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On Some One-Sided Dynamics of Cellular Automata

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Abstract

A dynamical system consists of a space of all possible world states and a transformation of said space. Cellular automata are dynamical systems where the space is a set of one- or two-way infinite symbol sequences and the transformation is defined by a homogenous local rule. In the setting of cellular automata, the geometry of the underlying space allows one to define one-sided variants of some dynamical properties; this thesis considers some such one-sided dynamics of cellular automata.

One main topic are the dynamical concepts of expansivity and that of pseudo-orbit tracing property. Expansivity is a strong form of sensitivity to the initial conditions while pseudo-orbit tracing property is a type of approximability. For cellular automata we define one-sided variants of both of these concepts. We give some examples of cellular automata with these properties and prove, for example, that right-expansive cellular automata are chain-mixing. We also show that left-sided pseudo-orbit tracing property together with right-sided expansivity imply that a cellular automaton has the pseudo-orbit tracing property.

Another main topic is conjugacy. Two dynamical systems are conjugate if, in a dynamical sense, they are the same system. We show that for one-sided cellular automata conjugacy is undecidable. In fact the result is stronger and shows that the relations of being a factor or a subsystem are undecidable, too.

Tiivistelmä

Dynaaminen systeemi muodostuu kaikkien maailmantilojen tila-avaruudesta ja tämän avaruuden transformaatiosta. Soluautomaatit ovat dynaamisia systeemejä, joiden tila-avaruus muodostuu yhteen tai kahteen suuntaan äärettömistä symbolien jonoista, ja joiden transformaation määrittelee homogeeninen paikallinen sääntö. Avaruuden geometriasta johtuen soluautomaateille voidaan joistain dynaamisista ominaisuuksista määritellä yksipuoleisia versioita; tässä väitöskirjassa tarkastellaan soluautomaattien yksipuoleista dynamiikkaa.

Yksi väitöskirjan pääteemoista ovat ekspansiivisuuden ja pseudoratojen jäljitysominaisuuden dynaamiset käsitteet. Ekspansiivisuus kuvaa systeemin voimakasta herkkyyttä muutoksiin alkutilassa. Pseudoratojen jäljitysominaisuus taas kuvaa systeemin tiettyä apporksimoitavuutta. Väitöskirjassa näistä ominaisuuksista määritellään yksipuoleiset versiot. Esitämme joitain esimerkkejä soluautomaateista, joilla nämä ominaisuudet ovat, ja todistamme muun muassa, että oikealle ekspansiiviset soluautomaatit ovat ketjusekoittavia. Näytämme myös, että soluautomaatilla jolla on vasen pseudoratojen jäljitysominaisuus ja oikea ekspansiivisuus on itse asiassa tavanomainen pseudoratojen jäljitysominaisuus.

Toinen pääteema on konjugaattisuus. Dynaamiset systeemit ovat konjugaatteja, jos ne ovat dynaamisessa mielessä sama systeemi. Näytämme, että yksipuoleisille soluautomaateille konjugaattisuus on ratkeamaton ominaisuus. Itse asiassa tulos on vahvempi ja näyttää samalla, että on ratkeamatonta onko soluautomaatti toisen tekijä tai alisysteemi.

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Riikka: Rakkaus on pikku juttu, se mahtuu väliin rivien; kuin ohimennen, niin itsestäänselvää – kuten kappaaleessa sanotaan.

the 12th of August, 2020

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List of original publications

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4. J. Jalonen, J. Kari: “On the Conjugacy Problem of Cellular Automata”. *Information and Computation* (2020) 104531, doi: 10.1016/j.ic.2020.104531

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Chapter 1

Introduction

We start with an informal introduction to the topic, proper mathematical rigour shall make its way into the thesis starting from the beginning of the next chapter. For now it shall be good enough to consider that we are studying infinite sequences of states (or symbols), such as this one

$$\dots 0000111010000101000 \dots$$

A sequence like this is called a configuration; we are interested in the set of all configurations. We usually have more states than just two (though always only finitely many), and even though we mostly use numbers as states, the numerical value is not important, and we could just as well define all configurations using, for example, letters or colours. A cellular automaton is a function that maps configurations to configurations in a special way: Every position updates its state into a new state uniquely, based on its own state and the states of its neighbors. Positions are called cells and in every cell we have a simple automaton determining its behaviour, hence the name *cellular automaton*. An example should make things clearer. Consider all infinite sequences of zeroes and ones. Update every position as follows: Add the value in the cell and in its immediate right and left neighbors together modulo two, so that the next value is 0 if the three consequent cells contain even number of ones and 1 otherwise. So, for example, the configuration above updates as follows

$$\begin{aligned} &\dots 0000111010000101000 \dots \\ &\dots 0001010011001101100 \dots \\ &\dots 0011011100110000010 \dots \\ &\dots 0100001011001000111 \dots \end{aligned}$$

where we have assumed that the state of every non-visible cell is zero. Figures 1.1 and 1.2 present some space-time diagrams of this cellular automa-

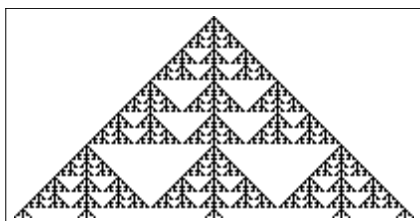


Figure 1.1: Running rule 150 from $\dots 00100\dots$. Time advances downwards, as it will in every figure of this thesis.

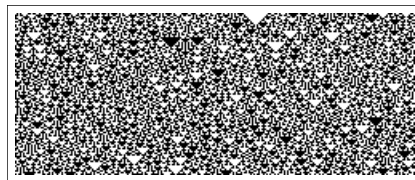


Figure 1.2: Running rule 150 from a random initial configuration.

ton. This cellular automaton is known as rule 150 based on the numbering scheme devised by Stephen Wolfram [63].

From the example above, let us try to infer some abstract reasons that make cellular automata worth studying. Notice that locally a cellular automaton is very simple. In fact, those in the know will recognize that locally a cellular automaton is defined by what is called a finite state automaton, a machine which is known to be computationally quite restricted. Notice also that this local behaviour is not only simple, it is also homogenous in the sense that the same rule is applied everywhere. John von Neumann, who originally presented cellular automata [59], was interested in a mathematical model that resembles natural computation. The usual model of computation, that is, Turing machines resemble an artificial computational device (a computer) in that it has two separate parts: A complex part that is responsible for carrying out the computation (think of the CPU of a computer) and a static memory that the complex part can access and modify. In a natural computational device (such as a brain) complexity arises from parallelism rather than having a centralized computational unit, in other words, natural computational devices are locally simple, at least when compared to their global behaviour (neurons are relatively simple when compared to the very complex behaviour their parallel interaction gives rise to). Cellular automata attempts to mimic this natural phenomenon: Locally simple, globally complex.

Cellular automata can simulate arbitrary Turing machines; this computational universality was shown for two-dimensional cellular automata by von Neumann [59], for one-dimensional cellular automata by Alvy Rey Smith III [57], and for reversible one-dimensional cellular automata by Kenichi Morita and Masateru Harao [49]. These are theoretically important in the sense that they show that cellular automata are capable of carrying out complex computations. However, it is somewhat unsatisfactory that

these constructions explicitly set the cellular automaton up to implement a known computationally universal machine rather than the complexity to arise more “naturally”. In this regard the computational universality of the famous Game of Life and that of the rule 110 (in the aforementioned Wolfram’s numbering scheme) are interesting as these are both very simple cellular automata neither of which was tailor made for the purpose. Game of Life was proven computationally universal by Elwyn Berlekamp, John Conway, and Richard Guy [5], and rule 110 was proven computationally universal by Matthew Cook [13].

It is worth noting that none of the constructions of cellular automata capable of universal computing benefit from the parallelism in any way, but rather work against it. The constructions are set up so that most of the configuration is in idle state while the computation is carried out in small portion of the space at any given time. This also leads to only considering certain suitable configurations, since most configurations are not valid encodings of the simulated machine (if one is directly simulating a Turing machine, then one does not, for example, want multiple Turing machine heads in one configuration). This leads one to ask whether we can study cellular automata more as a whole: What kind of behaviour they exhibit? What are their dynamics? The vagueness of these questions already points out that we are lacking a proper language to have this discussion. The necessary tools and language is provided by topological dynamics; this study was initiated by Gustav Hedlund [26]. To get a feel of how the viewpoint may change when we consider cellular automata as dynamical systems rather than just computational devices, consider undecidability results. Asking whether running a given cellular automaton from some fixed initial configuration produces some specific state in the zeroth coordinate seems reasonable from the computational point-of-view we started with, but feels unrelated to the dynamical systems point-of-view. On the other hand, asking for a fixed cellular automaton, if it is decidable whether a given a configuration is periodic, is somewhere in between, since this is a dynamical property but, on the other hand, considers only the behaviour of a single configuration. Asking whether it is decidable if all configurations are periodic would be even more natural decision problem from dynamical point-of-view. There are plenty of undecidability results about cellular automata as dynamical systems, for example, it is known that entropy (a certain complexity measure) is uncomputable (Lyman P. Hurd, Jarkko Kari, and Karel Culik [29]), that all non-trivial properties of the limit set (meaning the set of configurations that can appear arbitrarily late) are undecidable (Jarkko Kari [36]), and the aforementioned periodicity is also undecidable (Jarkko Kari and Nicholas Ollinger [38]), to mention a few.

Everything said thus far hopefully gives a rough feel of the field this thesis belongs to. Let us next go through the structure of the thesis and

explain what is accomplished. In order to keep this exposition light, we will not go into too much details.

In Chapter 2 we ground our notations and mention known results that will be useful in the later chapters. Our notations and terminology are standard where standards exist. Most of the work goes into presenting geometrical interpretations of various dynamical properties for cellular automata.

In Chapter 3 we present the notion of stripe shifts. Stripe shifts are obtained when configurations appearing in time-evolutions of a cellular automaton are required to satisfy certain spatial homogeneity. This is an interesting topic, but largely independent from the rest of the thesis (we only use one result about stripe shifts in a novel way in Chapter 5), thus we will not aim to explain stripe shifts further here.

A dynamical system consists of a space and a transformation of this space. When cellular automaton is interpreted as a dynamical system, the set of configurations is the space and the map defined by a local rule is the transformation of this space. Most of the concepts we consider in this thesis are general concepts of dynamical systems (for example, expansivity, pseudo-orbit tracing property, and conjugacy). However, the geometry of cellular automata allows considering also one-sided variants of some dynamical properties which cannot be defined for general dynamical systems. From Chapter 4 onwards word “one-sided” appears frequently.

In all simplicity one-sided refers to the fact that the configurations as described above were infinite both to the left and to the right, and then we may consider certain properties only for one of these directions. First of all, we can define one-sided cellular automata: This is nothing more than same kind of local rules as above, but now over configurations which are infinite only to the right, such as

$$0111010000101000 \cdots .$$

Notice that now the local rule must also be one-sided, since if the rule needs to look left, then we cannot define our map at the zeroth coordinate (for example, the rule described above is not one-sided, and we could not determine whether the first 0 should map to 0 or 1). In Chapter 4 we study cellular automata which are one-sided and reversible, i.e. there exists another one-sided cellular automaton which reverts its action.

Expansivity means strong sensitivity to the initial conditions, more specifically it states that every pair of points, no matter how close to each other they are, will at some point be far away from each other. For cellular automata this can be interpreted as horizontal determinism in the space-time diagrams, i.e. a cellular automaton is expansive if and only if a large enough vertical slice of a space-time diagram determines the entire space-time diagram. Pseudo-orbit tracing property means that if the system is

simulated in high-enough precision one cannot distinguish this simulation from reality. For cellular automata this can be interpreted by saying that a cellular automaton has pseudo-orbit tracing property if and only if even if one is allowed to make arbitrary changes to the configuration far enough from the zeroth cell on every time step, the sequence observed in the zeroth coordinate is a sequence appearing in some actual space-time diagram. In Chapter 5 we study expansivity and pseudo-orbit tracing property for cellular automata. Again the geometry allows us to define one-sided variants of these concepts. We prove that a cellular automaton which has left pseudo-orbit tracing property and is right-expansive actually has to have pseudo-orbit tracing property (not only one-sided).

In Chapter 6 we turn to conjugacy. Conjugacy of dynamical systems is a natural notion of “sameness”; dynamical systems that are conjugate are in topological dynamical sense the same system. First we prove that for one-sided cellular automata conjugacy is undecidable, i.e. there does not exist an algorithm that would take two cellular automata and output “yes” if they are conjugate and “no” if not. Then we prove this for reversible two-dimensional cellular automata. In fact we prove more general results which imply that some other common dynamical relations are also undecidable.

In Chapter 7 we wrap things up by stating some open problems.

For most parts the contents of this thesis have appeared in [31, 32, 33, 30]. Contents of Chapters 3 and 5 can be found in [32] and [33] with the exception of an example of an uncountable stripe shift, which has not been published before. Some quite simple reversible one-sided cellular automata are used in constructions of all of the aforementioned publications, and for this sake we have included here a slightly longer discussion about them than was possible in any of these publications; this is done in Chapter 4. The main undecidability results regarding conjugacy in Chapter 6 have appeared in [31] and [30], though the fixed alphabet variants presented here are new.

Chapter 2

Preliminaries

In this chapter we present notations, some basic definitions, and some well-known, or at least simple, results about symbolic dynamics and cellular automata in particular. Mainly we want to interpret topological dynamical concepts in the setting of symbolic dynamics; various properties tend to have nice visual interpretations in the setting of cellular automata. We assume that the reader is familiar with large portions of this chapter and in many topics we will only mention our conventions without going into more details.

The book [42] by Douglas Lind and Brian Marcus is a good comprehensive introduction to symbolic dynamics (and coding), and we will pick a few results from there. For an introduction to topological dynamical point of view to symbolic dynamics see, e.g, Petr Kůrka's book [40].

2.1 Basic Notations

We denote by \mathbb{N}, \mathbb{Z} and \mathbb{R} the sets of natural number, integers and reals, respectively. In order to avoid some repetition, we use \mathbb{M} to denote when both \mathbb{N} and \mathbb{Z} may be used. For us, $0 \in \mathbb{N}$. Let $i, j \in \mathbb{Z}$ such that $i \leq j$, then we denote by $[i, j] = [i, j+1) = (i-1, j] = \{i, i+1, \dots, j\}$. Let X, Y, Z be sets. We denote by Y^X the set of all functions $X \rightarrow Y$. For $f \in Y^X$ and $g \in Z^Y$ the composition of functions is written as gf and defined by $gf(x) = g(f(x))$ for all $x \in X$. For a function $f : X \rightarrow Y$ and a subset $X' \subseteq X$, the restriction of f to X' is denoted by $f|_{X'}$ or, in some cases, simply by $f_{X'}$. We denote the cardinality of a set X by $|X|$.

If we need to use superscripts as indices, we write the index in parenthesis to separate indices from exponents, i.e. $x^{(i)}$ denotes an element indexed by i while x^i denotes the i^{th} power of x .

2.2 Automata, Languages, and Graphs

Let A be, as it always will be in this thesis, a finite non-empty set. In the context of languages the set A is often called an *alphabet* and its elements *symbols* or *letters*, we will call them also *states*. A *word* of length $n \in \mathbb{N}$ over the alphabet A is an n -element sequence of letters from A . The word of length 0 is called the *empty word*. The empty word is often given a special symbol, but we will not need one. The set of all words of length $n \in \mathbb{N}$ over A is denoted by A^n . We also denote $A^* = \bigcup_{n \in \mathbb{N}} A^n$ and $A^+ = \bigcup_{n \in \mathbb{N} \setminus \{0\}} A^n$. The length of a word $u \in A^*$ is denoted by $|u|$. Any subset $L \subseteq A^*$ is called a *language*. If L is finite, the language is a *finite language*. If the language L is recognized by a finite state automaton (as defined, for example, in [28, §2.2.1]), it is a *regular language*.

A (*labeled directed*) *graph* is a quintuple $\mathcal{G} = (V, E, s, t, \lambda)$ where V is a finite set of *vertices*, E is a finite set of *edges* or *arrows*, s and t are functions $E \rightarrow V$ which should be understood as specifying the source and the terminal vertices of the edges, i.e. $e \in E$ is an arrow from $s(e)$ to $t(e)$, and λ is a labeling function $E \rightarrow A$ where A is some finite alphabet. We are not actually ever again going to bother being this formal and instead will define graphs as triplets (V, E, λ) and consider $E \subseteq V \times V$, but then E actually would need to be a multiset to allow multiple edges between same vertices. In practice no confusion should arise. We denote by $v \xrightarrow{a} v'$ that there are vertices $v, v' \in V$ and an edge $e \in E$ such that $s(e) = v$ and $t(e) = v'$ and $\lambda(e) = a$, we may leave the label out if it is not relevant for the considerations at hand. Note that loops, i.e. edges from a vertex to itself, are allowed. We extract two languages from a graph. First is a language over V of sequences of vertices that can follow each other in the graph, i.e. $v = v_0 v_1 \cdots v_{n-1} \in V^*$ is in this language if and only if there is a directed path $v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_{n-1}$ in the graph (note that the labels play no role here). The other one is defined by the labels of paths, i.e. a word $u = u_0 u_1 \cdots u_{n-1} \in A^*$ is in this language if and only if there exists a path $v_0 \xrightarrow{u_0} v_1 \xrightarrow{u_1} \cdots \xrightarrow{u_{n-1}} v_n$, where $u_i = \lambda((v_i, v_{i+1}))$, is in the graph. These are, of course, regular languages. One can define an “edge language” in the same way we defined “vertex language”, but we will not use this notion.

Graph is called *connected* if it is possible to reach any vertex from any other vertex by traversing along edges, irregardless of the direction of the edges (i.e. one is allowed to go from vertex v_0 to v_1 if there is an edge $v_1 \rightarrow v_2$ or an edge $v_2 \rightarrow v_1$). Graph is called *strongly connected* if there is a directed path from every vertex to every other vertex. A *connected component* is a maximal connected subgraph and a *strongly connected component* is a maximal strongly connected subgraph.

2.3 Topological Dynamics

A (*topological*) *dynamical system* is a pair (X, ϕ) where X is a compact metric space and f a continuous map $X \rightarrow X$. Let (X, ϕ) and (Y, φ) be two dynamical systems. A continuous map $\psi : X \rightarrow Y$ is a *homomorphism* if $\psi\phi = \varphi\psi$. If ψ is surjective, it is a *factor map*, and (Y, φ) is a *factor* of (X, ϕ) . If ψ is injective, it is an *embedding*, and (X, ϕ) is a *subsystem* of (Y, φ) . If ψ is a bijection, it is a *conjugacy*, and (X, ϕ) and (Y, φ) are *conjugate*. Conjugacy of (X, ϕ) and (Y, φ) is denoted by $(X, \phi) \cong (Y, \varphi)$ or just $\phi \cong \varphi$. Homomorphisms from (X, ϕ) to itself are called *endomorphisms*, and bijective homomorphisms (which are homeomorphisms, as the underlying space is compact and metric) from (X, ϕ) to itself are called *automorphisms*. A dynamical system (X, ϕ) is called *reversible* if ϕ is a homeomorphism. Let $\text{dist} : X \times X \rightarrow \mathbb{R}_+ \cup \{0\}$ be the metric considered. A sequence $(x_i)_{i \in \mathbb{Z}} \in X^{\mathbb{Z}}$ is a (*two-way*) *orbit* of ϕ if $\phi(x_i) = x_{i+1}$ for every $i \in \mathbb{Z}$. Let $x, y \in X$. There is an ε -*chain from x to y* if there exists $n > 0$ and a sequence $x = x_0, x_1, \dots, x_n = y \in X$ such that $\text{dist}(\phi(x_i), x_{i+1}) < \varepsilon$, for all $i \in \{0, 1, \dots, n-1\}$. Two-way infinite ε -chains are called ε -*pseudo-orbits*.

Let (X, ϕ) be a dynamical system.

- (X, ϕ) is *recurrent* if for every non-empty open set U there exists $n > 0$ such that $\phi^n(U) \cap U \neq \emptyset$.
- (X, ϕ) is *transitive* if for all non-empty open sets U, V there exists $n > 0$ such that $\phi^n(U) \cap V \neq \emptyset$.
- (X, ϕ) is *mixing* if for all non-empty open sets U, V there exists $N > 0$ such that for all $n \geq N$ it holds that $\phi^n(U) \cap V \neq \emptyset$.
- (X, ϕ) is *chain-recurrent* if for all $x \in X$ and $\varepsilon > 0$ there exists an ε -chain from x to x .
- (X, ϕ) is *chain-transitive* if for all $x, y \in X$ and $\varepsilon > 0$ there exists an ε -chain from x to y .
- (X, ϕ) is *chain-mixing* if for all $x, y \in X$ and $\varepsilon > 0$ there exists $N > 0$ such that for all $n \geq N$ there exists an ε -chain $x = x_0, x_1, \dots, x_n = y$ from x to y .
- (X, ϕ) has the *pseudo-orbit tracing property (POTP)*, often also called the *shadowing property*, if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any δ -pseudo-orbit $(x_i)_{i \in \mathbb{Z}}$ there exists an orbit $(y_i)_{i \in \mathbb{Z}}$ such that $\text{dist}(x_i, y_i) < \varepsilon$ for all $i \in \mathbb{Z}$.
- (X, ϕ) is *positively expansive* if ϕ is surjective and there exists $\varepsilon > 0$ such that for all $x, y \in X$ it holds that $x \neq y \implies \exists n \in \mathbb{N} : \text{dist}(\phi^n(x), \phi^n(y)) > \varepsilon$. Such an ε is called an *expansivity constant*.

- (X, ϕ) is *expansive* if ϕ is reversible and there exists $\varepsilon > 0$ such that for all $x, y \in X$ it holds that $x \neq y \implies \exists n \in \mathbb{Z} : \text{dist}(\phi^n(x), \phi^n(y)) > \varepsilon$. Such an ε is called an *expansivity constant*.

A point $x \in X$ is *ϕ -periodic* if there exists $n \in \mathbb{N} \setminus \{0\}$ such that $\phi^n(x) = x$. The set of all ϕ -periodic points is denoted by $\text{Per}_\phi(X)$.

A point $x \in X$ is an *equicontinuity* point of (X, ϕ) if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $y \in X$ it holds that $\text{dist}(x, y) < \delta \implies \text{dist}(\phi^n(x), \phi^n(y)) < \varepsilon$ for every $n \geq 1$. A dynamical system is *equicontinuous* if every point is an equicontinuity point.

2.4 Symbolic Dynamics and Cellular Automata

General dynamical systems are not the object of study of this thesis, rather we are interested in specific kinds of dynamical systems, namely, to some degree, *subshifts* and, especially, *cellular automata*. In this section we give an overview of these topics picking important, mostly very well-known, results from the literature, and also give interpretations for the various topological dynamical properties defined in the previous section.

Vast majority of this thesis considers one-dimensional cellular automata, and to accomodate this we first discuss one-dimensional symbolic dynamics. However, two-dimensional symbolic dynamics plays a minor role in two ways: First, the space-time diagrams of one-dimensional cellular automata are two-dimensional objects, and so some concepts of two-dimensional symbolic dynamics can aid understanding. Second, and more direct, way is the last chapter of the thesis, where we discuss conjugacy of two-dimensional cellular automata. Thus we end this section by defining higher-dimensional subshifts and cellular automata.

2.4.1 Shift Spaces

Let A , again, be a finite nonempty set. In this context we may also call A a set of *colours*. The set $A^{\mathbb{M}}$ is called the *full (A-)shift*. An element $c \in A^{\mathbb{M}}$ is called a *configuration*. In all the definitions we add the word *two-sided* when we want that $\mathbb{M} = \mathbb{Z}$ and the word *one-sided* when we want that $\mathbb{M} = \mathbb{N}$. The i^{th} element (or the color of the *cell* i), is usually denoted by $c_i = c(i)$. In pictures and in mental images the configurations are usually sequences of symbols such as

$$\dots 01210.11020 \dots$$

where the first letter to the right of the decimal point, is the zeroth element. We leave this decimal point out when knowing the zeroth cell is not important. For one-sided configurations marking the zeroth letter is unnecessary,

so it is left out:

$$01221001 \dots$$

We could also present these as colorings of rows of squares; the numbers here are considered solely as symbols, and their numerical value has no relevance. For a finite word $u \in A^n$ we denote $u^\omega \in A^{\mathbb{N}}$ the configuration defined by $u_i^\omega = u_{i \bmod n}$. Similarly we denote ${}^\omega u^\omega \in A^{\mathbb{Z}}$. We can also do concatenations such as ${}^\omega uvw^\omega$ which is understood as a configuration $\dots uvvwvw \dots$, and if the zeroth element is important, we denote it with a decimal point as above, for example, ${}^\omega u.vw^\omega$. If the configuration is given using letters, we use brackets to denote which part is to be repeated, for example,

$${}^\omega(01)2(010)^\omega = \dots 0101012010010010 \dots$$

The word *shift* appears in the name of the full shift because of the importance of the following simple function: The map $\sigma : A^{\mathbb{M}} \rightarrow A^{\mathbb{M}}$ defined by $\sigma(c)_i = c_{i+1}$ for every $i \in \mathbb{M}$ is called the *shift map*. Here is an example of the shift map σ applied to a two-sided configuration

$$\begin{aligned} &\dots 01210.11020 \dots \\ &\dots 12101.10200 \dots \\ &\dots 21011.02001 \dots \end{aligned}$$

and to a one-sided configuration

$$\begin{aligned} &01221001 \dots \\ &12210010 \dots \\ &22100100 \dots \end{aligned}$$

Already at this point one may remark an important difference: the shift map is a bijection over $A^{\mathbb{Z}}$ but not over $A^{\mathbb{N}}$ (in the non-trivial case when $|A| > 1$).

So far we have defined only full shifts; we would like to have other shift spaces too. Let $X \subseteq A^{\mathbb{M}}$. We say that a word $u \in A^n$ *appears* in X if there exists $c \in X$ and $i \in \mathbb{M}$ such that $c_{[i, i+n)} = u$. Now the *language* of X is the set of all words that appear in some configuration of X , i.e. $\mathcal{L}(X) = \{u \in A^* \mid \exists c \in X : \exists i \in \mathbb{M} : c_{[i, i+|u|)} = u\}$. What we want from our shift spaces is that they are defined by their languages, i.e. we say that $X \subseteq A^{\mathbb{M}}$ is a *shift space* or a *subshift* if for every $c \in A^{\mathbb{M}}$ it holds that if $\mathcal{L}(c) \subseteq \mathcal{L}(X)$ then $c \in X$. We want this since it gives us two important properties. First of all, X is then shift invariant, i.e. if $c \in X$ then also $\sigma^i(c) \in X$ for all $i \in \mathbb{M}$, since clearly $\mathcal{L}(\sigma^i(c)) \subseteq \mathcal{L}(c)$ for all $i \in \mathbb{M}$. Second one is a compactness property: for any configuration $c \in A^{\mathbb{M}} \setminus X$ there is always a finite segment $c_{[i, i+n)}$ for some $i \in \mathbb{M}, n \in \mathbb{N} \setminus \{0\}$ from which

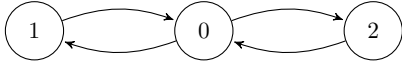


Figure 2.1: Example of a graph defining an 2-SFT, where every second letter is 0 and every second is either 1 or 2.

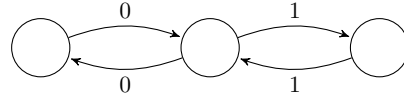


Figure 2.2: A graph defining a proper sofic shift: For any k , for example, the configuration ${}^\omega 10^{2k+1}1^\omega$ illustrates that the subshift defined is not an k -SFT

we see that $c \notin X$, i.e. $c_{[i,i+n)} \notin \mathcal{L}(X)$. Thus we cannot have a subshift $X \in \{0,1\}^{\mathbb{Z}}$ which would contain ${}^\omega 010^\omega$ but would not contain ${}^\omega 0^\omega$ since $\mathcal{L}({}^\omega 0^\omega) = 0^* \subseteq \mathcal{L}({}^\omega 010^\omega)$.

A subshift X is called *one-sided* if $X \subseteq A^{\mathbb{N}}$ and *two-sided* if $X \subseteq A^{\mathbb{Z}}$. As was mentioned before, the important difference is whether the shift map is reversible or not.

We denote by $\mathcal{L}_n(X) = \mathcal{L}(X) \cap A^n$, i.e. the words of length n that appear in X . For $n \in \mathbb{N} \setminus \{0\}$ we say that the subshift X is an *n -degree subshift of finite type (n -SFT)* if it is sufficient to check the words of length n that appear in c to determine whether it belongs to X or not, or more formally, X is an n -SFT if

$$\mathcal{L}_n(c) \subseteq \mathcal{L}_n(X) \implies c \in X.$$

A subshift X is an *subshift of finite type (SFT)* if it is an n -SFT for some $n \in \mathbb{N} \setminus \{0\}$. The 2-SFT's can be conveniently expressed as (directed) graphs (V, E) where the set of vertices $V = A$ is the alphabet and there is an edge $(a, b) \in E \subseteq V \times V$ if and only if $ab \in \mathcal{L}_2(X)$ (see Figure 2.1). The subshift X is a *sofic shift* if $\mathcal{L}(X)$ is a regular language, and thus sofic shifts can be expressed as labels of edges of (labeled directed) graphs (by using finite state automata). The sofic shift defined by the graph (automaton) of Figure 2.2 illustrates that sofic shifts need not be SFT's, such sofic shifts are called *proper sofic* if the distinction is important.

We call an *n -SFT approximation* of a subshift X the smallest n -SFT that contains X , and this is denoted by $\text{SFT}_n(X)$. In other words, $c \in \text{SFT}_n(X)$ if and only if $\mathcal{L}_n(c) \subseteq \mathcal{L}_n(X)$. For example 3-SFT approximation for the subshift defined by the graph of Figure 2.2 is a subshift where 101 and 010 are forbidden.

Subshifts can equally well be defined using *forbidden words*, i.e. by saying that X is a subshift if and only if there exists a language \mathcal{F} (of forbidden words) such that $c \in X$ if and only if for every $i, j \in \mathbb{M}$ such that $i < j$ holds that $c_{[i,j)} \notin \mathcal{F}$. A subshift is an SFT if and only if the language of forbidden words can be chosen finite and sofic if and only if the language of forbidden words can be chosen regular. A subshift defined by forbidding language \mathcal{F}

is denoted by $X_{\mathcal{F}}$. See [42, §1.3] for details.

2.4.2 Cellular Automata

Let $X \subseteq A^{\mathbb{M}}$ be a subshift, B an alphabet, and $a, m \in \mathbb{N}$ (if $\mathbb{M} = \mathbb{N}$ then we require that $m = 0$). Let $F_{loc} : \mathcal{L}_{m+a+1}(X) \rightarrow B$ be a map which we call a *local rule*. Local rule defines a *sliding block map* $F : X \rightarrow B^{\mathbb{M}}$ by $F(c)_i = f(c_{[i-m, i+a]})$; it is called a sliding block map since we can consider it as sliding a window of size $m + a + 1$ over the configuration and applying F_{loc} in each position to get a new configuration. Here m and a are *memory* and *anticipation* of the sliding block map, and $r = \max\{m, a\}$ is its *radius*. The smallest possible radius that can be used to define the sliding block map F is denoted by $r(F)$. Mostly we discuss specific kind of sliding block maps, where we in addition require that $F(X) \subseteq X$, such maps we call *cellular automata*. A cellular automaton over $X \subseteq A^{\mathbb{N}}$ is called *one-sided* and a cellular automaton over $X \subseteq A^{\mathbb{Z}}$ is called *two-sided*.

As an example, notice that the SFT of Figure 2.1 maps to the sofic shift of Figure 2.2 by a sliding block map defined by the local rule $F_{loc} : \{01, 10, 02, 20\} \subseteq A^{\{0,1\}} \rightarrow \{0, 1\}$ where $01, 10 \mapsto 0$ and $02, 20 \mapsto 1$, for example,

$$F(0101020102020 \dots) = 000011001111 \dots$$

If $m = 0$ we can interpret the sliding block map over both one- and two-sided subshifts. If $m > 0$ then, of course, the sliding block map can only be considered over a two-sided subshift.

Let (X, F) be a cellular automaton with radius $r = r(F)$. Then the local rule can be applied to any word that is at least $2r + 1$ letters long. We will denote the maps that cellular automata induce on finite words by lower case letters, i.e. f for F . So $f : \bigcup_{k \in \mathbb{N}} \mathcal{L}_{2r+k+1}(X) \rightarrow \mathcal{L}(X)$ is defined by $f(u_{[0, 2r+k]}) = F_{loc}(u_{[0, 2r]})F_{loc}(u_{[1, 2r+1]}) \cdots F_{loc}(u_{[k, 2r+k]})$.

2.4.3 Subshifts and Cellular Automata as Topological Dynamical Systems

Next we want to put subshifts and cellular automata into a topological dynamical setting and then interpret topological dynamical properties for them. After this section the general topological definitions will play only a minor role, if even that.

First we turn full shifts into compact metric spaces. This is achieved by defining metric $\text{dist} : A^{\mathbb{M}} \rightarrow \mathbb{R}_+ \cup \{0\}$ by setting

$$\text{dist}(c, e) = \begin{cases} 2^{-i}, & \text{if } i = \min\{|k| \mid c_k \neq e_k\} \text{ exists} \\ 0, & \text{if } c = e \end{cases}$$

for all $c, e \in A^{\mathbb{M}}$. It is well-known that this turns $A^{\mathbb{M}}$ into a compact metric space. Let $D \subseteq \mathbb{M}$ be a finite subset and $u \in A^D$ some finite pattern, then the set $[u] = \{c \in A^{\mathbb{M}} \mid c_D = u\}$ is called a *cylinder*. Cylinder sets form a countable clopen (closed and open) basis for the topology induced by the metric dist .

Now that $A^{\mathbb{M}}$ has been interpreted as a compact metric space we could study dynamical systems $(A^{\mathbb{M}}, \phi)$ where $\phi : A^{\mathbb{M}} \rightarrow A^{\mathbb{M}}$ is a continuous map. These are, however, not necessarily cellular automata, as for example the map $T : A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ defined by $T(c)_i = c_{-i}$ for all $i \in \mathbb{Z}$ can easily be seen continuous, yet cannot be defined by a sliding block map. The issue is that while cellular automata are spatially homogenous the metric we defined gives a special emphasis to the zeroth cell. Spatial homogeneity can be imposed by requiring that the maps commute with the shift map, i.e. we are going to study dynamical systems $(A^{\mathbb{M}}, F)$ where $\sigma F = F\sigma$. Equivalently we can say we are studying the endomorphisms of $(A^{\mathbb{M}}, \sigma)$, as $(A^{\mathbb{M}}, \sigma)$ is itself a dynamical system (σ is easily seen to be continuous). We can then also study subsystems, i.e. endomorphisms of (X, σ) where we obviously want that X is compact (i.e. closed) and shift invariant (i.e. $\sigma^i(X) \subseteq X$ for all $i \in \mathbb{M}$). Classical results state that these are precisely the correct requirements to consider subshifts and cellular automata as dynamical systems.

Proposition 2.4.1 ([40, Proposition 3.12]). *Let $X \subseteq A^{\mathbb{M}}$. Then X is a subshift if and only if X is closed and $\sigma^i(X) \subseteq X$ for all $i \in \mathbb{M}$.*

Theorem 2.4.2 (Curtis-Hedlund-Lyndon [26, Theorem 3.4]). *Let X be a shift space. Cellular automata on X are precisely the endomorphisms of (X, σ) .*

The shift map is usually only implicitly present, though we may talk about *shift-dynamical system* (X, σ) if we want to emphasize that we are considering the subshift as a dynamical system itself, not just as the underlying space. The above results will be used as alternative definitions for subshifts and cellular automata (i.e. we may, for example, prove that a set is a subshift by showing that it is closed and shift-invariant without referring to the proposition above).

