The metric dimension for resolving several objects

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Abstract

A set of vertices S is a *resolving set* in a graph if each vertex has a unique array of distances to the vertices of S. The natural problem of finding the smallest cardinality of a resolving set in a graph has been widely studied over the years. In this paper, we wish to resolve a set of vertices (up to ℓ vertices) instead of just one vertex with the aid of the array of distances. The smallest cardinality of a set S resolving at most ℓ vertices is called ℓ-set-metric dimension. We study the problem of the ℓ-set-metric dimension in two infinite classes of graphs, namely, the two dimensional grid graphs and the n-dimensional binary hypercubes.

Keywords: Resolving set, sets of vertices, metric dimension, binary hypercube, grid graph

1 Introduction

In this paper, a graph G is finite, undirected, simple and connected. As usual, we denote its vertex set by V and the set of edges by E. The distance between two vertices $u, v \in V$ (that is, the number of edges in any shortest path joining u and v) is denoted by $d(u, v) = d_G(u, v)$. Let $N(v) = \{u \in V \mid d(u, v) = 1\}$ for $v \in V$. The *Cartesian product* of graphs $G = (V, E)$ and $H = (V', E')$, denoted by $G \square H$, is the graph with vertex set $V \times V' = \{(a, b) \mid a \in V, b \in V'\}$, where (a, b) is adjacent to (u, v) if $a = u$ and the edge $\{b, v\} \in E'$, or $b = v$ and $\{a, u\} \in E$. The distance $d((a, b),(u, v)) = d_G(a, u) + d_H(b, v)$.

Let $S \subseteq V$ and denote its cardinality by $|S|$. Let us write S as an ordered set $S = (s_1, s_2, \ldots, s_{|S|})$. For any $x \in V$, we denote by

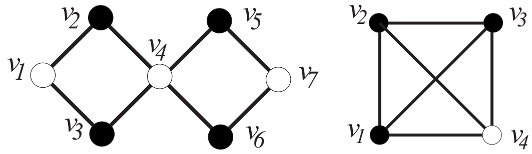
$$\mathcal{D}(x) = \mathcal{D}_S(x) = (d(x, s_1), d(x, s_2), \ldots, d(x, s_{|S|}))$$

the distance array of x with respect to S. If $\mathcal{D}_S(x) \neq \mathcal{D}_S(y)$ for any two distinct vertices x and y in V, then S is called a *resolving set*. The concept of a resolving set was introduced independently by Slater [15] and Harary and Melter [8]. Resolving sets are widely studied [5, 3, 4, 9, 1, 6, 13] and these sets have many connections to other diverse problems, see for example, network discovery and verification [2], robot navigation [10] and connected joins in graphs [14]. In [15], each $s_i \in S$ is considered as a site for a sonar station, and the location of an object (like an intruder in $x \in V$) is then uniquely determined using its distances to stations in $\mathcal{D}(x)$.

In this paper, we consider the situation where there can be several objects whose locations (the set $X \subseteq V$) we want to determine simultaneously. Naturally, here each sonar $s_i \in S$ measures the distance to the *closest* vertex in the object set $X \subseteq V$ (there can be several objects at that particular distance), but reveals no further information on the locations or the cardinality of X. Finding several objects has earlier been considered in other contexts of sensor networks, like in the case of identifying codes and locating-dominating sets, where the sensors can detect objects within a fixed radius, see [7, 12] and also the list in [11].

For any $X \subseteq V$ and $v \in V$, denote $d(v, X) = \min\{d(v, x) \mid x \in X\}$. Furthermore, for any $X \subseteq V$, let the distance array

$$\mathcal{D}(X) = \mathcal{D}_S(X) = (d(s_1, X), d(s_2, X), \ldots, d(s_{|S|}, X)).$$



(a) The graph G_7 (b) The graph $\mathcal{K}_4$

Figure 1: The set S consists of the black vertices.

We write in short, $\mathcal{D}(\{x_1, \ldots, x_k\}) = \mathcal{D}(x_1, \ldots, x_k)$. Hence $\mathcal{D}(x)$ means the same distance array as before.

Definition 1. Let $G = (V, E)$ be a finite, undirected, simple and connected graph. Let further ℓ be an integer such that $1 \le \ell \le |V|$. A subset $S \subseteq V$ is called an *ℓ-resolving set* (or an *ℓ-set resolving set*) if

$$\mathcal{D}(X) \ne \mathcal{D}(Y)$$

for any two distinct and nonempty subsets $X, Y \subseteq V$ with $|X| \le \ell$ and $|Y| \le \ell$.

The minimum cardinality of an ℓ-resolving set of G is called the *ℓ-set-metric dimension* of G and it is denoted by $\beta_\ell(G)$. An ℓ-resolving set of cardinality $\beta_\ell(G)$ is called an *ℓ-set-metric basis* of G. Clearly, a 1-resolving set is the usual resolving set, and the set $S = V$ is always an ℓ-resolving set for all $1 \le \ell \le |V|$.

