} \mapsto \mathbb{R}$ *satisfy for all* $x \in \mathcal{I}$, $x_0 > y$, *and* $g \in C(\mathcal{I})$ *the recursive (running present value) relations*

$$\begin{aligned} K_\rho(x) &= \mathbf{E}_x\left[e^{-r\tilde{\tau}(y)}\left(g(X^\rho_{\tilde{\tau}(y)}) + K_\rho(x_0)\right)\right] \\ \hat{K}_0(x) &= e^{-r\bar{t}(y)}\left(g(\tilde{X}_{\bar{t}(y)}) + \hat{K}_0(x_0)\right), \end{aligned}$$

where $\tilde{\tau}(y) = \inf\{t \geq 0 : X_t^\rho \leq y\}$ *denotes the first entrance time of the underlying diffusion to the exercise region* $(a, y]$ *and* $\bar{t}(y) = \inf\{t \geq 0 : \tilde{X}_t \leq y\} = (\ln\varphi_\rho(y) - \ln\varphi_\rho(x))^+/r$. *Then* $K_\rho(x) = \hat{K}_0(x)$ *and the values read as in (3.11).*

Proof. See Appendix C. □

In accordance with our previous findings on risk adjusted valuation, Theorem 3.4 demonstrates that the same risk adjustment which is needed in the single decision case is sufficient in the sequential case as well. It is worth emphasizing that Theorem 3.3 and Theorem 3.4 imply that the values of the sequential investment policies can alternatively be expressed as

$$F_\rho(x) = F_\rho(x_0) + (\psi_\rho(x) - \psi_\rho(x_0))H_\rho^a(\max(x, y)) \tag{3.13}$$

and

$$K_\rho(x) = K_\rho(x_0) + (\varphi_\rho(x) - \varphi_\rho(x_0))H_\rho^b(\min(x, y)) \tag{3.14}$$

showing how the value of the considered sequential (cyclical) policy can be decomposed into two parts. The first part captures the value accrued at the generic initial state while the second captures the value of the future investment (or harvesting) opportunities. Moreover, the assumed boundary behavior of the underlying diffusion implies that $\lim_{x\downarrow a}\psi_\rho(x) = \lim_{x\uparrow b}\varphi_\rho(x) = 0$ and, therefore, that

$$\lim_{x_0\downarrow a} F_\rho(x) = \begin{cases} g(x) & x \geq y \\ g(y)\frac{\psi_\rho(x)}{\psi_\rho(y)} & x < y \end{cases} \tag{3.15}$$

and

$$\lim_{x_0\uparrow b} K_\rho(x) = \begin{cases} g(y)\frac{\varphi_\rho(x)}{\varphi_\rho(y)} & x > y \\ g(x) & x \leq y \end{cases} \tag{3.16}$$

thereby demonstrating how the value of the sequential investment opportunity is connected with the single investment opportunity considered in Theorem 3.1 and Theorem 3.2. In other words, given the boundary specification of the considered class of diffusions the value of the sequential investment opportunity approaches the value of the single investment opportunity as the generic initial state tends to the boundary of the state space of the underlying stochastic dynamics.

Even though Theorem 3.1, Theorem 3.2, Theorem 3.3, and Theorem 3.4 present a certainty equivalent deterministic form for a considerably broad class of valuations subject to stochasticity, these theorems do not describe the sign of the required adjustment, nor do they describe how increased volatility affects the considered values in general. In order to accomplish this task, we first state the following auxiliary result partially extending the results on the strict convexity of the fundamental solutions stated in Alvarez (2003a, 2003b, 2004) and emphasizing the key role of the net appreciation rate $\theta(x) = \mu(x) - rx$ as the key determinant of the convexity properties of the fundamental solutions and, therefore, the values of the the considered class of investment strategies.

Theorem 3.5. *(A) Assume that the net appreciation rate $\theta(x) = \mu(x) - rx$ is non-increasing and that $\lim_{x\downarrow a} \mu(x)/\varphi_\rho(x) \leq 0$. Then, the increasing fundamental solution $\psi_\rho(x)$ is strictly convex and satisfies for all $x > y$, where $x, y \in \mathcal{I}$, and $\hat{\rho} > \rho$ the inequalities*

$$\frac{\psi_{\hat{\rho}}(x)}{\psi_{\hat{\rho}}(y)} \geq \frac{\psi_\rho(x)}{\psi_\rho(y)} \quad \text{and} \quad \frac{\psi'_{\hat{\rho}}(x)}{\psi_{\hat{\rho}}(x)} \leq \frac{\psi'_\rho(x)}{\psi_\rho(x)}.$$

Moreover, for all $x_0 < x$, where $x, x_0 \in \mathcal{I}$, and $\hat{\rho} > \rho$ we have

$$\frac{\psi'_{\hat{\rho}}(x)}{\psi_{\hat{\rho}}(x) - \psi_{\hat{\rho}}(x_0)} \leq \frac{\psi'_\rho(x)}{\psi_\rho(x) - \psi_\rho(x_0)}.$$

(B) Assume that $\theta(x) = \mu(x) - rx$ is non-increasing and that $\lim_{x\uparrow b} \mu(x)/\psi_\rho(x) \geq 0$. Then, the decreasing fundamental solution $\varphi_\rho(x)$ is strictly convex and satisfies for all $x > y$, where $x, y \in \mathcal{I}$, and $\hat{\rho} > \rho$ the inequalities

$$\frac{\varphi_{\hat{\rho}}(x)}{\varphi_{\hat{\rho}}(y)} \geq \frac{\varphi_\rho(x)}{\varphi_\rho(y)} \quad and \quad \frac{\varphi'_\rho(x)}{\varphi_\rho(x)} \leq \frac{\varphi'_{\hat{\rho}}(x)}{\varphi_{\hat{\rho}}(x)}.$$

Moreover, for all $x < x_0$, where $x, x_0 \in \mathcal{I}$, and $\hat{\rho} > \rho$ we have

$$\frac{\varphi'_{\hat{\rho}}(x)}{\varphi_{\hat{\rho}}(x) - \varphi_{\hat{\rho}}(x_0)} \geq \frac{\varphi'_{\rho}(x)}{\varphi_{\rho}(x) - \varphi_{\rho}(x_0)}.$$

Proof. See Appendix D. □

Theorem 3.5 states a set of conditions under which the fundamental solutions are strictly convex and, therefore, under which increased volatility unambiguously increases the present expected value of a unit of money accrued at the exercise date (i.e. the price of a zero coupon bond expiring at exercise). Interestingly, and in accordance with the findings of Alvarez (2003a, 2003b), Theorem 3.5 shows that the monotonicity of the net appreciation rate $\theta(x) = \mu(x) - rx$ is the principal determinant of the strict convexity of the fundamental solutions and, therefore, of the sign of the relationship between increased volatility and the value of the considered investment opportunities for a broad class of diffusion processes modelling the underlying state variable. An important consequence of Theorem 3.1 and Theorem 3.5 is now summarized in the following.

Corollary 3.6. *Assume that the conditions of part (A) or (B) of both Theorem 3.1 and Theorem 3.5 are satisfied. Then, $0 < k_\rho(x) < r$ for all $x \in \mathcal{I}$. Moreover, if the conditions of part (A) of both Theorem 3.1 and Theorem 3.5 are satisfied then $k_\rho(x) > \mu(x)/x$ for all $x \in \mathcal{I}$ as well.*

Proof. The inequality $0 < k_\rho(x) < r$ is a straightforward consequence of the strict convexity of the fundamental solutions $\psi_\rho(x)$ and $\varphi_\rho(x)$ and the assumptions on the drift $\mu(x)$. To establish that $\mu(x) < k_\rho(x)x$ whenever the conditions of part (A) of both Theorem 3.1 and Theorem 3.5 are satisfied we first observe that the strict convexity and monotonicity of the fundamental solution $\psi_\rho(x)$ and the boundary condition $\psi_\rho(0) = 0$ implies that $\psi'_\rho(x)x > \psi_\rho(x)$ and, therefore, that $k_\rho(x) = \mu(x)\psi'_\rho(x)/\psi_\rho(x) > \mu(x)/x$. □

Corollary 3.6 states the familiar result from utility theory that in order to attain indifference an investor should be compensated from investing in a project yielding a stochastic return instead of investing in the risk free project rendering a known rate of return. Put somewhat differently, Corollary 3.6 demonstrates that in order to attain indifference the opportunity costs of investing must be lower in the presence of uncertainty than in its absence. A second important implication of Theorem 3.2 and

Theorem 3.5 characterizing how increased volatility affects the required cash flow adjustment is now summarized in the following.

Corollary 3.7. *(A) Assume that the conditions of part (A) of both Theorem 3.2 and Theorem 3.5 are satisfied. Then,* $\mu(x) < \tilde{\mu}_\rho(x) < rx$ *for all* $x \in \mathcal{I}$.
(B) Assume that the conditions of part (B) of both Theorem 3.2 and Theorem 3.5 are satisfied. Then, $\tilde{\mu}_\rho(x) < \mu(x)$ *for all* $x \in \mathcal{I}$.

Proof. The alleged results are direct implications of Theorem 3.2 and Theorem 3.5. □

A third important consequence of Theorem 3.5 related to both the impulse control and the optimal stopping of the underlying linear diffusion is now summarized in our next corollary.

Corollary 3.8. *(A) Assume that the conditions of part (A) of Theorem 3.5 are satisfied, that the exercise payoff is non-decreasing on* $\mathcal{I}$ *and continuously differentiable outside a set* $\mathcal{D} \subset \mathcal{I}$ *of measure zero, that* $x_0 \in \mathcal{I}$ *is a known exogenously determined constant, and that* $\hat{\rho} > \rho$. *Then* $h^a_{\hat{\rho}}(x) \geq h^a_\rho(x)$ *for all* $x \in (x_0, b)\backslash\mathcal{D}$ *and* $\{x \in (x_0, b)\backslash\mathcal{D} : H^{a\prime}_\rho(x) > 0\} \subseteq \{x \in (x_0, b)\backslash\mathcal{D} : H^{a\prime}_{\hat{\rho}}(x) > 0\}$*, where*

$$h^a_\rho(x) = \frac{\psi_\rho(x) - \psi_\rho(x_0)}{\psi'_\rho(x)} g'(x) - g(x) \quad \textit{and} \quad H^a_\rho(x) = \frac{g(x)}{\psi_\rho(x) - \psi_\rho(x_0)}.$$

(B) Assume that the conditions of part (B) of Theorem 3.5 are satisfied, that the exercise payoff is non-increasing on $\mathcal{I}$ *and continuously differentiable outside a set* $\mathcal{D} \subset \mathcal{I}$ *of measure zero, that* $x_0 \in \mathcal{I}$ *is a known exogenously determined constant, and that* $\hat{\rho} > \rho$. *Then* $h^b_{\hat{\rho}}(x) \geq h^b_\rho(x)$ *for all* $x \in (a, x_0)\backslash\mathcal{D}$ *and* $\{x \in (a, x_0)\backslash\mathcal{D} : H^{b\prime}_{\hat{\rho}}(x) > 0\} \subseteq \{x \in (a, x_0)\backslash\mathcal{D} : H^{b\prime}_\rho(x) > 0\}$*, where*

$$h^b_\rho(x) = \frac{\varphi_\rho(x) - \varphi_\rho(x_0)}{\varphi'_\rho(x)} g'(x) - g(x) \quad \textit{and} \quad H^b_\rho(x) = \frac{g(x)}{\varphi_\rho(x) - \varphi_\rho(x_0)}.$$

Proof. See Appendix E. □

Corollary 3.8 states a set of typically satisfied conditions under which increased volatility unambiguously increases the value of the mappings $h^a_\rho(x)$ and $h^b_\rho(x)$ determining the monotonicity properties of the mappings $H^a_\rho(x)$ and $H^b_\rho(x)$. Thus, part (A) of Corollary 3.8 actually states a set of conditions under which we find that if the mapping $H^a_{\hat{\rho}}(x)$ attains a unique maximum, it attains it at a higher state than the mapping $H^a_\rho(x)$. On the other hand, part (B) of Corollary 3.8 states a set of conditions under which we find that if the mapping $H^b_{\hat\rho}(x)$ attains a unique maximum it attains it at a lower state than the mapping $H^a_\rho(x)$. It is also worth emphasizing that since $\lim_{x_0\downarrow a}\psi_\rho(x)=0$ and $\lim_{x_0\uparrow b}\varphi_\rho(x)=0$ by the assumed boundary behavior of the underlying diffusion, we find that the results of Corollary 3.8 can be extended to the case where the considered functional is either $g(x)/\psi_\rho(x)$ or $g(x)/\varphi_\rho(x)$ as well and, therefore, to the case where the considered policy can be exercised only once. As we will later observe, these results are closely related to the standard argument stating that increased volatility should postpone the exercise of an irreversible investment decision by increasing the value of waiting and, therefore, the required exercise premium associated with the investment opportunity.

