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Harmonic approximations of analytic functions

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ABSTRACT
This paper aims to introduce a measure of the non-univalency of a harmonic mapping. By using it, we find the best approximation of an analytic function by a univalent harmonic mapping.

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1. Introduction

The problem of finding the best approximation for a non-univalent analytic function, which shows how ‘far’ is an analytic function from a subclass of univalent functions, does not have a long history. Therefore, it has not been widely studied. First time, in 2012, this problem was studied by Pascu and Pascu [1]. They found the best approximation of an analytic function in a subclass of starlike functions by extending the *Karush–Kuhn–Tucker* (KKT) conditions to semi-infinite quadratic programming. Additionally, they solved this problem in 2014 for a certain subclass of convex functions, see Ref. [2]. Kargar et al. [3] studied, in 2017, the best approximation of an analytic function by a locally univalent normalized analytic function. Furthermore, Arora et al. [4] very recently considered non-vanishing analytic functions of the form z/f where f is a univalent function and studied their best approximations by functions z/g with g belonging to a certain subclass of starlike functions. Here, we note that Ponnusamy and Qiao [5] used a different method of approximating certain biharmonic mappings than the method used in the above works. Motivated by the above works, we shall consider this problem for certain subclasses of harmonic mappings. We will therefore present some characteristic information regarding harmonic mappings.

Let Ω be a domain and $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc in the complex plane $\mathbb{C}$. We say that an analytic and normalized ($f(0) = 0 = f'(0) - 1$) function f is *starlike* in $\mathbb{D}$ if $f(\mathbb{D})$ is a set that is starlike with respect to the origin. In other words, the straight line

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joining any point in $f(\mathbb{D})$ to the origin lies in $f(\mathbb{D})$. In addition, Ω is *convex* if any line segment joining two points in Ω also lies in Ω. An univalent function f is called convex if $f(\mathbb{D})$ is a convex set. A planar *harmonic* mapping $f : \Omega \to \mathbb{C}$ is a complex-valued function of the form $f = u(z) + iv(z)$, where $z = x + iy$, u and v are real harmonic functions. It is known that each complex-valued harmonic mapping f on $\mathbb{D}$ admits the decomposition $f = h + \overline{g}$, where both h and g are analytic in $\mathbb{D}$, see Ref. [6]. We denote by $\mathcal{H}$ the class of all normalized ($f(0) = 0$ and $f_z(0) = 1$) harmonic mappings f having the form

$$f = h + \overline{g}(g(0) = 0); h(z) := z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) := \sum_{n=1}^{\infty} b_n z^n. \tag{1}$$

The Jacobian of the mapping f is defined as $J_f(z) := |f_z(z)|^2 - |f_{\overline{z}}(z)|^2 = |h'(z)|^2 - |g'(z)|^2$. We notice that (see Ref. [7]) a harmonic mapping $f = h + \overline{g}$ is locally univalent and sense-preserving in $\mathbb{D}$ if, and only if, $J_f(z) > 0$. We denote by $\mathcal{S}_{\mathcal{H}}$ the class of functions $f \in \mathcal{H}$ which are *sense-preserving* and *univalent* in $\mathbb{D}$. A certain subclass of $\mathcal{S}_{\mathcal{H}}$ is considered here as:

$$S^0_{\mathcal{H}} := \{f \in \mathcal{S}_{\mathcal{H}} : b_1 = f_{\overline{z}}(0) = 0\}.$$

We note that when $f \in \mathcal{S}_{\mathcal{H}}$ and $g \equiv 0$, then the class $S^0_{\mathcal{H}}$ reduces to the well-known class $\mathcal{U}$ of univalent analytic functions in $\mathbb{D}$. We continue with a definitions.

A function $f \in \mathcal{H}$ is called harmonic *starlike* of order α, $\alpha \in [0, 1)$, if

$$\frac{\partial}{\partial\theta}\left(\arg f(re^{i\theta})\right) > \alpha \quad (z = re^{i\theta} \in \mathbb{D}, 0 \le r < 1, 0 \le \theta \le 2\pi).$$

The class of harmonic starlike functions f of order α in $\mathbb{D}$ will be denoted by $\mathcal{S}^*_{\mathcal{H}}(\alpha)$. Likewise, a function $f \in \mathcal{H}$ is called harmonic *convex* of order α, $\alpha \in [0, 1)$, if

$$\frac{\partial}{\partial\theta}\left\{\arg\left(\frac{\partial}{\partial\theta}\left(\arg f(re^{i\theta})\right)\right)\right\} > \alpha \quad (z = re^{i\theta} \in \mathbb{D}, 0 \le r < 1, 0 \le \theta \le 2\pi).$$

We denote by $\mathcal{K}_{\mathcal{H}}(\alpha)$ the class of all harmonic convex mappings f of order α in $\mathbb{D}$.

In order to find the best approximation, we need some lemmas.

Lemma 1.1 (see Ref. [8]): *Let $f = h + \overline{g}$ be given by (1) and $\alpha \in [0, 1)$. If*

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| + \sum_{n=1}^{\infty}(n+\alpha)|b_n| \le 1-\alpha, \tag{2}$$

then $f = h + \overline{g}$ is harmonic univalent in $\mathbb{D}$ and $f \in \mathcal{S}^*_{\mathcal{H}}(\alpha)$. The starlike harmonic mapping

$$f_1(z) := z + \sum_{n=2}^{\infty}\frac{1-\alpha}{n-\alpha}x_n z^n + \sum_{n=1}^{\infty}\frac{1-\alpha}{n+\alpha}\overline{y}_n \overline{z}^n,$$

where $\sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| = 1$ shows that the inequality is sharp.

Lemma 1.2 (see Ref. [9]): *Let $f = h + \overline{g}$ be given by (1) and $\alpha \in [0, 1)$. If*

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| + \sum_{n=1}^{\infty} n(n+\alpha)|b_n| \leq 1-\alpha, \tag{3}$$

then $f = h + \overline{g}$ is harmonic univalent in $\mathbb{D}$ and $f \in \mathcal{K}_{\mathcal{H}}(\alpha)$. The result is sharp for

$$f_2(z) := z + \sum_{n=2}^{\infty} \frac{1-\alpha}{n(n-\alpha)} x_n z^n + \sum_{n=1}^{\infty} \frac{1-\alpha}{n(n+\alpha)} \overline{y}_n \overline{z}^n,$$

where $\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1$.

Sufficient coefficient conditions of the type described in Lemmas 1.1 and 1.2 are very useful to derive similar results. See, for example, Refs. [10, Lemma 1] and [11, Lemma 2.1] and thus, the ideas of this article could be utilized to cover several other situations.

We will denote by $\mathcal{S}^{**}_{\mathcal{H}}(\alpha)$ and $\mathcal{K}^{*}_{\mathcal{H}}(\alpha)$ the subclasses of $\mathcal{S}^{*}_{\mathcal{H}}(\alpha)$ and $\mathcal{K}_{\mathcal{H}}(\alpha)$ consisting of functions f which satisfy (2) and (3), respectively.

2. Preliminary results

The first preliminary result of this section is as follows.

Lemma 2.1: *Let $f : \mathbb{D} \to \mathbb{C}$ be analytic in $\mathbb{D}$ and have a series representation of the form $f = h + \overline{g}$, where*

$$h(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n. \tag{4}$$

Then,

$$\int_{\mathbb{D}} |f(x+iy)|^2 \mathrm{d}x\mathrm{d}y = \sum_{n=0}^{\infty} \frac{\pi}{n+1} \left(|a_n|^2 + |b_n|^2\right).$$

Proof: It is well known (cf. Ref. [12, p. 113]) that

$$\int_0^{2\pi} |f(re^{i\theta})|^2 r\mathrm{d}\theta = 2\pi \sum_{n=0}^{\infty} \left(|a_n|^2 + |b_n|^2\right) r^{2n+1}$$

and the desired result is a consequence of Fubini's theorem by integrating the last relation with respect to r $(0 \leq r \leq R)$ and then allow $R \to 1^-$. ∎

Following, we give a definition denoted by $\mathrm{dist}(f, \mathcal{S}^{*}_{\mathcal{H}}(\alpha))$. This definition shows that how 'far' is a function f from the class $\mathcal{S}^{*}_{\mathcal{H}}(\alpha)$. Also, the same definition is true for $\mathrm{dist}(f, \mathcal{K}_{\mathcal{H}}(\alpha))$, $\mathrm{dist}(f, \mathcal{S}^{**}_{\mathcal{H}}(\alpha))$ and $\mathrm{dist}(f, \mathcal{K}^{*}_{\mathcal{H}}(\alpha))$.

Definition 2.1: For $f \in \mathcal{H}$, we define

$$\mathrm{dist}(f, \mathcal{S}^{*}_{\mathcal{H}}(\alpha)) := \inf_{F \in \mathcal{S}^{*}_{\mathcal{H}}(\alpha)} \left(\iint_{\mathbb{D}} |f(x+iy) - F(x+iy)|^2 \mathrm{d}x\mathrm{d}y \right)^{1/2}.$$

We note that $\mathrm{dist}(f,\mathcal{S}^*_{\mathcal{H}}(\alpha))$ is a measure and Theorem 2.1 shows that $\mathrm{dist}(\cdot,\mathcal{S}^*_{\mathcal{H}}(\alpha))$ is not a norm in $\mathcal{H}$.