Example 2. (i) Let us consider the graph of Figure 1(a). Take the set $S = \{v_2, v_3, v_5, v_6\}$. It is easy to check that S is a 1-resolving set, and $\mathcal{D}(v_4) = (1, 1, 1, 1)$. If we receive the distance array $(1, 1, 1, 1)$, we immediately conclude that the object (like an intruder) is in v_4. However, if there are *two* objects (intruders), say in v_1 and v_7, we can falsely make that decision and no intruder is found, since also $\mathcal{D}(v_1, v_7) = (1, 1, 1, 1)$.

(ii) Denote a path on $n \ge 2$ vertices by P_n and write the vertices as an ordered set $P_n = (v_1, v_2, \ldots, v_n)$. The set $S = \{v_1, v_n\}$ is a 2-resolving set as we will show next. Let $X \subseteq \{v_1, \ldots, v_n\}$ and $1 \le |X| \le 2$. Now $\mathcal{D}(X) = (a, b)$ for some $0 \le a, b \le n - 1$ (here $S = (v_1, v_n)$ is considered as an ordered set). If $a + b = n - 1$, then there X consists of one vertex, namely, v_{1+a}. On the other hand, if $a + b < n - 1$, then there are two vertices in X, namely, $X = \{v_{i+a}, v_{n-1-b}\}$. Consequently, $\beta_2(P_n) \le 2$. Moreover, the 2-set-metric dimension $\beta_2(P_n) = 2$. Indeed, if $S = \{v_i\}$ for some $1 \le i \le n$, then $\mathcal{D}(v_i) = (0) = \mathcal{D}(v_i, v_j)$ for any $j \ne i$, $j = 1, \ldots, n$.

(iii) Consider then the complete graph $\mathcal{K}_4$ of Figure 1(b). We will show that a set $S \ne V$ cannot be a 2-resolving set. Without loss of generality, say $v_4 \notin S$ for some 2-resolving set S. Notice that if we add vertices to a 2-resolving set, it remains 2-resolving. Hence we may assume that $S = \{v_1, v_2, v_3\}$. Since $\mathcal{D}(v_2) = (1, 0, 1) = \mathcal{D}(v_2, v_4)$, the set S is not 2-resolving. It follows that $\beta_2(\mathcal{K}_4) = 4$. By the same token, $\beta_2(\mathcal{K}_n) = n$ for all complete graphs $\mathcal{K}_n$, $n \ge 3$. This example shows that a 2-resolving set must not be confused with so-called *doubly* resolving set which is discussed, for instance, in [4] — there it is shown that the smallest doubly resolving set in $\mathcal{K}_n$ equals $n - 1$.

In this paper, we consider ℓ-resolving sets in two infinite families of graphs, namely, in the two dimensional grid graphs $P_p \square P_q$ and the n-dimensional binary hypercubes $\mathbb{F}^n$. For the usual (1-)resolving set, it has been shown that the metric dimension of the two dimensional grid graph equals two [10]. Section 2 shows that we can determine the 2-set-metric dimension in the grid graph using a helpful geometric flavour of the problem. In Section 3, we consider ℓ-resolving sets in the binary hypercubes $\mathbb{F}^n$. For the usual (1-)resolving sets it is known that $\beta_1(\mathbb{F}^n) \le n$ [5] and, asymptotically [14],

$$\lim_{n\to\infty} \beta_1(\mathbb{F}^n) \cdot \frac{\log n}{n} = 2.$$

2 On ℓ-resolving sets in a grid graph

In this section, we find the 2-set-metric dimension of the grid graph $P_p \square P_q$ and show that the only ℓ-resolving set for $3 \leq \ell \leq pq$ is the whole set of vertices $S = P_p \times P_q$. Recall that the path in Example 2(ii) can be interpreted as $P_n \square P_1$ where P_1 consists of a single vertex.

Theorem 3. *Let $p, q \geq 2$ be integers. Then we have $\beta_2(P_p \square P_q) = \min\{p, q\} + 2$.*

Proof. First we consider the lower bound $\beta_2(P_p \square P_q) \geq \min\{p, q\} + 2$. Let S be any 2-resolving set in the graph $P_p \square P_q$. Denote $P_p = (v_1, \ldots, v_p)$ and $P_q = (w_1, \ldots, w_q)$. The distance between two vertices (v_i, w_j) and $(v_{i'}, w_{j'})$ of $P_p \times P_q$ equals

$$|i - i'| + |j - j'|. \tag{1}$$

First we show that all the *corners* (v_1, w_1), (v_p, w_1), (v_1, w_q) and (v_p, w_q) necessarily belong to the 2-resolving set S. Assume to the contrary that $(v_1, w_1) \notin S$ (proceed analogously with the other corners). Consider now two sets $X = \{(v_2, w_2)\}$ and $Y = \{(v_1, w_1), (v_2, w_2)\}$. By (1), we see that any vertex in $P_p \square P_q$ apart from (v_1, w_1) has shorter (or equal) distance to (v_2, w_2) than to (v_1, w_1). Therefore, for any element $s \in S$, we get $d(s, Y) = d(s, (v_2, w_2)) = d(s, X)$. Consequently, $\mathcal{D}(X) = \mathcal{D}(Y)$, which is a contradiction, and we are done.