Having considered the nature of the required risk adjustment, we now plan to consider the comparative static properties of the considered certainty equivalent valuations. More precisely, we plan to analyze how increased volatility affects the required risk-adjustment. Our first set of results on this subject are now summarized in the following.

Theorem 3.9. *Assume that the conditions of either part (A) or part (B) of Theorem 3.1 and Theorem 3.5 are satisfied. Assume also that $g^{-1}(\mathbb{R}_+)\neq\emptyset$ and that $y\in g^{-1}(\mathbb{R}_+)$. Then, increased volatility increases the expected present value of the exercise payoff and decreases the risk adjusted discount factor. More precisely, if $\hat\rho>\rho$, then $k_\rho(x)\geq k_{\hat\rho}(x)$ and*

$$\begin{aligned} e^{-\int_0^{t(y)}k_\rho(\hat X_s)ds}g(\hat X_{t(y)}) &\leq e^{-\int_0^{t(y)}k_{\hat\rho}(\hat X_s)ds}g(\hat X_{t(y)}) \quad \textit{if } \mu(x)>0\\ e^{-\int_0^{\tilde t(y)}k_\rho(\hat X_s)ds}g(\hat X_{\tilde t(y)}) &\leq e^{-\int_0^{\tilde t(y)}k_{\hat\rho}(\hat X_s)ds}g(\hat X_{\tilde t(y)}) \quad \textit{if } \mu(x)<0. \end{aligned}$$

Proof. The alleged result is a straightforward implication of Theorem 3.1 and Theorem 3.5. □

Theorem 3.9 demonstrates the economically sensible result that given the conditions of our theorems 3.1 and 3.5 increased volatility decreases the risk adjusted discount rate and, therefore, increases the present expected present value of the exercise payoff. Consequently, it is clear that the incentives of holding such contracts alive (whenever exercise can be postponed) increases as the volatility of the underlying diffusion process increases. This finding is naturally a consequence of the strict convexity of the fundamental solutions, since as was demonstrated in Alvarez

(2003a) the strict convexity of these mappings imply that all r-excessive mappings for a linear diffusion are strictly convex on the continuation set where exercising a deferrable contract is suboptimal. Thus, Theorem 3.9 actually establishes that increased volatility can be discounted away by simply adjusting the discount rate to match the added volatility. The impact of increased volatility on the risk adjusted growth rate $\tilde{\mu}_\rho(x)$ is now characterized in the following.

Theorem 3.10. *(A) Assume that the conditions of part (A) of both Theorem 3.2 and Theorem 3.5 are satisfied and that $\hat{\rho} > \rho$. Then, $\tilde{\mu}_{\hat{\rho}}(x) \geq \tilde{\mu}_\rho(x)$ for all $x \in \mathcal{I}$.*
(B) Assume that the conditions of part (B) of both Theorem 3.2 and Theorem 3.5 are satisfied and that $\hat{\rho} > \rho$. Then, $\tilde{\mu}_{\hat{\rho}}(x) \leq \tilde{\mu}_\rho(x)$ for all $x \in \mathcal{I}$.

Theorem 3.10 states a set of conditions under which the impact of increased volatility on the risk adjusted growth rate can be unambiguously described. It is worth observing that in contrast to our findings on the risk adjusted discount rate the risk adjusted growth rate may be an increasing or a decreasing function of the underlying volatility depending on whether the considered adjustment is based on the increasing or the decreasing fundamental solution. The reason for this observation are naturally the terms $\psi_\rho''(x)/\psi_\rho'(x)$ and $\varphi_\rho''(x)/\varphi_\rho'(x)$ which have opposite signs whenever the fundamental solutions are convex. A set of results characterizing the impact of increased volatility on the value of sequential stopping problems is now summarized in the following.

Theorem 3.11. *Assume that the conditions of either part (A) or part (B) of Theorem 3.1 and Theorem 3.5 are satisfied. Assume also that $g^{-1}(\mathbb{R}_+) \neq \emptyset$ and that $y \in g^{-1}(\mathbb{R}_+)$. Then, increased volatility increases decreases the risk adjusted discount factor and increases the values $F_\rho(x)$ and $K_\rho(x)$. More precisely, if $\hat{\rho} > \rho$, then $k_\rho(x) \geq k_{\hat{\rho}}(x)$, $F_{\hat{\rho}}(x) \geq F_\rho(x)$ and $K_{\hat{\rho}}(x) \geq K_\rho(x)$ for all $x \in \mathcal{I}$.*

Proof. See Appendix F. □

Theorem 3.11 extends the results of Theorem 3.9 to the sequential case. Again we find that, given the strict convexity of the fundamental solutions, increased volatility unambiguously decreases the risk-adjusted discount rate and, therefore, increases the value of the sequential investment opportunities by increasing the present value of the exercise payoff.

It is at this point worth emphasizing that the introduction of potential liquidation risk has a strong impact on both the risk adjusted discount rate and on the risk adjusted growth rate. To illustrate this explicitly, assume that the underlying diffusion X_t^ρ is killed whenever it exits the set $(l, b) \subset \mathcal{I}$ (i.e. when an exogenously determined liquidation threshold l is introduced). In that case the increasing fundamental solution (given up to a multiplicative constant) of the ordinary second order differential equation $(\mathcal{A}_\rho v)(x) = rv(x)$ reads as $\hat{\psi}_\rho(x) = \psi_\rho(x) - \psi_\rho(l)\varphi_\rho(x)/\varphi_\rho(l)$. Therefore, in that case the risk adjusted discount factor and the risk adjusted growth rate read as

$$\hat{k}_\rho(x) = k_\rho(x) + \frac{\mu(x)\psi_\rho(l)BS'(x)}{\psi_\rho(x)\varphi_\rho(l)\hat{\psi}_\rho(x)}$$

and

$$\hat{\mu}_\rho(x) = \tilde{\mu}_\rho(x) - \frac{r\psi_\rho(l)BS'(x)}{\psi'_\rho(x)\varphi_\rho(l)\hat{\psi}'_\rho(x)} < \tilde{\mu}_\rho(x)$$

where $k_\rho(x) = \mu(x)\psi'_\rho(x)/\psi_\rho(x)$ and $\tilde{\mu}_\rho(x) = r\psi_\rho(x)/\psi'_\rho(x)$. It is now clear from these expressions that if $\mu(x) > 0$ then $\hat{k}_\rho(x) > k_\rho(x)$ demonstrating the intuitively clear result that *potential liquidation risk increases the risk adjusted discount rate*. In economic terms, *a rational investor will require a higher compensation for undertaking an investment subject to both value uncertainty and the risk of potential liquidation*. Similarly, if the underlying diffusion X_t^ρ is killed whenever it exits the set $(a, u) \subset \mathcal{I}$ then the decreasing fundamental solution of the ordinary second order differential equation $(\mathcal{A}_\rho v)(x) = rv(x)$ reads as $\hat{\varphi}_\rho(x) = \varphi_\rho(x) - \psi_\rho(x)\varphi_\rho(u)/\psi_\rho(u)$. In accordance with our findings above we now observe that the risk adjusted discount factor and the risk adjusted growth rate now read as

$$\hat{k}_\rho(x) = k_\rho(x) - \frac{\mu(x)\varphi_\rho(u)BS'(x)}{\varphi_\rho(x)\psi_\rho(u)\hat{\varphi}_\rho(x)}$$

and

$$\hat{\mu}_\rho(x) = \tilde{\mu}_\rho(x) + \frac{r\varphi_\rho(u)BS'(x)}{\varphi'_\rho(x)\psi_\rho(u)\hat{\varphi}'_\rho(x)} > \tilde{\mu}_\rho(x).$$

It is now clear from these expressions that if $\mu(x) < 0$ then $\hat{k}_\rho(x) > k_\rho(x)$ demonstrating that the introduction of liquidation risk increases the risk adjusted discount rate in this case as well.

4 Optimal Timing of a Single Investment Opportunity

We now plan to apply the general results of our previous section in the analysis of a broad class of optimal stopping problems appearing in the literature on the valuation and rational exercise of real investment opportunities and in the analysis of single rotation (or harvesting) opportunities. More precisely, we now plan to investigate the optimal timing problems

$$V_\rho(x) = \sup_{\tau} \mathbf{E}_x \left[e^{-r\tau} g(X_\tau^\rho) \right], \tag{4.1}$$

$$\tilde{V}_0(x) = \sup_{t \geq 0} \left[e^{-\int_0^t k_\rho(\hat{X}_s)ds} g(\hat{X}_t) \right], \tag{4.2}$$

and

$$\bar{V}_0(x) = \sup_{t \geq 0} \left[e^{-rt} g(\tilde{X}_t) \right], \tag{4.3}$$

where $r > 0$ denotes the exogenously determined risk free discount rate, $k_\rho : \mathcal{I} \mapsto \mathbb{R}_+$ is a known positive and continuous mapping, and $g : \mathcal{I} \mapsto \mathbb{R}$ is a sufficiently smooth mapping (at least continuous) measuring the exercise payoff accrued whenever the underlying process is endogenously stopped. It is worth emphasizing that this type of valuations arise typically in models considering irreversible decision making under uncertainty (for example, irreversible investment and Wicksellian single rotation problems) and in the determination of the price and rational exercise strategies of perpetual American contingent contracts. In accordance with the approach introduced in the previous section, we plan to present a certainty equivalent deterministic formulation of the stochastic valuation (4.1) in terms of a deterministic valuation (4.2) subject to a potentially state-dependent discount rate. Our first main result characterizing the relationship of the solutions of the optimal stopping problems (4.1) and (4.2) in the case the exercise payoff is non-decreasing is now summarized in the following.