Theorem 2.1: *Let $f\in\mathcal{H}$ be of the form (1). Then,* $\mathrm{dist}(f,\mathcal{S}^*_{\mathcal{H}}(\alpha))=0$ *if and only if $f\in\mathcal{S}^*_{\mathcal{H}}(\alpha)$.*

Proof: Let us assume that $\mathrm{dist}(f,\mathcal{S}^*_{\mathcal{H}}(\alpha))=0$. Then we may find a sequence $\{f_n\}_{n=1}^{\infty}\in\mathcal{S}^*_{\mathcal{H}}(\alpha)$ such that

$$\iint_{\mathbb{D}}|f(x+iy)-f_n(x+iy)|^2\mathrm{d}x\mathrm{d}y<\frac{\pi}{n}\quad(n=1,2,\ldots).\tag{5}$$

First, we will show that f_n converges to f uniformly on compact subsets of $\mathbb{D}$. Because $f\in\mathcal{H}$ and $\mathcal{S}^*_{\mathcal{H}}(\alpha)\subset\mathcal{H}$, f and f_n have the following series representation:

$$f(z)=\sum_{m=1}^{\infty}a_mz^m+\overline{\sum_{m=1}^{\infty}b_mz^m}\quad\text{and}\quad f_n(z)=\sum_{m=1}^{\infty}a_{n,m}z^m+\overline{\sum_{m=1}^{\infty}b_{n,m}z^m}\quad(z\in\mathbb{D}),\tag{6}$$

with $a_1=1=a_{n,1}$. Now from (5), (6) and Lemma 2.1, we get

$$\sum_{m=1}^{\infty}\frac{1}{m+1}\left(|a_m-a_{n,m}|^2+|b_m-b_{n,m}|^2\right)<\frac{1}{n}\quad(n=1,2,\ldots).$$

Let r be a fixed arbitrary number such that $0<r<1$ and $z_0\in\overline{\mathbb{D}}_r=\{z:|z|\le r\}$. Then by (6), the last inequality and the elementary inequalities

$$(a+b)^p\le(2\max\{a,b\})^p\le 2^p\max\{a^p,b^p\}\le 2^p(a^p+b^p)\quad(a,b\in\mathbb{R},p>0),$$

we obtain

$$\begin{aligned}
&|f(z_0)-f_n(z_0)|\\
&=\left|\sum_{m=1}^{\infty}(a_m-a_{n,m})z_0^m+\sum_{m=1}^{\infty}(\overline{b}_m-\overline{b}_{n,m})\overline{z_0}^m\right|\\
&\le\sum_{m=1}^{\infty}\left(|a_m-a_{n,m}|+|\overline{b}_m-\overline{b}_{n,m}|\right)|z_0|^m\le\sum_{m=1}^{\infty}\left(|a_m-a_{n,m}|+|\overline{b}_m-\overline{b}_{n,m}|\right)r^m\\
&\le\left(\sum_{m=1}^{\infty}\frac{\left(|a_m-a_{n,m}|+|\overline{b}_m-\overline{b}_{n,m}|\right)^2}{m+1}\right)^{\frac{1}{2}}\left(\sum_{m=1}^{\infty}(m+1)r^{2m}\right)^{1/2}\\
&\le\left(\sum_{m=1}^{\infty}\frac{2^2\left(|a_m-a_{n,m}|^2+|\overline{b}_m-\overline{b}_{n,m}|^2\right)}{m+1}\right)^{1/2}\left(\sum_{m=1}^{\infty}(m+1)r^{2m}\right)^{1/2}
\end{aligned}$$

$$= \left(\sum_{m=1}^{\infty} \frac{2^2 \left(|a_m - a_{n,m}|^2 + |b_m - b_{n,m}|^2\right)}{m+1} \right)^{1/2} \left(\sum_{m=1}^{\infty} (m+1) r^{2m} \right)^{1/2}$$

$$< \frac{2}{\sqrt{n}} \left(\sum_{m=2}^{\infty} (m+1) r^{2m} \right)^{1/2} \quad \to 0$$

provided that $n \to \infty$, because the series $\sum_{m=2}^{\infty}(m+1)r^{2m}$ converges for any $r \in (0,1)$. Therefore, f_n converges to f uniformly on $\overline{\mathbb{D}}_r$ for any $r \in (0,1)$, so f_n converges to f uniformly on compact subsets of $\mathbb{D}$. Since $f_n \in \mathcal{S}^*_{\mathcal{H}}(\alpha) \subset \mathcal{S}_{\mathcal{H}}$, f_n are univalent in $\mathbb{D}$. Consequently, the limit function f is either univalent or it is identically constant in $\mathbb{D}$. But the last possibility is not accepted, because the function f is normalized by $f(0) = 0 = f_z(0) - 1$. On the other hand, f_n for each n satisfies the following inequality:

$$\frac{\partial}{\partial \theta} \left(\arg f_n(re^{i\theta}) \right) > \alpha \quad (z = re^{i\theta} \in \mathbb{D}, 0 \le r < 1, 0 \le \theta \le 2\pi). \tag{7}$$

Since each f_n is a harmonic starlike function of order α, by Hurwitz's theorem, f is also harmonic starlike functions of order α. Also since f_n converges uniformly to f on compact subsets of $\mathbb{D}$, so $(f_n)_z$ converges to f_z. Now, if we allow n to ∞ in the above inequality (7), then we conclude that f satisfies

$$\frac{\partial}{\partial \theta} \left(\arg f(re^{i\theta}) \right) \ge \alpha \quad (z = re^{i\theta} \in \mathbb{D}, 0 \le r < 1, 0 \le \theta \le 2\pi). \tag{8}$$

Since $f \in \mathcal{H}$ is non-constant, the recent inequality is in fact a strict inequality and that means $f \in \mathcal{S}^*_{\mathcal{H}}(\alpha)$. The converse implication is obvious and thus the details are omitted. This is the end of the proof. ■

3. The best approximation

We begin this section by describing the KKT conditions. It is well known that (see Ref. [13, p. 244]) in mathematical optimization, the KKT conditions are the first-order necessary conditions for a solution in nonlinear programming to be optimal, provided that some regularity conditions are satisfied. Allowing inequality constraints, the KKT approach to nonlinear programming generalizes the method of Lagrange multipliers, which allows only equality constraints. The system of equations and inequalities corresponding to the KKT conditions is usually not solved directly, except in the few special cases where a closed-form solution can be derived analytically. In general, many optimization algorithms can be interpreted as methods for numerically solving the KKT system of equations and inequalities.

Let $x \in \mathbb{R}^n$ be a column vector, $Q \in \mathcal{M}_{n\times n}$ a symmetric matrix, $A \in \mathcal{M}_{m\times n}$, $b \in \mathcal{M}_{m\times 1}$ and $c \in \mathcal{M}_{1\times n}$. Furthermore, assume that a feasible solution exists and that the constraint region is bounded. Now, we consider the problem of minimizing

$$f(x) = x^T Qx + cx$$

under the conditions

$$Ax \le b \quad \text{and} \quad x \ge 0.$$

We remark that the above is a particular case of quadratic programming and it is known that when the objective function $f(x)$ is strictly convex for all feasible points, the problem has a unique local minimum which is also the global minimum (a sufficient condition which guarantees the strict convexity of the objective function f is that Q is a positive definite matrix).

The KKT conditions below are necessary conditions for a global minimum. If Q is positive definite, they are also sufficient for a global minimum. Let us denote the Lagrangian function L for the above quadratic programming problem as follows:

$$L := x^T Qx + cx + \mu(Ax - b).$$

Here, we recall that the KKT conditions are as follows:

$$\begin{aligned} \frac{\partial L}{\partial x_i} &\geq 0 \quad (i = 1, 2, \ldots, n); \\ \frac{\partial L}{\partial \mu_j} &\leq 0 \quad (j = 1, 2, \ldots, m); \\ x_i \frac{\partial L}{\partial x_i} &= 0 \quad (i = 1, 2, \ldots, n); \\ \mu(Ax - b) &= 0; \\ x_i &\geq 0 \quad (i = 1, 2, \ldots, n); \\ \mu_j &\geq 0 \quad (j = 1, 2, \ldots, m). \end{aligned}$$

From now on, for convenience, in the second series (2)–(3) above, we replace 'n' by 'm' for $n \geq 1$. In order to find the best approximation of an analytic function $f \in \mathcal{H}$ in the subclass $\mathcal{S}^{**}_{\mathcal{H}}(\alpha) \subset \mathcal{S}^{*}_{\mathcal{H}}(\alpha)$, by Lemma 2.1 and Definition 2.1, we consider the problem of finding

$$\inf \left\{ \sum_{n=2}^{\infty} \frac{(x_n - a_n)^2}{n+1} + \sum_{m=1}^{\infty} \frac{(y_m - b_m)^2}{m+1} \right\}, \tag{9}$$

where $(a_n)_{n\geq 2}$ and $(b_m)_{m\geq 1}$ are given sequences of non-negative real numbers and the infimum is taken over all non-negative sequences $(x_n)_{n\geq 2}$ and $(y_m)_{m\geq 1}$ of real numbers satisfying

$$\sum_{n=2}^{\infty} (n - \alpha)x_n + \sum_{m=1}^{\infty} (m + \alpha)y_m \leq 1 - \alpha. \tag{10}$$

We note that the above infimum (9) is 0 when $x_n = a_n$ $(n \geq 2)$ and $y_m = b_m$ $(m \geq 1)$. It is clear that the solution of the above problem is trivial if

$$\sum_{n=2}^{\infty} (n - \alpha)a_n + \sum_{m=1}^{\infty} (m + \alpha)b_m \leq 1 - \alpha.$$

We will therefore consider the following additional hypothesis on the sequences $(a_n)_{n\geq 2}$ and $(b_m)_{m\geq 1}$:

$$\sum_{n=2}^{\infty} (n - \alpha)a_n + \sum_{m=1}^{\infty} (m + \alpha)b_m > 1 - \alpha. \tag{11}$$

