UNIVERSITY OF TURKU

GEOMETRY OF INTRINSIC METRICS

Oona Rainio

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ABSTRACT

The study of intrinsic metrics is an interesting area of research in the geometric function theory, which is a subfield of mathematical analysis. The topics of research include quasiregular mappings, conformal capacity and boundary geometry of domains. I study here the inequalities between different hyperbolic type metrics, focusing especially on the properties of the triangular ratio metric.

This work consists of six original articles publicly available on arXiv.org. The first article introduces several sharp inequalities between the triangular ratio metric, the hyperbolic metric and other hyperbolic type metrics in an open sector of the complex plane. A new result describing the distortion of the triangular ratio metric under quasiconformal mappings is also given in this article.

In the second and the third articles, the so-called midpoint rotation is used to create inequalities for the triangular ratio metric. Namely, the second article defines the triangular ratio metric and the Möbius metric in an annular ring domain, explains how these metrics can be efficiently computed in this domain and also gives a new Möbius-invariant lower bound for the conformal capacity. In the third article, the value of the triangular ratio metric is estimated in the unit disk by using both the Euclidean and the hyperbolic midpoint rotations.

The fourth article concerns two intrinsic quasi-metrics, out of which one is already known and the other is first introduced in this paper, and shows how they offer upper and lower bounds for the triangular ratio metric. In the fifth article, the results of the third and the fourth articles are used to obtain new information about the distortion of the intrinsic metrics under conformal and quasiregular mappings. The sixth article deals with two domain functionals defined with the hyperbolic metric, uses them to study the uniform perfectness and gives a new lower bound for the conformal capacity.

These six articles offer the reader an advanced understanding of intrinsic metrics, the inequalities between them and their behaviour under different types of mappings. The results found here can be applied, for instance, to study the conformal capacity further or find new information about the intrinsic geometry of numerous domains. Studying the conjectures introduced in the articles can also provide ground for future research.

KEYWORDS: Conformal capacity, hyperbolic geometry, intrinsic metrics, quasiregular mappings, uniform perfectness

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TIIVISTELMÄ

Intrinsisten metriikoiden tutkimus on mielenkiintoinen osa geometrista funktioteoriaa, joka on puolestaan matemaattisen analyysin osa-alue. Tutkimusaiheisiin kuuluvat kvasisäännölliset kuvaukset, konforminen kapasiteetti ja metriikoiden määrittelyjoukon reunan geometria. Tutkin tässä työssä useiden hyperbolistyyppisten metriikoiden välisiä epäyhtälöitä keskittyen eritoten kolmisuhdemetriikan eri ominaisuuksiin.

Väitöskirjani koostuu kuudesta alkuperäisartikkelista, jotka ovat julkisesti saatavilla arXiv.org-nettisivustolla. Ensimmäinen artikkeli esittelee useita tarkkoja epäyhtälöitä kolmisuhdemetriikalle, hyperboliselle metriikalle ja muille hyperbolistyyppisille metriikoille kompleksitason avoimessa sektorissa. Artikkelissa annetaan myös uusi tulos kolmisuhdemetriikan käyttäytymisestä kvasikonformikuvauksissa.

Toinen ja kolmas artikkeli hyödyntävät kolmisuhdemetriikan tutkimuksessa uutta menetelmää, jossa tarkasteltavat pisteet kierretään niiden keskipisteen suhteen. Toinen artikkeli tutkii kolmisuhdemetriikkaa ja Möbius-metriikkaa renkaan muotoisessa joukossa, antaa tapoja näiden metriikoiden arvojen laskemiseksi ja esittelee uuden Möbius-invariantin alarajan konformiselle kapasiteetille. Kolmas artikkeli sen sijaan hyödyntää sekä euklidista että hyperbolista kiertoa, ja antaa uudet ylä- ja alarajat yksikkökiekossa määritellylle kolmisuhdemetriikalle.

Neljäs artikkeli käsittelee kahta kolmisuhdemetriikan tutkimuksen kannalta hyödyllistä intrinsistä kvasimetriikkaa, joista toinen on jo tunnettu ja toinen esitellään artikkelissa ensimmäistä kertaa. Viidennessä artikkelissa yhdistetään kolmannen ja neljännen artikkelin tuloksia, ja luodaan niiden pohjalta epäyhtälöitä intrinsisten metriikoiden arvoille konformisissa ja kvasisäännöllisissä kuvauksissa. Kuudes artikkeli esittelee kaksi hyperbolisen metriikan määrittelyjoukosta riippuvaa suuretta, tutkii niiden avulla uniformista perfektiyttä ja antaa uuden alarajan kapasiteetille.

Nämä kuusi artikkelia antavat lukijalle kokonaisvaltaisen kuvan intrinsistä metriikoista, niiden välisistä epäyhtälöistä ja niiden käyttäytymisestä erityyppisissä kuvauksissa. Artikkelien tuloksia voidaan käyttää esimerkiksi konformisen kapasiteetin arvioinnissa ja erilaisten joukkojen intrinsisen geometrian tutkimisessa. Tutkimuksessani esitellään myös muutama uusi konjektuuri, jotka antavat suuntaa ja ideoita jatkotutkimukselle.

ASIASANAT: Konforminen kapasiteetti, hyperbolinen geometria, intrinsiset metriikat, kvasisäännölliset kuvaukset, uniforminen perfektiys

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List of Original Publications

This dissertation is based on the following six original publications:

I O. Rainio and M. Vuorinen. Triangular ratio metric under quasiconformal mappings in sector domains. arXiv: 2005.11990.

II O. Rainio. Intrinsic metrics in ring domains. arXiv: 2105.01309.

III O. Rainio and M. Vuorinen. Triangular ratio metric in the unit disk. Complex Var. Elliptic Equ., (to appear), doi: 10.1080/17476933.2020.1870452, arXiv: 2009.00265.

IV O. Rainio. Intrinsic quasi-metrics. Bull. Malays. Math. Sci. Soc., 44:2873-2891, 2021, doi: 10.1007/s40840-021-01089-9, arXiv: 2011.02153.

V O. Rainio. Intrinsic metrics under conformal and quasiregular mappings. arXiv: 2103.04397.

VI O. Rainio, T. Sugawa and M. Vuorinen. Intrinsic geometry and boundary structure of plane domains. Sib. Math. J., 62(4):691-706, 2021, doi: 10.1134/S0037446621040121, arXiv: 2008.03457.

1 Introduction

During the thousands of years old history of geometry, scientists have been interested in its limitations. Why is the sum of angles in a triangle always π? Given a line and a point not on that line, is there truly just one line that passes through the point but does not intersect with the first line? Could one build a geometric system similar to the Euclidean one but with some fundamental difference?

These questions were answered in the 19th century when hyperbolic geometry was first introduced. It was not invented by just one mathematician but instead discoveries of several scholars were crucial for the groundwork of this geometric model. Still, the Russian mathematician Nikolai Ivanovich Lobachevsky (1792-1856) is often considered one of the most important contributors.

In hyperbolic geometry of the unit disk, lines are defined as length-minimizing curves that represent the shortest distances through their points when measured with the hyperbolic metric. This leads to unusual features in the geometric model: The angle sum of a triangle is less than π and the parallel postulate of the Euclidean geometry does not hold. In fact, there exist even such triangles whose all angles are arbitrarily small positive numbers.

However, there are certain issues related to the hyperbolic metric. This metric can be used to study countless of different plane domains but it often cannot be defined in proper subdomains of high-dimensional real spaces $\mathbb{R}^n$ with $n \geq 3$. Furthermore, while one of the key properties of the hyperbolic metric is its invariance under conformal mappings, it is not very well-suited for researching several other types of mappings.

Namely, there are a few vital classes of mappings that can be considered generalizations of conformal mappings in higher dimensions. Quasiconformal mappings were first introduced over 90 years ago by the German mathematician H. Grötzsch [1] and studied notably in the 1930s by both another German mathematician called O. Teichmüller [2] and the Finnish mathematician L.V. Ahlfors [3]. In the early 1960s, F.W. Gehring [4] and J. Väisälä [5] initiated the study of quasiconformal homeomorphisms in the case of the n-dimensional Euclidean spaces. This area of research was then extended further by Yu. G. Reshetnyak [6], who studied the case of the non-injective mappings called quasiregular mappings. Some of the key references include [7; 8; 9; 10; 11].

In the study of these aforementioned mappings, the hyperbolic metric has often been used as a model to define other metrics that share some of its properties but work better for this research purpose. For instance, F.W. Gehring studied in his works [12; 13; 14] several these kinds of metrics, such as the distance ratio metric and the quasihyperbolic metric. Unlike the hyperbolic metric, the hyperbolic type metrics can be defined generally in higher dimensions, and especially the quasihyperbolic metric is an important tool in the study of quasiconformal mappings.

Over the past three or four decades, numerous such metrics have been defined that have a significant role in the current research of the metric geometry [15; 16; 17; 18; 19]. One noteworthy example is the triangular ratio metric introduced by P. Hästö [20], which has recently been studied by M. Fujimura et al. [21], J. Chen et al. [22], and P. Hariri et al. [23]. Other important hyperbolic type metrics include the j^*-metric introduced in [23], the Barrlund metric studied in [24], and the Möbius metric researched by P. Seittenranta in [25].