Theorem 4.1. *Assume that the exercise payoff $g(x)$ is continuous and non-decreasing, and that the mapping $g(x)/\psi_\rho(x)$ attains a unique global maximum at $x_\rho^* \in \mathcal{I}$. Define the mapping $\hat{V}_\rho : \mathcal{I} \mapsto \mathbb{R}_+$ as*

$$\hat{V}_\rho(x) = \psi_\rho(x) \sup_{y \geq x} \left[\frac{g(y)}{\psi_\rho(y)} \right] = \begin{cases} g(x) & x \in [x_\rho^*, b) \\ g(x_\rho^*) \frac{\psi_\rho(x)}{\psi_\rho(x_\rho^*)} & x \in (a, x_\rho^*) \end{cases} \tag{4.4}$$

and assume that the mapping

$$h_a(x) = \frac{\hat{V}_\rho'(x)}{S_\rho'(x)}\psi_\rho(x) - \frac{\psi_\rho'(x)}{S_\rho'(x)}\hat{V}_\rho(x)$$

is non-increasing, continuously differentiable outside a countable set $\mathcal{D} \subset \mathcal{I}$*, and satisfies the condition* $|h_a'(x\pm)| < \infty$ *for all* $x \in \mathcal{D}$*. Then* $\hat{V}_\rho(x) = V_\rho(x)$ *and the optimal stopping date is* $\tau^* = \inf\{t \geq 0 : X_t^\rho \geq x_\rho^*\}$*. Moreover, if* $\mu(x) > 0$ *on* $\mathcal{I}$ *and* $k_\rho(x) = \mu(x)\frac{\psi_\rho'(x)}{\psi_\rho(x)}$*, then* $\hat{V}_\rho(x) = V_\rho(x) = \tilde{V}_0(x)$ *and if* $\tilde{\mu}_\rho(x) = r\frac{\psi_\rho(x)}{\psi_\rho'(x)}$*, then* $\hat{V}_\rho(x) = V_\rho(x) = \bar{V}_0(x)$*.*

Proof. See Appendix G. □

Theorem 4.1 presents a set of sufficient conditions under which both the value and the optimal exercise threshold of the optimal stopping problem (4.1) can be explicitly solved in terms of the increasing fundamental solution and the exercise payoff. In accordance with the findings of part (A) of Theorem 3.1 and part (A) of Theorem 3.2, Theorem 4.1 also states a set conditions under which the value of the optimal timing policy can be expressed in a certainty equivalent deterministic form. It is worth emphasizing that since this type of valuations typically arise in studies considering either the valuation and exercise of irreversible investment opportunities, optimal entry, or the valuation of perpetual American contingent contracts written on a dividend paying asset, our results essentially indicate that within that class of problems the optimal policy and its value can be typically derived by analyzing the associated deterministic risk adjusted problem and, therefore, by relying on ordinary techniques of differential calculus.

A set of sufficient conditions under which the optimal stopping problem (4.1) is solvable in the case of a non-increasing exercise payoff, arising typically in studies considering optimal exit and the valuation of perpetual American put options written on a dividend paying asset, is now summarized in our following theorem:

Theorem 4.2. *Assume that the exercise payoff* $g(x)$ *is continuous and non-increasing, and that the mapping* $g(x)/\varphi_\rho(x)$ *attains a unique global maximum at* $\tilde{x}_\rho \in \mathcal{I}$*. Define the mapping* $\hat{V}_\rho : \mathcal{I} \mapsto \mathbb{R}_+$ *as*

$$\hat{V}_\rho(x) = \varphi_\rho(x) \sup_{y \leq x} \left[\frac{g(y)}{\varphi_\rho(y)}\right] = \begin{cases} g(\tilde{x}_\rho)\frac{\varphi_\rho(x)}{\varphi_\rho(\tilde{x}_\rho)} & x \in (\tilde{x}_\rho, b) \\ g(x) & x \in (a, \tilde{x}_\rho] \end{cases} \tag{4.5}$$

and assume that the mapping

$$h_b(x) = \frac{\hat{V}'_\rho(x)}{S'_\rho(x)}\varphi_\rho(x) - \frac{\varphi'_\rho(x)}{S'_\rho(x)}\hat{V}_\rho(x)$$

is non-increasing, continuously differentiable outside a countable set $\mathcal{D} \subset \mathcal{I}$, and satisfies the condition $|h'_b(x\pm)| < \infty$ for all $x \in \mathcal{D}$. Then, $V_\rho(x) = \hat{V}_\rho(x)$ and the optimal stopping date is $\tau^* = \inf\{t \geq 0 : X_t^\rho \leq \tilde{x}_\rho\}$. Moreover, if $\mu(x) < 0$ on $\mathcal{I}$ and $k_\rho(x) = \mu(x)\frac{\varphi'_\rho(x)}{\varphi_\rho(x)}$, then $\hat{V}_\rho(x) = V_\rho(x) = \tilde{V}_0(x)$, and if $\tilde{\mu}_\rho(x) = r\frac{\varphi_\rho(x)}{\varphi'_\rho(x)}$, then $\hat{V}_\rho(x) = V_\rho(x) = \bar{V}_0(x)$.

Proof. The proof is analogous with the proof of Theorem 4.1. □

Having established a set of conditions under which the stochastic stopping problems can be solved by re-expressing the original valuations in a certainty-equivalent deterministic form subject to a potentially state-dependent discount rate, we now plan to analyze the impact of increased volatility on the required risk adjustment and on both the value and optimal exercise threshold of the considered valuations.

Theorem 4.3. *(A) Assume that the conditions of Theorem 4.1 and part (A) of Theorem 3.5 are satisfied. Then increased volatility increases the value and postpones rational exercise by increasing the optimal exercise threshold. More precisely, for all $\hat{\rho} > \rho$ we have that $V_{\hat{\rho}}(x) \geq V_\rho(x)$ and $x^*_{\hat{\rho}} > x^*_\rho$. Moreover, if $\mu(x) > 0$ for $x \in \mathcal{I}$, then $k_{\hat{\rho}}(x) \leq k_\rho(x)$, and if $\tilde{\mu}_\rho(x) = r\psi_\rho(x)/\psi'_\rho(x)$ for $x \in \mathcal{I}$, then $\tilde{\mu}_{\hat{\rho}}(x) \geq \tilde{\mu}_\rho(x)$ for $x \in \mathcal{I}$.*
(B) Assume that the conditions of Theorem 4.1 and part (B) of Theorem 3.5 are satisfied. Then increased volatility increases the value and postpones rational exercise by decreasing the optimal exercise threshold. More precisely, for all $\hat{\rho} > \rho$ we have that $V_{\hat{\rho}}(x) \geq V_\rho(x)$ and $\tilde{x}_{\hat{\rho}} < \tilde{x}_\rho$. Moreover, if $\mu(x) < 0$ for $x \in \mathcal{I}$, then $k_{\hat{\rho}}(x) \leq k_\rho(x)$, and if $\tilde{\mu}_\rho(x) = r\varphi_\rho(x)/\varphi'_\rho(x)$ for $x \in \mathcal{I}$, then $\tilde{\mu}_{\hat{\rho}}(x) \leq \tilde{\mu}_\rho(x)$ for $x \in \mathcal{I}$.

Proof. The alleged result are straightforward implications of Theorem 4.1, Theorem 3.5, and Theorem 3.9. □

Theorem 4.3 states a set of conditions under which the standard arguments stating that increased volatility should increase the value of an investment opportunity and postpone rational exercise are valid. Put somewhat differently, Theorem 4.3 states a set of conditions under which

$$x^*_{\hat{\rho}} = \operatorname{argmax}\left\{\frac{g(x)}{\psi_{\hat{\rho}}(x)}\right\} \geq \operatorname{argmax}\left\{\frac{g(x)}{\psi_\rho(x)}\right\} = x^*_\rho \qquad (4.6)$$

whenever the exercise payoff is non-decreasing and

$$\tilde{x}_{\hat{\rho}} = \operatorname{argmax}\left\{\frac{g(x)}{\varphi_{\hat{\rho}}(x)}\right\} \leq \operatorname{argmax}\left\{\frac{g(x)}{\varphi_{\rho}(x)}\right\} = \tilde{x}_{\rho} \qquad (4.7)$$

whenever the exercise payoff is non-increasing. This result, which clearly indicates that the ratio between the exercise payoff and the corresponding fundamental solution is the principal determinant of the required exercise premium can also be motivated by applying the results of Corollary 3.8. In line with Theorem 3.9, Theorem 4.3 again also demonstrates that increased volatility decreases the required risk-adjustment as well.

5 On the Timing of Sequential Irreversible Policies

In order to extend the analysis of the previous section to a broad class of impulse control problems arising in studies considering the optimal timing of sequential irreversible policies (for example, in Faustmannian ongoing rotation problems, in capital accumulation models subject to nonnegative gross investment rates, and in cash flow managements models considering rational dividend policies), assume that the dynamics of the underlying controlled diffusion process are described by the generalized Itô-equation

$$X_t^\nu = x + \int_0^t \mu(X_s^\nu)ds + \int_0^t \rho\sigma(X_s^\nu)dW_s + \sum_{\tau_k \le t} \varsigma\xi_k, \tag{5.1}$$

where $t \in [0, \tau(\mathcal{I}))$, $\tau(\mathcal{I}) = \inf\{t \ge 0 : X_t^\nu \notin \mathcal{I}\} \le \infty$ denotes the possibly finite first exit time from the state-space $\mathcal{I}$, $\varsigma \in \{-1, 1\}$ is a known parameter determining whether the implemented control increases or decreases the state of the controlled diffusion, and $\mu : \mathcal{I} \mapsto \mathbb{R}$ and $\sigma : \mathcal{I} \mapsto \mathbb{R}_+$ (i.e. $\sigma(x) > 0$ for all $x \in \mathcal{I}$) are known sufficiently smooth (at least continuous) mappings guaranteeing the existence of a solution for (2.1) (cf. Borodin and Salminen 2002, pp. 46–47). As in Alvarez 2004, an impulse control for the system (5.1) is a possibly finite sequence

$$\nu = (\tau_1, \tau_2, \ldots, \tau_k, \ldots; \xi_1, \xi_2, \ldots, \xi_k, \ldots)_{k \le N} \quad (N \le \infty),$$

where $\{\tau_k\}_{k \le N}$ is an increasing sequence of $\mathcal{F}_t$-stopping times for which $\tau_1 \ge 0$, and $\{\xi_k\}_{k \le N}$ denote a sequence of non-negative impulses (i.e. $\xi_k \ge 0$ for all $k \le N$) exerted at the corresponding intervention dates $\{\tau_k\}_{k \le N}$, respectively. In other words, the times τ_k can be interpreted as the dates at which the irreversible policies are exercised and ξ_k measures the size of the implemented irreversible policy. In line with various financial and economical applications of impulse control, we assume that whenever the irreversible policy is exercised, the system is instantaneously driven to a known generic initial state $x_0 \in \mathcal{I}$ and restarted from there. We denote as $\mathcal{V}$ the class of admissible policies and assume that $\tau_k \to \tau(\mathcal{I})$ almost surely for all admissible policies $\nu \in \mathcal{V}$ and all states $x \in \mathcal{I}$.

Given the stochastic system described in (5.1) and our assumptions, we now plan to determine the admissible impulse control policy $\nu^* \in$

$\mathcal{V}$ which maximizes the expected cumulative present value of all future exercise payoffs from the present up to a potentially infinitely distant future. More precisely, we now plan to consider the stochastic impulse control problem

$$J_\rho(x) = \sup_{\nu\in\mathcal{V}} \mathbf{E}_x \left[\sum_{k=1}^{N} e^{-r\tau_k} g(X^\nu_{\tau_k-})\right], \tag{5.2}$$

where $g : \mathcal{I} \mapsto \mathbb{R}$ is a continuous mapping representing the revenues accrued each time the irreversible policy is exercised. We assume throughout this section that $g(x_0) < 0$, that is, that the exercise payoff is negative at the generic initial state (thereby generating incentives to wait immediately after the irreversible policy has been exercised). In line with the notation in the previous section, we denote the associated deterministic processes as

$$\hat{X}^\nu_t = x + \int_0^t \mu(\hat{X}^\nu_s)ds + \sum_{t_k\leq t} \varsigma\xi_k, \quad 0 \leq t \leq T^\nu(\mathcal{I}),$$

and

$$\tilde{X}^\nu_t = x + \int_0^t \tilde{\mu}_\rho(\tilde{X}^\nu_s)ds + \sum_{t_k\leq t} \varsigma\xi_k, \quad 0 \leq t \leq \bar{T}^\nu(\mathcal{I}),$$

where $T^\nu(\mathcal{I}) = \inf\{t \geq 0 : \hat{X}^\nu_t \notin \mathcal{I}\}$ and $\bar{T}^\nu(\mathcal{I}) = \inf\{t \geq 0 : \tilde{X}^\nu_t \notin \mathcal{I}\}$ denote the first exit times of the underlying deterministic processes $\hat{X}^\nu_t$ and $\tilde{X}^\nu_t$ from the state-space $\mathcal{I}$, respectively. Accordingly, the value of the associated deterministic impulse control problems are now denoted as

$$\tilde{J}_0(x) = \sup_{\nu\in\mathcal{V}_0} \sum_{k=1}^{N} e^{-\int_0^{t_k} k_\rho(\hat{X}_s)ds} g(\hat{X}^\nu_{t_k-}), \tag{5.3}$$

and

$$\bar{J}_0(x) = \sup_{\nu\in\mathcal{V}_0} \sum_{k=1}^{N} e^{-rt_k} g(\tilde{X}^\nu_{t_k-}), \tag{5.4}$$

where ν denotes the implemented sequential policy and $\mathcal{V}_0$ denotes the class of admissible irreversible policies (i.e. a sequence of deterministic stopping times and non-negative impulses). We can now establish the following important result characterizing both the value and the optimal policy of the considered sequential timing problem.