If $p = 2$ or $q = 2$, this already gives the claim $\beta_2(P_p \square P_q) \geq 4$, so assume from now on that $p, q \geq 3$. We denote the *rows* (which are not intersecting the corners) by $R_k = \{(v_i, w_k) \mid i = 1, \ldots, p\}$, where $k = 2, \ldots, q-1$, and *columns* by $I_h = \{(v_h, w_j) \mid j = 1, \ldots q\}$, where $h = 2, \ldots, p-1$. Denote the *cross* (without the center (v_h, w_k)) by $C_{h,k} = (R_k \cup I_h) \setminus \{(v_h, w_k)\}$. We need the following fact:

- **Fact 1:** There exists at least one element of S in any cross $C_{h,k}$ where $h = 2, \ldots, p-1$ and $k = 2, \ldots, q-1$.

  In order to prove this, let us consider the sets $X = \{(v_h, w_{k+1}), (v_h, w_{k-1})\}$ and $Y = \{(v_{h-1}, w_k), (v_{h+1}, w_k)\}$. There must be an element of S in the cross $C_{h,k}$ if S is a 2-resolving set, since any vertex u outside the cross has $d(u, X) = d(u, Y)$. Indeed, suppose u is in the first quadrant when (v_h, w_k) is considered as the origin in the Euclidean plane. The vertices in that quadrant outside the cross are (v_h, w_k) and $U = \{(v_a, w_b) \mid a \geq h+1, b \geq k+1\}$. If $u = (v_h, w_k)$, we clearly have $d(u, X) = 1 = d(u, Y)$, so suppose that $u \in U$. Now the distance from u to (v_h, w_{k+1}) in X is the same as its distance to (v_{h+1}, v_k) in Y (and the distance to the other vertices of X and Y are larger). Therefore, $d(u, X) = d(u, Y)$. If u belongs to other quadrants, the argument is similar.

If all the rows R_k, $k = 2, \ldots, q-1$, contain at least one element of S, then $|S| \geq q-2+4 = q+2$. Suppose then that there is a row R_t for some $t \in \{2, \ldots, q-1\}$ which has no element of S in it. By Fact 1, we know, by considering the cross $C_{h,t}$, that every column I_h, $h = 2, \ldots, p-1$ must contain an element of S. There are $p-2$ such columns, so we get $|S| \geq p-2+4 = p+2$. Combining these two observations, we get the assertion $\beta_2(P_p \square P_q) \geq \min\{p+2, q+2\}$.

Next we give a construction achieving the lower bound. Suppose $p \leq q$ (the other case is analogous). Consider the set (see Figure 2(a))

$$S = \{(v_i, w_1) \mid i = 1, \ldots p\} \cup \{(v_1, w_q), (v_p, w_q)\}.$$

Clearly, $|S| = p + 2 = \min\{p, q\} + 2$. We will show that S is a 2-resolving set in $P_p \square P_q$. We do this by determining the set $X \subseteq P_p \times P_q$, where $1 \leq |X| \leq 2$, using only the distance array $\mathcal{D}(X)$. This implies that $\mathcal{D}(X) \neq \mathcal{D}(Y)$ for all distinct subsets $X, Y \subseteq P_p \times P_q$ with $1 \leq |X|, |Y| \leq 2$.

Denote the diagonal line (see Figure 2(a)) with slope 1 passing through the vertex $u = (v_i, w_j)$ by

$$L^+(u) = \{(v_{i+r}, w_{j+r}) \in P_p \times P_q \mid r \in \mathbb{Z}\}.$$

Analogously, the diagonal line with slope -1 is denoted by $L^-(u)$.

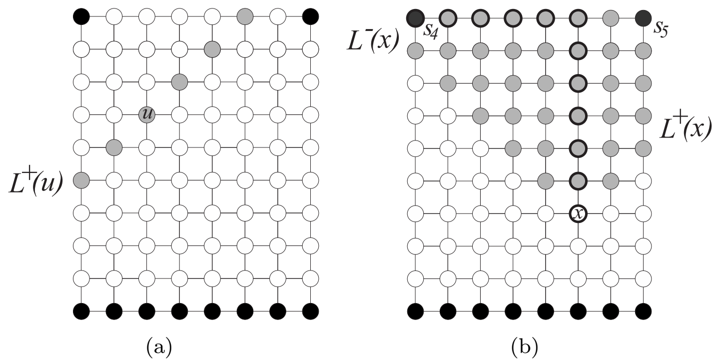


Figure 2: (a) The set S consists of the black vertices and the diagonal line $L^+(u)$ is marked in gray. (b) The top region R is illustrated by gray vertices and the vertices of the path are circled.

Now let $X \subseteq P_p \times P_q$, where $1 \leq |X| \leq 2$. Denote the elements of S on the bottom line of the grid by $S' = \{(v_i, w_1) \mid i = 1, \dots p\}$. Consider the distances $d(s, X)$ corresponding to vertices $s \in S'$ in $\mathcal{D}(X)$ and choose one with smallest value. Say $s_1 = (v_m, w_1)$ gives the minimal value $d(s_1, X) = k$ (there can be others giving the minimal value also). Then we know that $x = (v_m, w_k)$ belongs to X. Indeed, the other vertices in $P_p \square P_q$ which have the distance equal to k from s_1 are those in $L^+(x) \cup L^-(x)$ whose second coordinate w_h is such that $h < k$. But these vertices in X would imply that there were a smaller value than k in $\mathcal{D}(X)$ among vertices of S'.