Define the function

$$f(x,y) := \sum_{n=2}^{\infty} \frac{(x_n - a_n)^2}{n+1} + \sum_{m=1}^{\infty} \frac{(y_m - b_m)^2}{m+1}, \tag{12}$$

where x denotes the non-negative sequence $(x_n)_{n\geq 2}$ and y denotes the non-negative sequence $(y_m)_{m\geq 1}$. Applying Lemma 2.1 with considering the function $f(x,y)$, the above problem is a particular case of a semi-infinite quadratic problem (see Ref. [14]) with the corresponding Lagrangian given by

$$\begin{aligned} L := &\sum_{n=2}^{\infty} \frac{(x_n - a_n)^2}{n+1} + \sum_{m=1}^{\infty} \frac{(y_m - b_m)^2}{m+1} \\ &+ \mu \left(\sum_{n=2}^{\infty} (n-\alpha)x_n + \sum_{m=1}^{\infty} (m+\alpha)y_m - (1-\alpha) \right). \end{aligned}$$

The solution of the quadratic problems (9) and (10) is given by the KKT conditions (we let that the same conditions can be applied for an infinite instead of a finite number of variables, as detailed in Remark 3.2), which in this case become

$$\frac{\partial L}{\partial x_n} = 2\frac{x_n - a_n}{n+1} + \mu(n-\alpha) \geq 0 \quad (n \geq 2), \tag{13}$$

$$\frac{\partial L}{\partial y_m} = 2\frac{y_m - b_m}{m+1} + \mu(m+\alpha) \geq 0 \quad (m \geq 1), \tag{14}$$

$$\frac{\partial L}{\partial \mu} = \sum_{n=2}^{\infty} (n-\alpha)x_n + \sum_{m=1}^{\infty} (m+\alpha)y_n - 1 + \alpha \leq 0, \tag{15}$$

$$x_n \frac{\partial L}{\partial x_n} = x_n \left(2\frac{x_n - a_n}{n+1} + \mu(n-\alpha) \right) = 0 \quad (n \geq 2), \tag{16}$$

$$y_m \frac{\partial L}{\partial y_m} = y_m \left(2\frac{y_m - b_m}{m+1} + \mu(m+\alpha) \right) = 0 \quad (m \geq 1), \tag{17}$$

$$\mu \frac{\partial L}{\partial \mu} = \mu \left(\sum_{n=2}^{\infty} (n-\alpha)x_n + \sum_{m=1}^{\infty} (m+\alpha)y_m - (1-\alpha) \right) = 0, \tag{18}$$

$$x_n, y_m \geq 0 \quad (n \geq 2, m \geq 1), \tag{19}$$

$$\mu \geq 0. \tag{20}$$

From (18), we have that either $\mu = 0$ or

$$\sum_{n=2}^{\infty} (n-\alpha)x_n + \sum_{m=1}^{\infty} (m+\alpha)y_m = 1-\alpha.$$

We claim that $\mu \neq 0$. Otherwise, from (16) and (17), we get '$x_n = 0$ or $x_n = a_n$' and '$y_m = 0$ or $y_m = b_m$', respectively. Also (13) and (14) imply that if $\mu = 0$, then $x_n \geq a_n$ and $y_m \geq$

b_m, respectively, and we therefore conclude that $x_n = a_n$ $(n \geq 2)$ and $y_m = b_m$ $(m \geq 1)$. However by (11), we have

$$\sum_{n=2}^{\infty}(n-\alpha)x_n + \sum_{m=1}^{\infty}(m+\alpha)y_m = \sum_{n=2}^{\infty}(n-\alpha)a_n + \sum_{m=1}^{\infty}(m+\alpha)b_m > 1-\alpha,$$

which contradicts (15). Therefore, we have $\mu > 0$ and the above system (13)–(20) becomes in this case

$$2\frac{x_n - a_n}{n+1} + \mu(n-\alpha) \geq 0 \quad (n \geq 2), \tag{21}$$

$$2\frac{y_m - b_m}{m+1} + \mu(m+\alpha) \geq 0 \quad (m \geq 1), \tag{22}$$

$$\sum_{n=2}^{\infty}(n-\alpha)x_n + \sum_{m=1}^{\infty}(m+\alpha)y_m = 1-\alpha, \tag{23}$$

$$x_n\left(2\frac{x_n - a_n}{n+1} + \mu(n-\alpha)\right) = 0 \quad (n \geq 2), \tag{24}$$

$$y_m\left(2\frac{y_m - b_m}{m+1} + \mu(m+\alpha)\right) = 0 \quad (m \geq 1), \tag{25}$$

$$x_n, y_m \geq 0 \quad (n \geq 2, m \geq 1), \tag{26}$$

$$\mu > 0. \tag{27}$$

Equation (24) implies that either $x_n = 0$ or

$$x_n = a_n - \frac{1}{2}\mu(n+1)(n-\alpha) \quad (n \geq 2). \tag{28}$$

We will denote by I the set of indices $n \geq 2$ for which the equality (28) holds (therefore, $x_n = 0$ for $n \in I^c = \{2, 3, \ldots\} - I$). Also from (25), we get either $y_m = 0$ or

$$y_m = b_m - \frac{1}{2}\mu(m+1)(m+\alpha) \quad (m \geq 1). \tag{29}$$

We also will denote by J the set of indices $m \geq 1$ for which the equality (29) holds (therefore, $y_m = 0$ for $m \in J^c = \{1, 2, \ldots\} - J$). Let α_n and β_m be defined by

$$\alpha_n := \frac{2a_n}{(n+1)(n-\alpha)} \quad (n \geq 2, 0 \leq \alpha < 1) \tag{30}$$

and

$$\beta_m := \frac{2b_m}{(m+1)(m+\alpha)} \quad (m \geq 1, 0 \leq \alpha < 1), \tag{31}$$

respectively. From (21), it follows that

$$\mu \geq \alpha_n \quad (n \in I^c)$$

and from (26)

$$\mu \leq \alpha_n \quad (n \in I). \tag{32}$$

Also, (22) implies that

$$\mu \geq \beta_m \quad (m \in I^c)$$

and from (26), we get

$$\mu \leq \beta_m \quad (m \in I). \tag{33}$$

Therefore, we consider

$$\mu \geq \max\{\alpha_n, \beta_m\} \quad (n \in I^c, m \in J^c) \tag{34}$$

and

$$\mu \leq \max\{\alpha_n, \beta_m\} \quad (n \in I, m \in J). \tag{35}$$

We notice that if we impose the following additional hypothesis:

$$\lim_{n\to\infty} \frac{a_n}{n^2} = 0 = \lim_{m\to\infty} \frac{b_m}{m^2},$$

the inequality (32), (33) and (35) cannot hold for infinitely many indices n and m (i.e. I and J must be finite). Replacing (28) and (29) into (23), we obtain

$$\begin{aligned} 1-\alpha &= \sum_{n=2}^{\infty}(n-\alpha)x_n + \sum_{m=1}^{\infty}(m+\alpha)y_m \\ &= \sum_{n\in I}(n-\alpha)a_n + \sum_{m\in J}(m+\alpha)b_m \\ &\quad - \frac{1}{2}\mu\left[\sum_{n\in I}(n+1)(n-\alpha)^2 + \sum_{m\in J}(m+1)(m+\alpha)^2\right] \end{aligned}$$

and thus we must have

$$\mu = \frac{2\left[\sum_{n\in I}(n-\alpha)a_n + \sum_{m\in J}(m+\alpha)b_m - (1-\alpha)\right]}{\sum_{n\in I}(n+1)(n-\alpha)^2 + \sum_{m\in J}(m+1)(m+\alpha)^2} > 0. \tag{36}$$

The last relation (36) shows that the set of indices I and J must be finite, otherwise, since $\sum_{n\in I}(n+1)(n-\alpha)^2 = \infty$ and $\sum_{m\in J}(m+1)(m+\alpha)^2 = \infty$, we get $\mu = 0$, contradicting (27).

Remark 3.1: Let $(\gamma_n)_{n\geq 1}$ be a sequence of positive numbers with $\lim_{n\to\infty}\gamma_n = 0$. Then each of the interval $[1, +\infty)$ and $[1/(1+s), 1/s), s \geq 1$, contain only a finite number of the terms of the sequence. This led us to find a permutation $(i_n)_{n\geq 1}$ of the indices $\{1, 2, 3, \ldots\}$ such that $(\gamma_{i_n})_{n\geq 1}$ is a non-increasing sequence and $\lim_{n\to\infty}\gamma_{i_n} = 0$.

With this preparation, we are now ready to prove the following.

Theorem 3.1: *Let $(a_n)_{n\geq 2}$ and $(b_m)_{m\geq 1}$ be two sequences of positive real numbers with*

$$\sum_{n=2}^{\infty}(n-\alpha)a_n+\sum_{m=1}^{\infty}(m+\alpha)b_m>1-\alpha \tag{37}$$

and

$$\lim_{n\to\infty}\frac{a_n}{n^2}=0=\lim_{m\to\infty}\frac{b_m}{m^2}. \tag{38}$$

Then there exist integers $N\geq 2$ and $M\geq 1$ such that the minimum of the quadratic problems (9) and (10) is given by

$$\sum_{n\in I^c}\frac{a_n^2}{n+1}+\sum_{m\in J^c}\frac{b_m^2}{m+1}+\frac{\left[\sum_{n\in I}(n-\alpha)a_n+\sum_{m\in J}(m+\alpha)b_m-(1-\alpha)\right]^2}{\sum_{n\in I}(n+1)(n-\alpha)^2+\sum_{m\in J}(m+1)(m+\alpha)^2}$$

and is attained for the sequences $(x_n)_{n\geq 2}$ and $(y_m)_{m\geq 1}$, respectively, given by

$$x_n=\begin{cases} a_n-\frac{1}{2}\mu_{N,M}(n+1)(n-\alpha), & n\in I;\\ 0, & n\in I^c\end{cases}$$

and

$$y_m=\begin{cases} b_m-\frac{1}{2}\mu_{N,M}(m+1)(m+\alpha), & m\in J;\\ 0, & m\in J^c,\end{cases}$$

where

$$\mu_{N,M}:=\frac{2\left[\sum_{n\in I}(n-\alpha)a_n+\sum_{m\in J}(m+\alpha)b_m-(1-\alpha)\right]}{\sum_{n\in I}(n+1)\,(n-\alpha)^2+\sum_{m\in J}(m+1)\,(m+\alpha)^2}, \tag{39}$$

$n=2,3,\ldots,|\mathcal{P}|+1$, $m=1,2,\ldots,|\mathcal{Q}|$, $I=\{i_1,i_2,\ldots,i_N\}$ and $J=\{j_1,j_2,\ldots,j_M\}$. Here, $(i_n)_{n=2,\ldots,|\mathcal{P}|+1}$ and $(j_m)_{m=1,\ldots,|\mathcal{Q}|}$ are permutations of the indices in $\mathcal{P}=\{n\geq 2: a_n>0\}$ and respectively in $\mathcal{Q}=\{m\geq 1: b_m>0\}$ such that $(\alpha_n)_{n\geq 2}$ and $(\beta_m)_{m\geq 1}$ defined by (30) and (31) are non-increasing sequences, respectively.

