TURUN KAUPPAKORKEAKOULUN JULKAISUJA

PUBLICATIONS OF THE TURKU SCHOOL
OF ECONOMICS AND BUSINESS ADMINISTRATION

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OPTIMAL INVESTMENT FOR A PENSION INSURER WITH SOLVENCY CONSTRAINTS IN A ONE-PERIOD DIFFUSION MODEL

Sarja Keskustelua ja raportteja/
Series Discussion and Working Papers
2:2006

ISBN 951-564-344-9 (nid.) 951-564-345-7 (PDF)
ISSN 0357-4687 (nid.) 1459-7632 (PDF)
UDK 519.2
519.245
368.03

Esa Print Tampere, Tampere 2006

ABSTRACT

Investment decisions are especially crucial for an insurer with liabilities defined by law, since in this situation the management can control only the asset side of the balance sheet. We present in this paper a one-period diffusion model for assets and liabilities of a pension insurer managing a statutory pension scheme subject to statutory solvency constraints. By approximating probabilistic constraints suitably with piecewise polynomial functions, we construct a performance function which incorporates both the insurer's risk preferences and solvency considerations, and derive the optimal control policy for assets using the Hamilton–Jacobi–Bellman equation and the theory of forward–backward stochastic differential equations. We discuss some properties of the optimal policy and illustrate the model with a simulation example. As a special area of application we have in mind the Finnish statutory pension scheme.

Keywords: Asset–liability management; Monte Carlo simulation; Pension insurance; Solvency; Stochastic control; Stochastic differential equations

JEL Subject Classification: C15; C61; G11; G22

MSC Subject Classification: IM10; IM22; IE53; IB81

CONTENTS

1 INTRODUCTION

A pension insurance company managing a partly funded statutory pension scheme is responsible for investing some predetermined proportion of received contributions profitably while the rest of these contributions is used to finance current pensions on a pay-as-you-go basis. Excess profits from investment of assets corresponding to funded part of pensions may later be used to finance pensions in order to keep the contribution rate within reasonable bounds. The current Finnish statutory pension scheme is of this type.

As in a statutory scheme the company management has only restricted possibilities to control the liability side of the balance sheet, choosing a profitable and safe investment strategy is of crucial importance in controlling the financial status of the company. Safety in this context refers to the insurer's solvency position. Pension insurers' activities are usually heavily regulated and their solvency position is subject to public supervision: typically the insurer is required to hold a certain solvency margin over and above the liabilities and investments in some asset categories may be restricted or completely forbidden. If the insurer does not satisfy the statutory solvency requirements or does not comply with regulations, the supervising authorities may intervene, meaning that in such a situation the management's freedom to choose business strategies will be restricted considerably.

Because of preceding considerations it would be of interest to determine an investment strategy which maximizes expected utility of returns and simultaneously keeps the probability of ruin (i.e. an unacceptable solvency position) on a reasonably low level. We present in this study a simple diffusion model for assets and liabilities of a pension insurance company and derive the optimal control policy with respect to a performance function incorporating solvency considerations and the investor‘s risk preferences. This performance function is constructed using an utility function to represent the insurer's risk preferences and smooth approximations of probabilistic constraints to take the solvency position into account. As a special area of application the Finnish pension scheme is constantly kept in mind.

We consider separately two wealth processes: that of assets backing the liabilities and that of assets corresponding to solvency margin. Our goal is to control the ratios of these two monetary quantities to the monetary value of liabilities in such a way that at the terminal time T the value of assets backing liabilities exceeds the value of liabilities and the value of assets corresponding to solvency margin exceeds the required solvency margin with an acceptably high probability. This separation is motivated by the fact that regulations tend to require considerable amount of prudence from investment strategies of assets backing liabilities, which makes attaining higher returns difficult. Thus the assets corresponding to solvency margin offer a way to generate higher returns by allowing more risky strategies for these assets. It is worth mentioning that recently there has been much discussion in Finland about increasing the proportion of stock investments in the pension insurers' portfolios to obtain higher returns in order to lessen the need to increase contribution rates in the future.

In line with financial literature, we assume investment strategies to be self-financing. In practice, both inflow and outflow of assets will occur: there are yearly outflows in the form of bonuses given to the sponsors and dividends paid to the shareholders, as well as monthly or quarterly out- and inflows in the form of pension payments by the insurer and contributions by the sponsors. We deal with these aspects by introducing some additional stochastic fluctuation in the model to reflect the effect of inflows and outflows during a one-year period and by setting the terminal time T equal to one year. Decisions concerning distribution of profits and allocation of remaining assets are then made at the end of the period and these decisions determine the initial state of the insurer at the beginning of the next period. Liabilities in pension insurance can have a very long duration, and a planning horizon of some 20–30 years is not unreasonable. Our model can be used to determine the optimal policy for a single period at a time given the initial state. Decisions about distribution of profits and asset allocation link then the different periods together, and the effects of different rules determining the bonuses and dividends can be investigated by simulations proceeding year by year. It is useful to be able to compare the consequences of adopting different rules concerning the distribution of profits, provided that the alternatives to be investigated are chosen appropriately.

The contents of this study are as follows. In Section 2 the proposed diffusion model is presented. In the next section we define the form of our performance function as a sum of a certain utility function and an additional cost function reflecting solvency considerations. In Section 4 we derive the general form of optimal control policy in terms of the value function Φ using Hamilton–Jacobi–Bellman equation and obtain an expression for the value function Φ using the theory of forward–backward stochastic differential equations. The behavior of the optimal strategy and its implications for the applicability of the model are discussed in the following section. In Section 6 we present a simple illustrative simulation example. Concluding remarks are made in Section 7. For completeness some details concerning the solution of a one-dimensional Cauchy problem are presented in appendix A. Also, some details of Finnish solvency regulations for pension insurers can be found in appendix B.

2 PROPOSED MODEL FOR ASSETS AND LIABILITIES

We consider the usual Black–Scholes type market consisting of a risky asset S with a geometric Brownian motion price process

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

with $W(t)$ a standard Wiener process and a risk-free asset B with price process

$$dB(t) = rB(t)dt,$$

where r is the risk-free rate of return. The monetary value of the insurance company's total liabilities (the funded part of pensions) $L(t)$ is assumed to evolve as a geometric Brownian motion

$$dL(t) = \nu L(t)dt + \gamma L(t)d\hat{W}(t), \tag{2.1}$$

where $\hat{W}(t)$ is a Wiener process independent of $W(t)$. The drift ν need not be directly based on the observed evolution of technical reserves; instead, it may reflect the required return on technical reserves determined by the regulator (technical interest rate). In the latter case parameter γ reflects the uncertainty about the development of the technical interest rate. The insurer has some initial wealth $V(0)$, which is divided into two parts: assets backing liabilities $V_L(0)$ and solvency margin $V_S(0)$. Given an initial allocation $(V_L(0), V_S(0))$ and a terminal time T, the investment strategy consists of deciding the proportions of risky assets $(\theta^L(t), \theta^S(t)) \in [0,1]^2$ for every $t \in (0,T]$. Restriction $\theta \in [0,1]$ is natural as gearing the portfolio by short selling or borrowing is a very risky activity and is in general forbidden for pension insurers, which are expected to invest prudently. Both portfolios are assumed to be self-financing. No inflows or outflows occur during the interval $(0,T]$ and no transfers are made between the two portfolios.

In reality pension payments made to beneficiaries induce outflow of assets and contributions received from sponsors in turn induce inflow of assets. These transactions occur usually monthly or quarterly. Thus with terminal time T longer or equal to one year the self-financing assumption is unrealistic. Even if outflows and inflows balance each other on average, they will cause some fluctuations in the level of total asset value. To incorporate this into our model without giving up the mathematically convenient self-financing assumption we introduce as an additional source of uncertainty a Wiener process $\tilde{W}(t)$ affecting the value of assets backing liabilities. With this addition stochastic differential equations for value processes of the assets backing liabilities and the assets corresponding to solvency margin are

$$dV_L(t) = \big(r - (r-\mu)\theta^L(t)\big)\, V_L(t)dt + \sigma\theta^L(t)V_L(t)dW(t) + \tilde{\sigma}V_L(t)d\tilde{W}(t), \tag{2.2}$$

and

$$dV_S(t) = \big(r - (r-\mu)\theta^S(t)\big)\, V_S(t)dt + \sigma\theta^S(t)V_S(t)dW(t), \tag{2.3}$$

respectively. We point out that investment strategy θ^L has no influence on the $d\tilde{W}(t)$ term in equation 2.2. This stands to reason as the company's current investment policy can be expected to have no immediate effect on outgoing pension payments or incoming contributions.

We set $T = 1$ and restrict our attention to determining the optimal strategy for a given initial wealth and its allocation. Once this is done, several different rules determining the distribution of profits and allocation of remaining wealth at the end of a period can be tested and compared by simulating the model period by period. Anticipated future changes in economic environment can be taken into account by specifying different parameter values for different periods. Expert knowledge is needed in determining appropriate alternative rules to be considered and in specification of model parameters.

The processes V_L, V_S and L constitute a 3-dimensional system. We can reduce the dimension by considering instead of these money-valued processes the ratios $X := V_L/L$ and $Y := V_S/L$. Ratios of this type are commonly used in analyzing insurance business (see Daykin et al.