Now we have found one vertex of X, so we continue to look for the possible other vertex or conclude that there are no other vertices. The distance from $x \in X$ to the other vertices in S' is easy to calculate — for $s_2 = (v_i, w_1) \in S'$ we have $d(s_2, x) = k + |m - i|$. We separate two cases, depending on whether $d(s, X) = d(s, x)$ for all $s \in S'$ or not.

1) Assume first that there are elements $s \in S'$ with smaller $d(s, X)$ than $d(s, x)$. Say $s_3 = (v_f, w_1) \in S'$ gives the *smallest* value $d(s_3, X) = k'$ among the elements of S' which differ from $d(s, x)$. Then, reasoning as above, we know that $y = (v_f, w_{k'})$ belongs to X and we have determined $X = \{x, y\}$ completely.

2) Assume then that $d(s, X) = d(s, x)$ for all $s \in S'$. This implies that if there is another vertex y in X besides $x = (v_m, w_k)$, it must be in the top region bounded by the sets $L^+(x)$ and $L^-(x)$ as illustrated in Figure 2(b), that is, in the set

$$R = \{(v_a, w_b) \in P_p \times P_q \mid b > k, m - (b - k) \leq a \leq m + (b - k)\}.$$

Now we use the remaining two elements of S (apart from S'), namely, $s_4 = (v_1, w_q)$ and $s_5 = (v_p, w_q)$ to resolve X. Denote $d(s_4, x) = e_1$ and $d(x_5, x) = e_2$ (we can calculate these, since we know x). If $d(s_4, X) = e_1$ and $d(s_5, X) = e_2$, then X consists of a single point $X = \{x\}$. Otherwise, we can find y at the intersection of the following two diagonal lines. The first diagonal line is $L^+(u)$ where u is the vertex at distance $d(s_4, X)$ from s_4 on the following path (see Figure 2(b)) between s_4 and x:

$$\{(v_1, w_q), (v_2, w_q), \dots, (v_m, w_q), (v_m, w_{q-1}), \dots, (v_m, w_k)\}.$$

The second diagonal line is $L^-(u')$ where u' is the vertex at distance $d(s_5, X)$ from s_5 on the path between s_5 and x:

$$\{(v_p, w_q), (v_{p-1}, w_q), \dots, (v_m, w_q), (v_m, w_{q-1}), \dots, (v_m, w_k))\}.$$

We have determined X using $\mathcal{D}(X)$ in all cases which completes the proof. $\square$

We consider next ℓ-resolving sets when $\ell \geq 3$.

Theorem 4. *Let $p, q \geq 2$ be integers. For $3 \leq \ell \leq pq$, we have $\beta_\ell(P_p \square P_q) = pq$.*

Proof. We will show that the only ℓ-set-metric basis is the whole set of vertices $V = P_p \times P_q$ (trivially, this set is ℓ-resolving). Let S be any ℓ-resolving set for $3 \leq \ell \leq pq$ in $P_p \square P_q$. Assume to the contrary that there exists $u \in P_p \times P_q$ such that $u \notin S$. Since an ℓ-resolving set is also a 2-resolving set, we know from the proof of Theorem 3, that u cannot be any of the corners (v_1, w_1), (v_1, w_q) (v_p, w_1) or (v_p, w_q).

We divide the examination into two parts depending on where u lies — whether it is on the first or last row or column or it is not on one of them.

1) Suppose that $u = (v_i, w_j) \notin S$ is such that $2 \leq i \leq p-1$ and $2 \leq j \leq q-1$. Hence necessarily $p, q \geq 3$. Denote $y = (v_{i+1}, w_{j+1})$ and $z = (v_{i-1}, w_{j-1})$. It follows that $\mathcal{D}(y, z) = \mathcal{D}(u, y, z)$. Indeed, the distance of any vertex $s \in S$, $s \neq u$, to x is larger than or equal to its distance to the set $\{y, z\}$ (in other words, $d(s, \{y, z\}) = d(s, \{u, y, z\})$. Hence, if S does not contain u, it cannot be an ℓ-resolving set for $\ell \geq 3$.

