Group Decision and Negotiation
https://doi.org/10.1007/s10726-020-09697-9

The Incidence of Some Voting Paradoxes Under Domain Restrictions

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Abstract

Voting paradoxes have played an important role in the theory of voting. They typically say very little about the circumstances in which they are particularly likely or unlikely to occur. They are basically existence findings. In this article we study some well known voting paradoxes under the assumption that the underlying profiles are drawn from the Condorcet domain, i.e. a set of preference profiles where a Condorcet winner exists. The motivation for this restriction is the often stated assumption that profiles with a Condorcet winner are more likely than those without it. We further restrict the profiles by assuming that the starting point of our analysis is that the Condorcet winner coincides with the choice of the voting rule under scrutiny. The reason for making this additional restriction is that—intuitively—the outcomes that coincide with the Condorcet winner make those outcomes stable and, thus, presumably less vulnerable to various voting paradoxes. It will be seen that this is, indeed, the case for some voting rules and some voting paradoxes, but not for all of them.

Keywords Voting rule · Voting paradox · Condorcet domain · Profile restrictions

1 Introduction

Many, if not not all, voting rules can be seen as attempts to overcome specific problems, anomalies or puzzles observed in conducting elections or analyzing their results. Sometimes the problems faced with are so grave that they acquire the status of paradoxes. Voting paradoxes have, indeed, played an important role in the development of voting theory. The best known of these are known as Borda's and Condorcet's paradoxes, illustrated in Tables 1 and 2, respectively.

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Published online: 17 August 2020

Table 1 Borda's paradox Grazia (1953)

4 Voters	3 Voters	2 Voters
A	B	C
B	C	B
C	A	A

Table 2 Condorcet's paradox

1 Voter	1 Voter	1 Voter
A	B	C
C	A	B
B	C	A

In these tables the letters A, B and C stand for candidates (or policy alternatives, for that matter) and the columns represent the preference rankings of voters from top to bottom. Borda's paradox pertains to the plurality (or first-past-the-post) voting where the candidate receiving more votes than any of his/her competitors is the winner. Assuming that Table 1 depicts the preferences of the nine voters, A can be expected to win under plurality voting, but would be defeated by both B and C in pairwise majority voting. In fact, the plurality winner A is an absolute loser in the sense of being the last-ranked candidate of an absolute majority voters. Thus, the paradox boils down to an incompatibility of two intuitive views on what winning means, viz. one which determines the winner by looking at the distribution of first ranks among the alternatives and one based on pairwise majority contests.[1]

The latter intuition may lead to the other classic voting paradox, viz. Condorcet's or cyclic majority paradox exemplified in Table 2. Conducting pairwise majority comparisons in this profile results in a cycle: B defeats A, A defeats C, C defeats B etc. The paradox consists in observing that if individual complete and transitive preference orderings are being aggregated into a collective one by means of pairwise majority votes, the outcome may fail on transitivity.

These two paradoxes are just examples from a larger class of anomalies encountered in aggregating individual preferences into collective choices or rankings (for overviews, see e.g. Brams 1976; Felsenthal 2012; Kramer 1973; Nurmi 1999; Saari 2000a, b, 2001). The paradoxes are rule-related: some rules are vulnerable to some paradoxes, while others can lead to other kinds of paradoxes. No voting rule is invulnerable to all paradoxes. The way the vulnerability is demonstrated is by presenting a profile where the rule under scrutiny leads to a paradoxical outcome. Typically the profiles are selected from an unlimited set of profiles, that is, no constraints are

[1] Borda's proposal, currently known as the Borda count, would here result in B. In Borda count involving k candidates each voter assigns $k-1$ points to his/her first ranked candidate, $k-2$ points to the second ranked one etc, 0 points to the last ranked candidate. For each candidate the points assigned by all voters are summed to obtain the Borda score of that candidate. The winner(s) is (are) then the candidate(s) with the largest Borda score.

imposed on the types of profiles. Sometimes this may be implausible if it is known that the profiles where the rules are being applied always or very frequently satisfy a restriction that affects the vulnerability assessment. One such restriction is the presence of a Condorcet winner in the profiles under investigation.