(1994)) and are in fact very suitable for our current needs, as statutory solvency requirements often define the size of required solvency margin as a specified proportion of the value of liabilities. In our model solvency constraints can then be expressed in form $X > 1$ and $Y > \beta$, where β is the required solvency margin as a proportion of liabilities.

Remark 2.1. *Solvency requirements often attempt to take into account the risk structure of an insurer's asset portfolio and are dependent on the composition of the portfolio. This is the case in the Finnish pension scheme. In the example of Section 6 a prudent way to deal with this kind of requirement is illustrated.*

Derivation of explicit expressions for ratio processes X and Y is straightforward, since the processes $V_L(t)$, $V_S(t)$ and $L(t)$ are all of form

$$Z(t) = Z(0) \exp\left\{ \int_0^t \alpha_Z(t) dt \right\} M(t),$$

where Z is a generic notation for a stochastic process, $\alpha_Z(t)$ is the drift and $M(t) = \exp\{\xi B(t) - (1/2)\xi^T \xi t\}$ is an exponential martingale with ξ being the diffusion matrix and $B(t)$ the vector of driving Wiener processes. Stochastic differential equations for X and Y are

$$dX = \big(r - (r-\mu)\theta^L - \nu\big) X dt + \sigma\theta^L X dW + \tilde{\sigma} X d\tilde{W} - \gamma X d\hat{W}, \quad (2.4)$$

and

$$dY = \big(r - (r-\mu)\theta^S - \nu\big) Y dt + \sigma\theta^S Y dW - \gamma Y d\hat{W}, \quad (2.5)$$

respectively (in above equations argument t is omitted for brevity). These equations determine a two-dimensional diffusion process with drift vector

$$b = \big((r - (r-\mu)\theta^L - \nu)X, (r - (r-\mu)\theta^S - \nu)Y\big) \quad (2.6)$$

and diffusion matrix

$$\underline{\sigma} = \begin{pmatrix} \sigma\theta^L X & \tilde{\sigma} X & -\gamma X \\ \sigma\theta^S Y & 0 & -\gamma Y \end{pmatrix}. \quad (2.7)$$

3 CONSTRUCTION OF PERFORMANCE FUNCTION

We assume that the risk preferences of the insurer can be represented by a utility function U. Ideally we would like to maximize the expected utility $\mathbb{E}[U(x)]$ of values of ratios $X(T)$ and $Y(T)$ subject to probabilistic constraints

$$\mathbb{P}^{V_L(0)}(X(T) > 1) \geq \alpha_1 \text{ and } \mathbb{P}^{V_S(0)}(Y(T) > \beta) \geq \alpha_2, \tag{3.1}$$

where α_i, $i = 1, 2$, are the required probabilities of acceptable solvency position at time T. For a given initial allocation $(V_L(0), V_S(0))$ we have, due to the no transfers assumption, two independent controls θ^L, θ^S. As both ratios are required to satisfy their respective solvency requirements on their own and there are often special regulatory restrictions on investment strategies for assets backing the liabilities, this separation feature seems to be reasonable.

Our objective function and constraints are both concerned with the behavior of the ratio processes at terminal time T. We incorporate the constraints into the objective via a cost function and define the performance function

$$J^\theta(V(0)) = \mathbb{E}^{V(0)}\left[F(X(T), Y(T))\right], \tag{3.2}$$

where $F : \mathbb{R}_+^2 \to \mathbb{R}$ is a function describing the utility of state $(X(T), Y(T))$ and incorporating in some way the effect of terminal solvency position. The value function is defined as

$$\Phi(V(0)) := \sup_\theta J^\theta(V(0)) = J^{\theta^*}(V(0)), \tag{3.3}$$

where $\theta^* = (\theta^{L*}, \theta^{S*})$ is the optimal control. The solvency set

$$G = [0, T] \times [1, \infty) \times [\beta, \infty).$$

By the reasoning presented in the beginning of this section, it is natural to assume that F is additive with respect to X and Y, i.e. $F(x, y) =$

$F_1(x)+F_2(y)$. Specifically, we assume F to be of form $F = U(X(T)) + U(Y(T)) + g(X(T)) + h(Y(T))$, where U describes utility of terminal ratios and g and h are related to solvency constraints. This is known in economics as a separable utility function.

Our choice of utility function U is based on its risk aversion characteristics and its compatibility with the functional form of optimal control derived in Section 5. The choice $U(x) = (1/a)x^a$, $0 < a < 1$, gives us some flexibility in choosing the degree of risk aversion by specification of a. This is known as HARA-utility. Risk-aversion is reasonable, as pension insurance companies are expected to invest prudently. When $a \to 1$, U approaches the risk-neutral linear utility. Additional functions h and g are needed because constraints presented in equation 3.1 are difficult to handle due to lack of smoothness (though maybe not impossible, see remark 3.3). The basic idea is to construct well-behaving, smooth functions h and g so that they “punish“ the company for unacceptable solvency positions by reducing total utility in a way which creates incentives towards an acceptably safe strategy (in a probabilistic sense).

As a starting point for constructing h and g, consider the constraints determined by equation 3.1. A kind of Lagrange multiplier approach presented in Øksendal (2003) offers a method for handling probabilistic constraints of type

$$\mathbb{E}^{Z(0)}[M(Z(T))] = 0$$

on the terminal state of controlled process $Z(t)$, where M is a continuous function. Constraints given in equation 3.1 can be expressed as expectations of indicator functions $\chi_{[\beta,\infty)}$:

$$\mathbb{E}^{V(0)}[\chi_{[\beta,\infty)}(X(T)) - \alpha] = 0. \tag{3.4}$$

However, $\chi_{[\beta,\infty)}$ is not continuous. However, it follows from the next theorem that it can be approximated with arbitrary smoothness.

Theorem 3.1. *Let* $0 < r_1 < r_2$. *Then there exists a* C^∞ *function* f :

$\mathbb{R}^n \to [0,1]$ *such that* $f(x) = 1$, *if* $\| x \| \leq r_1$, *and* $f(x) = 0$, *if* $\| x \| \geq r_2$.

Proof. Define $g : \mathbb{R} \to \mathbb{R}$ as follows: $g(t) = 0$, if $t \leq 0$, and $g(t) = \exp(-1/t)$, if $t > 0$. It is seen by induction that if $t \in (0, \infty)$, then for each $k \in \mathbb{N}$ the kth derivative $g^{(k)}(t) = P_k(1/t)\exp(-1/t)$, where P_k is a kth degree polynomial. Then $\lim_{t \to 0+} g^{(k)}(t) = 0$ for each $k \in \mathbb{N}$ and thus $g \in C^\infty$. Define now a nonnegative C^∞ function h by $h(t) = g(t - r_1^2)g(r_2^2 - t)$. This function is identically zero if $t \notin (r_1^2, r_2^2)$. Thus $\phi(t) = \int_{-\infty}^{t} h(u)du$ is a C^∞ function such that $\psi(t) = 0$ if $t \leq r_1^2$ and $\psi(t)$ is a constant c if $t \geq r_2^2$. Multiplying ψ with a suitable $K > 0$, we may assume that $c = 1$. Now $f_0(x) := 1 - \psi(\| x \|^2)$ is the desired f. □

Using the notation of the previous proof, the indicator $\chi_{[\beta,\infty)}$, $\beta > 0$, can be approximated with arbitrary smoothness by choosing $r_1 < \beta < r_2$ and defining the approximation piecewise by $f^\beta_{r_1,r_2}(x) = f_0(-x)$ if $x \in (-\infty, 2\beta - r_1)$ and $f^\beta_{r_1,r_2}(x) = 1$, otherwise.

For simplicity, we take $r_1 = r_2 = n$ and use as an approximation a piecewise determined C^2 function $f^\beta_n(x)$, which on the interval $(\beta - 1/n, \beta + 1/n)$ is a polynomial and otherwise is equal to $\chi_{[\beta,\infty)}$. Lemma 3.2 says that if $\alpha < 1$, we can by choosing a sufficiently large value of n obtain a new constraint which is stronger than the original by replacing the indicator with f^β_n and α with $\alpha + \epsilon_n < 1$ in equation 3.4. This lemma applies to any approximation which takes values in $[0,1]$ and vanishes outside interval $(\beta - 1/n, \beta + 1/n)$.