2) If $p = q = 2$, then all the vertices are corners, so $S = V$ immediately. Therefore, we can assume that $p \geq 3$ or $q \geq 3$. Without loss of generality, say $q \geq 3$. Let $u = (v_i, w_j)$ be such that $i \in \{1, p\}$ or $j \in \{1, q\}$. Assume that $i = 1$ (the other cases are analogous and if $p = 2$ the case $j \in \{1, q\}$ is trivial). Since u is not a corner, we know that $2 \leq j \leq q-1$. Choose $y = (v_2, w_{j+1})$ and $z = (v_1, w_{j-1})$. Again it is easy to see that $\mathcal{D}(y, z) = \mathcal{D}(u, y, z)$ and we are done. $\square$

3 On ℓ-resolving sets in the binary hypercube

In this section, we examine ℓ-resolving sets in the n-dimensional binary hypercubes. Let $\mathbb{F} = \{0, 1\}$. Denote the Cartesian product $\mathbb{F}^n = \mathbb{F} \times \cdots \times \mathbb{F}$ (n times). The vertex set of the n-dimensional binary hypercube is $\mathbb{F}^n$ and two vertices $x = (x_1, \ldots, x_n) \in \mathbb{F}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{F}^n$ are adjacent if and only if they differ in exactly one coordinate position. The (Hamming) distance between two vertices is the number of coordinates in which they differ. As usually, the vertices in $\mathbb{F}^n$ are called *words*. The *support* of a word $x = (x_1, \ldots, x_n) \in \mathbb{F}^n$ is defined as $\mathrm{supp}(x) = \{i \mid x_i = 1, i = 1, 2, \ldots, n\}$. The *weight* $w(x)$ of x equals $|\mathrm{supp}(x)|$. Denote $\mathbf{0} = (0, \ldots, 0) \in \mathbb{F}^n$.

Next we will provide an analogous result to Theorem 4 for the binary hypercube.

Theorem 5. *Let $n \geq 3$ and $\ell \leq 2^n$. We have $\beta_\ell(\mathbb{F}^n) = 2^n$ if*

$$\ell \geq \begin{cases} \frac{n-1}{2} + 2 & \text{when } n \text{ is odd,} \\ \frac{n}{2} + 1 & \text{when } n \text{ is even.} \end{cases}$$

Proof. We show that the only ℓ-resolving set in the n-dimensional binary hypercube is $\mathbb{F}^n$.

First, let n be odd and S be an ℓ-resolving set with $\frac{n-1}{2} + 2 \leq \ell \leq 2^n$. Suppose to the contrary that there is $x \in \mathbb{F}^n$ such that $x \notin S$. Without loss of generality (the binary hypercube is vertex-transitive), we may assume that $x = \mathbf{0}$. Denote by y_i the word with support $\{2i-1, 2i\}$ for $i = 1, \ldots, (n-1)/2$. Denote further $z = 000\ldots01$. Since S is an ℓ-resolving set, we must have

$$\mathcal{D}(X) \neq \mathcal{D}(Y)$$

for the sets $X = \{y_1, \ldots, y_{(n-1)/2}, z\}$ and $Y = \{\mathbf{0}, y_1, \ldots, y_{(n-1)/2}, z\}$, because $|X|, |Y| \leq \ell$. Since $\mathcal{D}(X) \neq \mathcal{D}(Y)$ there must exist $s_1 \in S$ such that $d(s_1, X) \neq d(s_1, Y)$ (recall that s_1 cannot be $\mathbf{0}$). We need to have $d(s_1, \mathbf{0}) < d(s_1, X)$. Any shortest path from s_1 to $\mathbf{0}$ (the only vertex differing in the sets X and Y) has to go through (or begin at) a vertex with weight one, say a with $\mathrm{supp}(a) = \{j\}$ where $j = 1, 2, \ldots n$. If $j < n$, then the distance from s_1 to $\mathbf{0}$ is the same as its distance to y_i with $i = \lceil j/2 \rceil$. If $j = n$, then $d(s_1, \mathbf{0}) > d(s_1, z)$. This implies that $d(s_1, X) = d(s_1, Y)$, a contradiction.

The case n even goes similarly using the sets $X = \{y_1, \ldots, y_{n/2}\}$ and $Y = \{\mathbf{0}, y_1, \ldots, y_{n/2}\}$. $\square$

Notice that $\mathbb{F}^2$ is isomorphic to $P_2 \square P_2$, so, by Theorem 3, we get $\beta_2(\mathbb{F}^2) = 4$ and hence $\beta_\ell(\mathbb{F}^2) = 4$ for $\ell = 3, 4$. By the previous theorem, there are no non-trivial ℓ-resolving sets for $n = 3, 4$ when $\ell \geq 3$. The code $C = \mathbb{F}^3 \setminus \{000, 111\}$ (resp. $C = \mathbb{F}^4 \setminus \{0000, 0001, 0010, 0100, 0111, 1011\}$) gives $\beta_2(\mathbb{F}^3) \leq 6$ (resp. $\beta_2(\mathbb{F}^4) \leq 10$). It is easy to check using a computer that $\beta_2(\mathbb{F}^3) = 6$ and $\beta_2(\mathbb{F}^4) = 10$. Now we will show that the bound on ℓ of the previous theorem is optimal in the sense that if ℓ is smaller than that, there exists a set $S \neq \mathbb{F}^n$ such that it is ℓ-resolving.