Moreover, N and M can be taken to be equal to

$$N=\min\{n\geq 2:\max\{\alpha_{n+1},\beta_{m+1}\}\leq\mu_{n,m}\leq\max\{\alpha_n,\beta_m\}\ \forall m\geq 1\}$$

and

$$M=\min\{m\geq 1:\max\{\alpha_{n+1},\beta_{m+1}\}\leq\mu_{n,m}\leq\max\{\alpha_n,\beta_m\}\ \forall n\geq 2\},$$

where

$$\mu_{n,m}=\frac{2\left[\sum_{k=2}^{n}(i_k-\alpha)a_{i_k}+\sum_{k=1}^{m}(j_k+\alpha)b_{j_k}-(1-\alpha)\right]}{\sum_{k=2}^{n}(i_k+1)\,(i_k-\alpha)^2+\sum_{k=1}^{m}(j_k+1)(j_k+\alpha)^2}.$$

Proof: According to what we mentioned above, in order to find the solution of the problems (9) and (10), it remains to find the sets of indices I and J. To do this, we recall that μ given by (36) must satisfy (34) and (35) (then we can find the value of x_n and y_m). The choice of the sets I and J depends on whether all the terms of the sequences $(a_n)_{n\geq 2}$ and

$(b_m)_{m\geq 1}$ are positive or not, so we introduce the set of indices $\mathcal{P} = \{n \geq 2 : a_n > 0\}$ and $\mathcal{Q} = \{m \geq 1 : b_m > 0\}$ and distinguish the following cases.

Case 1 $\mathcal{P} = \{2, 3, \ldots\}$ and $\mathcal{Q} = \{1, 2, \ldots\}$.

Let α_n and β_m be defined by (30) and (31), respectively. The hypothesis (38) shows that α_n and β_m are sequences of positive numbers converging to 0. From Remark 3.1, we can choose permutations $(i_n)_{n\geq 2}$ and $(j_m)_{m\geq 1}$ of the indices in $\mathcal{P}$ and $\mathcal{Q}$, respectively, such that

$$\alpha_{i_n} = \frac{2a_{i_n}}{(i_n+1)(i_n-\alpha)} \quad (n \geq 2, 0 \leq \alpha < 1)$$

is non-increasing sequence, $\alpha_{i_n} \to 0$ as $n \to \infty$ and

$$\beta_{j_m} = \frac{2b_{j_m}}{(j_m+1)(j_m+\alpha)} \quad (m \geq 1, 0 \leq \alpha < 1)$$

is non-increasing sequence, $\beta_{j_m} \to 0$ as $m \to \infty$. Inequality (37) implies that

$$\sum_{n=2}^{\infty}(i_n-\alpha)a_{i_n} + \sum_{m=1}^{\infty}(j_m+\alpha)b_{j_m} = \sum_{n=2}^{\infty}(n-\alpha)a_n + \sum_{m=1}^{\infty}(m+\alpha)b_m > 1-\alpha.$$

Therefore, there exist integers $n_0 \geq 2$ and $m_0 \geq 1$ so that

$$\sum_{n=2}^{n_0}(i_n-\alpha)a_{i_n} + \sum_{m=1}^{m_0}(j_m+\alpha)b_{j_m} > 1-\alpha$$

and assume that n_0 and m_0 are the smallest indexes with this property. We show that $0 < \mu_{n_0,m_0} \leq \max\{\alpha_{n_0}, \beta_{m_0}\}$. Since

$$\sum_{n=2}^{n_0-1}(i_n-\alpha)a_{i_n} + \sum_{m=1}^{m_0-1}(j_m+\alpha)b_{j_m} \leq 1-\alpha,$$

we obtain

$$\begin{aligned}
\mu_{n_0,m_0} &= \frac{2\left[\sum_{k=2}^{n_0}(i_k-\alpha)a_{i_k} + \sum_{k=1}^{m_0}(j_k+\alpha)b_{j_k} - (1-\alpha)\right]}{\sum_{k=2}^{n_0}(i_k+1)(i_k-\alpha)^2 + \sum_{k=1}^{m_0}(j_k+1)(j_k+\alpha)^2} \\
&= \frac{2\left[\begin{array}{c}(i_{n_0}-\alpha)a_{i_{n_0}} + (j_{m_0}+\alpha)b_{j_{m_0}} + \sum_{k=2}^{n_0-1}(i_k-\alpha)a_{i_k} \\ + \sum_{k=1}^{m_0-1}(j_k+\alpha)b_{j_k} - (1-\alpha)\end{array}\right]}{\begin{array}{c}\left(i_{n_0}+1\right)(i_{n_0}-\alpha)^2 + (j_{m_0}+1)(j_{m_0}+\alpha)^2 \\ + \sum_{k=2}^{n_0-1}(i_k+1)(i_k-\alpha)^2 + \sum_{k=1}^{m_0-1}(j_k+1)(j_k+\alpha)^2\end{array}} \\
&\leq \frac{2\left[(i_{n_0}-\alpha)a_{i_{n_0}} + (j_{m_0}+\alpha)b_{j_{m_0}}\right]}{\begin{array}{c}\left(i_{n_0}+1\right)(i_{n_0}-\alpha)^2 + (j_{m_0}+1)(j_{m_0}+\alpha)^2 \\ + \sum_{k=2}^{n_0-1}(i_k+1)(i_k-\alpha)^2 + \sum_{k=1}^{m_0-1}(j_k+1)(j_k+\alpha)^2\end{array}} \\
&\leq \frac{2\left[(i_{n_0}-\alpha)a_{i_{n_0}} + (j_{m_0}+\alpha)b_{j_{m_0}}\right]}{\left(i_{n_0}+1\right)(i_{n_0}-\alpha)^2 + (j_{m_0}+1)(j_{m_0}+\alpha)^2}
\end{aligned}$$

$$\leq \max\left\{\frac{2a_{i_{n_0}}}{\left(i_{n_0}+1\right)\left(i_{n_0}-\alpha\right)}, \frac{2b_{j_{m_0}}}{\left(j_{m_0}+1\right)\left(j_{m_0}+\alpha\right)}\right\}$$
$$= \max\left\{\alpha_{i_{n_0}}, \beta_{j_{m_0}}\right\},$$

where we have used this fact that if $a, b, c, d > 0$, then $(a+b)/(c+d) \leq \max\{a/c, b/d\}$. In the case of $n_0 = 2$ and $m_0 = 1$, we also get

$$\mu_{2,1} = \frac{2\left[(i_2-\alpha)a_{i_2} + (j_1+\alpha)b_{j_1} - (1-\alpha)\right]}{(i_2+1)(i_2-\alpha)^2 + (j_1+1)(j_1+\alpha)^2}$$
$$\leq \frac{2\left[(i_2-\alpha)a_{i_2} + (j_1+\alpha)b_{j_1}\right]}{(i_2+1)(i_2-\alpha)^2 + (j_1+1)(j_1+\alpha)^2}$$
$$\leq \max\left\{\frac{2a_{i_2}}{(i_2+1)(i_2-\alpha)}, \frac{2b_{j_1}}{(j_1+1)(j_1+\alpha)}\right\}$$
$$= \max\{\alpha_{i_2}, \beta_{j_1}\}.$$

Therefore, we conclude that $0 < \mu_{n_0,m_0} \leq \max\{\alpha_{n_0}, \beta_{m_0}\}$. Using the non-increasing property of $(\alpha_{i_n})_{n=2}^{\infty}$ and $(\beta_{j_m})_{m=1}^{\infty}$, we have

$$\mu_{n_0,m_0} \leq \max\{\alpha_{i_{n_0}}, \beta_{j_{m_0}}\} \leq \max\{\alpha_{i_{n_0-1}}, \beta_{j_{m_0-1}}\} \leq \cdots \leq \max\{\alpha_{i_1}, \beta_{j_1}\}.$$

We shall find the smallest integers N and M such that

$$\max\{\alpha_{i_{N+1}}, \beta_{j_{M+1}}\} \leq \mu_{N,M} \leq \max\{\alpha_{i_N}, \beta_{j_M}\} \leq \max\{\alpha_{i_{N-1}}, \beta_{j_{M-1}}\} \leq \cdots \leq \max\{\alpha_{i_1}, \beta_{j_1}\}.$$

We consider two subcases as follows:

Case 1a: $\mu_{n_0,m_0} \geq \max\{\alpha_{n_0+1}, \beta_{m_0+1}\}$.

