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IMPRECISE MEASUREMENTS IN QUANTUM MECHANICS

by

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Contents

Abstract

In this thesis the structure and properties of imprecise quantum measurements are investigated. The starting point for this investigation is the representation of a quantum observable as a normalized positive operator measure.

A general framework to describe measurement inaccuracy is presented. Requirements for accurate measurements are discussed, and the relation of inaccuracy to some optimality criteria is studied.

A characterization of covariant observables is given in the case when they are imprecise versions of a sharp observable. Also the properties of such observables are studied.

The case of position and momentum observables is studied. All position and momentum observables are characterized, and the joint position-momentum measurements are discussed.

List of articles

This thesis consists of the introductory review part and the following six articles:

I T. Heinonen, P. Lahti, J.-P. Pellonpää, S. Pulmannova, K. Ylinen,
The norm-1-property of a quantum observable,
Journal of Mathematical Physics 44 (2003), 1998-2008.

II P. Busch, T. Heinonen, P. Lahti,
Noise and disturbance in quantum mechanics,
Physics Letters A 320 (2004), 261-270.

III T. Heinonen, P. Lahti, K. Ylinen,
Covariant fuzzy observables and coarse-graining,
Reports on Mathematical Physics 53 (2004), 425-441.

IV C. Carmeli, T. Heinonen, A. Toigo,
Position and momentum observables on $\mathbb{R}$ and on $\mathbb{R}^3$,
Journal of Mathematical Physics 45 (2004), 2526-2539.

V C. Carmeli, T. Heinonen, A. Toigo,
On the coexistence of position and momentum observables,
Journal of Physics A: Mathematical and General 38 (2005), 5253-5266.

VI T. Heinonen,
Optimal measurements in quantum mechanics,
Physics Letters A 346 (2005), 77-86.

Chapter 1

Introduction

Physics, being an empirical science, is ultimately based on measurements. Scientific measurements require high expertise and sophisticated devices. In large part the progress of physics has been connected to the development of measurement techniques.

Quantum theory has shown that measurements need also theoretical examination. Soon after the birth of quantum theory the deep nature of the theoretical issues related to measurements were generally recognized. It became clear that the concept of measurement is not at all innocuous, and an extensive mathematical and philosophical research has been concentrated on measurements.

Every real measurement is, more or less, imprecise. In some cases this fact can be passed in theoretical investigations, and, being aware of the idealization, one may concentrate on absolutely accurate measurements. However, in quantum mechanics imprecision has a fundamental role. For instance, there is no joint measurement for sharp position and momentum observables. These quantities can be measured together only if the measurement accuracy is not too high. Thus, in quantum mechanical studies one must be able to describe imprecise measurements.

As is well understood, the conventional representation of an observable as a selfadjoint operator is not sufficient to cover measurement inaccuracy. Selfadjoint operators describe ideal, absolutely accurate, measurements. It has been known for a long time that a more general representation of an observable as a normalized positive operator measure offers, among other things, a natural way to describe imprecise measurements. This is the basic mathematical framework in this thesis.

This thesis is based on a research which was conducted in order to receive a better understanding of the inaccuracy in quantum measurements. Part of the research was focused on some general issues related to the measurement imprecision. Covariant measurements have reserved special attention as they cover many important cases. Covariance conditions reflect symmetry properties of measurements, and they give a consistent way of finding mathematical descriptions for typical physical quantities, such as position, momentum, phase, and spin.

The introductory review part is organized as follows: Chapter 2 presents a general framework where imprecise measurements can be described and studied. In typical applications observables are identified by their symmetry properties. The necessary mathematical machinery for this purpose is discussed in Chapter 3. In Chapter 4 the example of position and momentum observables is treated in this framework. The question of joint measurability of position and momentum observables is analyzed, and some formulations of the uncertainty principle are discussed.

Chapter 2

Imprecise measurements

2.1 States and observables

The most basic situation in physics is the following: we have an object system under investigation, and we try to obtain information about it by making an experiment. As a result of the experiment, measurement outcomes are registered. Quantum mechanics predicts the probabilities of the measurement outcomes. In this section we recall the probability structure of quantum mechanics as presented, for instance, in [16], [21], [33], and [46].

Let Ω be the set of possible measurement outcomes in a given experiment. We make probability statements about possible *events*, which are subsets of Ω. An event $X \subseteq \Omega$ occurs if the measurement outcome belongs to X. We denote by $\mathcal{A}$ the collection of events, and we assume that $\mathcal{A}$ is a σ-algebra. We call the measurable space $(\Omega, \mathcal{A})$ an *outcome space*. The experiment defines a probability measure on $(\Omega, \mathcal{A})$ giving the measurement outcome statistics.

It is practical to divide an experiment into a preparation procedure and a measurement. In a given experiment this division may be quite arbitrary, but that is not a serious drawback. We simply assume that there is a set of possible preparations and a set of possible measurements, and any pair of a preparation and a measurement can be combined to an experiment. Hence, a preparation specifies a probability distribution for every possible measurement of the system. Two preparation procedures can be superficially quite different and yet lead to the same probability distribution in any chosen measurement. A *state* of the system is an equivalence class of preparation procedures which give the same probability distributions in all measurements. Similarly, an *observable* is an equivalence class of measurements which give the same probability distributions in all preparations. For a pair of a state T and an observable E, we denote by p_T^E the measurement outcome distribution of the associated experiment.

In quantum mechanics a system is described by a complex separable Hilbert space $\mathcal{H}$. The states of the system are represented as positive operators of trace one, and we denote the set of states by $\mathcal{S}(\mathcal{H})$. The set $\mathcal{S}(\mathcal{H})$ is convex: if $T_1, T_2 \in \mathcal{S}(\mathcal{H})$ and $0 \leq \lambda \leq 1$, then also the convex combination $\lambda T_1 + (1-\lambda)T_2$ belongs to $\mathcal{S}(\mathcal{H})$. This convex structure of $\mathcal{S}(\mathcal{H})$ corresponds to the possibility of mixing states. An extreme element of the convex set $\mathcal{S}(\mathcal{H})$ is called a *pure state*, and the pure states are the one-dimensional projections on $\mathcal{H}$. A state which is not pure is called a *mixed state*. It is natural to assume that the correspondence between the states and the observables is consistent with the convex structure of the states: if $T_1, T_2 \in \mathcal{S}(\mathcal{H})$ and $0 \leq \lambda \leq 1$, then

$$p^E_{\lambda T_1 + (1-\lambda)T_2} = \lambda p^E_{T_1} + (1-\lambda) p^E_{T_2} \tag{2.1}$$

for any observable E.