Theorem 6. *Let $n \geq 5$. For n odd and $\ell \leq (n-1)/2+1$, we have $\beta_\ell(\mathbb{F}^n) \leq 2^{n-1}$. For n even and $\ell \leq n/2$, we have similarly $\beta_\ell(\mathbb{F}^n) \leq 2^{n-1}$.*

Proof. Let $n \geq 5$ and

$$S = \{(c_1, \dots, c_n) \in \mathbb{F}^n \mid c_1 + c_2 + \cdots + c_n = 0\} \tag{2}$$

where the sum is taken modulo two, that is, S consists of all of the words in $\mathbb{F}^n$ which have even weight. Clearly, $|S| = 2^{n-1}$. We will show that this set gives an $((n-1)/2+1)$-resolving set for n odd and an $n/2$-resolving set for n even. This implies that there exists an ℓ-resolving set for all $\ell \leq (n-1)/2+1$ when n is odd and for all $\ell \leq n/2$ when n is even. We need to show that $\mathcal{D}(X) \neq \mathcal{D}(Y)$ for any distinct subsets $X, Y \subseteq \mathbb{F}^n$ with $1 \leq |X|, |Y| \leq \ell$ where $\ell = (n-1)/2+1$ if n is odd and $\ell = n/2$ if n is even.

Assume to the contrary that for some such sets X and Y we have

$$\mathcal{D}(X) = \mathcal{D}(Y). \tag{3}$$

Without loss of generality, we may assume that $|X| \geq |Y|$ and that there exists $x \in X$ which is not in Y. Moreover, due to (3), the word x cannot belong to S. Since $x \notin S$, the neighbours $N(x)$ must be in S by the definition of S.

Clearly, $d(s, X) \leq 1$ for all $s \in N(x)$ because $x \in X$. Moreover, $d(s, X) = 0$ if and only if $s \in X$. Suppose that k of the vertices in $N(x)$ are in X, where $0 \leq k \leq \ell - 1$. Denote these words by $v_1, \dots, v_k$ (if there are any). Due to (3) (recall that $v_1, \dots, v_k \in S$), these words must also belong to Y. Since $d(s, X) = 1$ for the rest of the neighbours $N(x)$, for each $s \in N(x) \setminus \{v_1, \dots, v_k\}$ there must be $y \in Y$ such that $d(s, y) = 1$. Because $x \notin Y$, clearly $d(x, y) = 2$. On the other hand, for any such y there can be *at most two* words in $N(x) \setminus \{v_1, \dots, v_k\}$, say s_1 and s_2, such that $d(s_1, y) = d(s_2, y) = 1$ — indeed, in the binary hypercube, any words at distance two have exactly two common neighbours. There are $n-k$ words in $N(x) \setminus \{v_1, \dots, v_k\}$ which all must have a word of Y at distance one from them. Since $\{v_1, \dots, v_k\} \subseteq Y$, there are at most $|Y| - k$ words available for that in Y. Because each such word in Y can take care of at most two of the $n-k$ words, the inequality

$$n - k \leq 2(|Y| - k) \tag{4}$$

must be satisfied when (3) holds. Notice that the inequality is not satisfied unless $|Y| = \ell$ (and hence also $|X| = \ell$ because $|X| \geq |Y|$).

Since $X \neq Y$ and $|X| = |Y| = \ell$, there must be a word in Y which is not in X. This, as we shall see below in cases 1) and 2), is a word $\tilde{y} \in Y \setminus X$ at distance two from x. Moreover, we will see that all the words in Y are at distance two or less from x. We separate the investigation into two parts depending on whether n is odd or even.

1) Let first n be odd. In this case $\ell = (n-1)/2+1$ and thus (4) with $|Y| = \ell$ gives $k \leq 1$.

- Assume first that $k = 1$. Thus Y consists of $v_1 \in N(x)$ and words y_i, $i = 1, \dots, (n-1)/2$, such that $d(x, y_i) = 2$ which *all* are needed to take care of the words in $N(x) \setminus \{v_1\}$. Notice that hence every word of Y is within distance two from the word x. Since $v_1 \in X \cap Y$, the word $\tilde{y}$ is one of the y_i's.
- Let then $k = 0$. Now again all words in the set Y are such that their distance to x is at most two (actually, now the distance is exactly two). Indeed, any $\ell - 1$ words $y_1, \dots, y_{(n-1)/2}$ cannot give $d(s, Y) = 1$ for all $s \in N(x)$ — recall that each y_i can take care of at most two of the words in $N(x)$. Therefore, all the ℓ words of Y must be at distance two form x and $\tilde{y}$ is found among them.

2) Assume then that n is even and $\ell = n/2$. By (4), we know that $k = 0$. Now all the words of Y are again within distance two form x and the promised $\tilde{y}$ exists.

Next we show how the word $\tilde{y}$ and the fact that the words of Y are within distance two from x can be used to get a contradiction with (3). By the definition of S and the fact that $x \notin S$, we know that $\tilde{y}$ is not in S and all the words in $N(\tilde{y})$ belong to S. In $N(\tilde{y})$, there are $n-2$ words which are at distance three from x (the two remaining words in $N(\tilde{y})$ are in $N(x)$). Denote this set of $n-2$ words by U. Now $U \subseteq S$ and $d(s,Y) \leq 1$ for every $s \in U$ — more precisely, $d(s,Y)=1$ because all the words in Y are within distance two from x. Due to (3), the same must be true with respect to X. In order to have $d(s,X)=1$ for $s \in U$ there must be a word in X, which is at distance four or two from x. However, a word at distance four, say $u' \in X$, is not possible, because then there would be a word $w \in N(u') \subseteq S$ at distance five from x such that $d(w,X) \leq 1$, but clearly $d(w,Y) > 1$ due to the fact that all the words in Y are within distance two from x. Consider then the case that we have a word $u' \in X$ which is at distance two form x. For each such word u' ($u' \neq \tilde{y}$) there is at most one word s in U such that $d(s,u')=1$. But $|U|=n-2$ and there are at most $\ell-1$ words in X at distance two from x. This gives the sought contradiction with (3), because there exists $s \in U$ such that $d(s,Y)=1$ in $\mathcal{D}(Y)$ but $d(s,X)>1$ in $\mathcal{D}(X)$. □