Remark 2.4.3. Since the systems we study, i.e. cellular automata, are spatially homogenous but the metric gives an emphasis to the zeroth coordinate, one may ask whether this is a good choice of metric. Indeed it can seem questionable at first that points ${}^\omega 0^\omega, {}^\omega 0.10^\omega \in \{0, 1\}^{\mathbb{Z}}$ are far away from each other even though these points are the same in almost every cell, and we are considering systems which treat every position equally. In this sense one could say that this is not a good *intrinsic* metric; it does not capture what is relevant about the space for the dynamical system itself. However, we consider cellular automata as computational devices, and use topological

dynamics for their study. From this point-of-view the metric above is justifiable, for if we consider cellular automaton as a system someone is actually observing, then this observer cannot observe the entire system and is forced to give some cell a special role of being the central cell. One could call this metric an *observer's metric*. For this outside observer it is natural to consider that configurations which are different far from the, arbitrarily chosen, central cell are close to each other. We refer to the article of Gianpiero Cattaneo, Enrico Formenti, Luciano Margara, and Jacques Mazoyer [11] for a candidate for better intrinsic (pseudo-)metric.

A cellular automaton (X, F) is called *reversible* if there exists another cellular automaton (X, G) such that $FG = GF = \text{id}$, where id is the identity map (which itself is clearly a cellular automaton). It turns out that it is sufficient to assume that F is bijective: Clearly the inverse is shift-commuting, and since X is a compact metric space, the inverse of F is also continuous, and then by the Curtis-Hedlund-Lyndon Theorem the inverse is a cellular automaton. For cellular automata over full shifts the following holds by John Myhill [50].

Proposition 2.4.4 ([50]). *Let $(A^{\mathbb{M}}, F)$ be a cellular automaton. If F is injective, it is surjective, and thus bijective, and so also reversible.*

We will use this in Chapter 6, but notice that this does not hold for cellular automata over arbitrary subshifts; see an example by Tullio Ceccherini-Silberstein and Michel Coornaert in their arXiv paper [12, Example 7.13].

Let (X, F) be a cellular automaton. In this setting we call orbits *space-time diagrams*. The set of all two-way space-time diagrams of (X, F) is denoted by

$$st_{\mathbb{Z}}(X, F) = \{(c^{(i)})_{i \in \mathbb{Z}} \in X^{\mathbb{Z}} \mid \forall i \in \mathbb{Z} : F(c^{(i)}) = c^{(i+1)}\}.$$

Naturally we then denote the set of one-way space-time diagrams of (X, F) by

$$st_{\mathbb{N}}(X, F) = \{(c^{(i)})_{i \in \mathbb{N}} \in X^{\mathbb{N}} \mid \forall i \in \mathbb{N} : F(c^{(i)}) = c^{(i+1)}\}.$$

We may also short-hand these notations to $st_{\mathbb{Z}}(F)$ and $st_{\mathbb{N}}(F)$ if there is no risk of confusion. Instead of a sequence of configurations we usually visualize these as labeled or coloured square lattices where each row represents a configuration and time advances downwards. The words *left*, *right*, *up*, and *down* should be understood accordingly. Considering the two-way infinite space-time diagrams means in practice that the space-time diagrams are considered over the *limit set* $\Lambda(X, F) = \bigcap_{i \in \mathbb{N}} F^i(X)$. Of course for surjective cellular automata $\Lambda(X, F) = X$.

An (*two-sided*) *n-trace* of a cellular automaton (X, F) where $X \subseteq A^{\mathbb{M}}$ is

$$\tau_{\mathbb{Z}, n}(X, F) = \{t \in (A^n)^{\mathbb{Z}} \mid \exists (c^{(i)})_{i \in \mathbb{Z}} \in st_{\mathbb{Z}}(F) : \forall i \in \mathbb{Z} : c_{[0, n]}^{(i)} = t_i\}.$$

Similarly we define $\tau_{\mathbb{N},n}(X, F)$. If the context allows, we may write this just as $\tau_n(F)$, and trust that both the space and whether we are considering one- or two-sided traces is clear from the context. Traces are just vertical slices of space-time diagrams, and easily seen to be subshifts. For two-sided cellular automata it is in some sense more natural to consider n -traces to be centralized around the zeroth cell, but in practice this makes no real difference, and thus we will choose the indexing of the trace subshift's alphabet by whatever is most convenient in any given context.

Entropy can be defined for an arbitrary dynamical system, but we settle for defining it only for one-dimensional cellular automata. Let (X, F) be a cellular automaton. Then *entropy* of (X, F) is

$$h(X, F) = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{n} \log_2 |\mathcal{L}_n(\tau_m(X, F))|.$$

For a shift-dynamical system (X, σ) we have that $|\mathcal{L}_n(\tau_{m+1})| \leq |A| \cdot |\mathcal{L}_n(\tau_m)|$ so we have that

$$h(X, \sigma) = \lim_{n \rightarrow \infty} \frac{1}{n} \log_2 |\mathcal{L}_n(\tau_1(X, \sigma))| = \lim_{n \rightarrow \infty} \frac{1}{n} \log_2 |\mathcal{L}_n(X)|$$

and thus we can also write for a cellular automaton (X, F) that

$$h(X, F) = \lim_{m \rightarrow \infty} h(\tau_m(X, F)).$$

Let us state some known results about entropy.

Proposition 2.4.5 ([40, Propositions 5.71 and 5.72]). *Let (X, F) be a cellular automaton where $X \subseteq A^{\mathbb{M}}$ and $r(F) = r$.*

1. $h(X, F) \leq 2r \cdot \log_2 |A|$.
2. *If $\mathbb{M} = \mathbb{N}$ then $h(X, F) = h(\tau_r(X, F))$.*

The following generalization is due to Keywon Koh Park, and says that analogous result to point 2 of the previous proposition holds in fact also for two-sided cellular automata.

Proposition 2.4.6 ([54, Lemma 1]). *Let $X \subseteq A^{\mathbb{Z}}$ and let (X, F) be a cellular automaton. Then $h(X, F) = h(\tau_{2r}(X, F))$.*

The most important thing about entropy is that it is a conjugacy invariant.

Proposition 2.4.7 ([42, Proposition 4.1.9 and Corollary 4.1.10]). *Let (X, F) be a cellular automaton and (Y, G) its factor. Then $h(Y, G) \leq h(X, F)$. In particular this implies that if (X, F) and (Y, G) are conjugate, then $h(X, F) = h(Y, G)$.*

For transitive sofic shifts the following holds.

Proposition 2.4.8 ([42, Corollary 4.4.9]). *Let X be a transitive sofic shift and $Y \subseteq X$ a subshift. If $h(Y, \sigma) = h(X, \sigma)$, then $X = Y$.*

The following proposition is used in Chapter 6 to obtain arbitrarily high entropy cellular automata from some positive entropy cellular automaton.

Proposition 2.4.9. *Let (X, F) and (Y, G) be two cellular automata. Then $h(X \times Y, F \times G) = h(X, F) + h(Y, G)$, and $h(X, F^n) = n \cdot h(X, F)$.*

In practice it is impossible to observe entirety of an infinite system, and traces can be considered as representing the finite precision that the observer can manage. This makes sense for observing, but not for simulating since we would need complete knowledge of the systems state in order to run the simulation for arbitrarily long. This viewpoint of simulating a system is captured by *pseudo-orbits*. Let $X \subseteq A^{\mathbb{Z}}$ be a two-sided subshift. Let (X, F) be a cellular automaton with radius r . For every $m \in \mathbb{N} \setminus \{0\}$ we define a (directed labeled multi-)graph $\mathcal{G}_m(X, F) = (V_m, E_m, \lambda_m)$ as follows:

- The set of vertices is $V_m = \mathcal{L}_m(X)$.
- For every $u \in V_m$ and $x, y \in \mathcal{L}_r(X)$ such that $xuy \in \mathcal{L}_{n+2r}(X)$ there is an edge $(u, f(xuy))$ whose label is $\lambda_m((u, f(xuy))) = xy$. We express this by saying that $u \xrightarrow{xy} f(xuy)$ is in E_m .

The graph $\mathcal{G}_m(X, F)$ defines an SFT $\mathcal{PO}_m(X, F)$ such that $\tau_m(X, F) \subseteq \mathcal{PO}_m(X, F) \subseteq (A^m)^{\mathbb{Z}}$ where $(u, v) \in (A^m)^2$ is forbidden if there is no edge $u \rightarrow v$ in the graph $\mathcal{G}_m(X, F)$. The points of $\bigcup_{i \in \mathbb{N} \setminus \{0\}} \mathcal{PO}_i(X, F)$ are essentially the pseudo-orbits of (X, F) . As per usual, we may simplify the notations by leaving the underlying space out of them. We could, again, define one- and two-way variants, but at no point in this thesis are we interested in one-way pseudo-orbits, so we will save ourselves the trouble of introducing notations that would go unused anyway. One can consider pseudo-orbits to represent simulations that only keep track of what is happening inside a window of width m and on each step of the simulation outside of this window arbitrary changes can occur. For one-sided cellular automata similar definitions can be made, but then naturally edges are labeled by $\mathcal{L}_r(X)$ instead; however, we use the same notations for one-sided cellular automata too.

The n -pseudo-orbits for a shift-dynamical system are easy to describe.

Proposition 2.4.10. *Let $X \subseteq A^{\mathbb{Z}}$ be a subshift. The set of n -pseudo-orbits $\mathcal{PO}_n(X, \sigma)$ is naturally conjugate to $\text{SFT}_{n+1}(X)$.*

Proof. It is easy to check that $\phi : \text{SFT}_{n+1}(X) \rightarrow \mathcal{PO}_n(X)$ defined by $\phi(c)_i = c_{[i, i+n)}$ defines a conjugacy between $\mathcal{PO}_n(X)$ and $\text{SFT}_{n+1}(X)$. \square

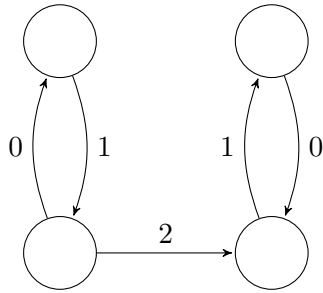


Figure 2.3: Example of a non-transitive sofic shift.

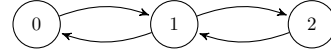


Figure 2.4: This is not mixing.

Similar result holds also for one-sided subshifts as long as one then considers only forward infinite pseudo-orbits, which we just promised we would not do at any point of this thesis.

Let us state some characterizations of topological properties for shift spaces.

Proposition 2.4.11. *Let $X \subseteq A^{\mathbb{N}}$ be a subshift.*

1. X is recurrent if and only if for every $u \in \mathcal{L}(X)$ there exists $w \in \mathcal{L}(X)$ such that $uwu \in \mathcal{L}(X)$.
2. X is transitive if and only if for every $u, v \in \mathcal{L}(X)$ there exists $w \in \mathcal{L}(X)$ such that $uwv \in \mathcal{L}(X)$.
3. X is mixing if and only if for every $u, v \in \mathcal{L}(X)$ there exists $N > 0$ such that for every $n \geq N$ there exists $w \in \mathcal{L}_n(X)$ such that $uwv \in \mathcal{L}(X)$.
4. X is chain-recurrent if and only if $SFT_n(X)$ is recurrent for every $n \in \mathbb{N}$.
5. X is chain-transitive if and only if $SFT_n(X)$ is transitive for every $n \in \mathbb{N}$.
6. X is chain-mixing if and only if $SFT_n(X)$ is mixing for every $n \in \mathbb{N}$.

Proof. It is enough to consider cylinder sets since these form a basis of the topology. Cylinder sets are in a correspondence with (finite sets of) finite words. These observations make the claims rather obvious. \square

From the previous proposition we see that for SFT's chain-recurrence, chain-transitivity, and chain-mixingness are equivalent to recurrence, transitivity, and mixingness (resp.). For sofic shifts this is not the case. For example, the graph in Figure 2.3 defines a proper sofic shift which is not transitive but is chain-transitive. Figure 2.4 shows its 2-SFT, i.e. the 1-pseudo-orbits,

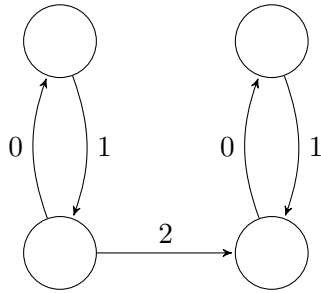


Figure 2.5: Another example of a non-transitive sofic shift.

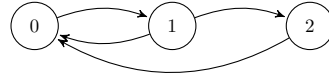


Figure 2.6: This is a mixing SFT.

which shows that this sofic shift is not chain-mixing. Similarly Figures 2.5 and 2.6 show that a non-transitive sofic shift can be chain-mixing (the latter figure only illustrates that 2-SFT's are mixing, and one should of course check that all n -SFT's are mixing). Similar examples were presented by Enrico Formenti and Petr Kůrka [22].

Let (X, σ) where $X \subseteq A^{\mathbb{M}}$ be a shift-dynamical system. For any $k \in \mathbb{N} \setminus \{0\}$ the k^{th} power (X, σ^k) is a dynamical system which is naturally conjugate to the shift-dynamical system $(X^{(k)}, \sigma)$ where $X^{(k)} = \{c \in (A^k)^{\mathbb{M}} \mid \exists e \in A^{\mathbb{M}} : \forall i \in \mathbb{M} : c_i = e_{[ik, (i+1)k]}\}$. Based on this we will call (X, σ^k) a shift-dynamical system itself. The following proposition allows reducing a problem about transitive SFT's to a problem about mixing SFT's in certain situations (this is used in Chapter 5).

Proposition 2.4.12 ([42, §4.5]). *Let (X, σ) be a transitive SFT. There exists $k \in \mathbb{N}$ such that (X, σ^k) is a finite union of disjoint mixing SFT's.*

Let us characterize subshifts which have POTP.

Proposition 2.4.13. *Subshift $X \subseteq A^{\mathbb{M}}$ has POTP if and only if X is an SFT.*

Proof. This is clear by using Proposition 2.4.10; this fact that for subshifts POTP is equivalent to being an SFT was originally proved by Peter Walters [60]. \square

Next proposition is well-known and says that all subshifts are expansive.

Proposition 2.4.14. *Every two-sided subshift is expansive and every one-sided subshift is positively expansive.*

Proof. Let $c, e \in A^{\mathbb{M}}$. If $c \neq e$ then there exists $i \in \mathbb{M}$ such that $c_i \neq e_i$ and so $\text{dist}(\sigma^i(c), \sigma^i(e)) = 1$. \square

The following gives a topological characterization of sofic shifts; we consider it as an alternative definition.

Proposition 2.4.15 ([42, Theorem 3.2.1]). *Let $X \subseteq A^{\mathbb{M}}, Y \subseteq B^{\mathbb{M}}$ be shift spaces and let $F : X \rightarrow Y$ be a sliding block code, i.e. a factor map. If X is an SFT, then $F(X)$ is a sofic shift. If X is a sofic shift, so is $F(X)$. In fact, sofic shifts are precisely the factors of SFT's.*

Next we interpret topological dynamical concepts for cellular automata.

Proposition 2.4.16. *Let $X \subseteq A^{\mathbb{M}}$ be a subshift and (X, F) a cellular automaton.*

1. *(X, F) is recurrent if and only if for every $n \in \mathbb{N} \setminus \{0\}$ the n -trace $\tau_{n, \mathbb{N}}(X, F)$ is recurrent.*
2. *(X, F) is transitive if and only if for every $n \in \mathbb{N} \setminus \{0\}$ the n -trace $\tau_{n, \mathbb{N}}(X, F)$ is transitive.*
3. *(X, F) is mixing if and only if for every $n \in \mathbb{N} \setminus \{0\}$ the n -trace $\tau_{n, \mathbb{N}}(X, F)$ is mixing.*
4. *(X, F) is chain-recurrent if and only if for every $n \in \mathbb{N} \setminus \{0\}$ the subshift $\mathcal{PO}_n(X, F)$ is recurrent, or equivalently, if for every $n \in \mathbb{N} \setminus \{0\}$ there exists a directed path from each vertex of $\mathcal{G}_n(X, F)$ back to itself.*
5. *(X, F) is chain-transitive if and only if for every $n \in \mathbb{N} \setminus \{0\}$ the subshift $\mathcal{PO}_n(X, F)$ is transitive, or equivalently, if for every $n \in \mathbb{N} \setminus \{0\}$ there exists a directed path from every vertex of $\mathcal{G}_n(X, F)$ to every other vertex of $\mathcal{G}_n(X, F)$.*
6. *(X, F) is chain-mixing if and only if for every $n \in \mathbb{N} \setminus \{0\}$ the subshift $\mathcal{PO}_n(X, F)$ is mixing, or equivalently, if there exists $N \in \mathbb{N}$ such that for every $n > N$ there exists a path of length n from every vertex of $\mathcal{G}_n(X, F)$ to every other vertex of $\mathcal{G}_n(X, F)$.*

Proof. It is enough to consider cylinder sets, since these form a basis of the topology.

1. Let (X, F) be such that $\tau_{n, \mathbb{N}}(X, F)$ is recurrent for all $n \in \mathbb{N} \setminus \{0\}$. Let $u \in \mathcal{L}(X)$; we need to show that there exists $n \in \mathbb{N} \setminus \{0\}$ such that $F^n([u]) \cap [u] \neq \emptyset$. According to the first point of Proposition 2.4.11 there exists $w \in \mathcal{L}(\tau_{|u|}(X, F))$ such that $uwu \in \mathcal{L}(\tau_{|u|}(X, F))$. But this already proves that (X, F) is recurrent.

Suppose next that (X, F) is recurrent. Let $n \in \mathbb{N} \setminus \{0\}$ and $\bar{u} = (u^{(i)})_{i \in [0, m]} \in \mathcal{L}(\tau_{n, \mathbb{N}}(X, F))$ be arbitrary. We need to show that there exists $\bar{w} = (w^{(i)})_{i \in [0, k]} \in \mathcal{L}(\tau_{n, \mathbb{N}}(X, F))$ such that $\bar{z} = \bar{u}\bar{w}\bar{u} \in \mathcal{L}(\tau_n(X, F))$. Since $\bar{u} \in \tau_{n, \mathbb{N}}(X, F)$ there exists $c \in X$ such that $F^i(c)_{[0, n]} = u^{(i)}$ for all $i \in [0, m]$. Let $r = r(F)$ and consider the cylinder $[c_{[-mr, n+mr]}] \subseteq X$. By recurrence,

there exists k such that $F^k([c_{[-mr, n+mr]}]) \cap c_{[-mr, n+mr]} \neq \emptyset$, which proves the claim.

The points 2. and 3. are similar.

4. This is a simple matter of rewriting the definitions.

(X, F) is chain-recurrent

$$\begin{aligned} &\iff \forall \varepsilon > 0 : \forall c \in X : \exists c = c^{(0)}, \dots, c^{(n)} = c \in X : \text{dist}(F(c^{(i)}), c^{(i+1)}) < \varepsilon \\ &\iff \forall k \in \mathbb{N} : \forall c \in X : \exists c = c^{(0)}, \dots, c^{(n)} = c \in X : F(c^{(i)})_{[-k, k]} = (c^{(i+1)})_{[-k, k]} \\ &\iff \forall k \in \mathbb{N} : \text{there is a directed path from each vertex of } \mathcal{G}_k(F) \text{ to itself} \\ &\iff \forall k \in \mathbb{N} : \mathcal{PO}_k(F) \text{ is recurrent.} \end{aligned}$$

Points 5. and 6. are similar. □

Let (X, F) be a cellular automaton. A word $u \in \mathcal{L}_{[-m, m] \cap \mathbb{M}}(X)$ is called a *blocking word* (for (X, F)) if for every $c, e \in X$ it holds that

$$c_{[-m, m] \cap \mathbb{M}} = u = e_{[-m, m] \cap \mathbb{M}} \implies \forall i \in \mathbb{N} : F^i(c)_0 = F^i(e)_0.^1$$

The following is due to Petr Kůrka.

Proposition 2.4.17 ([39, Theorem 4]). *A cellular automaton (X, F) is equicontinuous if and only if there exists $M \in \mathbb{N}$ such that all words in $\mathcal{L}_{[-M, M] \cap \mathbb{M}}(X)$ are blocking words. This is also equivalent to the existence of $p, n \in \mathbb{N}$ such that for all $c \in X$ it holds that $F^{n+p}(c) = F^n(c)$.*

A cellular automaton (X, F) is *periodic* if there exists p such that $F^p(c) = c$ for every $c \in X$; of course periodic cellular automata are reversible. From the previous proposition it follows that if (X, F) is reversible then it is equicontinuous if and only if it is periodic.

Let us introduce some natural projections. For any $i, j \in [0, n)$ where $i \leq j$ we denote by $\text{pr}_i : A^n \rightarrow A$ and $\text{pr}_{[i, j]} : A^n \rightarrow A^{j-i+1}$ the projections defined by $\text{pr}_i(a_0 a_1 \cdots a_{n-1}) = a_i$ and $\text{pr}_{[i, j]}(a_0 a_1 \cdots a_{n-1}) = a_i a_{i+1} \cdots a_j$. We overload the notation and also denote by pr_i and $\text{pr}_{[i, j]}$ the maps $(A^n)^{\mathbb{M}} \rightarrow A^{\mathbb{M}}$ and $(A^n)^{\mathbb{M}} \rightarrow (A^{j-i+1})^{\mathbb{M}}$ which are defined cellwise by pr_i and $\text{pr}_{[i, j]}$.

Let (X, F) be a cellular automaton and $\mathcal{PO}_m(X, F)$ its m -pseudo-orbits. We denote by ${}_i \Sigma_j^{(m)}(X, F) = \text{pr}_{[i, i+m]}(\mathcal{PO}_n(X, F))$ where $n = i + j + m$; it is natural to call these *pseudo-traces*. We may denote ${}_i \Sigma_j^{(m)} = {}_i \Sigma_j^{(m)}(X, F)$ if there is no risk of confusion in the air. The pseudo-traces are factors of pseudo-orbits, and since pseudo-orbits are SFTs, the pseudo-traces are sofic shifts. One way to think of pseudo-traces ${}_i \Sigma_j^{(m)}$ is as the set of stripes of

¹In the standard terminology this u would be called a *1-blocking word with offset m* , see [40, Definition 5.11].

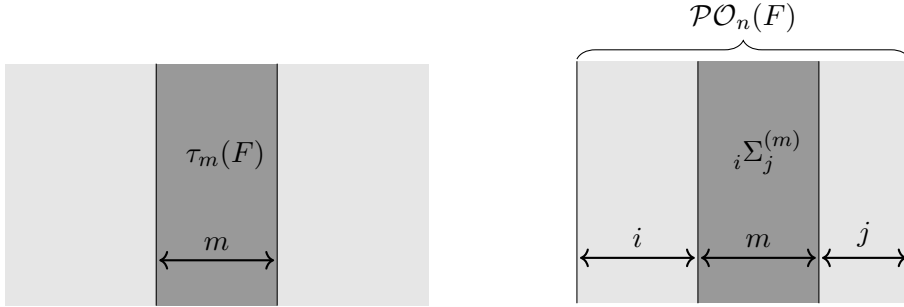


Figure 2.7: Traces are vertical stripes of space-time diagrams.

Figure 2.8: Pseudo-orbits are configurations where outside of the stripes of width n we allow anything. Pseudo-traces are factors of pseudo-orbits in a natural way.

width m which can be extended i columns to the left and j columns to the right without introducing violations of the local rule of F . See Figures 2.7 and 2.8 for visualizations of traces, pseudo-orbits, and pseudo-traces.

Since pseudo-traces are non-empty and ${}_{i+1}\Sigma_j^{(m)} \subseteq {}_i\Sigma_j^{(m)}$ we have, by the finite intersection property, that ${}_\infty\Sigma_j^{(m)} = \bigcap_{k \in \mathbb{N}} {}_k\Sigma_j^{(m)}$ is non-empty. Since ${}_\infty\Sigma_j^{(m)}$ is also closed and shift-invariant it is a subshift. In similar fashion we define subshifts ${}_i\Sigma_\infty^{(m)} = \bigcap_{k \in \mathbb{N}} {}_i\Sigma_k^{(m)}$ and ${}_\infty\Sigma_\infty^{(m)} = \bigcap_{k \in \mathbb{N}} {}_k\Sigma_k^{(m)}$. The following proposition shows that in the limit, pseudo-traces coincide with traces.

Proposition 2.4.18. *Let (X, F) be a cellular automaton. Then for every $m \in \mathbb{N} \setminus \{0\}$ it holds that $\tau_m(F) = {}_\infty\Sigma_\infty^{(m)}$.*

Proof. Let (X, F) be a cellular automaton with radius r and let $m \in \mathbb{N} \setminus \{0\}$ be arbitrary.

“ \subseteq ”: If $t \in \tau_m(F)$ then any space-time diagram that contains t shows that $t \in {}_k\Sigma_k^{(m)}$ for every $k \in \mathbb{N}$.

“ \supseteq ”: It is enough to show that $\mathcal{L}(\bigcap_{k \in \mathbb{N}} {}_k\Sigma_k^{(m)}) \subseteq \mathcal{L}(\tau_m(F))$. Suppose not, i.e. that there exists $u \in \mathcal{L}_n(\bigcap_{k \in \mathbb{N}} {}_k\Sigma_k^{(m)}) \setminus \mathcal{L}_n(\tau_m(F))$ for some $n \in \mathbb{N}$. Let

$$U = \{(v, w) \in A^{nr} \times A^{nr} \mid vu_0w \in \mathcal{L}_{2nr+m}(X) \text{ and if we consider } vu_0w \text{ as an element of } A^{[-nr, m+nr]} \text{ then } f^i(vu_0w)_{[0, m]} = u_i \text{ for all } i \in \{0, \dots, n-1\}\}.$$

Since $u \in \mathcal{L}_{nr}({}_{nr}\Sigma_{nr}^{(m)})$ the set U is non-empty. Since $u \notin \mathcal{L}_n(\tau_m(F))$ we have that for all $(v, w) \in U$ it holds that $vu_0w \notin \mathcal{L}(\Lambda(X, F))$ (the language of the

limit set of F). By compactness there exists $l \in \mathbb{N}$ such that vu_0w for every $(v, w) \in U$ is already forbidden in $F^l(X)$. But since $u \in \mathcal{L}_{((n+l)r, \Sigma_{(n+l)r}^{(m)})}$ there has to exist $(v, w) \in U$ such that $vu_0w \in \mathcal{L}_1(lr, \Sigma_{lr}^{(m+2nr)})$ so that it does appear in $F^l(X)$, thus reaching a contradiction. \square

We will return to pseudo-orbits and -traces in Chapter 5.

Next we show that a cellular automaton has POTP if and only if already finite precision pseudo-traces coincide with the traces.

Proposition 2.4.19. *Let (X, F) be a cellular automaton. The following are equivalent:*

i. F has POTP.

ii. For every $m \in \mathbb{N} \setminus \{0\}$ there exists $n \in \mathbb{N}$ such that $\tau_m(F) = {}_n\Sigma_n^{(m)}$.

Proof. “*i. \Rightarrow ii.*”: The POTP immediately implies that there exists $n \in \mathbb{N}$ such that the middle columns of $\mathcal{PO}_{m+2n}(F)$ are $\tau_m(F)$, i.e. that ${}_n\Sigma_n^{(m)} = \tau_m(F)$.

“*ii. \Rightarrow i.*”: If $\tau_m(F) = {}_n\Sigma_n^{(m)}$ then for pseudo-orbit $x \in \mathcal{PO}_{2n+m}(F)$ there exists an orbit $(c^{(i)})_{i \in \mathbb{Z}}$ such that $(\pi_{[n, n+m]}(c^{(i)}))_{i \in \mathbb{Z}} = \pi_{[n, n+m]}(x)$. \square

From this it follows that if (X, F) has POTP then $\tau_n(F)$ is sofic for every n , and that if $\tau_n(F)$ is an SFT for every n then (X, F) has POTP. These were already proved by Kůrka in [39] where also counterexamples for the converses were provided.

Cellular automaton (X, F) is *weak-mixing* if $(X \times X, F \times F)$ is transitive; clearly mixing implies weak-mixing which implies transitivity. In his doctoral dissertation Subrahmonian Moothathu proved that for a cellular automata over mixing subshifts weak mixing and transitivity are equivalent [48, Proposition 2.5.1]. It is an open problem whether transitivity and mixingness are equivalent for cellular automata. The following proposition allows us to show that chain-mixing and chain-transitivity are equivalent for cellular automata over mixing subshifts.

Proposition 2.4.20 ([42, Proposition 10]). *Let $X \subset A^{\mathbb{M}}$ be a transitive but non-mixing 2-SFT. Then there exists a partition $\{P_i\}_{i \in [0, k]}$ of A where $k > 1$ such that for all $ab \in \mathcal{L}_2(X)$ holds that if $a \in P_i$ then $b \in P_{i+1 \pmod k}$.*

Proposition 2.4.21. *Let X be a mixing subshift. A chain-transitive cellular automaton (X, F) is chain-mixing.*

Proof. Let (X, F) be a chain-transitive cellular automaton. Suppose for contradiction that for some $n \in \mathbb{N} \setminus \{0\}$ the subshift $\mathcal{PO}_n(F)$ is not mixing; it is, however, transitive. Then, by Proposition 2.4.20, there exists a partition

$\{L_i\}_{i \in \{0,1,\dots,k-1\}}$ of $\mathcal{L}_n(X)$ where $k > 1$ and if $(u, v) \in \mathcal{L}_2(\mathcal{PO}_n(F))$ and $u \in L_i$ then $v \in L_{i+1 \bmod k}$. Now let $u, v \in L_0$ and $w \in L_1$. Since X is mixing there exists $l \in \mathbb{N}$ and $x, y \in \mathcal{L}_l(X)$ such that $uxv, uyw \in \mathcal{L}_{2n+l}(X)$. But now $\mathcal{PO}_{2n+l}(F)$ cannot be transitive, since uyw cannot be reached from uxv since u, v will cycle through the partition $\{L_i\}_{i \in \{0,1,\dots,k-1\}}$ so if one can reach $uy'w'$ from uxv then $w' \in L_0$ also. \square

This cannot be generalized to cover cellular automata over transitive subshifts, since, for example, any transitive non-mixing SFT is chain-transitive but not chain-mixing.

Let (X, F) be a reversible cellular automaton where $X \subseteq A^{\mathbb{M}}$. It is expansive if and only if there exists $n \in \mathbb{N}$ such that for all space-time diagrams $(c^{(i)})_{i \in \mathbb{Z}}, (e^{(i)})_{i \in \mathbb{Z}} \in st_{\mathbb{Z}}(F)$ it holds that

$$c^{(0)} \neq e^{(0)} \implies (\exists k \in \mathbb{Z} : c_{[-n,n] \cap \mathbb{M}}^{(k)} \neq e_{[-n,n] \cap \mathbb{M}}^{(k)}).$$

Similarly, (X, F) is positively expansive if and only if there exists $n \in \mathbb{N}$ such that for all space-time diagrams $(c^{(i)})_{i \in \mathbb{N}}, (e^{(i)})_{i \in \mathbb{N}} \in st_{\mathbb{N}}(F)$ it holds that

$$c^{(0)} \neq e^{(0)} \implies (\exists k \in \mathbb{N} : c_{[-n,n] \cap \mathbb{M}}^{(k)} \neq e_{[-n,n] \cap \mathbb{M}}^{(k)}).$$

For any cellular automaton (X, F) let $\phi_m : X \rightarrow \tau_{\mathbb{M},m}(F)$ be the map defined by $\phi_m(c)_i = F^i(c)_{[0,m]}$ for all $i \in \mathbb{M}$ (where $\mathbb{M} = \mathbb{N}$ if (X, F) is not reversible).

Proposition 2.4.22. *Let (X, F) be a reversible (surjective) cellular automaton. Then (X, F) is expansive (positively expansive) if and only if there exists $m \in \mathbb{N} \setminus \{0\}$ such that ϕ_m is a conjugacy between (X, F) and $\tau_{\mathbb{Z},m}(F)$ ($\tau_{\mathbb{N},m}(F)$).*

Proof. We prove the case that (X, F) is reversible. Clearly ϕ_m is continuous and surjective for all m . Since the spaces at hand are compact and metric, it is sufficient to show that there exists m such that ϕ_m is injective if and only if (X, F) is expansive. But the claim is rather clear: If some ϕ_m is injective, this means that any difference is seen within a window of width m , i.e. all configurations, no matter how close to each other they are, have been or will be a fixed distance away from each other at some moment. On the other hand, if ϕ_m is non-injective for every m , we have pairs of points which are different but have always been and will always be arbitrarily close to each other. \square

For cellular automata the following characterization provides a useful visualization of expansivity.

Proposition 2.4.23. *A reversible cellular automaton (X, F) is expansive if and only if there exists $n \in \mathbb{N}$ such that for all configurations $c, e \in X$ the following holds*

$$\begin{aligned} (\forall i \in [-n, n] : F^i(c)_{[-n, n] \cap \mathbb{M}} = F^i(e)_{[-n, n] \cap \mathbb{M}}) \\ \implies c_{[-n-1, n+1] \cap \mathbb{M}} = e_{[-n-1, n+1] \cap \mathbb{M}}. \end{aligned} \quad (2.1)$$

Proof. We prove the case $\mathbb{M} = \mathbb{N}$; the case $\mathbb{M} = \mathbb{Z}$ goes similarly.

Suppose the claim does not hold and let $(c^{(i)})_{i \in \mathbb{N}}, (e^{(i)})_{i \in \mathbb{N}} \in X^{\mathbb{N}}$ be two sequences such that for any $n \in \mathbb{N}$ the pair $(c^{(n)}, e^{(n)})$ satisfies $F^i(c^{(n)})_{[0, n]} = F^i(e^{(n)})_{[0, n]}$ for every $i \in [-n, n]$ and $c_{n+1}^{(n)} \neq e_{n+1}^{(n)}$. Let $\varepsilon > 0$ be the expansivity constant. There exists $n_\varepsilon \in \mathbb{N}$ such that $\text{dist}(\sigma^k(c^{(n_\varepsilon+k)}), \sigma^k(e^{(n_\varepsilon+k)})) < \varepsilon$ for all $k \in \mathbb{N}$. Now, by compactness, there exists an infinite index set $\mathcal{I} \subseteq \mathbb{N}$ such that both sequences $(\sigma^k(c^{(n_\varepsilon+k)}))_{k \in \mathcal{I}}$ and $(\sigma^k(e^{(n_\varepsilon+k)}))_{k \in \mathcal{I}}$ converge, say, to points $c \in X$ and $e \in X$ (respectively). But now we have a contradiction: $c \neq e$ since $c_{n_\varepsilon+1} \neq e_{n_\varepsilon+1}$, but $\text{dist}(F^i(c), F^i(e)) < \varepsilon$ for all $i \in \mathbb{Z}$. \square

Similar result holds for positive expansivity.

Proposition 2.4.24. *A surjective cellular automaton (X, F) is positively expansive if and only if there exists $n \in \mathbb{N}$ such that for all configurations $c, e \in X$ the following holds*

$$\begin{aligned} (\forall i \in [0, n] : F^i(c)_{[-n, n] \cap \mathbb{M}} = F^i(e)_{[-n, n] \cap \mathbb{M}}) \\ \implies c_{[-n-1, n+1] \cap \mathbb{M}} = e_{[-n-1, n+1] \cap \mathbb{M}}. \end{aligned} \quad (2.2)$$

2.4.4 Higher-Dimensional Symbolic Dynamics

Symbolic dynamics can also be considered in higher dimensions. As we mostly consider one-dimensional symbolic dynamics, we have no need for a very long introduction to multidimensional symbolic dynamics. Since we are, however, going to prove some results about multidimensional cellular automata in Chapter 6, and since considering one-dimensional cellular automata as certain kinds of two-dimensional subshifts can be helpful for intuition, it is worth saying at least a few words.

