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SOME PROPERTIES OF A CLASS OF ELLIPTIC PARTIAL DIFFERENTIAL OPERATORS

SHAOLIN CHEN AND MATTI VUORINEN

Abstract. We prove Schwarz-Pick type estimates and coefficient estimates for a class of elliptic partial differential operators introduced by Olofsson. Then we apply these results to obtain a Landau type theorem.

1. Introduction and main results

Let $\mathbb{C}$ be the complex plane. For $a \in \mathbb{C}$, let $r > 0$ and $\mathbb{D}(a, r) = \{z : |z - a| < r\}$. In particular, we use $\mathbb{D}_r$ to denote the disk $\mathbb{D}(0, r)$ and $\mathbb{D}$, the open unit disk $\mathbb{D}_1$.

For a real 2×2 matrix, we will consider the matrix norm $\|A\| = \sup\{|Az| : |z| = 1\}$ and the matrix function $l(A) = \inf\{|Az| : |z| = 1\}$. For $z = x + iy \in \mathbb{C}$ with x and y real, we denote the *complex differential operators*

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \text{ and } \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

If we denote the formal derivative of $f = u + iv$ by

$$D_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

then $\|D_f\| = |f_z| + |f_{\overline{z}}|$ and $l(D_f) = \big||f_z| - |f_{\overline{z}}|\big|$, where u, v are real functions, $f_z = \partial f/\partial z$ and $f_{\overline{z}} = \partial f/\partial \overline{z}$. Throughout this paper, we denote by $\mathcal{C}^n(\mathbb{D})$ the set of all n-times continuously differentiable complex-valued functions in $\mathbb{D}$, where $n \in \{1, 2, \cdots\}$.

For $\alpha \in \mathbb{R}$ and $z \in \mathbb{D}$, let

$$T_\alpha = -\frac{\alpha^2}{4}(1 - |z|^2)^{-\alpha-1} + \frac{\alpha}{2}(1 - |z|^2)^{-\alpha-1}\left(z\frac{\partial}{\partial z} + \overline{z}\frac{\partial}{\partial \overline{z}}\right) + \frac{1}{4}(1 - |z|^2)^{-\alpha}\Delta$$

be the *second order elliptic partial differential operator*, where Δ is the usual complex *Laplacian operator*

$$\Delta := 4\frac{\partial^2}{\partial z \partial \overline{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

We consider the *Dirichlet boundary value problem* of distributional sense as follows

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$$(1.1)\qquad \begin{cases} T_\alpha(f) = 0 & \text{in } \mathbb{D},\\ f = f^* & \text{on } \partial\mathbb{D}. \end{cases}$$

Here, the boundary data $f^* \in \mathfrak{D}'(\partial\mathbb{D})$ is a *distribution* on the boundary $\partial\mathbb{D}$ of $\mathbb{D}$, and the boundary condition in (1.1) is interpreted in the distributional sense that $f_r \to f^*$ in $\mathfrak{D}'(\partial\mathbb{D})$ as $r \to 1-$, where

$$(1.2)\qquad f_r(e^{i\theta}) = f(re^{i\theta}),\ e^{i\theta} \in \partial\mathbb{D},$$

for $r \in [0,1)$ (see [21]).

In [21], Olofsson proved that, for parameter values $\alpha > -1$, a function $f \in \mathcal{C}^2(\mathbb{D})$ satisfies (1.1) if and only if it has the form of a *Poisson type integral*

$$(1.3)\qquad f(z) = \frac{1}{2\pi}\int_0^{2\pi} K_\alpha(ze^{-i\tau})f^*(e^{i\tau})d\tau,\ \text{for } z \in \mathbb{D},$$

where

$$K_\alpha(z) = c_\alpha \frac{(1-|z|^2)^{\alpha+1}}{|1-z|^{\alpha+2}},$$

$c_\alpha = \big(\Gamma(\alpha/2+1)\big)^2/\Gamma(1+\alpha)$ and $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$ for $s > 0$ is the standard Gamma function. If we take $\alpha = 2(n-1)$, then f is *polyharmonic* (or *n-harmonic*), where $n \in \{1, 2, \dots\}$ (cf. [1, 2, 5, 14, 18, 19, 20, 22]). In particular, if $\alpha = 0$, then f is harmonic.

For $a, b, c \in \mathbb{R}$ with $c \neq 0, -1, -2, \dots$, the *hypergeometric* function is defined by the power series

$$F(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!},\ |x| < 1,$$

where $(a)_0 = 1$ and $(a)_n = a(a+1)\cdots(a+n-1)$ for $n = 1, 2, \dots$ are the *Pochhammer* symbols. Obviously, for $n = 0, 1, 2, \dots$, $(a)_n = \Gamma(a+n)/\Gamma(a)$. In particular, for $a, b, c > 0$ and $a + b < c$, we have (cf. [3, 4])

$$(1.4)\qquad F(a,b;c;1) = \lim_{x\to 1} F(a,b;c;x) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} < \infty.$$

On the basis of Olofsson's research, we continue to investigate some properties of solutions to (1.1). The following is a Schwarz-Pick type estimate on the solutions to (1.1).