Lemma 3.2. *There is* $n_0 \in \mathbb{N}$ *such that if* $n \geq n_0$, *then*

$$\mathbb{E}^{X(0)}[f^\beta_n(X(T)) - \alpha - \epsilon_n] = 0 \Rightarrow \mathbb{E}^{X(0)}[\chi_{[\beta,\infty)}(X(T)) - \alpha] \geq 0.$$

Proof. For notational convenience we suppress the superscripts $X(0)$ in this proof (i.e. $\mathbb{E}^{X(0)} = \mathbb{E}$). Since $X(T)$ is the quotient of two lognormal random variables, it is lognormally distributed and consequently has a bounded C^∞ density $g_{X(T)}$. Then

$$\mathbb{P}\left(X(T) \geq \beta\right) = \mathbb{E}[\chi_{[\beta,\infty)}(X(T))] = \int_{\mathbb{R}_+} \chi_{[\beta,\infty)}(u) g_{X(T)}(u) du.$$

On the other hand $f_n^\beta = \chi_{[\beta,\infty)} + \chi_{[\beta-1/n,\beta)} f_n^\beta - \chi_{[\beta,\beta+1/n)}(1 - f_n^\beta)$ and thus the expectation $\mathbb{E}[f_n^\beta(X(T))] = \int_{\mathbb{R}_+} f_n^\beta(u) g_{X(T)}(u) du$ equals

$$\mathbb{P}(X(T) \geq \beta) + \int_{[\beta-1/n,\beta)} f_n^\beta(u) g_{X(T)}(u) du - \int_{[\beta,\beta+1/n)} (1 - f_n^\beta(u)) g_{X(T)}(u) du.$$

Because $0 \leq f_n^\beta \leq 1$, it follows that with C_T being the upper bound for $g_{X(T)}$ we have the inequalities

$$\mathbb{P}(X(T) \geq \beta) - 2C_T/n \leq \mathbb{E}[f_n^\beta(X(T))] \leq \mathbb{P}(X(T) \geq \beta) + 2C_T/n.$$

Then $\mathbb{E}[f_n^\beta(X(T))] = \alpha + \epsilon_n$ implies that $\mathbb{P}(X(T) \geq \beta) \geq \alpha + \epsilon_n - 2C_T/n$ for any $\epsilon_n > 0$ satisfying condition A: $\alpha + \epsilon_n \leq 1$. This condition is necessary, since $\mathbb{E}[f_n^\beta(X(T))] \leq 1$. Choosing $\epsilon_n = 2C_T/n$, we get $\mathbb{P}(X(T) \geq \beta) \geq \alpha$. However, in order to satisfy condition A we must have $n \geq 2C_T/(1-\alpha) := n_0$. □

Previous considerations suggest the following specification of F:

$$F(X,Y) = (1/a_1)X^{a_1} + (1/a_2)Y^{a_2} + \lambda_1(f^1(X) - \alpha_1) + \lambda_2(f^2(Y) - \alpha_2), \tag{3.5}$$

with $0 < a_1, a_2 < 1$, $\lambda_1, \lambda_2 \in \mathbb{R}$ and $f^1 = f_n^1$, $f^2 = f_m^\beta$. Unfortunately an explicit value for the upper bound C_T used in the proof of lemma 3.2 is hard to find analytically and simulations of the process seem to indicate that for reasonable time spans $C_T > 2$. For such values of C_T acceptable confidence levels ($\alpha \geq 0.95$) lead to large n_0 values ($n_0 \geq 40$). As will be seen in Section 5, a large value of n tends to make investment strategy less prudent and decreases possibilities of achieving low ruin probabilities with reasonable initial allocations (naturally with T fixed and initial wealth large enough, ruin probability will be low for almost any investment strategy). So while increasing n makes the approximation of the probabilistic constraint better, it will also at some point begin to make our investment policy myopic. As n values required for satisfying the probabilistic constraints are too large, the question is, whether choosing some smaller value of n can lead to a strategy which is acceptably safe from a practical point of view. This can be investigated with Monte Carlo simulations. The choice of n may be based on simulation experiments after the other model parameters have been specified, as it can be expected that reasonable n values to be investigated will be relatively few in number. See the example of Section 6 for an illustration.

Remark 3.3. *Concerning difficulties caused by discontinuous terminal conditions we point out that recently in Zhang (2005) some results have been obtained for a class of decoupled forward–backward stochastic differential equations with discontinuous terminal data.*

4 OPTIMAL CONTROL AND VALUE FUNCTION

Optimal control policy θ^* in terms of value function Φ is determined by the generator

$$L^\theta = \frac{\partial}{\partial t} + \sum_{i=1}^{2} b_i \frac{\partial}{\partial x_i} + \sum_{i,j=1}^{2} (\underline{\sigma}\,\underline{\sigma}^T)_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \tag{4.1}$$

of the diffusion defined by equations 2.4 and 2.5. Coefficients b_1 and b_2 in equation 4.1 are as in equation 2.6 and $\underline{\sigma}$ is defined in equation 2.7.

Assuming sufficient regularity of value function Φ, most notably finiteness and C^2 smoothness in G^0, see Øksendal (2003) for details, the optimal control satisfies Hamilton–Jacobi–Bellman equation

$$\begin{aligned}&\sup_\theta \big\{\Phi_t + (r - (r-\mu)\theta^L - \nu)x_1\Phi_1\Phi_t + (r - (r-\mu)\theta^S - \nu)x_2\Phi_2 + \\ &+\tfrac{1}{2}(\sigma^2(\theta^L)^2 + \tilde{\sigma}^2 + \gamma^2)x_1^2\Phi_{11} + (\sigma^2\theta^L\theta^S - \gamma^2)x_1x_2\Phi_{12} + \\ &+\tfrac{1}{2}(\sigma^2(\theta^S)^2 + \gamma^2)x_2^2\Phi_{22}\big\} = 0.\end{aligned} \tag{4.2}$$

Differentiation with respect to θ^L and θ^S and setting the obtained partial derivatives equal to zero yields the necessary conditions for supremum

$$\begin{cases} (\mu - r)\Phi_1 + \sigma^2\theta^L x_1\Phi_{11} + \sigma^2\theta^S x_2\Phi_{12} = 0 \\ (\mu - r)\Phi_2 + \sigma^2\theta^L x_1\Phi_{12} + \sigma^2\theta^S x_2\Phi_{22} = 0. \end{cases}$$

Solving this pair of equations gives the optimal control $(\theta^{L*}, \theta^{S*})$ in terms of value function Φ:

$$\left(\frac{(r-\mu)\Phi_2\Phi_{12} + (\mu - r)\Phi_1\Phi_{22}}{x_1\sigma^2(\Phi_{12}^2 - \Phi_{11}\Phi_{22})}, \frac{(r-\mu)\Phi_1\Phi_{12} + (\mu - r)\Phi_2\Phi_{11}}{x_2\sigma^2(\Phi_{12}^2 - \Phi_{11}\Phi_{22})}\right). \tag{4.3}$$

The general form of optimal control policy is defined by equation 4.3. Our choice of performance function will determine the functional form of value function Φ.

Next we derive an explicit expression for value function Φ. The chosen approach is to form a system of forward–backward stochastic differential equations (fbsdes for short) by using the control problem's adjoint equation as the backward equation and solve this system for the unique adapted solution $((X, Y), p, K)$ using the four step scheme presented in Ma et al. (1994). Then the backward component p is the spatial gradient of Φ and can be integrated to obtain the value function.

We consider first some results for general fbsdes. Let $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\})$ be a filtered probability space, W_t a d-dimensional Wiener process and $\mathcal{F}_t = \sigma\{W_s\,;\, 0 \le s \le t\}$ augmented by the $\mathbb{P}$-null sets in $\mathcal{F}$. The general form of a fbsde is

$$\begin{aligned} X_t &= X_0 + \int_0^t b(s, X_s, Y_s, Z_s)ds + \int_0^t \sigma(s, X_s, Y_s, Z_s)dW_s \\ Y_t &= g(X_T) + \int_t^T \hat{b}(s, X_s, Y_s, Z_s)ds + \int_t^T \hat{\sigma}(s, X_s, Y_s, Z_s)dW_s, \end{aligned} \tag{4.4}$$

with $X_t \in \mathbb{R}^n$ being the forward component and $Y_t \in \mathbb{R}^m$ the backward component. We are interested in obtaining adapted solutions as defined below.

Definition 4.1. *A $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$-valued triple of processes (X, Y, Z) is an* adapted solution *of fbsde 4.4 if it is $\{\mathcal{F}_t\}$-adapted, square integrable and satisfies equations 4.4 $\mathbb{P}$-a.s.*

It is the extra process $Z_t \in \mathbb{R}^{m \times d}$ which makes the existence of an adapted solution possible. Solvability of a fbsde over an arbitrarily prescribed time horizon T can be a fairly delicate matter: in Ma and Yong (1995), the solvability problem is converted to a problem of finding the nodal set of the viscosity solution to a certain Hamilton–Jacobi–Bellman equation by first allowing Z_t to be an adapted measure-valued process and allowing the underlying probability space to change when necessary.

Fortunately, it can be shown that in our model the fbsde of interest has a unique adapted solution. Unique means here and in the following $\mathbb{P}$-a.s. unique. The sde for the vector of controlled ratio processes (X, Y) is now the forward sde and the adjoint equation corresponding to this controlled process is the backward sde. By a version of Pontryagin's maximum principle for It diffusions a control θ such that the corresponding adjoint equation for controlled process X has a solution satisfying certain conditions is an optimal control (see Øksendal and Sulem

(2005)). The forward sde is given in equations 2.4 and 2.5, and the initial condition is $(X(0), Y(0)) = (X_0, Y_0)$. Denoting by ∂_i the partial derivative with respect to variable i, the adjoint equation is a linear bsde

$$\begin{pmatrix} dp_1(t) \\ dp_2(t) \end{pmatrix} = \begin{pmatrix} [\nu - r - (\mu - r)\theta^L]p_1(t) - \partial_X[\text{tr}(K^T\underline{\sigma})] \\ \left[\nu - r - (\mu - r)\theta^S\right] p_2(t) - \partial_Y[\text{tr}(K^T\underline{\sigma})] \end{pmatrix} dt + K d\mathbf{W}(t) \tag{4.5}$$

where K is an unknown matrix, $\underline{\sigma}$ is given in equation 2.7 and $\mathbf{W} = (W, \tilde{W}, \hat{W})$. The terminal condition for the adjoint equation is $p(T) = \nabla F(X(T), Y(T))$, and by the additivity assumption of Section 3, $\nabla F(x, y) = (D_x[F_1(x)], D_y[F_2(y)])$.