Since the sequences $(\alpha_n)_{n=2}^{\infty}$ and $(\beta_m)_{m=1}^{\infty}$ are non-increasing, we have

$$\mu_{n_0,m_0} \leq \max\{\alpha_{n_0}, \beta_{m_0}\} \leq \max\{\alpha_n, \beta_m\} \quad (n \in \{2, 3, \ldots, n_0\}, m \in \{1, 2, \ldots, m_0\})$$

and

$$\mu_{n_0,m_0} \geq \max\{\alpha_{n_0+1}, \beta_{m_0+1}\} \geq \max\{\alpha_n, \beta_n\} \quad (n \in \{n_0+1, n_0+2, \ldots\},$$
$$m \in \{m_0+1, m_0+2, \ldots\}).$$

In this case, we can choose $N = n_0$, $M = m_0$, $I = \{i_1, i_2, \ldots, i_{n_0}\}$ and $J = \{j_1, j_2, \ldots, j_{m_0}\}$, so $\mu = \mu_{n_0,m_0}$ satisfies (34) and (35) concluding the proof.

Case 1b: $\mu_{n_0,m_0} < \max\{\alpha_{n_0+1}, \beta_{m_0+1}\}$.

In this case using again the above observation, we have

$$\mu_{n_0,m_0} \leq \mu_{n_0+1,m_0+1} \leq \max\{\alpha_{i_{n_0+1}}, \beta_{m_0+1}\}.$$

Here we are faced with two modes as follows:

(i) $\mu_{n_0+1,m_0+1} \geq \max\{\alpha_{i_{n_0+2}}, \beta_{j_{m_0+2}}\}$. In this case, by the same argument as *Case 1a* above, we can choose $N = n_0 + 1$, $M = m_0 + 1$, $I = \{i_1, i_2, \ldots, i_{n_0+1}\}$ and $J = \{j_1, j_2, \ldots, j_{m_0+1}\}$, so $\mu = \mu_{n_0+1,m_0+1}$ satisfies (34) and (35), giving the solution.

(ii) $\mu_{n_0+1,m_0+1} < \max\{\alpha_{i_{n_0+2}}, \beta_{j_{m_0+2}}\}$. In this case, proceeding as above, we have

$$\mu_{n_0,m_0} \le \mu_{n_0+1,m_0+1} \le \mu_{n_0+2,m_0+2} \le \max\{\alpha_{i_{n_0+2}}, \beta_{j_{m_0+2}}\}$$

and proceeding inductively, either

$$\begin{aligned} 0 < \mu_{n_0,m_0} &\le \mu_{n_0+1,m_0+1} \le \mu_{n_0+2,m_0+2} \le \cdots \le \mu_{n_0+k,m_0+k} \\ &\le \max\{\alpha_{i_{n_0+k}}, \beta_{j_{m_0+k}}\} \quad (k \ge 0) \end{aligned} \tag{40}$$

or we can find an integer $k \ge 1$ for which

$$\max\{\alpha_{i_{n_0+k+1}}, \beta_{j_{m_0+k+1}}\} \le \mu_{n_0+k,m_0+k} \le \max\{\alpha_{i_{n_0+k}}, \beta_{j_{m_0+k}}\}. \tag{41}$$

Since $\alpha_n \to 0$ as $n \to \infty$ and $\beta_m \to 0$ as $m \to \infty$, the inequalities in (40) cannot hold for every $k \ge 0$, and therefore the first possibility above is ruled out. Thus we conclude that one can always find an integer k for which (41) holds true. In this case, proceeding as in *Case 1a* above, we can choose $N = n_0 + k$, $M = m_0 + k$, $I = \{i_1, i_2, \dots, i_{n_0+k}\}$ and $J = \{j_1, j_2, \dots, j_{m_0+k}\}$ from which it follows that $\mu = \mu_{n_0+k,m_0+k}$ satisfies (34) and (35) concluding the proof of the theorem in this case.

Case 2: $\mathcal{P} = \{n \ge 2 : a_n > 0\} \subsetneq \{2, 3, \dots\}$ and $\mathcal{Q} = \{m \ge 1 : b_m > 0\} \subsetneq \{1, 2, \dots\}$.

We consider the following subcases.

Case 2a: $\mathcal{P}$ and $\mathcal{Q}$ are infinite.

If $n \in \{2, 3, \dots\} - \mathcal{P}$, then $a_n = 0$ and if $m \in \{1, 2, \dots\} - \mathcal{Q}$, then $b_m = 0$. Therefore,

$$\sum_{n\in\mathcal{P}} (n-\alpha)a_n + \sum_{m\in\mathcal{Q}} (n+\alpha)b_m = \sum_{n=2}^{\infty} (n-\alpha)a_n + \sum_{m=1}^{\infty} (m+\alpha)b_m > 1-\alpha.$$

This led us to apply the argument in *Case 1* above to the sequences $(a_n)_{n\in\mathcal{P}}$ and $(b_m)_{m\in\mathcal{Q}}$ of positive numbers and to obtain a solution of the problem

$$\inf\left\{\sum_{n\in\mathcal{P}} \frac{(x_n-a_n)^2}{n+1} + \sum_{m\in\mathcal{Q}} \frac{(y_m-b_m)^2}{m+1}\right\},$$

where the infimum is taken over all non-negative sequences $(x_n)_{n\in\mathcal{P}}$ and $(y_m)_{m\in\mathcal{Q}}$ with

$$\sum_{n\in\mathcal{P}} (n-\alpha)a_n + \sum_{m\in\mathcal{Q}} (m+\alpha)b_m \le 1-\alpha.$$

It is easy to see that the solution of the above minimization problem is also a solution of the original minimization problems (9) and (10) (for $n \in \{2, 3, \dots\} - \mathcal{P} \subset I^c$ and $m \in \{1, 2, \dots\} - \mathcal{Q} \subset J^c$, we have $x_n = a_n = 0$ and $y_m = b_m = 0$, respectively).

Case 2b: $\mathcal{P}$ and $\mathcal{Q}$ are finite.

From (11), it follows that the sets $\mathcal{P}$ and $\mathcal{Q}$ cannot be empty, so we let $|\mathcal{P}| = p$ $(p \ge 2)$ and $|\mathcal{Q}| = q$ $(q \ge 1)$. Let $(\alpha_n)_{n\ge2}$ and $(\beta_m)_{m\ge1}$ be defined by (30) and (31), respectively. Also let $(i_n)_{n=2,\dots,p+1}$ and $(j_m)_{m=1,2,\dots,q}$ be two permutations of the indices in $\mathcal{P}$ and $\mathcal{Q}$, respectively. Proceeding as in Case 1 above, either we can find an integer $k \geq 0$ such that the index sets $I = \{i_2, i_3, \ldots, i_{n_0+k}\}$ and $J = \{j_1, j_2, \ldots, j_{m_0+k}\}$ give the solution or else

$$0 < \mu_{n_0,m_0} \leq \mu_{n_0+1,m_0+1} \leq \cdots \leq \mu_{p+1,q+1} \leq \max\{\alpha_{p+1}, \beta_{q+1}\}.$$

In this case, we can also choose $I = \{i_2, i_3, \ldots, i_{p+1}\}$ and $J = \{j_1, j_2, \ldots, j_q\}$ such that $\mu = \mu_{p+1,q+1}$ satisfies (34) and (35), giving the solution of the minimization problems (9) and (10) in this last case. Here the proof ends. ■

Remark 3.2: To complete the proof of Theorem 3.1, we have left to justify that we can use the KKT conditions for the quadratic programming problems (9) and (10), with an infinite (instead of a finite) number of variables. The reasoning being similar to Ref. [3, Remark 3.1], we will just briefly outline it. The idea is to observe that for any integer $\lambda \geq 2$ and $\nu \geq 1$, we have

$$\inf\left\{\sum_{n=2}^{\infty}\frac{(x_n - a_n)^2}{n+1} + \sum_{m=1}^{\infty}\frac{(y_m - b_m)^2}{m+1}\right\} \geq \inf\left\{\sum_{n=2}^{\lambda}\frac{(x_n - a_n)^2}{n+1} + \sum_{m=1}^{\nu}\frac{(y_m - b_m)^2}{m+1}\right\}, \tag{42}$$

where both infima are taken over all non-negative sequences $(x_n)_{n\geq 2}$ and $(y_m)_{m\geq 1}$ of real numbers with

$$\sum_{n=2}^{\infty}(n-\alpha)a_n + \sum_{m=1}^{\infty}(m+\alpha)b_m \leq 1-\alpha.$$

Since $x_{\lambda+1}, x_{\lambda+2}, \ldots$ and $y_{\nu+1}, y_{\nu+2}, \ldots$ do not appear in the objective function in the second infimum above, the second infimum is the same when taken over all finite truncated sequences $(x_n)_{n=2,\ldots,\lambda}$ and $(y_m)_{m=1,\ldots,\nu}$ with

$$\sum_{n=2}^{\lambda}(n-\alpha)a_n + \sum_{m=1}^{\nu}(m+\alpha)b_m \leq 1-\alpha.$$

Solving the KKT conditions for this finite-dimensional problem (the calculations are identical as in the proof above) and using the notation of Theorem 3.1, it follows that for $\lambda \geq \max\{i_n, \ldots, i_N\}$ and $\nu \geq \max\{j_n, \ldots, j_M\}$, the second infimum in (42) is attained for the sequence $(x_n)_{n=2,\ldots,\lambda}$ and $(y_m)_{m=1,\ldots,\nu}$ given by

$$x_n = \begin{cases} a_n - \dfrac{1}{2}\mu_{N,M}(n+1)(n-\alpha), & n \in I; \\ 0, & n \in I_\lambda^c := \{2, \ldots, \lambda\} - I \end{cases}$$

and

$$y_m = \begin{cases} b_m - \dfrac{1}{2}\mu_{N,M}(m+1)(m+\alpha), & m \in J; \\ 0, & m \in J_\nu^c := \{1, \ldots, \nu\} - J, \end{cases}$$

respectively, where $\mu_{N,M}$ is defined in (39). Combining with (42), we obtain

$$\inf\left\{\sum_{n=2}^{\infty}\frac{(x_n - a_n)^2}{n+1} + \sum_{m=1}^{\infty}\frac{(y_m - b_m)^2}{m+1}\right\}$$

$$\geq \sum_{n\in I_\lambda^c} \frac{a_n^2}{n+1} + \sum_{m\in J_\nu^c} \frac{b_m^2}{m+1} + \frac{1}{4}\mu_{N,M}^2 \left(\sum_{n\in I}(n+1)(n-\alpha)^2 + \sum_{m\in J}(m+1)(m+\alpha)^2 \right)$$

$$= \sum_{n\in I_\lambda^c} \frac{a_n^2}{n+1} + \sum_{m\in J_\nu^c} \frac{b_m^2}{m+1} + \frac{\left[\sum_{n\in I}(n-\alpha)a_n + \sum_{m\in J}(m+\alpha)b_m - (1-\alpha)\right]^2}{\sum_{n\in I}(n+1)(n-\alpha)^2 + \sum_{m\in J}(m+1)(m+\alpha)^2}.$$