Theorem 1. *For $\alpha > -1$, let $f \in \mathcal{C}^2(\mathbb{D})$ satisfy (1.1) and $\sup_{z\in\mathbb{D}}|f(z)| \leq M$, where M is a positive constant. Then, for $z \in \mathbb{D}$,*

$$(1.5)\qquad \left|f(z) - \frac{(1-|z|)^{\alpha+1}}{1+|z|}f(0)\right| \leq M\left[\frac{1}{2\pi}\int_0^{2\pi} K_\alpha(ze^{-it})dt - \frac{(1-|z|)^{\alpha+1}}{1+|z|}K_\alpha(0)\right]$$

and

$$(1.6)\qquad \|D_f(z)\| \leq \frac{M\mathcal{M}_\alpha(|z|)\big[2+\alpha+(4+3\alpha)|z|\big]}{1-|z|^2} \leq \frac{M\big[2+\alpha+(4+3\alpha)|z|\big]}{1-|z|^2},$$

where

$$(1.7)\quad \mathcal{M}_\alpha(r) = \frac{1}{2\pi}\int_0^{2\pi} K_\alpha(re^{i\epsilon})d\epsilon = \frac{\left[\Gamma\left(1+\frac{\alpha}{2}\right)\right]^2}{\Gamma(1+\alpha)} F\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; r^2\right),\ r \in [0,1).$$

Let f be a harmonic mapping of $\mathbb{D}$ onto $\mathbb{D}$ with $f(0) = 0$. In [15], Heinz showed that, for $z \in \mathbb{D}$,

$$\|D_f(z)\| \geq \frac{2}{\pi}.$$

By using Theorem 1, we get a Heinz type inequality on $\partial\mathbb{D}$ as follows.

Theorem 2. *For $\alpha \geq 0$, let $f \in \mathcal{C}^2(\overline{\mathbb{D}})$ satisfying* (1.1)*. Suppose that $f(0) = 0$, $f(\overline{\mathbb{D}}) = \overline{\mathbb{D}}$ and $f(\partial\mathbb{D}) = \partial\mathbb{D}$.*

(a) *If $\alpha = 0$, then, for $\theta \in [0, 2\pi]$,*

$$\|D_f(e^{i\theta})\| \geq \frac{2}{\pi};$$

(b) *If $\alpha > 0$, then, for $\theta \in [0, 2\pi]$,*

$$\|D_f(e^{i\theta})\| \geq \lim_{r\to 1-} \frac{d}{dr}\mathcal{M}_\alpha(r) = \frac{\alpha}{2},$$

where $\mathcal{M}_\alpha(r)$ is given by (1.7)*.*

The following result is the homogeneous expansion of solutions to (1.1).

Theorem A. ([21, Theorem 2.2]) *Let $\alpha \in \mathbb{R}$ and $f \in \mathcal{C}^2(\mathbb{D})$. Then f satisfies* (1.1) *if and only if it has a series expansion of the form*

$$(1.8)\qquad f(z) = \sum_{k=0}^{\infty} c_k F\left(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; |z|^2\right) z^k + \sum_{k=1}^{\infty} c_{-k} F\left(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; |z|^2\right) \overline{z}^k,\ z \in \mathbb{D},$$

for some sequence $\{c_k\}_{k=-\infty}^{\infty}$ of complex numbers satisfying

$$(1.9)\qquad \lim_{|k|\to\infty} \sup |c_k|^{\frac{1}{|k|}} \leq 1.$$

In particular, the expansion (1.8)*, subject to* (1.9)*, converges in $\mathcal{C}^\infty(\mathbb{D})$, and every solution f of* (1.1) *is $\mathcal{C}^\infty$-smooth in $\mathbb{D}$.*

For $\alpha = 0$, there are numerous discussions on coefficient estimates of harmonic mappings in the literature, see for example [7, 8, 9, 11, 13, 17, 24]. We investigate the problem of coefficient estimates on the solutions to (1.1) as follows.

Theorem 3. *For $\alpha > -1$, let $f \in \mathcal{C}^2(\mathbb{D})$ be a solution to* (1.1) *with the series expansion of the form* (1.8) *and $\sup_{z\in\mathbb{D}} |f(z)| \leq M$, where M is a positive constant. Then, for $k \in \{1, 2, \ldots\}$,*

$$(1.10)\qquad \left|c_k F\Big(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; 1\Big)\right| + \left|c_{-k} F\Big(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; 1\Big)\right| \leq \frac{4M}{\pi}$$

and

$$\left|c_0 F\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; 1\right)\right| \leq M.$$

In particular, if $\alpha = 0$*, then the estimate of* (1.10) *is sharp and all the extreme functions are*

$$f_k(z) = \frac{2\varepsilon M}{\pi} \mathrm{Im}\left(\log \frac{1+\vartheta z^k}{1-\vartheta z^k}\right),$$

where $|\varepsilon| = |\vartheta| = 1$*.*

The following result easily follows from Theorem 3 and [21, Proposition 1.4].

Corollary 1.1. *For* $\alpha > -1$*, let* $f \in \mathcal{C}^2(\mathbb{D})$ *be a solution to* (1.1) *with the series expansion of the form* (1.8) *and* $\sup_{z\in\mathbb{D}} |f(z)| \leq M$*, where* M *is a positive constant. Then, for* $k \in \{1, 2, \ldots\}$*,*

$$|c_k| + |c_{-k}| \leq \frac{4M\Gamma\left(1+\frac{\alpha}{2}\right)\Gamma\left(k+1+\frac{\alpha}{2}\right)}{k!\Gamma(\alpha+1)\pi}.$$

For $p \in (0, \infty]$, the *Hardy space* $\mathcal{H}^p$ consists of those functions $f: \mathbb{D} \to \mathbb{C}$ such that f is measurable, $M_p(r, f)$ exists for all $r \in (0, 1)$ and $\|f\|_p < \infty$, where

$$\|f\|_p = \begin{cases} \sup\limits_{0<r<1} M_p(r, f), & \text{if } p \in (0, \infty), \\ \sup\limits_{z\in\mathbb{D}} |f(z)|, & \text{if } p = \infty, \end{cases}$$

and

$$M_p^p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta.$$

The classical theorem of Landau shows that there is a $\rho = \frac{1}{M+\sqrt{M^2-1}}$ such that every function f, analytic in $\mathbb{D}$ with $f(0) = f'(0) - 1 = 0$ and $|f(z)| < M$ in $\mathbb{D}$, is univalent in the disk $\mathbb{D}_\rho$ and in addition, the range $f(\mathbb{D}_\rho)$ contains a disk of radius $M\rho^2$ (see [16]), where $M \geq 1$ is a constant. Recently, many authors considered Landau type theorem for planar harmonic mappings (see [6, 7, 8, 9]), biharmonic mappings (see [1]) and polyharmonic mappings (see [11]). Applying Theorems 1 and 3, we get the following Landau type theorem.