The drift vector in the adjoint equation can be simplified by performing the multiplication $K^T\underline{\sigma}$. The trace of the resulting matrix is (using notation $K = (k_{ij})$)

$$k_{11}\sigma\theta^L X + k_{21}\sigma\theta^S Y + k_{12}\tilde{\sigma}X - \gamma(k_{13}X + k_{23}Y),$$

which gives

$$\partial_X[\text{tr}(K^T\underline{\sigma})] = k_{11}\sigma\theta^L + k_{12}\tilde{\sigma} - \gamma k_{13}$$

and

$$\partial_Y[\text{tr}(K^T\underline{\sigma})] = k_{21}\sigma\theta^S - \gamma k_{23}.$$

Then the drift vector in equation 4.5 can be written as

$$\hat{b} := \begin{pmatrix} \left[\nu - r - (\mu - r)\theta^L\right] p_1(t) - k_{11}\sigma\theta^L - k_{12}\tilde{\sigma} + \gamma k_{13} \\ \left[\nu - r - (\mu - r)\theta^S\right] p_2(t) - k_{21}\sigma\theta^S + \gamma k_{23} \end{pmatrix}. \tag{4.6}$$

We see that in our model the forward and backward equations are decoupled, that is, the coefficients of the forward (resp. backward) sde do not depend on process (p, K) (resp. (X, Y)). We may prove the existence of a unique adapted solution $((X, Y), p, K)$ by proving separately the existence of a unique adapted square integrable solution (X, Y) of the forward sde and the existence of a unique adapted square integrable solution (p, K) of the bsde. Once the existence of a unique adapted solution is established, we may apply the four step scheme to obtain a solution candidate. If this candidate is an adapted solution in the sense of definition 4.1, it is then the desired solution.

In the following considerations $\| \cdot \|$ denotes for vectors $x \in \mathbb{R}^n$ the usual Euclidean norm and for $z \in \mathbb{R}^{n \times d}$ the norm defined by $\text{tr}(zz^T)^{1/2}$. A standard theorem stated in Rogers and Williams (2000b) can be applied to obtain the existence and uniqueness of adapted solution to a forward sde of form

$$X(t) = x + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s), \tag{4.7}$$

where $W(s)$ is a d-dimensional Wiener process.

Theorem 4.2 (It). *Consider equation 4.7. If there exists $K < \infty$ such that*

$$\| \sigma(t, x(t)) - \sigma(t, y(t)) \| \leq K \sup \{ \| x(s) - y(s) \| : \ s \leq t \}$$

and

$$\| b(t, x(t)) - b(t, y(t)) \| \leq K \sup \{ \| x(s) - y(s) \| : \ s \leq t \}$$

for all $t \geq 0$, $x(s), y(s) \in \mathbb{R}^n$ (Lipschitz condition), and if for each $T > 0$ there is some C_T such that

$$\| \sigma(s, 0) \| + \| b(s, 0) \| \leq C_T$$

whenever $s \leq T$, then there is exactly one (modulo indistinguishability) square integrable semimartingale $X(t)$ satisfying sde 4.7.

Since a semimartingale is always adapted (see Protter (2004) for more information), the semimartingale $X(t)$ of theorem 4.2 is the unique adapted square integrable solution of equation 4.7. The forward sde defined by equations 2.4 and 2.5 is of form 4.7 and satisfies the conditions of theorem 4.2.

As our bsde is the familiar adjoint equation of stochastic control theory, we restrict our attention to bsdes with $\hat{\sigma}(t, x, y, z) = z$. We consider the following bsde

$$Y(t) = \xi + \int_t^T f(s, Y(s), Z(s))ds + \int_t^T Z(s)dB(s), \tag{4.8}$$

where the driver $f(s, y, z)$ is assumed to be continuous in (y, z) for almost all (y, z). Furthermore, we assume that there exist constants $M > 0$ and $\alpha \in [0, 1]$ such that for a.e. $t \in [0, T]$

$$\| f(t, y, z) \| \leq M(1 + \| y \|^\alpha + \| z \|^\alpha), \ \mathbb{P}\text{-a.s.}$$

Following Bahlali (2002), we make two definitions.

Definition 4.3. *Suppose that there exists $L > 0$ such that for a.e. $t \in [0, T]$*

$$\| f(t, y, z) - f(t, y', z') \| \leq L(\| y - y' \| + \| z - z' \|), \ \mathbb{P}\text{-a.s.}$$

Then we say that f is Lipschitz.

Definition 4.4. *Suppose that for every $N \in \mathbb{N}$ there exists $L_N > 0$ such that for a.e. $t \in [0, T]$*

$$\| f(t, y, z) - f(t, y', z') \| \leq L_N(\| y - y' \| + \| z - z' \|), \ \mathbb{P}\text{-a.s.}$$

whenever $\| y \| \leq N$, $\| y' \| \leq N$, $\| z \| \leq N$ and $\| z' \| \leq N$. Then we say that f is locally Lipschitz.

Now we state a theorem proved in Bahlali (2002).

Theorem 4.5. *Let f be locally Lipschitz and ξ be a square integrable random variable. Assume that there exists $L > 0$ such that $L_N = L + \sqrt{\log N}$. Then equation 4.8 has a unique adapted square integrable solution.*

This result for bsdes with a locally Lipschitz driver contains as a special case bsdes with a Lipschitz driver. It follows from the mean value theorem that an autonomous function with bounded derivatives is Lipschitz. In our case the driver $f(t, (X, Y), p, K) = \hat{b}(p, K)$ is autonomous and linear in (p, K), which implies that it has bounded derivatives in (p, K). Thus by theorem 4.5 the bsde defined by equation 4.5 has a unique adapted square integrable solution, since the terminal condition $\nabla F(X, Y)$ is square integrable.

We have now verified the existence of a unique adapted solution $((X,Y),p,K)$ to the fbsde defined by equations 2.4, 2.5 and 4.5. We apply now the four step scheme to obtain a solution candidate.

Step 1. We define $z:[0,T]\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^{2\times3}\to\mathbb{R}^{2\times3}$ by

$$z(t,x,q,m) = -\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} \sigma\theta^L x_1 & \tilde{\sigma}x_1 & -\gamma x_1 \\ \sigma\theta^S x_2 & 0 & -\gamma x_2 \end{pmatrix}.$$

Step 2. Using the function z obtained in step 1, we solve the following two-dimensional parabolic system for $u(t,\mathbf{x}) = (u^1(t,x_1,x_2),u^2(t,x_1,x_2))$:

$$u^i_t + L^i u^i + \hat{b}^i\left(t,\mathbf{x},u,z(t,\mathbf{x},u,J(u))\right) = 0,\ i=1,2, \tag{4.9}$$

where $\hat{b}^i$ is the ith component of drift vector in 4.6, $\underline{\sigma}$ is the diffusion matrix in 2.7, $J(u)$ is the Jacobian of u and

$$L^i v = (1/2)\text{tr}(H(v)\underline{\sigma\sigma}^T) + b(t,\mathbf{x})\cdot\nabla_{\mathbf{x}}v$$

with b being the drift vector of the forward sde and $H(v)$ being the Hessian of v. We have thus a two-dimensional quasilinear Cauchy problem defined by equation 4.9 and the terminal condition

$$u(T,\mathbf{x}) = \nabla F(\mathbf{x}),\ \mathbf{x}\in\mathbb{R}^2, \tag{4.10}$$