Multidimensional (more specifically d -dimensional) symbolic dynamics considers the set $A^{\mathbb{Z}^d}$ where $d \in \mathbb{N} \setminus \{0, 1\}$. There has been very little study of one-sided multidimensional symbolic dynamics, i.e. of the set $A^{\mathbb{N}^d}$. We are also not going to consider these, but will point out some open problems in one-sided multidimensional symbolic dynamics.

First we give $A^{\mathbb{Z}^d}$ a metric; it is a generalization of the metric given for $A^{\mathbb{Z}}$. Let $\text{dist} : A^{\mathbb{Z}^d} \times A^{\mathbb{Z}^d} \rightarrow \mathbb{R}$ be defined by

$$\text{dist}(c, e) = \begin{cases} \frac{1}{2^i} & \text{where } i = \min\{\|\mathbf{x}\| \mid c_{\mathbf{x}} \neq e_{\mathbf{x}}\}, \\ 0 & \text{if } c = e \end{cases},$$

where $\|\mathbf{x}\| = \sum_{i=1}^d |x_i|$ for $\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{Z}^d$. With this metric, $A^{\mathbb{Z}^d}$ is a compact metric space. For finite $D \subseteq \mathbb{Z}^d$ the set $[u] = \{c \in A^{\mathbb{Z}^d} \mid c_D = u\}$ for any $u \in A^D$ is again called a *cylinder*.

For every $\mathbf{x} \in \mathbb{Z}^d$ we denote by $\sigma_{\mathbf{x}}$ the shift map defined by $\sigma_{\mathbf{x}}(c)_{\mathbf{y}} = c_{\mathbf{x}+\mathbf{y}}$. We generalize the topological characterization of subshifts to a definition of multidimensional subshifts: A set $X \subseteq A^{\mathbb{Z}^d}$ is a (*d-dimensional subshift*) if X is closed and $\sigma_{\mathbf{x}}(X) \subseteq X$ for all $\mathbf{x} \in \mathbb{Z}^d$. Let $\mathcal{C}_n = \{\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{Z}^d \mid \forall i \in [1, d] : |x_i| \leq n\}$. Denote by $\mathcal{L}(X) = \{c|_{\mathcal{C}_n} \mid c \in X\}$ the hypercubic patterns that appear in X . As in one-dimensional case, X is an SFT if it is sufficient to check validity of configurations through some finite window, i.e. if there exists $n \in \mathbb{N}$ such that

$$(\forall \mathbf{x} \in \mathbb{Z}^d : \sigma_{\mathbf{x}}(c)|_{\mathcal{C}_n} \in \mathcal{L}(X)) \implies c \in X.$$

A *d-dimensional cellular automaton* is a dynamical system (X, F) where $X \subseteq A^{\mathbb{Z}^d}$ such that F commutes with the shift maps, i.e. $F\sigma_{\mathbf{x}} = \sigma_{\mathbf{x}}F$ for all $\mathbf{x} \in \mathbb{Z}^d$. The Curtis-Hedlund-Lyndon Theorem holds in higher dimensions too, and so we could have equivalently defined *d-dimensional cellular automata* using local rules.

Notice that if $X \subseteq A^{\mathbb{Z}}$ is an *k-SFT* and (X, F) is a (one-dimensional) cellular automaton, then $st_{\mathbb{Z}}(X, F)$ is a two-dimensional SFT. Indeed, if $r = r(F)$, then $s \in A^{\mathbb{Z}^2}$ is in $st_{\mathbb{Z}}(X, F)$ if and only if for all $\mathbf{x} \in \mathbb{Z}^d$ it holds that

$$\sigma_{\mathbf{x}}(s)|_{\{(0,0), (0,1), \dots, (0, k-1)\}} \in \mathcal{L}_k(X)$$

and

$$f(s_{\mathbf{x}+\{(-r,0), (-r+1,0), \dots, (0,0), \dots, (r-1,0), (r,0)\}}) = s_{\mathbf{x}+(0,1)}.$$

Now comparing the second condition with expansivity, in particular with the geometrical interpretation of Proposition 2.4.23, we see that the condition is very similar; contents of a finite rectangle uniquely determines the content of a cell next to it. Both of these describe determinism in some direction for the SFT, vertically by the local rule and horizontally by expansivity. Then one can also consider determinism in other directions, indeed, we could shift our perspective and overall just consider two-dimensional SFT's with different deterministic directions. This point-of-view was initiated by Mike Boyle and Douglas Lind [10].

2.4.5 Computability

We do not define Turing machines, and incidentally we are going to discuss computability questions only in informal terms. The reason we do this is that even a short exposition would end up being quite long, and since our results are based on high-level Turing reductions we do not actually need to discuss Turing machines directly. Any introductory textbook on automata and formal language theory should cover everything necessary for our considerations, for example, [28, Chapters 8 and 9] by John Hopcroft, Rajeev Motwani, and Jeffrey Ullman. To emphasize: Rest of this section should be considered informal.

In computability related questions we are going to consider only cellular automata over full shifts, so we may as well use them as examples. A *decision problem* about cellular automata is a “yes” or “no” question that asks whether a given cellular automaton possesses some given property or not. For example, whether a cellular automaton is periodic. Here cellular automata are the *instances* of the decision problem, as they are the items for which the problem is asked. An instance is called a *positive instance* if it does have the property and a *negative instance* if it does not. A decision problem is *decidable* if there exists an algorithm that solves it, i.e. a program that for every instance will answer correctly either “yes” or “no” in finite time. Otherwise a decision problem is called *undecidable*. A problem is *semi-decidable* if there exists an algorithm that will answer “yes” in finite time for every positive instance, but for negative instances it may run forever (however, it must not answer incorrectly “yes” for any negative instances).

Any decision problem that has only finite set of instances is decidable, since then there are only finitely many functions from the set of instances to the set {“yes”, “no”} and then one of these functions is an algorithm for the problem. This means that very difficult problems can be trivial as decision problems, for example, any existing conjecture in mathematics yields a trivial decision problem “Does conjecture C hold?”, though it can be tricky to say which of the two possible algorithms is correct.

With proper definitions and a short explanation it would be apparent that undecidable problems exist since, informally, there are more problems than algorithms. This is, however, a nonconstructive fact, i.e. it does not provide examples of undecidable problems, it simply states that some exist. Famously, Alan Turing proved that the explicit problem of deciding whether a Turing machine (or in this informal setting, a computer program) halts is undecidable. Knowing one undecidable problem one can prove other problems undecidable using what are called *Turing reductions*. The idea of Turing reductions is simple and goes as follows. Let P be a problem that is known to be undecidable and Q be a problem that we want to show is undecidable. Instead of directly showing that Q is undecidable we do the

contrapositive and assume that Q is decidable, i.e. that there exists an algorithm that solves Q . Now we use this hypothetical algorithm for Q as a black box and solve P . If we can do this, it shows that our assumption was wrong and Q must instead be undecidable since we know that P is.

The above described method of Turing reductions (which we will call also just *reductions*) is what allows us to be informal in this section. We are not going to need explicit constructions using Turing machines but rather just reduce existing undecidable problems to our problems.

Lastly we define (informally) *recursive inseparability* as this is the form in which our main results in Chapter 6 are stated. Let I be a set of instances (for example, the set of all cellular automata). Formally, any subset of I is called a *property* of I , but often one rather considers that a property defines a subset (for example, the set of all periodic cellular automata would formally be a property, but one tends to consider that the property of being periodic defines a subset). Let $p_1, p_2 \subseteq I$ be two disjoint properties of I , i.e. $p_1 \cap p_2 = \emptyset$. Now these properties are *recursively inseparable* if it is undecidable whether $i \in I$ is in p_1 . Alternatively one can consider that for any partition $I = I_1 \cup I_2$ such that $p_1 \subseteq I_1$ and $p_2 \subseteq I_2$ it is undecidable whether $i \in I$ is in I_1 .

Chapter 3

Stripe Shifts

In this short and relatively independent chapter we introduce *stripe shifts*. Stripe shift is obtained by restricting a cellular automaton to a subshift on which the cellular automaton obeys certain spatial homogeneity; we do not require the strictest spatial homogeneity, i.e. that all cells have the same symbol, but a looser one where all cells of a configuration are required to contain elements from the same part of some prefixed partition of the alphabet. This forces each row of a space-time diagram to be constant if the elements are projected with respect to said partition (see Figure 3.1). We aim to study what kind of “stripes” can be obtained this way.

This relates to the study of projective subdynamics that we mentioned in the preliminaries. Nathalie Aubrun and Mathieu Sablik [4], and Bruno Durand, Andrei Romashchenko, and Alexander Shen [19] proved independently that given any effective one-dimensional subshift $X \subseteq A^{\mathbb{Z}}$ one can define a two-dimensional SFT $Y \subseteq B^{\mathbb{Z}^2}$ which has the following properties: There is a map $\iota : B \rightarrow A$ such that $\iota(c_{(i,j)}) = \iota(c_{(i+1,j)})$ for all $c \in Y$ and $i, j \in \mathbb{Z}$, and by extending ι to $B^{\mathbb{Z}^2}$ cell-wise, it holds that $\iota(Y) = X$ (since rows of $\iota(Y)$ are constant, $\iota(Y)$ can be considered as a one-dimensional subshift). These improve result by Michael Hochman [27] where effective one-dimensional subshifts are obtained in the same way from three dimensional SFT’s. In his doctoral dissertation Charalampos Zinoviadis [64] studied *extremely expansive* SFT’s, i.e. SFT’s where all but one direction are expansive and asks whether all effective subshifts can be obtained from an extremely expansive SFT in this projective way. Our result can be seen as complementary to this question, as it states that for SFT’s which are deterministic perpendicular to the direction of the stripes that draw the one-dimensional subshift, the above result definitely does not hold.

While in the above sense the study of stripe shifts may prove interesting in its own right, for us the main purpose of this chapter is, however, that there is a novel way in which we use the stripe shifts (or more precisely the

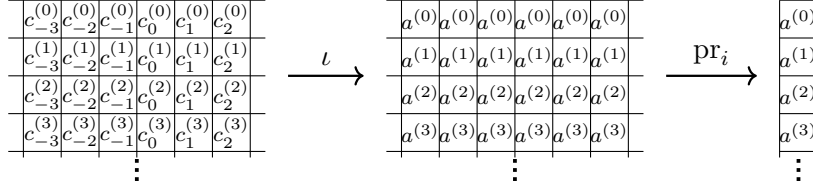


Figure 3.1: For stripe shifts we consider cellular automaton only on those configurations that on every time-step contain letters from only one part of a prefixed partition. Then projecting according to said partition (ι denotes this projection in the figure) leaves every row constant. These “stripes” can be considered as one-dimensional configurations, since projecting with any pr_i leads to the same configuration.

fact that certain kinds of stripe shifts do not exist) when we prove one of the main results in Chapter 5.

3.1 Definition and the Stripe Lemma

Let A be a finite set, and visualize this set as a set of colours. Now consider that some colours appear in different shades, say that our alphabet is $A = \cup_{i \in [0, k)} A_i$ where $P = \{A_i\}_{i \in [0, k)}$ is a partition of A , and each A_i contains only different shades of the same color. Let $(A^{\mathbb{Z}}, F)$ be a cellular automaton. We say that F respects P at $c \in A^{\mathbb{Z}}$ if for every time step the configuration $F^i(c)$ contains only different shades of the same color, i.e. if for every $i \in \mathbb{N}$ there exists $j \in [0, k)$ such that $F^i(c) \in A_j^{\mathbb{Z}}$. Let R denote the set of all configurations where F respects P . Clearly R is a subshift, since it is easily seen to be closed and shift-invariant, and (R, F) is a cellular automaton since $F(R) \subseteq R$. Now each horizontal row of every forward orbit $s \in \text{st}_{\mathbb{N}}(R, F)$ has only different shades of the same color. If we forget the shades, these rows become constant, and the forward orbits can be considered as a one-dimensional one-sided subshift. Now we ask: Which subshifts can arise this way?

To be more exact, let $(A^{\mathbb{Z}}, F)$ be a cellular automaton and $P = \{A_i\}_{i \in [0, k)}$ a partition of A . Let $\iota : A \rightarrow [0, k)$ be the projection defined by $\iota(a) = j$ if $a \in A_j$. Now the *stripe shift defined by $(A^{\mathbb{Z}}, F)$ and P* is

$$\Xi_P(F) = \{t \in [0, k)^{\mathbb{N}} \mid \exists c \in A^{\mathbb{Z}} : \forall i \in \mathbb{N} : \forall j \in \mathbb{Z} : \iota(F^i(c)_j) = t_i\}.$$

A one-sided subshift X is called a *stripe shift* if there exists a cellular automaton $(A^{\mathbb{Z}}, F)$ and a partition P of A such that $\Xi_P(F) = X$. Let us illustrate with a simple example that there are at least some stripe shifts.

Example 3.1.1. Two natural places to start are the trivial partitions, either $P = \{A_0\}$ where $A_0 = A$ or $P = \{A_i\}_{i \in [0, |A|]}$ where each A_i is a singleton set. The first one defines a one-point subshift $\{0^\omega\}$, so it is deeply uninteresting. The second case is at least somewhat more interesting, as it also leads to a general fact about stripe shifts. So let $(A^\mathbb{Z}, F)$ be a cellular automaton and let $P = \{\{a\} \mid a \in A\}$. Let R be the set of points where F respects P . Clearly $c \in R$ implies that for all $i, j \in \mathbb{Z}$ it holds that $c_i = c_j$, i.e. that $c = {}^\omega a^\omega$ for some $a \in A$. Since F maps constant configurations to constant configurations this is also sufficient property, and so we have that $R = \{{}^\omega a^\omega \mid a \in A\}$. We see that $\Xi_P(F) \subseteq A^\mathbb{N}$ (here the projection ι can be ignored) is an 2-SFT defined by $ab \in \mathcal{L}_2(\Xi_P(F))$ if and only if $F({}^\omega a^\omega) = {}^\omega b^\omega$. This is of course eventually periodic, i.e. there exists $k, p \in \mathbb{N}$ such that $\sigma^{k+p}(s) = \sigma^k(s)$ for every $s \in \Xi_P(F)$.

The previous example leads to the following observation.

Proposition 3.1.2. *Let $(A^\mathbb{Z}, F)$ be a cellular automaton and $P = \{A_i\}_{i \in [0, k)}$ a partition of A . The stripe shift $\Xi_P(F)$ is non-empty.*

Proof. The cellular automaton F respects P at ${}^\omega a^\omega$ for any $a \in A$. This proves the claim. \square

So we have seen that stripe shifts are always non-empty, and that at least some very simple shifts are stripe shifts. Let us now state the first interesting fact about stripe shifts.

Lemma 3.1.3 (Stripe Lemma). *The binary full shift $\{0, 1\}^\mathbb{N}$ is not a stripe shift.*

Proof. We can simplify the situation and assume that the local rule of F is a function $A^{[0, 1]} \rightarrow A$ by composing F with a suitable power of the shift map and using a suitable grouping map $\text{gr}_n : A^\mathbb{Z} \rightarrow (A^n)^\mathbb{Z}$, $(\text{gr}_n(c))_i = c_{in}c_{in+1} \cdots c_{in+n-1}$. This can be done since shifting does not change the stripe shift and if the original cellular automaton would give the full shift as stripe shift, definitely we could partition the grouped alphabet so that it would also give the full shift.

Suppose there exists a cellular automaton $(A^\mathbb{Z}, F)$ and a partition $P = \{A_0, A_1\}$ such that $\Xi_P(F) = \{0, 1\}^\mathbb{N}$. Let R be the set of configurations that respect the partition $\{A_0, A_1\}$. For every $l \in \mathbb{N} \setminus \{0\}$ and $u \in \{0, 1\}^l$ we denote

$$R_u = \{v \in \mathcal{L}_l(R) \mid F^j(v)_0 \in A_{u_j} \text{ for all } j \in [0, l)\}.$$

Since the stripe shift is the full shift, all of these R_u sets must be non-empty. Let $|A| = k$. We show that no matter how large k is, it will not be large enough.

Let $u, v \in \{0, 1\}^l$ for some $l \in \mathbb{N} \setminus \{0\}$. Now consider a word $w \in R_{uv}$. Let $w = w'w''$ where $w', w'' \in A^l$. From the definition of R_{uv} we have that both w' and w'' must be in R_u , so we have $R_{uv} \subseteq R_u R_u$ and so

$$\bigcup_{v \in \{0,1\}^l} R_{uv} \subseteq R_u R_u.$$

From this we get the inequality $\sum_{v \in \{0,1\}^l} |R_{uv}| \leq |R_u|^2$, and then we have that

$$\min_{v \in \{0,1\}^l} |R_{uv}| \leq \frac{|R_u|^2}{2^l}. \quad (3.1)$$

From this we get an upper bound for the size of the smallest set $R_{u_1 \dots u_{2^n}}$ for every $n \in \mathbb{N}$:

$$\min_{u_1 u_2 \dots u_{2^n} \in \{0,1\}^{2^n}} |R_{u_1 u_2 \dots u_{2^n}}| \leq z_n,$$

where z_n is defined by the recursive formula

$$z_{n+1} = \frac{z_n^2}{2^{2^n}}, \quad z_0 = \frac{k}{2}.$$

We get this from the fact that $\min\{|R_0|, |R_1|\} \leq \frac{k}{2}$, and then using inequality (3.1). Solving this recursion yields

$$z_n = \frac{k^{2^n}}{2^{2^n + n \cdot 2^{n-1}}}.$$

But now we see that the sequence $(z_n)_{n \in \mathbb{N}}$ converges to zero, and so for some $N \in \mathbb{N}$ and $u \in \{0, 1\}^N$ we have that $R_u = \emptyset$ reaching a contradiction and proving the claim. \square

Before we move onwards let us point something out. Instead of considering one-dimensional cellular automata $(A^{\mathbb{Z}}, F)$ we could consider higher-dimensional cellular automata $(A^{\mathbb{Z}^d}, F)$ where $d > 1$. We can still define stripe shifts in a similar fashion:

$$\Xi_P(F) = \{t \in [0, k)^{\mathbb{N}} \mid \exists c \in A^{\mathbb{Z}^d} : \forall i \in \mathbb{N} : \forall \mathbf{j} \in \mathbb{Z}^d : \iota(F^i(c))_{\mathbf{j}} = t_i\},$$

where $P = \{A_i\}_{i \in [0, k)}$ is a partition of A . Now, perhaps surprisingly, the proof of the Stripe Lemma, mutatis mutandis, does not work anymore. Going through a similar reasoning one is led to a recursive formula

$$z_{n+1} = \frac{z_n^4}{2^{2^n}}, \quad z_0 = \frac{k}{2}.$$

(We have used such grouping and shifting that our neighborhood has four cells.) Solving this leads to

$$z_n = \frac{k^{2^{2^n}}}{2^{2^{2^n} + (2^n - 1) \cdot 2^{n-1}}}$$

but this sequence does not converge!

Question 3.1.4. *Does Stripe Lemma hold for higher-dimensional cellular automata?*

3.2 Characterization of Sofic Stripe Shifts

The fact that stripe shifts are closed under union will be useful when characterizing sofic stripe shifts, and is an interesting fact by itself.

Proposition 3.2.1. *Let X_1, \dots, X_l be stripe shifts. Then $X = \cup_{i=1}^l X_i$ is a stripe shift.*

Proof. It is enough to show that the union of two stripe shifts is a stripe shift. Let X and Y be stripe shifts and assume that $X \cup Y \subseteq [0, n]^{\mathbb{N}}$ and $\mathcal{L}_1(X \cup Y) = [0, n]$. Let $(A^{\mathbb{Z}}, F)$ be a cellular automaton and $\{A_i\}_{i \in [0, n]}$ a collection of subsets of A such that $P_A = \{A_i \mid i \in [0, n], A_i \neq \emptyset\}$ is a partition of A such that $\Xi_{P_A}(F) = X$. Let $(B^{\mathbb{Z}}, G)$, $\{B_i\}_{i \in [0, n]}$ and P_B be defined in a similar way so that $\Xi_{P_B}(G) = Y$. We can assume that A and B are disjoint. Now let $P = \{A_i \cup B_i\}_{i \in [0, n]}$, which is a partition of $A \cup B$ (A_i and B_i cannot both be empty for any $i \in [0, n]$ since all letters appear in $X \cup Y$). Our goal is to define a cellular automaton $((A \cup B)^{\mathbb{Z}}, H)$ such that $\Xi_P(H) = X \cup Y$. We can assume that the local rules of $(A^{\mathbb{Z}}, F)$ and $(B^{\mathbb{Z}}, G)$ both have neighborhood $[-r, r]$ for some $r \in \mathbb{N}$. Let $a_0 \in A_0 \cup B_0$ and $a_1 \in A_1 \cup B_1$ be some letters. The local rule h of H is the map $(A \cup B)^{[-r, r]} \rightarrow A \cup B$ defined by

$$h(x_{-r} \cdots x_{-1} x_0 x_1 \cdots x_r) = \begin{cases} f(x_{-r} \cdots x_r), & \text{if } x_{-r} \cdots x_r \in A^{[-r, r]} \\ g(x_{-r} \cdots x_r), & \text{if } x_{-r} \cdots x_r \in B^{[-r, r]} \\ a_0, & \text{if } x_0 \in A \text{ and } x_{-1} \text{ or } x_1 \text{ in } B \\ a_1, & \text{otherwise} \end{cases}$$

Clearly, from the first two lines, we have that $X \cup Y \subseteq \Xi_P(H)$. On the other hand, if a configuration $c \in (A \cup B)^{\mathbb{Z}}$ contains letters from both A and B then the third and fourth lines guarantee that $H(c)$ contains a_0 and a_1 so that H does not respect P at c . Thus $\Xi_P(H) = X \cup Y$. \square

Natural question arises.

Question 3.2.2. *Are stripe shifts closed under intersection?*

We have seen that at least every finite subshift is a stripe shift; this follows from Example 3.1.1 using the fact that finite subshifts are those where every point is eventually periodic (i.e. of form uv^ω for some finite words u, v). On the other hand we saw that the full shift is not a stripe shift. We are facing a risk that stripe shifts are so trivial that they are simply not interesting at all. The following example shows that at least a bit more complex stripe shifts exist.

Example 3.2.3. The infinite firing squad cellular automaton $(B^{\mathbb{Z}}, G)$ presented by Jarkko Kari [35] has the following property: There exists $f \in B$ and $(c^{(i)})_{i \in \mathbb{Z}} \in st_{\mathbb{Z}}(G)$ such that $c^{(0)} = {}^{\omega}f^{\omega}$ and for all $i \in \mathbb{Z} \setminus \{0\}$ and $j \in \mathbb{Z}$ we have that $c_j^{(i)} \neq f$. Now we use $P = \{\{f\}, B \setminus \{f\}\}$ as the partition of B and see that the *sunny side up* subshift $X_{\leq 1} = \{c \in \{0, 1\}^{\mathbb{N}} \mid c \text{ has at most one } 1\}$ is a stripe shift (it is obvious that $X_{\leq 1} \subseteq \Xi_P(G)$ and easy to see that $\Xi_P(G) \subseteq X_{\leq 1}$, though the latter requires one to look up the actual definition of the firing squad cellular automaton from [35]).

Now we can characterize which sofic shifts are stripe shifts. Stripe Lemma allows us to show that positive entropy sofic shifts are not stripe shifts, and for any zero entropy sofic shift we can, with the help of the firing squad cellular automaton, construct a cellular automaton which using a suitable partition defines said subshift.

Proposition 3.2.4. *Let X be a sofic shift. If X has positive entropy then no stripe shift can contain it, and complementarily, if X has zero entropy then it is a stripe shift.*

Proof. For sofic shifts, having positive entropy is equivalent to being uncountable. Let the number of letters that appear in X be n .

Suppose that X is an uncountable sofic shift and that $(A^{\mathbb{Z}}, F)$ is a cellular automaton and $P = \{A_i\}_{i \in [0, n)}$ a partition such that $X \subseteq \Xi_P(F)$. Then there exists $u, v \in \mathcal{L}(X)$ such that $u_0 \neq v_0$ and $\{u, v\}^{\mathbb{N}} \subseteq X$. Let $\tilde{u} = uv$ and $\tilde{v} = vu$, so that $|\tilde{u}| = |\tilde{v}|$. Of course also $\{\tilde{u}, \tilde{v}\}^{\mathbb{N}} \subseteq X$. Now let $P' = \{A'_0, A'_1\}$ be a partition of A such that P is a refinement of P' and $\iota^{-1}(u_0) \subseteq A'_0$ and $\iota^{-1}(v_0) \subseteq A'_1$ (where ι is the projection $A \rightarrow [0, n)$ according to the partition P). Now the stripe shift defined by $F^{|\tilde{u}|}$ and P' is $\{0, 1\}^{\mathbb{N}}$ contradicting Lemma 3.1.3.

Next let X be a countable sofic shift and let \mathcal{G} be a labeled directed graph such that the labels of the one-way infinite paths of \mathcal{G} are the points of X . According to Ronnie Pavlov and Michael Schraudner [56, Lemma 4.8.] we can assume that the connected components of \mathcal{G} consist of some number of (directed) cycles $C(1), \dots, C(k)$ and (directed) paths $P(1), \dots, P(k-1)$ such that $P(i)$ is a path from $C(i)$ to $C(i+1)$. According to Proposition 3.2.1 we can assume that there is only one connected component.

Denote the edge set of \mathcal{G} by $E_{\mathcal{G}}$ and define $F_1 : E_{\mathcal{G}} \rightarrow E_{\mathcal{G}}$ so that if e has a unique follower edge in \mathcal{G} then $F_1(e)$ is that edge, otherwise $F_1(e)$ is the follower edge which is on the same cycle as e (the only edges where the follower is not unique are the ones on cycles where there is a choice to either continue along the cycle or start along the path connecting to the next cycle). Similarly we define $F_2 : E_{\mathcal{G}} \rightarrow E_{\mathcal{G}}$ but F_2 does the opposite choice than F_1 in the edges where there are two possible ways to continue. Let $(B^{\mathbb{Z}}, G)$ be the firing squad cellular automaton of Example 3.2.3. We

define a cellular automaton $((E_{\mathcal{G}} \times B \times \cdots \times B)^{\mathbb{Z}}, F)$, where we have $k - 1$ copies of B , by $F(c, e^{(1)}, \dots, e^{(k-1)}) = (c', G(e^{(1)}), \dots, G(e^{(k-1)}))$ where

$$c'_i = \begin{cases} F_1(c_i) & \text{if } e_i^{(j)} \neq f \text{ for every } j \in \{1, 2, \dots, k-1\} \\ F_2(c_i) & \text{if } e_i^{(j)} = f \text{ for some } j \in \{1, 2, \dots, k-1\} \end{cases}.$$

Let $P = \{E_x \times B \times \cdots \times B\}_{x \in \mathcal{L}_1(X)}$ where $E_x = \{a \in E_{\mathcal{G}} \mid \text{label of } a \text{ is } x\}$. We claim that $X = \Xi_P(F)$.

“ \supseteq ”: Let $c \in \Xi_P(F)$. By definitions, there exists $t \in \tau_{\mathbb{N},1}(F)$ such that $\lambda(\pi_E(t)) = c$ where λ is the labeling function of \mathcal{G} (applied cellwise) and π_E is the natural projection $(E_{\mathcal{G}} \times B \times \cdots \times B)^{\mathbb{N}} \rightarrow E_{\mathcal{G}}^{\mathbb{N}}$. The definition of F using F_1 and F_2 guarantees that the $E_{\mathcal{G}}$ -layer of every column of any space-time diagram describes a directed path on \mathcal{G} , and so $c \in X$.

“ \subseteq ”: For every $i \in \mathbb{N}$ let $g^{(i)} \in B^{\mathbb{Z}}$ denote a configuration such that $G^i(g^{(i)}) = \omega f \omega$ and $G^j(g^{(i)}) \in (B \setminus \{f\})^{\mathbb{Z}}$ for all $j \in \mathbb{N} \setminus \{i\}$, and let $g^{(\infty)} \in B^{\mathbb{Z}}$ denote a configuration such that for all $j \in \mathbb{N}$ it holds that $G^j(g^{(\infty)}) \in (B \setminus \{f\})^{\mathbb{Z}}$. Let $c \in X$ and $q = (q_i)_{i \in \mathbb{N}} \in E_{\mathcal{G}}^{\mathbb{N}}$ such that $\lambda(q) = c$. For every $i \in [1, k]$ let $P(i)_1$ denote the first edge on the path $P(i)$, and let $I = \{i_j\}_{j \in [1, l]} \subset \mathbb{N}$ be the set of indices such that q is just about to start the transition from one cycle to the next one, i.e. $q_{l+1} \in \{P(i)_1 \mid i \in [1, k]\}$ if and only if $l \in I$; it must be that $l \leq k$. Now, by definitions, the configuration $(\omega q_0^\omega, g^{(i_1)}, g^{(i_2)}, \dots, g^{(i_l)}, g^{(\infty)}, \dots, g^{(\infty)})$ shows that $c \in \Xi_P(F)$. \square

As remarked in the beginning of the proof, we could equally well formulate the above proposition with the condition of X being uncountable. We used the entropy condition instead since in the next section we show that uncountability in general is possible for stripe shifts and, on the other hand, it seems at the very least plausible that positive entropy in general is impossible for stripe shifts.

3.3 An Uncountable Stripe Shift

In the first section of this stripe shift themed chapter we saw finite examples of stripe shifts. In the previous section the firing squad cellular automaton provided an infinite stripe shift which, however, was only countable. In this section we construct an uncountable stripe shift. This example relies heavily upon the firing squad cellular automaton.

The cellular automaton which we describe here has three layers called the *firing squad*-layer, the *Toeplitz*-layer, and the *signal*-layer. First we describe each of these layers individually and then we fit them together. Note that we are only trying to prove that there exists an uncountable stripe shift, this allows us to leave out some details about the cellular automaton, since it is sufficient to describe how the cellular automaton works on the set of

configurations that provides uncountably many points to our stripe shift, and on the other configurations the cellular automaton can act arbitrarily since it definitely will not decrease the size of the stripe shift.

First we describe the relevant properties of the infinite firing squad cellular automaton, henceforth known just as the *firing squad*. We have to do very minor modifications to the original one in order to get our timings right. Based on the original firing squad defined in [35] it is clear that we can define cellular automaton $(A^{\mathbb{Z}}, F)$ which has the following properties (for details we refer to [35]):

1. The alphabet A contains a state $_$ called a *blank*, state $\#$ called a *general*, states f_1, f_2 called *reloading* states, and f called a *firing* state. These are not the only states, but the only ones we give names. (When the original cellular automaton from [35] goes to the firing state, our cellular automaton goes to f_1 .)
 - Denote by $\#_i$ the configuration $c \in A^{\mathbb{Z}}$ such that $c_j = \#$ if $i \mid j$ and $c_j = _$ otherwise.
2. For all $i > 1$ it holds that $F^{2^i}(\#_{2^i}) = \#_{2^{i-1}}$, and also $F^2(\#_2) = {}^\omega f_1^\omega$. Furthermore, the states f_1, f_2 , and f do not appear in $F^k(\#_{2^i})$ for any $i \geq 1, k \in \{0, \dots, 2^i - 1\}$.
3. It holds that $F({}^\omega f_1^\omega) = {}^\omega f_2^\omega$ and $F({}^\omega f_2^\omega) = {}^\omega f^\omega$. (The states f_2 and f are the new states compared to the firing squad of [35]; for technical reasons we need to prolong the firing.)
 - Notice that for any $i \geq 1$ we have that $F^{2^{i+1}}(\#_{2^i}) = {}^\omega f^\omega$, and f does not appear in the space-time diagram before this. This is the property we use, and the only thing about the firing squad we refer to below.

In our construction below we use the cellular automaton $((A \cup \tilde{A})^{\mathbb{Z}}, \mathcal{F})$ where $\tilde{A} = \{\tilde{a} \mid a \in A \setminus \{_\}\} \cup \{_\}$ (in other words a copy of A , except we have the same blank symbol in both alphabets), and \mathcal{F} behaves as the firing squad for any configuration which is in $A^{\mathbb{Z}} \cup \tilde{A}^{\mathbb{Z}}$. We do not care how \mathcal{F} behaves on configurations that are not valid, i.e. which do not appear in a space-time diagram for some shift of some $\#_{2^i}$ or $\tilde{\#}_{2^i}$. This will be the firing squad -layer

Let $(\{0, 1, a, b, _\}^{\mathbb{Z}}, t)$ be the identity cellular automaton. Denote the state set by $T = \{0, 1, a, b\}$. This will be the Toeplitz-layer.

Let $(\{\overset{x}{\leftarrow}, \overset{x}{\leftarrow}_w, x \leftrightarrow^y, x \leftrightarrow_w^y, \overset{y}{\rightarrow}, \overset{y}{\rightarrow}_w, _ \mid x, y \in \{1, a\}\}^{\mathbb{Z}}, s)$ be the cellular automaton that moves speed half signals to the left and to the right. These signals also carry a name, either 1 or a . Denote the state set by S . The label w tells that the signal is *waiting*. Again, it is enough to describe how the

Now we are ready to describe how \mathcal{G} works on the valid configurations, meaning the configurations that appear in space-time diagrams of valid initial configurations (reader may want to refer to Figure 3.2). If the firing squad -layer is not firing, i.e. the local rule sees no occurrences of f or \tilde{f} , then $\mathcal{G}(c_f, c_t, c_s) = (\mathcal{F}(c_f), t(c_t), s(c_s))$, so that the layers are completely independent of each other. If the firing squad -layer is firing, i.e. an occurrence of f or \tilde{f} is seen (remember that we are only considering a certain subset of configurations, and within this set f or \tilde{f} appears in some cell if and only if it appears in every cell), then $(c'_f, c'_t, c'_s) = \mathcal{G}(c_f, c_t, c_s)$ is defined as follows:

- The firing squad -layer is defined as follows: If $c_s(i) = \overset{1}{\leftrightarrow}_w^1$ then $c'_f(i) = \#$, if $c_s(i) = \overset{a}{\leftrightarrow}_w^a$ then $c'_f(i) = \tilde{\#}$, otherwise $c'_f(i) = _$.
- The Toeplitz-layer is defined as follows: If $c_s(i) = \overset{x}{\leftrightarrow}_w^x$ where $x \in \{1, a\}$, then $c'_t(i) = 0$ if $c_t(i) = 1$ and $c'_t(i) = 1$ if $c_t(i) = 0$, and similarly $c'_t(i) = b$ if $c_t(i) = a$ and $c'_t(i) = a$ if $c_t(i) = b$, in other words we flip 0's and 1's, and a 's and b 's. If $c_s(i) \neq \overset{x}{\leftrightarrow}_w^x$, then $c'_t(i) = _$.
- The signal-layer is defined as follows: If $c'_t(i) = 1$ then $c'_s(i) = \overset{1}{\leftrightarrow}_w^1$, if $c'_t(i) = a$ then $c'_s(i) = \overset{a}{\leftrightarrow}_w^a$, and otherwise $c'_s(i) = _$.

Now let us partition $((A \cup \tilde{A}) \times T \times S)$ as follows: $P_1 = \{(f, x, y) \mid x \in T, y \in S\}$, $P_2 = \{(\tilde{f}, x, y) \mid x \in T, y \in S\}$, and $P_0 = ((A \cup \tilde{A}) \times T \times S) \setminus (P_1 \cup P_2)$. Call this partition P . By construction it holds that for any sequence $(a_i)_{i \in \mathbb{N}} \in \{1, 2\}^{\mathbb{N}}$ we have that $\prod_{i \in \mathbb{N}} (0^{2^{i+1}} a_i) \in \Xi_P(\mathcal{G})$. In particular $\Xi_P(\mathcal{G})$ is uncountable.

This is all we know about stripe shifts. Obviously questions remain. For example the following.

Question 3.3.1. *Do stripe shifts always have zero entropy?*

Notice that the minimal subshifts of every stripe shift we have presented are periodic.