Let $\mathcal{L}(\mathcal{H})$ be the set of bounded linear operators on $\mathcal{H}$, and denote the zero operator and the identity operator by O and I, respectively. An observable with the outcome space $(\Omega, \mathcal{A})$ is represented as a mapping $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ such that

(i) $E(\cup_{i=1}^{\infty} X_i) = \sum_{i=1}^{\infty} E(X_i)$ for any disjoint sequence $(X_i) \subseteq \mathcal{A}$;

(ii) $E(X) \geq O$ for any $X \in \mathcal{A}$;

(iii) $E(\Omega) = I$.

The first condition means that E is σ-additive with respect to the weak operator topology, and therefore, it is an operator measure. The second condition states that the operators in the range of E are positive, and the last condition is a normalization condition. In conclusion, the conditions (i), (ii) and (iii) mean that E is a *normalized positive operator measure.* The probability measure p^E_T is given by the trace formula

$$p^E_T(X) = \mathrm{tr}[TE(X)], \qquad X \in \mathcal{A}. \tag{2.2}$$

In this way each observable E defines a mapping $T \mapsto p^E_T$ from the set of states to the set of probability measures, and this mapping satisfies condition (2.1). Any such mapping arises in this way. This is a consequence of the fact that the dual space of $\mathcal{T}(\mathcal{H})$, the Banach space of trace class operators, is isomorphic to $\mathcal{L}(\mathcal{H})$.

The values of observables are selfadjoint operators A on $\mathcal{H}$ satisfying $O \leq A \leq I$, and they are called *effects.* The usual addition of operators defines a partial binary operation $\oplus$ on $\mathcal{E}(\mathcal{H})$. Namely, for any $A, B \in \mathcal{E}(\mathcal{H})$,

$A \oplus B$ is defined when $A + B \leq I$ and then $A \oplus B := A + B$. Thus, $A \oplus B$ exists exactly when there is an observable E and disjoint events X and Y such that $E(X) = A$ and $E(Y) = B$, and in that case $A \oplus B = E(X \cup Y)$.

For each state T, the formula

$$f_T(A) = \mathrm{tr}[TA], \qquad A \in \mathcal{E}(\mathcal{H}), \tag{2.3}$$

defines a mapping f_T from the set of effects $\mathcal{E}(\mathcal{H})$ to the interval $[0, 1]$. This mapping has the following properties:

$$f_T(A_1 \oplus A_2 \oplus \ldots) = f_T(A_1) + f_T(A_2) + \ldots . \tag{2.4}$$

whenever $A_1 \oplus A_2 \oplus \ldots$ exists[1], and

$$f_T(I) = 1. \tag{2.5}$$

Every mapping with these properties has the form (2.3) for some state T. Indeed, a mapping $f : \mathcal{E}(\mathcal{H}) \to [0, 1]$ with the additivity property (2.4) extends uniquely to a positive linear functional on $\mathcal{L}(\mathcal{H})$, which is normal[2]. It follows that $f = f_T$ for some positive trace class operator T as shown, for instance, in [33, Lemma 1.6.1]. The normalization condition (2.5) implies that $\mathrm{tr}[T] = 1$, and thus, the operator T is a state.

We conclude that the above framework for states and observables is coherent in the sense that if the representation of the one is given, the representation of the other follows from natural assumptions.

Remark 1. The structures of the sets of states and observables have been investigated also in more general frameworks than discussed here. The convex structure of the set of states is the starting point in the convex, or operational, approach. For a discussion and further references, we refer to [42]. The set of effects with the partial sum has motivated investigations where an abstract effect algebra is taken as a primitive structure; see, for instance, [6] and [43].

2.2 Sharpness of an observable

Let E be an observable with the outcome space $(\Omega, \mathcal{A})$, and let us assume that each $E(X)$ is a projection operator, i.e., E is a projection valued measure (also called a spectral measure). These kinds of observables have certain

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[1] The sum $A_1 \oplus A_2 \oplus \ldots$ exists if the finite sums $A_1 \oplus \ldots \oplus A_n$ exist for every n. In this case the increasing sequence of the finite sums has the least upper bound; see e.g. [8, Proposition 1]

[2] For a detailed calculation, see e.g. [14].

ideal properties, and we call them *sharp observables.* To distinguish sharp observables from general observables, we reserve the letter Π for a sharp observable.

Let $\mathcal{B}(\mathbb{R})$ be the Borel σ-algebra of the real line $\mathbb{R}$, and let Π be a sharp observable with the outcome space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. The observable Π defines a selfadjoint operator A in $\mathcal{H}$ by

$$A = \int x \, d\Pi(x). \tag{2.6}$$

By the spectral theorem, equation (2.6) gives a one-to-one correspondence between selfadjoint operators[3] and sharp observables with the outcome space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Therefore, the definition of an observable as a normalized positive operator measure generalizes the usual textbook definition of an observable as a selfadjoint operator.

It is convenient to call projection valued measures sharp observables since they possess some ideal properties which observables in general do not have. However, it is questionable to say that only the projection valued measures describe accurate or precise measurements. The issue seems to be more delicate.

Let us first list three properties of sharp observables which refer to their ideal accuracy. Let Π be a sharp observable with the outcome space $(\Omega, \mathcal{A})$.

(a) For each $\Pi(X) \neq O$, there is a state T such that

$$p_T^{\Pi}(X) = 1. \tag{2.7}$$

(b) For any $X \in \mathcal{A}$, we have

$$\Pi(X) \wedge \Pi(X') = O, \tag{2.8}$$

where $\Pi(X) \wedge \Pi(X')$ denotes the infimum of $\Pi(X)$ and $\Pi(X')$ in the partially ordered set $\mathcal{E}(\mathcal{H})$.

Moreover, if the outcome space of Π is $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, then it has the following feature:

(c) For any $\epsilon > 0$, there exists a state T such that

$$\mathrm{Var}(p_T^{\Pi}) < \epsilon. \tag{2.9}$$

---

[3] Generally, A is a densely defined unbounded operator. The operator A is bounded exactly when $\Pi(X) = I$ for some bounded Borel set X in $\mathbb{R}$.

Property (a) means that given an event X with $\Pi(X) \neq O$, we can choose a state T such that the measurement outcome will belong to the set X with probability one. To confirm that the sharp observable Π has this property, take T to be a pure state defined by an eigenvector of $\Pi(X)$ with eigenvalue 1. With this choice equation (2.7) obviously holds.