We see that this argument can be applied for any arbitrary $\lambda \geq \max\{i_n, \ldots, i_N\}$ and $\nu \geq \max\{j_n, \ldots, j_M\}$, so passing to the limit with $\lambda, \nu \to \infty$, we obtain

$$\inf \left\{ \sum_{n=2}^{\infty} \frac{(x_n - a_n)^2}{n+1} + \sum_{m=1}^{\infty} \frac{(y_m - b_m)^2}{m+1} \right\}$$

$$\geq \lim_{\lambda\to\infty} \sum_{n\in I_\lambda^c} \frac{a_n^2}{n+1} + \lim_{\nu\to\infty} \sum_{m\in J_\nu^c} \frac{b_m^2}{m+1}$$

$$+ \frac{1}{4}\mu_{N,M}^2 \left(\sum_{n\in I}(n+1)(n-\alpha)^2 + \sum_{m\in J}(m+1)(m+\alpha)^2 \right)$$

$$= \sum_{n\in I^c} \frac{a_n^2}{n+1} + \sum_{m\in J^c} \frac{b_m^2}{m+1} + \frac{\left[\sum_{n\in I}(n-\alpha)a_n + \sum_{m\in J}(m+\alpha)b_m - (1-\alpha)\right]^2}{\sum_{n\in I}(n+1)(n-\alpha)^2 + \sum_{m\in J}(m+1)(m+\alpha)^2},$$

which is just the value of the objective function f defined by (12) for the sequences $(x_n)_{n\geq 2}$ and $(y_m)_{m\geq 1}$ defined in Theorem 3.1. It follows that the infimum of the quadratic problems (9) and (10) is attained for the sequences in the statement of Theorem 3.1, completing the argument used in the proof.

4. Applications

As an application of Theorem 3.1, we will determine the best approximation of a harmonic mapping f in the subclass $\mathcal{S}_{\mathcal{H}}^{**}(\alpha)$, that is, we will find

$$\operatorname{dist}(f, \mathcal{S}_{\mathcal{H}}^{**}(\alpha)) = \inf_{F\in\mathcal{S}_{\mathcal{H}}^{**}(\alpha)} \left(\iint_{\mathbb{D}} |f(x+iy) - F(x+iy)|^2 \mathrm{d}x\mathrm{d}y \right)^{1/2}$$

for a given function $f \in \mathcal{H}$, and we will determine the extremal function $F \in \mathcal{S}_{\mathcal{H}}^{**}(\alpha)$ for which the minimum is attained. The result is the following.

Theorem 4.1: *Let the function $f \in \mathcal{H}$ have the series expansion (1) and*

$$\lim_{n\to\infty} \frac{|a_n|}{n^2} = 0 = \lim_{n\to\infty} \frac{|b_m|}{m^2}.$$

(i) If

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| + \sum_{m=1}^{\infty}(m+\alpha)|b_m| \leq 1-\alpha,$$

then

$$\operatorname{dist}(f, \mathcal{S}^{**}_{\mathcal{H}}(\alpha)) = 0,$$

which is attained for $F = f \in \mathcal{S}^{**}_{\mathcal{H}}(\alpha) \subset \mathcal{S}^{*}_{\mathcal{H}}(\alpha)$.

(ii) If

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| + \sum_{m=1}^{\infty}(m+\alpha)|b_m| > 1-\alpha,$$

then

$$\begin{aligned}
&\operatorname{dist}(f, \mathcal{S}^{**}_{\mathcal{H}}(\alpha)) \\
&= \sqrt{\pi}\left(\sum_{n\in I^c}\frac{a_n^2}{n+1} + \sum_{m\in J^c}\frac{b_m^2}{m+1}\right. \\
&\left. + \frac{\left[\sum_{n\in I}(n-\alpha)|a_n| + \sum_{m\in J}(m+\alpha)|b_m| - (1-\alpha)\right]^2}{\sum_{n\in I}(n+1)(n-\alpha)^2 + \sum_{m\in J}(m+1)(m+\alpha)^2}\right)^{1/2},
\end{aligned}$$

where I and J are given by Theorem 3.1. Moreover, the infimum value of $\operatorname{dist}(f, \mathcal{S}^{**}_{\mathcal{H}}(\alpha))$ *above is attained for the function F given by*

$$F(z) = H(z) + \overline{G(z)}, \tag{43}$$

where

$$H(z) = z + \sum_{n=2}^{\infty} A_n z^n, \qquad G(z) = \sum_{m=1}^{\infty} B_m z^m,$$

$$A_n = \begin{cases} \left(|a_n| - \frac{1}{2}\mu_{N,M}(n+1)(n-\alpha)\right)e^{i\arg(a_n)}, & n \in I; \\ 0, & n \in I^c \end{cases}$$

and

$$B_m = \begin{cases} \left(|b_m| - \frac{1}{2}\mu_{N,M}(m+1)(m+\alpha)\right)e^{i\arg(b_m)}, & m \in J; \\ 0, & m \in J^c, \end{cases}$$

where $\mu_{N,M}$ *is given by (39) with* a_n *and* b_m *replaced by* $|a_n|$ *and* $|b_m|$*, respectively.*

Proof: (i) The claim is clear. ii) Let F be defined as (43). Lemma 2.1 and the triangle inequality imply that

$$\begin{aligned}
\operatorname{dist}(f, \mathcal{S}^{**}_{\mathcal{H}}(\alpha)) &= \inf_{F\in\mathcal{S}^{**}_{\mathcal{H}}(\alpha)}\left(\iint_{\mathbb{D}} |f(x+iy) - F(x+iy)|^2 \mathrm{d}x\mathrm{d}y\right)^{1/2} \\
&= \sqrt{\pi}\left(\inf\left\{\sum_{n=2}^{\infty}\frac{|a_n - A_n|^2}{n+1} + \sum_{m=1}^{\infty}\frac{|b_m - B_m|^2}{m+1}\right\}\right)^{1/2}
\end{aligned} \tag{44}$$

$$\geq \sqrt{\pi}\left(\inf\left\{\sum_{n=2}^{\infty}\frac{(|a_n|-|A_n|)^2}{n+1}+\sum_{m=1}^{\infty}\frac{(|b_m|-|B_m|)^2}{m+1}\right\}\right)^{1/2} \tag{45}$$

$$= \sqrt{\pi}\left(\inf\left\{\sum_{n=2}^{\infty}\frac{(|a_n|-x_n)^2}{n+1}+\sum_{m=1}^{\infty}\frac{(|b_m|-y_m)^2}{m+1}\right\}\right)^{1/2}, \tag{46}$$

where both infima in (44) and (45) are taken over all sequences $(A_n)_{n\geq 2}$ and $(B_m)_{m\geq 1}$ of complex numbers satisfying

$$\sum_{n=2}^{\infty}(n-\alpha)|A_n|+\sum_{m=1}^{\infty}(m+\alpha)|B_m|\leq 1-\alpha,$$

and the infimum in (46) is taken over all non-negative sequences $(x_n)_{n\geq 1}$ and $(y_m)_{m\geq 1}$ of real numbers satisfying

$$\sum_{n=2}^{\infty}(n-\alpha)x_n+\sum_{m=1}^{\infty}(m+\alpha)y_m\leq 1-\alpha.$$

Applying Theorem 3.1 with $|a_n|$ and $|b_m|$ instead of a_n and b_m, respectively, we obtain the infimum above (46) attained for the sequences x_n and y_m

$$x_n=\begin{cases} |a_n|-\frac{1}{2}\mu_{N,M}(n+1)(n-\alpha), & n\in I;\\ 0, & n\in I^c\end{cases}$$

and

$$y_m=\begin{cases} |b_m|-\frac{1}{2}\mu_{N,M}(m+1)(m+\alpha), & m\in J;\\ 0, & m\in J^c,\end{cases}$$

respectively, where $\mu_{N,M}$ is given by (39). Since the inequality $|z-w|\geq ||z|-|w||$ is sharp if $\arg(z)=\arg(w)$, it follows that

$$\operatorname{dist}(f,\mathcal{S}_{\mathcal{H}}^{**}(\alpha))=\sqrt{\pi}\left(\inf\left\{\sum_{n=2}^{\infty}\frac{|a_n-A_n|^2}{n+1}+\sum_{m=1}^{\infty}\frac{|b_m-B_m|^2}{m+1}\right\}\right)^{1/2}$$

is attained for the sequences $(A_n)_{n\geq 2}$ and $(B_m)_{m\geq 1}$ of complex numbers with

$$A_n=x_ne^{i\arg(a_n)}\quad (n\geq 2)\quad\text{and}\quad B_m=y_me^{i\arg(b_m)}\quad (m\geq 1).$$