2 The Condorcet Domain

Our focus is on voting rules that can be described as social choice functions mapping a set of n individual preference relations over a set A of candidates into the set of subsets of A. The preference relations are assumed to be complete (or connected) and transitive. Thus, for any n- tuple of individual complete and transitive preference relations, the voting rule specifies a subset of candidates, viz. the winners in the sense of this particular rule in the specified profile. The evaluation of voting rules typically involves a consideration of the selected winners vis-à-vis the individual preference relations.[2] For example, the rule may yield as the winner a candidate that would in pairwise majority comparisons be defeated by all its competitors if the voters voted according to their preference relations. As an example of such a rule and profile we can refer to the plurality voting and Table 1. While one such profile is sufficient to demonstrate the failure of the plurality rule to guarantee the exclusion of an eventual absolute loser, it doesn't say anything about the general conditions under which such an anomaly emerges. Yet it is not difficult to envision a sufficient condition for avoiding the anomaly: the profile contains an absolute winner. Thus, in a domain satisfying this condition the plurality voting excludes the absolute loser.[3]

Given the intuitive difficulty of constructing examples of various voting paradoxes it is worth studying the profiles types under which the paradoxes occur. In particular, if it is known that some types are far more common than others, we may question the relevance of general vulnerability results and focus on finding the incidences of paradoxes under those domains which are most likely to be encountered in practice. This focus is by no means new. In fact, it was discussed already before Arrow's theorem appeared in the literature. Duncan Black introduced the nowadays well-known profile restriction, viz. single-peakedness in late 1940's (Black 1948). Based on spatial characterization of decision alternatives this and related restrictions soon gained much scholarly attention. Our focus is on a non-spatial restriction on preference relations, viz. one where the defining feature is the presence of a Condorcet winner. Profiles satisfying this restriction constitute the Condorcet domain

[2] Social choice functions as here defined were not the focus of Arrow in his classic general possibility theorem (Arrow 1963). Rather his interest was in social welfare functions mapping preference profiles into the set of collective preference rankings. The distinction is important in characterizing voting rules [e.g. the consistency of Kemeny's median (Fishburn 1977; Young and Levenglick 1978)], but since most voting paradoxes deal with choice sets, our focus is on voting rules as choice functions.

[3] The condition is sufficient, but not necessary for the avoidance of the plurality choice of an absolute loser. Consider a small modification of Table 1: let the 4-voter group consist of only 2 voters, *ceteris paribus*. The resulting profile would have seven voters, an absolute loser (A) and no absolute winner. There B would become the plurality winner and thus the choice of the absolute loser is avoided.

Table 3 A non-single-peaked profile with a Condorcet winner

2 Voters	2 Voters	2 Voters	2 Voters	1 Voter
A	A	B	C	B
B	C	C	B	A
C	B	A	A	C

of profiles (Campbell and Kelly 2015). This domain contains the single-peakedness profiles as single-peakedness is a sufficient condition for there being a Condorcet winner. It is, however, not necessary as shown by Table 3. There B is the Condorcet winner and yet there is no unanimity regarding which candidate is the worst, best or middle-ranked one. Hence the Table 3 profile is not single-peaked. It is not even value-restricted since this property would require that unanimity prevails regarding which candidate is not the best one or which candidate is not the middle-ranked one.

So, the Condorcet domain contains as a proper subset the profiles with single-peaked preferences. Are the profiles in the Condorcet domain more common than those without a Condorcet winner? In some circumstances, e.g. in one-dimensional policy spaces where the voters' preferences can be represented by ellipsoidal indifference curves, the answer is clearly yes. However, by Kramer's (1973) result, introducing more policy dimensions makes the existence of single-peaked preferences unlikely. Moreover, in many-dimensional policy-spaces, the core[4] or similar equilibrium conditions do not in general exist (Kramer 1973; Saari 1997; Schofield 1983). Thus, even though probabilistic and simulation results seem to suggest that the probability of profiles with a Condorcet winner is quite high in small alternative settings and tends to become larger with increasing number of voters (Gehrlein and Lepelley 2011, p. 21) and (Gehrlein and Lepelley 2011, p. 55) , the question of prevalence of which types of profiles—those with a Condorcet winner or those without it—cannot be resolved *in abstracto*.[5] Nevertheless, the Condorcet domain constitutes a class of profiles of interest if for no other reason than because its profiles contain an alternative that many authors deem an obvious winner.[6] So, the question is: does

[4] The core is the set of majority undominated alternatives. It coincides with the Condorcet winner whenever the latter exists.