Even if property (a) has a clear intuitive meaning, it can hardly be tested experimentally. This is due to the fact that probability one and probability almost one cannot be distinguished in a real experiment. A relaxation of (a) for an observable E taking this fact into account is the following:

(a') For each $E(X) \neq O$ and for any $\delta < 1$, there exists a state T such that

$$p_T^E(X) > \delta. \tag{2.10}$$

An observable satisfying (a') is called *approximately sharp*.

Let us first note that if inequality (2.10) holds for some state T, then it also holds for some pure state T_1. This follows from the fact that T can be given as a σ-convex decomposition of pure states. Therefore, inequality (2.10) holds for some state T if and only if the spectrum of the operator $E(X)$ extends above δ. We conclude that an observable E has property (a') exactly when the spectrum of every nonzero operator $E(X)$ contains 1.

Obviously, if an observable E has property (a), then also condition (a') is satisfied. If the dimension of $\mathcal{H}$ is finite, then properties (a) and (a') are equivalent. Indeed, in this case every operator on $\mathcal{H}$ has a pure point spectrum, and thus, (a') is equivalent to the fact that every nonzero operator $E(X)$ has eigenvalue 1. Generally, however, (a') does not imply (a). A physically relevant situation demonstrating this fact is discussed in Example 1 below.

Condition (b) means that $\Pi(X)$ and $\Pi(X')$ are mutually exclusive. This condition is satisfied only by sharp observables. A generalization of (b) for an observable E would be the requirement that $E(X)$ and $E(X')$ are approximately exclusive in some appropriate sense. Let us first note that (b) contains two statements: the infimum of $\Pi(X)$ and $\Pi(X')$ exists and is zero. The infimum of the effects $E(X)$ and $E(X')$ may not exist[4]. In fact, as shown in [41], it exists if and only if the spectrum of $E(X)$ is contained either in $\{0\} \cup [\frac{1}{2}, 1]$ or in $[0, \frac{1}{2}] \cup \{1\}$. Also, the operational meaning of the (non)existence of the infimum is not clear. For these reasons we proceed differently.

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[4] There is always a maximal $C \in \mathcal{E}(\mathcal{H})$ such that $C \leq E(X)$ and $C \leq E(X')$, but it may not be unique; see [58].

For any two effects $A, B \in \mathcal{E}(\mathcal{H})$, we denote by $lb(A, B)$ the set of their lower bounds, that is,

$$lb(A, B) := \{C \in \mathcal{E}(\mathcal{H}) \mid C \leq A, C \leq B\}.$$

Condition (b) is equivalent to the fact that for any $X \in \mathcal{A}$, we have

$$lb(\Pi(X), \Pi(X')) = \{O\}. \tag{2.11}$$

For a general observable E, the set $lb(E(X), E(X'))$ contains also nonzero effects. If $E(X)$ and $E(X')$ should be exclusive in some approximate sense, a necessary requirement is that neither $E(X)$ nor $E(X')$ belongs to the set $lb(E(X), E(X'))$. Otherwise one of them is included in the other. Hence, we have the following relaxation of (b):

(b') For any $X \in \mathcal{A}$ such that $O \neq E(X) \neq I$, neither $E(X) \leq E(X')$ nor $E(X) \leq E(X')$.

Observables having property (b') are called *regular*. It is a direct observation that condition (b') is satisfied if and only if the spectrum of each nonzero effect $E(X)$ extends above $\frac{1}{2}$. Thus, an observable E is regular exactly when for each nonzero effect $E(X)$, there exists a state T such that

$$p_T^E(X) > \frac{1}{2}. \tag{2.12}$$

This shows that an approximately sharp observable is regular. Furthermore, regularity is equivalent to the condition that the range of an observable E is a Boolean lattice with respect to the order and complement restricted to itself; see e.g. [51].

Condition (c) means that with a suitable choice of a state T, the variance of the probability measure p_T^Π can be made arbitrarily small; for a proof of this fact, see e.g. [7, Section 3.1.4]. This indicates that the sharp observable Π has no intrinsic imprecision.

Let E be an observable with the outcome space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and suppose that E is bounded, meaning that $E(X) = I$ for some bounded Borel set $X \subset \mathbb{R}$. Let Π be a sharp observable such that the mean values of the probability distributions p_T^Π and p_T^E are the same in every state T. In this case we may write

$$\mathrm{Var}(p_T^E) = \mathrm{Var}(p_T^\Pi) + \eta(E; T), \tag{2.13}$$

where

$$\eta(E; T) = \int x^2 dp_T^E(x) - \left(\int x dp_T^E(x)\right)^2 \geq 0. \tag{2.14}$$

Therefore, the variance of p_T^E cannot be less than the variance of p_T^Π, and the additional term $\eta(E;T)$ can be thought of as an indication of a measurement noise. However, in some situations this kind of reasoning may be problematic. It is shown in Article I that if the observable E is approximately sharp, then it also has property (c). In these cases the variance as a measure of noise may be misleading.

Example 1. A physically relevant example of an observable satisfying conditions (a'), (b') and (c) but not being sharp is the canonical phase observable. The canonical phase observable is canonically conjugate to the number observable of a single-mode field, and it can be singled out from all covariant phase observables by optimality requirements; see, for instance, [16], [46], and [59]. As shown, for instance, in [22], the canonical phase observable does not satisfy condition (a). The fact that it is approximately sharp is proved in [22] and Article I.

2.3 Measurement imprecision and fuzzy sets

Fuzzy sets have been used to model imprecision, uncertainty, and vagueness in numerous situations. They have found applications, for instance, in engineering, economics and sociology. Fuzzy sets are also suitable for describing measurement imprecision. We briefly recall the basic terminology of fuzzy sets as introduced by L. Zadeh in his seminal articles [74] and [75]. With those concepts we formulate the definition of a fuzzification of an observable.

Let Ω be a set. A *fuzzy set* in Ω is a function $\widetilde{X}$ from Ω to the interval $[0,1]$. The value $\widetilde{X}(\omega)$ is interpreted as the grade of membership of a point ω in $\widetilde{X}$. The values 0 and 1 correspond to full membership and full nonmembership, respectively. The values between 0 and 1 indicate a partial membership. An ordinary set $X \subseteq \Omega$ is identified with the characteristic function χ_X of X taking only values 0 and 1. In this way the set X can be thought of as a special kind of fuzzy set.