Because of additivity of $F(x_2,x_2)$, we have actually two separate terminal conditions $u^1(T,\mathbf{x}) = D_{x_1}F_1(x_1)$ and $u^2(T,\mathbf{x}) = D_{x_2}F_2(x_2)$. If a solution of pde 4.9 is assumed to be of the form $v(t)w(x_1,x_2)$, then separation of variables and the superposition principle yield that the general solution is $\int_{\xi\in\mathbb{R}} a(t,\xi)b(x_1,x_2,\xi)d\xi$, with $a(t,\xi)$ an exponential function of t. We see that matching the terminal condition consisting of a finite number of terms $d_i x_i^s$, $s\in\mathbb{R}$, is possible only if each of the b functions corresponding to appropriate ξ values depends only on x_1 or only on x_2. Thus, also for $t<T$ u^i is a function of x_i only. Then we have two one-dimensional quasilinear Cauchy problems of type

$$\begin{aligned} &u_t(t,x) + (1/2)A^i x^2 u_{xx}(t,x) + B^i x u_x(t,x) + C^i u(t,x) = 0 \\ &u(T,x) = F_x(x),\ x\in\mathbb{R}^2, \end{aligned} \tag{4.11}$$

where now the coefficients are

$$A^1 = \sigma^2(\theta^L)^2 + \tilde{\sigma}^2 + \gamma^2, \quad B^1 = r - \nu + (\mu - r)\theta^L + \sigma^2(\theta^L)^2 + \tilde{\sigma}^2 + \gamma^2,$$
$$C^1 = \nu - r - (\mu - r)\theta^L$$

for u^1 and

$$A^2 = \sigma^2(\theta^S)^2 + \gamma^2, \qquad B^2 = r - \nu + (\mu - r)\theta^S + \sigma^2(\theta^S)^2 + \gamma^2,$$
$$C^2 = \nu - r - (\mu - r)\theta^S$$

for u^2. The pdes of these Cauchy problems can be solved by applying standard methods, i.e. separation of variables and superposition property (see Appendix A for details). The general solutions are (in integral notation) of form

$$u(t, x_i) = \int_{\xi \in K} c^i_{1,\xi} \exp\left\{-(C^i + \xi)t\right\} \left[c^i_{2,\xi} x_i^{m_1(\xi)} + c^i_{3,\xi} x_i^{m_2(\xi)}\right] d\xi, \tag{4.12}$$

where $K = (-(A^i/2)(B^i/A^i - 1/2)^2, \infty)$ and $m_1(\xi)$, $m_2(\xi)$ are roots of equation

$$m^2 + \left(2A^i/B^i - 1\right) m - 2\xi/A^i = 0. \tag{4.13}$$

This solution contains as a special case any series solution

$$u(t, x_i) = \sum_{\xi \in \Xi} c^i_{1,\xi} \exp\left\{-(C^i + \xi)t\right\} \left[c^i_{2,\xi} x_i^{m_1(\xi)} + c^i_{3,\xi} x_i^{m_2(\xi)}\right], \tag{4.14}$$

where Ξ is a set of discrete points in K. Now the Cauchy problem is readily solved if the boundary condition, determined by the choice of performance function $F(\mathbf{x})$, is of form $\sum_{s \in S^i} d^i_s x_i^s$ with $S^i \subset \mathbb{R}$ finite and d^i_s a constant for each s. Then we can for each $s \in S^i$ choose such a value of ξ that $m_1(\xi) = s$. Setting then $c^i_{3,\xi} = 0$, $c^i_{1,\xi} = \exp\{(C^i + \xi)T\}$ and $c^i_{2,\xi} = d^i_s$ gives a function $u^i(t, x_i)$ such that it solves problem 4.11.

As our function $F_i(x_i)$ is the sum of an utility function $(1/a^i)x_i^{a^i}$ and a piecewise polynomial function $h(x_i)$ (or $g(x_i)$), $D_{x_i}[F_i(x_i)]$ is now of desired form if $x_i \in (\beta - 1/n, \beta + 1/n)$. Outside this interval $h(x_i)$ ($g(x_i)$) is constant (meaning that $d^i_s = 0$ for all $s \in S^i \setminus \{a^i - 1\}$) and thus we have a boundary condition in which the d^i_s coefficients are piecewise constant.

Step 3. Using u and z, we solve the fsde

$$X(t) = X_0 + \int_0^t b_{mod}(s, X(s))ds + \int_0^t \sigma_{mod}(s, X(s))dW(s),$$

where $b_{mod}(t,x) = b(t, x, u(t,x), z(t, x, u(t,x), u_x(t,x)))$ and $\sigma_{mod}(t,x) = \sigma(t, x, u(t,x))$. Since now b and σ are independent of u, this is trivial and the solution $(X(t), Y(t))$ is

$$\begin{cases} X_0 \exp\left\{\int_0^t \alpha(\theta^L)ds + \sigma\theta^L W(t) + \tilde{\sigma}\tilde{W}(t) - \gamma\hat{W}(t) - (1/2)\left(\sigma^2(\theta^L)^2 + \tilde{\sigma}^2 + \gamma^2\right)t\right\} \\ Y_0 \exp\left\{\int_0^t \alpha(\theta^S)ds + \sigma\theta^S W(t) - \gamma\hat{W}(t) - (1/2)\left(\sigma^2(\theta^S)^2 + \gamma^2\right)t\right\} \end{cases}$$

where $\alpha(\theta) = r + (\mu - r)\theta - \nu$.

Step 4. Finally, we set $(p_1(t), p_2(t))^T := (u^1(t, X(t)), u^2(t, Y(t)))^T$, which equals

$$\begin{pmatrix} \sum_{\xi\in\Xi_1} c^1_{1,\xi} \exp\{-(C^1+\xi)t\}\left[c^1_{2,\xi}X(t)^{m^1_1(\xi)} + c^1_{3,\xi}X(t)^{m^1_2(\xi)}\right] \\ \sum_{\xi\in\Xi_2} c^2_{1,\xi} \exp\{-(C^2+\xi)t\}\left[c^2_{2,\xi}Y(t)^{m^2_1(\xi)} + c^2_{3,\xi}Y(t)^{m^2_2(\xi)}\right] \end{pmatrix},$$

and $K(t) := z(t, (X(t), Y(t)), u(t, (X(t), Y(t))), J(u(t, (X(t), Y(t)))))$ with $J(u)$ being the Jacobian of vector u. Taking now into account the terminal conditions as outlined in Step 2 we may write $p(t)$ in a more explicit form

$$\begin{pmatrix} \sum_{s\in S^1} \exp\{(C^1 + \xi(s))(T-t)\} d^1_s X(t)^s \\ \sum_{s\in S^2} \exp\{(C^2 + \xi(s))(T-t)\} d^2_s Y(t)^s \end{pmatrix},$$

where $\xi(s) = (A^i/2)\left(s^2 + 2s(B^i/A^i - 1/2)\right)$ with A^i and B^i as in equation 4.11. Solution $((X(t), Y(t)), p(t), K(t))$ is $\{\mathcal{F}_t\}$-adapted and square integrable and is consequently the unique adapted solution in the sense of definition 4.1.

We have thus obtained an explicit expression for the spatial gradient of value function Φ. Integration of p yields

$$\begin{aligned}\Phi(t,X,Y) \quad &= \textstyle\sum_{s\in S_1} \left(\exp\left\{(C^1+\xi(s))(T-t)\right\} d_s X(t)^{s+1}\right)/(s+1)+ \\ &+ \textstyle\sum_{s\in S_2} \left(\exp\left\{(C^2+\xi(s))(T-t)\right\} d_s Y(t)^{s+1}\right)/(s+1).\end{aligned} \tag{4.15}$$

With Φ defined as in equation 4.15, the expression for optimal control in 4.3 reduces to

$$\left(-\frac{(\mu-r)\Phi_1}{X(t)\sigma^2\Phi_{11}}, -\frac{(\mu-r)\Phi_2}{Y(t)\sigma^2\Phi_{22}}\right). \tag{4.16}$$

5 ON THE BEHAVIOR OF THE OPTIMAL STRATEGY

Next we present some observations concerning the behavior of our optimal strategy. As our goal is a model which may offer some guidance in making investment decisions in a normal situation where the particular insurer's actions are not restricted by supervisors, we restrict our attention to the solvency set $G = [1, \infty) \times [\beta, \infty)$. States outside G represent unacceptable solvency positions, which in real world require special measures and are in this paper regarded as being outside the normal course of events (even though insurers occasionally do end up insolvent).

From equations 4.16 and 4.15 we can see why polynomial constraints approximating the probabilistic constraints 3.1 very accurately lead to short-sighted strategies. If we denote

$$P_i := \{s \in S \,:\, s \neq a^i - 1\} \subset \mathbb{N},$$

$h^i(t) := \exp\{[C^i + \xi(a^i - 1)](T - t)\}$ and $k^i_j(t) := \exp\{[C^i + \xi(j)](T - t)\}$, the value function 4.15 can be written as

$$\Phi(t, x, y) = h_1(t)U(x) + h_2(t)U(y) + \lambda_1(g^1_n(t, x) - \alpha_1) + \lambda_2(g^2_n(t, y) - \alpha_2), \tag{5.1}$$

with $g^i_n(t, x) = \sum_{j \in P_i} k^i_j(t) d^i_j x^{j+1}/(j + 1)$ and thus the optimal proportions are of form

$$-\frac{(\mu - r)\left[h_i(t)U'(z) + \lambda_i D_z[g^i_n(t, z)]\right]}{z\sigma^2\left[h_i(t)U''(z) + \lambda_i D^2_z[g^i_n(t, z)]\right]}, \tag{5.2}$$

with $i = 1, 2$, $z = x, y$. It should be noted that $g^i_n(t, x) = g^i_n(x)$ is similarly piecewise determined as f^i_n and is constant when $Z(t) \notin (\beta, \beta + 1/n) =: I^i_n$. When $Z(t) \in [\beta + 1/n, \infty)$ the function $g^i_n(Z(t))$ will have no effect on the proportion of risky asset, as it is constant and both the first and the second derivatives vanish. The time-dependent functions $h_i(t)$ will also disappear. In this case the proportions of risky asset can be expressed as

$$\left(-\frac{(\mu-r)U'(X(t))}{\sigma^2 X(t)U''(X(t))}, -\frac{(\mu-r)U'(Y(t))}{\sigma^2 Y(t)U''(Y(t))}\right), \tag{5.3}$$

and since utility $U(x) = (1/a)x^a$ has constant relative Arrow–Pratt risk aversion $A(x) = -xU''(x)/U'(x) = 1-a$, this means that the proportions are constant $((\mu-r)/[\sigma^2(1-a^1)], (\mu-r)/[\sigma^2(1-a^2)])$. Hence, the function g_n^i has an effect on the optimal control only if the value of the controlled process lies in the interval I_n^i. As n increases, this interval becomes thinner. In practical terms this means that we do not react to adverse development until ruin is already imminent – at which point it is probably too late. So if we want to have an investment strategy which takes into account the obvious increase in insolvency risk as the value of portfolio approaches the solvency border, we cannot have a very large value of n.