Theorem 5.1. *Assume that $\varsigma = -1$, that the exercise payoff $g(x)$ is continuous and non-decreasing, and that the mapping $H^a_\rho(x)$ attains on the set (x_0, b) a unique maximum at the state $y^*_\rho \in (x_0, b)$. Define the mapping $\hat{J}_\rho : \mathcal{I} \mapsto \mathbb{R}_+$ as*

$$\begin{aligned}\hat{J}_\rho(x) &= \hat{J}_\rho(x_0) + (\psi_\rho(x) - \psi_\rho(x_0)) \sup_{y \geq x \vee x_0} \{H^a_\rho(y)\} \\ &= \begin{cases} g(x) + H^a_\rho(y^*_\rho)\psi_\rho(x_0) & x \in [y^*_\rho, b) \\ H^a_\rho(y^*_\rho)\psi_\rho(x) & x \in (a, y^*_\rho) \end{cases}\end{aligned} \tag{5.5}$$

and assume that $H^a_\rho(y^*_\rho)(\psi_\rho(x) - \psi_\rho(x_0)) - g(x)$ is non-increasing on (a, y^*_ρ) and that

$$c^a_\rho(x) = \frac{\hat{J}'_\rho(x)}{S'_\rho(x)}\psi_\rho(x) - \frac{\psi'_\rho(x)}{S'_\rho(x)}\hat{J}_\rho(x)$$

is non-increasing, continuously differentiable outside a countable set $\mathcal{D} \subset \mathcal{I}$, and satisfies the condition $|c^{a\prime}_\rho(x\pm)| < \infty$ for all $x \in \mathcal{D}$. Then, $J_\rho(x) = \hat{J}_\rho(x)$ and the optimal impulse control is to instantaneously take the process X^ν_t to the generic initial state x_0 whenever the process X^ν_t hits the threshold $\hat{y}^*_\rho$ and restart the process from x_0. Thus, the optimal impulse dates are $\tau_1 = \inf\{t \geq 0 : X^\nu_t \geq y^*_\rho\}$, and $\tau_{k+1} = \inf\{t \geq \tau_k : X^\nu_t \geq y^*_\rho\}$, $k \geq 1$, and the associated impulses are $\xi_k = \max(x, y^*_\rho) - x_0$, $k \geq 1$. Moreover, if $\mu(x) > 0$ and $k_\rho(x) = \mu(x)\frac{\psi'_\rho(x)}{\psi_\rho(x)}$, then $\hat{J}_\rho(x) = J_\rho(x) = \tilde{J}_0(x)$, and if $\tilde{\mu}_\rho(x) = r\psi_\rho(x)/\psi'_\rho(x)$, then $\hat{J}_\rho(x) = J_\rho(x) = \bar{J}_0(x)$.

Proof. See Appendix H. □

Theorem 5.1 states a set of weak conditions under which the considered stochastic impulse control problem (5.2) can be solved explicitly. Along the lines indicated by our findings on the optimal timing problems, Theorem 5.1 also presents a set of conditions under which the value of the considered sequential stochastic control problem coincides with the values of the associated certainty equivalent deterministic impulse control problems (5.3) and (5.4). It is worth pointing out that although the value and the optimal policies can be described in an entirely analogous way, the actual timing of the irreversible policies are naturally not identical due to the randomness of the intervention dates in the sequential stochastic control case. An interesting implication of Theorem 5.1 is now summarized.

Corollary 5.2. *Assume that $\varsigma = -1$, that the exercise payoff $g(x)$ is non-decreasing, that $g \in C^1(\mathcal{I}) \cap C^2(\mathcal{I}\backslash\mathcal{D})$, where $\mathcal{D}$ is a countable set of points in $\mathcal{I}$, and that $g''(x\pm) < \infty$ for all $x \in \mathcal{D}$. Assume also that the mapping $g(x)/\psi_\rho(x)$ attains a unique global maximum at $x_\rho^* \in \mathcal{I}$, and that $g'(x)/\psi_\rho'(x)$ is decreasing on $\mathcal{I}$. Then, the results of Theorem 5.1 are valid. More precisely, there is a unique threshold $y_\rho^* \in (x_0, b)$ maximizing the auxiliary mapping $H_\rho^a(x)$, the value $J_\rho(x)$ reads as in (5.5), and the optimal impulse control can be characterized by the sequence $\nu^* = \{(\tau_k, \zeta_k)\} = \{(\inf\{t \geq \tau_{k-1} : X_t^\nu \geq y_\rho^*\}, \max(x, y_\rho^*) - x_0)\}$. Moreover, if $\mu(x) > 0$ and $k_\rho(x) = \mu(x)\frac{\psi_\rho'(x)}{\psi_\rho(x)}$, then $\hat{J}_\rho(x) = J_\rho(x) = \tilde{J}_0(x)$, and if $\tilde{\mu}_\rho(x) = r\psi_\rho(x)/\psi_\rho'(x)$, then $\hat{J}_\rho(x) = J_\rho(x) = \bar{J}_0(x)$.*

Proof. See Appendix I. □

Corollary 5.2 states a set of slightly stronger conditions under which the considered class of sequential timing problems can be explicitly solved and under which the value of the optimal policy coincides with the value of the associated certainty equivalent formulations of these valuations. Moreover, in accordance with the findings of Alvarez (2004) and our findings (3.15) and (3.16) we also find the following analogy between the stochastic sequential timing problem (5.2) and the optimal timing problem (4.1).

Corollary 5.3. *Assume that the conditions of Theorem 5.1 are met. Then $J_\rho(x) \geq V_\rho(x)$ and $y_\rho^* \leq x_\rho^*$. Moreover, $J_\rho(x) \downarrow V_\rho(x)$ and $\lim_{x_0 \downarrow a} y_\rho^* \uparrow x_\rho^*$ as $x_0 \downarrow a$.*

Proof. The value function $J_\rho \in C^1(\mathcal{I}) \cap C^2(\mathcal{I}\backslash(\mathcal{D} \cup \{y_\rho^*\}))$ satisfies the variational inequalities $J_\rho(x) \geq J_\rho(x_0) + g(x) \geq g(x)$ for all $x \in \mathcal{I}$ and $(\mathcal{A}_\rho J_\rho)(x) - rJ_\rho(x) \leq 0$ for all $x \in \mathcal{I}\backslash(\mathcal{D}\cup\{y_\rho^*\})$. Moreover, $J_\rho''(x\pm) < \infty$ for all $x \in \mathcal{D}\cup\{y_\rho^*\}$ implying that the conditions of Theorem 10.4.1 in Øksendal (1998) are met and, therefore, that $J_\rho(x) \geq V_\rho(x)$. Since $0 = g'(y_\rho^*)(\psi_\rho(y_\rho^*) - \psi_\rho(x_0)) - g(y_\rho^*)\psi_\rho'(y_\rho^*) \leq g'(y_\rho^*)\psi_\rho(y_\rho^*) - g(y_\rho^*)\psi_\rho'(y_\rho^*)$, the inequality $y_\rho^* \leq x_\rho^*$ follows directly from the analysis of part (A) of Corollary 3.8. The rest of the results are direct implications of Theorem 5.1 and the limit $\lim_{x \downarrow a} \psi_\rho(x) = 0$ following from the assumed boundary behavior of the underlying diffusion at the lower boundary a. □

Corollary 5.3 demonstrate that there is a close connection between the stochastic impulse control problem (5.2) and the optimal stopping problem (4.1). More precisely, Corollary 5.3 shows that given the conditions of Theorem 5.1, the value and optimal exercise threshold of the considered sequential irreversible policy tends towards the value and exercise threshold of the associated optimal timing problem of a single irreversible policy (4.1) as the generic initial state x_0 tends towards the lower boundary a of the state space of the diffusion modelling the underlying state variable. However, the value of the impulse control problem dominates the value of the associated optimal stopping problem. The reason for this finding is naturally the fact that in the sequential stopping case, the decision maker does not lose the option to exercise the policy once again later in the future. Hence, the required exercise premium is lower in the impulse control than in the optimal stopping case and, accordingly, the exercise threshold of the opportunity is lower in the sequential stopping case than in the single stopping case. In terms of forest economics, Corollary 5.3 demonstrates that the expected present value of the future harvests is higher and the optimal rotation threshold is lower in the ongoing rotation problem than in the single rotation case (cf. Alvarez 2004).

In line with the analysis of the optimal stopping problem (4.1) in the presence of a decreasing exercise payoff, we can now establish the following.

Theorem 5.4. *Assume that $\varsigma = 1$, that the exercise payoff is continuous and non-increasing, and that the mapping $H_\rho^b(x)$ attains on the set (a, x_0) a maximum at the state $\tilde{y}_\rho \in (a, x_0)$. Define the mapping $\hat{J}_\rho : \mathcal{I} \mapsto \mathbb{R}_+$ as*

$$\hat{J}_\rho(x) = \hat{J}_\rho(x_0) + (\varphi_\rho(x) - \varphi_\rho(x_0)) \sup_{y \le x \wedge x_0} \{H_\rho^b(y)\}$$
$$= \begin{cases} g(x) + H_\rho^b(\tilde{y}_\rho)\varphi_\rho(x_0) & x \in (a, \tilde{y}_\rho] \\ H_\rho^b(\tilde{y}_\rho)\varphi_\rho(x) & x \in (\tilde{y}_\rho, b) \end{cases} \tag{5.6}$$

and assume that $H_\rho^b(\tilde{y}_\rho)(\varphi(x) - \varphi(x_0)) - g(x)$ is non-decreasing on $(\tilde{y}_\rho, b)$ and that

$$c_\rho^b(x) = \frac{\hat{J}_\rho'(x)}{S_\rho'(x)}\varphi_\rho(x) - \frac{\varphi_\rho'(x)}{S_\rho'(x)}\hat{J}_\rho(x)$$

is non-increasing, continuously differentiable outside a countable set $\mathcal{D} \subset \mathcal{I}$, and satisfies the condition $|c_\rho^{b\prime}(x\pm)| < \infty$ for all $x \in \mathcal{D}$. Then, $\hat{J}_\rho(x) = J_\rho(x)$ and the optimal impulse control is to instantaneously take the process X_t^ν to the generic initial state x_0 whenever the process X_t^ν hits the threshold $\tilde{y}_\rho$ and restart the process from x_0.