Theorem 4. *For* $\alpha \in (-1, 0]$*, let* $f \in \mathcal{C}^2(\mathbb{D})$ *be a solution to* (1.1) *satisfying* $f(0) = |J_f(0)| - \lambda = 0$ *and* $f \in \mathcal{H}^p$*, where* λ *is a positive constant and* J_f *is the Jacobian of* f*. Then* f *is univalent in* $\mathbb{D}_{\gamma_0\rho_0}$*, where* ρ_0 *satisfies the following equation*

$$\frac{\lambda}{M^*(2+\alpha)} - \frac{4M^*\rho_0}{\pi}\left[\frac{2-\rho_0}{(1-\rho_0)^2} + \frac{2\rho_0}{(1-\rho_0)(1-\rho_0^2)^2}\right] = 0,$$

where $\mu(\gamma) = \left(1+\gamma\right)^{\frac{\alpha+1}{p}} / \left[\gamma\left(1-\gamma\right)^{\frac{1}{p}}\right]$, $\mu(\gamma_0) = \min_{0<\gamma<1} \mu(\gamma)$ *and* $M^* = c_\alpha^{\frac{1}{p}} \|f\|_p \mu(\gamma_0)$*. Moreover,* $f\left(\mathbb{D}_{\gamma_0\rho_0}\right)$ *contains a univalent disk* $\mathbb{D}_{\gamma_0 R_0}$ *with*

$$R_0 \geq \frac{2\rho_0}{3}\left[\frac{\lambda}{M^*(2+\alpha)} - \frac{M^*\rho_0(2-\rho_0)}{\pi(1-\rho_0)^2}\right].$$

We remark that Theorem 4 is a generalization of [6, Theorem 2] and [10, Theorem 5].

The proofs of Theorems 1, 2 and 3 will be presented in Section 2, and the proof of Theorem 4 will be given in Section 3.

2. Schwarz-Pick type estimates and coefficient estimates

Proof of Theorem 1. We first prove (1.5). By the assumption, we see that $f_r \to f^*$ in $\mathfrak{D}'(\partial\mathbb{D})$ as $r \to 1-$, where f_r is given by (1.2) for $r \in [0,1)$. By (1.3), for $z = re^{i\theta} \in \mathbb{D}$, we have

$$\begin{aligned}
&\left| f(z) - \frac{(1-|z|)^{\alpha+1}}{1+|z|} f(0) \right| \\
= &\ \frac{1}{2\pi} \left| \int_0^{2\pi} K_\alpha(ze^{-it}) f^*(e^{it}) dt - \frac{(1-|z|)^{\alpha+1}}{1+|z|} \int_0^{2\pi} K_\alpha(0) f^*(e^{it}) dt \right| \\
= &\ \frac{1}{2\pi} \left| \int_0^{2\pi} \Big(K_\alpha(ze^{-it}) - \frac{(1-|z|)^{\alpha+1}}{1+|z|} K_\alpha(0) \Big) f^*(e^{it}) dt \right| \\
\leq &\ \frac{1}{2\pi} \int_0^{2\pi} \Big(K_\alpha(ze^{-it}) - \frac{(1-|z|)^{\alpha+1}}{1+|z|} K_\alpha(0) \Big) |f^*(e^{it})| dt \\
\leq &\ M \left[\frac{1}{2\pi} \int_0^{2\pi} K_\alpha(ze^{-it}) dt - \frac{(1-|z|)^{\alpha+1}}{1+|z|} K_\alpha(0) \right].
\end{aligned}$$

Next we prove (1.6). By the proof of [21, Theorem 3.1], we observe that

$$\mathcal{M}_\alpha(r) = \frac{1}{2\pi} \int_0^{2\pi} K_\alpha(re^{i\epsilon}) d\epsilon = \frac{\left[\Gamma\left(1+\frac{\alpha}{2}\right)\right]^2}{\Gamma(1+\alpha)} F\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; r^2\right) \tag{2.1}$$

and $\mathcal{M}_\alpha(r)$ is increasing on $r \in [0,1)$ with

$$\lim_{r\to 1-} \mathcal{M}_\alpha(r) = 1.$$

By elementary calculations, for $z \in \mathbb{D}$, we have

$$\frac{\partial}{\partial z} K_\alpha(ze^{-it}) = c_\alpha \frac{(1-|z|^2)^\alpha \left[\left(1+\frac{\alpha}{2}\right)e^{-it}(1-\overline{z}e^{it})(1-|z|^2) - (\alpha+1)\overline{z}|1-ze^{-it}|^2\right]}{|1-ze^{-it}|^{4+\alpha}}$$

and

$$\frac{\partial}{\partial \overline{z}} K_\alpha(ze^{-it}) = c_\alpha \frac{(1-|z|^2)^\alpha \left[\left(1+\frac{\alpha}{2}\right)e^{it}(1-ze^{-it})(1-|z|^2) - (\alpha+1)z|1-ze^{-it}|^2\right]}{|1-ze^{-it}|^{4+\alpha}},$$