But how exactly does the approximating constraint affect the optimal proportion in the "reaction zone", i.e. when $Z(t) \in I_n^i$? This will depend on λ_i and n. For fixed (t,x) the proportion of risky asset is a function of λ:

$$M(\lambda) = -\frac{(\mu-r)\left[h(t)U'(x) + \lambda D_x[g_n^i(t,x)]\right]}{x\sigma^2\Big[h(t)U''(x) + \lambda D_x^2[g_n^i(t,x)]\Big]} =: -\frac{(\mu-r)}{\sigma^2}N(\lambda). \tag{5.4}$$

In other words the optimal proportion is the risky asset's Sharpe's ratio multiplied by $(1/\sigma)N(\lambda)$ with $N(\lambda) = -\big(a(t,x)\lambda + b(t,x)\big)/\big(c(t,x)\lambda + d(t,x)\big)$, where $a(t,x) = D_x[g_n^i(t,x)]$, $b(t,x) = h(t)U'(x)$, $c(t,x) = D_x^2[g_n^i(t,x)]$ and $d(t,x) = h(t)U''(x)$.

A reasonable range for $N(\lambda)$ is then $[-\sigma^2/(\mu-r), 0]$, as this will lead to a portfolio weight $\theta \in [0,1]$. The function $N(\lambda)$ will have an asymptote at the point $\lambda = -d(t,x)/c(t,x) \neq 0$ and as $\lambda \to \infty$, $N(\lambda) \to a(t,x)/c(t,x)$. Thus choosing a large enough λ yields a control $\theta^*(t,x)$ which in the reaction zone behaves approximately as $-[(\mu - r)/\sigma^2][a(t,x)/c(t,x)]$. Due to our choice of using a piecewise polynomial approximation, we know that for $x \in I_n^i$ we have $D_x[f_n^\beta(x)] \geq 0$

and $D_x^2[f_n^\beta(x)] \leq 0$ and consequently $g_n^i(T,x)$ behaves similarly. It is not obvious, however, that this property is in general preserved when $t < T$, since functions $k_j^i(t)$ affect each term of $g_n^i(t,x)$ differently.

6 AN ILLUSTRATIVE EXAMPLE

To illustrate the model presented above we consider a hypothetical simulation example incorporating in a simplified form some aspects of the Finnish pension scheme. This example is meant to be an illustration and determination of parameter values is consequently performed in a simplistic fashion, relying mostly on the Finnish solvency regulations. In Finnish regulations assets are divided into seven categories (I–VII) with specified returns and standard deviations (see appendix B for numerical values). As an approximation we combine categories I–III into one "risk-free" portfolio using constant and equal weights for each category. Similarly, we approximate the risky asset by combining categories IV–VII.

The model parameters are assumed to be as follows:

$\mu = 0.10$, which is approximately equal to the mean return of a portfolio with equal proportions $(1/4)$ invested in investment categories IV, V, VI and VI.

$\sigma = 0.15$, which is approximately equal to the standard deviation of a portfolio with equal proportions $(1/4)$ invested in investment categories IV, V, VI and VI.

$r = 0.03$ is the risk-free interest rate (which is actually equal to the current (2005) actuarial interest rate, which determines the minimum level of technical interest rate in the Finnish pension scheme).

$\nu = 0.04$ is the technical interest rate.

$\tilde{\sigma} = \gamma = 0.02$, which are assumptions reflecting the fluctuations in the value of asset portfolio caused by the stream of pension payments and contributions (which are mainly determined by demographic factors and wage development) on the other hand and the fluctuation of the technical interest rate (which in the Finnish pension scheme is determined semiannually by the Ministry of Social Affairs and Health).

$(\alpha_1, \alpha_2) = (0.5, 1)$, which correspond to limits laid down in solvency regulations.

$(a_1, a_2) = (0.1, 0.95)$, which means that the insurer is assumed to be quite risk-averse when investing assets backing technical reserves even in a good solvency position, whereas for solvency margin assets risk aversion is lesser.

Concerning these parameter value assumptions it should be noted that using the insurer's own assessments (based on market data, for example) instead of values prescribed in regulations could be more sensible from a practical point of view, as regulations will inevitably lag behind reality and thus the parameter values given in regulations may not be appropriate.

The solvency border in Finnish pension system is determined by the insurer's asset allocation. This can be taken into account in a prudent way by calculating the maximal proportion of risky assets for the whole asset portfolio and determining the value of the solvency border from this. For this, we assume that the value of total assets is 1.5 times the value of liabilities (which in the case of Finnish pension system can be considered prudent, as currently for pension insurers a typical value of this ratio is about 1.25). Now suppose that maximal percentage $100\alpha_1$ % of assets backing the liabilities are invested in risky assets, and that the proportion of risky assets in the solvency margin assets is $\theta^{S*} = (\mu - r)/[\sigma(1 - a_2)]$. Then the maximal proportion of risky assets is $(1/1.5)(\alpha_1 + 0.5\theta^{S*})$ and in this example this equals 0.4979. The solvency border is calculated from the formula

$$\beta = 0.90 \left(-1.08 \sum_{i=1}^{7} \beta_i m_i + 1.98 \sqrt{\sum_{i=1,j=1}^{7} \beta_i \beta_j s_i s_j r_{ij}} \right), \tag{6.1}$$

where m_i, s_i and r_{ij} are return of asset category i, standard deviation of asset category i and correlation between categories i and j, respectively (numerical values are given in Appendix B). β_i is the proportion of assets invested in category i. As an approximation, we calculate the returns

S_1, S_2 and standard deviations M_1, M_2 of the two combined portfolios defined in the beginning of this section and their correlation coefficient R_{12}. Plugging these into the modified formula for solvency border

$$\beta = 0.90 \left(-1.08 \sum_{i=1}^{2} \beta_i M_i + 1.98 \sqrt{\sum_{i=1,j=1}^{2} \beta_i \beta_j S_i S_j R_{ij}} \right) \qquad (6.2)$$

yields $\beta = 0.106$.

We need to simulate a process of form

$$X_i(t) = X_i(0) \exp\left\{ (\mu_i - (1/2)\underline{\sigma}_i \underline{\sigma}_i^T)t + \underline{\sigma}_i \underline{W}(t) \right\},$$

$i = 1, 2$, with $\underline{W}(t)$ a three-dimensional driving Wiener process. A simulation of a continuous stochastic process must be based on generating a discrete series of (pseudo)random shocks. For simplicity, we consider in this presentation only discretizations with equidistant grid points. We apply the following algorithm.

Step 1. Generate 3 series of N standard normal random numbers W^1, W^2 and W^3 and define $W := (W^1, W^2, W^3)^T$.

Step 2. Set $\Delta t := T/N$ and for $i = 1, 2$ set

$$X_i(t_j) = X_i(t_{j-1}) \exp\left\{ \left(\mu_i - (1/2)(\underline{\sigma}_i \underline{\sigma}_i^T)\right) \Delta t + \underline{\sigma}_i^T W_j \sqrt{\Delta t} \right\}.$$

with $t_j = t_{j-1} + \Delta t$, $j = 1, \ldots, N$ and $t_0 = 0$.

The value of N gives the number of discrete random shocks occurring during the period $[0, T]$. To obtain a satisfactory approximation the discretization should not be too crude, i.e. we must have a reasonably large value of N. It should be noted that now both the drifts μ_i and diffusion coefficients σ_{ik} used in the algorithm when calculating the state at time t_j are functions of the portfolio weight $\theta(t_{j-1})$ and the state of the process $X(t_{j-1})$ at previous time point.

In reality, it is not possible to adjust the investment strategy continuously. It is also obvious that for a discretization with a large number N of grid points adjusting the portfolio at each time point t_j will not be feasible. This necessitates a decision concerning a suitable re-balancing period. For practical purposes a fixed length of re-balancing period may be convenient, as the problem is then simplified to determination of the optimal period length. A very long re-balancing period makes sustained deviations from optimality possible, but on the other hand shortening the period increases the transaction costs (in theory, for a diffusion process continuous re-balancing with proportional transaction costs implies infinite transaction costs). In our model transaction costs are assumed to be zero, as the assumption of negligible transaction costs may be a valid approximation for pension insurers, as these are usually large investors. Thus in this example the decision about re-balancing period is exogenous and is taken as given in the model.