Question 3.3.2. *Can a stripe shift contain a non-periodic minimal subshift?*

We could also restrict to reversible cellular automata, in which case it is natural to consider two-sided stripe shifts (it may be reasonable to consider two-sided stripe shifts for non-reversible cellular automata too).

Question 3.3.3. *What are the stripe shifts of reversible cellular automata?*

To this last question we can remark that still, obviously, full shifts are impossible. Further, we can say that sofic shifts can be handled similarly since there exists a reversible variant of the firing squad cellular automaton by Ville Lukkarila [44]. However, the construction of an uncountable stripe shift cannot in any obvious way be turned into a reversible one.

Chapter 4

Reversible One-Sided Cellular Automata

In this chapter we study reversible one-sided cellular automata over full shifts. It is well-known that reversible two-sided cellular automata are computationally complex (Kenichi Morita and Masateru Harao showed that reversible cellular automata can simulate Turing machines [49]). Some of this known complexity can be, quite literally, shifted to one-sided cellular automata since composing a two-sided cellular automaton with a suitable power of the shift map yields a one-sided cellular automaton. However, it is usually not possible to compose a reversible two-sided cellular automaton with a power of the shift map in such a way that the result would be a *reversible* one-sided cellular automaton.

Informally, one of the difficulties in constructing examples of reversible one-sided cellular automata is that one cannot employ signals. We will not attempt to formalize the concept of signals, they are simply something that carry information through space over time. The reason that signals are impossible in reversible one-sided cellular automata is that if a signal would exist and would carry information, say, from right to left, it would eventually reach the zeroth coordinate and then vanish, but then we would have lost information and reversibility would be impossible! In [16] Martin Delacourt and Nicholas Ollinger proved that reversible one-sided cellular automata are capable of universal computing, which can be considered an one-sided analogue of Dubacq's result mentioned above, though there are no similarities between the constructions. Delacourt and Ollinger also proved the first undecidability results about reversible one-sided cellular automata; they showed that given a reversible one-sided cellular automaton and a configuration with a simple description, it is undecidable whether the said configuration is periodic or not. The problem of not having usual signals is circumvented by, in some sense, exchanging the roles of space and time; the

vertical stripes contain the state of the computation on a given moment and time (of the computation) increases as one goes left in the space-time diagram. They also conjecture that the periodicity problem is undecidable for reversible one-sided cellular automata (this is known for two-sided cellular automata by Jarkko Kari and Nicholas Ollinger [38]).

For computational questions it is sufficient to consider only such reversible one-sided cellular automata that both the cellular automaton itself and its inverse have radius one, which we call *elementary reversible one-sided cellular automata*. This can be done since the inverse rule is computable (according to Serafino Amoroso, and Yale Patt [3]) and then we can use the grouping of cells to obtain an elementary reversible one-sided cellular automaton which is conjugate to the original one.

We start this chapter by doing some combinatorial considerations on elementary reversible one-sided cellular automata; these are largely the same considerations that Pablo Dartnell, Alejandro Maass, and Fernando Schwartz have done [15]. We show that every elementary reversible one-sided cellular automaton is a product of at most four involutive elementary reversible one-sided cellular automata. Next we present some examples, most importantly an elementary reversible one-sided cellular automaton we have named the *zot* cellular automaton; this is important for us since it is useful in constructions in Chapters 5 and 6. We conclude by some considerations on the periodicity of reversible one-sided cellular automata. We give a necessary condition for a reversible one-sided cellular automaton to be periodic, and show with an example that this, however, is unfortunately not a sufficient condition, providing only a new semi-algorithm for periodicity (and there of course exists a trivial semi-algorithm for testing periodicity).

4.1 Reversible One-Sided Cellular Automata

For one-sided cellular automata and reversible one-sided cellular automata we shall use the acronyms *OCA* and *ROCA*, respectively. In this chapter we consider cellular automata only over full shifts.

Let $(A^{\mathbb{N}}, F)$ be an OCA, where $|A| = n$ and $r(F) = 1$, i.e. the local rule is a function $F_{loc} : A^{[0,1]} \rightarrow A$. This local rule can be viewed as a set of n maps defined by $f_x : A \rightarrow A$ where $f_x(a) = F_{loc}(ax)$. Reminder: For cellular automata over full shifts, injectivity and reversibility are equivalent by Proposition 2.4.4. Now a necessary requirement for $(A^{\mathbb{N}}, F)$ to be reversible is that for every x the function f_x is a permutation, since if $f_x(a) = f_x(b)$ for some $a, b \in A$ where $a \neq b$, then $F(ax^\omega) = F(bx^\omega)$. We write $\pi_a = f_a$ for all $a \in A$ when all the functions f_a are permutations. Let $(\{0, 1\}^{\mathbb{N}}, \text{xor})$ be an OCA defined by permutations $\pi_0(0) = 0, \pi_0(1) = 1, \pi_1(0) = 1, \pi_1(1) = 0$. Now $\text{xor}(0^\omega) = \text{xor}(1^\omega)$, and we see that it is not sufficient for reversibility

that all the defining maps are permutations. However, if all the defining maps are permutations, then the OCA is surjective, as is well-known and easy to prove.

Proposition 4.1.1. *Let $(A^{\mathbb{N}}, F)$ be an OCA with $r(F) = 1$. Then $(A^{\mathbb{N}}, F)$ is surjective if and only if the functions f_a for $a \in A$ defining it are all permutations.*

Proof. Let $c \in A^{\mathbb{N}}$ and $a \in A$ be arbitrary and assume that $f_a = \pi_a$ are permutations for all $a \in A$. Consider the sequence $(e^{(i)})_{i \in \mathbb{N}} \in (A^{\mathbb{N}})^{\mathbb{N}}$ where $e_j^{(i)} = a$ for $j > i$ and the rest of $e^{(i)}$ is defined recursively by starting from the i^{th} cell and advancing towards zero using $e_k^{(i)} = \pi_{e_{k+1}^{(i)}}^{-1}(c_k)$ for every $k \in \{0, 1, \dots, i\}$. By compactness, this sequence has a converging subsequence $(e^{(i)})_{i \in \mathcal{I}}$ where $\mathcal{I} \subseteq \mathbb{N}$ is some infinite set. Let e be the limit of this sequence. By continuity $(F(e^{(i)}))_{i \in \mathcal{I}}$ converges to $F(e)$ and, on the other hand, by definition it also converges to c . Thus $F(e) = c$, proving the claim. \square

We call $(A^{\mathbb{N}}, F)$ an *elementary ROCA* if it is reversible and $r(F) = r(F^{-1}) = 1$. Of course for an elementary ROCA the inverse cellular automaton $(A^{\mathbb{N}}, F^{-1})$ must also be defined by permutations. The permutations of the forward rule F we denote by π_a for $a \in A$, and the permutations of the inverse rule we denote by ρ_a for $a \in A$. We represent the local rule of F as a two-dimensional array $[x_{ij}]_{i,j \in A}$ where $x_{i,j} = \pi_i(j)$. We also denote this array by $[\pi_a]_{a \in A}$. In the same fashion we denote permutation $\pi : [0, k) \rightarrow [0, k)$ by $[\pi(0)\pi(1) \dots \pi(k-1)]$ (see Example 4.1.4 for the notations).

Let $(A^{\mathbb{N}}, F)$ be an elementary ROCA defined by $[\pi_a]_{a \in A}$, then there is a simple way to obtain permutations $[\rho_a]_{a \in A}$ defining the inverse elementary ROCA $(A^{\mathbb{N}}, F^{-1})$.

Proposition 4.1.2. *Let $(A^{\mathbb{N}}, F)$ be an elementary ROCA defined by $[\pi_a]_{a \in A}$ and $(A^{\mathbb{N}}, F^{-1})$ its inverse defined by $[\rho_a]_{a \in A}$. Then*

$$\rho_a = \pi_{\pi_b^{-1}(a)}^{-1}$$

for every $a, b \in A$.

Proof. Let $x, y, z \in A$. Applying f to xyz yields $\pi_y(x)\pi_z(y)$. Applying the local rule of F^{-1} to this gives $\rho_{\pi_z(y)}(\pi_y(x)) = x$ since $F^{-1}F = \text{id}$. From this we get that $\pi_y(x) = \rho_{\pi_z(y)}^{-1}(x)$ for every x , and so $\pi_y = \rho_{\pi_z(y)}^{-1}$, i.e. $\rho_{\pi_z(y)} = \pi_y^{-1}$ for any $y, z \in A$. Let $a = \pi_z(y)$. Then $y = \pi_z^{-1}(a)$ and we have $\rho_a = \pi_{\pi_z^{-1}(a)}^{-1}$. Here a goes through all values of A as y goes through all values of A since π_z is a permutation. This proves the claim. \square

In [15] Dartnell, Maass, and Schwartz gave the following simple description of elementary ROCA's.

Proposition 4.1.3. *Let $(A^{\mathbb{N}}, F)$ be an OCA with $r(F) = 1$ defined by $[\pi_a]_{a \in A}$. Then $(A^{\mathbb{N}}, F)$ is an elementary ROCA if and only if it holds that*

$$\pi_x(a) = \pi_y(b) \implies \pi_a = \pi_b$$

for every $x, y, a, b \in A$.

Proof. Let $(A^{\mathbb{N}}, F)$ be an elementary ROCA. For contradiction, suppose that there exists $x, y, a, b \in A$ such that $\pi_x(a) = \pi_y(b)$ and $\pi_a \neq \pi_b$. Let $z = \pi_x(a) = \pi_y(b)$. By Proposition 4.1.2 we have that $\rho_z = \pi_{\pi_w^{-1}(z)}^{-1}$ for every $w \in A$. In particular $\rho_z = \pi_{\pi_x^{-1}(z)}^{-1} = \pi_a^{-1}$ and $\rho_z = \pi_{\pi_y^{-1}(z)}^{-1} = \pi_b^{-1}$ which is a contradiction since $\pi_a \neq \pi_b$.

For the other direction, assume that $\pi_x(a) = \pi_y(b) \implies \pi_a = \pi_b$ for all $x, y, a, b \in A$. For every $z \in A$ define $\rho_z = \pi_{\pi_x^{-1}(z)}^{-1}$ for some $x \in A$. Notice that any choice of x leads to the same permutation since even if $a = \pi_x^{-1}(z) \neq \pi_y^{-1}(z) = b$, we have that $\pi_x(a) = \pi_y(b)$ and so $\pi_a = \pi_b$, and also $\pi_a^{-1} = \pi_b^{-1}$. As was seen in Proposition 4.1.2, this defines the inverse of $(A^{\mathbb{N}}, F)$. \square

Example 4.1.4. Let π_0 be the permutation $\pi_0 : \{0, 1, 2\} \rightarrow \{0, 1, 2\}$ defined by $\pi_0(0) = 0, \pi_0(1) = 2, \pi_0(2) = 1$ which we denote by $\pi_0 = [021]$. Let $\pi_1 = [120]$ and $\pi_2 = \pi_0$. These define an OCA

$$\begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}.$$

From this notation it is easy to check that the permutations define an elementary ROCA; according to Proposition 4.1.3 it is sufficient to check that if some columns contain the same element, then the corresponding rows are the same. In this example zeroth and second column contain 0, and thus we need to check that $\pi_0 = \pi_2$, as they are. It is also easy to produce the inverse rule (once it has been verified that the rule defines an elementary ROCA):

$$\begin{bmatrix} 0 & 2 & 1 \\ 1 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{\text{reorder}} \begin{bmatrix} 0 & 2 & 1 \\ 0 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix} \xrightarrow{\text{invert}} \begin{bmatrix} 0 & 2 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

First we reorder the rows using any of the permutations π_a , and then we invert every permutation. If the original OCA is not an elementary ROCA, then the result of this procedure depends on which permutation is used in the reordering of the rows.

Example 4.1.5. Of course an OCA may be reversible without being an elementary ROCA. For example the cellular automaton $(A^{\mathbb{N}}, F)$ defined by $\pi_0 = [0213] = \pi_2, \pi_1 = [1203]$, and $\pi_3 = [0123]$ is reversible with inverse radius $r(F^{-1}) = 2$.

4.1.1 Elementary ROCA's as Products of Involutions

Next we show that every elementary ROCA can be written as a composition of at most four involutions. This is a corollary to Dartnell, Maass, and Schwartz's result [15, Proposition 3.1], though we prove everything necessary.

Let $(A^{\mathbb{N}}, F)$ be an elementary ROCA. Let $k = |\{\pi_a\}_{a \in A}|$. Let us define *permutation partition* of A as the partition $pp_F(A) = \{\mathcal{I}_i\}_{i \in [1, k]}$ of A defined by

1. $a, b \in \mathcal{I}_i$ implies that $\pi_a = \pi_b$ and
2. for $i \neq j$ it holds that $a \in \mathcal{I}_i$ and $b \in \mathcal{I}_j$ implies that $\pi_a \neq \pi_b$.

Lemma 4.1.6. *Let $(A^{\mathbb{N}}, F)$ be an elementary ROCA. Let $pp_F(A) = \{\mathcal{I}_i\}_{i \in [1, k]}$ and $pp_{F^{-1}}(A) = \{\mathcal{J}_j\}_{j \in [1, k]}$ where $k = |\{\pi_a\}_{a \in A}|$ (which is also the number of permutations for F^{-1}). Then for every $a \in A$ and $i \in \{1, 2, \dots, k\}$ there exists a unique $j \in \{1, 2, \dots, k\}$ such that $\pi_a(\mathcal{I}_i) = \mathcal{J}_j$.*

Proof. Suppose for contradiction that for some $a \in A$ there exists \mathcal{I}_i such that there are $i_1, i_2 \in \mathcal{I}_i$ such that $\pi_a(i_1) = j_1 \in \mathcal{J}_j$ and $\pi_a(i_2) = j_2 \in \mathcal{J}_{j'}$ where $j \neq j'$. Then $\rho_{j_1} \neq \rho_{j_2}$, i.e. $\pi_{\pi_a^{-1}(j_1)}^{-1} \neq \pi_{\pi_a^{-1}(j_2)}^{-1}$. But then $\pi_{i_1}^{-1} \neq \pi_{i_2}^{-1}$ and so $\pi_{i_1} \neq \pi_{i_2}$ which is a contradiction as $i_1, i_2 \in \mathcal{I}_i$. \square

So we have that $\pi_a(pp_F(A)) = pp_{F^{-1}}(A)$ for any $a \in A$. We can prove the following sufficient condition for periodicity.

Proposition 4.1.7. *Let $(A^{\mathbb{N}}, F)$ be an elementary ROCA. If $pp_F(A) = pp_{F^{-1}}(A)$ then F is periodic.*

Proof. Assume that $pp_F(A) = pp_{F^{-1}}(A)$. Suppose that there are $a, b \in A$ such that $\pi_a(\mathcal{I}_i) \neq \pi_b(\mathcal{I}_i)$ for some i . This means that for $x \in \mathcal{I}_i$ we have that $\pi_{\pi_a(x)} \neq \pi_{\pi_b(x)}$. But since $pp_F(A) = pp_{F^{-1}}(A)$, this implies that $\rho_{\pi_a(x)} \neq \rho_{\pi_b(x)}$. But this implies that $\pi_x \neq \pi_x$, which is a contradiction. Thus we must have that $\pi_a(\mathcal{I}_i) = \pi_b(\mathcal{I}_i)$ for all $a, b \in A$. But this means that all words of length two are blocking, since each permutation determines which permutation it becomes in the next time step. \square

The previous proposition gives, for example, the following corollary.

Corollary 4.1.8. *Let $(A^{\mathbb{N}}, F)$ be an elementary ROCA. If for every $a \in A$ there exists $b \in A$ such that $\pi_b(a) = a$, then $(A^{\mathbb{N}}, F)$ is periodic.*

Proof. Let $a \in A$ be arbitrary. We know that there exists $b \in A$ such that $\pi_b(a) = a$. Now we have $\rho_a = \pi_{\pi_b^{-1}(a)}^{-1} = \pi_a^{-1}$. But then $pp_F(A) = pp_{F^{-1}}(A)$ and the claim follows. \square

As a special case of the above corollary we have that if $(A^{\mathbb{N}}, F)$ is an elementary ROCA such that $\pi_a = \text{id}$ for some $a \in A$, then the said elementary ROCA is periodic. We also immediately get the following.

Corollary 4.1.9. *[15, Proposition 3.1] Any elementary ROCA is a composition of two periodic elementary ROCA's, and the other one can be chosen to be a symbol permutation.*

Proof. For example $F = \pi_0 \circ F_1$ (here π_0 is understood as extended cell-wise to a function $A^{\mathbb{N}} \rightarrow A^{\mathbb{N}}$) where F_1 is defined by $[\pi_0^{-1}\pi_a]_{a \in A}$. Since F is an elementary ROCA, so is F_1 . Clearly π_x is periodic and F_1 is periodic according to Corollary 4.1.8. \square

Following lemma is well-known.

Lemma 4.1.10. *Let A be a finite set and $\pi : A \rightarrow A$ a permutation. Then π can be written as a composition of two involutions.*

Proof. It is enough to show that any cyclic permutation can be written as a composition of two involutions. Let $\pi : \{0, 1, \dots, n-1\} \rightarrow \{0, 1, \dots, n-1\}$ be a cyclic permutation defined by $\pi(i) = i + 1 \pmod n$. Define two involutions δ_1 and δ_2 as follows: $\delta_1(i) = (n-1) - i$ for $i \in \{0, 1, \dots, n-1\}$ and $\delta_2(0) = 0$ and $\delta_2(i) = n - i$ for $i \in \{1, 2, \dots, n-1\}$. Now $\delta_2\delta_1(i) = \delta_2(n-1-i)$ which is 0 if $i = n-1$ and $n - (n-1-i) = i+1$ otherwise, i.e. $\pi = \delta_2\delta_1$. \square

Corollary 4.1.11. *Every elementary ROCA can be written as a composition of at most four involutive elementary ROCA's.*

Proof. According to Corollary 4.1.9 and Lemma 4.1.10 it is sufficient to show that an elementary ROCA $(A^{\mathbb{N}}, F)$ defined by $[\pi_a]_{a \in A}$ where $\pi_0 = \text{id}$ can be written as a composition of two involutive elementary ROCA's.

Let $\delta_{a,1}$ and $\delta_{a,2}$ be involutions such that $\pi_a = \delta_{a,2}\delta_{a,1}$, and we also assume that these involutions are such that if $\delta_{a,l}(b) = c$ ($l \in \{1, 2\}$) then for some k it holds that $\pi_a^k(b) = c$ and if $\pi_a = \pi_b$ then also the corresponding involutions are the same. Now we claim that $F_l = (\delta_{a,l})_{a \in A}$ ($l \in \{1, 2\}$) define elementary ROCA's such that $F = F_2F_1$. The fact that these are elementary ROCA's follows from the fact that $\pi_0 = \text{id}$ and the additional assumption we posed on the involutions $\delta_{a,l}$: If $\delta_{x,l}(a) = \delta_{y,l}(b)$ then there

exists k_x and k_y such that $\pi_x^{k_x}(a) = \pi_y^{k_y}(b)$, but since $\pi_0 = \text{id}$ we have that $\pi_w = \pi_{\pi_z(w)}$ for any $z, w \in A$ so this implies that $\pi_a = \pi_b$ and so $\delta_{a,l} = \delta_{b,l}$. This proves that F_l 's are elementary ROCA's by Proposition 4.1.3. Now for $F = F_2F_1$ we want to show that

$$\delta_{\delta_{y,1}(x),2} \circ \delta_{x,1} = \pi_x.$$

It is sufficient to show that $\pi_{\delta_{y,1}(x)} = \pi_x$. Again, if $\delta_{y,1}(x) = z$ then there exists k such that $\pi_y^k(x) = z$ and since $\pi_0 = \text{id}$ we have that $\pi_z = \pi_x$, which proves the claim. \square

It is known that some elementary ROCA's cannot be written as a product of two involutions [23, Theorem 2], which raises the question: Are all elementary ROCA's compositions of at most three involutions?

4.2 Examples

Next we present some examples of elementary ROCA's. Most of the focus is put on the *zot* cellular automaton which is useful in chapters to come.

Trivial Examples

Over alphabets of size 1 or 2 there are only trivial elementary ROCA's (actually only trivial ROCA's as proved by Gustav Hedlund in [26, Theorem 6.9.]). Trivial here means that they have radius zero. These are the elementary ROCA's where every letter defines the same permutation.

Example by Hedlund

This is perhaps the first elementary ROCA to appear in the literature; it was presented by Gustav Hedlund in the paper [26] which laid significant groundwork for studying cellular automata as dynamical systems. In [26, Theorem 6.9] Hedlund proves that there are only trivial reversible one-sided cellular automata over the binary alphabet. The following elementary ROCA is then given as an example, that this claim holds true no more, if the alphabet size is increased:

$$\begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

By Proposition 4.1.3 this is an elementary ROCA and since $\pi_0 = \text{id}$ we know by Corollary 4.1.8 that this defines a periodic cellular automaton. In fact one easily sees that this cellular automaton is an involution.

A Less Trivial Periodic Elementary Cellular Automaton

Let $(A^{\mathbb{N}}, F)$ be an elementary ROCA defined by

$$\begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 & 1 \\ 3 & 2 & 0 & 1 \\ 2 & 3 & 0 & 1 \\ 2 & 3 & 0 & 1 \end{bmatrix} \text{ whose inverse is } \begin{bmatrix} \rho_0 \\ \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 0 & 1 \\ 2 & 3 & 0 & 1 \\ 2 & 3 & 1 & 0 \\ 2 & 3 & 1 & 0 \end{bmatrix}.$$

We see that $pp_F(A) = pp_{F^{-1}}(A)$ so that according to Proposition 4.1.7 the cellular automaton must be periodic. The trivial cellular automata with radius zero are periodic since every letter uniquely determines which letter will follow it. Periodicity of cellular automata with $pp_F(A) = pp_{F^{-1}}(A)$ is due to similar reason; while now letter does not uniquely define the following letter, each permutation uniquely determines which permutation follows it. In this example permutation π_0 (which both 0 and 1 define) is always followed by permutation π_2 (since $\pi_3 = \pi_2$), and similarly π_2 is always followed by π_0 . This means that every column in the space-time diagram is, if we forget the actual letters and focus only on which permutation they define, a repetition of $\pi_0\pi_2$. Since $\pi_2\pi_0 = [1023]$ we have that $(\pi_2\pi_0)^2 = \text{id}$ and we see that $F^4 = \text{id}$.

Periodic but $pp_F(A) \neq pp_{F^{-1}}(A)$

According to Proposition 4.1.7 it holds that if $pp_F(A) = pp_{F^{-1}}(A)$ then F is periodic, which we used in the previous example. Next examples shows that this is, however, not a necessary condition.

Let $A = \{0, 1, 2, 3, 4\}$ and define $(A^{\mathbb{N}}, F)$ by

$$\begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 4 & 3 \\ 0 & 1 & 3 & 4 & 2 \\ 1 & 0 & 3 & 4 & 2 \\ 0 & 1 & 3 & 4 & 2 \end{bmatrix}.$$

Here $\pi_0(pp_F(A)) \neq pp_F(A)$ so $pp_F(A) \neq pp_{F^{-1}}(A)$. However, F is periodic. This can be seen by observing that every word of length three (or more) is a blocking word: Let $abc \in A^3$. Since letters 0 and 1 can only map to letters 0 and 1, and these define the same permutation, we have that if $b \in \{0, 1\}$ or $c \in \{0, 1\}$ then abc is a blocking word. But if $b, c \in \{2, 3, 4\}$ then already bc is a blocking word since 2, 3 and 4 can only map to 2, 3 and 4 and these define the same permutation when restricted to the set $\{2, 3, 4\}$.

As a sidenote we remark that this cellular automaton has period 12, and this is the largest period amongst the elementary ROCA's over an alphabet of size five (which the reader may verify on his/hers leisure time by going through all such elementary ROCA's).

The Zot Cellular Automaton

In intuitive terms a question arises: Since ROCAs cannot move information through space, does this mean that perhaps they are all periodic? (Un)fortunately this is not the case. The cellular automaton we present next was first considered by Pablo Dartnell, Alejandro Maass, and Fernando Schwartz in [15, §4.1] (at least as far as we know, but due to the simplicity of this example anyone considering reversible one-sided cellular automata is likely to end up considering it).

Since this cellular automaton is useful in some constructions to come, we give it a name and call it the *zot* cellular automaton. It is the cellular automaton $(A^{\mathbb{N}}, \mathcal{Z})$ where $A = \{0, 1, 2\}$ defined by

$$\begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}.$$

We compute the trace of this elementary ROCA, which turns out to be a proper sofic shift, and also gives us the entropy of *zot*. We also show that *zot* is chain-transitive and thus, by Proposition 2.4.21, also chain-mixing, but not transitive (since, as proved in [15], it has blocking words). Lastly we point out that *zot* does not have POTP.

According to Proposition 4.1.2 the inverse of *zot* is defined by

$$\begin{bmatrix} \rho_0 \\ \rho_1 \\ \rho_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix}.$$

Drawing some space-time diagrams of *zot* suggests some of its dynamics; Figure 4.1 hints that *zot* is not periodic,. However, Figure 4.2 suggests that there exists some equicontinuity points. These facts are explicitly proven in [15, §4.1]. Before computing *zot*'s trace we prove the following simple lemma.

Lemma 4.2.1. *Let $(A^{\mathbb{N}}, \mathcal{Z})$ be the *zot* cellular automaton.*

- *Let $s \in st_{\mathbb{Z}}(A^{\mathbb{N}}, \mathcal{Z})$ and $\mathbf{a} = a_1a_2 = s_{i,j}s_{i,j+1}$, $\mathbf{b} = b_1b_2 = s_{i,j+2}s_{i,j+3}$, $\mathbf{c} = c_1c_2 = s_{i+1,j+1}, s_{i+1,j+2}$. Then the following hold:*

1. *If $\mathbf{a} = 00$ and $\mathbf{b} = 00$ then $\mathbf{c} = 00$.*
2. *If $\mathbf{a} = 00$ and $\mathbf{b} = 12$ then $\mathbf{c} = 12$.*
3. *If $\mathbf{a} = 12$ and $\mathbf{b} = 00$ then $\mathbf{c} = 12$.*
4. *If $\mathbf{a} = 12$ and $\mathbf{b} = 12$ then $\mathbf{c} \in \{00, 01, 20, 21\}$, and all of these are possible.*

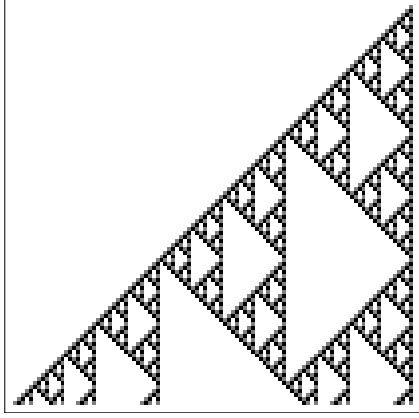


Figure 4.1: Running zot from $\omega 010^\omega$.

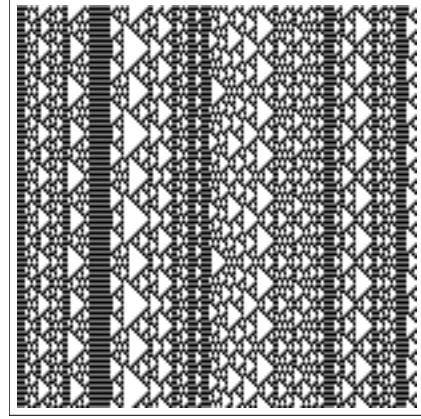


Figure 4.2: Running zot from a random initial configuration.

- If $20^k 1 \in \mathcal{L}(\tau_1(\mathcal{Z}))$, then k is even.
- Let $s \in A^{\mathbb{N} \times \mathbb{Z}}$. Let $D = \{(0, 0), (0, 1)\}$. If for every $(i, j) \in \mathbb{N} \times \mathbb{Z}$ where i, j are either both odd or both even it holds that $s_{D+(i,j)} \in \{00, 12\}$ and $s_{D+(i,j)} \oplus s_{D+(i,j+2)} = s_{D+(i+1,j+1)}$ where 00 is interpreted as 0, 12 is interpreted as 1, and \oplus denotes addition modulo two, then $s \in st_{\mathbb{Z}}(A^{\mathbb{N}}, F)$.

Proof.

- (See Figure 4.3) These can be checked simply by going through all possibilities. For example consider the first case where $\mathbf{a} = 00$ and $\mathbf{b} = 00$. Now we have that $\pi_{c_1}(0) = 0$ and $\rho_{c_1}(0) = 0$ which leave only the option that $c_1 = 0$ possible.
- (See Figure 4.4) We see that $201 \notin \mathcal{L}(\tau_1(A^{\mathbb{N}}, \mathcal{Z}))$ since $\pi_x(0) = 1$ implies $x = 1$ but $\rho_1(0) = 0 \neq 1$. Let $k > 1$ and consider $20^k 1$. Suppose this appears in $\tau_1(A^{\mathbb{N}}, \mathcal{Z})$. Let $s \in st_{\mathbb{Z}}(A^{\mathbb{N}}, \mathcal{Z})$ such that $s_{0,0}s_{0,1} \cdots s_{0,k+1} = 20^k 1$. Then the first property forces $s_{1,1}s_{1,2} \cdots s_{1,k} = 20^{k-2} 1$. If k was odd, we could repeat this until we obtained 201 which, as we saw, cannot be in $\mathcal{L}(\tau_1(A^{\mathbb{N}}, \mathcal{Z}))$.
- (See Figure 4.5) According to the first point such a configuration has no violations of the local rule of \mathcal{Z} , and thus it must be a valid space-time diagram.

□

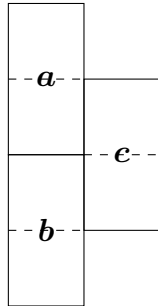


Figure 4.3: If the pattern formed by \mathbf{a} , \mathbf{b} and \mathbf{c} is a valid pattern in $st_{\mathbb{Z}}(\mathcal{Z})$ and two of the three \mathbf{a} , \mathbf{b} , \mathbf{c} are in $\{00, 12\}$, but $\mathbf{ab} \neq 1212$, then also the third is in $\{00, 12\}$ and it is determined by $\mathbf{a} \oplus \mathbf{b} = \mathbf{c}$ where we identify 00 with 0 and 12 with 1, and \oplus is addition modulo two. In case $\mathbf{ab} = 1212$ then $\mathbf{c} = 00$ gives a pattern that appears in $st_{\mathbb{Z}}(\mathcal{Z})$, but there are also other valid ways to choose \mathbf{c} .

1		
2	1	
0	2	1
0	0	2
0	0	0
0	0	1
0	1	2
1	2	
2		

Figure 4.4: The traces of \mathcal{Z} cannot have a word $20^n 1$ where n is odd. This can be seen by applying the modulo two addition as described in Figure 4.3, which leads to the word 201, which is invalid since since only $\pi_1(0) = 1$ but $\rho_1(0) \neq 2$.

1			
2	0		
1	0	1	
2	1	2	0
0	2	1	0
0	0	2	
0	0		
0			

Figure 4.5: Fill the leftmost column in an arbitrary way using the blocks 00 and 12. Fill the next column using the modulo two addition as described in Figure 4.3. Notice that we get no violations of the local rule of \mathcal{Z} doing this. Repeat.

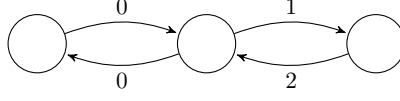


Figure 4.6: Graph defining $\tau_1(\mathcal{Z})$.

Next we compute $\tau_1(\mathcal{Z})$.

Proposition 4.2.2. *Let $(A^{\mathbb{N}}, \mathcal{Z})$ be the zot cellular automaton. Then $\tau_1(\mathcal{Z}) = X_{\mathcal{F}}$ where $\mathcal{F} = \{02, 10, 11, 22\} \cup \{20^{2k+1}1 \mid k \in \mathbb{N}\}$. This is the proper sofic shift defined by the graph of Figure 4.6.*

Proof. Notice that every configuration in $X_{\mathcal{F}}$ is a concatenation of words 00 and 12 and all concatenations of 00 and 12 are in $X_{\mathcal{F}}$ (we are being vague about the origin).

Take an arbitrary configuration $t^{(0)} \in X_{\mathcal{F}}$. Inductively define $t^{(i)} \in \{00, 12\}^{\mathbb{Z}}$ using the modulo two addition to $t^{(i-1)}$ as in Lemma 4.2.1 (interpreting 00 as 0 and 12 as 1, and shifting $t^{(i)}$ so that the two-cell rectangles are not aligned, see Figure 4.5). Then the configuration $s \in A^{\mathbb{N} \times \mathbb{Z}}$ where $s_{x,y} = t_y^{(x)}$ is in $st_{\mathbb{Z}}(A^{\mathbb{N}}, \mathcal{Z})$ according to the third point of Lemma 4.2.1. Thus $X_{\mathcal{F}} \subseteq \tau_1(\mathcal{Z})$. On the other hand the second point of Lemma 4.2.1 forbids $20^{2k+1}1$, so that we have $\tau_1(\mathcal{Z}) = X_{\mathcal{F}}$. \square

Previous proposition and Proposition 2.4.5 allows us to compute zot's entropy.

Proposition 4.2.3. *The entropy of $(A^{\mathbb{N}}, \mathcal{Z})$ is $h(A^{\mathbb{N}}, \mathcal{Z}) = \frac{1}{2}$.*

Proof. According to Proposition 4.2.2 we have that $|\mathcal{L}_{2n}(\tau_1(\mathcal{Z}))| = 3 \cdot 2^n - 1$. By Proposition 2.4.5 we have that $h(\{0, 1, 2\}^{\mathbb{N}}, \mathcal{Z}) = h(\tau_1(\mathcal{Z})) = \frac{1}{2}$. \square

It turns out that if we restrict \mathcal{Z} to finite words in $\{0, 1, 2\}^n$ in such a way that in the rightmost cell, in which we cannot use the local rule of \mathcal{Z} , we instead always use π_1 , then this defines a cyclic permutation of $\{0, 1, 2\}^n$. This gives an unconventional way to enumerate finite words of length n .

Proposition 4.2.4. *Let $A = \{0, 1, 2\}$ and $(A^{\mathbb{N}}, \mathcal{Z})$ be the zot cellular automaton as above. Let π be the map*

$$\begin{aligned} \bigcup_{n \in \mathbb{N} \setminus \{0\}} A^n &\xrightarrow{\pi} \bigcup_{n \in \mathbb{N} \setminus \{0\}} A^n \\ a_0 \cdots a_{n-1} &\longmapsto \pi_{a_1}(a_0) \cdots \pi_1(a_{n-1}). \end{aligned}$$

Then for every $n \in \mathbb{N} \setminus \{0\}$ the restriction $\pi|_{A^n}$ is a cyclic permutation of A^n .

Proof. Proof goes by induction on n : For $n = 1$ the claim is clear as π_1 is a cyclic permutation of A .

Suppose that the claim holds for some $n \in \mathbb{N} \setminus \{0\}$. It is enough to show that the map $x \mapsto \pi^{3^n}(x0^n)_0$ is a cyclic permutation of A .

Notice that in $(\pi^i(x0^n))_{i \in \mathbb{Z}}$ patterns of form of Figure 4.3 are valid, i.e. do not violate the local rule of \mathcal{Z} . Clearly for every $j \in \{1, \dots, n\}$ the configuration $t_j = (\pi^i(x0^n)_j)_{i \in \mathbb{Z}}$ is periodic. Applying π to 0^{n-j+1} until we return back to 0^{n-j+1} it is straightforward to see that a word of form $0^{n-j+1}t'_j0^{n-j}$, for some finite word t'_j , is a smallest period in t_j . We claim that the words t'_j cannot contain words 20^k1 where k is odd. Suppose that such a word does exist. If $k \geq 2(n-j) + 1$ then using the argument in Figure 4.3 we see that this was not the smallest period as we have returned to 0^{n-j+1} . If $k < 2(n-j) + 1$ then using the same argument we reach a contradiction with the observation that the patterns of Figure 4.3 are valid in $(\pi^i(x0^n))_{i \in \mathbb{Z}}$. It is also clear that t'_j starts with 1 and ends with 2.