which, together with (1.3) and (2.1), imply that

$$\begin{aligned}
\|D_f(z)\| &= \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial z} K_\alpha(ze^{-it}) f^*(e^{it}) dt \right| + \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial \overline{z}} K_\alpha(ze^{-it}) f^*(e^{it}) dt \right| \\
&\leq \frac{Mc_\alpha}{\pi} \int_0^{2\pi} \frac{(1-|z|^2)^\alpha \left[(1+\alpha)|z||1-ze^{-it}|^2 + \left(1+\frac{\alpha}{2}\right)|1-ze^{-it}|(1-|z|^2)\right]}{|1-ze^{-it}|^{4+\alpha}} dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{Mc_\alpha}{\pi}\left[\int_0^{2\pi}\frac{(1+\alpha)|z|(1-|z|^2)^{\alpha}}{|1-ze^{-it}|^{2+\alpha}}dt+\int_0^{2\pi}\frac{\left(1+\frac{\alpha}{2}\right)(1-|z|^2)^{\alpha+1}}{|1-ze^{-it}|^{3+\alpha}}dt\right]\\
&\leq \frac{Mc_\alpha}{\pi}\left[\frac{(1+\alpha)|z|}{(1-|z|^2)}\int_0^{2\pi}\frac{(1-|z|^2)^{\alpha+1}}{|1-ze^{-it}|^{2+\alpha}}dt+\frac{\left(1+\frac{\alpha}{2}\right)}{(1-|z|)}\int_0^{2\pi}\frac{(1-|z|^2)^{\alpha+1}}{|1-ze^{-it}|^{2+\alpha}}dt\right]\\
&= \left[\frac{2M(1+\alpha)|z|}{1-|z|^2}+\frac{M(2+\alpha)}{1-|z|}\right]\frac{1}{2\pi}\int_0^{2\pi}K_\alpha(ze^{-it})dt\\
&= \left[\frac{2M(1+\alpha)|z|}{1-|z|^2}+\frac{M(2+\alpha)}{1-|z|}\right]\mathcal{M}_\alpha(|z|)\\
&= \frac{M\mathcal{M}_\alpha(|z|)\big[2+\alpha+(4+3\alpha)|z|\big]}{1-|z|^2}\\
&\leq \frac{M\big[2+\alpha+(4+3\alpha)|z|\big]}{1-|z|^2}.
\end{aligned}
$$

The proof of this theorem is complete. □

Proof of Theorem 2. Since (a) easily follows from the inequality (15) in [15], we only need to prove (b). Let $\alpha>0$. By Theorem 1 (1.5), we have

$$
\begin{aligned}
\frac{|f(e^{i\theta})-f(re^{i\theta})|}{1-r} &\geq \frac{1-|f(re^{i\theta})|}{1-r}\\
&\geq \frac{1-\frac{1}{2\pi}\int_0^{2\pi}K_\alpha(ze^{-it})dt+\frac{(1-|z|)^{\alpha+1}}{1+|z|}K_\alpha(0)}{1-r}. \qquad (2.2)
\end{aligned}
$$

where $z=re^{i\theta}\in\mathbb{D}$ and $\theta\in[0,2\pi)$.

Applying [21, Theorem 3.1], we get

$$
\frac{1}{2\pi}\lim_{|z|\to1-}\int_0^{2\pi}K_\alpha(ze^{-it})dt=\lim_{r\to1-}\mathcal{M}_\alpha(r)=1,
$$

which, together with L'Hopital's rule and (2.2), yield that

$$
\begin{aligned}
\|D_f(e^{i\theta})\| &\geq \left(\left|\frac{\partial f(re^{i\theta})}{\partial r}\right|\right)_{r=1}\\
&= \lim_{r\to1-}\frac{|f(e^{i\theta})-f(re^{i\theta})|}{1-r}\\
&\geq \lim_{r\to1-}\frac{1-\frac{1}{2\pi}\int_0^{2\pi}K_\alpha\big(re^{i(\theta-t)}\big)dt+\frac{(1-r)^{\alpha+1}}{1+r}K_\alpha(0)}{1-r}\\
&= \lim_{r\to1-}\frac{1-\frac{1}{2\pi}\int_0^{2\pi}K_\alpha\big(re^{i\eta}\big)d\eta+\frac{(1-r)^{\alpha+1}}{1+r}K_\alpha(0)}{1-r}
\end{aligned}
$$

$$= \frac{1}{2\pi}\lim_{r\to 1-}\frac{d}{dr}\int_0^{2\pi} K_\alpha\big(re^{i\eta}\big)d\eta$$
$$= \lim_{r\to 1-}\frac{d}{dr}\mathcal{M}_\alpha(r).$$

where $\mathcal{M}_\alpha(r)$ is given by (1.7). It follows from the proof of [21, Theorem 3.1] that

$$\mathcal{M}_\alpha(r) = \frac{\big[\Gamma\big(1+\frac{\alpha}{2}\big)\big]^2}{\Gamma(1+\alpha)}F\left(-\frac{\alpha}{2},-\frac{\alpha}{2};1;r^2\right) = \frac{\big[\Gamma\big(1+\frac{\alpha}{2}\big)\big]^2}{\Gamma(1+\alpha)}\sum_{n=0}^{\infty}\frac{\big[\big(-\frac{\alpha}{2}\big)_n\big]^2}{(n!)^2}r^{2n},$$

which yields that

$$\frac{d}{dr}\mathcal{M}_\alpha(r) = \frac{\alpha^2}{2}\frac{\big[\Gamma\big(1+\frac{\alpha}{2}\big)\big]^2}{\Gamma(1+\alpha)}rF\left(1-\frac{\alpha}{2},1-\frac{\alpha}{2};2;r^2\right),$$

where $r\in(0,1)$. By (1.4), for $\alpha>0$, we see that

$$\begin{aligned}\lim_{r\to 1-}\frac{d}{dr}\mathcal{M}_\alpha(r) &= \frac{\alpha^2}{2}\frac{\big[\Gamma\big(1+\frac{\alpha}{2}\big)\big]^2}{\Gamma(1+\alpha)}F\left(1-\frac{\alpha}{2},1-\frac{\alpha}{2};2;1\right)\\ &= \frac{\alpha^2}{2}\frac{\big[\Gamma\big(1+\frac{\alpha}{2}\big)\big]^2}{\Gamma(1+\alpha)}\frac{\Gamma(2)\Gamma(\alpha)}{\big[\Gamma\big(1+\frac{\alpha}{2}\big)\big]^2}\\ &= \frac{\alpha^2}{2}\frac{\Gamma(\alpha)}{\Gamma(1+\alpha)} = \frac{\alpha}{2}.\end{aligned}$$