Thus, the optimal impulse dates are $\tau_1 = \inf\{t \geq 0 : X_t^\nu \leq \tilde{y}_\rho\}$*, and* $\tau_{k+1} = \inf\{t \geq \tau_k : X_t^\nu \leq \tilde{y}_\rho\}$*,* $k \geq 1$*, and the associated impulses are* $\xi_k = x_0 - \min(x, y_\rho^*)$*,* $k \geq 1$*. Moreover, if* $\mu(x) < 0$ *and* $k_\rho(x) = \mu(x)\frac{\varphi_\rho'(x)}{\varphi_\rho(x)}$*, then* $J_\rho(x) = \hat{J}_\rho(x) = \tilde{J}_0(x)$*, and if* $\tilde{\mu}_\rho(x) = r\varphi_\rho(x)/\varphi_\rho'(x)$*, then* $\hat{J}_\rho(x) = J_\rho(x) = \bar{J}_0(x)$*.*

Proof. The proof is analogous with the proof of Theorem 5.1. □

Theorem 5.4 states a set of conditions under which the stochastic impulse control problem (5.2) can be solved in terms of the decreasing fundamental solution when the exercise payoff is decreasing. Such configurations arise typically in models considering irreversible lumpy capital investments and inventory control. An interesting implication of Theorem 5.4 is now summarized.

Corollary 5.5. *Assume that* $\varsigma = 1$*, that the exercise payoff* $g(x)$ *is non-increasing, that* $g \in C^1(\mathcal{I}) \cap C^2(\mathcal{I}\backslash\mathcal{D})$*, where* $\mathcal{D}$ *is a countable set of points in* $\mathcal{I}$*, and that* $g''(x\pm) < \infty$ *for all* $x \in \mathcal{D}$*. Assume also that the mapping* $g(x)/\varphi_\rho(x)$ *attains a unique global maximum at* $\tilde{x}_\rho \in \mathcal{I}$*, and that* $g'(x)/\varphi_\rho'(x)$ *is increasing on* $\mathcal{I}$*. Then, the results of Theorem 5.4 are valid. More precisely, there is a unique threshold* $\tilde{y}_\rho \in (a, x_0)$ *maximizing the auxiliary mapping* $H_\rho^b(x)$*, the value* $J_\rho(x)$ *reads as in (5.6), and the optimal policy can be characterized by the sequence* $\tilde{\nu} = \{(\tau_k, \zeta_k)\} = \{(\inf\{t \geq \tau_{k-1} : X_t^\nu \leq \tilde{y}_\rho\}, x_0 - \min(x, \tilde{y}_\rho))\}$*. Moreover, if* $\mu(x) < 0$ *and* $k_\rho(x) = \mu(x)\frac{\varphi_\rho'(x)}{\varphi_\rho(x)}$*, then* $J_\rho(x) = \hat{J}_\rho(x) = \tilde{J}_0(x)$*, and if* $\tilde{\mu}_\rho(x) = r\varphi_\rho(x)/\varphi_\rho'(x)$*, then* $\hat{J}_\rho(x) = J_\rho(x) = \bar{J}_0(x)$*.*

Proof. The proof is analogous with the proof of Corollary 5.2. □

As in the case of Theorem 5.1, we can now establish a connection between between the stochastic sequential irreversible investment problem (5.2) and the optimal timing problem of a single investment opportunity (4.1). This connection is established in the following.

Corollary 5.6. *Assume that the conditions of Theorem 5.4 are met. Then* $J_\rho(x) \geq V_\rho(x)$ *and* $\tilde{y}_\rho \geq \tilde{x}_\rho$*. Moreover,* $\lim_{x_0 \uparrow b} J_\rho(x) = V_\rho(x)$ *and* $\lim_{x_0 \uparrow b} y_\rho^* = \tilde{x}_\rho$*.*

Proof. The proof is analogous with the proof of Corollary 5.3. □

In accordance with the finding of Corollary 5.3, Corollary 5.6 demonstrates that there is again a close connection between the impulse control and the optimal stopping of a linear diffusion. Having established a set of sufficient conditions under which the considered stochastic impulse control problems are explicitly solvable, we now consider the impact of increased volatility on the value and exercise threshold of the optimal policy. In accordance with the findings of Theorem 4.3 we now find the following.

Theorem 5.7. *(A) Assume that the conditions of Theorem 5.1 and part (A) of Theorem 3.5 are satisfied. Then increased volatility increases the value and postpones rational exercise by increasing the optimal exercise threshold. More precisely, for all $\hat{\rho} > \rho$ we have that $J_{\hat{\rho}}(x) \geq J_{\rho}(x)$ and $y^*_{\hat{\rho}} > y^*_{\rho}$. Moreover, if $\mu(x) > 0$ for $x \in \mathcal{I}$, then $k_{\hat{\rho}}(x) \leq k_{\rho}(x)$ and if $\tilde{\mu}_{\rho}(x) = r\psi_{\rho}(x)/\psi'_{\rho}(x)$ then $\tilde{\mu}_{\hat{\rho}}(x) \geq \tilde{\mu}_{\rho}(x)$.*
(B) Assume that the conditions of Theorem 5.4 and part (B) of Theorem 3.5 are satisfied. Then, increased volatility increases the value and postpones rational exercise by decreasing the optimal exercise threshold. More precisely, for all $\hat{\rho} > \rho$ we have that $J_{\hat{\rho}}(x) \geq J_{\rho}(x)$ and $\tilde{y}_{\hat{\rho}} < \tilde{y}_{\rho}$. Moreover, if $\mu(x) < 0$ for $x \in \mathcal{I}$, then $k_{\hat{\rho}}(x) \leq k_{\rho}(x)$ and if $\tilde{\mu}_{\rho}(x) = r\varphi_{\rho}(x)/\varphi'_{\rho}(x)$ then $\tilde{\mu}_{\hat{\rho}}(x) \leq \tilde{\mu}_{\rho}(x)$.

Proof. The alleged result are straightforward implications of Theorem 4.1, Theorem 3.5, and Theorem 3.9. □

Theorem 5.7 extends the results of Theorem 4.3 and states a set of conditions under which increased volatility unambiguously increases the value of an investment opportunity and postpones its rational exercise by increasing the required exercise premium. In line with the analysis of Theorem 4.3, we observe that Theorem 5.7 states a set of conditions under which

$$y^*_{\hat{\rho}} = \operatorname{argmax}\left\{H^a_{\hat{\rho}}(x)\right\} \geq \operatorname{argmax}\left\{H^a_{\rho}(x)\right\} = y^*_{\rho} \tag{5.7}$$

whenever the exercise payoff is non-decreasing and

$$\tilde{y}_{\hat{\rho}} = \operatorname{argmax}\left\{H^b_{\hat{\rho}}(x)\right\} \leq \operatorname{argmax}\left\{H^b_{\hat{\rho}}(x)\right\} = \tilde{y}_{\rho} \tag{5.8}$$

whenever the exercise payoff is non-increasing. This result, which clearly indicates that the ratio between the exercise payoff and the corresponding fundamental solution is the principal determinant of the required exercise premium can also be motivated by applying the results of Corollary 3.8. In line with Theorem 3.9, Theorem 4.3 also demonstrates that increased volatility decreases the required risk-adjustment in this case as well.

6 Explicit Illustrations

6.1 Geometric Brownian Motion

In order to illustrate our results explicitly in a parametrized and typically applied framework, assume that the underlying stochastic dynamics $\{X_t^\sigma; t \geq 0\}$ are described by the stochastic differential equation

$$dX_t^\sigma = \mu X_t^\sigma dt + \sigma X_t^\sigma dW_t, \quad X_0^\sigma = x, \tag{6.1}$$

where $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}_+$ are known exogenously determined constants and W_t denotes a standard Wiener process. It is now clear that in the present example the fundamental solutions read as $\psi_\sigma(x) = x^{\alpha_\sigma}$ and $\varphi_\sigma(x) = x^{\beta_\sigma}$, where

$$\alpha_\sigma = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 0$$

denotes the positive and

$$\beta_\sigma = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} < 0$$

denotes the negative root of the characteristic equation $\sigma^2 z(z-1)+2\mu z-2r = 0$. Given these observations, we find that the results of Theorem 3.1 can be expressed in the present case as follows.

Corollary 6.1. *(A) Assume that* $\mu > 0$ *and that* $k_\sigma(x) = \mu\alpha_\sigma = r - \frac{1}{2}\sigma^2\alpha_\sigma(\alpha_\sigma - 1)$. *Then we have for all* $g \in C(\mathbb{R}_+)$ *and* $x, y \in \mathbb{R}_+$ *that*

$$e^{-\mu\alpha_\sigma t(y)} g(\hat{X}_{t(y)}) = \mathbf{E}_x\left[e^{-r\tau(y)} g(X^\sigma_{\tau(y)})\right] = \begin{cases} g(x) & x \geq y \\ g(y)(x/y)^{\alpha_\sigma} & x < y, \end{cases}$$

where $\hat{X}_t = xe^{\mu t}$, $t(y) = \inf\{t \geq 0 : \hat{X}_t \geq y\} = \frac{1}{\mu}(\ln y - \ln x)^+$ *and* $\tau(y) = \inf\{t \geq 0 : X_t^\sigma \geq y\}$.
(B) Assume that $\mu < 0$ *and that* $k_\sigma(x) = \mu\beta_\sigma = r - \frac{1}{2}\sigma^2\beta_\sigma(\beta_\sigma - 1)$. *Then, we have for all* $g \in C(\mathbb{R}_+)$ *and* $x, y \in \mathbb{R}_+$ *that*

$$e^{-\mu\beta_\sigma \tilde{t}(y)} g(\hat{X}_{\tilde{t}(y)}) = \mathbf{E}_x\left[e^{-r\tilde{\tau}(y)} g(X^\sigma_{\tilde{\tau}(y)})\right] = \begin{cases} g(y)(x/y)^{\beta_\sigma} & x > y \\ g(x) & x \leq y, \end{cases}$$

where $\hat{X}_t = xe^{\mu t}$, $\tilde{t}(y) = \inf\{t \geq 0 : \hat{X}_t \leq y\} = -\frac{1}{\mu}(\ln x - \ln y)^+$ *and* $\tilde{\tau}(y) = \inf\{t \geq 0 : X_t^\sigma \leq y\}$.

It is worth noticing that if the conditions of part (A) of Corollary 6.1 are satisfied, then

$$\frac{\partial k_\sigma(x)}{\partial \sigma} = \mu \frac{2\alpha_\sigma(1-\alpha_\sigma)}{\sigma(\alpha_\sigma - \beta_\sigma)} \gtreqqless 0, \quad \mu \gtreqqless r.$$

That is, as was already indicated by our Theorem 3.5, the required risk adjusted discount rate is a decreasing function of the underlying volatility only if $\mu < r$, that is, only if the net appreciation rate $\theta(x) = \mu x - rx$ is decreasing. However, in case the conditions of part (B) of Corollary 6.1 are satisfied the net appreciation rate is always decreasing and, therefore, in that case

$$\frac{\partial k_\sigma(x)}{\partial \sigma} = \mu \frac{2\beta_\sigma(\beta_\sigma - 1)}{\sigma(\alpha_\sigma - \beta_\sigma)} < 0.$$

The results of Theorem 3.2 are illustrated in the present example in the following.