My work in this thesis continues the research of different intrinsic and hyperbolic type metrics. By an intrinsic metric, I mean such a metric that considers how the points in a domain are positioned with respect to the boundary, which is one of the first properties of a hyperbolic type metric listed in [26, pp. 191-192]. This topic of study is a research area in the geometric function theory, which is one of the fields of the mathematical analysis.

2 Definitions and Notations

Firstly, let us introduce the necessary definitions. For any n-dimensional real space $\mathbb{R}^n$, define the extended real space $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$. Note that if $n = 2$, we use the notations of the two-dimensional real plane $\mathbb{R}^2$ and the complex plane $\mathbb{C}$ interchangeable, and $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Denote the set of unit vectors by $\{e_1, ..., e_n\}$ and fix the upper half-space as $\mathbb{H}^n = \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_n > 0\}$.

For any three distinct points $x, y, z \in \mathbb{R}^n$, let $L(x, y)$ be the Euclidean line passing through the points x and y, $[x, y]$ the Euclidean line segment between x and y, and $\measuredangle XYZ$ either the smaller angle between the lines $L(x, y)$ and $L(y, z)$, or the value of this angle, depending on the context. Use the notation $\measuredangle XOY$ for the angle between lines $L(0, x)$ and $L(0, y)$. For any angle $\theta \in (0, 2\pi)$, denote the open sector of the complex plane $\mathbb{C}$ by $S_\theta = \{x \in \mathbb{C} \setminus \{0\} \mid 0 < \arg(x) < \theta\}$. Here, the argument of a complex number is always chosen from $[0, 2\pi)$.

For all points $x \in \mathbb{R}^n$ and any positive number $r > 0$, define the Euclidean open ball $B^n(x, r) = \{y \in \mathbb{R}^n \mid |x - y| < r\}$, its closure $\overline{B}^n(x, r) = \{y \in \mathbb{R}^n \mid |x - y| \leq r\}$ and its boundary sphere $S^{n-1}(x, r) = \{y \in \mathbb{R}^n \mid |x - y| = r\}$. In the special case $x = 0$ and $r = 1$, use the simplified notations $\mathbb{B}^n = B^n(0, 1)$ and $S^{n-1} = S^{n-1}(0, 1)$. For $n \geq 2$, denote the $(n-1)$-dimensional surface area of S^{n-1} by ω_{n-1}.

Denote the hyperbolic sine, cosine and tangent here by sh, ch and th, respectively. The *hyperbolic metric* can be now defined as [26, (4.8), p. 52]

$$\text{ch}\rho_{\mathbb{H}^n}(x, y) = 1 + \frac{|x - y|^2}{2d_{\mathbb{H}^n}(x)d_{\mathbb{H}^n}(y)}, \quad x, y \in \mathbb{H}^n$$

in the upper half-plane $\mathbb{H}^n$ and [26, (4.14), p. 55]

$$\text{sh}^2\frac{\rho_{\mathbb{B}^n}(x, y)}{2} = \frac{|x - y|^2}{(1 - |x|^2)(1 - |y|^2)}, \quad x, y \in \mathbb{B}^n$$

in the Poincaré unit ball $\mathbb{B}^n$. If $n = 2$, these notations can be simplified to

$$\text{th}\frac{\rho_{\mathbb{H}^2}(x, y)}{2} = \left|\frac{x - y}{x - \overline{y}}\right|, \quad \text{th}\frac{\rho_{\mathbb{B}^2}(x, y)}{2} = \left|\frac{x - y}{1 - x\overline{y}}\right| = \frac{|x - y|}{A[x, y]},$$

where $\overline{y}$ is the complex conjugate of y and the Ahlfors bracket $A[x, y]$ is defined as $\sqrt{|x - y|^2 + (1 - |x|^2)(1 - |y|^2)}$ [26, (3.17) p. 39]. For all $x, y \in \mathbb{B}^2$, the *hyper-*

bolic midpoint q of x and y is given by [27, Thm 1.4, p. 3]

$$q = \frac{y(1-|x|^2)+x(1-|y|^2)}{1-|x|^2|y|^2+A[x,y]\sqrt{(1-|x|^2)(1-|y|^2)}}.$$

Furthermore, because of the invariance properties of the hyperbolic metric, its value can be computed in the sector S_θ with the following formula:

$$\text{th}\frac{\rho_{S_\theta}(x,y)}{2} = \text{th}\frac{\rho_{\mathbb{H}^2}(x^{\pi/\theta},y^{\pi/\theta})}{2} = \left|\frac{x^{\pi/\theta}-y^{\pi/\theta}}{x^{\pi/\theta}-\overline{y}^{\pi/\theta}}\right|, \quad x,y\in S_\theta.$$

Other than the hyperbolic metric, there are several other significant metrics in this area of study. Suppose next that G is a proper domain in $\mathbb{R}^n$, until otherwise specified. For all points $x \in G$, denote the Euclidean distance from the point x to the boundary ∂G by $d_G(x) = \inf\{|x-z| \mid z \in \partial G\}$. The *quasihyperbolic metric* introduced in 1976 by Gehring and Palka [13] is the function $k_G : G \times G \to [0,\infty)$, [26, (5.2), p. 68]

$$k_G(x,y) = \inf_{\gamma\in\Gamma_{xy}} \int_\gamma \frac{|dz|}{d_G(z)},$$

where Γ_{xy} consists of all the rectifiable curves in G joining x and y. The *distance ratio metric* was also originally presented in [13] and then slightly modified by M. Vuorinen [11] into the form $j_G : G \times G \to [0,\infty)$, [11, (2.26), p. 78]

$$j_G(x,y) = \log\left(1+\frac{|x-y|}{\min\{d_G(x),d_G(y)\}}\right).$$

This metric can be used to derive the *j^*-metric* $j^*_G : G \times G \to [0,1)$, [23, 2.2, p. 1123 & Lemma 2.1, p. 1124]

$$j^*_G(x,y) = \text{th}\frac{j_G(x,y)}{2} = \frac{|x-y|}{|x-y|+2\min\{d_G(x),d_G(y)\}},$$

as noted in 2015 by P. Hariri et al. [23].

The most important metric in my study is the *triangular ratio metric* $s_G : G \times G \to [0,1]$, [22, (1.1), p. 683]

$$s_G(x,y) = \frac{|x-y|}{\inf_{z\in\partial G}(|x-z|+|z-y|)},$$

which was originally introduced in 2002 by P. Hästö [20]. To compute the value of this metric, one needs to find a boundary point z that gives the infimum of the denominator, which is a trivial task if the boundary of the domain G consists of a line, several line segments, or separate points. However, solving the triangular ratio distance is a more difficult problem in some other domains, such as the unit disk.

In the study of the triangular ratio metric, there are a few different quasi-metrics that can be quite useful. The *point pair function* $p_G : G \times G \to [0,1)$, [23, 2.4, p. 1124]

$$p_G(x,y) = \frac{|x-y|}{\sqrt{|x-y|^2 + 4d_G(x)d_G(y)}}$$

can be used to create upper bounds for the triangular ratio metric in convex domains $G \subset \mathbb{R}^n$. If we suppose that the domain G is convex, we can also define the w*-quasi-metric* $w_G : G \times G \to [0,1)$, [IV, Def. 4.1, p. 8]

$$w_G(x,y) = \frac{|x-y|}{\min\{\inf_{\widetilde{y}\in\widetilde{Y}} |x-\widetilde{y}|, \inf_{\widetilde{x}\in\widetilde{X}} |y-\widetilde{x}|\}} \quad \text{with}$$
$$\widetilde{X} = \{\widetilde{x} \in S^{n-1}(x, 2d_G(x)) \mid (x+\widetilde{x})/2 \in \partial G\}.$$

Furthermore, the triangular ratio metric is a special case of the *Barrlund metric* $b_{G,p} : G \times G \to [0,\infty)$, [24, (1), p. 1]

$$b_{G,p}(x,y) = \sup_{z\in\partial G} \frac{|x-y|}{(|x-z|^p + |z-y|^p)^{1/p}} \quad \text{for some} \quad p \geq 1.$$

Indeed, $s_G(x,y) = b_{G,1}(x,y)$ for all $x, y \in G$.

For all distinct points $x, y \in \overline{\mathbb{R}}^n$, define the *spherical (chordal) metric* $q : \overline{\mathbb{R}}^n \times \overline{\mathbb{R}}^n \to [0,1]$, [26, (3.6), p. 29]

$$q(x,y) = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}} \quad \text{if} \quad x, y \in \mathbb{R}^n, \quad q(x,\infty) = \frac{1}{\sqrt{1+|x|^2}}.$$

With this definition, the expression of the *cross-ratio* can be written for any four distinct points $a, b, c, d \in \overline{\mathbb{R}}^n$ as [26, (3.10), p. 33]

$$|a,b,c,d| = \frac{q(a,c)q(b,d)}{q(a,b)q(c,d)} \left(= \frac{|a-c||b-d|}{|a-b||c-d|}, \quad \text{if } a,b,c,d \in \mathbb{R}^n \right).$$

If G is such a domain in $\overline{\mathbb{R}}^n$ that its complement $\overline{\mathbb{R}}^n \backslash G$ contains at least two points, the *Möbius metric* can be defined in G as the function $\delta_G : G \times G \to [0,\infty)$, [25, Def. 1.1, p. 511]

$$\delta_G(x,y) = \sup_{a,b\in\partial G} \log(1 + |a,x,b,y|).$$

Let us next introduce a few types of mappings.