Now let $x = 0$. Let $t = (\pi^i(0^{n+1})_0)_{i \in [0, 3^n - n]}$. By the observations of the previous paragraph we know that the vertical word to the right of t is $0^n t'_1$ where $t'_1 \in \{00, 12\}^+$ (understood here as a word over alphabet $\{0, 1, 2\}$). Then t is determined by repeatedly applying the reasoning in Figure 4.3. By the induction hypothesis we know that all words of length n appear exactly once in $(\pi^i(0^n))_{i \in [0, 3^n]}$. This means that in the right side of t there will be 1 exactly 3^{n-1} times. In particular this is an odd number of times. Every time we see 1 the modulo two addition will swap t between a stream of 00's and a stream of 12's. Since we start with 00 and swap for odd number of times, we must end in 12. So we have that if n is even, then $\pi^{3^n}(00^n)_0 = 1$, and if n is odd, then $\pi^{3^n}(00^n)_0 = 2$. It is now enough that we show that if n is even, then $\pi^{3^n}(10^n)_0 = 2$, and if n is odd, then $\pi^{3^n}(20^n)_0 = 1$.

Suppose that n is even and $x = 1$. The only way the stream of 12's in t (defined as before) swaps to a stream of 00's is if 2 in t is aligned with a 1 in t'_1 . Before t'_1 starts we have n times a 0 which map 1 to 2 and 2 back to 1. As we noted, $t'_1 \in \{00, 12\}^+$ and begins with 1, so since n is even we see that 2 is never aligned with 1 and thus t will only swap between 1 and 2 for 3^n steps. Since we do this odd number of times, we end up with a 2.

The case n is odd and $x = 2$ goes similarly. □

From the inverse rule of zot, the elementary ROCA defined by

$$\begin{bmatrix} \rho_0 \\ \rho_1 \\ \vdots \\ \rho_{2k} \\ \rho_{2k+1} \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 4 & 3 & \dots & 2k & 2k-1 \\ 0 & 2 & 1 & 4 & 3 & \dots & 2k & 2k-1 \\ & & & \vdots & & & \vdots & \\ 0 & 2 & 1 & 4 & 3 & \dots & 2k & 2k-1 \\ 2k & 0 & 1 & 2 & 3 & \dots & 2k-2 & 2k-1 \end{bmatrix}$$

can be considered a generalization of the inverse of zot to higher odd sized alphabets. From small computer simulations it seems like these may possess similar property as described above, i.e. that using local rule of the cellular automaton where possible and ρ_{2k-1} in the last symbol enumerates all words in $\{0, 1, \dots, 2k-1\}^n$ for any $n \in \mathbb{N} \setminus \{0\}$. Elementary ROCAs defined by

$$\begin{bmatrix} \pi_0 \\ \pi_1 \\ \vdots \\ \pi_{2k} \\ \pi_{2k+1} \end{bmatrix} = \begin{bmatrix} 2k & 0 & 1 & \dots & 2k-3 & 2k-2 & 2k-1 \\ 2k & 2k-1 & 2k-2 & \dots & 2 & 1 & 0 \\ 2k & 2k-1 & 2k-2 & \dots & 2 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2k & 2k-1 & 2k-2 & \dots & 2 & 1 & 0 \end{bmatrix}$$

also seem to have this property for permutation π_0 .

Question 4.2.5. *Does analogous result to Proposition 4.2.4 hold for any cellular automata over an alphabet of even size?*

From what we have seen so far it is easy now to deduce the following properties of zot.

Proposition 4.2.6. *The zot cellular automaton is chain-mixing, but not transitive, and does not have POTP.*

Proof. Let $(A^{\mathbb{N}}, \mathcal{Z})$ be the zot cellular automaton.

Zot is chain-mixing: According to Proposition 2.4.21 it is sufficient to show that zot is chain-transitive. Clearly zot is chain-transitive according to Proposition 4.2.4.

Zot is not transitive: This holds since zot has blocking words, such as 11.

Zot does not have POTP: Not having POTP can be seen by observing that when one starts applying π and π^{-1} to 0^n the word $20^{2n-1}1$ appears to the leftmost column, so $20^{2n-1}1 \in \mathcal{L}(0\Sigma_n^1)$ and since, according to the third point of Lemma 4.2.1, word $20^{2n-1}1$ is not in $\mathcal{L}(\tau_1(\mathcal{Z}))$ we have that zot does not have POTP. \square

An Expansive Elementary ROCA

The zot cellular automaton is not expansive since it has blocking words. Let us now present a simple expansive elementary ROCA.

Let $(A^{\mathbb{N}}, F)$ be an elementary ROCA defined by

$$\begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 & 1 \\ 1 & 3 & 2 & 0 \\ 1 & 3 & 2 & 0 \\ 0 & 2 & 3 & 1 \end{bmatrix}.$$

The inverse of this is

$$\begin{bmatrix} \rho_0 \\ \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 1 & 2 \\ 0 & 3 & 1 & 2 \\ 3 & 0 & 2 & 1 \\ 3 & 0 & 2 & 1 \end{bmatrix}.$$

This is expansive since for any $abc \in \tau_1(F)$ it holds that $|\{x \in A \mid \pi_x(b) = c \text{ and } \rho_x(b) = a\}| = 1$, and then by Proposition 2.4.23 we have that $(A^{\mathbb{N}}, F)$ is expansive. This cellular automaton has POTP, in fact every expansive ROCA has POTP as was proven by Masakazu Nasu [52]; a result we are going to reprove in Chapter 5

4.3 Periodicity and the Diagonal Cellular Automata

We already saw that $pp_F(A) = pp_{F^{-1}}(A)$ implies periodicity, but this is a very specific case. In this section we provide a new necessary and sufficient condition for a (elementary) ROCA to be periodic. This condition is described as an procedure which starts repeating itself only for periodic cellular automata. Further more, if it runs only finitely long, the cellular automaton at hand is not periodic. The hope was that this could perhaps provide an algorithm for periodicity if it turned out that the procedure is finite for all non-periodic elementary ROCA's. However, we end this section by giving an example that this is not the case.

If $(A^{\mathbb{N}}, F)$ is periodic, then according to Proposition 2.4.17 there exists $M \in \mathbb{N}$ such that every word of length $n > M$ is a blocking word; using the language of Dartnell, Maass, and Schwartz we call the smallest such M the *equicontinuity constant* of $(A^{\mathbb{N}}, F)$. In [15] Darntell, Maass, and Schwartz gave a characterization of periodic elementary ROCA's with equicontinuity constant equal to one (i.e. all words of length two are blocking). Their justification for considering only equicontinuity constant one is that any equicontinuous cellular automaton is conjugate to an equicontinuous cellular automaton with equicontinuity constant one via the usual grouping of cells; the same argument that allows reducing much of the considerations of ROCAs to considering elementary ROCAs. However, if one is interested in algorithmic questions about one-sided cellular automata, then the latter grouping is justified while the first one is not, as we do not know whether there exists any computable bound on the equicontinuity constant (as we do not know whether periodicity is decidable). This is the reason we consider arbitrary equicontinuity constants.

Remark 4.3.1. Let $(A^{\mathbb{N}}, F)$ be a cellular automaton. Then $(A^{\mathbb{N}}, F)$ is periodic if and only if $(A^{\mathbb{Z}}, F)$ is periodic. This is sometimes useful when we want to argue that $(A^{\mathbb{N}}, F)$ is not periodic, as this is sometimes easy to do

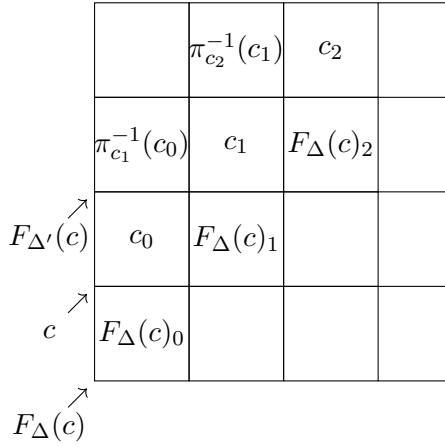


Figure 4.7: Diagonal cellular automaton F_{Δ} is defined only if the upwards going cellular automaton $F_{\Delta'}$ is reversible.

by noticing that there exists $c \in A^{\mathbb{Z}}$ such that $F^i(c) \neq c$ for all $i \in \mathbb{N}$. The reason that considering one-sided configurations as witnesses of non-periodicity is less convenient is that every configuration of form $uv^{\omega} \in A^{\mathbb{N}}$ for any $u, v \in A^+$ is periodic for any ROCA.

Let $(A^{\mathbb{N}}, F)$ be an elementary ROCA defined by permutations $[\pi_a]_{a \in A}$ and with an inverse rule defined by $[\rho_a]_{a \in A}$. We define, when possible, the *diagonal cellular automaton* of $(A^{\mathbb{N}}, F)$ as follows: First define an auxiliary cellular automaton as $(A^{\mathbb{N}}, F_{\Delta'})$ by $[\pi_a^{-1}]_{a \in A}$. Since $f(xa) = \pi_a(x)$ this $(A^{\mathbb{N}}, F_{\Delta'})$ cellular automaton simply does the local inverse of F , and thus it maps southwest-northeast diagonals upwards (see Figure 4.7). Now if $(A^{\mathbb{N}}, F_{\Delta'})$ happens to be a ROCA (not necessarily elementary) then the diagonal cellular automaton of $(A^{\mathbb{N}}, F)$ is defined as $(A^{\mathbb{N}}, F_{\Delta})$ where $F_{\Delta} = F_{\Delta'}^{-1}$. We observe the following.

Proposition 4.3.2. *Elementary ROCA $(A^{\mathbb{N}}, F)$ is periodic if and only if $(A^{\mathbb{N}}, F_{\Delta})$ exists and is periodic. Furthermore, in this case $(A^{\mathbb{N}}, F)$ and $(A^{\mathbb{N}}, F_{\Delta})$ have the same equicontinuity constant.*

Proof. The case that $(A^{\mathbb{N}}, F_{\Delta})$ is not defined has a nice visual interpretation. That $(A^{\mathbb{N}}, F_{\Delta'})$ is not reversible means that there has to exist configurations $c, e \in A^{\mathbb{N}}$ such that $c \neq e$ and $F_{\Delta'}(c) = F_{\Delta'}(e)$. Suppose $c_i \neq e_i$. Then since $F_{\Delta'}(c)_i = F_{\Delta'}(e)_i$ we have that $c_{i+1} \neq e_{i+1}$ since otherwise c_{i+1} and e_{i+1} would define the same permutation and could not map the different letters c_i and e_i to the same letter. Thus $c_j \neq e_j$ for all $j \geq i$. Thus we may assume that $c_i \neq e_i$ for all $i \in \mathbb{N}$ by taking a shifted version of c and e if necessary. This means that for all $n \in \mathbb{N}$ there exists a pair $(x^{(n)}, y^{(n)}) \in A^{\mathbb{N}} \times A^{\mathbb{N}}$

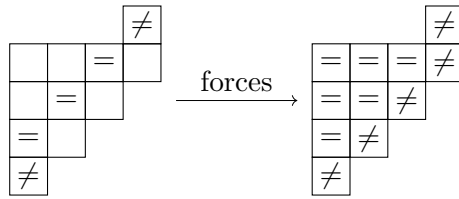


Figure 4.8: If $r(F_\Delta) = 3$ there has to be words of length 4 which agree on the first three letters but differ on the fourth such that their images differ also (the left image). The local rule of F demands that the equalities spread above and the differences move along the diagonal (since the same permutation cannot map the same letter differently).

such that $x_{[0,n]}^{(n)} = y_{[0,n]}^{(n)}$ and $F^n(x^{(n)})_0 \neq F^n(y^{(n)})_0$, which is the fastest possible speed for a perturbation to advance for an elementary ROCA. We have seen that these “speed-of-light” perturbations exist if and only if F_Δ is not defined.

The claim itself is clear: The cellular automaton $(A^\mathbb{N}, F)$ is periodic if and only if there exists $p \in \mathbb{N}$ such that for every $s \in st_{\mathbb{Z}}(F)$ and $i \in \mathbb{N}, j \in \mathbb{Z}$ it holds that $s_{i,j} = s_{i,j+p}$. The same holds for $(A^\mathbb{N}, F_\Delta)$. The claim about the equicontinuity constant is also clear, for if there exists a word $u \in A^n$ which is not a blocking word for $(A^\mathbb{N}, F)$ then one finds a word $v \in A^n$ which is not a blocking word for $(A^\mathbb{N}, F_\Delta)$, i.e. for example $v_i = F^{n-i-1}(ua^\omega)_i$ for any $a \in A$. Similarly for the other direction. \square

Let us state a simple lemma here.

Lemma 4.3.3. *If $(A^\mathbb{N}, F)$ is an equicontinuous elementary ROCA, then $M \geq r(F_\Delta)$ where M is the equicontinuity constant.*

Proof. The radius of F_Δ immediately implies the existence of a word of length r which is not blocking (see Figure 4.8). \square

In [15, Theorem 3.5] a characterization of elementary ROCA’s with equicontinuity constant $M = 1$ is given. The following proposition gives a characterization of these cellular automata using the diagonal cellular automaton.

Proposition 4.3.4. *Let $(A^\mathbb{N}, F)$ be an elementary ROCA. If the diagonal cellular automaton has radius ≤ 1 then it is an elementary ROCA and can be diagonalized itself. The elementary cellular automaton $(A^\mathbb{N}, F)$ is periodic with equicontinuity constant $M = 1$ if and only if this diagonalization process can be repeated indefinitely.*

Proof. If the diagonalization process cannot be repeated indefinitely then either at some point the diagonal cellular automaton is not defined, and then, by Proposition 4.3.2, the original cellular automaton is not periodic, or the radius of some diagonal cellular automaton is larger than 1 and then by, Proposition 4.3.3, there are words of length 2 which are not blocking.

Suppose next that the diagonalization process can be repeated indefinitely. This means that for any $a, b \in A$ and $c \in A^{\mathbb{N}}$ we can determine $F^n(abc)_0$ from ab alone, since $F(abc)_0 = \pi_b(a)$, $F^2(abc)_0 = \pi_b^{\Delta_1}(\pi_b(a))$, $F^3(abc)_0 = \pi_b^{\Delta_2}(\pi_b^{\Delta_1}(\pi_b(a)))$, and so on, where $\pi_b^{\Delta_i}$ denotes the permutation defined by b for the i^{th} diagonal cellular automata. In other words, every word of length two is a blocking word. \square

Since there are only finitely many elementary ROCAs over a fixed alphabet the above procedure either ends or starts repeating itself. The following example shows that, unsurprisingly, F_{Δ} may be an elementary ROCA even if F is not periodic.

Example 4.3.5. Let $(A^{\mathbb{N}}, F)$ be the elementary ROCA defined by

$$\begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 0 & 4 & 1 & 5 \\ 2 & 3 & 0 & 4 & 1 & 5 \\ 3 & 2 & 1 & 5 & 0 & 4 \\ 3 & 2 & 1 & 5 & 0 & 4 \\ 3 & 2 & 1 & 5 & 0 & 4 \\ 3 & 2 & 1 & 5 & 0 & 4 \end{bmatrix}.$$

Then F_{Δ} is defined by

$$\begin{bmatrix} 3 & 2 & 1 & 5 & 0 & 4 \\ 3 & 2 & 1 & 5 & 0 & 4 \\ 2 & 3 & 0 & 4 & 1 & 5 \\ 3 & 2 & 1 & 5 & 0 & 4 \\ 2 & 3 & 0 & 4 & 1 & 5 \\ 3 & 2 & 1 & 5 & 0 & 4 \end{bmatrix} \quad \begin{array}{l} \text{which is an} \\ \text{elementary ROCA} \\ \text{so its inverse, } F_{\Delta}, \\ \text{is defined by:} \end{array} \quad \begin{bmatrix} 3 & 2 & 1 & 5 & 0 & 4 \\ 3 & 2 & 1 & 5 & 0 & 4 \\ 2 & 3 & 0 & 4 & 1 & 5 \\ 3 & 2 & 1 & 5 & 0 & 4 \\ 2 & 3 & 0 & 4 & 1 & 5 \\ 3 & 2 & 1 & 5 & 0 & 4 \end{bmatrix}.$$

However, $(A^{\mathbb{N}}, F)$ is not periodic. (It is easy to check that the two-sided configuration ${}^{\omega}(430)0^{\omega}$ is not periodic.)

Time to generalize: Let $(A^{\mathbb{N}}, F)$ be an elementary ROCA. Suppose that the diagonal cellular automaton $(A^{\mathbb{N}}, F_{\Delta})$ exists but has $r(F_{\Delta}) = r > 1$. Then we can group the cells into supercells of size r and this new cellular automaton is an elementary ROCA (the inverse of F_{Δ} has radius 1 so we do not need to worry about that when we do the grouping). Now we can diagonalize this grouped version of F_{Δ} . As long as the diagonal cellular automaton, after grouping, is again diagonalizable we can repeat this procedure. Let

$F_{\Delta(i)}$ denote the n^{th} diagonal cellular automaton (before grouping the cells). Let us define the *diagonal-sequence* of $(A^{\mathbb{N}}, F)$ as $\underline{\Delta}(A^{\mathbb{N}}, F) = (r(F_{\Delta(i)}))_{i \in \mathcal{I}}$ where $\mathcal{I} \in \{\emptyset\} \cup (\bigcup_{k \in \mathbb{N} \setminus \{0\}} \{[1, k]\}) \cup \{\mathbb{N} \setminus \{0\}\}$ depending on how many times the described procedure can be done starting from $(A^{\mathbb{N}}, F)$. Using this sequence we can characterize all periodic elementary ROCA's.

Proposition 4.3.6. *An elementary ROCA $(A^{\mathbb{N}}, F)$ is periodic if and only if $\underline{\Delta}(A^{\mathbb{N}}, F)$ is in $\mathbb{N}^{\mathbb{N} \setminus \{0\}}$ and has only finitely many non-1 entries.*

Proof. " \Leftarrow ": Let $(A^{\mathbb{N}}, F)$ be an elementary ROCA. According to Proposition 4.3.2, it is sufficient to show that if $\underline{\Delta}(A^{\mathbb{N}}, F) = (1, 1, 1, \dots)$ then $(A^{\mathbb{N}}, F)$ is periodic. But then the claim follows from Proposition 4.3.4.

" \Rightarrow ": For contradiction, suppose that $(A^{\mathbb{N}}, F)$ is a periodic elementary ROCA such that $\underline{\Delta}(A^{\mathbb{N}}, F)$ is not eventually $(1, 1, 1, \dots)$ (we do, however, assume it is infinite, since otherwise the first part of the proof of Proposition 4.3.2 gives the claim). Since $\underline{\Delta}(A^{\mathbb{N}}, F)$ is not eventually $(1, 1, 1, \dots)$ we have that the alphabet will get arbitrarily large as we keep repeating the diagonalization. But then, essentially by Proposition 4.3.3, we have that $(A^{\mathbb{N}}, F)$ is not periodic. □

In particular, the proposition above says that if $\underline{\Delta}(A^{\mathbb{N}}, F)$ is finite, then $(A^{\mathbb{N}}, F)$ is not periodic. Since finiteness of $\underline{\Delta}(A^{\mathbb{N}}, F)$ is clearly semi-decidable and so is periodicity of $(A^{\mathbb{N}}, F)$, it is worth pointing out that there are non-periodic elementary ROCA's with infinite diagonal-sequences. Example 4.3.7 below has $\underline{\Delta}(A^{\mathbb{N}}, F) = \underline{\Delta}(A^{\mathbb{N}}, F^{-1}) = (2, 2, 2, \dots)$ and $|A| = 7$; over an alphabet of size 6 one can find a non-periodic elementary ROCA $(A^{\mathbb{N}}, F)$ such that $\underline{\Delta}(A^{\mathbb{N}}, F)$ is infinite, but not such that both $\underline{\Delta}(A^{\mathbb{N}}, F)$ and $\underline{\Delta}(A^{\mathbb{N}}, F^{-1})$ are infinite, and over an alphabet of size 5 or less infiniteness of $\underline{\Delta}(A^{\mathbb{N}}, F)$ is equivalent to periodicity.

Example 4.3.7. In this example it is important to keep in mind, that in our drawings vertical-axis increases downwards, meaning that whenever we "go downwards" in a space-time diagram, the y -coordinate increases.

Let $A = \{0, 1, 2, 3, 4, \#_1, \#_2\}$. Let $(A^{\mathbb{N}}, F)$ be the elementary ROCA defined by

$$\begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \\ \pi_{\#_1} \\ \pi_{\#_2} \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 & 4 & 1 & \#_2 & \#_1 \\ 0 & 2 & 3 & 4 & 1 & \#_2 & \#_1 \\ 1 & 2 & 3 & 4 & 0 & \#_2 & \#_1 \\ 0 & 2 & 3 & 4 & 1 & \#_2 & \#_1 \\ 0 & 2 & 3 & 4 & 1 & \#_2 & \#_1 \\ 0 & 2 & 3 & 4 & 1 & \#_2 & \#_1 \\ 2 & 1 & 3 & 0 & 4 & \#_2 & \#_1 \end{bmatrix}.$$

We claim that $\underline{\Delta}(A^{\mathbb{N}}, F) = \underline{\Delta}(A^{\mathbb{N}}, F^{-1}) = (2, 2, 2, \dots)$. It is sufficient to prove that $\underline{\Delta}(A^{\mathbb{N}}, F) = (2, 2, 2, \dots)$ since renaming the alphabet A according to $2 \leftrightarrow 3, 1 \leftrightarrow 4, \#_1 \leftrightarrow \#_2$ actually gives the inverse rule, and thus the same proof works in both cases.

Let us denote by

$$C_n = \{(x, -y) \mid x \in \{0, 1, \dots, 2^{n-1} - 1\}, y = \sum_{i=0}^{\infty} \lfloor \frac{x}{2^i} \rfloor\} \subset \mathbb{Z}^2.$$

Assuming our claim is true, this is the form of one cell of the n^{th} diagonal cellular automata (see Figure 4.9). Now what we need to show is that for every n the following claims hold for the space-time diagrams of $(A^{\mathbb{N}}, F)$:

1. The contents of C_n and $C_n + (2^{n-1}, -2^n + 1)$ do not uniquely determine the content of $C_n + (0, 1)$, i.e. that $F_{\Delta(n)}$ is not an elementary ROCA.
2. The contents of $C_n, C_n + (2^{n-1}, -2^n + 1)$ and $C_n + (2^n, -2^{n+1} + 2)$ do determine uniquely the contents of $C_n + (0, 1)$, i.e. that $F_{\Delta(n)}$ is actually defined and has radius 2.

It is useful to notice that $C_n \cup (C_n + (2^{n-1}, -2^n + 1)) = C_{n+1}$

Proof of 1): To see this, consider the space-time diagram of $c = {}^\omega 0 \#_2 0^\omega$ (this is two-sided, but the conclusions will still be valid) where $c_1 = \#_2$. The right side of $\#_2$ is irrelevant for us; it will just remain 0^ω and affects the left side in no way since $\#_2$ blocks its influence. Now we prove that if we position C_{n+1} where $n \geq 1$ so that the rightmost cell is at $(0, 0)$ then we see only zeroes through the window of shape C_{n+1} , but if we lower the left half of C_{n+1} (which has the shape of C_n) with one cell, then it contains a 1. This proves 1. since the space-time diagram of ${}^\omega 0^\omega$ is all-zero, and thus there are two different ways all-zero configuration should be mapped by the diagonal cellular automaton. See Figure 4.9 for visual aid.

Let us denote by $s \in st_{\mathbb{Z}}(A^{\mathbb{Z}}, F)$ the space-time diagram generated by $c = {}^\omega 0 \#_2 0^\omega$, i.e. we have that $s(i, j) = F^j(c)_i$. First we note that the sequence $(s(-k, 2k))_{k \in \mathbb{N}}$ is $(0, 1, 1, 1, \dots)$ and that $(s(-k, 2k - 1))_{k \in \mathbb{N}} = (0, 0, 0, \dots)$. These can be seen as follows. By definition of F we have that $(s(1, i))_{i \in \mathbb{N}} = (\#_2 \#_1)^\omega$ and $\#_2$ does not appear in any cell $s(x, y)$ where $x < 1$. Again by definition $(s(0, i))_{i \in \mathbb{N}} = (023)^\omega$. Now for the rest of the columns it is sufficient to observe from the local rule that $\pi_0(0) = 0, \pi_2(0) = 1, \pi_1(0) = 0$, and $\pi_x(1) = 2$ for all $x \neq \#_2$; using these one sees by starting from cell $(0, 0)$ and repeatedly moving one step left and two steps down that “stairs” of 12’s appear where to the left of the stairs everything is zero (see Figure 4.9 again). Now position C_{n+1} so that rightmost cell is at $(0, 0)$, i.e. consider the set $D = C_{n+1} - (2^n - 1, -2^{n+1} + n + 2)$. Since

$$\sum_{i=0}^{\infty} \lfloor \frac{x}{2^i} \rfloor < \sum_{i=0}^{\infty} \frac{x}{2^i} = 2x$$

0	0	0	0	0	0	0	0	0	0	0	0	0	# ₁	0		
0	0	0	0	0	0	0	0	0	0	0	0	0	0	# ₂	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	2	# ₁	0	
0	0	0	0	0	0	0	0	0	0	0	0	1	3	# ₂	0	
0	0	0	0	0	0	0	0	0	0	0	0	2	0	# ₁	0	
0	0	0	0	0	0	0	0	0	0	0	0	1	3	2	# ₂	0
0	0	0	0	0	0	0	0	0	0	0	2	4	3	# ₁	0	
0	0	0	0	0	0	0	0	0	1	3	1	0	# ₂	0	0	
0	0	0	0	0	0	0	0	0	0	2	4	2	2	# ₁	0	
0	0	0	0	0	0	0	0	0	1	3	0	3	3	# ₂	0	
0	0	0	0	0	0	0	0	0	2	4	0	4	0	# ₁	0	
0	0	0	0	0	0	0	1	3	1	0	1	0	# ₂	0	0	
0	0	0	0	0	0	0	2	4	2	0	2	2	# ₁	0	0	
0	0	0	0	0	1	3	0	3	1	3	3	# ₂	0	0	0	
0	0	0	0	0	2	4	0	4	2	4	0	# ₁	0	0	0	
0	0	0	0	1	3	1	0	0	3	1	0	# ₂	0	0	0	
0	0	0	0	2	4	2	0	0	4	2	2	# ₁	0	0	0	
0	0	0	1	3	0	3	0	0	0	3	3	# ₂	0	0	0	
0	0	0	2	4	0	4	0	0	0	4	0	# ₁	0	0	0	
0	0	1	3	1	0	1	0	0	0	1	0	# ₂	0	0	0	
0	0	2	4	2	0	2	0	0	0	2	2	# ₁	0	0	0	

Figure 4.9: Colors highlight the supercells of $F_{\Delta(3)}$. Four cells form one supercell of $F_{\Delta(3)}$. Taking only two supercells does not uniquely determine the next time step, as is illustrated by the supercell highlighted with blue; all-zero neighborhood would have to map both to 0001 and to 0000. With red is highlighted that the issue disappears when we take three supercells, though this, of course, only shows that the issue disappears in this instance; Figure 4.10 aims to explain why three supercells is, indeed, enough.

we see that s_D is an all-zero word. Now consider the rightmost cell of the left half of D , i.e. the cell whose first component is -2^{n-1} . If we lower this cell by one, its second component becomes

$$-\sum_{i=0}^{\infty} \lfloor \frac{2^{n-1}}{2^i} \rfloor + (2^{n+1} - n - 2) + 1 = 2^n,$$

i.e. it is the cell $(-2^{n-1}, 2^n)$, and we have seen that $s(-2^{n-1}, 2^n) = 1$. This concludes the proof of 1).

Proof of 2): For this proof, viewing Figure 4.10 may be useful.

First we prove a small lemma: Let $(B^{\mathbb{Z}}, G)$ be a cellular automaton defined by $B = \{0, 1, 2, 3, 4\}$ and $G = F|_{\{0,1,2,3,4\}^{\mathbb{N}}}$. It is easy to check that

$(B^{\mathbb{Z}}, G)$ is an elementary ROCA and that G_{Δ} is also an elementary ROCA. We claim that from this it follows that in the space-time diagrams of $(B^{\mathbb{Z}}, G)$ the contents of C_n and $C_n + (2^{n-1}, -2^n + 1)$ uniquely determine the content of $C_n + (0, 1)$. The proof goes by induction. When $n = 1$ the claim holds as G_{Δ} is an elementary ROCA. Now assume that the claim holds for n . We claim that the content of $C_{n+1} \cup (C_{n+1} + (2^n, -2^{n+1} + 1))$ uniquely determines the content of $C_{n+1} + (0, 1)$. Now $C_{n+1} = C_n \cup (C_n + (2^{n-1}, -2^n + 1))$ as was remarked above. Then, by induction hypothesis, the content of $C_{n+1} + (2^n, -2^{n+1} + 1)$ determines the content of $C_n + (2^n, -2^{n+1} + 2)$. Now we know the contents of $C_{n+1} \cup (C_n + (2^n, -2^{n+1} + 2))$ and according to the induction hypothesis this determines the content of $C_{n+1} + (0, 1)$, which concludes the proof of this lemma.

Now let

$$D_n = C_n \cup (C_n + (2^{n-1}, -2^n + 1)) \cup (C_n + (2^n, -2^{n+1} + 2)),$$

its width is $3 \cdot 2^{n-1}$. The claim is that the content of D_n determines the content of $C_n + (0, 1)$ for space-time diagrams $st_{\mathbb{Z}}(A^{\mathbb{Z}}, F)$. Let $s \in st_{\mathbb{Z}}(A^{\mathbb{Z}}, F)$.

Let us consider three cases: 1) There is a cell $(x, y) \in D_n$ such that $x \in [2^{n-1}, 3 \cdot 2^{n-1})$ and $s(x, y) \in \{\#_1, \#_2\}$. 2) For all $(x, y) \in D_n$ it holds that $s(x, y) \notin \{\#_1, \#_2\}$. 3) There is a cell $(x, y) \in D_n$ such that $x \in [0, 2^{n-1})$ and $s(x, y) \in \{\#_1, \#_2\}$. The order of these cases may seem odd, but after proving the claim for 1) and 2), case 3) will be obvious.

1) Now suppose that s_{D_n} contains either $\#_1$ or $\#_2$ in, say, the cell $(x, y) \in D_n$ where $x \in [2^{n-1}, 3 \cdot 2^{n-1})$. Then we know the entire column $(s(x, y'))_{y' \in \mathbb{Z}}$ since $\#_1$ always maps to $\#_2$ and vice versa. And so we will actually know the entire stripe $(s(x', y'))_{x' \in \{0, 1, \dots, x\}, y' \in \mathbb{Z}}$. Thus the claim holds in this case.

2) Suppose s_{D_n} does not contain $\#_1$ or $\#_2$. Consider the triangular shape $T = \{(x, y) \mid x \in [0, 3 \cdot 2^{n-1}), y \in (-3 \cdot 2^{n-1} + x, 3 \cdot 2^{n-1} - x)\}$. Notice that $s_{\{(x, 0) \mid x \in [0, 3 \cdot 2^{n-1})\}}$ determines s_T . Notice also that the shape $(C_n \cup (C_n + (2^{n-1}, -2^n + 1)) \cup (C_n + (0, 1))) + (0, 3 \cdot 2^{n-1} - 2)$ is entirely within T . Since s_T by our assumption cannot contain any occurrences of symbols $\#_1$ or $\#_2$ we know that the same pattern s_T appears also in $st_{\mathbb{Z}}(B^{\mathbb{N}}, G)$. But above we saw that in $st_{\mathbb{Z}}(B^{\mathbb{N}}, G)$ the content of $C_n \cup (C_n + (2^{n-1}, -2^n + 1))$ determines the content of $C_n + (0, 1)$ which proves the claim.

3) At this point this should be obvious.

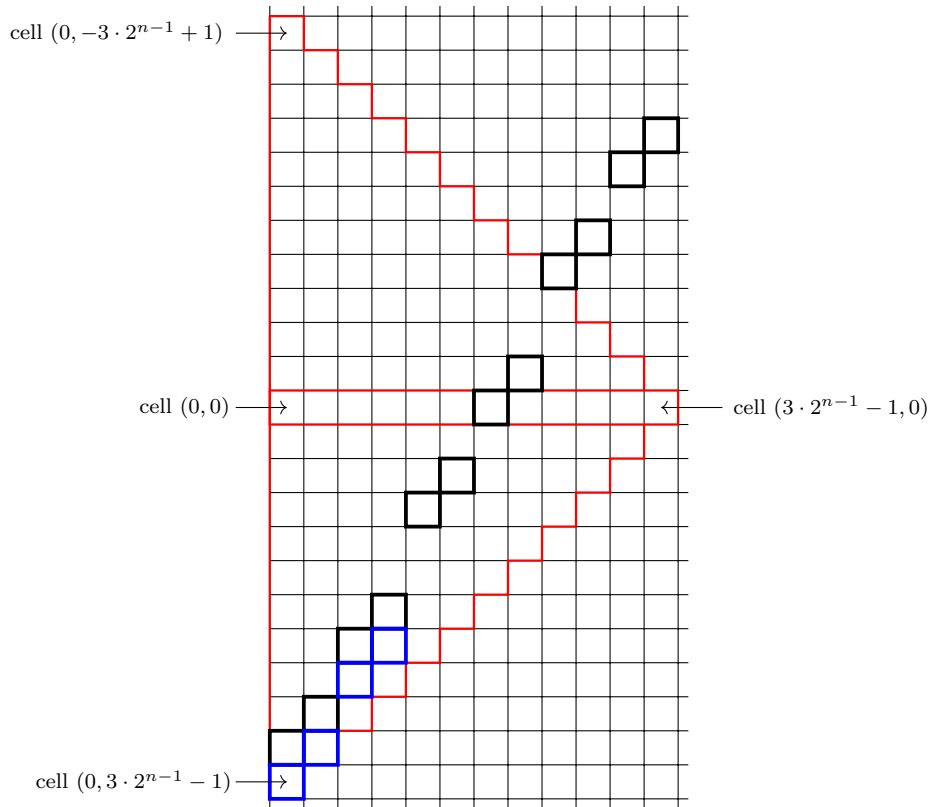


Figure 4.10: This illustrates the situation for $n = 3$. If any of the cells highlighted with black contain $\#_1$ or $\#_2$ then the entire space-time diagram to the left of this cell is determined by the contents of the black cells, since $\#_1$ and $\#_2$ are blocking words. If the black cells contain neither $\#_1$'s nor $\#_2$'s then the entire red triangle contains no $\#_1$'s or $\#_2$'s and so the red triangle appears in a space-time diagram for the restricted cellular automaton $(B^{\mathbb{N}}, G)$ and thus, by the lemma proved for the restriction of $(B^{\mathbb{N}}, G)$, we know that the contents of black cells within the red triangle determine the contents of the blue cells.

Chapter 5

One-Sided Ultimate Expansivity and One-Sided Pseudo-Orbit Tracing Property

In this chapter we further discuss expansivity and pseudo-orbit tracing property. In Chapter 2 we defined expansivity as a strong sensitivity to initial conditions, but so that expansivity was defined only for reversible cellular automata and positive expansivity only for surjective cellular automata. In this chapter we consider a more general notion of expansivity which we call ultimate expansivity (but which some, for example Masakazu Nasu, have called just expansivity; the terminology regarding expansivity is not quite standardized). Further, we define one-sided variants of ultimate expansivity, which correspond to perturbations spreading possibly only to one direction. All of the expansivity concepts can be considered as certain deterministic directions in the space-time diagrams.