Here we notice that if $a_n=0$ and $b_m=0$, then from the proof of Theorem 3.1, it follows that $n\in I^c$ and $m\in J^c$, respectively. Thus we have $x_n=0$ and $y_m=0$, consequently $A_n=0$ and $B_m=0$. This means that $F(z)\equiv 0$ that is unambiguously defined. Since $A_n=0$ and $B_m=0$ for $n\in I^c$ and $m\in J^c$, respectively, so $|A_n|=x_n\geq 0$ for $n\in I$ and $|B_m|=y_m\geq 0$ for $m\in J$. Therefore, we obtain

$$\sum_{n=2}^{\infty}(n-\alpha)|A_n|+\sum_{m=1}^{\infty}(m+\alpha)|B_m|$$

$$= \sum_{n=2}^{\infty}(n-\alpha)\left[|a_n| - \frac{1}{2}\mu_{N,M}(n+1)(n-\alpha)\right]$$

$$+ \sum_{m=1}^{\infty}(m+\alpha)\left[|b_m| - \frac{1}{2}\mu_{N,M}(m+1)(m+\alpha)\right]$$

$$= \sum_{n=2}^{\infty}(n-\alpha)|a_n| + \sum_{m=1}^{\infty}(m+\alpha)|b_m|$$

$$- \frac{1}{2}\mu_{N,M}\left[\sum_{n\in I}(n+1)(n-\alpha)^2 + \sum_{m\in J}(m+1)(m+\alpha)^2\right]$$

$$= 1-\alpha,$$

which implies that F given by (43) belongs to the class $\mathcal{S}^{**}_{\mathcal{H}}(\alpha)$ and

$$\left(\iint_{\mathbb{D}} |f(x+iy) - F(x+iy)|^2 \mathrm{d}x\mathrm{d}y\right)^{1/2}$$

$$= \sqrt{\pi}\left(\sum_{n=2}^{\infty}\frac{|a_n - A_n|^2}{n+1} + \sum_{m=1}^{\infty}\frac{|b_m - B_m|^2}{m+1}\right)^{1/2}$$

$$= \sqrt{\pi}\left(\sum_{n=2}^{\infty}\frac{(|a_n| - |A_n|)^2}{n+1} + \sum_{m=1}^{\infty}\frac{(|b_m| - |B_m|)^2}{m+1}\right)^{1/2}$$

$$= \sqrt{\pi}\left(\sum_{n\in I^c}\frac{a_n^2}{n+1} + \sum_{m\in J^c}\frac{b_m^2}{m+1}\right.$$

$$\left. + \frac{\left[\sum_{n\in I}(n-\alpha)|a_n| + \sum_{m\in J}(m+\alpha)|b_m| - (1-\alpha)\right]^2}{\sum_{n\in I}(n+1)(n-\alpha)^2 + \sum_{m\in J}(m+1)(m+\alpha)^2}\right)^{1/2}$$

$$= \mathrm{dist}(f, \mathcal{S}^{**}_{\mathcal{H}}(\alpha))$$

as desired. Here the proof ends. ■

Remark 4.1: If we take $\alpha = 0$ and $g \equiv 0$ in Theorem 4.1, we get Theorem 5 in Ref. [1].

By the same argument, we have the following theorem:

Theorem 4.2: *Let the function $f \in \mathcal{H}$ have the series expansion (1) and*

$$\lim_{n\to\infty}\frac{|a_n|}{n^3} = 0 = \lim_{m\to\infty}\frac{|b_m|}{m^3}.$$

(i) If

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| + \sum_{m=1}^{\infty} m(m+\alpha)|b_m| \le 1-\alpha,$$

then

$$\operatorname{dist}(f, \mathcal{K}_{\mathcal{H}}^*(\alpha)) = 0,$$

which is attained for $F = f \in \mathcal{K}_{\mathcal{H}}^*(\alpha) \subset \mathcal{K}_{\mathcal{H}}(\alpha)$.

(ii) If

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| + \sum_{m=1}^{\infty} m(m+\alpha)|b_m| > 1-\alpha,$$

then

$$\operatorname{dist}(f, \mathcal{K}_{\mathcal{H}}^*(\alpha)) = \sqrt{\pi}\left(\sum_{n\in I^c}\frac{a_n^2}{n+1} + \sum_{m\in J^c}\frac{b_m^2}{m+1} + \frac{\left[\sum_{n\in I} n(n-\alpha)|a_n| + \sum_{m\in J} m(m+\alpha)|b_m| - (1-\alpha)\right]^2}{\sum_{n\in I} n^2(n+1)(n-\alpha)^2 + \sum_{m\in J} m^2(m+1)(m+\alpha)^2}\right)^{1/2},$$

where I and J are given by Theorem 3.1. Moreover, the infimum value of $\operatorname{dist}(f, \mathcal{K}_{\mathcal{H}}^*(\alpha))$ *above is attained for the function F given by*

$$F(z) = H(z) + \overline{G(z)},$$

where

$$H(z) = z + \sum_{n=2}^{\infty} A_n z^n, \qquad G(z) = \sum_{m=1}^{\infty} B_m z^m,$$

$$A_n = \begin{cases} \left(|a_n| - \frac{1}{2}\mu_{N,M}(n)(n+1)(n-\alpha)\right) e^{i\arg(a_n)}, & n \in I; \\ 0, & n \in I^c \end{cases}$$

and

$$B_m = \begin{cases} \left(|b_m| - \frac{1}{2}\mu_{N,M}(m)(m+1)(m+\alpha)\right) e^{i\arg(b_m)}, & m \in J; \\ 0, & m \in J^c, \end{cases}$$

where $\mu_{N,M}$ *is given by (39) with* a_n *and* b_m *replaced by* $|a_n|$ *and* $|b_m|$*, respectively.*

Remark 4.2: Letting $\alpha = 0$ and $g \equiv 0$ in Theorem 4.2, we get Theorem 5 in Ref. [2].

5. Examples

In this section as an application of the previous theorems in Section 4, we consider the following examples. We recall that the simplest example of harmonic mappings that need not be conformal are the *affine* mappings $f(z) = az + b + c\bar{z}$ with $|a| \neq |c|$. Affine mappings with $b = 0$ are linear mappings. Following, in the first example, we consider a linear mapping.

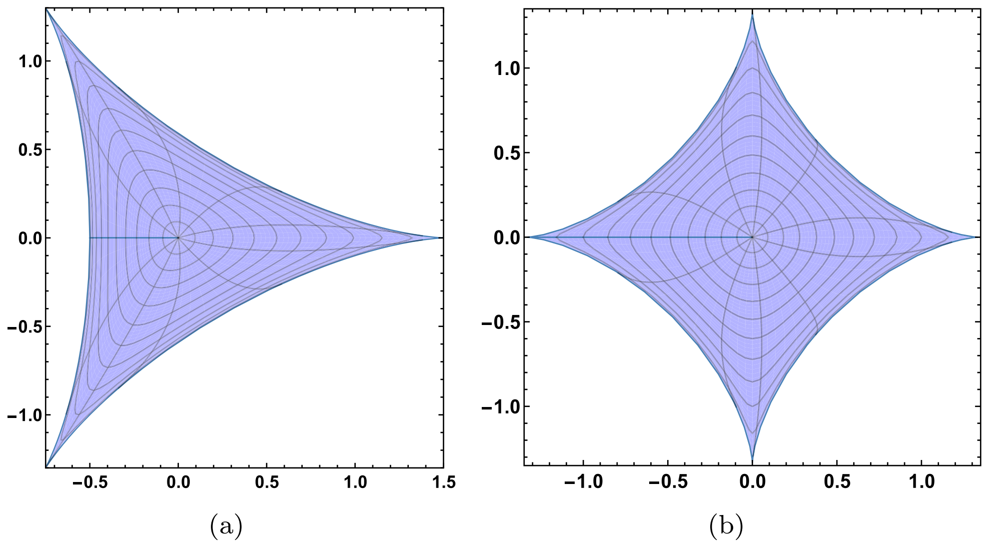


Figure 1. (a) The image of $\mathbb{D}$ under f_2 and (b) the image of $\mathbb{D}$ under f_3.

Example 5.1: Consider the function $f_\gamma : \mathbb{D} \to \mathbb{C}$ defined by $f_\gamma(z) = z + \overline{\gamma z}$ $(|\gamma| \neq 1)$, where $\gamma \in \mathbb{C}$ is a constant. We see that $(|\overline{\gamma}| = |\gamma|)$

(1) if $|\gamma| \leq (1-\alpha)/(1+\alpha)$, $0 \leq \alpha < 1$, then $f_\gamma \in \mathcal{S}^{**}_{\mathcal{H}}(\alpha)$, and thus $\operatorname{dist}(f_\gamma, \mathcal{S}^{**}_{\mathcal{H}}(\alpha)) = 0$.
(2) if $|\gamma| > (1-\alpha)/(1+\alpha)$, then applying Theorem 4.1 it follows that $\mathcal{Q} = \{1\}$, $M = 1$ and $J = \{j_1\} = \{1\}$. Therefore,

$$\operatorname{dist}(f_\gamma, \mathcal{S}^{**}_{\mathcal{H}}(\alpha)) = \sqrt{\frac{\pi}{2}}\left(|\gamma| - \frac{1-\alpha}{1+\alpha}\right) \quad (0 \leq \alpha < 1)$$

is attained for the function $F_{\alpha,\gamma} \in \mathcal{S}^{**}_{\mathcal{H}}(\alpha)$ defined by

$$F_{\alpha,\gamma}(z) = z + \left(\frac{1-\alpha}{1+\alpha} e^{i \arg \overline{\gamma}}\right) \overline{z} \quad (0 \leq \alpha < 1, z \in \mathbb{D}).$$

Another important example that we are interested to study is a generalization of the function