Let $\mathcal{A}$ be a σ-algebra of subsets of Ω, so that $(\Omega, \mathcal{A})$ is an outcome space. A fuzzy set $\widetilde{X}$ is called a *fuzzy event* if it is a measurable function. We denote the collection of fuzzy events by $\widetilde{\mathcal{A}}$. If m is a probability measure and $\widetilde{X} \in \widetilde{\mathcal{A}}$, the probability $m(\widetilde{X})$ is defined as

$$m(\widetilde{X}) = \int \widetilde{X}(\omega)\ dm(\omega). \tag{2.15}$$

Each event $X \in \mathcal{A}$ is also a fuzzy event (when identified with the characteristic function χ_X), and in this case equation (2.15) gives the usual probability $m(X)$.

We model measurement imprecision by a mapping Λ from events to fuzzy events,

$$\Lambda : \mathcal{A} \to \widetilde{\mathcal{A}}, \quad X \mapsto \Lambda(X).$$

The mapping Λ should have some kind of consistency properties. We require that

(i) $\Lambda(X') = \chi_\Omega - \Lambda(X)$;

(ii) $\sum_{i=1}^{\infty} \Lambda(X_i) = \chi_\Omega$ whenever $(X_i) \subseteq \mathcal{A}$ is a sequence of pairwise disjoint sets such that $\cup_{i=1}^{\infty} X_i = \Omega$.

Condition (i) means that complements are mapped to fuzzy complements while (ii) means that a partition is mapped to a fuzzy partition. We call a mapping Λ with the properties (i) and (ii) a *confidence mapping.*

Let Λ be a confidence mapping and m a probability measure. The properties (i) and (ii) imply that the composite mapping $m \circ \Lambda$ is also a probability measure.

Definition 1. Let E and F be observables with the outcome space $(\Omega, \mathcal{A})$. If there exists a confidence mapping $\Lambda : \mathcal{A} \to \widetilde{\mathcal{A}}$ such that for any state $T \in \mathcal{S}(\mathcal{H})$,

$$p_T^F = p_T^E \circ \Lambda, \tag{2.16}$$

we say that F is a *fuzzification* of E (or a *fuzzy version* of E).

Another form of (2.16) can be written using a Markov kernel. A mapping $\lambda : \Omega \times \mathcal{A} \to [0, 1]$ is a *Markov kernel* if

(i) for every $\omega \in \Omega$, the mapping $\lambda(\omega, \cdot)$ is a probability measure;

(ii) for every $X \in \mathcal{A}$, the mapping $\lambda(\cdot, X)$ is measurable.

Thus, λ is a Markov kernel exactly when the mapping $X \mapsto \lambda(\cdot, X)$ is a confidence mapping. Therefore, F is a fuzzy version of E if and only if there exists a Markov kernel λ such that

$$F(X) = \int \lambda(\omega, X)\ dE(\omega), \qquad X \in \mathcal{A}. \tag{2.17}$$

Some relevant properties of the Markov kernels and their use in the description of measurement inaccuracy have been discussed, for instance, in [9], [10], and [44].

2.4 Fuzzy outcome space

The distinction between the concepts of a mathematical point and a physical point has been realized for a long time. Influential applications of this idea in quantum mechanics were developed in the articles of Ali and Doebner [3], Ali and Emch [4], and Ali and Prugovečki [5]. This formalism and its applications have also been discussed in [1] and [61].

Intuitively, close points in a physical space cannot be distinguished with certainty. Therefore, the actual location of a point is not known precisely. This uncertainty depends on the imprecision of the measurement in question. This idea is formalized in the following way. For any point $\omega \in \Omega$, we define a probability measure ρ_ω on Ω to describe the uncertainty in the location of ω. The actual location of a point ω is in a set X with the probability $\rho_\omega(X)$. We also assume that, for each $X \in \mathcal{A}$, the mapping $\omega \mapsto \rho_\omega(X)$ is measurable. The set of the pairs (ω, ρ_ω) is a *fuzzy outcome space*. The ideal case of absolute precision corresponds to the choice of the point measures δ_ω.

Let E be an observable with the outcome space $(\Omega, \mathcal{A})$. The observable E can be written in the form

$$E(X) = \int \delta_\omega(X)\ dE(\omega). \tag{2.18}$$

Assume then that more inaccuracy is involved. If the imprecision is characterized by the fuzzy outcome space $\{(\omega, \rho_\omega) \mid \omega \in \Omega\}$, then the point measures δ_ω in (2.18) are replaced with the probability measures ρ_ω. As a result we get an observable F, defined as

$$F(X) = \int \rho_\omega(X)\ dE(\omega). \tag{2.19}$$

We conclude that the idea of a fuzzy outcome space leads to the same formulation as we had in Section 2.3.

2.5 Standard model of measurement theory

From the point of view of quantum measurement theory, the fuzzification formalism described in the last two sections would not be so interesting without a useful measurement theoretical model. One of the most important measurement models, also called *the standard model of quantum measurement theory*, leads to the format of Sections 2.3 and 2.4. This kind of measurement model can be traced back to the book of J. von Neumann [69]. Since then it has been used in various applications. The properties of the standard model are investigated in [16, Section II.3], [19], and [20], which also contain further references. For a detailed exposition of quantum measurement theory, we refer to [21].

Suppose that our aim is to measure a sharp observable Π of the object system, described by a Hilbert space $\mathcal{H}$. We assume that Π has the outcome space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, and we denote by A the selfadjoint operator corresponding to Π. The object system is coupled with a measurement apparatus, or a probe, which we describe by a Hilbert space $\mathcal{K}$. The measurement interaction is given by a unitary operator U on $\mathcal{H} \otimes \mathcal{K}$, defined as

$$U = e^{iA \otimes B}, \tag{2.20}$$

where B is a selfadjoint operator in $\mathcal{K}$. Let T_i and S_i be the initial states of the object system and the apparatus, respectively, and let Z be the pointer observable being applied on the apparatus. We assume that, prior to measurement, the system and the apparatus are independent of each other, and therefore, the initial state of the compound system is $T_i \otimes S_i$. The final state of the compound system after the measurement interaction is $U(T_i \otimes S_i)U^*$, and the final state S_f of the apparatus is the reduced state, obtained by the partial trace over $\mathcal{H}$.