First main result of this chapter says that ultimately right-expansive cellular automaton over a mixing sofic shift is chain-mixing. This improves Mike Boyle's result [7, Corollary 4.3] which states that expansive cellular automata (in the sense of Chapter 2) over a mixing sofic shift are chain-mixing. Then we use this to prove the second main result which states that ultimately right-expansive surjective cellular automata with left pseudo-orbit tracing property over a transitive SFT have the pseudo-orbit tracing property. This result has been proved before by Masakazu Nasu [53, Theorem 6.3] with the additional assumption of chain-recurrence. Nasu proves his results by using his textile system theory, which is an alternative approach to two-dimensional symbolic dynamics. Note that one can get rid of the chain-recurrence assumption in the following way: Nasu remarks that "It

is an open problem whether an onto endomorphism ϕ of a mixing SFT has ϕ -periodic points dense [...]. If the answer is affirmative, then the chain recurrence condition on ϕ in Theorem 6.3 and that on $\tilde{\phi}$ in Corollary 6.4 can be removed.” [53, pp. 185], but there is no need to solve this longstanding open problem as denseness of recurrent points implies chain-recurrence, and this follows from the Poincaré recurrence theorem. Alternatively, the first half of the proof of our Theorem 5.1.12 proves exactly what Nasu uses chain-recurrence to prove.

After we have proved that left pseudo-orbit tracing property and ultimate right-expansivity together imply soficness of the trace subshifts, we illustrate with examples that neither ultimate right-expansivity nor left pseudo-orbit tracing property alone is enough to guarantee soficness of the trace subshifts.

5.1 Ultimate One-Sided Expansivity

In Chapter 2 we introduced the notions of expansivity and positive expansivity. In Proposition 2.4.23 we saw that for cellular automata these properties can equivalently be defined as determinism in the horizontal direction of the space-time diagrams. For two-sided cellular automata the usual definition of expansivity leads to determinism both to the left and to the right. Next we define one-sided expansivity where perturbations need to propagate only to one direction; geometrically this corresponds to having determinism only in one direction for the space-time diagrams. Let us simultaneously also generalize our definition to cover all cellular automata (not just reversible for expansive, and surjective for positively expansive); this is achieved by considering only two-way infinite orbits.

A cellular automaton (X, F) is *ultimately right-expansive* if there exists $\varepsilon > 0$ such that for all space-time diagrams $(c^{(i)})_{i \in \mathbb{Z}}, (e^{(i)})_{i \in \mathbb{Z}} \in st_{\mathbb{Z}}(F)$ it holds that

$$(\exists i > 0 : c_i^{(0)} \neq e_i^{(0)}) \implies (\exists n \in \mathbb{Z} : d(F^n(c), F^n(e)) > \varepsilon). \quad (5.1)$$

Ultimately left-expansive is defined by replacing “ $\exists i > 0$ ” with “ $\exists i < 0$ ”. A cellular automaton is *ultimately expansive* if it is both ultimately left- and ultimately right-expansive. Ultimate expansivity can be generalized for arbitrary dynamical systems: Dynamical system (X, ϕ) is *ultimately expansive* if there exists $\varepsilon > 0$ such that for all orbits $(x^{(i)})_{i \in \mathbb{Z}}$ and $(y^{(i)})_{i \in \mathbb{Z}}$ of (X, ϕ) it holds that

$$x^{(0)} \neq y^{(0)} \implies \exists i \in \mathbb{Z} : \text{dist}(x^{(i)}, y^{(i)}) > \varepsilon.$$

By definition, positively expansive cellular automata and expansive cellular automata are ultimately expansive. Next examples show that ultimate expansivity covers cases which expansivity and positive expansivity do not.

Example 5.1.1. A cellular automaton $(A^{\mathbb{Z}}, F)$ is *nilpotent* if there exists $q \in A$ and $n \in \mathbb{N}$ such that for every $c \in A^{\mathbb{Z}}$ we have that $F^n(c) = {}^\omega q^\omega$. Then $st_{\mathbb{Z}}(F)$ is a singleton and it follows that $(A^{\mathbb{Z}}, F)$ is ultimately expansive.

Notice that since nilpotency is undecidable (Jarkko Kari [34]) it easily follows that so is ultimate expansivity. Though more interesting question is whether ultimate expansivity is undecidable for surjective cellular automata.

Since ultimate expansivity only considers two-way infinite orbits, it considers only configurations in the limit set $\Lambda(X, F) = \bigcap_{i \in \mathbb{N}} F^i(X)$ of the cellular automaton. As all cellular automata are surjective over their limit sets, it would be reasonable to define ultimate expansivity only for surjective cellular automata. In this sense also the above example does not diverge from the usual definition, since it describes a cellular automaton which is expansive (and positively expansive, too) when restricted to the limit set. Notice also that there cannot be a reversible cellular automaton that would be ultimately expansive but not expansive, as ultimate expansivity and reversibility together are equivalent to expansivity. However, there are surjective ultimately expansive cellular automata which are neither expansive nor positively expansive, as the following examples illustrate.

Example 5.1.2. Let $A = \{0, 1\}$, $\sigma : A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ be the shift map and $\mathcal{X}_{\mathbb{N}} : A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ be the one-sided XOR -cellular automaton, that is, the cellular automaton defined by $\mathcal{X}_{\mathbb{N},loc} : A^{[0,1]} \rightarrow A$, $\mathcal{X}_{\mathbb{N},loc}(ab) = a \oplus b$, where \oplus denotes addition modulo 2. Consider the cellular automaton $(A^{\mathbb{Z}}, \sigma \mathcal{X}_{\mathbb{N}})$. This is not reversible and thus not expansive. This is also not positively expansive since all $c, e \in A^{\mathbb{Z}}$ such that $c_i = e_i$ for all $i > -n$ have that $\text{dist}(F^k(c), F^k(e)) \leq \frac{1}{2^n}$ for all $k \in \mathbb{N}$. This cellular automaton is, however, ultimately expansive: Take any $t \in \tau_{\mathbb{Z},2}(\sigma \mathcal{X})$ and let $s \in st_{\mathbb{Z}}(\sigma \mathcal{X})$ be a space-time diagram such that for all $i \in \mathbb{Z}$ we have that $t_i = s(0, i)s(1, i)$. Then this s is actually uniquely determined by t since $s(i-1, j) = s(i, j-1) \oplus s(i+1, j-1)$ meaning t determines the space-time diagram to the left, and also $s(i+1, j) = s(i-1, j+1) \oplus s(i, j)$ so that t also determines the space-time diagram to the right. (Any positively right-expansive cellular automaton can be composed with a suitable power of the shift map to yield an ultimately expansive cellular automaton which is neither expansive nor positively expansive.)

Example 5.1.3. Let $A = \{0, 1\}$, $\sigma : A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ be the shift map and $\mathcal{X}_{\mathbb{Z}} : A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ be the two-sided XOR -cellular automaton, that is, the cellular automaton defined by $\mathcal{X}_{\mathbb{Z},loc} : A^{[-1,1]} \rightarrow A$, $\mathcal{X}_{\mathbb{Z},loc}(abc) = a \oplus c$. The shift map σ is expansive, and so also ultimately expansive. The XOR -cellular automaton cannot be expansive as it is not reversible. However, it is positively expansive, and so also ultimately expansive. Consider the direct product of these, that is the cellular automaton $\sigma \times \mathcal{X}_{\mathbb{Z}} : A^{\mathbb{Z}} \times A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}} \times A^{\mathbb{Z}}$

defined by $(\sigma \times \mathcal{X}_{\mathbb{Z}})(c, e) = (\sigma(c), \mathcal{X}_{\mathbb{Z}}(e))$. This is neither expansive (not even reversible) nor positively expansive (since σ is not). However, $\sigma \times \mathcal{X}_{\mathbb{Z}}$ is ultimately expansive.

The following proposition is a one-sided variant of Proposition 2.4.23, i.e. it states that ultimate right-expansivity means determinism from left to right in the space-time diagrams.

Proposition 5.1.4. *A cellular automaton (X, F) is ultimately right-expansive if and only if there exists $m, n \in \mathbb{N}$ such that for all space-time diagrams $(c^{(j)})_{j \in \mathbb{Z}}$ and $(e^{(j)})_{j \in \mathbb{Z}}$ the following holds*

$$(\forall j \in \{0, \dots, 2n\} : c_{[0, m]}^{(j)} = e_{[0, m]}^{(j)}) \implies c_m^{(n)} = e_m^{(n)}. \quad (5.2)$$

It is worth being slightly careful when discussing expansivity in context where cellular automata may be one- or two-sided. In the following we denote by $X_{\mathbb{N}}$ and $X_{\mathbb{Z}}$ the one- and two-sided subshifts which are otherwise essentially the same. We require that $\sigma(X_{\mathbb{N}}) = X_{\mathbb{N}}$ in order to avoid the ambiguity that examples such as $X_{\mathbb{N}} = \{10^\omega, 0^\omega\}$ cause.

Proposition 5.1.5. *Let F_{loc} be a memoryless local rule and X a subshift such that $\sigma(X_{\mathbb{N}}) = X_{\mathbb{N}}$. If $(X_{\mathbb{N}}, F)$ is expansive (ultimately expansive), then $(X_{\mathbb{Z}}, F)$ is right-expansive (ultimately right-expansive) but usually not expansive (ultimately expansive). If $(X_{\mathbb{Z}}, F)$ is ultimately expansive, then $(X_{\mathbb{N}}, F)$ is ultimately expansive.*

Notice that if $(A^{\mathbb{N}}, F)$ is expansive and $|A| > 1$ then $(A^{\mathbb{Z}}, F)$ is right-expansive but never expansive (though the cellular automaton $(\{0^\omega\}, F)$ is expansive and so is $(\{\omega 0^\omega\}, F)$). In [2] Luigi Acerbi, Alberto Dennunzio and Enrico Formenti consider how certain properties are affected when comparing memoryless two-sided cellular automaton with the one-sided cellular automaton defined by the same local rule.

Let (X, F) be an ultimately right-expansive cellular automaton. Then Proposition 5.1.4 says that for all large enough $m \in \mathbb{N}$ we can define a cellular automaton $(\tau_m(F), \vec{F}_m)$ such that for every $t \in \tau_m(F)$ we have that $\vec{F}_m(t) \in \tau_m(F)$ is the unique configuration such that $\pi_{[1, m]}(t) = \pi_{[0, m-1]}(\vec{F}_m(t))$ and the last column of $\vec{F}_m(t)$ is the column defined by (5.2) (Figure 5.1). Then $\{(\pi_0(\vec{F}_m^i(t)))_{i \in \mathbb{N}} \mid t \in \tau_m(F)\}$ is the set of right halves of $st_{\mathbb{Z}}(F)$.

Notice that if F is surjective and ultimately expansive then (X, F) is a factor of $(\tau_m(F), \sigma)$. On the other hand, let $\psi : (A^m)^{\mathbb{Z}} \rightarrow (A^m)^{\mathbb{N}}$ be the map defined by $\psi(\dots c_{-1}c_0c_1\dots) = c_0c_1\dots$. Then $(\psi(\tau_m(F)), \sigma)$ is a factor of (X, F) . If (X, F) is expansive, it is conjugate to $(\tau_m(F), \sigma)$. If (X, F) is positively expansive, it is conjugate to $(\psi(\tau_m(F)), \sigma)$.

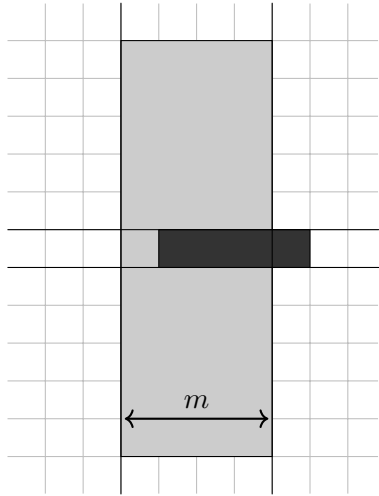


Figure 5.1: An ultimately right-expansive cellular automaton defines a cellular automaton \vec{F}_m which draws the (right halves) of the space-time diagrams. The figure illustrates how \vec{F}_m is defined; assuming that the grid has a valid space-time diagram of F , then \vec{F}_m maps the pattern in the light gray rectangle to the pattern in the dark gray rectangle.

5.1.1 One-Sided Pseudo-Orbit Tracing Property

Let us recall some definitions from Chapter 2. Let (X, F) be a cellular automaton with radius r , where $X \subseteq A^{\mathbb{Z}}$. We defined (labeled multi-)graphs $\mathcal{G}_m(X, F) = (V_m, E_m, \lambda_m)$ by

- The set of vertices is $V_m = \mathcal{L}_m(X)$.
- For every $u \in V_m$ and $x, y \in \mathcal{L}_r(X)$ such that $xuy \in \mathcal{L}_{n+2r}(X)$ there is an edge $(u, f(xuy))$ whose label is $\lambda_m((u, f(xuy))) = xy$. We express this compactly by saying that $u \xrightarrow{xy} f(xuy)$ is in E_m .

The set of infinite paths of this graph were denoted by $\mathcal{PO}_m(X, F)$, and these correspond to the pseudo-orbits of (X, F) . Pseudo-traces of (X, F) were defined as ${}_i\Sigma_j^{(m)}(X, F) = \text{pr}_{[i, i+m)}(\mathcal{PO}_n(X, F))$ where $n = i + j + m$. As usual, we may drop “ X ” or “ (X, F) ” entirely from the notations, if it does not cause confusion. We also defined ${}_\infty\Sigma_j^{(m)} = \bigcap_{k \in \mathbb{N}} {}_k\Sigma_j^{(m)}$ which is non-empty by the finite intersection property. In similar fashion we defined subshifts ${}_i\Sigma_\infty^{(m)} = \bigcap_{k \in \mathbb{N}} {}_i\Sigma_k^{(m)}$ and ${}_\infty\Sigma_\infty^{(m)} = \bigcap_{k \in \mathbb{N}} {}_k\Sigma_k^{(m)}$.

Let us now define one-sided variant of POTP.

According to Propositions 2.4.18 and 2.4.19 we have that (X, F) has POTP if and only if for every $m \in \mathbb{N} \setminus \{0\}$ there exists $n \in \mathbb{N}$ such that ${}_n\Sigma_n^{(m)} = {}_\infty\Sigma_\infty^{(m)}$. This leads to a natural definition of one-sided POTP: We

say that F has the *left pseudo-orbit tracing property (left-POTP)* if for every m there exists i, j

$${}_i\Sigma_j^{(m)} = {}_\infty\Sigma_j^{(m)}.$$

The *right pseudo-orbit tracing property (right-POTP)* is defined analogously. We see that for cellular automata over SFT's this definition behaves as one-sided variants are expected to:

Proposition 5.1.6. *Let X be an SFT and (X, F) a cellular automaton. Then (X, F) has POTP if and only if (X, F) has left- and right-POTP.*

Proof. “ \Rightarrow ”: Immediate from Propositions 2.4.18 and 2.4.19.

“ \Leftarrow ”: It is enough to show that for large enough m it holds that there exists n such that $\tau_m(F) = {}_n\Sigma_n^{(m)}$. Let l be large enough so that there exists a set of forbidden words $S \subseteq A^l$ such that $X = X_S$ and let $m \geq \max\{l, 2r\}$ where $r = r(F)$. Then left- and right-POTP say that we have n such that ${}_\infty\Sigma_n^{(m)} = {}_n\Sigma_n^{(m)} = {}_n\Sigma_\infty^{(m)}$. Now consider $t \in {}_n\Sigma_n^{(m)}$. It can be extended infinitely to the left without introducing violations of the local rule of F , and also to the right. If we take any valid extension to the left and glue it together with any valid extension to the right, we have a valid space-time diagram since m was chosen large enough so that the patterns checking the validity of the space-time diagram cannot see both sides of the stripe of width m . Thus ${}_n\Sigma_n^{(m)} = \tau_m(F)$ and by Proposition 2.4.19 we are done. \square

The following proposition shows that memorylessness is a special case of left-POTP.

Lemma 5.1.7. *Let X be an SFT and (X, F) be a memoryless cellular automaton. Then there exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$ and for all $n \in \mathbb{N}$ it holds that ${}_0\Sigma_m^{(n)} = {}_\infty\Sigma_m^{(n)}$.*

Proof. Let X be an SFT and $l \in \mathbb{N}$ such that there exists $S \subseteq A^l$ such that $X = X_S$. Let (X, F) be a cellular automaton with neighborhood $[0, r]$ where $r \in \mathbb{N}$. Let $m \geq m_0 = \max\{l, r\}$. Take any configuration $t \in \mathcal{PO}_m(F)$ and a sequence $(a_i)_{i \in \mathbb{Z}} \in A^{\mathbb{Z}}$ such that $a_i t_i \in \mathcal{L}_{m+1}(X)$ for all $i \in \mathbb{Z}$. There is no reason why $(a_i t_i)_{i \in \mathbb{Z}}$ should be in $\mathcal{PO}_{m+1}(F)$. However, we can construct a valid extension as follows: For every $j \in \mathbb{N}$ define a new sequence $(a_i^{(j)})_{i \in \mathbb{Z}}$ by setting $a_i^{(j)} = a_i$ for $i < -j$ and the rest of the sequence is defined inductively by

$$a_{-j}^{(j)} = a_{-j} \text{ and } a_{k+1}^{(j)} = f(a_k^{(j)} t_k)_0 \text{ for } k \geq -j.$$

Notice that by the choice of m we have that if $x, y \in A, u, v \in A^m, w \in A^r$ such that $xu \in \mathcal{L}_{m+1}(X), uw \in \mathcal{L}_{m+r}(X)$ and $F(xu)_0 = y, F(uw) = v$, then $xuw \in \mathcal{L}_{m+r+1}(X)$ (since $m \geq l$) and $F(xuw) = yv$ (since $m \geq$

r). Thus the configuration $(a_i^{(j)} t_i)_{i \in \mathbb{Z}}$ looks like a valid configuration of $\mathcal{PO}_{m+1}(F)$ for all $i \geq -j$. By compactness the sequence $((a_i^{(j)})_{i \in \mathbb{Z}})_{j \in \mathbb{N}}$ has a converging subsequence, say $((a_i^{(j)})_{i \in \mathbb{Z}})_{j \in \mathcal{I}}$ where $\mathcal{I} \subseteq \mathbb{N}$ is an infinite subset. Let $(b_i)_{i \in \mathbb{Z}}$ be the limit of this subsequence. Now the configuration $(b_i t_i)_{i \in \mathbb{Z}} \in \mathcal{PO}_{m+1}(F)$ shows that t can be extended to the left with one column. We can repeat the process and extend t to the left as much as we will. This shows that for every $n \in \mathbb{N}$ we have that ${}_0\Sigma_m^{(n)} = {}_\infty\Sigma_m^{(n)}$. \square

We get the following corollaries.

Corollary 5.1.8. *Let X be an SFT and (X, F) be a memoryless cellular automaton. Then (X, F) has left-POTP.*

Proof. Immediate from Lemma 5.1.7. \square

Corollary 5.1.9. *Let X be an SFT. If (X, F) is memoryless, then $\tau_m(F) = \bigcap_{i \in \mathbb{N}} {}_0\Sigma_i^{(m)}$.*

Proof. From Lemma 5.1.7 it follows that for large enough n we have ${}_0\Sigma_n^{(m)} = {}_n\Sigma_n^{(m)}$, and so the claim follows from Proposition 2.4.18. \square

When discussing expansivity one needs to be careful whether the underlying space is assumed one- or two-sided. For POTP one does not need to worry as much.

Proposition 5.1.10. *Let F_{loc} be a memoryless local rule and X a subshift such that $\sigma(X_{\mathbb{N}}) = X_{\mathbb{N}}$. Then $(X_{\mathbb{N}}, F)$ has POTP if and only if $(X_{\mathbb{Z}}, F)$ has POTP.*

Proof. This is clear: Since $\sigma(X_{\mathbb{N}}) = X_{\mathbb{N}}$ we have that $\tau_m(X_{\mathbb{N}}, F) = \tau_m(X_{\mathbb{Z}}, F)$. \square

5.1.2 Right-Expansivity Implies Chain-Mixingness

Let (X, F) be a cellular automaton. We call a set $U \subseteq X$ *inward* if $F(U) \subseteq U$.

Next we want to show that an ultimately right-expansive surjective cellular automaton (X, F) over a mixing sofic shift X is chain-mixing. Proposition 2.4.21 states that chain-mixing is equivalent to chain-transitivity in this setting, thus it is sufficient to prove chain-transitivity. This means that we want to show that the graphs $\mathcal{G}_n(F)$ are transitive (Proposition 2.4.16). We are one auxiliary result away from having all the necessary components for our proof; we still need to show that for (X, F) it holds that if a clopen set U is inward, then $F(U) = U = F^{-1}(U)$. This states that connected components of $\mathcal{G}_n(F)$ are strongly connected. Using this our proof

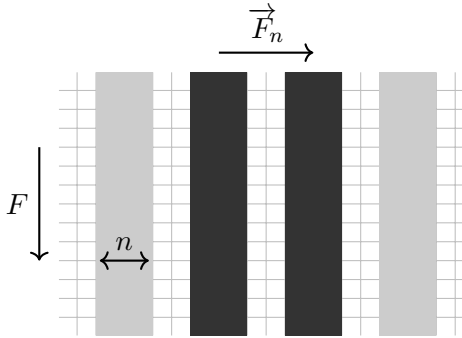


Figure 5.2: If F is not chain-transitive, then for some n graph $\mathcal{G}_n(F)$ is not strongly connected, and then, by Proposition 5.1.11, not even connected. Then F draws stripes (illustrated with gray and black), since columns of width n in $st_{\mathbb{Z}}(F)$ must be configurations of the shift defined by $\mathcal{G}_n(F)$. Since X is a mixing sofic shift we can draw complex configurations with these vertical stripes. This leads to a contradiction, since \vec{F}_n could now draw a stripe shift which is more complex than Proposition 3.2.4 allows.

goes, intuitively, as follows (see Figure 5.2). Suppose $\mathcal{G}_n(F)$ is not transitive. Then there are two disjoint strongly connected components in $\mathcal{G}_n(F)$. This means that F draws vertical stripes. Since X is mixing sofic shift we can draw quite complex configurations with these vertical stripes. But this contradicts our main result about stripe shifts (Proposition 3.2.4), since now right-expansivity would define a cellular automaton that would be able to draw complex horizontal stripes (the vertical stripes of F are horizontal stripes for \vec{F}_n).

In [32] we proved these results for cellular automata over full shifts, and then the required property that $F(U) = U = F^{-1}(U)$ for inward clopen sets follows easily from the classical balancedness property proved by Akira Maruoka and Masayuki Kimura [45]. In [33] we generalized the results over mixing sofic shifts, and needed a replacement for this result. This more general version hinges on results from ergodic theory.

Proposition 5.1.11. *Let X be a mixing sofic shift. Then (X, F) is surjective if and only if for every clopen set $U \subseteq X$ it holds that if U is inward then $F(U) = U = F^{-1}(U)$.*

Proof. “ \Leftarrow ” The whole space X is itself clopen, and clearly $F(X) \subseteq X$. Then $F(X) = X$, i.e. F is surjective.

“ \Rightarrow ” This direction requires some ergodic theory. As ergodic theory plays no further role, we will not go into details. The following outlines known results that can be used to conclude the claim:

According to Benjamin Weiss [61], X has a unique σ -invariant measure of maximal entropy μ (known as the *Parry measure*, originally presented in [55] for SFT's by Bill Parry). From the definition one sees that $\mu(U) > 0$ for every non-empty open set $U \subseteq X$.

The *push-forward* measure of μ under F is defined by setting $F(\mu)(U) = \mu(F^{-1}(U))$ for any clopen set $U \subseteq X$. According to [46, Theorem 3.3.] every σ -invariant measure of maximal entropy is the push-forward measure under F of some σ -invariant measure of maximal entropy. Since μ is the unique σ -invariant measure of maximal entropy, we have that $F(\mu) = \mu$.

Now suppose that $F(U) \subseteq U$ for some clopen set $U \subseteq X$. Then $U \subseteq F^{-1}(U)$. Now we have that $\mu(F^{-1}(U) \setminus U) = \mu(F^{-1}(U)) - \mu(U) = 0$, and since $F^{-1}(U) \setminus U$ is open we get that $F^{-1}(U) \setminus U = \emptyset$. So we have that $F^{-1}(U) \subseteq U$, and thus $F^{-1}(U) = U$. Since F is surjective we also have that $F(U) = U$. □

Following result is a generalization of Mike Boyle's result [7, Corollary 4.3] where the cellular automaton is assumed to be expansive rather than ultimately right-expansive.

Theorem 5.1.12. *Let X be a mixing sofic shift. A surjective ultimately right-expansive cellular automaton (X, F) is chain-mixing.*

Proof. Let us show that (X, F) is chain-transitive. For contradiction, suppose that (X, F) is not chain-transitive. Then, by Proposition 2.4.16, there exists m such that $\mathcal{G}_m(F)$ is not strongly connected. We may assume that m is large enough so that we have a cellular automaton $(\tau_m(F), \vec{F}_m)$ as defined by Proposition 5.1.4. Let $\mathcal{G}_m(F)_1, \dots, \mathcal{G}_m(F)_k$ be the strongly connected components of $\mathcal{G}_m(F)$. There has to exist a strongly connected component which has no arrows to other strongly connected components (if every connected component could be left, there would have to exist a cycle which would visit two different connected components, which is a contradiction); we may assume that $\mathcal{G}_m(F)_1$ is such. Let $V_1 \subseteq \mathcal{L}_m(X)$ be the vertex set of $\mathcal{G}_m(F)_1$ and $V_1^c = \mathcal{L}_m(X) \setminus V_1$. We denote with V_1 and V_1^c also the clopen sets of X which the vertex sets naturally define. Since V_1 has no arrows pointing outwards, we have that $F(V_1) \subseteq V_1$. Then according to Proposition 5.1.11 we have that $F(V_1) = V_1$ and $F^{-1}(V_1) = V_1$. According to $F^{-1}(V_1) = V_1$ there are no arrows pointing from V_1^c to V_1 and so $F(V_1^c) \subseteq V_1^c$. Again, by Proposition 5.1.11, we have that $F(V_1^c) = V_1^c$.

Define a partition P of A^m as follows:

$$\begin{aligned} P_1 &= V_1 \\ P_0 &= A^m \setminus V_1, \end{aligned}$$

and let $\iota : A^m \rightarrow \{0, 1\}$ be the projection defined by this partition. Of course $V_1^c \subseteq P_0$. As we saw above $\mathcal{PO}_m(F) \subseteq P_0^{\mathbb{Z}} \cup P_1^{\mathbb{Z}}$. Then we also must have that $\tau_m(F) \subseteq P_0^{\mathbb{Z}} \cup P_1^{\mathbb{Z}}$. Take one vertex $u \in V_1 = P_1$ and one $v \in V_1^c \subseteq P_0$. Since X is a mixing sofic shift, we have $K \in \mathbb{N}$ and words $w_{uu}, w_{uv}, w_{vv}, w_{vu} \in A^K$ such that

$$Y = \{\cdots x_{-1} w_{x_{-1}x_0} x_0 w_{x_0x_1} x_1 \cdots \mid x_i \in \{u, v\} \text{ for all } i \in \mathbb{Z}\} \subseteq X.$$

Now extend $(\tau_m(F), \vec{F}_m)$ arbitrarily into a cellular automaton $((A^m)^{\mathbb{Z}}, \vec{F}'_m)$. But now the stripe shift defined by \vec{F}'_m and P would contain an uncountable sofic shift: For $x, y \in \{u, v\}$ define $z_{x,y} \in \{0, 1\}^{|u|+K-1}$ as $z_{x,y} = \iota((xw_{xyy})_{[0,m)}) \iota((xw_{xyy})_{[1,m+1)}) \cdots \iota((xw_{xyy})_{[|u|+K-1, m+|u|+K)})$, then

$$\begin{aligned} \{i_0 z_{x_0, y_0} i_1 z_{x_1, y_1} \cdots \mid i_j \in \{0, 1\} \text{ and } \iota(x_j) = i_j \text{ and} \\ \iota(y_j) = i_{j+1} \text{ for all } i, j \in \mathbb{N}\} \subseteq \Xi_P(\vec{F}'_m). \end{aligned}$$

This contradicts the Stripe Lemma (more specifically its corollary, Proposition 3.2.4).

Now, by Proposition 2.4.21, (X, F) is chain-mixing. \square

Remark 5.1.13. In Theorem 5.1.12 mixing sofic shift X cannot be replaced by a transitive sofic shift X : Take two (reversible) expansive cellular automata $(A^{\mathbb{M}}, F)$ and $(B^{\mathbb{M}}, G)$ where $\mathbb{M} = \mathbb{N}$ or $\mathbb{M} = \mathbb{Z}$, and A and B are disjoint. For convenience assume that the local neighborhood is $\{-1, 0, 1\} \cap \mathbb{M}$. Let $X \subseteq (A \cup B)^{\mathbb{M}}$ be a transitive SFT defined by the set of forbidden patterns $\mathcal{F} = \{xy \mid x, y \in A\} \cup \{xy \mid x, y \in B\}$. Now define a cellular automaton (X, H) by a local rule with neighborhood $\{-2, 0, 2\} \cap \mathbb{M}$. Within this neighborhood the local rule sees letters only from A or only from B . This local neighborhood is mapped according to F or G depending on whether the local rule sees letters from A or from B . This (X, H) is expansive since $(A^{\mathbb{M}}, F)$ and $(B^{\mathbb{M}}, G)$ are, but not chain-transitive.

Theorem 5.1.12 has the following immediate corollary.

Corollary 5.1.14. *Let X be a mixing sofic shift and let (X, F) be a surjective ultimately right-expansive cellular automaton. Then ${}_i \Sigma_j^{(m)}$ is a mixing sofic shift for every $i, j \in \mathbb{N}, m \in \mathbb{N} \setminus \{0\}$.*

Proof. The pseudo-traces ${}_i \Sigma_j^{(m)}$ are mixing sofic shifts as factors of mixing SFT's. \square

5.2 Left-POTP and Ultimate Right-Expansive Cellular Automata Have POTP

Let (X, F) be a ultimately right-expansive cellular automaton where $X \subseteq A^{\mathbb{Z}}$. Let m be large enough so that ultimate right-expansivity defines a

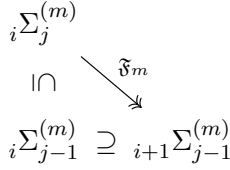


Figure 5.3: For large enough $i, j, m \in \mathbb{N}$ this diagram holds.

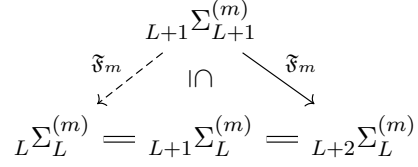


Figure 5.4: The bottom row are equal due to left-POTP for large enough L .

cellular automaton $(\tau_m(F), \vec{F}_m)$ and $r = r(\vec{F}_m)$. For all $i, j \in \mathbb{N}$ let $X_i^{(j)}$ denote the SFT of $(A^j)^\mathbb{Z}$ defined by forbidding $(A^j)^i \setminus \mathcal{L}_i(\tau_j(F))$. Now \vec{F}_m can be extended to $X_{2r+1}^{(m)}$ by using the same local rule of radius r ; Let \mathfrak{F}_m denote the extension of \vec{F}_m to $X_{2r+1}^{(m)}$. It does not necessarily hold that $\mathfrak{F}_m(X_{2r+1}^{(m)}) \subseteq X_{2r+1}^{(m)}$ so this is not necessarily a cellular automaton but it is a sliding block map.

Let us prove a simple lemma.

Lemma 5.2.1 (Figure 5.3). *Let (X, F) be a surjective right-expansive cellular automaton. Let $m \in \mathbb{N}$ be large enough so that expansivity defines a cellular automaton $(\tau_m(F), \vec{F}_m)$. Let $r = r(F)$ and $r' = r(\vec{F}_m)$, then*

$$\forall i, j \in \mathbb{N} : i \geq 2r'r \wedge j \geq 2r'r + 1 \implies \mathfrak{F}_m(i\Sigma_j^{(m)}) = i+1\Sigma_{j-1}^{(m)}.$$

Proof. Let $r = r(F)$ and $r' = r(\vec{F}_m)$. Now ${}_{2r'r}\Sigma_{2r'r}^{(m+1)}$ is the set of $m+1$ wide middle columns of $\mathcal{PO}_{4r'r+m+1}(F)$. Due to the width and surjectivity we have that $\mathcal{L}_{2r'+1}({}_{2r'r}\Sigma_{2r'r}^{(m+1)}) = \mathcal{L}_{2r'+1}(\tau_{m+1}(F))$. But then we have that

$$\mathfrak{F}_m({}_{2r'r}\Sigma_{2r'r+1}^{(m)}) = {}_{2r'r+1}\Sigma_{2r'r}^{(m)}$$

which proves the claim. \square

We can now prove that surjective ultimately right-expansive cellular automaton with left-POTP has POTP. Our proof is inspired by Siamak Taati's proof that a cellular automaton (over the full shift) which is reversible over its limit set is stable [58], i.e. reaches the limit set in finite time. The analogy is that left-POTP corresponds to the forward rule, the trace subshift $\tau_m(F)$ corresponds to the limit set, and the map defined by right-expansivity corresponds to the inverse rule defined on the limit set.

Theorem 5.2.2. *Let $X \subseteq A^\mathbb{Z}$ be a transitive SFT and let (X, F) be a surjective ultimately right-expansive cellular automaton with left-POTP. Then F has POTP and $\tau_k(F)$ is a sofic shift for every k . If F is memoryless, then*

$\tau_m(F)$ is an SFT for every large enough m . Also, if X is a mixing SFT, then $\tau_k(F)$ is mixing for every k .

Proof. By Proposition 2.4.12 there exists n such that (X, σ^n) is a finite union of disjoint mixing SFT's. As (X, F) is a surjective cellular automaton, some power of F is a cellular automaton when restricted to any of these mixing SFT's. If this power of F has POTP on each of these disjoint mixing SFT's then the original cellular automaton also has POTP. This is why it is enough to prove the claim with the additional assumption of X being a mixing SFT. This same argument has already been made by Nasu [53, end of Section 2] and also by Boyle [7, proof of Theorem 5.5].

Let (X, F) be a surjective ultimately right-expansive cellular automaton where X is a mixing SFT. We will show that for all large enough $L, m \in \mathbb{N}$ it holds that $\tau_m(F) = {}_L\Sigma_L^{(m)}$ so that (X, F) has POTP.

Let m be large enough so that ultimate right-expansivity defines a cellular automaton $(\tau_m(F), \vec{F}_m)$. Let l be large enough so that left-POTP says that ${}_l\Sigma_l^{(m)} = {}_\infty\Sigma_l^{(m)}$. Let r be a radius of F and let r' be a radius of \vec{F}_m . Notice that since F is surjective we have that $\mathcal{L}_k({}_{kr}\Sigma_{kr}^{(m)}) = \mathcal{L}_k(\tau_m(F))$ for every $k \in \mathbb{N}$. In particular ${}_{l'}\Sigma_{l'}^{(m)} \subseteq X_{2r+1}^{(m)}$ for every $l' \geq (2r' + 1)r$. Let $L \geq \max\{l, (2r' + 1)r\}$.

Now we claim that $\mathfrak{F}_m({}_{L+1}\Sigma_{L+1}^{(m)}) = {}_L\Sigma_L^{(m)}$. According to Lemma 5.2.1 we have that $\mathfrak{F}_m({}_{L+1}\Sigma_{L+1}^{(m)}) = {}_{L+2}\Sigma_L^{(m)}$ and, by left-POTP, ${}_{L+2}\Sigma_L^{(m)} = {}_L\Sigma_L^{(m)}$ (see Figure 5.4), which together prove this claim.