[5] In simulation studies one creates randomized electorates by generating for each voter a preference ranking over candidates using some probability distribution as the starting point. One then determines, for each generated electorate, whether a Condorcet winner exists or not. In probability models the probability estimates for the existence of the Condorcet winner are expressed in closed formulae. The estimates give theoretical likelihoods, in various types of electorates, for encountering an election setting with a Condorcet winner. The main point of simulations and probability models is, however, not to predict how often Condorcet winners are found in real world elections, but to assess the significance of the changes in profile parameters (e.g. number of voters, number of candidates, probability distribution of preference rankings) for the voting outcomes (Gehrlein and Lepelley 2017).

[6] Two assumptions have to be made in order to plausibly argue that the Condorcet winner is an obvious one. First, the majority principle leads always to the correct outcome. Second, a candidate that wins all those pairwise contests in which he/she participates is the socially best one. This view is defended i.a. in Felsenthal and Machover (1992a) and Risse (2001). It is, however, by no means uncontested, see esp. Saari (1995, 2003). For a non-technical exposition, see Nurmi (1999, pp. 31–40).

the existence of a Condorcet winner do away with important paradoxes in social choices? I.e. is the presence of a Condorcet winner a guarantee against unpleasant surprises in the form of voting paradoxes? If it is and those profiles are deemed more common than others, then we can optimistically dismiss those paradoxes as applicable only in rare circumstances.

3 The Paradoxes and Rule Types

Voting paradoxes are counterintuitive outcomes resulting from applying voting rules. Often the surprising aspect pertains to the relationship between the voter preferences and the outcomes of voting rules. Paradoxes can also be expressed as failures of procedures to satisfy specific desiderata. There are various possibilities to classify paradoxes. In the following we consider three classes: Condorcet incompatibility, monotonicity-related and subset choice paradoxes. The first class consists of paradoxes showing that the Condorcet winner is not elected or the Condorcet loser is elected under a given procedure. The monotonicity paradoxes include two major sub-classes, viz. those focusing on a given electorate where specific types of modifications take place and those where the electorate is augmented with new voters with specific preferences or where the electorate is diminished by removing voters with specific preferences. Choice set variance paradoxes, in turn, look at the possible changes in outcomes when some subsets of the candidate or voter set are considered.[7] We shall provide an overview of these paradoxes in both unrestricted and Condorcet domains.

3.1 Condorcet Incompatibility Paradoxes

This class consists of paradoxes where one intuition of winning clashes with another. Not all such clashes qualify as paradoxes. E.g. that plurality voting occasionally results in a different outcome than the Borda count is not usually deemed particularly dramatic. Indeed, which differences in outcomes are called paradoxes and which not is often in the eye of the beholder. Two incompatibilities are often regarded as particularly serious. Both are associated with Marquis de Condorcet. The first refers to the possibility that a Condorcet winner is not elected by the procedure under scrutiny. If such an instance is found, then it is said that the procedure violates the Condorcet winning criterion. An instance of this incompatibility is Table 3 where plurality voting results in A, whereas B is the Condorcet winner. Thus, plurality voting is vulnerable to this sort of incompatibility or paradox. Fishburn and Brams (1983) call this the thwarted-majorities paradox. The second incompatibility is the mirror image of the first one: a profile is found where the Condorcet loser *is* elected by the procedure under study. The procedure then violates the Condorcet losing criterion. An instance can be found in Table 1 where A, the

[7] Felsenthal discusses a long list of voting paradoxes in Felsenthal (2012). A somewhat more encompassing classification is to be found in Nurmi (1999).