The probability reproducibility condition

$$p^F_{T_i}(X) = p^Z_{S_f}(X), \qquad X \in \mathcal{B}(\mathbb{R}), \tag{2.21}$$

required to hold for all possible initial states T_i of the object system, defines the actually measured observable F. A direct computation shows that

$$F(X) = \int \mathrm{tr}[e^{iaB} S_i e^{-iaB} Z(X)] \, d\Pi(a). \tag{2.22}$$

Thus, the observable F is a fuzzification of the sharp observable Π. The properties of F depend on the actual choices of the initial state S_i, the operator B and the pointer observable Z. The ideal case $F = \Pi$ can be achieved only if Π is a discrete[5] observable; see e.g. [20].

2.6 Coarse-graining

The concept of coarse-graining means, informally, a reduction in the statistical description of a system. It can be formulated in a general way, also

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[5] An observable E with the outcome space $(\Omega, \mathcal{A})$ is discrete if there is a countable set $\Omega_0 \subseteq \Omega$ such that $E(\Omega_0) = I$. The sharp observable Π is discrete exactly when the operator A has a complete set of eigenvectors.

between different statistical theories as presented, for instance, in [23] and [62]. Here we use a restricted definition suitable for the present purposes.

An observable E with the outcome space $(\Omega, \mathcal{A})$ defines a linear mapping V_E from the set of trace-class operators $\mathcal{T}(\mathcal{H})$ to the set of complex measures $M(\Omega, \mathcal{A})$ by the formula

$$V_E(T)(X) = \mathrm{tr}[TE(X)], \qquad X \in \mathcal{A}. \tag{2.23}$$

In particular, if T is a state, then $V_E(T)$ is the probability measure p_T^E. The mapping V_E determines the observable E in the sense that for another observable F, the equality $V_E = V_F$ holds exactly when $E = F$. For more about the properties of the mapping V_E, we refer to [15], [29], and [64].

Definition 2. Let E and F be observables with outcome spaces $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$, respectively. The observable F is a *coarse-graining* of E if there exists a linear mapping $\Psi : M(\Omega_1, \mathcal{A}_1) \to M(\Omega_2, \mathcal{A}_2)$ such that

(i) for any probability measure m on $(\Omega_1, \mathcal{A}_1)$, $\Psi(m)$ is a probability measure on $(\Omega_2, \mathcal{A}_2)$;

(ii) V_F is the composite mapping of V_E and Ψ, that is, $V_F = \Psi \circ V_E$.

In Example 2 we show that every fuzzification is an instance of the coarse-graining procedure. Another case of coarse-graining, which we encounter later, is discussed in Example 3.

Example 2. Let E and F be observables with the outcome space $(\Omega, \mathcal{A})$. Assume that F is a fuzzification of E, and let λ be a corresponding Markov kernel on $\Omega \times \mathcal{A}$. We define a mapping Ψ on $M(\Omega, \mathcal{A})$ by

$$\Psi(m)(X) = \int \nu(\omega, X)\, dm(\omega), \qquad m \in M(\Omega, \mathcal{A}),\ X \in \mathcal{A}. \tag{2.24}$$

Clearly, Ψ is linear, and equation (2.17) implies that condition (ii) holds. We conclude that whenever F is a fuzzification of E, then F is also a coarse-graining of E.

Example 3. Let E and F be observables with outcome spaces $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$, respectively. Assume that F is a function of E, meaning that there is a measurable function $f : \Omega_1 \to \Omega_2$ such that

$$F(X) = E(f^{-1}(X)), \qquad X \in \mathcal{A}_2. \tag{2.25}$$

We define a linear mapping $\Psi : M(\Omega_1, \mathcal{A}_1) \to M(\Omega_2, \mathcal{A}_2)$ by

$$\Psi(m)(X) = m(f^{-1}(X)) \qquad m \in M(\Omega_1, \mathcal{A}_1),\ X \in \mathcal{A}_2. \tag{2.26}$$

Then conditions (i) and (ii) hold, and thus, F is a coarse-graining of E.

A simple but important fact is that a coarse-graining of an observable E cannot give more information than E. The information given by an observable is here understood as its ability to distinguish states. An observable E distinguishes two states T_1 and T_2 if the corresponding probability distributions $p^E_{T_1}$ and $p^E_{T_2}$ are different. This leads to the following definitions [32], [60].

Definition 3. Let E and F be two observables.

(a) If E distinguishes every pair of states which are distuinguished by F, then the *state distinction power* of E is greater than or equal to that of F. This means that for all states $T_1, T_2 \in \mathcal{S}(\mathcal{H})$,

$$p^E_{T_1} = p^E_{T_2} \implies p^F_{T_1} = p^F_{T_2}. \tag{2.27}$$

(b) The observables E and F are *informationally equivalent* if they distinguish exactly the same states, i.e., for all states $T_1, T_2 \in \mathcal{S}(\mathcal{H})$,

$$p^E_{T_1} = p^E_{T_2} \iff p^F_{T_1} = p^F_{T_2}. \tag{2.28}$$

If F is a coarse-graining of E, then for any state T, the probability measure p^F_T is the composite mapping $\Psi \circ p^E_T$. Obviously, in this case condition (2.27) holds and hence, the state distinction power of E is greater than or equal to that of F.

2.7 Optimal observables

Let us consider a typical situation where the aim is to measure some quantity of the system as accurately as possible. The intended measurement determines the measurement outcome space $(\Omega, \mathcal{A})$. Let $\mathcal{O}(\Omega, \mathcal{A}, \mathcal{H})$ denote the set of observables with the outcome space $(\Omega, \mathcal{A})$. Typically, we have some requirements and presumptions for the intended measurement, and therefore, only a restricted class $\mathcal{O} \subseteq \mathcal{O}(\Omega, \mathcal{A}, \mathcal{H})$ of observables are relevant. The problem is then to find an optimal observable from the set $\mathcal{O}$.

Naturally, an observable can be optimal in various ways. The state distinction power of an observable is frequently used as a criterion for optimality. The concepts of fuzzification and coarse-graining provide other formulations of optimality criteria. Indeed, let E and F be two observables with the outcome space $(\Omega, \mathcal{A})$. We denote

(i) $F \preccurlyeq_f E$, if F is a fuzzification of E;

(ii) $F \preccurlyeq_c E$, if F is a coarse-graining of E.

Then $\preccurlyeq_f$ and $\preccurlyeq_c$ are reflexive and transitive relations on $\mathcal{O}(\Omega, \mathcal{A}, \mathcal{H})$, and hence, they define partial orderings in the respective sets of equivalence classes; see Article VI for details. Optimality is formulated as the following maximality condition.