2.0.1. Conformal mappings. [26, Def. 3.1, p. 25] Let $f : G \to G'$ be a homeomorphism between domains $G, G' \subset \mathbb{R}^n$. In other words, the function f is a continuous bijection. Now, this function f is *conformal* if (1) its derivative f' exists and is continuous, (2) its Jacobian determinant $J_f(x)$ is non-zero at every point $x \in G$, and (3) $|f'(x)h| = |f'(x)||h|$ for all $x \in G$ and $h \in \mathbb{R}^n$. Furthermore, f is *sense-preserving* if $J_f(x) > 0$ for all points $x \in G$, and *sense-reversing* if $J_f(x) < 0$ instead.

The hyperbolic metric is invariant under all conformal mappings and, while none of the other metrics introduced here share this property, the Möbius metric is invariant under the following subclass of the class of conformal mappings.

2.0.2. Möbius transformations. [26, Ex. 3.2, pp. 25-26 & Def. 3.6, p. 27] For any $t > 0$ and $u \in \mathbb{R}^n \backslash \{0\}$,

$$P(u,t) = \{x \in \mathbb{R}^n \mid x \cdot u = t\} \cup \{\infty\}$$

is the hyperplane perpendicular to the vector u and at distance $t/|u|$ from the origin. The reflection in this hyperplane is defined by the function $g : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$,

$$g(x) = x - 2(x \cdot u - t)\frac{u}{|u|^2}, \quad g(\infty) = \infty,$$

and the inversion in the sphere $S^{n-1}(v,r)$ is $h : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$,

$$h(x) = v + \frac{r^2(x-v)}{|x-v|^2}, \quad h(v) = \infty, \quad h(\infty) = v.$$

A *Möbius transformation* is any function $f : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ that can be written as a function composition $f = f_1 \circ \cdots \circ f_m$, where each f_j is either a reflection in some hyperplane or an inversion in a sphere, and $m \geq 1$.

Note that, by *Liouville's theorem*, for a conformal mapping $f : G \to f(G) \subset \mathbb{R}^n$ in a domain $G \subset \mathbb{R}^n$, $n \geq 3$, there is a Möbius transformation h in $\overline{\mathbb{R}}^n$ such that $h(x) = f(x)$ [26, Rmk 3.44, p. 47].

2.0.3. K-quasiregular mappings. [26, pp. 289-288] Suppose that a function $f : G \to \mathbb{R}^n$ defined for a domain $G \subset \mathbb{R}^n$ is ACL^n, see definition for this from [26, p. 150]. If there is a constant $K \geq 1$ such that

$$|f'(x)|^n \leq K J_f(x), \quad |f'(x)| = \max_{|h|=1} |f'(x)h|$$

a.e. in G, f is now *quasiregular* and the smallest $K \geq 1$ fulfilling this inequality is called the *outer dilatation* of f, denoted by $K_O(f)$. Similarly, the *inner dilatation* $K_I(f)$ of f is the smallest $K \geq 1$ such that the inequality

$$J_f(x) \leq K\ell(f'(x))^n, \quad \ell(f'(x)) = \min_{|h|=1} |f'(x)h|,$$

holds a.e. in G. The function f is K-*quasiregular*, if $\max\{K_I(f), K_O(f)\} \leq K$.

Denote the *conformal modulus* defined in [26, (7.1), p. 104] by M, and consider our final class of mappings that is very closely related to the K-quasiregular mappings.

2.0.4. K-quasiconformal mappings. [26, Rmk 15.30 & (15.6), p. 289], [10, p. VI] If G, G' are domains in $\mathbb{R}^n$, a homeomorphism $f : G \to G' = f(G)$ is K-*quasiconformal* if

$$\mathsf{M}(\Gamma)/K \le \mathsf{M}(f(\Gamma)) \le K\mathsf{M}(\Gamma)$$

for all curve families Γ in G. If the homeomorphism f is sense-preserving, it is K-quasiconformal if and only if it is K-quasiregular and injective. Consequently, the sense-preserving K-quasiconformal mappings form a subclass of the K-quasiregular mappings, but sense-reversing quasiconformal mappings are not quasiregular and non-injective quasiregular mappings are not quasiconformal.

One of the crucial concepts in the geometric function theory is the condenser capacity [28], which can often be used to study several problems in mathematical physics introduced in [29]. For two non-empty sets $F_0, F_1 \subset \mathbb{R}^n$, use $\Delta(F_0, F_1; \mathbb{R}^n)$ to denote the family of all such closed non-constant curves in $\mathbb{R}^n$ that join F_0 and F_1. If $G \subset \overline{\mathbb{R}}^n$ is a domain and E is its compact non-empty subset, then the pair (G, E) is called a *condenser* and the *conformal capacity* of this condenser is [26, Def. 9.2, p. 150 & Thm 9.6, p. 152]

$$\text{cap}\,(G, E) = \mathsf{M}(\Delta(E, \partial G; G)) = \inf_u \int_G |\nabla u|^n dm,$$

where infimum is taken over all functions $u \in C_0^\infty(G)$, $u : G \to [0, \infty)$ defined as in [26, p. ix] with $u(x) \ge 1$ for all $x \in E$ and dm stands for the n-dimensional Lebesgue measure.

Capacity can also be defined for a ring domains. A *ring* is any domain $D \subset \overline{\mathbb{R}}^n$ whose complement $\overline{\mathbb{R}}^n \backslash D$ consists of exactly two components C_0 and C_1, the capacity of this ring is [26, p. 120]

$$\text{cap}\,(D) = \mathsf{M}(\Delta(C_0, C_1; \overline{\mathbb{R}}^n)),$$

and the conformal modulus of this ring is [26, (7.16), p. 120]

$$\text{mod}(D) = \left(\frac{\text{cap}\,(D)}{\omega_{n-1}}\right)^{1/(1-n)}.$$

One example of a ring is the two-dimensional annular ring $R(r, 1) = \{z \in \mathbb{C} \mid r < |z| < 1\}$ with $0 < r < 1$.

In order to understand results related to quasiregular mappings and capacity, one needs to know several different constants and special functions. The *Grötzsch capacity* is the decreasing homeomorphism $\gamma_n : (1, \infty) \to (0, \infty)$, [26, (7.17), p. 121]

$$\gamma_n(s) = \mathsf{M}(\Delta(\overline{\mathbb{B}}^n, [se_1, \infty); \mathbb{R}^n)), \quad s > 1.$$

In the special case $n = 2$, consider the explicit formulas [26, (7.18), p. 122]

$$\gamma_2(1/r) = \frac{2\pi}{\mu(r)}, \quad \mu(r) = \frac{\pi}{2}\frac{\mathcal{K}(\sqrt{1-r^2})}{\mathcal{K}(r)}, \quad \mathcal{K}(r) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}}$$

with $0 < r < 1$. For $K > 0$, define then an increasing homeomorphism $\varphi_{K,n} : [0,1] \to [0,1]$, [26, (9.13), p. 167]

$$\varphi_{K,n}(r) = \frac{1}{\gamma_n^{-1}(K\gamma_n(1/r))} \quad \text{if} \quad 0 < r < 1, \quad \varphi_{K,n}(0) = 0, \quad \varphi_{K,n}(1) = 1.$$

Fix [26, 7.1.3, p. 114]

$$c_n = \omega_{n-2}\left(2\int_0^{\pi/2}(\sin t)^{(2-n)/(n-1)}dt\right)^{1-n} \geq \omega_{n-2}(\pi(n-1))^{1-n}, \quad c_2 = \frac{2}{\pi},$$

and [26, (9.5) p. 157 & (9.6), p. 158]

$$\log\lambda_n = \lim_{t\to\infty}((\gamma_n(t)/\omega_{n-1})^{1/(1-n)} - \log t).$$

Here, $4 \leq \lambda_n < 2e^{n-1}$ for all $n \geq 2$ and $\lambda_2 = 4$. Furthermore, denote [26, Thm 16.39, p. 313]

$$c(K) = 2\text{arth}(\varphi_{K,2}(\text{th}(1/2))) \leq \log(2(1+\sqrt{1-1/e^2}))(K-1) + K,$$

and note that $c(K) \to 1$ when $K \to 1$.