Corollary 6.2. *(A) Assume that $\tilde{\mu}_\sigma(x) = rx/\alpha_\sigma = (\mu + \frac{1}{2}(\alpha_\sigma - 1)\sigma^2)x$. Then we have for all $g \in C(\mathbb{R}_+)$ and $x, y \in \mathbb{R}_+$ that*

$$e^{-r\hat{t}(y)} g(\tilde{X}_{\hat{t}(y)}) = \mathbf{E}_x \left[e^{-r\tau(y)} g(X^\sigma_{\tau(y)}) \right] = \begin{cases} g(x) & x \geq y \\ g(y)(x/y)^{\alpha_\sigma} & x < y, \end{cases}$$

where $\tilde{X}_t = xe^{(r/\alpha_\sigma)t}$, $\hat{t}(y) = \inf\{t \geq 0 : \tilde{X}_t \geq y\} = \frac{\alpha_\sigma}{r}(\ln y - \ln x)^+$ and $\tau(y) = \inf\{t \geq 0 : X^\sigma_t \geq y\}$.
(B) Assume that $\tilde{\mu}_\sigma(x) = rx/\beta_\sigma = (\mu - \frac{1}{2}(1 - \beta_\sigma)\sigma^2)x$. Then, we have for all $g \in C(\mathbb{R}_+)$ and $x, y \in \mathbb{R}_+$ that

$$e^{-r\bar{t}(y)} g(\tilde{X}_{\bar{t}(y)}) = \mathbf{E}_x \left[e^{-r\tilde{\tau}(y)} g(X^\sigma_{\tilde{\tau}(y)}) \right] = \begin{cases} g(y)(x/y)^{\beta_\sigma} & x > y \\ g(x) & x \leq y, \end{cases}$$

where $\tilde{X}_t = xe^{(r/\beta_\sigma)t}$, $\hat{t}(y) = \inf\{t \geq 0 : \tilde{X}_t \leq y\} = -\frac{\beta_\sigma}{r}(\ln x - \ln y)^+$, and $\tilde{\tau}(y) = \inf\{t \geq 0 : X^\sigma_t \leq y\}$.

It is now clear from Corollary 6.2 that if $\tilde{\mu}_\sigma(x) = rx/\alpha_\sigma$ then

$$\frac{\partial \tilde{\mu}_\sigma(x)}{\partial \sigma} = \frac{2rx(\alpha_\sigma - 1)}{\alpha_\sigma \sigma(\alpha_\sigma - \beta_\sigma)} \gtreqqless 0, \quad r \gtreqqless \mu$$

proving that the required risk adjusted growth rate is an increasing function of the underlying volatility as long as the net appreciation rate $\theta(x) =$

$\mu x - rx$ is decreasing. As in the case of the risk adjusted discount rate, we now find that if $\tilde{\mu}_\sigma(x) = rx/\beta_\sigma$ then

$$\frac{\partial \tilde{\mu}_\sigma(x)}{\partial \sigma} = \frac{2rx(1-\beta_\sigma)}{\beta_\sigma \sigma(\alpha_\sigma - \beta_\sigma)} < 0.$$

That is, as was established in Theorem 3.10, the risk adjusted growth rate is a decreasing function of the underlying volatility coefficient.

6.2 Mean Reverting Diffusion

To illustrate our results in a more complex case, consider the diffusion X_t^σ described on $(0, \gamma^{-1})$, where $\gamma > 0$ is a known exogenously determined parameter, by the stochastic differential equation (logistic growth subject to a stochastic intrinsic growth rate)

$$dX_t^\sigma = \mu X_t^\sigma(1-\gamma X_t^\sigma)dt + \sigma X_t^\sigma(1-\gamma X_t^\sigma)dW_t, \quad X_0^\sigma = x. \tag{6.2}$$

In order to illustrate the results of our first four sections, we first have to determine the fundamental solutions $\psi_\sigma(x)$ and $\varphi_\sigma(x)$. As was established in Alvarez 2000, we can now establish the following.

Lemma 6.3. *The fundamental solutions read as*

$$\psi_\sigma(x) = \left(\frac{\gamma x}{1-\gamma x}\right)^{\alpha_\sigma} F\left(a, b, c; -\frac{\gamma x}{1-\gamma x}\right)$$

and

$$\varphi_\sigma(x) = \left(\frac{\gamma x}{1-\gamma x}\right)^{\beta_\sigma} F\left(a, b, c; -\frac{\gamma x}{1-\gamma x}\right)$$

where F denotes the standard hypergeometric function, the parameters $\alpha_\sigma > 0$ and $\beta_\sigma < 0$ are defined as in the previous section, $c = 1 - \beta_\sigma + \alpha_\sigma$,

$$a = 1 + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} - \sqrt{\left(\frac{1}{2} + \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}},$$

and

$$b = 1 + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} + \sqrt{\left(\frac{1}{2} + \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}.$$

Proof. See Lemma 5 in Alvarez 2000. □

It is now clear from the findings of Lemma 6.3 that given the complexity of the fundamental solutions in the present example, deriving the explicit representation of the risk adjusted discount rate and expected growth rate is uninformative. Hence, in order to characterize the form of the risk adjustments we illustrate the nature of the adjustments numerically. Since the net appreciation rate reads in the present case as $x(\mu - r - \mu\gamma x)$ we find that whenever the growth rate is smaller than the discount rate, that is, when $\mu \leq r$, the conditions of Theorem 3.5 are satisfied and, therefore, that $\psi_\sigma(x)$ is strictly convex and that increased volatility decreases the risk adjusted discount rate $k_\sigma(x)$ and increases the risk adjusted growth rate $\tilde{\mu}_\sigma(x)$. In Figure 1 the risk adjusted discount rate $k_\sigma(x)$ is illustrated for various values of the volatility coefficient $\sigma = 0.1, 0.15, 0.2$ under the assumption that $\gamma = 0.01$, $\mu = 0.025$, and $r = 0.035$. Figure 1 clearly indicates that in the present case the risk adjusted discount rate is not only a decreasing function of the volatility coefficient, it is also a decreasing function of the underlying state. For example, the numerical illustration presented in Figure 1 indicates that for a volatility of 10% the risk adjusted discount rate ranges from 3.3% to 2.5%. Similarly, for a volatility of 20% the risk adjusted discount rate ranges from 3% to 1.5%.

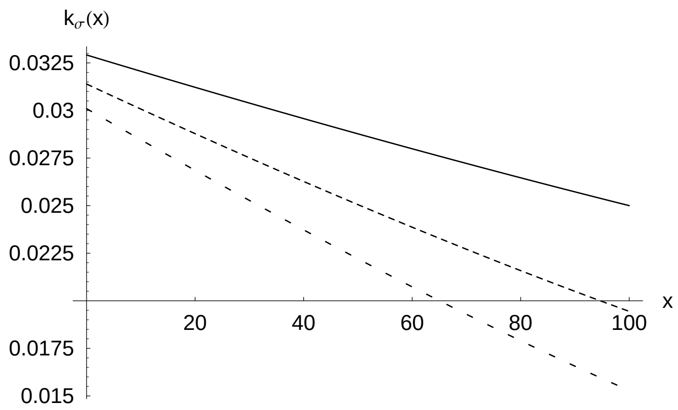

Figure 1: The risk adjusted discount rate $k_\sigma(x)$.

Figure 2 in turn illustrates the risk adjusted expected growth rate $\mu_\sigma(x) = r\psi_\sigma(x)/\psi'_\sigma(x)$ for various values of the volatility coefficient σ under the assumption that $\gamma = 0.01$, $\mu = 0.025$, and $r = 0.035$. As was established in Theorem 3.10 increased volatility increases the risk adjusted growth rate $\tilde{\mu}_\sigma(x)$. As Figure 2 clearly indicates, the risk adjusted growth rate $\tilde{\mu}_\sigma(x)$ is, in accordance with the the underlying mean reverting dynamics, hump-shaped and that the impact of increased volatility is relatively weak close to the boundaries of the state space and strong in a neighborhood of the maximal attainable growth rate.

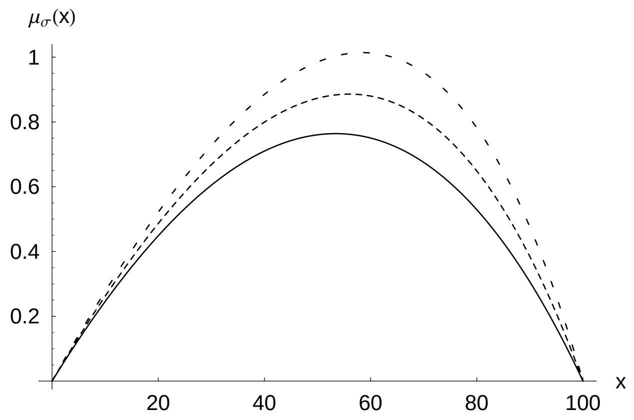

Figure 2: The risk adjusted expected growth rate $\mu_\sigma(x)$.

It is now clear that if $\mu > r$ then the increasing fundamental solution is not convex on the entire state space and, therefore, in that case the impact of increased volatility is ambiguous and depends on the state of the underlying process. The impact of increased volatility on the risk adjusted discount rate is illustrated in Figure 3 for various values of the volatility coefficient σ under the assumption that $\gamma = 0.01$, $\mu = 0.045$, and $r = 0.03$. As Figure 3 clearly indicates, the impact of increased volatility on the risk adjusted discount rate becomes ambiguous and that the sign of the relationship between increased volatility and $k_\sigma(x)$ can be actually reversed depending on the current state of the underlying diffusion.

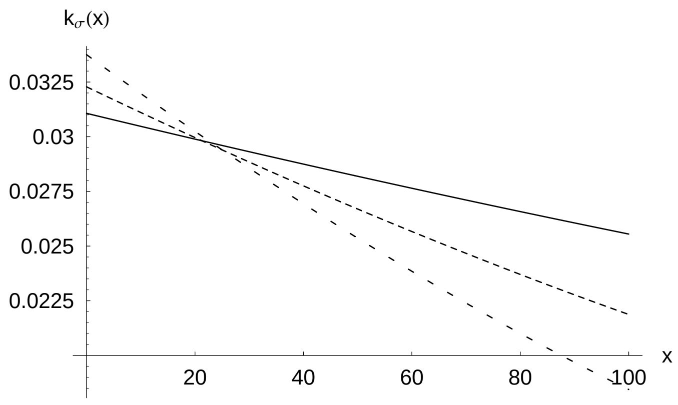

Figure 3: The risk adjusted discount rate $k_\sigma(x)$.

7 Conclusions

In this paper we have considered the certainty equivalent characterization of a broad class of valuations arising in the literature on irreversible decision making (for example, in the literature on irreversible investment and on the rational management of renewable resources). We presented a set of conditions under which the original valuation subject to a stochastic underlying state variable can be expressed in a deterministic certainty equivalent form by risk-adjusting either the interest rate at which future cash flows are discounted or the infinitesimal growth rate at which the randomly fluctuating underlying value process is expected to grow. Given the volatility-dependence of these adjustments, we stated a set of typically satisfied conditions under which increased volatility unambiguously increases the risk premium associated with the considered class of irreversible decision making problems and, therefore, tends to postpone the rational exercise of irreversible decisions. We characterized the comparative static properties of the risk-adjustments and found that the impact of increased volatility on the risk-adjusted discount rate is typically positive while its impact on the risk-adjusted growth rate may be positive or negative depending on the precise form of the considered valuation. We also showed that the risk adjustments are typically state-dependent and that the risk of potential liquidation strengthens the effect of volatility on both the risk-adjusted discount and growth rate.

Although the conclusions of this study are considerably general (in terms of the considered class of valuations) there are two possible directions to which the analysis could naturally be extended. First, although the assumed potential perpetuity of the considered planning horizon is acceptable for models considering deferrable decisions subject to stable market conditions (in the sense that the objective remains unchanged) it is not clear whether our conclusions would remain valid within a finite-horizon setting where the irreversible decision can be postponed only up to a known date at which the opportunity is expired. Second, it is not clear whether the conclusions of our study remain valid in a multidimensional framework where the underlying stochastic state-variables have a dynamic and potentially correlated stochastic structure. Although the results of the seminal study McDonald and Siegel (1986) clearly indicate that a certainty equivalent formulation is possible in that specific case, it is not clear whether this conclusion can be extended to a more general multidimensional setting. Such extensions naturally require a different and more general mathematical analysis which is out of the scope of the present study and, therefore, left for future research.