Table 4 Condorcet criteria summary (Felsenthal and Nurmi 2019)

Domain Procedure	Unrestricted		Condorcet	
	C-winning	C-losing	C-winning	C-losing
Amendment	1	1	1	1
Copeland	1	1	1	1
Dodgson	1	0	1	1
Kemeny	1	1	1	1
Minimax	1	0	1	1
Schwartz	1	1	1	1
Young	1	1	1	1
Borda	0	1	0	1
Plurality	0	0	0	0
Baldwin	1	1	1	1
Black	1	1	1	1
Bucklin	0	0	0	0
Coombs	0	1	0	1
Hare	0	1	0	1
Nanson	1	1	1	1
pl. runoff	0	1	0	1
Approval	0	0	0	0
Majority j.	0	0	0	0
Range voting	0	0	0	0

Condorcet loser, is elected by the plurality voting. Table 4 summarizes the compatibility of 20 voting procedures with the Condorcet winning criterion (C-winning) and the Condorcet losing (C-losing) criterion under the unrestricted (columns 2 and 3) (Nurmi 1983, 1987) and the Condorcet domain (Felsenthal and Nurmi 2019). The incompatibility (compatibility, respectively) of the procedure with the criterion is indicated by 0 (1).

The procedures are listed from pairwise through positional and multi-stage ones to procedures with non-standard voter input. The latter are procedures that require more information than the preference ranking of voters to yield a winner. Each class of systems is separated with a horizontal line from the following class.[8] For definitions of the procedures the reader is referred to Felsenthal and Nurmi (2019, pp. 5–13).

Most procedures in Table 4 are well-known. It will be recalled e.g. that the majority judgment method of Balinski and Laraki is based on ordinal grade (utility) values assigned by each voter to each candidate. For each candidate the median score is determined and the candidate with the highest median score is declared the winner (Balinski and Laraki 2011). So, in contradistinction to the ranking-based procedures,

[8] The assignment of procedures to classes is not unambiguous. E.g. the Borda count can be implemented as a purely binary system.

for each voter there may be rank positions that are assigned to no candidate or there may be rank positions occupied by several candidates. Range voting, in turn, allows voters to assign cardinal utility values to candidates and the sum of values given to each candidate determines the winner.[9]

Even a cursory inspection of Table 4 reveals that the domain restriction from universal to Condorcet makes very little difference in terms of performance with respect to the Condorcet winning and losing.[10] One should bear in mind that on each row a ‘1’ in the second (or third, respectively) column implies the identical entry in the fourth (or fifth) column as well. Of the binary systems which are all Condorcet extensions, only Dodgson’ rule and the minimax procedure exhibit different—improved—performance under the Condorcet domain. That the Dodgson rule and the minimax method may elect a Condorcet loser has been shown in Nurmi (2004a, p. 10) and Felsenthal and Nurmi (2018, p. 82). In fact, both methods may elect an absolute loser. This can only happen when no Condorcet winner exists in a profile since when one does exist, all Condorcet extensions (including Dodgson and minimax) end up with the Condorcet winner. The latter obviously cannot be a Condorcet loser.

Overall, it is not surprising that both Condorcet criteria are satisfied by nearly all binary procedures. After all, they are mainly attempts to extend the Condorcet winning idea to situations where no such winner exists. Table 1 demonstrates that the failure of the plurality rule on the Condorcet winning criterion also applies in the Condorcet domain. It also shows that this rule may end up with a Condorcet loser even in the restricted domain. The Borda count obviously satisfies the Condorcet losing criterion in the Condorcet domain since it does so in the unrestricted one. The failure of the Borda count on Condorcet’s winning criterion is shown in Table 5 where a (strong) Condorcet winner A exists (and thus the profile is in the Condorcet domain), but the Borda count elects B.

Baldwin’s (aka Borda elimination) rule is a Condorcet extension that runs in several rounds eliminating on each round the candidate with the lowest Borda score. This guarantees that the eventual Condorcet winner is not eliminated and that the eventual Condorcet loser will be eliminated on the final round or before it. The same

[9] There are several other procedures that have been proposed for specific types of decision settings, e.g. sequential voting by veto (Mueller 1978), successive procedure (Rasch 1995), sequential choice (Felsenthal and Machover 1992b; Mueller 1978; Stefánsson 2019), the Janeček method (Janeček 2018; Oreský 2020). The last-mentioned system allows the voters to cast both positive and negative votes. For each candidate the voter may give one positive vote or no vote. In addition, the voter can cast a negative vote to some candidates. The system is quite flexible with regard to the number of positive and negative votes, but it is suggested that the number of the latter be no more than half of the number of positive votes cast by the voter. In single-winner elections this procedure gives each voter two or three positive and one negative vote to distribute among the candidates. The winner is the candidate with the largest sum of votes (Janeček 2018). Since these procedures require information about voter opinions that is much richer than just the ranking of the candidates, we shall not discuss them here. N.B. the approval voting also requires more information than preference rankings, but since it is often included in similar comparisons we shall conform to this usage.