For n odd, the set found in the previous proof is actually a $((n-1)/2+1)$-set-metric basis.

Theorem 7. *Let $n \geq 5$ be odd. Then the $\beta_{(n-1)/2+1}(\mathbb{F}^n) = 2^{n-1}$.*

Proof. The previous theorem shows that there exists an ℓ-resolving set of cardinality 2^{n-1} for $\ell = (n-1)/2+1$. It suffices to show that there does not exist a smaller one. Suppose S is any $((n-1)/2+1)$-resolving set. Then it must be able to distinguish between the two sets $X = \{y_1, \dots, y_{(n-1)/2}\}$ and $Y = \{\mathbf{0}, y_1, \dots, y_{(n-1)/2}\}$ where y_i is the word with support $\mathrm{supp}(y_i) = \{2i-1, 2i\}$, $i = 1, \dots, (n-1)/2$. Notice that $|X|, |Y| \leq \ell$, so necessarily $\mathcal{D}(X) \neq \mathcal{D}(Y)$. We will show next that this implies that the pair $A = \{\mathbf{0}, z\}$, where $z = 00\dots01$, must contain an element of S.

Suppose to the contrary that $A \cap S = \emptyset$. Since $\mathcal{D}(X) \neq \mathcal{D}(Y)$ and the sets X and Y differ only in $\mathbf{0}$, there must be an element $s \in S$ such that $d(s,\mathbf{0}) < d(s,Y)$. Notice that $s \notin A$. Any shortest path between s and $\mathbf{0}$ goes through (or starts from) a word of weight one, say v with $\mathrm{supp}(v) = \{k\}$. If $k < n$, then $d(s,\mathbf{0}) \geq d(s, y_{\lceil k/2 \rceil})$ leading to $d(s,\mathbf{0}) \geq d(s,Y)$, so this is impossible. Assume then that $k = n$. Now $s \notin A$, so s must be a word of weight two or more. Hence the shortest path must go through some vertex $w \in N(z) \setminus A$. Say, w has the support $\{h, n\}$. But then $d(s,\mathbf{0}) \geq d(s,y_i)$ where $i = \lceil h/2 \rceil$. This gives a contradiction, so the set A must contain an element of S.

Since the binary hypercube is vertex-transitive, the fact that A contains an element of S implies that every pair $\{x0, x1\}$, $x \in \mathbb{F}^{n-1}$, which differ in the last coordinate must contain also an element of S. Since these pairs partition the binary hypercube $\mathbb{F}^n$ (because $\mathbb{F}^n = \mathbb{F}^{n-1} \times \mathbb{F}$), we obtain the assertion that $|S| \geq 2^{n-1}$. Hence the set of (2) is an $((n-1)/2+1)$-set-metric basis. □

We show next that the 2-set-metric dimension has an upper bound of order $\sim n^2$.

Theorem 8. *Let $n \geq 6$. In the n-dimensional binary hypercube we have*

$$\beta_2(\mathbb{F}^n) \leq \frac{1}{2}n^2 + \frac{3}{2}n + 2.$$

Proof. Let S be the set which consists of all the words of weight $0, 1, 2, n-1$ and n. We will verify that S is a 2-resolving set in the binary hypercube. Denote the words of weight two (resp. $n-1$) in S by S_2 (resp. S_{n-1}). Clearly, $|S| = \binom{n}{2} + 2(n+1) = \frac{1}{2}n^2 + \frac{3}{2}n + 2$.

In order to show that S is 2-resolving, we determine X with the aid of $\mathcal{D}(X)$, when $1 \leq |X| \leq 2$. We divide the consideration into two cases depending on whether or not the distance array $\mathcal{D}(X)$ contains zeros.

1) Suppose first that $\mathcal{D}(X)$ does not contain any zeros. Consequently, the weights of the words of X are between 3 and $n-2$. Consider first the smallest value in $\mathcal{D}(X)$ among the words of S_2 (of course, there can be many words in S_2 giving that value). Denote the smallest value by $d_{\min}$. Clearly, there is a word (maybe two) in X of weight $d_{\min}+2$. Similarly, let $d'_{\min}$ be the minimum value, which the words in S_{n-1} have in $\mathcal{D}(X)$. Then there is at least one word in X of weight $n - d'_{\min} - 1$.

$$X\colon \begin{array}{l} 0\underline{1}0\mathbf{11}0\mathbf{1}00\underline{10}00\underline{001} \\ 0\underline{0}0\mathbf{11}0\mathbf{1}00\underline{01}00\underline{110} \end{array}, \quad X'\colon \begin{array}{l} 0\underline{0}0\mathbf{11}0\mathbf{1}00\underline{11}00\underline{100} \\ 0\underline{1}0\mathbf{11}0\mathbf{1}00\underline{00}00\underline{011} \end{array}$$

Figure 3: Two sets of words. The coordinates in U' are in bold and in W underlined.