***Theorem* 2.0.5.** *[26, Thm 16.2, p. 300 & Thm 16.39, p. 313] If $G, G' \in \{\mathbb{H}^n, \mathbb{B}^n\}$, $f : G \to f(G) \subset G'$ is a non-constant K-quasiregular mapping with the inner dilation of $K_I(f)$ and $\alpha = K_I(f)^{1/(1-n)}$, then*

$$(1) \quad \text{th}\frac{\rho_{G'}(f(x),f(y))}{2} \leq \varphi_{K,n}\left(\text{th}\frac{\rho_G(x,y)}{2}\right) \leq \lambda_n^{1-\alpha}\left(\text{th}\frac{\rho_G(x,y)}{2}\right)^{\alpha},$$

$$(2) \quad \rho_{G'}(f(x),f(y)) \leq K_I(f)(\rho_G(x,y) + \log 4)$$

holds for all $x, y \in G$ and, if $n = 2$, the inequality

$$(3) \quad \rho_{G'}(f(x),f(y)) \leq c(K)\max\{\rho_G(x,y), \rho_G(x,y)^{1/K}\}$$

holds by the conformal invariance of the hyperbolic metric for any two simply connected planar domains G and G'.

Let us yet define a few necessary concepts.

2.0.6. Uniform perfectness. [26, Def. 8.14, p. 141] A closed set $E \subset \overline{\mathbb{R}}^n$ containing at least two points is α-*uniformly perfect* if there is no ring $D \subset \overline{\mathbb{R}}^n \backslash E$ separating E such that $\mathrm{mod}(D) > \alpha$, and E is *uniformly perfect* if it is α-uniformly perfect for some $\alpha > 0$.

The uniform perfectness of a domain G can be also studied with certain quantities. Denote the hyperbolic diameter of a compact subset E of a two-dimensional domain G by $\rho_G(E) = \sup_{x,y \in E} \rho_G(x,y)$, let

$$J_G(E) = \sup_{x,y,z \in E} \log\left(1 + \frac{|x-y|}{d_G(z)}\right)$$

and define the domain functionals

$$c(G) = \inf_{x,y \in G,\, x \neq y} \frac{\rho_G(x,y)}{k_G(x,y)} \quad \text{and} \quad \kappa(G) = \inf_E \frac{\rho_G(E)}{J_G(E)}.$$

Note that $J_G(E)$ here is not equivalent to the diameter of the set E defined by the distance ratio metric and a uniformly perfect set differs from a uniform set: A domain $G \subset \mathbb{R}^n$ is called *uniform* if there exists a number $A \geq 1$ such that $k_G(x,y) \leq A j_G(x,y)$ for all $x, y \in G$ [26, Def. 6.1, p. 84].

3 Properties of Intrinsic Metrics

In my study, one of the key concepts is an *intrinsic distance*, which measures the distance between two points x and y in a domain G by considering not only how close x is to y but also how these points are located with respect to the boundary of G. The hyperbolic metric is commonly used to study these distances but, even in the preceding chapter, eight other intrinsic metrics and quasi-metrics are defined. This often raises the question why newer metrics are introduced when one could focus on the study of the already existing metrics instead.

To answer this question, let us consider first the triangular ratio metric. It is an example of a *hyperbolic type metric* because it fulfills all the properties listed in [26, pp. 191-192]: The triangular ratio metric s_G is monotonic with respect to the domain so that $s_E(x,y) \geq s_G(x,y)$ for points x,y in a subdomain E of a domain G, it is sensitive to the boundary variation because $s_{G\backslash F}(x,y) > s_G(x,y)$ if $x,y \in G\backslash F$ are close enough to a compact subset $F \subset G$ and $G\backslash F$ is a domain, and the closure of the triangular ratio metric balls $B_s(x,r)$ is compact in their domain G when $x \in G$ and $r < 1$. In other words, the triangular ratio metric shares several important properties of the hyperbolic metric.

However, computing the value of the triangular ratio metric is often much simpler than finding the hyperbolic distance. Due to its invariance properties, the hyperbolic metric has an explicit formula in any such domain that can be conformally mapped onto the unit ball but, if this is not the case, we need to find the infimum over all integrals of a certain hyperbolic density, see [30, Def. 7.3, p. 125]. Because of this, computing the hyperbolic metric can be sometimes difficult, but the triangular ratio distance $s_G(x,y)$ is always found just by locating the point z giving the infimum $\inf_{z\in\partial G}(|x-z|+|z-y|)$, see Figure 1.

In order to understand how the value of the triangular ratio metric is computed, consider now the following result which can be used to build an algorithm for finding the triangular ratio distance analytically in any such domain whose boundary is formed out of lines, half-lines and line segments.

***Lemma* 3.0.1** (I, Lemma 2.4, p. 3). *(Heron's shortest distance problem) Given $x,y \in \mathbb{H}^2$, the Heron point $w = L(\overline{x},y) \cap \mathbb{R}$ minimizes the sum $|x-z|+|z-y|$ where $z \in \mathbb{R}$, and therefore* $\inf_{z\in\mathbb{R}}(|x-z|+|z-y|) = |\overline{x}-y|$.

Suppose that we have points x,y in a domain $G \subsetneq \mathbb{R}^2$ whose boundary consist of

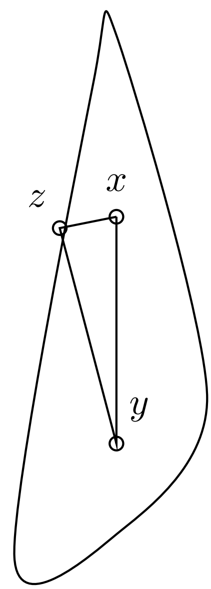

Figure 1. The point z giving the infimum $\inf_{z\in\partial G}(|x-z|+|z-y|)$ in the triangular ratio distance $s_G(x,y)$.

n parts l_i, $i=1,...,n$, out of which each is a line, a line segment or a half-line. If the intersection $[x,y]\cap\partial G$ is not empty, then the distance $s_G(x,y)$ equals 1. Otherwise, we can compute this distance by collecting a list of all the Heron's points z_i of each line L_i for which $l_i\subseteq L_i$ such that $z_i\in\partial G$ and all the vertices of the boundary ∂G, and then checking which of these candidates for the extreme point z gives the minimum for the sum $|x-z|+|z-y|$. Because the boundary of a sector consists of two half-lines, this method works in a sector domain, see Figure 2.

***Theorem* 3.0.2** (I, Thm 2.5, p. 3)*. For all $\theta\in(0,2\pi)$ and $x,y\in S_\theta$, there is an analytical solution to the value of $s_{S_\theta}(x,y)$.*

If the boundary of the domain G is an origin-centered sphere, the extremal point z of the triangular ratio distance $s_G(x,y)$ is positioned so that the line $L(0,z)$ bisects the angle $\measuredangle XZY$, as can be seen from Figure 3. Finding this point z is, in fact, a very old optimization problem [21]. Note that while there is an explicit solution for this point z only in terms of a quartic equation even if the domain G is the unit disk, this equation still helps us to find the value of the triangular ratio metric relatively quickly in more complicated domains. For instance, the triangular ratio metric can be computed in a ring domain with the following result, even though finding the exact value of the hyperbolic metric would be difficult in this non-simply connected domain.

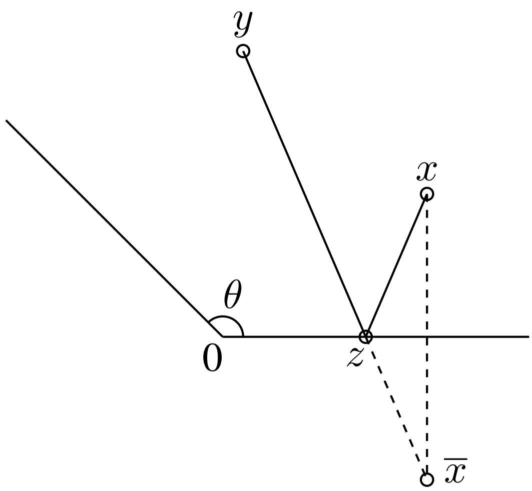

Figure 2. The point z giving the infimum $\inf_{z\in\partial S_\theta}(|x-z|+|z-y|)$ in the sector S_θ with an angle $\theta = 3\pi/4$.

***Theorem* 3.0.3** (II, Thm 3.2, p. 5)**.** *Consider the annular ring domain $R(r,1)$ with $0<r<1$. Let $x,y\in R(r,1)$ and choose z from the boundary of $R(r,1)$ so that it gives the infimum $\inf_{z\in\partial R(r,1)}(|x-z|+|z-y|)$. Then $z\in[x,y]\cap S^1(0,r)$ if $[x,y]\cap\overline{B}^2(0,r)\neq\varnothing$, and otherwise z fulfills the equality*

$$\overline{xy}z^4 - j^2(\overline{x}+\overline{y})z^3 + j^4(x+y)z - j^4xy = 0$$

with either $j=r$ or $j=1$.

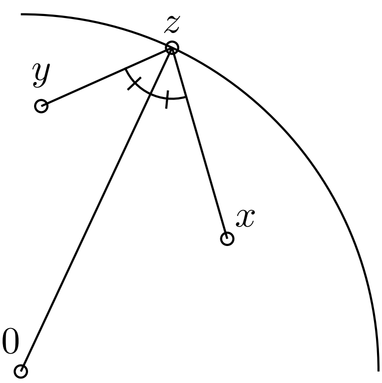

Figure 3. If the point z gives the infimum $\inf_{z\in S^1}(|x-z|+|z-y|)$ for $x,y\in\mathbb{B}^2$, then the line $L(0,z)$ bisects the angle $\measuredangle XZY$.