[10] The results reported here have been collected from a number of sources, i.a. Straffin (1980), Nurmi (1983, 1987), Tideman (2006), Felsenthal (2012), Janeček (2018), Felsenthal and Nurmi (2018, 2019), Oreský (2020).

Table 5 Borda count does not elect the strong Condorcet winner

7 Voters	4 Voters
A	B
B	C
C	A

Table 6 Bucklin's procedure in the Condorcet domain

1 Voter	10 Voters	11 Voters	11 Voters	11 Voters	1 Voter
A	A	B	B	C	C
B	C	A	C	A	B
C	B	C	A	B	A

applies to Nanson's rule which eliminates candidates with at most the average Borda scores. Bucklin's performance in the unrestricted domain has been established by Tideman (see Felsenthal 2012, pp. 50, 55, Tideman 2006, p. 197). From his examples we can infer that this procedure violates the Condorcet winning criterion in the Condorcet domain as well. However, its failure on the Condorcet losing criterion has not been established. Table 6 does that using a minor modification of Tideman's example. The profile contains a Condorcet winner, B, and a Condorcet loser, A. Yet, A is not excluded as the Condorcet losing criterion would require, but is elected along with C.[11]

That Coombs's procedure fails on Condorcet-winning criterion on the Condorcet domain as well as in the unrestricted one has been shown by Tideman (Felsenthal 2012, p. 50) and since it satisfies the Condorcet losing criterion in general, it does so in the Condorcet domain as well. The same argument applies to the Hare system (aka alternative vote procedure or the single transferable vote in single-member constituencies). As the Hare system and the plurality runoff are equivalent in three-candidate races and as the examples referred to above involve just three candidates, we can make the same conclusions regarding the plurality runoff system.

The last four procedures in Table 4 require a different voter input than the preceding ones to determine the winner and, thus, their evaluation using the same criteria may be deemed inappropriate. In any event, all of them fail on both kinds of Condorcet criteria both in unrestricted and in Condorcet domains (Felsenthal 2012; Oreský 2020).

In general, the restriction of the profile domain is accompanied with very few changes in the performance of voting procedures in terms of Condorcet-related criteria: what holds for the incidence of paradoxes under no domain restrictions mainly seems to hold in the Condorcet domain as well.

[11] So, the only candidate not elected by Bucklin's procedure in the Condorcet domain is B, the Condorcet winner which may be somewhat surprising.

3.2 Monotonicity-Related Paradoxes

A cornerstone of democratic rule is that the opinions of the electorate make a difference and that the difference they make is in a non-perverse direction. Several non-equivalent specifications of this principle have been introduced in the literature (for a brief exposition, see e.g. Nurmi 2004b). A common feature in each of them is that additional support, *ceteris paribus*, should never lead to an outcome that is to the disadvantage of the candidate. We consider three variants of the additional support and the associated paradoxical changes in the voting outcomes:

1. Monotonicity failure in a fixed electorate. We distinguish two sub-types of these failures. If either of these failures can occur, the procedure under investigation is non-monotonic:
   - *Upward monotonicity failure* Given a preference profile, procedure and a winner x, the profile is modified so that the position of x is improved in some individual preference rankings, *ceteris paribus*. A procedure is (upward) monotonic if such a modification never makes x a non-winner.
   - *Downward monotonicity failure* Given a preference profile, procedure and a winner x, suppose that the ranking of some other candidate y is lowered, *ceteris paribus*, and as a result y becomes the new winner. In this case we have an instance of the downward monotonicity failure.
2. Given a preference profile, procedure and a winner x, the electorate is augmented with a group of voters with an identical preference ranking where x is ranked first, *ceteris paribus*. A procedure satisfies strong positive involvement (Pérez 2001) or top property (Kasper et al. 2019) if x remains the winner after any such augmentation.
3. Given a preference profile, procedure and a winner y, the electorate is augmented with a group of voters with an identical preference ranking where z is ranked last. A procedure satisfies the strong negative involvement (Pérez 2001) or bottom property (Kasper et al. 2019) if z cannot become the winner after such a change.