Therefore, for $\theta\in[0,2\pi]$,

$$\|D_f(e^{i\theta})\| \geq \lim_{r\to 1-}\frac{d}{dr}\mathcal{M}_\alpha(r) = \frac{\alpha}{2},$$

where $\alpha>0$. The proof of this theorem is complete. ☐

Proof of Theorem 3. For $r\in[0,1)$, let

$$A_k(r,\alpha) = c_kF\Big(-\frac{\alpha}{2},k-\frac{\alpha}{2};k+1;r^2\Big)$$

and

$$B_k(r,\alpha) = c_{-k}F\Big(-\frac{\alpha}{2},k-\frac{\alpha}{2};k+1;r^2\Big),$$

where $r=|z|$. Then

$$A_k(r,\alpha)r^k = \frac{1}{2\pi}\int_0^{2\pi} f(z)e^{-ik\theta}d\theta$$

and

$$B_k(r,\alpha)r^k = \frac{1}{2\pi}\int_0^{2\pi} f(z)e^{ik\theta}d\theta,$$

which imply that

$$|A_k(r,\alpha)|r^k = \frac{1}{2\pi}\int_0^{2\pi} f(z)e^{-ik\theta}e^{-i\arg A_k(r,\alpha)}d\theta \tag{2.3}$$

and

$$(2.4)\qquad |B_k(r,\alpha)|r^k = \frac{1}{2\pi}\int_0^{2\pi} f(z)e^{ik\theta}e^{-i\arg B_k(r,\alpha)}d\theta\,,$$

where $A_k(r,\alpha) = |A_k(r,\alpha)|e^{i\arg A_k(r,\alpha)}$, $B_k(r,\alpha) = |B_k(r,\alpha)|e^{i\arg B_k(r,\alpha)}$ and $z = re^{i\theta}$. By (2.3), (2.4) and [12, Lemma 1], we have

$$
\begin{aligned}
(2.5)\qquad & \Big|\big(|A_k(r,\alpha)| + |B_k(r,\alpha)|\big)r^k\Big| \\
= & \left|\frac{1}{2\pi}\int_0^{2\pi} f(z)\left[e^{-i\left(k\theta+\arg A_k(r,\alpha)\right)} + e^{i\left(k\theta-\arg B_k(r,\alpha)\right)}\right]d\theta\right| \\
\leq & \frac{1}{2\pi}\int_0^{2\pi} |f(z)|\left|e^{-i\left(k\theta+\arg A_k(r,\alpha)\right)} + e^{i\left(k\theta-\arg B_k(r,\alpha)\right)}\right|d\theta \\
\leq & \frac{M}{2\pi}\int_0^{2\pi} \left|e^{-i\left(k\theta+\arg A_k(r,\alpha)\right)} + e^{i\left(k\theta-\arg B_k(r,\alpha)\right)}\right|d\theta \\
= & \frac{M}{2\pi}\int_0^{2\pi} \left|1 + e^{i\left(2k\theta+\arg A_k(r,\alpha)-\arg B_k(r,\alpha)\right)}\right|d\theta \\
= & \frac{M}{\pi}\int_0^{2\pi} \left|\cos\left(k\theta + \frac{\arg A_k(r,\alpha) - \arg B_k(r,\alpha)}{2}\right)\right| d\theta \\
= & \frac{4M}{\pi}.
\end{aligned}
$$

By letting $r \to 1-$ on (2.5), we obtain

$$|A_k(1,\alpha)| + |B_k(1,\alpha)| \leq \frac{4M}{\pi}.$$

On the other hand, for $k = 0$, we have

$$
\begin{aligned}
(2.6)\qquad \frac{1}{2\pi}\lim_{r\to 1-}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \;=\; & \left|c_0 F\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; 1\right)\right|^2 \\
& + \sum_{k=1}^{\infty}\left(\left|c_k F\left(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; 1\right)\right|^2\right. \\
& \left. + \left|c_{-k} F\left(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; 1\right)\right|^2\right) \\
\leq\; & M^2,
\end{aligned}
$$

where $r \in [0,1)$. It follows from (2.6) that

$$\left|c_0 F\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; 1\right)\right| \leq M.$$

If $\alpha = 0$, then the sharpness part follows from [11, Lemma 1]. The proof of this theorem is complete. □

3. The Landau type theorem

Lemma 1. *For $x \in [0,1)$, let*

$$\varphi(x)=\frac{\delta}{M(2+\alpha)}-\frac{4Mx}{\pi}\left[\frac{(2-x)}{(1-x)^2}+\frac{2x}{(1-x)(1-x^2)^2}\right],$$

where $\alpha>-2$, $\delta>0$ and $M>0$ are constant. Then φ is strictly decreasing and there is an unique $x_0\in(0,1)$ such that $\varphi(x_0)=0$.

Proof. For $x \in [0,1)$, let

$$f_1(x)=\frac{4M}{\pi}\frac{x(2-x)}{(1-x)^2} \text{ and } f_2(x)=\frac{4M}{\pi}\frac{2x^2}{(1-x)(1-x^2)^2}.$$