Definition 4. An observable $E \in \mathcal{O}$ is *optimal* in $\mathcal{O}$ with respect to preordering $\preccurlyeq$ (or $\preccurlyeq$-*optimal* in $\mathcal{O}$), if for any $F \in \mathcal{O}$, the condition $E \preccurlyeq F$ implies that $F \preccurlyeq E$.

Optimality criteria corresponding to $\preccurlyeq_f$, $\preccurlyeq_c$, and some similar preorderings have been investigated in [11], [34], [36], [57], and Article VI. In certain cases the optimal observables have been chracterized. We do not report the results of these articles here. Instead, we describe the connection of optimality criteria and the property of an observable being approximately sharp.

Let E be an observable with the outcome space $(\Omega, \mathcal{A})$. Each event $X \in \mathcal{A}$ defines an observable E_X corresponding to a measurement where we register only if the measurement outcome belongs to X or not. We index these two possibilities by 1 and 0, and hence, the observable E_X is defined by

$$E_X(\{1\}) = E(X), \quad E_X(\{0\}) = E(X'). \tag{2.29}$$

These kinds of two valued observables are called 1-0 observables.

Proposition 1. *Let E be an observable with the outcome space $(\Omega, \mathcal{A})$. The following conditions are equivalent:*

(i) E is approximately sharp;

(ii) for every $X \in \mathcal{A}$ with $O \neq E(X) \neq I$, the observable E_X is $\preccurlyeq_f$-optimal in the set of 1-0 observables;

(iii) for every $X \in \mathcal{A}$ with $O \neq E(X) \neq I$, the observable E_X is $\preccurlyeq_c$-optimal in the set of 1-0 observables.

The proof of Proposition 1 is given in Section 5.1 of Article VI.

Chapter 3

Covariance and imprecision

3.1 Symmetry in quantum mechanics

Symmetry has many different roles in science, ranging from a tool in practical calculations to a guideline in the foundations of philosophy of science. It can be convincingly argued that symmetry is the basic principle in theoretical science; see, for instance, [67].

The mathematical theory of symmetry, group theory, has been extensively used in many different ways in quantum mechanics. Group theory has become an indispensable tool both in concrete problems and in the study of the conceptual structure of quantum mechanics. This fundamental role of group theory in quantum mechanics is well illustrated, for instance, in the classical books of H. Weyl [73] and G. Mackey [55].

For the purposes of this chapter, we briefly recall the formulation of symmetry in quantum mechanics. A detailed account of this subject is presented in [26].

An automorphism of a set is, generally speaking, a structure preserving bijective mapping. The relevant structure depends on the context. The Hilbert space formulation of quantum mechanics bears several relevant structures. The convex structure of the set of states and the partial sum structure of the set of effects are both physically important structures, as discussed in Section 2.1. There are also other relevant structures as, for instance, the set of pure states with the transition probability structure. Each of them defines the corresponding automorphism group. However, the automorphism groups related to these different structures are all naturally isomorphic and homeomorphic; see, for instance, [26, Chapter 2]. Therefore, any one of them can be chosen to describe symmetry without a bias.

A *state automorphism* is a bijective mapping on $\mathcal{S}(\mathcal{H})$ preserving the convex combinations of states. Let $\mathcal{U}(\mathcal{H})$ denote the set of unitary operators, $\overline{\mathcal{U}}(\mathcal{H})$ the set of antiunitary operators, and $\mathbb{T}$ the set of complex numbers of modulus one. The union $\mathcal{U}(\mathcal{H}) \cup \overline{\mathcal{U}}(\mathcal{H})$ is a topological group with respect to the operator multiplication and the strong operator topology. An operator $U \in \mathcal{U}(\mathcal{H}) \cup \overline{\mathcal{U}}(\mathcal{H})$ defines a state automorphism by the formula

$$T \mapsto UTU^*, \qquad T \in \mathcal{S}(\mathcal{H}). \tag{3.1}$$

As shown, for instance, in [26, Chapter 2], every state automorphism can be written in this form. Two operators U_1 and U_2 define the same state automorphism exactly when $U_1 = zU_2$ for some $z \in \mathbb{T}$. Therefore, the group of state automorphisms is isomorphic to the quotient group $\Sigma(\mathcal{H}) := \mathcal{U}(\mathcal{H}) \cup \overline{\mathcal{U}}(\mathcal{H})/\mathcal{Z}$, where $\mathcal{Z} := \{zI \mid z \in \mathbb{T}\}$. The group $\Sigma(\mathcal{H})$ is called *the automorphism group of quantum mechanics.*

3.2 Covariant observables

The behavior of an observable under appropriate symmetry transformations determines to a large extent its structure. The symmetry properties of an observable have, in many cases, physical interpretations and they can be taken as defining properties. In this kind of approach observables need not be ideal and absolutely accurate, but also the cases with finite measurement precision can be investigated.

Suppose that $\mathcal{G}$ is a group describing some symmetry properties of the system. The associated symmetry in the set of states of the system is described by a *symmetry action*, that is, a group homomorphism $\sigma : g \mapsto \sigma_g$ from $\mathcal{G}$ to the automorphism group $\Sigma(\mathcal{H})$. Two states T_1 and T_2 are equivalent with respect to the symmetry action σ if, for some $g \in \mathcal{G}$,

$$T_1 = \sigma_g(T_2). \tag{3.2}$$

In the following discussion we assume that the symmetry group $\mathcal{G}$ is a topological group and that the symmetry action σ is a continuous mapping.

Let E be an observable with the outcome space $(\Omega, \mathcal{A})$. In the simplest case the measurement outcome distribution of the observable E does not change at all when the symmetry action σ is applied. This means that for every $T \in \mathcal{S}(\mathcal{H})$ and $g \in \mathcal{G}$,

$$p^E_{\sigma_g(T)} = p^E_T. \tag{3.3}$$

This motivates the following definition.

Definition 5. An observable E is *invariant* with respect to a symmetry action σ if

$$\sigma_g(E(X)) = E(X) \tag{3.4}$$

for every $g \in \mathcal{G}$ and $X \in \mathcal{A}$.