It is worth observing that monotonicity thus defined is a fixed electorate property, i.e. the changes investigated involve an electorate of a fixed size and a fixed number of candidates. The top and bottom properties, in turn, are variable electorate properties, i.e. they focus on changes occurring in voting outcomes that result from augmenting (or diminishing) a given electorate with specific types of voters.[12] The seminal result by Moulin amounts to stating the incompatibility of the Condorcet winning criterion and the invulnerability to the no show paradox when there are at least four candidates (Moulin 1988). The result has subsequently been refined and extended (Brandt et al. 2017; Duddy 2014; Pérez 2001).

[12] For an analysis of the relationships between the no show paradox and monotonicity, see Núñez and Sanver (2017).

Table 7 Invulnerability to some monotonicity-related paradoxes under unrestricted and Condorcet domain

Property	Monotonicity		Top		Bottom	
Domain	Unrestr.	Cond.	Unrestr.	Cond.	Unrestr.	Cond.
Amendment	1	1	0	1	0	0
Copeland	1	1	0	1	0	0
Dodgson	0	1	0	1	0	0
Kemeny	1	1	0	1	0	0
Minimax	1	1	0	1	1	1
Schwartz	1	1	0	1	0	0
Young	1	1	0	1	1	1
Borda	1	1	1	1	1	1
Plurality	1	1	1	1	1	1
Baldwin	0	1	1	1	0	0
Black	1	1	0	1	0	1
Bucklin	1	1	0	0	0	0
Coombs	0	0	0	0	1	1
Hare	0	0	0	1	0	0
Nanson	0	1	0	1	0	0
pl. runoff	0	0	0	1	0	0
Approval	1	1	1	1	1	1
Majority j.	1	1	0	0	0	0
Range voting	1	1	1	1	1	1

Table 7 summarizes the findings collected for the most part from Felsenthal and Nurmi (2017, 2018, 2019), Felsenthal and Tideman (2013), Felsenthalm and Tideman (2014). ‘1’ (‘0’, respectively) in the table indicates that the paradox in question cannot (can) occur when using the procedure under the domain indicated by the column. Naturally, if a procedure is is invulnerable to a paradox under unrestricted domain, it is *eo ipso* invulnerable to it also under the Condorcet domain. Similarly, if a procedure is vulnerable to a paradox in the Condorcet domain, it is also vulnerable to it under unrestricted domain.

Table 7 shows some important changes in performance of various rules in unrestricted vs. Condorcet domains. Most differences between the two domains are associated with the Condorcet extensions and the paradoxes in variable electorates. While the top property is satisfied by all Condorcet extensions in the Condorcet domain, the bottom property characterizes the same procedures in unrestricted and Condorcet domains. In fact, the existence of a Condorcet winner in a profile does not change the vulnerability of any system (of those under scrutiny here) to the negative involvement paradox.

Table 8 Invulnerability to subset choice and consistency paradoxes under unrestricted and DSF domain

Paradox	Subset choice		Consistency	
Domain	Unrestr.	DSF	Unrestr.	DSF
Amendment	0	1	0	1
Copeland	0	1	0	1
Dodgson	0	1	0	1
Kemeny	0	1	0	1
Minimax	0	1	0	1
Schwartz	0	1	0	1
Young	0	1	0	1
Borda	0	0	1	1
Plurality	0	0	1	1
Baldwin	0	1	0	1
Black	0	1	0	1
Bucklin	0	0	0	0
Coombs	0	0	0	0
Hare	0	0	0	0
Nanson	0	1	0	1
pl. runoff	0	0	0	0
Approval	1	1	1	1
Majority j.	1	1	0	0
Range voting	1	1	1	1