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A Proof of Theorem 3.1

Proof. (A) The positivity and monotonicity of the increasing fundamental solution $\psi_\rho(x)$ implies that the mapping $k_\rho(x) = \mu(x)\psi'_\rho(x)/\psi_\rho(x)$ is well-defined and does not diverge in the interior of the state-space $\mathcal{I}$. Thus, standard differentiation yields

$$\frac{\frac{d}{dt}\left[e^{-\int_0^t k_\rho(\hat{X}_s)ds}\psi_\rho(\hat{X}_t)\right]}{e^{-\int_0^t k_\rho(\hat{X}_s)ds}} = \psi'_\rho(\hat{X}_t)\mu(\hat{X}_t) - k_\rho(\hat{X}_t)\psi_\rho(\hat{X}_t) = 0.$$

Hence, the assumption $\mu(x) > 0$ on $\mathcal{I}$ implies that the first exit time of the process $\hat{X}_t$ from any open set $(a, y) \subset \mathcal{I}$ is finite and, therefore, that

$$e^{-\int_0^{t(y)} k_\rho(\hat{X}_s)ds}\psi_\rho(\hat{X}_{t(y)}) = e^{-\int_0^{t(y)} k_\rho(\hat{X}_s)ds}\psi_\rho(y) = \psi_\rho(x).$$

On the other hand, since the mapping $\psi_\rho(x)$ is r-harmonic for the diffusion X_t^ρ and the lower boundary is either natural, exit or killing (implying that $\psi_\rho(a) = 0$), we find that

$$\mathbf{E}_x\left[e^{-r\tau(y)}\psi_\rho(X^\rho_{\tau(y)})\right] = \mathbf{E}_x\left[e^{-r\tau(y)}\right]\psi_\rho(y) = \psi_\rho(x).$$

Given these observations, let us now consider the identity (3.2). Since $\tau(y) = t(y) = 0$ for all $x \geq y$, we find that if $x \leq y$ then

$$\mathbf{E}_x\left[e^{-r\tau(y)}g(X^\rho_{\tau(y)})\right] = g(y)\mathbf{E}_x\left[e^{-r\tau(y)}\right] = g(y)\frac{\psi_\rho(x)}{\psi_\rho(y)}$$

and

$$e^{-\int_0^{t(y)} k_\rho(\hat{X}_s)ds}g(\hat{X}_{t(y)}) = g(y)e^{-\int_0^{t(y)} k_\rho(\hat{X}_s)ds} = g(y)\frac{\psi_\rho(x)}{\psi_\rho(y)}$$

completing the proof of part (A). The proof of part (B) is completely analogous. □

B Proof of Theorem 3.2

Proof. In light of the proof of Theorem 3.1 it is sufficient to establish that

$$e^{-r\hat{t}(y)}g(\tilde{X}_{\hat{t}(y)}) = \begin{cases} g(x) & x \in [y, b) \\ g(y)\frac{\psi_\rho(x)}{\psi_\rho(y)} & x \in (a, y). \end{cases}$$

Since $\hat{t}(y) = 0$ for all $x \in [y, b)$ and $g(x)$ is continuous, we find that $e^{-r\hat{t}(y)} g(\tilde{X}_{\hat{t}(y)}) = g(x)$ for all $x \in [y, b)$. On $(a, y))$ we find that

$$e^{-r\hat{t}(y)} g(\tilde{X}_{\hat{t}(y)}) = g(y) e^{-r\hat{t}(y)} = g(y) d_y(x),$$

where the mapping $d_y(x)$ satisfies the ordinary first order differential equation $\tilde{\mu}_\rho(x) d_y'(x) - r d_y(x) = 0$ subject to the boundary condition $d_y(y) = 1$. Choosing $\tilde{\mu}_\rho(x) = r\psi_\rho(x)/\psi_\rho'(x)$ then yields that $d_y(x) = \psi_\rho(x)/\psi_\rho(y)$ which completes the proof of part (A) of our Theorem. Proving part (B) is completely analogous. □

C Proof of Theorem 3.3

Proof. (A) The recursive definition of the mapping $F_\rho(x)$ implies that

$$F_\rho(x) = (g(y) + F_\rho(x_0)) \frac{\psi_\rho(x)}{\psi_\rho(y)} \tag{C.1}$$

for all $x < y$. Letting $x \to x_0$ and solving $F_\rho(x_0)$ from the resulting equation then implies that

$$F_\rho(x_0) = \frac{g(y)\psi_\rho(x_0)}{\psi_\rho(y) - \psi_\rho(x_0)}. \tag{C.2}$$

Plugging (C.2) into (C.1) then yields that $F_\rho(x) = u_{x_0}(y)\psi_\rho(x)$ for all $x < y$. Since $\tau(y) = 0$ for all $x \geq y$ we find that $F_\rho(x) = g(x) + F_\rho(x_0) = g(x) + u_{x_0}(y)\psi_\rho(x_0) = g(x) - g(y) + u_{x_0}(y)\psi_\rho(y)$ for all $x \geq y$ and, therefore, that $F_\rho(x)$ can be expressed as in (3.9). The identity $F_\rho(x) = \tilde{F}_0(x)$ is now a straightforward implication of part (A) of Theorem 3.1. Proving part (B) is entirely analogous. □

D Proof of Theorem 3.5

Proof. (A) As in Alvarez (2003a, 2003b, 2004), assume that $a < l < u < b$ (implying that $(l, u) \subset \mathcal{I}$) and let $\bar{\tau} = \inf\{t \geq 0 : X_t^\rho \notin (l, u)\}$ denote the first exit time of the underlying diffusion from the open interval (l, u). Invoking Dynkin's theorem yields

$$\mathbf{E}_x \left[e^{-r\bar{\tau}} X_{\bar{\tau}}^\rho \right] = x + \mathbf{E}_x \int_0^{\bar{\tau}} e^{-rs} \theta(X_s^\rho) ds. \tag{D.1}$$

On the other hand, solving the boundary value problems representing the expected values appearing in (D.1) yields

$$\mathbf{E}_x\left[e^{-r\bar{\tau}}X^{\rho}_{\bar{\tau}}\right] = l\frac{\tilde{\varphi}(x)}{\tilde{\varphi}(l)} + u\frac{\tilde{\psi}(x)}{\tilde{\psi}(u)}$$

and

$$\begin{aligned}\mathbf{E}_x\int_0^{\bar{\tau}} e^{-rs}\theta(X^{\rho}_s)ds &= \tilde{B}^{-1}\tilde{\varphi}(x)\int_l^x \tilde{\psi}(y)\theta(y)m'_{\rho}(y)dy \\ &+ \tilde{B}^{-1}\tilde{\psi}(x)\int_x^u \tilde{\varphi}(y)\theta(y)m'_{\rho}(y)dy,\end{aligned}$$

where $m'_{\rho}(x) = 2/(\rho^2\sigma^2(x)S'_{\rho}(x))$ denotes the density of the speed measure of X^{ρ}_t, $\tilde{\varphi}(x) = \varphi_{\rho}(x) - \varphi_{\rho}(u)\psi_{\rho}(x)/\psi_{\rho}(u)$, $\tilde{\psi}(x) = \psi_{\rho}(x) - \psi_{\rho}(l)\varphi_{\rho}(x)/\varphi_{\rho}(l)$, and

$$\tilde{B} = \left(1 - \frac{\psi_{\rho}(a)\varphi_{\rho}(b)}{\psi_{\rho}(b)\varphi_{\rho}(a)}\right)B$$

denotes the Wronskian of the solutions $\tilde{\varphi}(x)$ and $\tilde{\psi}(x)$. Inserting these equations into (D.1) and differentiating then yields that

$$\begin{aligned}l\frac{\tilde{\varphi}'(x)}{\tilde{\varphi}(l)} + u\frac{\tilde{\psi}'(x)}{\tilde{\psi}(u)} = 1 &+ \tilde{B}^{-1}\tilde{\varphi}'(x)\int_l^x \tilde{\psi}(y)\theta(y)m'_{\rho}(y)dy \\ &+ \tilde{B}^{-1}\tilde{\psi}'(x)\int_x^u \tilde{\varphi}(y)\theta(y)m'_{\rho}(y)dy.\end{aligned}$$

Dividing this expression with $\tilde{\psi}'(x)$ and reordering terms yields

$$\begin{aligned}\frac{1}{\tilde{\psi}'(x)} = \frac{l\tilde{\varphi}'(x)}{\tilde{\psi}'(x)\tilde{\varphi}(l)} + \frac{u}{\tilde{\psi}(u)} &- \tilde{B}^{-1}\frac{\tilde{\varphi}'(x)}{\tilde{\psi}'(x)}\int_l^x \tilde{\psi}(y)\theta(y)m'_{\rho}(y)dy \\ &- \tilde{B}^{-1}\int_x^u \tilde{\varphi}(y)\theta(y)m'_{\rho}(y)dy.\end{aligned}$$

Differentiating this equation and multiplying the resulting identity with the factor $-(\tilde{\psi}'(x))^2$ then implies that

$$\tilde{\psi}''(x) = \frac{2S'_{\rho}(x)}{\sigma^2(x)}\left[r\int_l^x \tilde{\psi}(y)\theta(y)m'_{\rho}(y)dy - \theta(x)\frac{\tilde{\psi}'(x)}{S'_{\rho}(x)} - \frac{rlB}{\varphi_{\rho}(l)}\right].$$

Invoking now the assumed monotonicity of the mapping $\theta(x)$ then finally yields that

$$\tilde{\psi}''(x) \geq -\frac{2S'_{\rho}(x)B}{\sigma^2(x)}\frac{\mu(l)}{\varphi_{\rho}(l)}.$$

Letting $l \downarrow a$, invoking the inequality $\lim_{x\downarrow a} \mu(x)/\varphi_\rho(x) \leq 0$, and observing that $\lim_{l\downarrow a} \tilde{\psi}''(x) = \psi_\rho''(x)$ then finally implies that $\psi_\rho''(x) \geq 0$ for all $x \in \mathcal{I}$, proving the alleged convexity of the solution. The result that $\psi_{\hat{\rho}}(x)/\psi_{\hat{\rho}}(y) \geq \psi_\rho(x)/\psi_\rho(y)$ follows from Corollary 3 in Alvarez 2003a. Thus, it remains to establish that the percentage growth rate (i.e. logarithmic derivative) of the increasing fundamental solution $\psi_\rho(x)$ dominates the percentage growth rate of the increasing fundamental solution $\psi_{\hat{\rho}}(x)$. To observe that this is indeed the case, we notice that the inequality $\psi_{\hat{\rho}}(x)/\psi_{\hat{\rho}}(y) \geq \psi_\rho(x)/\psi_\rho(y)$ implies

$$\int_x^y \frac{\psi_{\hat{\rho}}'(s)}{\psi_{\hat{\rho}}(y)} ds \leq \int_x^y \frac{\psi_\rho'(s)}{\psi_\rho(y)} ds$$

from which the alleged result follows by the mean value theorem for integrals. In order to prove the last alleged result, we follow the approach introduced in Alvarez 2004 and define the continuous mapping $u_\rho : \mathcal{I} \mapsto \mathbb{R}$ for all $x \in \mathcal{I}$ and $y > x_0$ (with $(x_0, y) \subset \mathcal{I}$) as

$$u_\rho(x) = \frac{\psi_\rho(x)}{\psi_\rho(y)} - \frac{\psi_\rho(x) - \psi_\rho(x_0)}{\psi_\rho(y) - \psi_\rho(x_0)} = \frac{\psi_\rho(x_0)(\psi_\rho(y) - \psi_\rho(x))}{\psi_\rho(y)(\psi_\rho(y) - \psi_\rho(x_0))}.$$