Consequently, it depends on the domain how the value of the triangular ratio metric is computed efficiently. Clearly, in order to study the intrinsic geometry of some specific domain, one can choose the metric that works the best in that particular situation. However, we must take into account that these metrics are also different in such ways that are not related to their computation. With the exception of the spherical metric, every metric introduced in the previous chapter is intrinsic because their values depend on the position of the points $x, y \in G$ with respect to the boundary ∂G, but they all do not necessarily have the other properties of a hyperbolic type metric defined in [26, pp. 191-192].

Furthermore, some of the functions we study here are not always metrics, because they do not fulfill the triangle inequality. Both the point pair function p_G and the new quasi-metric w_G introduced in [IV] are quasi-metrics with a constant less than or equal to $\sqrt{2}$ in all domains G where they are defined [IV, Lemma 3.1, p. 5 & Cor. 4.8, p. 10]. Still, the point pair function is a metric in all sectors S_θ with an angle $\pi \leq \theta < 2\pi$ [IV, Thm 3.3, p. 5] and, according to the computer tests, it would seem that so is the w-quasi-metric in the unit ball [IV, Conj. 5.6, p. 12].

Also, the metrics have different invariance properties under mappings. As mentioned, the hyperbolic metric is invariant under all the conformal mappings and the Möbius metric is invariant under the Möbius transformations. While the other intrinsic metrics do not share these properties, they are still invariant under the stretching $z \mapsto rz$ by a factor $r > 0$ and all the reflections and rotations that preserve the domain G [I, Remark 3.3, p. 5].

Thus, it is important to have several intrinsic metrics because they have unique advantages and are suited for different purposes. New metrics can be applied to discover such intricate features of geometric entities that would not be detected otherwise. Behaviour of conformal and quasiregular mappings, geometry of the domain and properties of the other metrics can often be best explained by comparing the values of these metrics.

4 Inequalities for Triangular Ratio Metric

During my study, I have found several new inequalities between intrinsic metrics and quasi-metrics. These inequalities are important in this area of research because they can be used not only to estimate the values of such metrics whose exact values would be difficult to compute but also to study the distortion of these metrics under different types of mappings. Especially, many of the inequalities introduced here offer upper and lower bounds for the triangular ratio metric in the unit disk.

Consider first the following inequality that improves the upper bound of [23, Lemma 2.5(1), p. 1126], and the new inequalities between different hyperbolic type metrics in a sector domain.

***Theorem* 4.0.1** (I, Thm 3.6, p. 6). *For a domain $G \subsetneq \mathbb{R}^n$, the sharp inequality*

$$\frac{1}{\sqrt{2}}p_G(x,y) \leq s_G(x,y) \leq \sqrt{2}p_G(x,y)$$

holds for all $x, y \in G$.

***Theorem* 4.0.2** (I, Thm 3.23, pp. 13-14). *For a fixed angle $\theta \in (0, 2\pi)$, the following inequalities hold:*
(1) $j^*_{S_\theta}(x,y) \leq p_{S_\theta}(x,y) \leq \sqrt{2}j^*_{S_\theta}(x,y)$ if $\theta \in (0, 2\pi)$,
(2) $j^*_{S_\theta}(x,y) \leq s_{S_\theta}(x,y) \leq \sqrt{2}j^*_{S_\theta}(x,y)$ if $\theta \in (0, \pi]$,
(3) $j^*_{S_\theta}(x,y) \leq s_{S_\theta}(x,y) \leq 2\sin(\theta/4)j^*_{S_\theta}(x,y)$ if $\theta \in (\pi, 2\pi)$,
(4) $(\sqrt{2}\cos(\theta/4))^{-1}p_{S_\theta}(x,y) \leq s_{S_\theta}(x,y) \leq p_{S_\theta}(x,y)$ if $\theta \in (0, \pi]$,
(5) $p_{S_\theta}(x,y) \leq s_{S_\theta}(x,y) \leq \sqrt{2}\sin(\theta/4)p_{S_\theta}(x,y)$ if $\theta \in (\pi, 2\pi)$.
Furthermore, the constants are sharp in each case.

***Theorem* 4.0.3** (I, Cor. 4.9, p. 17). *For a fixed angle $\theta \in (0, 2\pi)$ and for all $x, y \in S_\theta$, the following results hold:*
(1) $s_{S_\theta}(x,y) \leq \text{th}(\rho_{S_\theta}(x,y)/2) \leq (\pi/\theta)\sin(\theta/2)s_{S_\theta}(x,y)$ if $\theta \in (0, \pi)$,
(2) $s_{S_\theta}(x,y) = \text{th}(\rho_{S_\theta}(x,y)/2)$ if $\theta = \pi$,
(3) $(\pi/\theta)s_{S_\theta}(x,y) \leq \text{th}(\rho_{S_\theta}(x,y)/2) \leq s_{S_\theta}(x,y)$ if $\theta \in (\pi, 2\pi)$.
Furthermore, these bounds are also sharp.

Because $j^*_G(x,y) \leq s_G(x,y)$ for all points x, y in all domains $G \subsetneq \mathbb{R}^n$ [23, Lemma 2.1, p. 1124], the j^*-metric works as a lower bound for the triangular ratio metric. However, according to my results, the new quasi-metric w is a better lower bound for the triangular ratio metric than j^*-metric in all proper convex domains G. In fact, computer tests suggest that this quasi-metric has values very close to the triangular ratio metric in the unit disk $\mathbb{B}^2$.

***Theorem* 4.0.4** (IV, Cor. 4.9, p. 10). *For any convex domain $G \subsetneq \mathbb{R}^n$ and all $x, y \in G$,*

$$j^*_G(x,y) \le w_G(x,y) \le s_G(x,y) \le p_G(x,y).$$

***Theorem* 4.0.5** (IV, Cor. 4.10, p. 10). *For all $x, y \in G \in \{\mathbb{H}^n, \mathbb{B}^n\}$,*

$$\operatorname{th}\frac{\rho_{\mathbb{H}^n}(x,y)}{4} \le j^*_{\mathbb{H}^n}(x,y) \le w_{\mathbb{H}^n}(x,y) = s_{\mathbb{H}^n}(x,y) = p_{\mathbb{H}^n}(x,y) = \operatorname{th}\frac{\rho_{\mathbb{H}^n}(x,y)}{2},$$
$$\operatorname{th}\frac{\rho_{\mathbb{B}^n}(x,y)}{4} \le j^*_{\mathbb{B}^n}(x,y) \le w_{\mathbb{B}^n}(x,y) \le s_{\mathbb{B}^n}(x,y) \le p_{\mathbb{B}^n}(x,y) \le \operatorname{th}\frac{\rho_{\mathbb{B}^n}(x,y)}{2}.$$

***Lemma* 4.0.6** (IV, Lemma 5.15, p. 16). *For all $x, y \in \mathbb{B}^2$ such that $|x| = |y|$ and $\measuredangle XOY = \pi/2$, the inequality*

$$s_{\mathbb{B}^2}(x,y) \le c \cdot w_{\mathbb{B}^2}(x,y)$$

holds with the sharp constant

$$c = \sqrt{\frac{h_0^2 - 2h_0 + 2}{2h_0^2 - 2\sqrt{2}h_0 + 2}} \approx 1.07313, \quad h_0 = \frac{1 - \sqrt{9 - 6\sqrt{2}}}{2 - \sqrt{2}}.$$

***Conjecture* 4.0.7** (IV, Conj. 5.19, p. 17). *For all $x, y \in \mathbb{B}^2$,*

$$s_{\mathbb{B}^2}(x,y) \le c \cdot w_{\mathbb{B}^2}(x,y),$$

where the constant c is as in Lemma 4.0.6.

Also, the Barrlund metric can be used to bound the triangular ratio metric in the unit disk.

***Lemma* 4.0.8** (III, Lemma 3.5, p. 5). *For all $x, y \in \mathbb{B}^2$, the following inequality holds and contains the best possible constants:*

$$\frac{1}{\sqrt{2}} b_{\mathbb{B}^2,2}(x,y) \le s_{\mathbb{B}^2}(x,y) \le b_{\mathbb{B}^2,2}(x,y).$$

Note that while there is no explicit formula for the value of the triangular ratio metric for all points in the unit disk, there are two special situations where this problem becomes trivial. Namely, if the points x, y are either collinear with origin or $|x| = |y|$, the distance $s_{\mathbb{B}^2}(x,y)$ can be computed with the known formulas in [26, 11.2.1(1) p. 205] and [16, Thm 3.1, p. 276], respectively. Interestingly, these formulas can be used to create bounds for the value of the triangular ratio metric because this metric fulfills certain inequalities for the points x, y rotated around their midpoint by using either Euclidean or hyperbolic geometry.