3.3 Choice Set Variance Paradoxes

By choice set variance paradoxes we refer to counterintuitive changes in voting outcomes resulting (1) from considering subsets of candidates vis-à-vis the superset, and (2) from considering the choices made in subsets of voters vis-à-vis the choice made by the electorate *in toto*. More specifically, the former paradoxes occur whenever, given a profile over a set A of candidates, a procedure and a winner, say $x \in A$ determined by the procedure, the same x is not the winner in all subsets of A containing x.[13] The paradox related to (2) occurs when, given a partitioning of voters and a profile such that the same candidate, $x \in A$, is the winner in each subset of voters when a given procedure is being applied, then some other candidate y is the winner when the procedure is applied to the entire electorate (Young 1974). We shall call the former the subset choice and the latter the consistency paradox. In Table 8 we report the possibility (0) or impossibility (1) of encountering these paradoxes under unrestricted domain and in a procedure-specific sub-domain of the Condorcet domain, viz. one where initial profile has a Condorcet winner and—moreover—this

[13] The property that is violated when this kind of paradox occurs has many names, e.g. heritage, heredity condition, subset choice condition, property α or Chernoff's condition (Aizerman and Malishevski 1981; Aleskerov 1999; Fishburn 1974; Sen 1970; Chernoff 1954).

candidate is elected by the procedure under investigation. This sub-domain will be called DSF domain in recognition of Dan S. Felsenthal who introduced the concept and studied it in detail. Obviously, the DSF domain coincides with the Condorcet domain for all Condorcet extensions. In other procedures the performance may differ in the Condorcet and DSF domains, since the latter is a proper sub-domain of the former. Thus, for example, the profile of Table 1 belongs to the DSF domain of the Borda count, but not to the DSF domain of the plurality rule. The point of this domain restriction is to provide an intuitive stable starting outcome, viz. one where there is a Condorcet winner which coincides with the choice ensuing from the procedure under investigation. The question addressed then is: to what extent does this initial stability guarantee invulnerability of the procedure against voting paradoxes?

A couple of observations on Table 8 are in order. First, the domain restriction does, indeed. make a difference when it comes to Condorcet extensions. While none of them exhibits invulnerability to subset choice and consistency paradoxes in general, they all avoid those paradoxes in the DSF domain. Second, the two positional procedures, Borda and plurality, are unaffected by the domain restriction: they are vulnerable to the subset choice paradox both in the unrestricted and DSF domains, but are invulnerable to the consistency paradox in both domains. Third, two of the more recent voting rules that require non-standard voter input—approval and range voting—seem to do quite well in terms of the two paradoxes, while the majority judgment falls behind them on consistency.

4 Conclusion

The voting theory community has long been divided into two camps: those advocating Condorcet's view of pairwise majority winners as the 'real' winners and those of more positional persuasion. This paper reports some findings that are unlikely to resolve the difference in favour of one view over the other. Yet it is hoped that a somewhat more nuanced picture emerges from the preceding. The Condorcet domain represents a profile restriction that seems plausible in some contexts. Hence, it is worthwhile to find out whether the properties of various voting rules change essentially when it is assumed that the procedures are mainly used in the Condorcet domain. In particular, it is of some interest to see whether the vulnerability to some important voting paradoxes depends essentially on the assumption that the profiles of interest are in the unrestricted domain rather than, say, in the Condorcet one. The no show paradox or the failure on the bottom property that, by Moulin's result (Moulin 1988), is a known drawback of Condorcet extensions seems a particularly persistent flaw of these procedures, surviving even in the Condorcet domain. In contrast, the more-is-less paradox or failure on the top property is a nonexistent possibility for those procedures in this domain. The subset choice and consistency paradoxes vanish in Condorcet extensions in the DSF domain. The Borda count and plurality voting fail in both domains on subset choice, but are consistent in these domains. The best performance in terms of the subset choice and consistency criteria is exhibited by the approval and range voting procedures.

Acknowledgements Open access funding provided by University of Turku (UTU) including Turku University Central Hospital. The author is profoundly indebted to the late Dan S. Felsenthal for numerous conversations in which the problems touched upon in this article were discussed. Those conversations culminated in the joint monographs (Felsenthal and Nurmi 2017, 2018, 2019). The author thanks two anonymous referees for constructive comments on an earlier version.

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