Since, for $x\in[0,1)$,

$$f_1'(x)=\frac{8M}{\pi}\frac{1}{(1-x)^3}>0,$$

we see that f_1 is continuous and strictly increasing in $[0,1)$. We observe that f_2 is also continuous and strictly increasing in $[0,1)$. Then

$$\varphi(x)=\frac{\delta}{M(2+\alpha)}-f_1(x)-f_2(x)$$

is continuous and strictly decreasing in $[0,1)$, which, together with

$$\lim_{x\to 0}\varphi(x)=\frac{\delta}{M(2+\alpha)}>0 \text{ and } \lim_{x\to 1-}\varphi(x)=-\infty,$$

imply that there is an unique $x_0\in(0,1)$ such that $\varphi(x_0)=0$. □

Lemma 2. *For $\alpha\in(-1,0]$, let $f\in\mathcal{C}^2(\mathbb{D})$ be a solution to (1.1) satisfying $f(0)=|J_f(0)|-\beta=0$ and $\sup_{z\in\mathbb{D}}|f(z)|\leq M$, where M, β are positive constants and J_f is the Jacobian of f. Then f is univalent in $\mathbb{D}_{\rho_0}$, where ρ_0 satisfies the following equation*

$$\frac{\beta}{M(2+\alpha)}-\frac{4M\rho_0}{\pi}\left[\frac{2-\rho_0}{(1-\rho_0)^2}+\frac{2\rho_0}{(1-\rho_0)(1-\rho_0^2)^2}\right]=0.$$

Moreover, $f\big(\mathbb{D}_{\rho_0}\big)$ contains a univalent disk $\mathbb{D}_{R_0}$ with

$$R_0\geq\frac{2\rho_0}{3}\left[\frac{\beta}{M(2+\alpha)}-\frac{M\rho_0(2-\rho_0)}{\pi(1-\rho_0)^2}\right].$$

Proof. By Theorem A, we can assume that

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty}c_kF\left(-\frac{\alpha}{2},k-\frac{\alpha}{2};k+1;|z|^2\right)z^k \\ &\quad+\sum_{k=1}^{\infty}c_{-k}F\left(-\frac{\alpha}{2},k-\frac{\alpha}{2};k+1;|z|^2\right)\overline{z}^k,\ z\in\mathbb{D}, \end{aligned}$$

for some sequence $\{c_k\}_{k=-\infty}^{\infty}$ of complex numbers satisfying

$$\lim_{|k|\to\infty}\sup|c_k|^{\frac{1}{|k|}}\leq 1.$$

For $\alpha\in(-1,0]$, by [21, Proposition 1.4], we observe that

$$F\Big(-\frac{\alpha}{2},k-\frac{\alpha}{2};k+1;r^2\Big)=\sum_{n=0}^{\infty}\frac{\big(-\frac{\alpha}{2}\big)_n\big(k-\frac{\alpha}{2}\big)_n}{(k+1)_n}\frac{r^{2n}}{n!}\geq 0$$

is bounded and increasing on $r\in[0,1)$, which imply that

$$\begin{aligned}(3.1)\qquad &\big(|c_k|+|c_{-k}|\big)F\Big(-\frac{\alpha}{2},k-\frac{\alpha}{2};k+1;r^2\Big)\\ \leq\ &\big(|c_k|+|c_{-k}|\big)F\Big(-\frac{\alpha}{2},k-\frac{\alpha}{2};k+1;1\Big)\\ \leq\ &\frac{4M}{\pi},\end{aligned}$$

where $r=|z|$ and $k\in\{1,2,\ldots\}$.

By (3.1) and Theorem 3, we see that, for each $k\in\{1,2,\ldots\}$,

$$(3.2)\qquad (|c_k|+|c_{-k}|)\frac{\big(-\frac{\alpha}{2}\big)_n\big(k-\frac{\alpha}{2}\big)_n}{(k+1)_n}\frac{1}{n!}\leq\frac{4M}{\pi},$$

where $n\in\{1,2,\ldots\}$.

Since $c_0=f(0)=0$, we see that

$$\begin{aligned}(3.3)\qquad f_z(z)-f_z(0)\ =\ &\sum_{k=2}^{\infty}kc_kF\left(-\frac{\alpha}{2},k-\frac{\alpha}{2};k+1;w\right)z^{k-1}\\ &+\sum_{k=1}^{\infty}c_k\frac{d}{dw}F\left(-\frac{\alpha}{2},k-\frac{\alpha}{2};k+1;w\right)z^k\overline{z}\\ &+\sum_{k=1}^{\infty}c_{-k}\frac{d}{dw}F\left(-\frac{\alpha}{2},k-\frac{\alpha}{2};k+1;w\right)\overline{z}^{k+1}\end{aligned}$$

and

$$\begin{aligned}(3.4)\qquad f_{\overline{z}}(z)-f_{\overline{z}}(0)\ =\ &\sum_{k=2}^{\infty}kc_{-k}F\left(-\frac{\alpha}{2},k-\frac{\alpha}{2};k+1;w\right)\overline{z}^{k-1}\\ &+\sum_{k=1}^{\infty}c_k\frac{d}{dw}F\left(-\frac{\alpha}{2},k-\frac{\alpha}{2};k+1;w\right)z^{k+1}\\ &+\sum_{k=1}^{\infty}c_{-k}\frac{d}{dw}F\left(-\frac{\alpha}{2},k-\frac{\alpha}{2};k+1;w\right)\overline{z}^kz,\end{aligned}$$

where $w=|z|^2$.

Applying (3.2), (3.3) and (3.4), we obtain

$$(3.5)\qquad \begin{aligned} & |f_z(z)-f_z(0)|+|f_{\overline{z}}(z)-f_{\overline{z}}(0)| \\ \leq\ & \sum_{k=2}^{\infty} k\big(|c_k|+|c_{-k}|\big)F\left(-\frac{\alpha}{2},k-\frac{\alpha}{2};k+1;w\right)|z|^{k-1} \\ & +2\sum_{k=1}^{\infty}\big(|c_k|+|c_{-k}|\big)\frac{d}{dw}F\left(-\frac{\alpha}{2},k-\frac{\alpha}{2};k+1;w\right)|z|^{k+1} \\ \leq\ & \frac{4M}{\pi}\sum_{k=2}^{\infty}k|z|^{k-1}+2\sum_{k=1}^{\infty}\left[\frac{4M}{\pi}\sum_{n=1}^{\infty}n|z|^{2(n-1)}\right]|z|^{k+1} \\ =\ & \frac{4M}{\pi}\frac{|z|(2-|z|)}{(1-|z|)^2}+\frac{8M}{\pi}\sum_{k=1}^{\infty}\frac{|z|^{k+1}}{(1-|z|^2)^2} \\ =\ & \frac{4M}{\pi}\frac{|z|(2-|z|)}{(1-|z|)^2}+\frac{8M}{\pi}\frac{|z|^2}{(1-|z|)(1-|z|^2)^2}. \end{aligned}$$