As was pointed out in Section 5, the values of coefficients λ_i, $i = 1, 2$, should be chosen large enough. Based on some simulation experiments, it seems that the choice $(\lambda_1, \lambda_2) = (10, 10)$ is appropriate as increasing the values from these does not lead to significant changes in results.

We take a time horizon of one year ($T = 1$) and set $N = 250$, corresponding roughly to one random shock per trading day. Re-balancing is assumed to take place 25 times a year, which corresponds roughly to re-balancing after every two weeks. Initial ratio of total wealth to liabilities is 1.25 and initial ratio of assets corresponding to solvency margin is 0.15. Asset allocation $(\theta^L(0), \theta^S(0))$ is assumed to be $(0.3, 0.6)$. It should be noted that if the state of the process is outside the solvency region at some re-balancing point, the proportion of risky asset is set to zero. The idea here is that an insolvent insurer is not allowed to take any risks at all. Furthermore, since it is possible that the portfolio weight θ^* given by the expression 5.4 may be outside $[0, 1]$,we set $\theta = 0$ if $\theta^* < 0$ and $\theta = 1$, if $\theta^* > 1$.

We can now simulate the controlled process, if we specify the values of parameters n_1 and n_2. Figure 1 shows an example of a bundle of 100 realizations of the controlled processes X and Y with $n_1 = 8$,

$n_2 = 3$. To find the best values of parameters n_i, we perform Monte

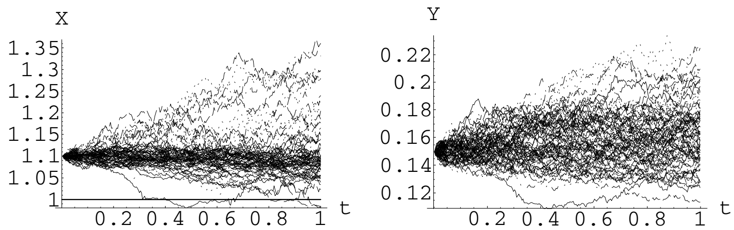

Figure 1: *A simulation of the controlled processes $X(t)$ (left) and $Y(t)$ (right) with $n_1 = 8$, $n_2 = 3$ (100 realizations).*

Carlo simulations with different values of n_i. In Figure 2 the empirical densities of the terminal values of assets backing the liabilities and assets corresponding to solvency margin are plotted. These densities are based on 100 simulations, each consisting of 100 realizations. It turns out that the choices $n_1 = 8$ and $n_2 = 6$ seem to give good results in terms of highest returns among simulations of 10000 realizations with observed proportions of ruins smaller than 1 % for both processes. Allowing the two processes to have different n_i values makes sense also from the point of view that the value of assets corresponding to solvency margin is now initially about 1.5 times the required amount, whereas the value of assets backing the liabilities is initially only 1.1 times the required amount. So it could be expected that the suitable width of the reaction zone is not the same for both processes. The resulting Monte Carlo estimates for terminal ratios are $(1.10079, 0.15903)$, and their variances are $(0.002597, 0.000552)$. Estimated actual returns on investment are then 4.12 % for assets backing the liabilities and 10.27 % for solvency margin. Estimates for variances of these returns are 0.00182 and 0.0261 and estimated probabilities of an unacceptable solvency position are 0.0055 and 0.0042. An estimate of the upper limit for probability of ruin (in the sense used in this model) is obtained by summation of these estimates and in this case the estimate is $0.0096 < 0.01$, which is acceptable if we consider a survival probability larger than 99 % to be satisfactory. The estimated mean return for assets backing liabilities

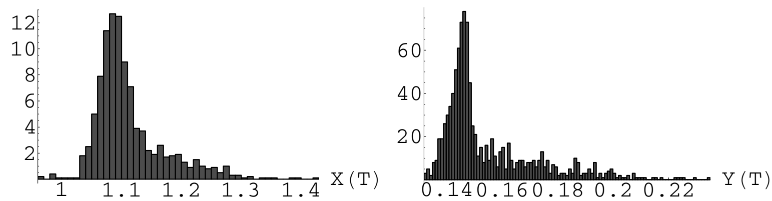

Figure 2: *Empirical densities of the terminal values of controlled processes* $X(t)$ *and* $Y(t)$ *based on* 100 *simulations of* 100 *realizations, with* $n_1 = 8$ *and* $n_2 = 22$.

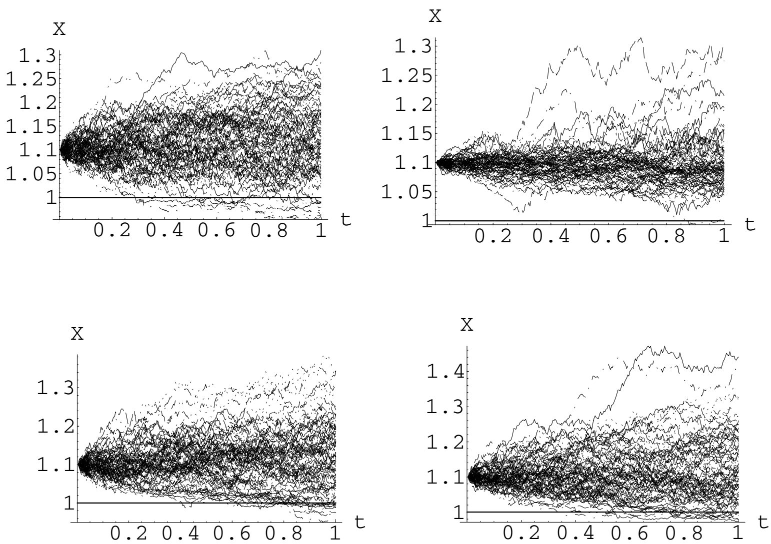

Figure 3: *Simulations of* 100 *realizations of process* X *with* $n_1 = 3, 8, 18, 24$ *(up left, up right, below left, below right).*

achieves the required return determined by the technical interest rate and the simulations indicate that despite the considerable risk taken in the investment of assets corresponding to solvency margin, the estimated probability of ruin remains acceptably small. Moreover, our estimate for the ruin probability is an upper limit based on the assumption that insolvencies of X and Y do not coincide, but actually it can be expected that X and Y have a tendency to develop in the same direction and the insolvencies of the processes do coincide.

Realization bundles of the X process with $n_1 = 3, 8, 18, 24$ are shown in Figure 3. It illustrates the obvious: increasing n_1 first shrinks the fluctuation range of the process as the approximating constraint tightens, but at some point the reaction zone becomes “too thin“ and the proportion of ruins increases again, approaching the situation with performance function equal to $U(X(T))$.

7 POSSIBILITIES FOR FUTURE RESEARCH

The model presented in preceding sections is useful in assessing the impact of alternative profit distribution policies. After a set of possible policy rules has been specified, the development of the insurer's position can be simulated year by year for each policy and the obtained results can be compared. Expert views on future changes in economic environment can be incorporated in the model by specifying different parameter values for different periods. However, for some other uses developing different modifications of the model could be worthwhile. Some modification possibilities are presented here as subjects for future research.

As the distribution of excess profits as dividends on one hand and as bonuses on the other is a significant management decision, it would be of interest to be able to determine a combined optimal investment strategy and profit distribution policy. Our intention is to consider this problem in future research and we discuss here briefly two possibilities for attacking the problem.

One possibility would be to model dividend and bonus stream as a singular control, which in the model might not be too difficult to handle from a mathematical point of view. However, a potential difficulty in the practical implementation of this kind of model is that in reality, dividends and bonuses are paid yearly, whereas a singular control typically is a rule for intervening (i.e. distributing profits) when the state of the controlled process enters some specified region. In the present example this would probably result into a rule where dividends are paid out only when the reserves are sufficiently high.

Another possibility would be to apply standard backward dynamic programming. That is, we would first specify a planning horizon N in years, determine the set of possible states in terms of wealth V_{N-1} and its allocation (V_{N-1}^L, V_{N-1}^S) at the beginning of year N and obtain the optimal strategy for each state using the simple model. Then we would move to the beginning of year $N-1$ and similarly determine the optimal strategies for each possible state taking into account the information obtained in the previous step concerning year N. At this point we would also decide the amounts of bonuses and dividends distributed at the end of year $N-1$. We would proceed in this way until the beginning of year 1, obtaining thus the optimal strategy. The problem with this approach is that the set of possible states at each step is innumerably infinite and has to be approximated in some way by a finite discrete set of states to achieve a numerically computable problem. Question is, whether any useful approximation leads to a problem of computationally manageable size. In this approach, obtaining an analytical solution is extremely difficult, if possible at all.

The current model does not take into account transaction costs. This could be remedied by also allowing the control of investments to be singular. However, this would complicate the model and would require a decision to be made about whether to allow transfers of wealth between solvency margin and assets backing the liabilities or not. Furthermore, since pension insurers are usually large investors, it may not be too far from reality to assume negligible transaction costs. Of course, especially if investment takes place on a small market (such as that of Finland), the implicit assumption made in the model that the insurer's actions have no influence on the asset prices may be too restrictive as well. In this case the effect of insurer's investment decisions on the asset prices should be incorporated into the model.