Definition 4.0.9 (III, Def. 4.1, p. 10)**.** *Euclidean midpoint rotation.* Choose distinct points $x, y \in \mathbb{B}^2$. Let $k = (x+y)/2$, and $l = |x-k| = |y-k|$. Let $x_0, y_0 \in S^1(k,l)$, $x_0 \neq y_0$, so that $|x_0| = |y_0|$ and the points x_0, k, y_0 are collinear. Fix then $x_1, y_1 \in S^1(k,l)$ so that x_1, k, y_1 are collinear, $|x_1| = |k| + l$ and $|y_1| = |k| - l$. Note that $x_0, y_0, y_1 \in \mathbb{B}^2$ always but x_1 is not necessarily in $\mathbb{B}^2$. See Figure 4.

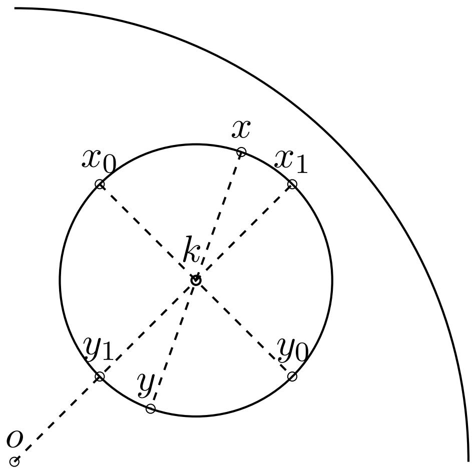

Figure 4. Euclidean midpoint rotation.

***Theorem* 4.0.10** (III, Thm 4.11, p. 15 & Thm 4.12, p. 17)**.** *For all points $x, y \in \mathbb{B}^2$ with x_0, y_0, x_1, y_1 as in Definition 4.0.9,*

$$\frac{|x-y|}{\sqrt{|x-y|^2 + (2-|x+y|)^2}} \leq s_{\mathbb{B}^2}(x_0, y_0) \leq s_{\mathbb{B}^2}(x, y),$$

and, if $x_1 \in \mathbb{B}^2$ here so that $s_{\mathbb{B}^2}(x_1, y_1)$ is well-defined, also the inequality

$$s_{\mathbb{B}^2}(x, y) \leq s_{\mathbb{B}^2}(x_1, y_1) = \frac{|x-y|}{2-|x+y|}$$

holds.

Definition 4.0.11 (III, Def. 5.1, p. 17)**.** *Hyperbolic midpoint rotation.* Choose distinct points $x, y \in \mathbb{B}^2$. Let q be their hyperbolic midpoint and $R = \rho_{\mathbb{B}^2}(x, q) = \rho_{\mathbb{B}^2}(y, q)$. Let $x_2, y_2 \in S^1_\rho(q, R)$ so that $|x_2| = |y_2|$ but $x_2 \neq y_2$. Fix then $x_3, y_3 \in S^1_\rho(q, R)$ so that x_3, y_3 are collinear with the origin and $|y_3| < |q| < |x_3|$. See Figure 5.

***Theorem* 4.0.12** (III, Thm 5.3, p. 20; Thm 5.11, p. 22 & Thm 5.12, p. 23)**.** *For all points $x, y \in \mathbb{B}^2$ with q, x_2, y_2, x_3, y_3 as in Definition 4.0.11 and the number*

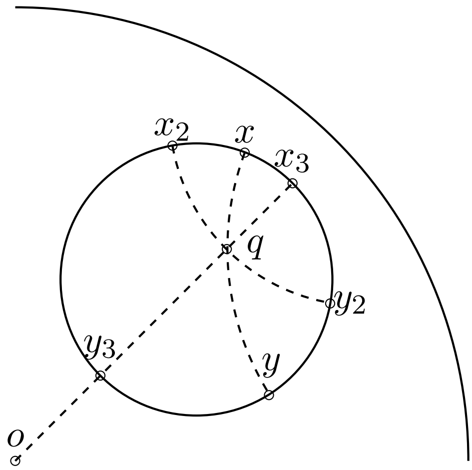

Figure 5. Hyperbolic midpoint rotation.

$t = \text{th}(\rho_{\mathbb{B}^2}(x,y)/4)$,

$$\sqrt{\frac{|q|^2+t^2}{1+|q|^2t^2}} \leq s_{\mathbb{B}^2}(x_2,y_2) \leq s_{\mathbb{B}^2}(x,y) \leq s_{\mathbb{B}^2}(x_3,y_3) = \frac{(1+|q|)t}{1+|q|t^2}, \quad \textit{where}$$

$$s_{\mathbb{B}^2}(x_2,y_2) = \sqrt{\frac{|q|^2+t^2}{1+|q|^2t^2}} \quad \textit{if } |q| < t^2 \quad \textit{and}$$

$$s_{\mathbb{B}^2}(x_2,y_2) = \frac{t(1+|q|)}{\sqrt{(1+t^2)(1+|q|^2t^2)}} \quad \textit{if } |q| \geq t^2 \quad \textit{instead.}$$

The Euclidean midpoint rotation can also be extended to the annular ring $R(r,1)$ with $0 < r < 1$ because it is similarly symmetric with respect to the origin as the unit disk.

Definition 4.0.13 (II, Def. 3.9, p. 8)**.** *Euclidean midpoint rotation with respect to the origin.* Choose distinct point $x, y \in \mathbb{R}^2$. Denote k, x_0, y_0, x_1, y_1 as in Definition 4.0.9. Note that if $x, y \in R(r,1)$, then $x_0, y_0 \in R(r,1)$ always, but it might be so that $y_1 \in \overline{B}^2(0,r)$ or $x_1 \notin \mathbb{B}^2$.

The Euclidean midpoint rotation above fulfills a corresponding version of the inequality of Theorem 4.0.10, when defined for the triangular ratio metric in the annular ring:

***Theorem* 4.0.14** (II, Thm 3.12, p. 9)**.** *For all distinct points $x, y \in R(r,1)$, fix x_0, y_0, x_1, y_1 as in Definition 4.0.13. If $x_1, y_1 \in R(r,1)$, the distance $s_{R(r,1)}(x_1,y_1)$ is well-defined and the inequality*

$$s_{R(r,1)}(x_0,y_0) \leq s_{R(r,1)}(x,y) \leq s_{R(r,1)}(x_1,y_1)$$

holds. Otherwise, only the first part of this inequality holds and the points x, y can be rotated around their Euclidean midpoint into such points x', y' that $s_{R(r,1)}(x', y') \to 1^-$.

Consider then the Möbius metric whose values in the annular ring $R(r, 1)$ can be computed by using the following result together with a single-variable optimization function.

***Theorem* 4.0.15** (II, Thm 4.1, p. 10)**.** *For all $x, y \in R(r, 1)$ with $|y| \leq |x|$, the Möbius distance $\delta_{R(r,1)}(x, y)$ is*

$$\max\left\{\rho_{\mathbb{B}^2}(x,y),\ \rho_{\mathbb{B}^2}\left(\frac{rx}{|x|^2},\frac{ry}{|y|^2}\right),\ \sup_{v\in[\mu,\pi]}\log(1+|e^{-u(v)i},|x|,re^{vi},|y|e^{\mu i}|)\right\},$$

where $\mu \in (0, \pi)$ is the value of the angle $\measuredangle XOY$, and

$$u(v) = \arcsin\left(\frac{-c_2+\sqrt{c_2^2-4c_1c_3}}{2c_1}\right) \quad \textit{with}$$
$$c_1 = |x|^2(1+r^2)^2 + r^2(1+|x|^2)^2 - 2r|x|(1+r^2)(1+|x|^2)\cos(v),$$
$$c_2 = 4r|x|\sin(v)(|x|(1+r^2) - r(1+|x|^2)\cos(v)),$$
$$c_3 = -r^2\sin(v)^2(1-|x|^2)^2.$$

While the general case is quite complicated, there are explicit formulas for the value of the Möbius metric in the following special case:

***Corollary* 4.0.16** (II, Cor. 4.2, p. 12)**.** *For all points $x, y \in R(r, 1)$ collinear with the origin such that $|y| \leq |x|$, the value of $\text{th}(\delta_{R(r,1)}(x, y)/2)$ is*

$$\max\left\{\frac{|x|-|y|}{1-|x||y|},\ \frac{r(|x|-|y|)}{|x||y|-r^2},\ \frac{(|x|-|y|)(1-r)}{2(1-|x|)(|y|-r)+(|x|-|y|)(1-r)}\right\},$$

if the value of the angle $\measuredangle XOY$ is 0, and

$$\max\left\{\frac{|x|+|y|}{1+|x||y|},\ \frac{r(|x|+|y|)}{|x||y|+r^2},\ \frac{(|x|+|y|)(1+r)}{2(1-|x|)(|y|-r)+(|x|+|y|)(1+r)}\right\},$$

if the value of the angle $\measuredangle XOY$ is π.

The result above for collinear points is useful because, according to the numerical tests, the Möbius metric fulfills the same inequality as the triangular ratio metric does in Theorem 4.0.14 and this result could therefore be used to find bounds for the value of the Möbius metric.