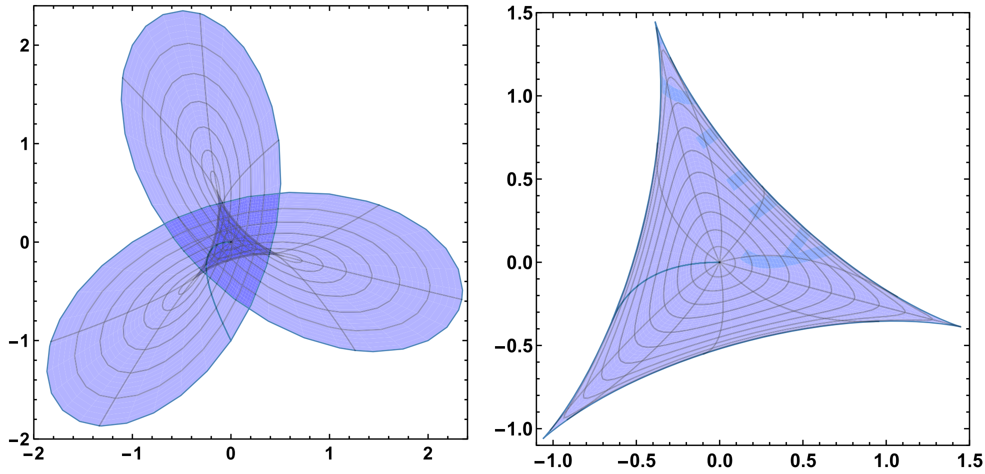
$$f_n(z) = z + \frac{1}{n}\overline{z}^n \quad (n \geq 2),$$

which maps the open unit disk $\mathbb{D}$ onto the region inside a hypocycloid of $n+1$ cups inscribed in the circle $|\omega| = (n+1)/n$. The function f_n is univalent for each $n \geq 2$, see Ref. [12, p. 3]. Figure 1 shows the image of $\mathbb{D}$ under f_2 and f_3.

Example 5.2: Define the function $f_{n,\gamma}$ as follows:

$$f_{n,\gamma}(z) = z + \overline{\gamma z}^n \quad (n \geq 1, \gamma \in \mathbb{C}). \tag{47}$$

Then for $n \geq 2$ (the case $n = 1$ is investigated in Example 5.1 above), we have:



(a) (b)

Figure 2. (a) The image of $\mathbb{D}$ under $f_{2,1+i}$ (non-starlike) and (b) the image of $\mathbb{D}$ under $F^0_{2,0,1+i}$ (the best starlike approximation of $f_{2,1+i}$).

- if

$$|\gamma| \le \frac{1-\alpha}{n+\alpha} \quad (\alpha \in [0,1), n \ge 2),$$

then $f_{n,\gamma} \in \mathcal{S}^{**}_{\mathcal{H}}(\alpha)$, therefore $\operatorname{dist}(f_{n,\gamma}, \mathcal{S}^{**}_{\mathcal{H}}(\alpha)) = 0$.

- if $|\gamma| > (1-\alpha)/(n+\alpha)$, from Theorem 4.1 we obtain that $M = 1$ and $J = \{j_1\} = \{n\}$. Then,

$$\operatorname{dist}(f_{n,\gamma}, \mathcal{S}^{**}_{\mathcal{H}}(\alpha)) = \sqrt{\frac{\pi}{n+1}} \left(|\gamma| - \frac{1-\alpha}{n+\alpha} \right) \quad (0 \le \alpha < 1, n \ge 2, \gamma \in \mathbb{C})$$

is attained for the function $F_{n,\alpha,\gamma} \in \mathcal{S}^{**}_{\mathcal{H}}(\alpha)$ defined by

$$F^0_{n,\alpha,\gamma}(z) = z + \left(\frac{1-\alpha}{n+\alpha} e^{i \arg \overline{\gamma}} \right) \overline{z}^n \quad (0 \le \alpha < 1, n \ge 2, \gamma \in \mathbb{C}, z \in \mathbb{D}).$$

Figures 2 and 3 show a comparison between the images of $\mathbb{D}$ under the function $f_{n,\gamma}$ and its best starlike approximation function $F^0_{n,\alpha,\gamma}$ for some values γ, $n = 2$ and $\alpha = 0$. Notice that in all cases, the minimum of $\operatorname{dist}(f_{2,\gamma}, \mathcal{S}^{**}_{\mathcal{H}}(\alpha))$ is attained for $f_{2,0.5}$ and its rotations, and that $f_{2,0.5}$ is starlike univalent while $f_{2,1+i}$ and $f_{2,0.5+0.3i}$ are not starlike univalent mappings.

- If

$$|\gamma| \le \frac{1-\alpha}{n(n+\alpha)} \quad (\alpha \in [0,1), n \ge 2),$$

then $f_{n,\gamma} \in \mathcal{K}^{*}_{\mathcal{H}}(\alpha)$, therefore $\operatorname{dist}(f_{n,\gamma}, \mathcal{K}^{*}_{\mathcal{H}}(\alpha)) = 0$.

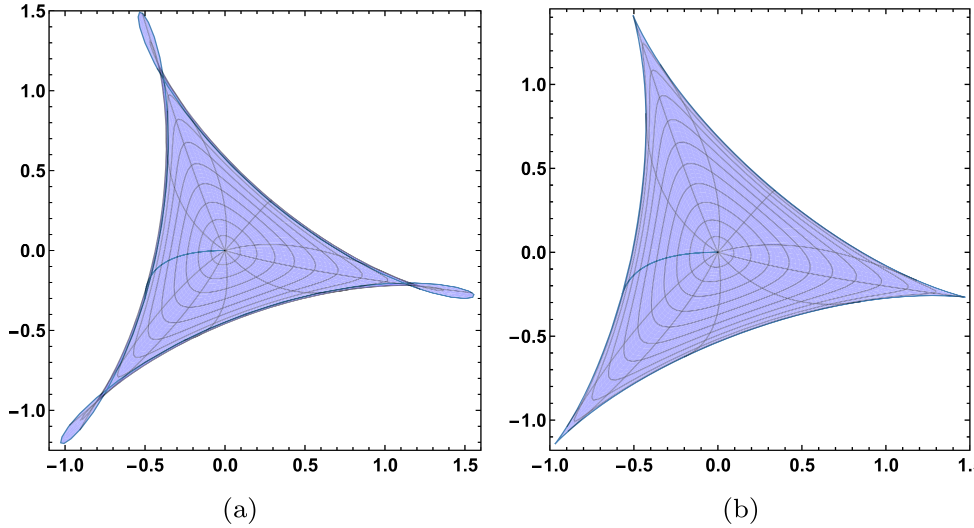


Figure 3. (a) The image of $\mathbb{D}$ under $f_{2,0.5+0.3i}$ (non-starlike) and (b) the image of $\mathbb{D}$ under $F^0_{2,0,0.5+0.3i}$ (the best starlike approximation of $f_{2,0.5+0.3i}$).

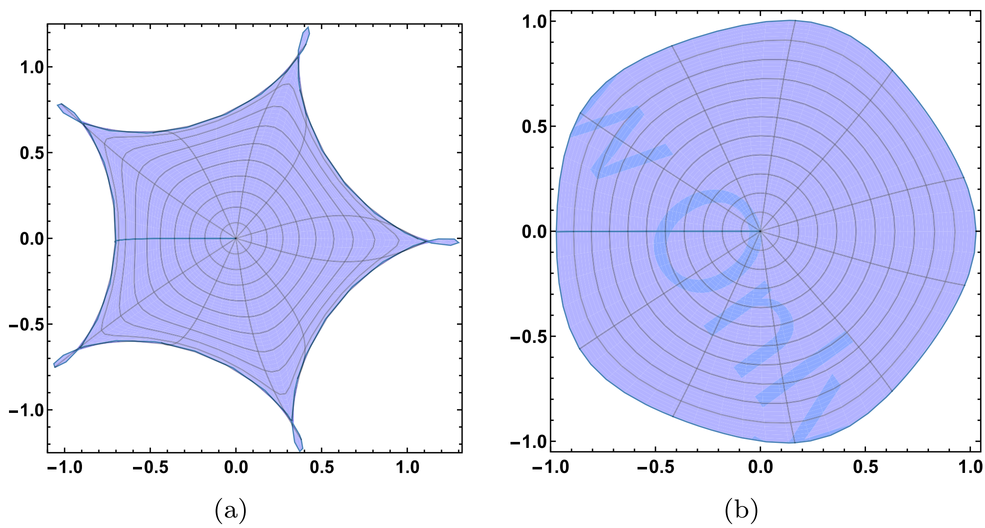


Figure 4. (a) The image of $\mathbb{D}$ under $f_{4,0.3+0.025i}$ (non-convex) and (b) the image of $\mathbb{D}$ under $F^1_{4,1/2,0.3+0.025i}$ (the best convex approximation of $f_{4,0.3+0.025i}$).

- If $|\gamma| > (1-\alpha)/n(n+\alpha)$, then applying Theorem 4.2 we get $M = 1$ and $J = \{j_1\} = \{n\}$. Therefore,

$$\operatorname{dist}(f_{n,\gamma}, \mathcal{S}^{**}_{\mathcal{H}}(\alpha)) = \sqrt{\frac{\pi}{n+1}}\left(|\gamma| - \frac{1-\alpha}{n(n+\alpha)}\right) \quad (0 \le \alpha < 1, n \ge 2, \gamma \in \mathbb{C}),$$

which is attained for the function $F^1_{n,\alpha,\gamma} \in \mathcal{S}^{**}_{\mathcal{H}}(\alpha)$ defined by

$$F^1_{n,\alpha,\gamma}(z) = z + \left(\frac{1-\alpha}{n(n+\alpha)} e^{i \arg \overline{\gamma}}\right) \overline{z}^n \quad (0 \leq \alpha < 1, n \geq 2, \gamma \in \mathbb{C}, z \in \mathbb{D}).$$

Figure 4 shows a comparison between the images of $\mathbb{D}$ under the function $f_{4,0.3+0.025i}$ and under its best convex approximation function $F^1_{4,1/2,0.3+0.025i}$ (here we note that for $\gamma = 0.3 + 0.025i$, we have $|\gamma| = 0.30104 > (1-1/2)/(4(4+1/2)) = 1/36 = 0.028$).

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