With regard to specification of model's parameters, expert views are necessary to complement estimates obtained from historical data when the environment is expected to change in the future, see Koivu et al. (2005). If expert knowledge is to be incorporated in the model, the dynamic programming approach may be more flexible, as it is still possible to specify different model parameters for each period if necessary. In the singular control approach this is not possible.

By the so-called mutual fund theorems it may be argued that having only two assets, a risk-free one and a risky one, is not too restrictive as asset classes can be combined into portfolios. However, from the point of view of obtaining useful guidelines for investment strategy, it could perhaps be desirable to have somewhat more than two assets. This is especially desirable if legal constraints for investment strategy are defined in terms of upper limits for investments in some asset classes specified in the regulations. For example, in Finnish insurance legislation the assets in which a pension insurer may invest are divided into seven broad categories, and at least with respect to assets backing liabilities proportional upper limits for investments exist.

While convenient from a mathematical point of view, geometric Brownian motions are not a perfect model for asset prices, as they do not allow jumps. Asset prices may sometimes behave in a way closely resembling a discontinuous jump (bursting of a speculative bubble, for example), and this aspect can be significant when assessing risk. Furthermore, stochastic processes in insurance are often naturally jump processes, and thus it might be useful to relax the assumption that the liabilities' value evolves as a geometric Brownian motion. These considerations would suggest replacing geometric Brownian motions with more general geometric Lvy processes. Such an extension is out of the scope of the present study and is, therefore, left for future research.

Acknowledgement: The author is grateful to *Luis Alvarez* for insightful comments.

REFERENCES

Asetus työeläkevakuutusyhtiön vakavaraisuusrajan laskemisesta, 1281/1999, <http://www.finlex.fi/fi/laki/alkup/1999/19991281>, retrieved 14.6.2005.

Bahlali, K. (2002) *Existence and uniqueness of solutions for bsdes with locally Lipschitz coefficient, Electronic Communications in Probability*, 7, 169–179.

Börsch-Supan, A. (2005) *The pension reform in Finland, Finnish Centre for Pensions Working Papers* 2005:1.

Bass, R. (1998) *Diffusions and elliptic operators.* Springer-Verlag.

Daykin, C., Pentikäinen, T., Pesonen, M. (1994) *Practical risk theory for actuaries.* Chapman & Hall.

Hilli, P., Koivu, M., Pennanen, T., Ranne, A., 2005. *A stochastic programming model for asset liability management of a Finnish pension company, Annals of Operations Research*, to appear.

Ilmakunnas, S., Vanne, R. (2004) *Funding and portfolio management in the Finnish earnings-related pension scheme, Finnish Centre for Pensions Working Papers* 11.

Koivu, M., Pennanen, T., Ranne, A. (2005) *Modeling assets and liabilities of a Finnish pension insurance company: a VEqC approach, Scandinavian Actuarial Journal*, 1, 46–76.

Ma, J., Protter, P., Yong, J. (1994) *Solving forward–backward stochastic differential equations explicitly – a four step scheme, Probability Theory and Related Fields*, 98, 339–359.

Ma, J., Yong, J. (1995) *Solvability of forward–backward stochastic differential equations and the nodal set of Hamilton–Jacobi–Bellman equations, Chinese Annals of Mathematics*, 16B(3), 279–298.

Øksendal, B. (2003) *Stochastic differential equations. An introduction with applications.* Springer-Verlag, 6th edition.

Øksendal, B., Sulem, A. (2005) *Applied stochastic control of jump diffusions.* Springer-Verlag.

Protter, P. (2004) *Stochastic integration and differential equations.* Springer-Verlag, 2nd edition.

Rogers, L., Williams, D. (2000a) *Diffusions, Markov processes and martingales. Volume 1: Foundations.* Cambridge University Press, 2nd edition.

Rogers, L., Williams, D. (2000b) *Diffusions, Markov processes and martingales. Volume 2: It Calculus.* Cambridge University Press, 2nd edition.

Zhang, J. (2005) *Representation of solutions to bsdes associated with a degenerate fsde, Annals of Applied Probability*, 15, 1798–1831.

Zwillinger, D. (1998) *Handbook of differential equations.* Academic Press, 3rd edition.

A SOLUTION OF THE ONE-DIMENSIONAL CAUCHY PROBLEM

We wish to solve the Cauchy problem of equation 4.11. Suppose that the solution of the pde is $u(t,x) = A(t)B(x)$. Substituting this into the pde and dividing by u one gets

$$A^{'}(t)/A(t) + (A/2)x^2 B^{''}(x)/B(x) + BxB^{'}(x)/B(x) = -C.$$

This can be separated into

$$A^{'}(t)/A(t) = \eta \text{ and } (A/2)x^2 B^{''}(x)/B(x) + BxB^{'}(x)/B(x) = \xi,$$

where $\eta + \xi = -C$. Solution $A(t) = c_1 \exp\{\eta t\}$ is immediate, and the equation for $B(x)$ is in fact the familiar Cauchy differential equation, which can be transformed via $x = \exp(s)$ into the Euler equation, which has the solution

$$B(s) = c_2 \exp(m_1 s) + c_3 \exp(m_2 s)$$

with $\{m_1, m_2\} = (1/2) - (B/A) \pm \sqrt{(B/A - 1/2)^2 + 2\xi/A}$. Transformation back to original variables yields three different solutions depending on the values of m_is. If $m_1 \neq m_2$ and both are real, the solution is $B(x) = c_2 x^{m_1} + c_3 x^{m_2}$. If $m_1 = m_2 = m$, the solution is $B(x) = c_2 x^m + c_3 x^m \ln x$. Because we now wish to match a boundary condition consisting of terms of form dx^s, $s \in \mathbb{R}$, we are not interested in the third case when $m_1, m_2 \in \mathbb{C}$. This case corresponds to $\xi < -(A/2)(B/A - 1/2)^2 \leq 0$.

Noting that $\eta = -C - \xi$, we have for $\xi > -(A/2)(B/A - 1/2)^2$ the solution

$$u_\xi(t,x) = c_{1,\xi} \exp\{-(C+\xi)t\}[c_{2,\xi} x^{m_1(\xi)} + c_{3,\xi} x^{m_2(\xi)}],$$

and any combination $\sum_{\xi \in \Xi} u_\xi(t,x)$, with $\Xi \subset (\xi < -(A/2)(B/A - 1/2)^2, \infty)$ is also a solution by superposition principle.

To match the boundary condition we must choose such ξs that

$$m_1(\xi) = 1/2 - B/A + \sqrt{(B/A - 1/2)^2 + 2\xi/A} = s \tag{A.1}$$

for all s such that the boundary function has a term of form dx^s. Equation A.1 is equivalent to

$$\xi = (A/2)\left(s^2 + 2s(B/A - 1/2)\right).$$

For the solution to be of correct form, this expression must be larger than or equal to $-(A/2)(B/A - 1/2)^2$, that is,

$$(A/2)\left(s^2 + 2s(B/A - 1/2)\right) \geq -(A/2)(B/A - 1/2)^2.$$

This, however, is equivalent to $(s + B/A - 1/2)^2 \geq 0$, which is always true, and so we can match terms of form dx^s for any $s \in \mathbb{R}$ by choosing the constants $c_{i,\xi}$ suitably.

B INVESTMENT CATEGORIES IN FINNISH REGULATIONS

In Finnish regulations concerning the calculation of solvency border for pension insurers managing the statutory pension scheme the investments are classified into seven categories, which are shortly described here:

I Short-term liabilities of reliable institutions (such as governments, municipalities, and financial institutions subject to public supervision) and special loans to sponsors based on pension contributions.

II Euro-denominated bonds and long-term liabilities issued or guaranteed by institutions specified in (I) and certain homeowner property financed by Finnish government.

III Other than euro-denominated bonds and long-term liabilities issued or guaranteed by public institutions specified in (I), euro-denominated bonds traded on an exchange or issued by traded companies, short-term liabilities traded on an exchange or issued by traded companies excluding short-term liabilities mentioned in (I).

IV Other than euro-denominated bonds and long-term liabilities issued or guaranteed by private institutions specified in (I) or traded companies, other than euro-denominated traded bonds and several types of property investments.

V All other property investments.

VI Traded stocks.

VII All other investments.

All investments in countries that are not members of the OECD belong to category VII. For these categories the law specifies expected returns (as excess returns with respect to the technical interest rate), standard deviations and correlations according to Table 1. For more details see Asetus työeläkevakuutusyhtiön vakavaraisuusrajan laskemisesta (1999)

Group	m_i	s_i	r_{ij}						
I	0.1	1.0	1	-0.1	-0.2	0	0	-0.1	-0.1
II	0.6	3.5	-0.1	1	0.4	-0.1	-0.1	0.1	0.1
III	0.6	4.4	-0.2	0.4	1	-0.1	-0.1	0.1	0.1
IV	3.7	8.2	0	-0.1	-0.1	1	0.7	0.3	0.3
V	3.7	15.0	0	-0.1	-0.1	0.7	1	0.3	0.3
VI	6.2	21.4	-0.1	0.1	0.1	0.3	0.3	1	0.7
VII	6.2	29.9	-0.1	0.1	0.1	0.3	0.3	0.7	1

Table 1: *Excess returns (m_i), standard deviations (s_i) and correlations (r_{ij}) specified in Finnish solvency regulations for pension insurers.*

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