The positivity, monotonicity and continuity of the increasing fundamental solution clearly implies that $u_\rho(x) > 0$ for all $x < y$ and that $\lim_{x\uparrow y} u_\rho(x) = 0$. Moreover, since $\psi_\rho(x)/\psi_\rho(y) \leq \psi_{\hat{\rho}}(x)/\psi_{\hat{\rho}}(y)$ for all $x \leq y$ and the mapping $u_\rho(x)$ is continuous, we find that for all $\varepsilon > 0$ there is an open neighborhood $(y - \delta, y)$ of y such that for all $x \in (y - \delta, y)$ we have that

$$\frac{\psi_{\hat{\rho}}(x) - \psi_{\hat{\rho}}(x_0)}{\psi_{\hat{\rho}}(y) - \psi_{\hat{\rho}}(x_0)} > \frac{\psi_{\hat{\rho}}(x)}{\psi_{\hat{\rho}}(y)} - \varepsilon > \frac{\psi_\rho(x)}{\psi_\rho(y)} - \varepsilon > \frac{\psi_\rho(x) - \psi_\rho(x_0)}{\psi_\rho(y) - \psi_\rho(x_0)} - \varepsilon.$$

However, since $\varepsilon > 0$ is arbitrary and $\lim_{x\uparrow y}(\psi_{\hat{\rho}}(x) - \psi_{\hat{\rho}}(x_0))/(\psi_{\hat{\rho}}(y) - \psi_{\hat{\rho}}(x_0)) = \lim_{x\uparrow y}(\psi_\rho(x) - \psi_\rho(x_0))/(\psi_\rho(y) - \psi_\rho(x_0)) = 1$, we find that

$$\frac{\psi_{\hat{\rho}}'(y)}{\psi_{\hat{\rho}}(y) - \psi_{\hat{\rho}}(x_0)} \leq \frac{\psi_\rho'(y)}{\psi_\rho(y) - \psi_\rho(x_0)}$$

for all $y \in (x_0, \infty)$. Proving part (B) is completely analogous. □

E Proof of Corollary 3.8

Proof. Consider the mappings $h_\rho^a(x)$ and $h_\rho^b(x)$ defined on $\mathcal{I}\backslash\mathcal{D}$. It is now clear that under the conditions of part (A) of Theorem 3.5 $h_{\hat{\rho}}^a(x) \geq$

$h^a_\rho(x)$ for all $x \in \mathcal{I}\backslash\mathcal{D}$ proving the first claim of part (A) of our corollary. Standard differentiation of the mapping $H^a_{\hat{\rho}}(x)$ yields that for all $x \in (x_0, b)\backslash\mathcal{D}$ we have

$$H^{a\prime}_{\hat{\rho}}(x) = \frac{\psi'_{\hat{\rho}}(x)}{(\psi_{\hat{\rho}}(x) - \psi_{\hat{\rho}}(x_0))^2} h^a_{\hat{\rho}}(x) \geq \frac{\psi'_{\hat{\rho}}(x)}{(\psi_{\hat{\rho}}(x) - \psi_{\hat{\rho}}(x_0))^2} h^a_\rho(x)$$

proving the second claim of part (A) of our corollary. Establishing the validity of part (B) is entirely analogous. □

F Proof of Theorem 3.11

Proof. Consider first the mapping $F_\rho(x)$ and assume that $\hat{\rho} > \rho$. Our assumptions now imply that $\psi_{\hat{\rho}}(x)/\psi_{\hat{\rho}}(y) \geq \psi_\rho(x)/\psi_\rho(y)$ for all $x \leq y$ and, therefore, that

$$F_\rho(x_0) = \frac{g(y)}{1 - \frac{\psi_\rho(x_0)}{\psi_\rho(y)}} - g(y) \leq \frac{g(y)}{1 - \frac{\psi_{\hat{\rho}}(x_0)}{\psi_{\hat{\rho}}(y)}} - g(y) = F_{\hat{\rho}}(x_0)$$

demonstrating that increased volatility increases the value at least locally at x_0. Combining this local observation with the recursive formulation of $F_\rho(x)$ now implies that

$$F_\rho(x) = (g(y) + F_\rho(x_0))\frac{\psi_\rho(x)}{\psi_\rho(y)} \leq (g(y) + F_{\hat{\rho}}(x_0))\frac{\psi_{\hat{\rho}}(x)}{\psi_{\hat{\rho}}(y)} = F_{\hat{\rho}}(x)$$

for all $x \in (a, y)$. Since $F_\rho(x) = g(x) + F_\rho(x_0) \leq g(x) + F_{\hat{\rho}}(x_0) = F_{\hat{\rho}}(x)$ on $[y, b)$ we find that increased volatility increases the value $F_\rho(x)$ on the entire state space $\mathcal{I}$. Proving this result for the value $H_\rho(x)$ is entirely analogous. □

G Proof of Theorem 4.1

Proof. Since

$$\hat{V}_\rho(x) = \mathbf{E}_x \left[e^{-r\tau^*} g(X^\rho_{\tau^*})\right]$$

we find that $\hat{V}_\rho(x) \leq V_\rho(x)$. To establish the opposite inequality we first observe that the conditions of the theorem imply that $\hat{V}_\rho \in C^1(\mathcal{I}) \cap C^2(\mathcal{I}\backslash\mathcal{D})$, that $\hat{V}''_\rho(x\pm) < \infty$ for all $x \in \mathcal{D}$, and that $\hat{V}_\rho(x) \geq \max(g(x), 0)$ for all $x \in \mathcal{I}$. Moreover, since $(\mathcal{A}\hat{V}_\rho)(x) = r\hat{V}_\rho(x)$ on (a, x^*_ρ) and

$$h'_a(x) = ((\mathcal{A}\hat{V}_\rho)(x) - r\hat{V}_\rho(x))\psi_\rho(x)m'_\rho(x) \leq 0$$

for all $x \in (x_\rho^*, b)\backslash\mathcal{D}$, we find that $(\mathcal{A}\hat{V}_\rho)(x) \leq r\hat{V}_\rho(x)$ for all $x \in \mathcal{I}\backslash\mathcal{D}$. Thus, $\hat{V}_\rho(x)$ constitutes an r-excessive majorant of the exercise payoff $g(x)$ for the underlying diffusion X_t^ρ. However, since $V_\rho(x)$ is the least of these majorants we find that $\hat{V}_\rho(x) \geq V_\rho(x)$ and, therefore, that $\hat{V}_\rho(x) = V_\rho(x)$. The latter part of our theorem is now a straightforward implication of part (A) of Theorem 3.1 and part (A) of Theorem 3.2. □

H Proof of Theorem 5.1

Proof. It is now clear that since the proposed value is attained by applying an admissible impulse control strategy we have that $\hat{J}_\rho(x) \leq J_\rho(x)$ for all $x \in \mathcal{I}$. In order to prove the opposite inequality we first observe that $\hat{J}_\rho \in C^1(\mathcal{I}) \cap C^2(\mathcal{I}\backslash\{y_\rho^*\})$, that $\hat{J}_\rho(x) \geq 0$, and that $\hat{J}_\rho(x) = \hat{J}_\rho(x_0) + g(x)$ for all $x \geq y_\rho^*$. However, since

$$\hat{J}_\rho(x) - \hat{J}_\rho(x_0) - g(x) = H_\rho^a(y_\rho^*)(\psi_\rho(x) - \psi_\rho(x_0)) - g(x)$$

for all $x \in (a, y_\rho^*)$ and $H_\rho^a(y_\rho^*)(\psi_\rho(x) - \psi_\rho(x_0)) - g(x)$ was assumed to be non-increasing on (a, y_ρ^*), we observe that $\hat{J}_\rho(x) \geq \hat{J}_\rho(x_0) + g(x)$ for all $x \in \mathcal{I}$. Moreover, since $(\mathcal{A}_\rho\hat{J}_\rho)(x) - r\hat{J}_\rho(x) = 0$ on (a, y_ρ^*) and by assumption

$$c_\rho^{a\prime}(x) = ((\mathcal{A}_\rho\hat{J}_\rho)(x) - r\hat{J}_\rho(x))\psi_\rho(x)m'(x) \leq 0$$

for all $x \in (y_\rho^*, b)\backslash\mathcal{D}$ we find that $(\mathcal{A}_\rho\hat{J}_\rho)(x) - r\hat{J}_\rho(x) \leq 0$ for all $x \in \mathcal{I}\backslash(\mathcal{D} \cup \{y_\rho^*\})$ and, therefore, that $\hat{J}_\rho(x) \geq J_\rho(x)$ for all $x \in \mathcal{I}$, thus completing the proof of the first claim of our theorem. The latter part of our theorem follows directly from Theorem 3.1 and Corollary 3.8. □

I Proof of Corollary 5.2

Proof. Consider first the mapping

$$h_\rho^a(x) = \frac{g'(x)}{\psi_\rho'(x)}(\psi_\rho(x) - \psi_\rho(x_0)) - g(x)$$

Our assumptions imply that $h_\rho^a(x)$ is decreasing and satisfies the conditions $h_\rho^a(x_0) = -g(x_0) > 0$ and $h_\rho^a(x_\rho^*) = -g'(x_\rho^*)\psi_\rho(x_0)/\psi_\rho'(x_\rho^*) < 0$ demonstrating that $H_\rho^a(x)$ attains a unique maximum on (x_0, b). It is, therefore, now sufficient to demonstrate that the proposed value function satisfies the standard quasi-variational inequalities guaranteeing the optimality of the proposed policy. The assumed monotonicity of the mapping $g'(x)/\psi'_\rho(x)$ implies that

$$(\mathcal{A}g)(x) - rg(x) \leq r\left(\frac{g'(x)}{\psi'_\rho(x)}\psi_\rho(x) - g(x)\right)$$

for all $x \in \mathcal{I}\backslash\mathcal{D}$. Thus, we observe that on (y^*_ρ, b)

$$\begin{aligned}\Delta_1(x) &= (\mathcal{A}\hat{J}_\rho)(x) - r\hat{J}_\rho(x) \\ &\leq r\left(\frac{g'(x)}{\psi'_\rho(x)}\psi_\rho(x) - \frac{g'(y^*_\rho)}{\psi'_\rho(y^*_\rho)}\psi_\rho(y^*_\rho)\right) - r(g(x) - g(y^*_\rho)).\end{aligned}$$

Since $\Delta_1(y^*_\rho) \leq 0$ and $\Delta_1(x)$ is decreasing by the monotonicity of $g'(x)/\psi'_\rho(x)$, we find that $(\mathcal{A}\hat{J}_\rho)(x) - r\hat{J}_\rho(x) \leq 0$ for all $x \in \mathcal{I}\backslash\mathcal{D}$. Consider now for all $x \in (a, y^*_\rho)$ the difference

$$\Delta_2(x) = \hat{J}_\rho(x) - \hat{J}_\rho(x_0) - g(x) = \frac{g'(y^*_\rho)}{\psi'_\rho(y^*_\rho)}(\psi_\rho(x) - \psi_\rho(x_0)) - g(x)$$

Since $\Delta_2(y^*_\rho) = 0$ and $\Delta_2(x)$ is decreasing by the monotonicity of $g'(x)/\psi'_\rho(x)$, we find that $\Delta_2(x) \geq 0$ for all $x \in (a, y^*_\rho)$ and, therefore, that $\hat{J}_\rho(x) \geq \hat{J}_\rho(x_0) + g(x)$ for all $x \in \mathcal{I}$. Since $g''(x\pm) < \infty$ for all $x \in \mathcal{D}$, we find that the conditions of Theorem 5.1 are satisfied thus completing the proof of our corollary. $\square$