Now we have that ${}_L\Sigma_L^{(m)}$ is a factor of ${}_{L+1}\Sigma_{L+1}^{(m)}$, so the entropy of ${}_L\Sigma_L^{(m)}$ is at most the entropy of ${}_{L+1}\Sigma_{L+1}^{(m)}$ (Proposition 2.4.7). But we also have that ${}_{L+1}\Sigma_{L+1}^{(m)} \subseteq {}_L\Sigma_L^{(m)}$, and so ${}_{L+1}\Sigma_{L+1}^{(m)}$ and ${}_L\Sigma_L^{(m)}$ have the same entropy. According to Proposition 2.4.8 we have that ${}_{L+1}\Sigma_{L+1}^{(m)} = {}_L\Sigma_L^{(m)}$, since ${}_L\Sigma_L^{(m)}$ is transitive (Corollary 5.1.14), and so ${}_L\Sigma_L^{(m)} = {}_\infty\Sigma_\infty^{(m)}$. Now (X, F) has POTP according to Proposition 2.4.19, and POTP always implies soficness of the trace subshifts.

Next let F be memoryless, i.e. it has a local neighborhood $[0, r]$. According to Lemma 5.1.7 we now have that ${}_0\Sigma_d^{(m)} = \tau_m(F)$ for some d . Let $x \in X_{2r'd+2}^{(m)}$ be arbitrary and $y \in (A^{m+d})^{\mathbb{Z}}$ be the unique configuration defined by $\pi_{[i, i+m)}(y) = \mathfrak{F}_m^i(x)$ for all $i \in \{0, 1, \dots, d\}$. Now $\pi_{[0, m+1)}(y) \in X_{2r'(d-1)+2}^{(m+1)}$ since $x = \pi_{[0, m)}(y) \in X_{2r'd+2}^{(m)}$ and $\pi_{[1, m)}(y)$ is defined using the local rule of \vec{F}_m . We can repeat this for d times and we get that $y \in X_2^{(m+d)}$, i.e. $y \in \mathcal{PO}_{m+d}(F)$. But then, since ${}_0\Sigma_d^{(m)} = \tau_m(F)$, we have that $x \in \tau_m(F)$. Of course $\tau_m(F) \subseteq X_{2r'd+2}^{(m)}$ and so we are done. This reasoning holds for any m large enough so that \vec{F}_m is defined.

Lastly, if X is mixing, then by Corollary 5.1.14 also $\tau_k(F)$ is mixing (we can skip the first paragraph of this proof). \square

Remark 5.2.3. Theorem 5.2.2 implies the following:

- If $(A^{\mathbb{N}}, F)$ is surjective and positively expansive, then $\tau_m(F)$ is an SFT for all large enough m (proved by Francois Blanchard, and Alejandro Maass in [6, Theorem 3.3], and by Mike Boyle, Doris Fiebig, and Ulf-Reiner Fiebig in [8]).
- If $(A^{\mathbb{N}}, F)$ is reversible and expansive, then $\tau_m(F)$ is an SFT for all large enough m (by Masakazu Nasu [52, Theorem 1.3])
- If (X, F) , where $X \subseteq A^{\mathbb{Z}}$ is a transitive SFT, is surjective, ultimately right-expansive, memoryless, and chain-recurrent, then (X, F) has POTP (by Masakazu Nasu [53, Theorem 6.3 (i)], the last assumption is not actually needed, see introduction of this chapter or the theorem above).

It is also known that if $(A^{\mathbb{Z}}, F)$ is positively expansive, then $\tau_m(F)$ is conjugate to a full shift (proved independently by Petr Kůrka [39], Masakazu Nasu [51], and Fabio Fagnani and Luciano Margara [21]).

5.3 Right-Expansive Cellular Automaton with Non-Sofic Traces

Next we show that while it might be possible to replace the assumption of left-POTP in Theorem 5.2.2 with a weaker assumption, it cannot be dropped entirely. Using the construction by Jarkko Kari and Ville Lukkarila [37] we give an example of a right-expansive cellular automaton that has a non-sofic trace. The next paragraphs summarize what we need, but for a more detailed presentation we refer the reader to [37, 34, 43].

A set of *Wang tiles* T is a set of squares with each edge coloured using finite colour set A . Let us fix that $(x, y, z, w) \in A^4$ is a Wang tile such that the colours are presented in the order W-N-E-S (west-north-east-south). A *tiling* by T is an assignment $\mathbb{Z}^2 \rightarrow T$. A tiling is *valid* if the adjacent edges have the same colour. The set T is called *NW-deterministic* if for all $a, b \in A$ we have that $|(\{a\} \times \{b\} \times A \times A) \cap T| \leq 1$. Other XY-determinism's are defined analogously. If T is both NW- and SE-deterministic, then T is called *two-way deterministic*. Any two-way deterministic tile set can be *completed* in the sense that we can add tiles T^C so that for all $a, b \in A$ holds that $|(\{a\} \times \{b\} \times A \times A) \cap (T \cup T^C)| = 1 = |(A \times A \times \{a\} \times \{b\}) \cap (T \cup T^C)|$. This can be done since we must be missing the same number of NW- and SE-pairs; in T^C we just match these arbitrarily. This means that a two-way

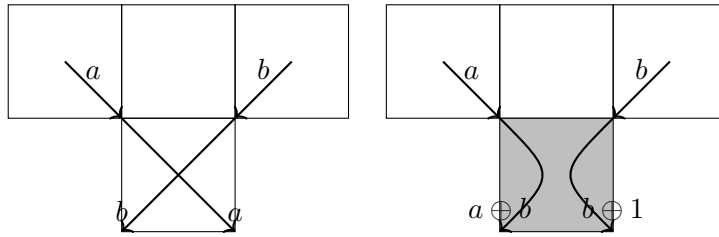


Figure 5.5: Here $a, b \in \{0, 1\}$, the light gray represents a tiling error, and \oplus denotes addition modulo 2. The label of the arrow pointing to the lower left corner is the first bit and the label of the arrow pointing to the lower right corner is the second bit.

deterministic set of Wang tiles T can be used to define a reversible cellular automaton $F_T : (T \cup T^C)^{\mathbb{Z}} \rightarrow (T \cup T^C)^{\mathbb{Z}}$ where configurations represent SW-NE-diagonals of valid tilings with $T \cup T^C$.

A tiling c by a Wang tile set T is *periodic* if there exists $(x, y) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ such that for every $(i, j) \in \mathbb{Z}^2$ we have that $c_{i,j} = c_{i+x,j+y}$. A Wang tile set T is *aperiodic* if it admits a valid tiling, but none of the valid tilings is periodic. We need the fact that there exists an aperiodic two-way deterministic Wang tile set. Such do exist: The Wang tile set derived from Amman's aperiodic tile set [25] is such, and the tile set constructed in [43] is even 4-way deterministic. Details of the chosen tile set are irrelevant for us.

Proposition 5.3.1. *There exists a right-expansive cellular automaton over a full shift such that for all $m \geq 2r$ it holds that m -trace is a non-sofic shift, where r is a radius of the cellular automaton.*

Proof. Following paragraph shortly describes what we get from a construction in [37]:

Let T be a two-way deterministic aperiodic set of Wang tiles. Define $F_T : (T \cup T^C) \rightarrow (T \cup T^C)$ as above; the tiles in T^C are considered to be tiling errors. For large enough k it holds that $\sigma^k F_T$ is expansive. Denote $F = \sigma^k F_T$. Next signals are added: Let $G : ((T \cup T^C) \times \{0, 1\}^2)^{\mathbb{Z}} \rightarrow ((T \cup T^C) \times \{0, 1\}^2)^{\mathbb{Z}}$ where the tiling-layer is mapped by F and the signal layer is mapped as illustrated by Figure 5.5. As noted in [37] this G is right-expansive and further has the property that if there is a tiling error somewhere, then every column right of the tiling error contains both zero- and one-signals.

Let r be the radius of G and suppose that $\tau_m(G)$ is sofic for some $m \geq 2r$. Take a space-time diagram which on the tiling-layer contains only states from T and on the signal-layer all the signals are zeroes. Such exists as T admits

valid tilings and then setting every signal to zero gives a suitable space-time diagram. Let $t \in \tau_m(G)$ be a vertical stripe of this space-time diagram. By sofiness it follows that there exists $i \in \mathbb{Z}, n \in \mathbb{N}$ such that $u = t_{[i, i+n]}$ is such that $\cdots uuu \cdots \in \tau_m(G)$. Now since the tiling-layer is expansive, this implies that in the space-time diagram that has $\cdots uuu \cdots$ as a column, the tiling layer must be periodic. But since T is an aperiodic tile set, it then has to be that there are tiling errors densely; that is to say that there exists $k \in \mathbb{N}$ such that every $k \times k$ square in the space-time diagram has a cell whose tiling-layer is in a state from T^C . In particular there has to be tiling errors left of the column $\cdots uuu \cdots$. But this is a contradiction, since if there is a tiling error left of the column, then the column's signal-layer has both zeroes and ones, but $\cdots uuu \cdots$ has only zeroes. \square

5.4 Left-POTP Cellular Automaton with Non-Sofic Traces

Next we give an example of a reversible cellular automaton over one-sided full shift with non-sofic traces, and so a reversible cellular automaton with left-POTP and with non-sofic traces.

Let $X \subseteq A^{\mathbb{Z}}$ be a subshift. The set of *isolated* points of X is

$$\text{Iso}(X) = \{c \in X \mid \exists n \in \mathbb{N} : [c_{-n} \cdots c_n] \cap X = \{c\}\}.$$

Sofic shifts have only finitely many isolated periodic points.

Lemma 5.4.1. *If $X \subseteq A^{\mathbb{Z}}$ is a sofic shift, then $|\text{Iso}(X) \cap \text{Per}_\sigma(X)| < \infty$.*

Proof. Suppose X is sofic but the intersection of its isolated and periodic points is infinite. Let \mathcal{G} be (labeled directed) graph that defines X ; we can assume that every vertex has both incoming and outgoing edges. Let $\{c_i\}_{i \in \mathbb{N}} \subseteq \text{Iso}(X) \cap \text{Per}_\sigma(X)$ be an infinite subset such that if $i \neq j$ then for all k it holds that $\sigma^k(c_i) \neq c_j$. For every $i \in \mathbb{N}$ let u_i be the shortest word such that $c_i = \omega u_i^\omega$. Since for every i any repetition of u_i appears, there has to exist a cycle in \mathcal{G} whose labels read u_i some number of times. Let $i, j \in \mathbb{N}$ be arbitrary but different. Let \mathcal{G}_i and \mathcal{G}_j be cycles in \mathcal{G} whose labels read u_i and u_j (resp.) some number of times. The cycles \mathcal{G}_i and \mathcal{G}_j must be separate in the sense that there cannot be a directed path from \mathcal{G}_i to \mathcal{G}_j or from \mathcal{G}_j to \mathcal{G}_i , since otherwise there would exist a word $w \in A^+$ such that $\omega u_i w u_j^\omega$ or $\omega u_j w u_i^\omega$ would be in X contradicting the isolation of c_i . Having an own separate cycle for infinitely many points contradicts the finiteness of \mathcal{G} . \square

Now we are ready to present a reversible one-sided cellular automaton with non-sofic traces.

Proposition 5.4.2. *There exists a reversible one-sided cellular automaton whose traces are non-sofic.*

Proof. Let $(A^{\mathbb{N}}, F)$ be an elementary ROCA where $A = \{0, 1, 2, 3\}$ and

$$\begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 2 & 0 & 3 \\ 0 & 2 & 1 & 3 \\ 1 & 2 & 0 & 3 \end{bmatrix}.$$

We will show that $\tau_1(F)$ is non-sofic, it is then easy to see that also $\tau_m(F)$ for any $m > 1$ is non-sofic.

First notice that 3 is always mapped to 3 so that every $c \in \tau_1(F)$ is either $\omega 3^\omega$ or has no appearances of the letter 3. Notice also that if there is a column $\omega 3^\omega$ in a space-time diagram of $(A^{\mathbb{N}}, F)$ then every column to the left of it has to be periodic. This can be seen, for example, by noticing that π_3 can be extended into a permutation $\pi_{3,n} : A^n \rightarrow A^n$ for any n by $\pi_{3,n}(u) = F(u3)$ where $u \in A^n$. Now since 3 is fixed, the two-way infinite sequences $(\pi_{3,n}^i(u))_{i \in \mathbb{Z}}$ are precisely the elements of $\tau_n(F)$ obtained by fixing 3 into the cell n . Now the order of the permutation $\pi_{3,n}$ gives an upper bound for the period of the n^{th} column to the left of $\omega 3^\omega$.

Notice that $(\{0, 1, 2\}^{\mathbb{N}}, F|_{\{0,1,2\}^{\mathbb{N}}})$ is the zot cellular automaton defined in Chapter 5, Section 4.2. According to Proposition 4.2.2 we know that $20^{2k+1}1$ cannot appear as a vertical word in any space-time diagram that does not contain 3's. We also know, by Lemma 4.2.1, that if the word $20^{2k+1}1$ appears in $\tau(A^{\mathbb{N}}, F)$ then it forces an appearance of $20^{2k-1}1$ to the right of it. Eventually this leads to the word 201. This was the reason why $20^{2k+1}1$ cannot appear in the trace of the zot cellular automaton, as the only letter that maps 0 to 1 is 1, but 1 does not map 0 to 2 when going backwards in time. However, the added letter 3 does exactly this: $\pi_3(0) = 1$ and $\rho_3(0) = 2$. The letter 3 itself is fixed, and we see that $20^{2k+1}1$ does appear in $\tau_1(F)$, and for each k it defines an isolated periodic point (see Figure 5.6). According to Lemma 5.4.1 this shows that $\tau_1(F)$ cannot be sofic and concludes the proof. □

Since any memoryless cellular automaton over a full shift has left-POTP (Lemma 5.1.7) we have the following.

Corollary 5.4.3. *There exists a reversible cellular automaton over a full shift with left-POTP whose traces are non-sofic.*

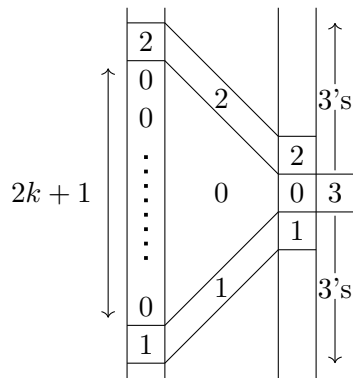


Figure 5.6: Word $20^{2k+1}1$ forces the word $20^{2k-1}1$ next to it. Eventually this leads to the word 201. Next to this there can only be a 3. Since 3's are fixed, the whole column is a constant 3. This uniquely determines the column containing $20^{2k+1}1$ and forces it to be periodic.

Chapter 6

Conjugacy

All cellular automata in this chapter are over full shifts.

As a last topic of this thesis we study conjugacy of cellular automata. Dynamical systems that are (topologically) conjugate share the same (topological dynamical) properties. This can often help in the study of dynamical systems, as we may freely use conjugacies to transform system we are studying into another one which may be easier to analyze. While discussing one-sided cellular automata in Chapter 4 we used this fact to justify studying cellular automata with radius one, as the grouping of cells is a conjugacy.

Naturally it could be useful to have an algorithm that would decide whether given two cellular automata are conjugate or not. In this chapter we show that such algorithm does not exist. First we prove this for one-sided one-dimensional cellular automata. In fact we prove a more general inseparability result which concerns stronger form of conjugacy (where the conjugacy map itself is also required to be shift-commuting) and immediately also gives that being a factor or being a subsystem are also undecidable properties. Then we prove the same inseparability result for reversible two-dimensional two-sided cellular automata. There are some obvious holes in the knowledge even after we are done: What about reversible one-dimensional (either one- or two-sided) cellular automata? or reversible two-dimensional one-sided cellular automata?

The first proofs we give for the above results use arbitrarily large alphabets. In the last section we improve these results by fixing the alphabet size. These can be considered as algebraic variants of the undecidability results as the set of all cellular automata over a fixed full shift form a monoid while reversible cellular automata over a fixed full shift form a group.

6.1 One-Dimensional Case

In his doctoral dissertation [20], Jeremias Epperlein proved that the conjugacy of periodic one-dimensional cellular automata is decidable. He also conjectured that for general one-dimensional cellular automata conjugacy is undecidable [20, Conjecture 5.19]. In this section we prove this conjecture. Actually we prove a result that is stronger in a couple of ways: We prove a recursive inseparability result which immediately implies that conjugacy, being a factor, being a subsystem, and the shift-commuting variants of all of these are undecidable for both one- and two-sided one-dimensional cellular automata. After the main result we mention some related problems.

6.1.1 Conjugacy of One-Dimensional One-Sided Cellular Automata Is Undecidable

Let $(A^{\mathbb{M}^d}, F)$ and $(A^{\mathbb{M}^d}, G)$ be two cellular automata. Let $H : A^{\mathbb{M}^d} \rightarrow B^{\mathbb{M}^d}$ be a conjugacy between $(A^{\mathbb{M}^d}, F)$ and $(B^{\mathbb{M}^d}, G)$ such that H also commutes with the shift maps. In papers [31] and [30], in which most of the results of this chapter originally appeared, we called such conjugacy a *strong* conjugacy. Since this is not a very descriptive term, we will change the terminology in this matter and call shift-commuting conjugacies, shift-commuting factor maps, and shift-commuting embeddings, *block conjugacies*, *block factor maps*, and *block embeddings*, respectively. If $(A^{\mathbb{M}^d}, F)$ and $(A^{\mathbb{M}^d}, G)$ are block conjugate, we denote this by $(A^{\mathbb{M}^d}, F) \cong_b (B^{\mathbb{M}^d}, G)$ or shortly just $F \cong_b G$.

We give the following definitions and results for one-sided cellular automata, as we do not need the two-sided variants (which do exist).

For a cellular automaton $(A^{\mathbb{N}}, F)$ a state $q \in A$ is *quiescent* if $F(q^\omega) = q^\omega$. A cellular automaton is *nilpotent* if there exists a quiescent state q such that for every $c \in A^{\mathbb{N}}$ there exists $n \in \mathbb{N}$ such that $F^n(c) = q^\omega$. It is known that for cellular automata nilpotency implies uniform nilpotency.

Proposition 6.1.1. ([14]) *Let $(A^{\mathbb{N}}, F)$ be a nilpotent cellular automaton. Then there exists $n \in \mathbb{N}$ such that for all $c \in A^{\mathbb{N}}$ it holds that $F^n(c) = q^\omega$.*

Let $(A^{\mathbb{N}}, F)$ be a cellular automaton whose local neighborhood contains cells 0 and 1. Then a state $s \in A$ is *spreading* if the local rule maps every neighborhood containing s to s . Such a state spreads in the sense that if $c_i = s$ for some $c \in A^{\mathbb{N}}$ and $i \in \mathbb{N} \setminus \{0\}$ then $F(c)_i = s$ and $F(c)_{i-1} = s$. Clearly a spreading state is quiescent. We need the following result, which follows from a simple compactness argument.

Proposition 6.1.2. *Let $(A^{\mathbb{N}}, F)$ be a cellular automaton that is not nilpotent, and let $s \in A$ be a spreading state. Then there exists $c \in A^{\mathbb{N}}$ such that $F^i(c)_j \neq s$ for all $i, j \in \mathbb{N}$.*

Proof. For every $n \in \mathbb{N}$ there exists $c^{(n)} \in A^{\mathbb{N}}$ such that $F^i(c^{(n)})_j \neq s$ for every $(i, j) \in \{(x, y) \in \mathbb{N}^2 \mid x, y \leq n\}$, since otherwise the appearing states s would spread and F would be nilpotent. By compactness the sequence $(c^{(n)})_{n \in \mathbb{N}}$ has a converging subsequence $(c^{(i)})_{i \in \mathcal{I}}$, and the limit of this sequence, say c , has that $F^i(c)_j \neq s$ for all $i, j \in \mathbb{N}$ as was claimed. \square

Our proof relies on the following undecidability result.

Theorem 6.1.3. ([34],[1]) *Nilpotency of one-dimensional one-sided cellular automata with a spreading state and radius 1 is undecidable.*

We are ready to prove the following inseparability result.

Theorem 6.1.4. *The following two sets of pairs of one-dimensional one-sided cellular automata are recursively inseparable:*

(i) *pairs where the first cellular automaton has strictly higher entropy than the second one, and*

(ii) *pairs that are block conjugate and both have zero topological entropy.*

Proof. We reduce the decision problem of Theorem 6.1.3 to this problem, which proves our claim.

Let $(B^{\mathbb{N}}, H)$ be an arbitrary given one-sided cellular automaton with neighborhood radius 1 and a spreading quiescent state $q \in B$. Let $k \in \mathbb{N}$ be such that $k > \log_2(|B|)$, $(A^{\mathbb{N}}, \mathcal{Z}_{2k})$ be the $2k$ -fold cartesian product of zot cellular automaton \mathcal{Z} from Chapter 5 Section 4.2, so that $A = \{0, 1, 2\}^{2k}$. This choice is done to have high enough entropy down the line. Now we are ready to define cellular automata \mathcal{F} and \mathcal{G} such that

$$\begin{aligned} H \text{ is not nilpotent} &\implies h(\mathcal{F}) > h(\mathcal{G}) \\ H \text{ is nilpotent} &\implies \mathcal{F} \cong_b \mathcal{G} \text{ and } h(\mathcal{F}) = h(\mathcal{G}) = 0. \end{aligned}$$

Both of these new cellular automata work on two tracks $\mathcal{F}, \mathcal{G} : (A \times B)^{\mathbb{N}} \rightarrow (A \times B)^{\mathbb{N}}$. The cellular automaton \mathcal{G} is simply $\text{id}_A \times H$, i.e.

$$\mathcal{G}((a_0, b_0)(a_1, b_1)) = (a_0, H(b_0b_1)),$$

for all $a_0, a_1 \in A$, $b_0, b_1 \in B$. The cellular automaton \mathcal{F} also acts as H on the B -track. On the A -track \mathcal{F} acts as \mathcal{Z}_{2k} when the B -track is not q , and as id_A when the B -track is q , i.e.

$$\mathcal{F}_{loc}((a_0, b_0)(a_1, b_1)) = \begin{cases} (\mathcal{Z}_{2k,loc}(a_0a_1), H_{loc}(b_0b_1)), & \text{if } b_0 \neq q \\ (a_0, q), & \text{if } b_0 = q, \end{cases}$$

for all $a_0, a_1 \in A$, $b_0, b_1 \in B$.

(i) **Suppose that H is not nilpotent.** The entropy of \mathcal{G} is

$$h((A \times B)^{\mathbb{N}}, \mathcal{G}) = h(A^{\mathbb{N}}, \text{id}_A) + h(B^{\mathbb{N}}, H) = h(B^{\mathbb{N}}, H),$$

since $\mathcal{G} = \text{id}_A \times H$. On the other hand, by Proposition 6.1.2, there exists a configuration $e \in B^{\mathbb{Z}}$ such that for all $i, j \in \mathbb{N}$ we have that $H^i(c)_j \neq q$. But then we have that

$$h((A \times B)^{\mathbb{N}}, \mathcal{F}) \geq h(A^{\mathbb{N}}, \mathcal{Z}_{2k}) > \log_2(|B|) \geq h(B^{\mathbb{N}}, H),$$

according to Proposition 4.2.3 and the choice of k . Overall we have that

$$h((A \times B)^{\mathbb{N}}, \mathcal{F}) > h((A \times B)^{\mathbb{N}}, \mathcal{G}),$$

as was claimed.

(ii) **Suppose that H is nilpotent.** Let us first explain informally why we now have that $\mathcal{F} \cong_b \mathcal{G}$. Both \mathcal{F} and \mathcal{G} behave identically on the B -track, so the conjugacy maps this track simply by identity. Nilpotency of H guarantees that for all configurations the B -track will be q^ω after some constant time n (Proposition 6.1.1). By the definition of \mathcal{F} this means that after n steps \mathcal{F} does nothing on the A -track. Since \mathcal{G} never does anything on the A -track, we can use this fact to define the conjugacy on the A -track simply with \mathcal{F}^n . That this is in fact a conjugacy follows since \mathcal{F} is, informally, reversible on the A -layer for a fixed B -layer.

Let us be exact. First we will define a continuous map $\phi : (A \times B)^{\mathbb{N}} \rightarrow (A \times B)^{\mathbb{N}}$ such that $\phi\mathcal{F} = \mathcal{G}\phi$. This ϕ will be a cellular automaton. Then we show that ϕ is injective, which implies reversibility (by Proposition 2.4.4), and so we will have $((A \times B)^{\mathbb{N}}, \mathcal{F}) \cong_b ((A \times B)^{\mathbb{N}}, \mathcal{G})$.

Let $\pi_A : A^{\mathbb{N}} \times B^{\mathbb{N}} \rightarrow A^{\mathbb{N}}$ be the projection $\pi_A(c, e) = c$ for all $c \in A^{\mathbb{N}}$ and $e \in B^{\mathbb{N}}$. Define $\pi_B : A^{\mathbb{N}} \times B^{\mathbb{N}} \rightarrow B^{\mathbb{N}}$ similarly.

Let $n \in \mathbb{N}$ be a number such that for all $c \in B^{\mathbb{N}}$ we have $H^n(c) = q^\omega$. Such n exists according to Proposition 6.1.1, since H is nilpotent. Because \mathcal{F} and \mathcal{G} act identically on the B -track, ϕ maps this layer simply by identity, i.e.

$$\pi_B\phi(c, e) = e,$$

for all $c \in A^{\mathbb{N}}, e \in B^{\mathbb{N}}$. After n steps \mathcal{F} does nothing on the A -track, i.e. acts the same way as \mathcal{G} does. Because of this we define

$$\pi_A\phi = \pi_A\mathcal{F}^n.$$

Now ϕ is a cellular automaton, since it is continuous and shift-commuting. Let us show that ϕ is a homomorphism. Of course we have that

$$\phi\mathcal{F} = \mathcal{G}\phi \iff (\pi_A\phi\mathcal{F} = \pi_A\mathcal{G}\phi \text{ and } \pi_B\phi\mathcal{F} = \pi_B\mathcal{G}\phi).$$

It is immediate from the definitions that $\pi_B \phi \mathcal{F} = \pi_B \mathcal{G} \phi$. For the equality on the A -layer notice first that $\pi_A \mathcal{G} = \pi_A$, and then compute:

$$\begin{aligned}
\pi_A \phi \mathcal{F} &\stackrel{\text{def.}}{=} (\pi_A \mathcal{F}^n) \mathcal{F} \\
&= \pi_A \mathcal{F} \mathcal{F}^n && \parallel \text{ after } n \text{ steps } \mathcal{F} \\
&= \pi_A \mathcal{G} \mathcal{F}^n && \text{behaves as } \mathcal{G} \\
&= \pi_A \mathcal{F}^n \\
&\stackrel{\text{def.}}{=} \pi_A \phi \\
&= \pi_A \mathcal{G} \phi.
\end{aligned}$$

So we have that $\phi \mathcal{F} = \mathcal{G} \phi$.

To prove that ϕ is a block *conjugacy* it is enough to show that ϕ is an injection. As the B -layer is mapped by identity, we only need to show that for a fixed $e \in B^{\mathbb{N}}$ we have that for all $c \in A^{\mathbb{N}}$ there exists a unique $c' \in A^{\mathbb{N}}$ such that $\phi(c', e) = (c, e)$. By the definition of ϕ this holds if

$$\begin{aligned}
\pi_A \mathcal{F}^n(-, e) : A^{\mathbb{N}} &\longrightarrow A^{\mathbb{N}} \\
c &\longmapsto \pi_A \mathcal{F}^n(c, e)
\end{aligned}$$

is a bijection for every $e \in B^{\mathbb{N}}$. This holds if the map

$$\begin{aligned}
\pi_A \mathcal{F}(-, e) : A^{\mathbb{N}} &\longrightarrow A^{\mathbb{N}} \\
c &\longmapsto \pi_A \mathcal{F}(c, e)
\end{aligned}$$

is a bijection for every $e \in B^{\mathbb{N}}$. Let $x \in A^{\mathbb{N}}$ be arbitrary. Define configuration c as follows:

- If $e_i = q$, then $c_i = x_i$.
- If $e_i \neq q$ but $e_{i+1} = q$, then $c_i = \pi_{x_{i+1}}^{-1}(x_i)$ (here $\pi_{x_{i+1}}$ is defined by the permutations of zot when extended in a natural way for \mathcal{Z}_{2k}).
- If $e_i \neq q$ and $e_{i+1} \neq q$, then $c_i = \mathcal{Z}_{2k}^{-1}(x)_i$.

Now $\mathcal{F}(c, e) = (x, H(e))$ and so the map above is surjective. On the other hand, it is clear from the definition of \mathcal{F} that this is the only possible preimage.

To complete the proof we observe that

$$h(\mathcal{F}) = h(\mathcal{G}) = h(\text{id}_A) + h(H) = 0,$$

since $\mathcal{F} \cong_b \mathcal{G} = \text{id}_A \times H$, and H is nilpotent.

□

Since Theorem 6.1.4 can be reduced to the two-sided case, also the two-sided variant is undecidable. We also get the following corollary.

Corollary 6.1.5. *Let $\mathbb{M} = \mathbb{N}$ or $\mathbb{M} = \mathbb{Z}$. Let $(A^{\mathbb{M}}, F)$ and $(A^{\mathbb{M}}, G)$ be two given cellular automata. Then the following hold:*

1. *It is undecidable whether F and G are (block) conjugate.*
2. *It is undecidable whether F is a (block) factor of G .*
3. *It is undecidable whether F is a (block) subsystem of G .*

Proof. 1. The pairs in the set (i) of Theorem 6.1.4 cannot be (block) conjugate, and the pairs in (ii) have to be. Thus deciding (block) conjugacy would separate these sets.

2. One of the cellular automata in the pair from the set (i) has strictly higher entropy than the other, so it cannot be a (block) factor of the other. On the other hand cellular automata of pairs from the set (ii) are (block) factors of each other. So checking whether both cellular automata of a pair is a (block) factor of the other would separate the sets of Theorem 6.1.4.

3. In a similar way, since a subsystem cannot have higher entropy.

□

6.1.2 Restricted Cases

We have seen that (block) conjugacy is undecidable in general for one-dimensional cellular automata. A natural follow-up question is whether (block) conjugacy remains undecidable even if we restrict to some natural subsets of cellular automata. When restricted to periodic cellular automata, the following is known by Jeremias Epperlein.

Theorem 6.1.6. *([20, Corollary 5.17.]) Conjugacy of periodic cellular automata on one- or two-sided subshifts of finite type is decidable.*

Periodic cellular automata are the least sensitive to changes in the initial configuration, and are precisely the reversible equicontinuous cellular automata. Naturally one could ask what happens if the requirement of reversibility is dropped, i.e. is conjugacy of eventually periodic cellular automata decidable ([20, Question 8.6.]), or the block conjugacy of either.

In the other end of the sensitivity scale are the cellular automata that are the most sensitive to initial conditions, i.e. positively expansive ones. Positively expansive cellular automata are quite extensively studied which allows us to deduce the following result.

Proposition 6.1.7. *Conjugacy of positively expansive cellular automata on one- or two-sided full shifts is decidable.*

Proof. Let $(A^{\mathbb{M}}, F)$ and $(B^{\mathbb{M}}, G)$ be two positively expansive cellular automata. Due to the positive expansivity, F and G are conjugate to $\tau_{\mathbb{N},k}(F)$ and $\tau_{\mathbb{N},k}(G)$ (resp.) for large enough k . These subshifts are conjugate to subshifts of finite type (see Mike Boyle and Bruce Kitchens [9] for one-sided case, and Masakazu Nasu [51] for two-sided case). According to Pietro di Lena [17, Theorem 36] we can effectively compute these subshifts. The claim follows, as the conjugacy of one-sided subshifts of finite type is decidable by Robert Williams [62]. \square

We know, for example, that expansive one-sided cellular automata are conjugate to two-sided SFT's (these were discussed in Chapter 5), yet the above proof fails since it is not known whether conjugacy of two-sided SFT's is decidable. Though perhaps more interesting questions to start with is whether expansivity of one-sided cellular automata is decidable.

6.1.3 Short Note on Conjugacy of Subshifts

Let $X \subseteq A^{\mathbb{M}}, Y \subseteq B^{\mathbb{M}}$ be two subshifts. Some well-known open problems in symbolic dynamics consider conjugacy of subshifts. Conjugacy of one-sided subshifts of finite type is known to be decidable (Robert Williams [62], we used this in the previous section), but the same problem for two-sided subshifts of finite type and for one- and two-sided sofic shifts is unknown. From the undecidability of block conjugacy we get the following undecidability result regarding (even one-sided) subshifts of finite type.

Proposition 6.1.8. *Let $X, Y \subseteq (A \times A)^{\mathbb{M}}$ be two subshifts of finite type, both conjugate to $A^{\mathbb{M}}$. It is undecidable whether X and Y are conjugate via a conjugacy of the form $\phi \times \phi$.*

Proof. The proof is a direct reduction from the undecidability of block conjugacy of cellular automata. Suppose it is decidable whether $X, Y \subseteq (A \times A)^{\mathbb{M}}$ that are both conjugate to $A^{\mathbb{M}}$ are conjugate via a conjugacy of the form $\phi \times \phi$. Let $(A^{\mathbb{M}}, F)$ and $(A^{\mathbb{M}}, G)$ be two cellular automata. Let $X = \{(c, F(c)) \mid c \in A^{\mathbb{M}}\}$ and $Y = \{(c, G(c)) \mid c \in A^{\mathbb{M}}\}$. Clearly these subshifts are conjugate to $A^{\mathbb{M}}$. Now X and Y are conjugate via a conjugacy of form $\phi \times \phi$ if and only if $(A^{\mathbb{M}}, F) \cong_b (A^{\mathbb{M}}, G)$:

Suppose X and Y are conjugate via some $\phi \times \phi$: There exists a conjugacy $\phi \times \phi : X \rightarrow Y$. Then ϕ commutes with the shift and for every $c \in A^{\mathbb{M}}$ we have that $(\phi(c), \phi F(c)) = (e, G(e))$, where e has to be $\phi(c)$, and so $\phi F(c) = G\phi(c)$ for all $c \in A^{\mathbb{M}}$. In other words ϕ is a block conjugacy of $(A^{\mathbb{M}}, F)$ and $(A^{\mathbb{M}}, G)$.

Suppose $(A^{\mathbb{M}}, F) \cong_b (A^{\mathbb{M}}, G)$: Let ϕ be a block conjugacy from $(A^{\mathbb{M}}, F)$ to $(A^{\mathbb{M}}, G)$. Then $\phi \times \phi$ is a conjugacy between X and Y . \square

6.2 Two-Dimensional Case

We have seen that conjugacy is undecidable for cellular automata, and one immediate follow-up question is to ask whether this still holds when restricted to reversible cellular automata. We are unable to answer this question in one-dimensional case and for this reason turn to two-dimensional cellular automata. We prove that for two-dimensional two-sided reversible cellular automata conjugacy, being a factor, being a subsystem, and the “block” variants of all of these are undecidable. The fact that conjugacy is undecidable was already proved by Jeremias Epperlein [20] even when restricted to periodic cellular automata with period two.

Notice that in this section we will only consider two-sided cellular automata, and simply call them cellular automata. There has not been much study on the one-sided cellular automata beyond the one-dimensional case. Our proof borrows a lot from the proof that reversibility is undecidable for two-dimensional cellular automata, but whether this is true for one-sided two-dimensional cellular automata is not known. Notice, however, that for example surjectivity is undecidable also for one-sided two-dimensional cellular automata as this follows from the two-sided case.