Generally, a symmetry action has an influence on a measurement outcome distribution. This change can be consistent, reflecting the symmetry properties of the observable. To formulate this idea, suppose that the outcome space of the observable E is $(\Omega, \mathcal{B}(\Omega))$, where Ω is a topological space and $\mathcal{B}(\Omega)$ is its Borel σ-algebra. We assume that the symmetry group $\mathcal{G}$ is a *transformation group* on Ω. This means that for every $g \in \mathcal{G}$, there is a homeomorphism τ_g on Ω such that the mapping $g \mapsto \tau_g$ is a group homomorphism and the mapping $(g, \omega) \mapsto \tau_g(\omega)$ from $\mathcal{G} \times \Omega$ to Ω is continuous. In a transformation corresponding to a $g \in \mathcal{G}$, an event $X \in \mathcal{B}(\Omega)$ is mapped to an event $\tau_g(X) := \{\tau_g(\omega) \mid \omega \in X\}$. A symmetry requirement for the observable E is that the probability distribution for a changed state is the same as the original probability distribution for a transformed event, i.e., for every $T \in \mathcal{S}(\mathcal{H})$ and $X \in \mathcal{B}(\Omega)$,

$$p^E_{\sigma_g(T)}(X) = p^E_T(\tau_g(X)). \tag{3.5}$$

This leads to the following definition.

Definition 6. An observable E is *covariant* with respect σ and τ if

$$\sigma_g(E(X)) = E(\tau_g(X)) \tag{3.6}$$

for every $g \in \mathcal{G}$ and $X \in \mathcal{B}(\Omega)$.

The idea of covariance as a defining property of an observable has been applied in the analysis of localization observables, phase space observables, screen observables, time observables, phase observables and various other cases. For a review, we refer to [16] and [46].

3.3 Transitive systems of covariance

For an important class of covariant observables satisfying certain additional assumptions, a powerful mathematical theory has been developed. Here we confine ourselves only to a sketchy summary of some basic facts.

Let us begin with the situation of Definition 6, where $\mathcal{G}$ is a transformation group on Ω, σ is a symmetry action of $\mathcal{G}$ and E is a covariant observable with the outcome space $(\Omega, \mathcal{B}(\Omega))$.

We assume that the transformation group $\mathcal{G}$ is *transitive* on Ω. This means that for any two points $\omega_1, \omega_2 \in \Omega$, there is a $g \in \mathcal{G}$ such that $\tau_g(\omega_1) = \omega_2$. Moreover, we require that $\mathcal{G}$ is locally compact, separable and metrizable. It follows that there is a closed subgroup $\mathcal{G}_0$ of $\mathcal{G}$ such that Ω can be homeomorphically identified with the quotient space $\mathcal{G}/\mathcal{G}_0$. Indeed, fix $\omega_0 \in \Omega$ and denote the associated stability subgroup by $\mathcal{G}_0$, that is, $\mathcal{G}_0 = \{g \in \mathcal{G} \mid \tau_g(\omega_0) = \omega_0\}$. The mapping $g \mapsto \tau_g(\omega_0)$ from G to Ω is continuous and constant in every left coset of $\mathcal{G}_0$. In addition, as $\mathcal{G}$ is transitive on Ω, the mapping is surjective. Therefore, the equation $\theta(g\mathcal{G}_0) = \tau_g(\omega)$ defines a continuous and bijective mapping θ from $\mathcal{G}/\mathcal{G}_0$ to Ω. Since the group $\mathcal{G}$ is σ-compact, the mapping θ is a homeomorphism; see, for instance, [39, Section 2.6]. From now on we identify Ω with the quotient space $\mathcal{G}/\mathcal{G}_0$. In this identification the mapping τ_g is identified with the left multiplication by g, which we denote by $\omega \mapsto g \cdot \omega$.

We also assume that there is a unitary representation[1] U of $\mathcal{G}$ such that the symmetry action σ can be given in the form

$$\sigma_g(T) = U_g T U_g^*, \qquad g \in \mathcal{G}, T \in \mathcal{S}(\mathcal{H}). \tag{3.7}$$

Under the previous assumptions Definition 6 leads to the following concept.

Definition 7. The pair (U, E) of a unitary representation U of $\mathcal{G}$ and an observable E with the outcome space $(\Omega, \mathcal{B}(\Omega))$ is a *transitive system of covariance* if

$$U_g E(X) U_g^* = E(g \cdot X) \tag{3.8}$$

for every $g \in \mathcal{G}$ and $X \in \mathcal{B}(\Omega)$.

If E is a sharp observable in Definition 7, then the pair (U, E) is known as a *transitive system of imprimitivity.* The concept of a transitive system of imprimitivity originates from the representation theory of finite groups. In the case of separable locally compact groups and infinite dimensional unitary representations, transitive systems of imprimitivity were systematically studied by G. Mackey, and the main results were presented in his articles [53] and [54]. An exposition of the theory and its applications is given in [56].

The imprimitivity theorem states that there exists a transitive system of imprimitivity (U, Π) if and only if the unitary representation U is equivalent to a representation induced from some unitary representation of the subgroup $\mathcal{G}_0$. Each induced representation has an associated sharp observable having a canonical form and they constitute a canonical system of imprimitivity. Any transitive system of imprimitivity is unitarily equivalent to such

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[1] By a unitary representation we mean a weakly (equivalently, strongly) continuous group homomorphism from $\mathcal{G}$ to $\mathcal{U}(\mathcal{H})$.

a canonical system of imprimitivity. Thus, if the representation U is fixed, there exists either none or, up to unitary equivalence, a unique transitive system of imprimitivity. A comprehensive treatment of transitive systems of imprimitivity and their applications in quantum mechanics is given in the book of V. Varadarajan [68].

Generalizations of the imprimitivity theorem for transitive systems of covariance have been investigated, for instance, in [25], [30], [31], [32], and [63]. For a review, we refer to [2]. The basic result related to transitive systems of covariance, known as the generalized imprimitivity theorem, is a covariant version of the Naimark dilation theorem. According to the generalized imprimitivity theorem there exists a transitive system of covariance (U, E) if and only if U is equivalent to a subrepresentation of a representation induced from a unitary representation of the subgroup $\mathcal{G}_0$. In this case there exists a Hilbert space $\mathcal{K}$, an isometric mapping $\Phi : \mathcal{H} \to \mathcal{K}$ and a canonical system of imprimitivity $(\widetilde{U}, \Pi)$ acting on $\mathcal{K}$ such that

$$\Phi U_g = \widetilde{U}_g \Phi, \qquad g \in G, \tag{3.9}$$

$$E(X) = \Phi^* \Pi(X) \Phi, \qquad X \in \mathcal{B}(\Omega). \tag{3.10}$$

Moreover, there exists a minimal dilation in the sense that the linear hull of the set

$$\{\Pi(X)\Phi\psi \mid X \in \mathcal{B}(\Omega), \psi \in \mathcal{H}\} \tag{3.11}$$

is dense in $\mathcal{K}$, and in this case the pair $(\widetilde{U}, \Pi)$ is unique up to a unitary equivalence.