Applying Theorem 1 (1.6), we get

$$\beta=|J_f(0)|=|\det D_f(0)|=\|D_f(0)\|l\big(D_f(0)\big)\leq M(2+\alpha)l\big(D_f(0)\big),$$

which gives that

$$(3.6)\qquad l\big(D_f(0)\big)\geq\frac{\beta}{M(2+\alpha)}.$$

In order to prove the univalence of f in $\mathbb{D}_{\rho_0}$, we choose two distinct points z_1, $z_2\in\mathbb{D}_{\rho_0}$ and let $[z_1,z_2]$ denote the segment from z_1 to z_2 with the endpoints z_1 and z_2, where ρ_0 satisfies the following equation

$$\frac{\beta}{M(2+\alpha)}-\frac{4M\rho_0}{\pi}\left[\frac{2-\rho_0}{(1-\rho_0)^2}+\frac{2\rho_0}{(1-\rho_0)(1-\rho_0^2)^2}\right]=0.$$

By (3.5), (3.6) and Lemma 1, we have

$$\begin{aligned} |f(z_2)-f(z_1)| &= \left|\int_{[z_1,z_2]}f_z(z)dz+f_{\overline{z}}(z)d\overline{z}\right| \\ &= \left|\int_{[z_1,z_2]}f_z(0)dz+f_{\overline{z}}(0)d\overline{z}\right| \\ &\quad -\left|\int_{[z_1,z_2]}\big(f_z(z)-f_z(0)\big)dz+\big(f_{\overline{z}}(z)-f_{\overline{z}}(0)\big)d\overline{z}\right| \\ &\geq l(D_f)(0)|z_2-z_1| \\ &\quad -\int_{[z_1,z_2]}\big(|f_z(z)-f_z(0)|+|f_{\overline{z}}(z)-f_{\overline{z}}(0)|\big)|dz| \end{aligned}$$

$$
\begin{aligned}
&> |z_2 - z_1|\Bigg\{\frac{\beta}{M(2+\alpha)} \\
&\quad -\frac{4M\rho_0}{\pi}\left[\frac{2-\rho_0}{(1-\rho_0)^2}+\frac{2\rho_0}{(1-\rho_0)(1-\rho_0^2)^2}\right]\Bigg\} \\
&= 0.
\end{aligned}
$$

Thus, $f(z_2) \neq f(z_1)$. The univalence of f follows from the arbitrariness of z_1 and z_2. This implies that f is univalent in $\mathbb{D}_{\rho_0}$.

Now, for any $\zeta' = \rho_0 e^{i\theta} \in \partial\mathbb{D}_{\rho_0}$, we obtain that

$$
\begin{aligned}
|f(\zeta') - f(0)| &= \left|\int_{[0,\zeta']} f_z(z)dz + f_{\overline{z}}(z)d\overline{z}\right| \\
&= \left|\int_{[0,\zeta']} f_z(0)dz + f_{\overline{z}}(0)d\overline{z}\right| \\
&\quad -\left|\int_{[0,\zeta']} \big(f_z(z)-f_z(0)\big)dz + \big(f_{\overline{z}}(z)-f_{\overline{z}}(0)\big)d\overline{z}\right| \\
&\geq l(D_f)(0)\rho_0 - \int_{[0,\zeta']} \big(|f_z(z)-f_z(0)| + |f_{\overline{z}}(z)-f_{\overline{z}}(0)|\big)|dz| \\
&\geq l(D_f)(0)\rho_0 - \frac{4M\rho_0^2}{\pi}\int_0^1\left[\frac{t(2-\rho_0 t)}{(1-\rho_0 t)^2}+\frac{2\rho_0 t^2}{(1-\rho_0 t)(1-\rho_0^2 t^2)^2}\right]dt \\
&\geq \frac{\beta\rho_0}{M(2+\alpha)} - \frac{4M\rho_0^2}{\pi}\left[\frac{(2-\rho_0)}{(1-\rho_0)^2}\int_0^1 t\,dt\right. \\
&\quad \left.+\frac{2\rho_0}{(1-\rho_0)(1-\rho_0^2)^2}\int_0^1 t^2 dt\right] \\
&= \rho_0\left\{\frac{\beta}{M(2+\alpha)} - \frac{4M\rho_0}{\pi}\left[\frac{2-\rho_0}{2(1-\rho_0)^2}+\frac{2\rho_0}{3(1-\rho_0)(1-\rho_0^2)^2}\right]\right\} \\
&= \frac{2\rho_0}{3}\left[\frac{\beta}{M(2+\alpha)} - \frac{M\rho_0(2-\rho_0)}{\pi(1-\rho_0)^2}\right].
\end{aligned}
$$

Hence $f\big(\mathbb{D}_{\rho_0}\big)$ contains a univalent disk $\mathbb{D}_{R_0}$ with

$$
R_0 \geq \frac{2\rho_0}{3}\left[\frac{\beta}{M(2+\alpha)} - \frac{M\rho_0(2-\rho_0)}{\pi(1-\rho_0)^2}\right].
$$

The proof of this lemma is complete. □

Let us recall the following result which is referred to as *Jensen's inequality* (cf. [23]).