6.2.1 Conjugacy of Reversible Two-Dimensional Cellular Automata

Denote $\mathcal{C}_n = [0, n]^2$. Let $\mathcal{A} \subseteq A^{\mathcal{C}_n}$ be a set of patterns, considered here to be valid, and define a *direction function* $\delta : A \rightarrow \{(\pm 1, 0), (0, \pm 1)\}$. A sequence $(p_1, p_2, \dots, p_k) \in (\mathbb{Z}^2)^k$ is a δ -path on $c \in A^{\mathbb{Z}^2}$ if $p_{i+1} = p_i + \delta(c_{p_i})$ for all $i \in \{1, \dots, k-1\}$. A δ -path (p_1, \dots, p_k) is (\mathcal{A}) -valid if

1. for every $i \in \{1, \dots, k\}$ we have that $\sigma_{p_i}(c)_{\mathcal{C}_n} \in \mathcal{A}$, and
2. for all $q \in \mathbb{Z}^2$ we have that

$$p_i + \delta(c_{p_i}) = q + \delta(q) \implies q = p_i.$$

In other words, a path is valid if every pattern along the path is valid and the path does not branch backwards. A position $p \in \mathbb{Z}^2$ is *valid* if it is part of a valid path and *invalid* otherwise. A pair (\mathcal{A}, δ) is an *orientation* on the full shift $A^{\mathbb{Z}^2}$. An orientation (\mathcal{A}, δ) is *acyclic* if every \mathcal{A} -valid δ -path contains no cycles. Let $c \in A^{\mathbb{Z}^2}$ and $p \in \mathbb{Z}^2$ be a valid position. Then p is a *beginning* of a valid path if (q, p) is not a valid path for any $q \in \mathbb{Z}^2$. Similarly, p is an *end* of a valid path if (p, q) is not a valid path for any $q \in \mathbb{Z}^2$. Lastly p is *in the middle* of a valid path if it is neither an end nor a beginning of a path. Notice that being an end, a beginning, or in the middle of a valid path is a local property. Directly from [35] we get the following result.

Proposition 6.2.1 ([35]). *Given an acyclic orientation (\mathcal{A}, δ) on $A^{\mathbb{Z}^2}$, it is undecidable whether there exists an infinite \mathcal{A} -valid δ -path.*

Notice that, by compactness, we have that if all valid paths are finite, then there is a global bound on the length of the valid paths.

Using similar notations as Tom Meyerovitch in [47] we write $I(c)$ for the maximal number of pairwise disjoint infinite valid paths in c , and for an orientation (\mathcal{A}, δ) we denote $I(\mathcal{A}, \delta) = \sup_{c \in A^{\mathbb{Z}^2}} I(c)$. Considerations of [47, Section 4] say that only a bounded number of infinite valid paths that the acyclic orientation in [35] defines can fit in any one configuration. Combining this with Proposition 6.2.1 above we get the following.

Proposition 6.2.2. *Given an acyclic orientation (\mathcal{A}, δ) on $A^{\mathbb{Z}^2}$ such that $I(\mathcal{A}, \delta) < \infty$, it is undecidable whether $I(\mathcal{A}, \delta) = 0$ or not.*

Now we proceed to defining cellular automata by adding a layer on top of $A^{\mathbb{Z}^2}$ in a similar fashion as in [35] and [47]. For an acyclic orientation (\mathcal{A}, δ) with $I(\mathcal{A}, \delta) < \infty$ we want to define two reversible cellular automata such that if $I(\mathcal{A}, \delta) = 0$ then the cellular automata are block conjugate and have zero entropy, and if $I(\mathcal{A}, \delta) \neq 0$, then one of the cellular automata has strictly larger entropy than the other one.

On top of $A^{\mathbb{Z}^2}$ we put another layer $B^{\mathbb{Z}^2}$ on which we simulate one-dimensional cellular automata $B^{\mathbb{Z}} \rightarrow B^{\mathbb{Z}}$ on the valid paths of $A^{\mathbb{Z}^2}$. We want to simulate the shift map σ . However, the simple shift map alone leads to non-reversible cellular automata on finite valid paths as information is either lost or has to be made up at the beginnings and ends of valid paths. To avoid this we take B to be $B_1 \times B_1$ for some finite set B_1 and the map $\hat{\sigma} : B^{\mathbb{Z}} \rightarrow B^{\mathbb{Z}}$ that shifts the first track to the left and the second one to the right, i.e. $\hat{\sigma}(c, e)_i = (c_{i+1}, e_{i-1})$ for all $(c, e) \in B^{\mathbb{Z}} = (B_1 \times B_1)^{\mathbb{Z}}$. In the beginnings and ends of valid paths we simply move the content from one track to the other forming a cycle. For technical reasons we further take that $B_1 = B_2 \times B_2$ for a finite set B_2 with at least two elements. Now our one-dimensional cellular automaton is a map $(B_2^4)^{\mathbb{Z}} \rightarrow (B_2^4)^{\mathbb{Z}}$ but this really should be considered as one two-track tape, one track moving to left and the other to right. The choice $B_1 = B_2 \times B_2$ is done so that we can give two different ways to restrict $\hat{\sigma}$ to the finite valid paths.

We define two maps, ν and μ , on finite words $(B_2 \times B_2)^+$ as follows:

$$\nu \begin{pmatrix} u_0 & u_1 & \cdots & u_{n-2} & u_{n-1} \\ v_0 & v_1 & \cdots & v_{n-2} & v_{n-1} \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & \cdots & u_{n-1} & u_0 \\ v_1 & v_2 & \cdots & v_{n-1} & v_0 \end{pmatrix},$$

and μ as

$$\mu \begin{pmatrix} u_0 & u_1 & \cdots & u_{n-2} & u_{n-1} \\ v_0 & v_1 & \cdots & v_{n-2} & v_{n-1} \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & \cdots & u_{n-1} & v_0 \\ v_1 & v_2 & \cdots & v_{n-1} & u_0 \end{pmatrix},$$

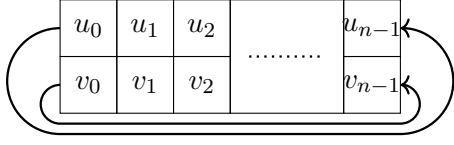


Figure 6.1: Illustration of ν .

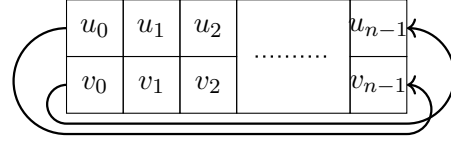


Figure 6.2: Illustration of μ .

where $\begin{smallmatrix} u_i \\ v_i \end{smallmatrix} \in B_2 \times B_2$ for all $i \in \{0, 1, \dots, n-1\}$. The map ν is obtained by taking a finite word, gluing the ends together, and applying the shift map $\sigma : (B_2 \times B_2)^{\mathbb{Z}} \rightarrow (B_2 \times B_2)^{\mathbb{Z}}$ locally (Figure 6.1). One can also consider μ to be obtained from σ by gluing the ends of finite words together, but this time the tape is also flipped to form a Möbius strip (Figure 6.2). Notice that if we restrict ν and μ to the words of even length $2n$, then we have a bijection ϕ such that $\nu = \phi^{-1}\mu^2\phi$, namely

$$\phi \begin{pmatrix} u_0 & u_1 & \cdots & u_{2n-2} & u_{2n-1} \\ v_0 & v_1 & \cdots & v_{2n-2} & v_{2n-1} \end{pmatrix} = \begin{pmatrix} u_0 & v_0 & \cdots & u_{n-1} & v_{n-1} \\ u_n & v_n & \cdots & u_{2n-1} & v_{2n-1} \end{pmatrix}. \quad (6.1)$$

Let (\mathcal{A}, δ) be an acyclic orientation of $A^{\mathbb{Z}^2}$ with $I(\mathcal{A}, \delta) < \infty$. We define two cellular automata $F_\nu, F_\mu : (A \times B)^{\mathbb{Z}^2} \rightarrow (A \times B)^{\mathbb{Z}^2}$ where $B = (B_2 \times B_2)^2$ as was defined above. Both map the A -layer by identity. On the B -layer we use ν for F_ν and μ for F_μ . To be more exact: Let $c \in A^{\mathbb{Z}^2}$, $e \in B^{\mathbb{Z}^2}$, $p_1 \in \mathbb{Z}^2$, and $e_{p_1} = (a_1, b_1, x_1, y_1)$. We define F_ν and F_μ in cases:

- If p_1 is not part of a valid path in c , then

$$F_\nu(c, e)_{p_1} = F_\mu(c, e)_{p_1} = (c, e)_{p_1}.$$

- If p_1 is a beginning of a valid path, and there exists $p_2 \in \mathbb{Z}^2$ such that (p_1, p_2) is valid (so that p_1 is not also an end), and let $e_{p_2} = (a_2, b_2, x_2, y_2)$, then

$$F_\nu(c, e)_{p_1} = F_\mu(c, e)_{p_1} = (c_{p_1}, (a_2, b_2, a_1, b_1)).$$

- If p_1 is in the middle of a valid path, say (p_0, p_1, p_2) is valid, and $e_{p_0} = (a_0, b_0, x_0, y_0)$, $e_{p_2} = (a_2, b_2, x_2, y_2)$, then

$$F_\nu(c, e)_{p_1} = F_\mu(c, e)_{p_1} = (c_{p_1}, (a_2, b_2, x_0, y_0)).$$

We are left with the cases where F_ν and F_μ behave differently, namely, at the ends of valid paths.

- If p_1 is an end of a valid path, $p_0 \in \mathbb{Z}^2$ such that (p_0, p_1) is valid, and $e_{p_0} = (a_0, b_0, x_0, y_0)$, then

$$F_\nu(c, e)_{p_1} = (c_{p_1}, (x_1, y_1, x_0, y_0)) \quad \text{and} \quad F_\mu(c, e)_{p_1} = (c_{p_1}, (y_1, x_1, x_0, y_0)).$$

- If p_1 is both the beginning and the end of a valid path, then

$$F_\nu(c, e)_{p_1} = (c_{p_1}, (x_1, y_1, a_1, b_1)) \quad \text{and} \quad F_\mu(c, e)_{p_1} = (c_{p_1}, (y_1, x_1, a_1, b_1)).$$

All this is to say that F_ν and F_μ simulate ν and μ (resp.) on the valid paths. Notice that F_μ^2 simulates μ^2 on the valid paths, so it is natural to define $F_{\mu^2} = F_\mu^2$.

We are ready to prove the following result.

Theorem 6.2.3. *The following two sets of pairs of reversible two-dimensional cellular automata are recursively inseparable:*

- (i) *pairs where the first cellular automaton has strictly higher entropy than the second one, and*
- (ii) *pairs that are block conjugate and both have zero entropy.*

Proof. Suppose these sets are separable, we use this to decide the decision problem of Proposition 6.2.2.

Let (\mathcal{A}, δ) be a given acyclic orientation on $A^{\mathbb{Z}^2}$ such that $I(\mathcal{A}, \delta) < \infty$. Construct cellular automata $((A \times B)^{\mathbb{Z}^2}, F_\nu)$ and $((A \times B)^{\mathbb{Z}^2}, F_{\mu^2})$ as described above. We claim that

$$\begin{aligned} I(\mathcal{A}, \delta) > 0 &\implies h(F_{\mu^2}) > h(F_\nu) \\ I(\mathcal{A}, \delta) = 0 &\implies F_\nu \cong_b F_{\mu^2} \text{ and } h(F_\nu) = h(F_{\mu^2}) = 0. \end{aligned}$$

Suppose that $0 < I(\mathcal{A}, \delta) < \infty$: The claim follows from the reasoning of Tom Meyerovitch [47, Lemma 3.2., Lemma 3.3., Theorem 3.4.]; let us outline this reasoning. Consider the cellular automaton $((A \times B)^{\mathbb{Z}^2}, F_\nu)$ (the same reasoning applies to $((A \times B)^{\mathbb{Z}^2}, F_{\mu^2})$). Let $d \in A^{\mathbb{Z}^2}$ be arbitrary and define $C_d = \{(c, e) \in (A \times B)^{\mathbb{Z}^2} \mid c = d\} \subseteq (A \times B)^{\mathbb{Z}^2}$, i.e. the configurations with a fixed background. Clearly C_d is closed, and thus compact, and also clearly $F_\nu(C_d) \subseteq C_d$, i.e. (C_d, F_ν) is a dynamical system (not necessarily a cellular automaton though, since C_d is not in general shift-invariant). In other words $((A \times B)^{\mathbb{Z}^2}, F_\nu)$ is a disjoint union of dynamical systems $\bigcup_{d \in A^{\mathbb{Z}^2}} (C_d, F_\nu)$. Then we have that $h((A \times B)^{\mathbb{Z}^2}, F_\nu) = \sup_{d \in A^{\mathbb{Z}^2}} \{h(C_d, F_\nu)\}$ according to T. N. T. Goodman [24, Corollary 1] (this is a corollary to the famous variational principle which relates topological and measure theoretical entropies). Now the entropy of any (C_d, F_ν) is directly tied to the entropy of ν as F_ν simulates ν on the valid paths. Notice that since the one-dimensional cellular automata we simulate are two-sided unlike the one-sided cellular automata simulated by Meyerovitch, an explicit formula for the entropy of F_ν is not necessary exactly as in [47]. However, it is obvious that $h(C_d, F_\nu) > 0$ for some $d \in A^{\mathbb{Z}^2}$, and if $h(C_d, F_\nu) > 0$ then

$h(C_d, F_\nu) < h(C_d, F_{\mu^2})$, and if $d' \in A^{\mathbb{Z}^2}$ is such that $h(C_{d'}, F_\nu) > h(C_d, F_\nu)$ then also $h(C_{d'}, F_{\mu^2}) > h(C_d, F_{\mu^2})$, and these are enough to conclude that

$$h((A \times B)^{\mathbb{Z}^2}, F_\nu) < h((A \times B)^{\mathbb{Z}^2}, F_{\mu^2}).$$

Suppose that $I(\mathcal{A}, \delta) = 0$: Now there can be only finite valid paths. By compactness we have a global bound $M \in \mathbb{N}$ such that for any valid path (p_1, p_2, \dots, p_k) it holds that $k < M$. Of course there then also exists $m \in \mathbb{N}$ such that any valid path fits inside a suitably positioned \mathcal{C}_m . We define a block conjugacy H_ϕ of $((A \times B)^{\mathbb{Z}^2}, F_\nu)$ and $((A \times B)^{\mathbb{Z}^2}, F_{\mu^2})$ based on the map ϕ defined by (6.1) above. The local rule of H_ϕ has domain $(A \times B)^{[-m, m+n]^2}$ where n is such that $\mathcal{A} \subseteq A^{\mathcal{C}^n}$. This domain guarantees that for any $c \in (A \times B)^{\mathbb{Z}^2}$ and $p \in \mathbb{Z}^2$ we can recognize the entire valid path that p is part of. Let $(c, e) \in (A \times B)^{\mathbb{Z}^2}$. We define $H_\phi(c, e)_p$ for an arbitrary $p \in \mathbb{Z}^2$. If p is not part of a valid path, then $H_\phi(c, e)_p = (c, e)_p$. Suppose p is part of a valid path and let (p_1, p_2, \dots, p_k) be the valid path such that p_1 is the beginning of the path, p_k the end of the path, and $p = p_i$ for some $i \in \{1, \dots, k\}$. As pointed out, the local neighborhood is large enough so that the local rule sees this entire valid path and can verify its validity on each position of the valid path. Denote $e_{p_j} = (a_j, b_j, x_j, y_j)$ for all $j \in \{1, \dots, k\}$. Now we define

$$H_\phi(c, e)_p = \left(c_p, \left(\phi \left(\begin{array}{cccccc} x_k & \cdots & x_1 & a_1 & \cdots & a_k \\ y_k & \cdots & y_1 & b_1 & \cdots & b_k \end{array} \right)_{k+i-1}, \right. \right. \\ \left. \left. \phi \left(\begin{array}{cccccc} x_k & \cdots & x_1 & a_1 & \cdots & a_k \\ y_k & \cdots & y_1 & b_1 & \cdots & b_k \end{array} \right)_{i-1} \right) \right).$$

Since ϕ is a bijection on words of even length we get that $((A \times B)^{\mathbb{Z}^2}, F_\nu) \cong_b ((A \times B)^{\mathbb{Z}^2}, F_{\mu^2})$. Since there are only finite valid paths, F_ν and F_{μ^2} are periodic, and so they have zero entropy. \square

The same way we got Corollary 6.1.5 from Theorem 6.1.4, we now get the following corollary.

Corollary 6.2.4. *Let $(A^{\mathbb{Z}^2}, F)$ and $(A^{\mathbb{Z}^2}, G)$ be two reversible cellular automata. Then the following hold:*

1. *It is undecidable whether F and G are (block) conjugate.*
2. *It is undecidable whether F is a (block) factor of G .*
3. *It is undecidable whether F is a (block) subsystem of G .*

Remark 6.2.5. Here is an alternative construction which uses one-sided reversible cellular automata, which allows using the construction and arguments of [47] more directly. Let $\mathcal{Z} : \{0, 1, 2\}^{\mathbb{N}} \rightarrow \{0, 1, 2\}^{\mathbb{N}}$ be the zot cellular automaton, and $A^{\mathbb{Z}^2}$ a full shift with an orientation (\mathcal{A}, δ) . We simulate \mathcal{Z} on valid paths as we did ν and μ . Since \mathcal{Z} is one-sided, we do not need multiple tracks in this construction. More precisely $F_{\mathcal{Z}} : (A \times \{0, 1, 2\})^{\mathbb{Z}^2} \rightarrow (A \times \{0, 1, 2\})^{\mathbb{Z}^2}$ is defined by $F_{\mathcal{Z}}((c, e))_p = (c_p, \mathcal{Z}(e_p e_{p+\delta(c_p)}))$ on beginnings and middle points of valid paths, and by identity on invalid positions. We still have to define $F_{\mathcal{Z}}$ on the ends of valid paths: the value at the end of a valid path is permuted by π_1 . In other words, $F_{\mathcal{Z}}$ uses the map π from Proposition 4.2.4 on finite (or left-infinite) paths.

Suppose there exists only finite valid paths. Then there is an upper bound $n \in \mathbb{N}$ for the length of the valid paths. By Proposition 4.2.4 we know that on a finite path of length $k \leq n$ the cellular automaton $F_{\mathcal{Z}}$ enumerates all finite words in $\{0, 1, 2\}^k$. The same holds also for $F_{\mathcal{Z}}^2$ since 3^n is odd. Now we can define a conjugacy ϕ which fixes the $A^{\mathbb{Z}^2}$ -layer and all letters on invalid paths on the $\{0, 1, 2\}^{\mathbb{Z}^2}$ -layer, and, slightly informally, on valid paths of length k we define $\phi : \{0, 1, 2\}^k \rightarrow \{0, 1, 2\}^k$ by setting $\phi(\pi^i(0^k)) = \pi^{2i}(0^k)$ for all $i \in \{0, 1, \dots, 3^k - 1\}$. This shows that $F_{\mathcal{Z}}$ and $F_{\mathcal{Z}}^2$ are block conjugate.

Suppose there exists infinite valid paths. Notice that in this construction we should let $I(\mathcal{A}, \delta)$ denote the maximal number of *forward* infinite paths (as in [47]), since only the forward infinite paths contribute to the entropy. Then $F_{\mathcal{Z}}^2$ has larger entropy than $F_{\mathcal{Z}}$ since \mathcal{Z}^2 has higher entropy than \mathcal{Z} . This concludes the proof.

From this remark we get the following variant.

Corollary 6.2.6. *The following sets of two-dimensional reversible cellular automata are recursively inseparable:*

- (i) *cellular automata $(A^{\mathbb{Z}^2}, F)$ such that F^2 has strictly higher entropy than F , and*
- (ii) *cellular automata $(A^{\mathbb{Z}^2}, F)$ such that $F \cong_b F^2$, and $h(A^{\mathbb{Z}^2}, F) = 0$.*

Notice that Epperlein's result [20, Corollary 5.19.] says that conjugacy is undecidable even among two-periodic cellular automata, while all our results restrict only to reversible cellular automata. Strengthening the undecidability of block conjugacy to periodic cellular automata seems plausible using the cellular automata from [20, Example 7.6.] in our construction instead of ν and μ . Example [20, Example 7.6.] presents two one-dimensional cellular automata which are (temporally) periodic and conjugate on (spatially) periodic configurations but not conjugate in general, and thus not block conjugate either. Using these we would still have that if all valid paths are

finite then the constructed cellular automata are conjugate. However, the entropy argument does not work in the case that also infinite valid paths exist, since the entropy of a periodic cellular automaton is zero. It is not clear that even though the one-dimensional cellular automata simulated are not conjugate that the two-dimensional cellular automata could not be.

Lastly we note that in [47, Question 6.1.] Meyerovitch asked whether for $d > 1$ there exists an injective d -dimensional cellular automaton which has finite non-zero entropy. Either of the constructions given above explicitly gives a positive answer to this question for $d = 2$. Simulating the one-dimensional cellular automata presented here on Meyerovitch's d -dimensional oriented full shifts, one also gets a positive answer for any larger d .

Proposition 6.2.7. *For any $d \in \mathbb{N} \setminus \{0\}$ there exists a reversible cellular automaton $(A^{\mathbb{Z}^d}, F)$ such that $0 < h(A^{\mathbb{Z}^d}, F) < \infty$.*

6.3 Fixing the Alphabet

In the previous section we proved some undecidability results for cellular automata as dynamical systems. A small change in the point of view leads to asking algebraic variants of these questions. More specifically, cellular automata over a fixed alphabet form a monoid while reversible cellular automata over a fixed alphabet form a group (function composition as the product). However, the undecidability results of the previous sections utilized arbitrarily large alphabets in order to achieve the undecidability results. In this section we show that the main results of the previous sections remain true even over fixed alphabets.

6.3.1 Fixing the Alphabet for One-Dimensional Cellular Automata

Looking at the proof of Theorem 6.1.4 one notices that there are two places where we used arbitrarily large alphabets: The underlying alphabet of the cellular automaton whose nilpotency we wanted to decide, and the alphabet of \mathcal{Z}_{2k} which was used to increase the entropy. Since we are no longer allowed to increase the size of the alphabet we want to replace these by instead increasing the size of the neighborhood. We use the fixed alphabet variant of nilpotency problem proved by Bruno Durand, Enrico Formenti, and Georges Varouchas [18], and instead of cartesian product we take powers of \mathcal{Z} , which also increases the entropy.

Our proof relies on the following result.

Theorem 6.3.1. *[18, Proposition 2.4] Given a cellular automaton $(\{0, 1\}^{\mathbb{N}}, F)$*

such that $r(F) = 2t$ for some $t \in \mathbb{N}$, and if 0^t appears in $u \in \{0, 1\}^{2t+1}$ then $F_{loc}(u) = 0$, it is undecidable whether F is nilpotent or not.

Using this we can prove the following fixed alphabet version of Theorem 6.1.4.

Theorem 6.3.2. *The following two sets of pairs of one-dimensional one-sided cellular automata over the alphabet $\{0, 1, 2\} \times \{0, 1\}$ are recursively inseparable:*

(i) *pairs where the first cellular automaton has strictly higher entropy than the second one, and*

(ii) *pairs that are strongly conjugate and both have zero topological entropy.*

Proof. Let $(\{0, 1\}^{\mathbb{N}}, H)$ be an instance of the decision problem of Theorem 6.3.1 with some $t \in \mathbb{N}$. Let $(A^{\mathbb{N}}, \mathcal{Z})$ be the zot cellular automaton. The cellular automaton $(\{0, 1\}^{\mathbb{N}}, H)$ will again draw the background and zot will be drawn on top of this, as long as the long sequences of zeroes which represent quiescent states do not appear. Again, we need the zot-layer to overpower the background layer in regards of entropy. This time, however, we cannot use the cartesian product trick, since it would increase the size of the alphabet. Instead, we use the fact that $h(A^{\mathbb{N}}, \mathcal{Z}^k) = k \cdot h(A^{\mathbb{N}}, \mathcal{Z})$; it is sufficient to take k to be larger than $4t$.

We define cellular automata $((A \times \{0, 1\})^{\mathbb{N}}, \mathcal{F})$ and $((A \times \{0, 1\})^{\mathbb{N}}, \mathcal{G})$. Again, $((A \times \{0, 1\})^{\mathbb{N}}, \mathcal{G})$ is just the cartesian product $\mathcal{G} = \text{id} \times H$. Before defining $((A \times \{0, 1\})^{\mathbb{N}}, \mathcal{F})$ let us introduce a notation. For any $u \in A^*$, $a \in A$ let $\mathcal{Z}^k(u\dot{a}) \in A^{|u|+1}$ denote the word which is obtained by applying \mathcal{Z} for k times to u with a fixed a in the end, i.e. it is defined as $\mathcal{Z}^k(u\dot{a}) = \mathfrak{z}(\mathfrak{z}(\cdots(\mathfrak{z}(ua)a)\cdots)a)a$ where \mathfrak{z} is the restriction of \mathcal{Z} to finite words. Now let $(c, e) \in (A \times \{0, 1\})^{\mathbb{N}}$ and define

$$\mathcal{F}(c, e)_0 = \begin{cases} (\mathcal{Z}^k(c)_0, H(e)_0), & \text{if } e_{[0, 2k+t]} \text{ does not contain } 0^t \\ (\mathcal{Z}^k(c_0 \cdots \dot{c}_i)_0, H(e)_0), & i = \min\{j \mid j \in [0, 2k], e_{[j, j+t]} = 0^t\} \end{cases}.$$

(i) **Suppose $(\{0, 1\}^{\mathbb{N}}, H)$ is not nilpotent.** Entropy of $((A \times \{0, 1\})^{\mathbb{N}}, \mathcal{G})$ is

$$h((A \times \{0, 1\})^{\mathbb{N}}, \mathcal{G}) = h(\{0, 1\}^{\mathbb{N}}, H) + h(A^{\mathbb{N}}, \text{id}) = h(\{0, 1\}^{\mathbb{N}}, H).$$

There exists a configuration $c \in \{0, 1\}^{\mathbb{N}}$ such that no appearances of 0^t occur in the space-time diagram of c under H . On this background $((A \times \{0, 1\})^{\mathbb{N}}, \mathcal{F})$ simulates the k^{th} power of zot on the A -layer. Thus we have

$$h((A \times \{0, 1\})^{\mathbb{N}}, \mathcal{F}) \geq h(A^{\mathbb{N}}, \mathcal{Z}^k) = \frac{k}{2} > 2t \geq h(\{0, 1\}^{\mathbb{N}}, H).$$

This proves the claim.

(ii) Suppose $(\{0, 1\}^{\mathbb{N}}, H)$ is nilpotent: We do as in the proof of Theorem 6.1.4. Only point that deserves to be reconsidered in this fixed alphabet variant is that the map

$$\begin{aligned} \pi_A \mathcal{F}^n(-, e) : A^{\mathbb{N}} &\longrightarrow A^{\mathbb{N}} \\ c &\longmapsto \pi_A \mathcal{F}^n(c, e) \end{aligned}$$

is again a bijection for every $e \in \{0, 1\}^{\mathbb{N}}$. Again, it is enough that $A^{\mathbb{N}} \rightarrow A^{\mathbb{N}}, c \mapsto \pi_A \mathcal{F}(c, e)$ is a bijection for an arbitrary $e \in B^{\mathbb{N}}$. Notice that $\theta : A^n \rightarrow A^n$ defined by $\theta(ua) = \mathcal{Z}^k(ua)$ is a bijection. Let $x \in A^{\mathbb{N}}$ and define c as follows:

- If there exists $j \in [i, i + 2k]$ such that $e_{[j, j+t]} = 0^t$ then let $c_i = \theta^{-1}(x_{[i, j]})_0$.
- If 0^t does not appear in $e_{[i, i+2k+t]}$ then let $c_i = \mathcal{Z}^{-1}(x)_i$.

Again it is clear that this is a preimage of x under the map defined above, and also that no other preimages can exist. □

6.3.2 Fixing the Alphabet for Two-Dimensional Cellular Automata

In the proof of Theorem 6.2.3, the two-dimensional inseparability result, obtaining a difference in entropy did not require increasing the size of the alphabet. However, the original proof of Proposition 6.2.2 (whether an acyclic orientation with $I(\mathcal{A}, \delta) < \infty$ has any infinite valid paths at all) has instances with arbitrarily large alphabets. This can be turned into a binary version, as is done in the following proof (we simultaneously embed the simulations of \mathcal{Z} to the binary encoding, so that overall we only need a binary alphabet).

Theorem 6.3.3. *The following sets of two-dimensional reversible cellular automata are recursively inseparable:*

- (i) cellular automata $(\{0, 1\}^{\mathbb{Z}^2}, F)$ such that $(\{0, 1\}^{\mathbb{Z}^2}, F^2)$ has strictly higher entropy than $(\{0, 1\}^{\mathbb{Z}^2}, F)$, and
- (ii) cellular automata $(\{0, 1\}^{\mathbb{Z}^2}, F)$ such that $(\{0, 1\}^{\mathbb{Z}^2}, F) \cong_b (\{0, 1\}^{\mathbb{Z}^2}, F^2)$, and $h(\{0, 1\}^{\mathbb{Z}^2}, F) = 0$.

Proof. Suppose these sets are separable; we reduce the decision problem of Proposition 6.2.2 to this problem. Let (\mathcal{A}, δ) be a given acyclic orientation on $A^{\mathbb{Z}^2}$ such that $I(\mathcal{A}, \delta) < \infty$. Let $l > \log_2(|A|)$ and let $\iota_1 : \{0, 1, 2\} \rightarrow \{0, 1\}^2$ and $\iota_2 : A \rightarrow \{0, 1\}^l$ be injections (the map ι_1 is used to embed the ternary

$$(a, x) \longrightarrow \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & \boxed{\iota_1(x)} & \boxed{\iota_2(a)} & 0 & & & \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline \end{array}$$

Figure 6.3: Substitution s in the case that $l = 3$.

alphabet of the zot cellular automaton, and ι_2 is used to embed the alphabet A). Let $D = [0, l + 5) \times [0, 4)$ and define a map $s : A \times \{0, 1, 2\} \rightarrow \{0, 1\}^D$ by

$$\begin{aligned} s(a, x)_{(i,j)} &= 1, \text{ if } i = 0 \text{ or } j = 0 \\ s(a, x)_{(i,j)} &= 0, \text{ if } i \neq 0 \text{ and } j \neq 0 \text{ and } (i, j) \notin [2, l + 4) \times \{2\} \\ s(a, x)_{(i,j)} &= \iota_1(x)_{i-2}, \text{ if } (i, j) \in \{(2, 2), (3, 2)\} \\ s(a, x)_{(i,j)} &= \iota_2(a)_{i-4}, \text{ if } (i, j) \in [4, l + 4) \times \{2\}. \end{aligned}$$

for all $a \in A$ (see Figure 6.3). We extend the definition of s to any finite rectangle: For arbitrary $w, w', h, h' \in \mathbb{N}$, where $w < w', h < h'$, we have that s defines a map $(A \times \{0, 1, 2\})^{[w, w') \times [h, h')} \rightarrow \{0, 1\}^{[w(l+5), w'(l+5)) \times [4h, 4h')}$ so that the element in position (i, j) in the original pattern is mapped to a pattern on $[i(l + 5), (i + 1)(l + 5)) \times [4j, 4(j + 1))$. Since s is an injection we can define a map s_{inv} on the image of s such that $s_{inv}s(c) = c$ for all $c \in (A \times \{0, 1, 2\})^{[w, w') \times [h, h')}$. For convenience we extend s_{inv} into a map $\{0, 1\}^{[w(l+5), w'(l+5)) \times [4h, 4h')} \rightarrow (A \times \{0, 1, 2\} \cup \{\#\})^{[w, w') \times [h, h')}$ where any invalid encoding is mapped to $\#$.

Let $F_{\mathcal{Z}}$ be as in Remark 6.2.5 and let $n \in \mathbb{N}$ be such that $F_{\mathcal{Z},loc}$ is a map $(A \times \{0, 1, 2\})^{[-n, n]^2} \rightarrow A \times \{0, 1, 2\}$. We define a new cellular automaton $(\{0, 1\}^{\mathbb{Z}^2}, G_{\mathcal{Z}})$ which simulates $F_{\mathcal{Z}}$ over a binary alphabet. Let $P \in \{0, 1\}^{[-n(l+5), (n+1)(l+5)) \times [-4n, 5n)}$ so that $s_{inv}(P) \in (A \times \{0, 1, 2\} \cup \{\#\})^{[-n, n] \times [-n, n]}$. First $G_{\mathcal{Z},loc}$ maps the pattern P to $s_{inv}(P)$. If no symbols $\#$ appear, $G_{\mathcal{Z},loc}$ proceeds by mapping $s_{inv}(P)$ to $F_{\mathcal{Z},loc}(s_{inv}(P))$, and then back to the binary alphabet, i.e. to $s(F_{\mathcal{Z},loc}(s_{inv}(P)))$. Here we described how $G_{\mathcal{Z},loc}$ replaces the $[0, l + 5) \times [0, 4)$ rectangle, but this can clearly also be done locally since any cell can recognize locally both whether it is part of a pattern which s_{inv} maps to a pattern containing no symbols $\#$ and also which cell in the central rectangle it is. If $\#$ does appear, local rule is identity.

Now we have that $(\{0, 1\}^{\mathbb{Z}^2}, G_{\mathcal{Z}}) \cong_b (\{0, 1\}^{\mathbb{Z}^2}, G_{\mathcal{Z}}^2)$ if and only if $((A \times \{0, 1, 2\})^{\mathbb{Z}^2}, F_{\mathcal{Z}}) \cong_b ((A \times \{0, 1, 2\})^{\mathbb{Z}^2}, F_{\mathcal{Z}}^2)$. □

Chapter 7

Open Problems

In Chapter 3 we defined stripe shifts and proved some preliminary results about them; for example, we proved that sofic stripe shifts are exactly the zero entropy sofic shifts. We also saw that there are uncountable (non-sofic) stripe shifts. Many questions were left open, we mention the following which could be quite attainable.

Question 7.0.1. *Do stripe shifts have zero entropy? Can the higher-than-one-dimensional cellular automata draw full shift as stripes?*

In Chapter 4 we considered reversible one-sided one-dimensional cellular automata over full shifts (ROCA's). There are no known undecidability results regarding the dynamics of ROCA's.

Question 7.0.2. *Is expansivity, pseudo-orbit tracing property (POTP), or periodicity decidable for ROCA's?*

Periodicity was conjectured to be undecidable for ROCA's by Delacourt and Ollinger [16]. Notice that if POTP is decidable then so are periodicity and expansivity, since both periodic and expansive ROCA's have POTP (this is obvious for any periodic cellular automaton, and follows from Theorem 5.2.2 for expansive ROCA's), and amongst ROCA's with POTP periodicity and expansivity are decidable.

About the topics of Chapter 5 the following question is the obvious one to state; this question has been raised before, even as a conjecture by Kůrka [41].

Question 7.0.3. *Do expansive two-sided cellular automata (over full shifts) have POTP or equivalently are they conjugate to SFT's?*

We do not have an example of a right-expansive cellular automaton which would not have right-POTP. Such an example would be interesting, since if right-expansivity implies right-POTP then the above problem would be positively answered. This can be formulated also in the following way.

Question 7.0.4. *Let $(A^{\mathbb{M}}, F)$ be a right-expansive cellular automaton (it makes sense to start with the full shifts). Does there exist $i \in \mathbb{N}, m \in \mathbb{N}$ such that $\mathfrak{F}_m({}_i\Sigma_i^{(m)}) \subseteq {}_i\Sigma_i^{(m)}$?*

Main results of Chapter 6 were that block conjugacy of one-dimensional one-sided cellular automata is undecidable as is also the block conjugacy of reversible two-dimensional two-sided cellular automata. There are some obvious questions that arise.

Question 7.0.5. *Is conjugacy of reversible one-dimensional (one- or two-sided) cellular automata undecidable? What about conjugacy of reversible two-dimensional one-sided cellular automata?*

Our proof that conjugacy of reversible two-dimensional cellular automata is undecidable used key parts from the proof of undecidability of reversibility of two-dimensional cellular automata, the same question is unknown for two-dimensional one-sided cellular automata

Question 7.0.6. *Is reversibility undecidable for two-dimensional one-sided cellular automata?*

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