If $d_{\min} + 2 \neq n - d'_{\min} - 1$, then we can immediately reconstruct the words of X. Namely, the word giving the weight $d_{\min} + 2$ (call this x) can be found by looking at all the words in S_2 which provide the value $d_{\min}$ to $\mathcal{D}(X)$ — the union of their supports gives the support of x, so we have determined x. On the other hand, the word of X of weight $n - d'_{\min} - 1$, say y, can be reconstructed by looking at the words of S_{n-1} with minimal value $d'_{\min}$ in $\mathcal{D}(X)$ — these reveal all the coordinates with zero in y (there is a zero in y at the coordinate where a word of S_{n-1} with $d'_{\min}$ has the unique zero). Therefore, y is also reconstructed and we know X.

Assume then that $d_{\min} + 2 = n - d'_{\min} - 1$, that is, the words of X have the same weight or there is just one word in X. Denote the weight by $w = d_{\min} + 2$. Let U be the set of the words (if any) in S_{n-1} such that they give the *maximal* value $d_{\max} > d'_{\min}$ in $\mathcal{D}(X)$ among the values provided by the words in S_{n-1}. This value $d_{\max} = d(s, X)$ is obtained if $s \in S_{n-1}$ has (its unique) 0 in a coordinate where all the words of X have a 1 (if $|X| = 2$, this means a common 1 for the words of X in this coordinate).

If $|U| = w$, then X consists of just one word and its ones are at the positions where the words of U have zeros. Consequently, we have determined the set X in this case.

Suppose then that $|U| < w$ (obviously, we cannot have $|U| > w$). Now there must be at least two words in X, say $X = \{x, y\}$. The coordinates where both x and y have common 1's (if any) are the positions where the words of U have zeros. Denote the set of coordinates with common 1's by U'. We separate three cases depending on the size of U'. Let T denote the set of words in S_2 which correspond to the minimum value $d_{\min}$ in $\mathcal{D}(X)$.

- Assume first that the size $|U'| = w - 1$. Now we can reconstruct x and y by looking at the words in T. Take one coordinate $i \in U'$ and test all the coordinates $j \notin U'$ to find out whether or not the word with support $\{i, j\}$ belongs to T. There must be exactly two such coordinates j_1 and j_2. Now (say) x is the word with 1's exactly at the coordinates in U' and one more 1 at coordinate j_1, and y is the word with 1's in U' and at the coordinate j_2.

- Assume then that $1 \leq |U'| \leq w-2$. Now take as previously the coordinates $i \in U'$ and $j \notin U'$ and test whether the word with support $\{i, j\}$ belongs to T. This reveals the coordinate positions outside U' where either x has 1 or y has 1 (but not both). Denote this set of coordinate positions by W. Notice that we do not know how these 1's are distributed among x and y (see Figure 3 for two different sets X and X' with the same U' and W). Take a coordinate $i' \in W$. Name the words x and y so that x is the word with 1 at the coordinate i' (and y has 0). Going through the words in T with support $\{i', j'\}$ where $j' \in W \setminus \{i'\}$ reveals all the coordinates j' with 1 in x. Indeed, only those words where there is 1 in x at the coordinate position j' belong to T. The rest of the coordinates of W are such that y has 1 and x has 0. Consequently, we have determined the set X.

- Let $|U'| = 0$. Now take any word in T. Suppose it has the support $\{i, j\}$. Now either x or y has 1's in these coordinates i and j (and the other has 0's). Call x the one with 1's at i and j. Going through all the words in T with support $\{i, j'\}$ where $j' \neq i$ we find all the coordinates of x with 1 in it. Hence we can determine x (and we know $\mathrm{supp}(x)$). After that take another word in T, whose support $\{h, k\}$ is such that $\mathrm{supp}(x) \cap \{h, k\} = \emptyset$ (there exists such a word because $|U'| = 0$). We can now find $\mathrm{supp}(y)$ in the same way as we did find $\mathrm{supp}(x)$. Consequently, we have determined X.

2) Assume then that $\mathcal{D}(X)$ contains zeros. If there are two zeros in $\mathcal{D}(X)$, then X consists of the two words of S giving the zeros in $\mathcal{D}(X)$. Now suppose there is exactly one zero. Again we know

one word of X immediately within S. If the zero in $\mathcal{D}(X)$ is given by a word of weight at least $n-1$ (resp. of weight at most 2), then we can find out (as above) using the words in S_2 (resp. S_{n-1}) whether or not there is another word in X of weight between 3 and $n-2$, and which it is. This completes the proof. $\square$

Actually, in the previous proof, we would only need words of weight 2 and $n-1$, but our choice made the argument shorter and we are only interested in the order of growth of S.

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