***Conjecture* 4.0.17** (II, Conj. 4.3 & Lemma 4.4, p. 13)**.** *For all distinct points $x, y \in R(r,1)$ such that $x_0, y_0, x_1, y_1 \in R(r,1)$, when these points are as in Definition 4.0.13, the Möbius metric fulfills*

$$\delta_{R(r,1)}(x_0, y_0) \leq \delta_{R(r,1)}(x, y) \leq \delta_{R(r,1)}(x_1, y_1).$$

Alternatively, the following inequality can be used to estimate the value of the Möbius metric in the annular ring.

***Theorem* 4.0.18** (II, Thm 4.8, p. 15)**.** *For all $x, y \in R(r,1)$,*

$$\frac{1}{2}s_{R(r,1)}(x,y) \leq j^*_{R(r,1)}(x,y) \leq \text{th}\frac{\delta_{R(r,1)}(x,y)}{2} \leq 2j^*_{R(r,1)}(x,y) \leq 2s_{R(r,1)}(x,y),$$

where the constants $1/2$ and 1 are sharp when $r \to 0^+$, and the constants 2 are sharp for all values of $r \in (0,1)$.

5 Distortion of Intrinsic Metrics

Next, let us study how much the distances measured with a certain metric can distort under conformal, K-quasiregular and K-quasiconformal mappings. Namely, we can use the information from the inequalities of the former chapter to create upper and lower bounds for this sort of distortion. This is an interesting focus of research because our metrics behave slightly different under these mappings and analysing these differences can give us a better idea how these mappings actually work.

The first result follows from the conformal invariance of the hyperbolic metric.

***Lemma* 5.0.1** (V, Lemma 4.5, p. 13)**.** *Suppose that* $G, G' \in \{\mathbb{H}^n, \mathbb{B}^n\}$*,* $f : G \to G' = f(G)$ *is a conformal mapping and* $\eta_G \in \{j^*_G, w_G, s_G, p_G\}$*. Then, for all* $x, y \in G$*,*

$$\eta_G(x,y)/2 \leq \eta_{G'}(f(x), f(y)) \leq 2\eta_G(x,y).$$

Furthermore, for $\eta_G \in \{w_G, s_G, p_G\}$*,*

$$\begin{aligned} &\eta_G(x,y)/2 \leq \eta_{G'}(f(x), f(y)) \leq \eta_G(x,y) \quad \text{if} \quad G = \mathbb{H}^n,\, G' = \mathbb{B}^n, \quad \text{and} \\ &\eta_G(x,y) \leq \eta_{G'}(f(x), f(y)) \leq 2\eta_G(x,y) \quad \text{if} \quad G = \mathbb{B}^n,\, G' = \mathbb{H}^n. \end{aligned}$$

The bounds above can be improved in the case of the unit disk if we can limit the absolute values of the points considered.

***Theorem* 5.0.2** (V, Thm 4.8, p. 14)**.** *If* $f : \mathbb{B}^n \to \mathbb{B}^n = f(\mathbb{B}^n)$ *is a conformal mapping,* $x, y \in \mathbb{B}^n$ *and* $r_l, r_u, R_l, R_u \in [0,1)$ *such that* $|x|, |y| \in [r_l, r_u]$ *and* $|f(x)|, |f(y)| \in [R_l, R_u]$*, then*

$$(1) \quad \frac{2(1+R_l)}{(1+r_u)\sqrt{5+2R_l+R_l^2}} \leq \frac{j^*_{\mathbb{B}^n}(f(x), f(y))}{j^*_{\mathbb{B}^n}(x,y)} \leq \frac{(1+R_u)\sqrt{5+2r_l+r_l^2}}{2(1+r_l)},$$

$$(2) \quad \frac{\sqrt{1-2r_u+2r_u^2}(1+R_l)}{1+r_u^2} \leq \frac{p_{\mathbb{B}^n}(f(x), f(y))}{p_{\mathbb{B}^n}(x,y)} \leq \frac{1+R_u^2}{(1+r_l)\sqrt{1-2R_u+2R_u^2}},$$

$$(3) \quad \frac{\sqrt{1+R_l^2}}{1+r_u} \leq \frac{b_{\mathbb{B}^n,2}(f(x), f(y))}{b_{\mathbb{B}^n,2}(x,y)} \leq \frac{1+R_u}{\sqrt{1+r_l^2}},$$

$$(4) \quad \frac{2(1+R_l)\sqrt{1-2r_u+2r_u^2}}{(1+r_u^2)\sqrt{5+2R_l+R_l^2}} \leq \frac{s_{\mathbb{B}^n}(f(x),f(y))}{s_{\mathbb{B}^n}(x,y)}$$
$$\leq \frac{(1+R_u^2)\sqrt{5+2r_l+r_l^2}}{2(1+r_l)\sqrt{1-2R_u+2R_u^2}},$$

The hyperbolic midpoint rotation of Definition 4.0.11 can also be used to bound the value of the triangular ratio metric.

***Theorem* 5.0.3** (V, Cor. 4.18, p. 17). *For any conformal mapping $f:\mathbb{B}^n\to f(\mathbb{B}^n)=\mathbb{B}^n$ and for all distinct points $x,y\in\mathbb{B}^n$ with a hyperbolic midpoint q and the distance $t=\text{th}(\rho_{\mathbb{B}^n}(x,y)/4)$,*

$$\frac{1+|q|t^2}{1+|q|} \leq \frac{s_{\mathbb{B}^n}(f(x),f(y))}{s_{\mathbb{B}^n}(x,y)} \leq u(|q|,t),$$

where the upper bound $u:[0,1]\times[0,1]\to\mathbb{R}^+$ is defined as

$$u(|q|,t)=\frac{2t}{1+t^2}\sqrt{\frac{1+|q|^2t^2}{|q|^2+t^2}} \quad \text{if } |q|<t^2,$$
$$u(|q|,t)=\frac{2}{1+|q|}\sqrt{\frac{1+|q|^2t^2}{1+t^2}} \quad \text{otherwise.}$$

The following generalizations of the Schwarz lemma give us new information about the distortion of the intrinsic metrics under K-quasiregular mappings.

***Theorem* 5.0.4** (V, Thm 5.5, p. 21). *If $f:\mathbb{B}^n\to f(\mathbb{B}^n)\subset\mathbb{B}^n$ is a K-quasiregular mapping with the inner dilatation $K_I(f)$, then for all $x,y\in\mathbb{B}^n$ and any constant $\alpha\leq K_I(f)^{1/(1-n)}$,*

$$\eta_{\mathbb{B}^n}(f(x),f(y)) \leq \varphi_{K,n}\left(\frac{2\eta_{\mathbb{B}^n}(x,y)}{1+\eta_{\mathbb{B}^n}(x,y)^2}\right) \leq \lambda_n^{1-\alpha}\left(\frac{2\eta_{\mathbb{B}^n}(x,y)}{1+\eta_{\mathbb{B}^n}(x,y)^2}\right)^{\alpha},$$

where $\eta_G\in\{j_G^*,w_G,s_G,p_G\}$.

***Theorem* 5.0.5** (V, Cor. 5.9, p. 22). *For all $x,y\in\mathbb{B}^2$, any $\eta_G\in\{j_G^*,w_G,s_G,p_G\}$ and every K-quasiregular mapping $f:\mathbb{B}^2\to f(\mathbb{B}^2)\subset\mathbb{B}^2$,*

$$\eta_{\mathbb{B}^2}(f(x),f(y)) \leq \varphi_{2K,2}(\eta_{\mathbb{B}^2}(x,y)^2) \leq 4^{1-1/(2K)}\eta_{\mathbb{B}^2}(x,y)^{1/K},$$

and the constant $4^{1-1/(2K)}$ here is sharp for $K=1$.

***Theorem* 5.0.6** (V, Cor. 5.11, p. 23)**.** *If* $f:\mathbb{B}^n\to f(\mathbb{B}^n)\subset\mathbb{B}^n$ *is a* K*-quasiregular mapping with inner dilatation of* $K_I(f)$ *such that* $f(0)=0$*, then for all* $x\in\mathbb{B}^n$ *and* $\alpha\leq K_I(f)^{1/(1-n)}$*,*

$$\frac{|f(x)|}{2-|f(x)|}\leq\varphi_{K,n}\left(\frac{|x|(2-|x|)}{|x|^2-2|x|+2}\right)\leq\lambda_n^{1-\alpha}\left(\frac{|x|(2-|x|)}{|x|^2-2|x|+2}\right)^{\alpha}.$$

Since every K-quasiconformal mapping is either a K-quasiregular mapping or a function composition of a K-quasiregular mapping and a reflection, the results concerning the distortion under K-quasiregular mappings can be also directly extended to K-quasiconformal mappings. In other words, we can see from Theorem 5.0.5 that the triangular ratio metric is Hölder continuous under both K-quasiregular and K-quasiconformal mappings defined in the unit disk. However, the next result shows this Hölder continuity of the triangular ratio metric under quasiconformal mappings more clearly.