Lemma B. *Let (Ω, A, μ) be a measure space such that $\mu(\Omega) = 1$. If g is a real-valued function that is μ-integrable, and if χ is a convex function on the real line, then*

$$\chi\left(\int_{\Omega} g\, d\mu\right) \leq \int_{\Omega} \chi \circ g\, d\mu.$$

Proof of Theorem 4. For $z \in \mathbb{D}_r$, we have

$$f(z) = \frac{c_\alpha}{2\pi r^\alpha} \int_0^{2\pi} \frac{\left(r^2 - |z|^2\right)^{\alpha+1}}{|r - ze^{-it}|^{2+\alpha}} f(re^{it}) dt,$$

where $r \in (0, 1)$. Let

$$\phi_z(r) = \frac{c_\alpha}{2\pi r^\alpha} \int_0^{2\pi} \frac{\left(r^2 - |z|^2\right)^{\alpha+1}}{|r - ze^{-it}|^{2+\alpha}} dt,$$

where $z \in \mathbb{D}_r$. Applying [21, Theorem 3.1], we see that, for $z \in \mathbb{D}$,

$$\phi_z(1) \leq \lim_{|z| \to 1-} \phi_z(1) = 1. \tag{3.7}$$

By using Jensen's inequality (see Lemma B), for $p \geq 1$, we get

$$\begin{aligned}
\left|\frac{f(z)}{\phi_z(r)}\right|^p &= \left|\frac{1}{2\pi} \int_0^{2\pi} \frac{c_\alpha}{r^\alpha \phi_z(r)} \frac{\left(r^2 - |z|^2\right)^{\alpha+1}}{|r - ze^{-it}|^{2+\alpha}} f(re^{it}) dt\right|^p \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{c_\alpha}{r^\alpha \phi_z(r)} \frac{\left(r^2 - |z|^2\right)^{\alpha+1}}{|r - ze^{-it}|^{2+\alpha}} |f(re^{it})|^p dt \\
&\leq \frac{c_\alpha}{r^\alpha \phi_z(r)} \frac{\left(r^2 - |z|^2\right)^{\alpha+1}}{\left(r - |z|\right)^{2+\alpha}} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt\right) \\
&\leq \frac{c_\alpha \|f\|_p^p}{r^\alpha \phi_z(r)} \frac{\left(r + |z|\right)^{\alpha+1}}{\left(r - |z|\right)},
\end{aligned}$$

which implies that

$$|f(z)| \leq \left[\frac{c_\alpha \|f\|_p^p \left(\phi_z(r)\right)^{p-1}}{r^\alpha}\right]^{\frac{1}{p}} \frac{\left(r + |z|\right)^{\frac{\alpha+1}{p}}}{\left(r - |z|\right)^{\frac{1}{p}}},$$

where $z \in \mathbb{D}_r$. By letting $r \to 1-$ and (3.7), for $z \in \mathbb{D}$, we have

$$|f(z)| \leq \left[c_\alpha \|f\|_p^p \left(\phi_z(1)\right)^{p-1}\right]^{\frac{1}{p}} \frac{\left(1 + |z|\right)^{\frac{\alpha+1}{p}}}{\left(1 - |z|\right)^{\frac{1}{p}}} \leq c_\alpha^{\frac{1}{p}} \|f\|_p \frac{\left(1 + |z|\right)^{\frac{\alpha+1}{p}}}{\left(1 - |z|\right)^{\frac{1}{p}}}. \tag{3.8}$$

For $\zeta \in \mathbb{D}$, let $Q(\zeta) = f(\gamma\zeta)/\gamma$, where $\gamma \in (0, 1)$. It is not difficult to know that $Q(0) = |J_Q(0)| - \lambda = 0$. By (3.8), for $\zeta \in \mathbb{D}$, we obtain

$$|Q(\zeta)| = \frac{|f(\gamma\zeta)|}{\gamma} \leq c_\alpha^{\frac{1}{p}} \|f\|_p \frac{\left(1 + \gamma\right)^{\frac{\alpha+1}{p}}}{\gamma\left(1 - \gamma\right)^{\frac{1}{p}}},$$

which gives that

$$|Q(\zeta)| \leq c_{\alpha}^{\frac{1}{p}} \|f\|_p \min_{0<\gamma<1} \mu(\gamma),$$

where

$$\mu(\gamma) = \left(1+\gamma\right)^{\frac{\alpha+1}{p}} / \left[\gamma\left(1-\gamma\right)^{\frac{1}{p}}\right].$$

Let $\gamma_0 \in (0,1)$ satisfy

$$\mu(\gamma_0) = \min_{0<\gamma<1} \mu(\gamma).$$

By using Lemma 2, we observe that Q is univalent in $\mathbb{D}_{\rho_0}$, where ρ_0 satisfies the following equation

$$\frac{\lambda}{M^*(2+\alpha)} - \frac{4M^*\rho_0}{\pi}\left[\frac{2-\rho_0}{(1-\rho_0)^2} + \frac{2\rho_0}{(1-\rho_0)(1-\rho_0^2)^2}\right] = 0,$$

where $M^* = c_{\alpha}^{\frac{1}{p}} \|f\|_p \mu(\gamma_0)$. Moreover, $Q\big(\mathbb{D}_{\rho_0}\big)$ contains a univalent disk $\mathbb{D}_{R_0}$ with

$$R_0 \geq \frac{2\rho_0}{3}\left[\frac{\lambda}{M^*(2+\alpha)} - \frac{M^*\rho_0(2-\rho_0)}{\pi(1-\rho_0)^2}\right].$$

Hence f is univalent in $\mathbb{D}_{\gamma_0\rho_0}$ and $f\big(\mathbb{D}_{\gamma_0\rho_0}\big)$ contains a univalent disk $\mathbb{D}_{\gamma_0 R_0}$. The proof of this theorem is complete. □

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Shaolin Chen, Department of Mathematics and Computational Science, Hengyang Normal University, Hengyang, Hunan 421008, People's Republic of China.
E-mail address: `mathechen@126.com`

Matti Vuorinen, Department of Mathematics and Statistics, University of Turku, Turku 20014, Finland.
E-mail address: `vuorinen@utu.fi`