***Theorem* 5.0.7** (I, Cor. 5.10, p. 21)**.** *Let* G *and* G' *be simply-connected domains in* $\mathbb{R}^2$ *and* $f:G\to G'=f(G)$ *a* K*-quasiconformal homeomorphism. Suppose that there exist* $A,B\in(0,\infty)$ *so that* $As_{G'}(u,v)\leq\text{th}(\rho_{G'}(u,v)/2)$ *for all* $u,v\in G'$ *and* $\text{th}(\rho_G(x,y)/2)\leq Bs_G(x,y)$ *for all* $x,y\in G$*. Then, for all* $x,y\in G$*,*

$$s_{G'}(f(x),f(y))\leq\frac{c(K)B^{1/K}}{A}s_G(x,y)^{1/K}.$$

The result above can be combined, for instance, with the inequalities between the triangular ratio metric and the hyperbolic metric in a sector in order to obtain the following theorem.

***Theorem* 5.0.8** (I, Cor. 5.11, p. 21)**.** *If* $\alpha,\beta\in(0,2\pi)$ *and* $f:S_\alpha\to S_\beta=f(S_\alpha)$ *is a* K*-quasiconformal homeomorphism, the following inequalities hold for all* $x,y\in S_\alpha$*:*

(1) $$\frac{\beta}{c(K)^K\pi\sin(\beta/2)}s_{S_\alpha}(x,y)^K\leq s_{S_\beta}(f(x),f(y))\leq c(K)\left(\frac{\pi}{\alpha}\sin\left(\frac{\alpha}{2}\right)\right)^{1/K}s_{S_\alpha}(x,y)^{1/K}\quad \text{if } \alpha,\beta\in(0,\pi],$$

(2) $$\frac{1}{c(K)^K}s_{S_\alpha}(x,y)^K\leq s_{S_\beta}(f(x),f(y))\leq\frac{c(K)\beta}{\pi}\left(\frac{\pi}{\alpha}\sin\left(\frac{\alpha}{2}\right)\right)^{1/K}s_{S_\alpha}(x,y)^{1/K}\quad \text{if } \alpha\in(0,\pi) \text{ and } \beta\in(\pi,2\pi),$$

(3) $$\left(\frac{\pi}{c(K)\alpha}\right)^K s_{S_\alpha}(x,y)^K\leq s_{S_\beta}(f(x),f(y))\leq\frac{c(K)\beta}{\pi}s_{S_\alpha}(x,y)^{1/K}$$
if $\alpha,\beta\in[\pi,2\pi)$.

We can also find sharp bounds for the distortion of the triangular ratio metric under a specific mapping, as can be seen below.

***Lemma* 5.0.9** (I, Lemma 5.15, p. 22)**.** *If $\alpha, \beta \in (0, \pi]$ and $f : S_\alpha \to S_\beta$, $f(z) = z^{(\beta/\alpha)}$, then for all $x, y \in S_\alpha$*

$$s_{S_\alpha}(x,y) \leq s_{S_\beta}(f(x),f(y)) \leq \frac{\beta\sin(\alpha/2)}{\alpha\sin(\beta/2)} s_{S_\alpha}(x,y) \quad \text{if } \alpha \leq \beta,$$
$$\frac{\beta\sin(\alpha/2)}{\alpha\sin(\beta/2)} s_{S_\alpha}(x,y) \leq s_{S_\beta}(f(x),f(y)) \leq s_{S_\alpha}(x,y) \quad \text{otherwise,}$$

and the constants here are sharp.

Furthermore, the triangular ratio metric can be used to bound the distortion of the Euclidean metric.

***Theorem* 5.0.10** (III, Thm 6.1, p. 24)**.** *For a K-quasiconformal mapping $f : \mathbb{B}^2 \to \mathbb{B}^2 = f(\mathbb{B}^2)$, the inequality*

$$|f(x) - f(y)| \leq 2^{3-1/K} \left(\frac{s_{\mathbb{B}^2}(x,y)}{1 + s_{\mathbb{B}^2}(x,y)^2} \right)^{1/K},$$

holds for all $x, y \in \mathbb{B}^2$.

6 Capacity and Uniform Perfectness

Finally, let us focus on the applications of the metrics. Especially, we study the conformal capacity by using domain functionals defined with the hyperbolic type metrics. However, let us first show what these functionals can tell us about the domain and its boundary geometry.

Suppose below that $G \subsetneq \mathbb{C}$ is a domain whose complement has at least three points so that the hyperbolic metric can be defined in G.

***Theorem* 6.0.1** (VI, Thm A, p. 692; Thm 1.4, p. 693 & Thm 1.5, p. 693)**.** *The functionals $c(G)$ and $\kappa(G)$ defined for a domain $G \subset \mathbb{C}$ fulfill the following inequalities:*
(1) $c(G) \leq 1$ where the equality holds if and only if G is convex,
(2) $c(G) > 0$ if and only if the boundary ∂G is uniformly perfect,
(3) $c(G)/2 \leq \kappa(G) \leq c(G)$ where $\kappa(G) > 0$ if and only if ∂G is uniformly perfect, and
(4) $\kappa(G) \leq \kappa(\mathbb{H}^2)$ where the equality holds if and only if G is convex.

Consider the following theorem and see Figure 6 related to it.

***Theorem* 6.0.2** (VI, Thm 1.6, p. 693)**.** *For a convex domain $G \subsetneq \mathbb{C}$, there exists a compact subset $E \subset G$ satisfying*

$$\kappa(G) = \rho_G(E)/J_G(E)$$

if and only if G is a half-plane. In fact, there exists a three-point set $E^* = \{i, x, -\overline{x}\}$ in $\mathbb{H}^2$ with $\mathrm{Im}(x) > 1$ such that

$$\kappa(\mathbb{H}^2) = \rho_{\mathbb{H}^2}(E^*)/J_{\mathbb{H}^2}(E^*).$$

Furthermore, this three-point set constitutes a hyperbolic equilateral triangle and is unique up to similarities keeping $\mathbb{H}^2$ invariant.

Next, let us introduce new lower bounds for the conformal capacity by using our domain functionals.

***Theorem* 6.0.3** (VI, Cor. 1.8, p. 693 & Thm 5.2, p. 705)**.** *Let E be a continuum in a simply connected domain $G \subsetneq \mathbb{C}$. Denote $\Phi(x) = \gamma_2(\tanh(x/2))$ for $x \in$*

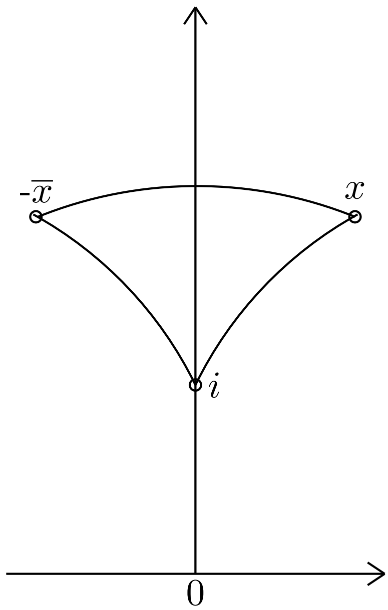

Figure 6. The three-point set $E^* = \{i, x, -\overline{x}\}$ of Theorem 6.0.2 consists of the vertices of a hyperbolic equilateral triangle.

$(0,\infty)$, *$\kappa_1 = \kappa(\mathbb{B}^2)$ and $\kappa_0 = \inf_F \kappa(F)$, where the infimum is taken over all simply connected proper subdomains of $\mathbb{C}$. Now,*

$$\text{cap}\,(G,E) \geq \Phi(\kappa(G)J_G(E)) \geq \Phi(\kappa_0 J_G(E))$$

where $1/4 \leq \kappa_0 < 0.4251605$ and, if G is convex, then

$$\text{cap}\,(G,E) \geq \Phi(\kappa_1 J_G(E))$$

where $\kappa_1 > 0.87509875$.

The following result gives us another new lower bound for the capacity but, unlike the bounds above, this bound is Möbius invariant because of the invariance properties of the Möbius metric.

***Lemma* 6.0.4** (II, Cor. 5.3, p. 18 & Lemma 5.7, p. 20)**.** *If $D \subset \overline{\mathbb{R}}^n$ is a ring and E, F are the components of its complement, then the capacity of D has a symmetric and Möbius invariant lower bound:*

$$\text{cap}\,(D) \geq \frac{1}{2}c_n\delta(E,F), \quad \textit{where}$$
$$\delta(E,F) = \sup_{x,y\in E} \delta_{\overline{\mathbb{R}}^n\setminus F}(x,y) = \sup_{x,y\in F} \delta_{\overline{\mathbb{R}}^n\setminus E}(x,y).$$

Consider yet the connection between the the distance ratio metric and the hyperbolic metric.

***Theorem* 6.0.5** (VI, Thm 1.2, p. 692)**.** *For a domain $G \subsetneq \mathbb{C}$, there is a constant $c > 0$ such that $cj_G(x, y) \leq \rho_G(x, y)$ for all $x, y \in G$ if and only if the boundary ∂G is uniformly perfect in $\overline{\mathbb{C}}$.*

It follows from the result above that the hyperbolic metric ρ_G defined in a domain $G \subsetneq \mathbb{C}$ is comparable with the distance ratio metric j_G if and only if G is uniform and has uniformly perfect boundary [VI, Cor. 1.3, p. 692].

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