In some cases transitive systems of covariance have been classified. In the case of a finite dimensional representation of a compact group, a characterization is given in [32]. For a general representation of a compact group, a characterization is presented in [48]. The case when the symmetry group $\mathcal{G}$ is Abelian and $\Omega = \mathcal{G}$ is treated in [47]. The general case of an Abelian symmetry group is solved in [28]. The case when U is an irreducible representation and $\mathcal{G}_0$ is a central subgroup is treated in [27]. These latter two cases are also discussed in [66].

3.4 Uniform fuzzy outcome space

In this section we combine the discussion of fuzzy outcome spaces from Section 2.4 with the concept of covariant observables.

Let us assume that $\mathcal{G}_0$ is a normal subgroup of $\mathcal{G}$, and Ω is the quotient group $\mathcal{G}/\mathcal{G}_0$. The group structure of Ω provides a simple way to model situations where the imprecision related to different points of Ω are similar.

We recall from Section 2.4 that a fuzzy outcome space is a set of the pairs (ω, ρ_ω), where ρ_ω is a probability measure describing the uncertainty in the location of a point ω. Let $\rho \equiv \rho_e$ be the probability measure on $\mathcal{B}(\Omega)$ describing the imprecision associated with the identity element e. We require that the probability measure ρ_ω related to a point ω is the translated probability measure $X \mapsto \rho(\omega^{-1}X)$. For a fixed $X \in \mathcal{B}(\Omega)$, the mapping $\omega \mapsto \rho(\omega^{-1}X)$ is measurable; see e.g. Lemma 1 in Article III. Thus, the probability measure ρ defines a fuzzy outcome space. Since the imprecision related to any point in Ω is the same, we call this kind of construction a *uniform fuzzy outcome space.*

Let (U, E) be a transitive system of covariance and let F be a fuzzification of E, corresponding to a uniform fuzzy outcome space defined by a probability measure ρ. Then F is given by the formula

$$F(X) = \int \rho(\omega^{-1}X)\ dE(\omega), \qquad X \in \mathcal{B}(\Omega). \tag{3.12}$$

For a state $T \in \mathcal{S}(\mathcal{H})$, the probability measure p_T^F can be written in the form

$$p_T^F = p_T^E * \rho, \tag{3.13}$$

where $p_T^E * \rho$ is the convolution of the measures p_T^E and ρ. The covariance of E implies that also F is covariant. Indeed, for every $T \in \mathcal{S}(\mathcal{H})$, $g \in \mathcal{G}$, and $X \in \mathcal{B}(\Omega)$, we have

$$\begin{aligned} p^F_{U_g^* T U_g}(X) &= \int \rho(\omega^{-1}X)\ dp^E_{U_g^* T U_g}(\omega) = \int \rho((g^{-1}\cdot\omega)^{-1}X)\ dp_T^E(\omega) \\ &= \int \rho(\omega^{-1}(g\cdot X))\ dp_T^E(\omega) = p_T^F(g\cdot X). \end{aligned}$$

We conclude that every probability measure ρ defines a fuzzy version F of E, and (U, F) is a transitive system of covariance.

A special instance of the preceding construction appears when ρ is the point measure δ_{ω_0} of some $\omega_0 \in \Omega$. The uniform fuzzy outcome space associated to the point measure δ_{ω_0} does not introduce actual imprecision; rather, points are just translated by ω_0. The corresponding fuzzification F of E is given by

$$F(X) = E(X\omega_0^{-1}), \qquad X \in \mathcal{B}(\Omega), \tag{3.14}$$

and the observable F is called a *translation of* E. The observables E and F are equivalent in the sense that E is a fuzzification of F.

3.5 Covariant fuzzifications of a sharp observable

Let us take a closer look at uniform fuzzy outcome spaces in the case of a sharp observable. Let (U, Π) be a transitive system of imprimitivity. As discussed in Section 3.4, every probability measure ρ on $\mathcal{B}(\Omega)$ defines a covariant fuzzification of the sharp observable Π. In the present case also the contrary holds, namely, every covariant fuzzification has this form.

Proposition 2. *For a covariant observable F, the following conditions are equivalent:*

- *(i) F is a fuzzification of Π;*
- *(ii) F is a coarse-graining of Π;*
- *(iii) there is a probability measure ρ such that*

$$F(X) = \int \rho(\omega^{-1}X)\ d\Pi(\omega), \qquad X \in \mathcal{B}(\Omega). \tag{3.15}$$

The proof of Proposition 2 is given in Section 4 of Article III, and it is based on Wendel's result on left centralizers [70].

Next we discuss some properties of a covariant fuzzification F of Π. The proofs of Propositions 3, 4 and 5 are given in Section 5 of Article III.

Obviously, every function of a sharp observable is also sharp. In particular, every translation of Π is a sharp observable. There are no other covariant sharp observables that are fuzzifications of Π. Indeed, we have the following result.

Proposition 3. *Let F be a covariant observable which is a fuzzification of Π. Then F is approximately sharp if and only if F is a translation of Π.*

Apart from the translations of Π, there are fuzzifications of Π which are regular observables. Namely, let $\omega_1, \omega_2 \in \Omega$ be distinct points, and let F be a convex combination of the translations of Π,

$$F(X) = t\Pi(X\omega_1^{-1}) + (1-t)\Pi(X\omega_2^{-1}), \qquad X \in \mathcal{B}(\Omega). \tag{3.16}$$

If $t \neq 0, 1$, then F is not a sharp observable. Moreover, if $t \neq \frac{1}{2}$, then F is a regular observable; see Example 4 in Article III for details. Proposition 4 shows that, after all, in many cases regularity is not attained.

Proposition 4. *Let* Ω *be non-discrete group and* F *a covariant fuzzification of* Π*, corresponding to a probability measure* ρ*. If* ρ *is absolutely continuous with respect to the Haar measure* μ_Ω *of* Ω*, then* F *is not regular.*

To formulate the next proposition, we assume that the quotient group Ω is Abelian. We denote by $\widehat{\Omega}$ the dual group of Ω, and the Fourier-Stieltjes transform of a probability measure ρ is denoted by $\widehat{\rho}$.

Proposition 5. *Let* F *be a covariant fuzzy version of* Π*, corresponding to a probability measure* ρ*. The observables* Π *and* F *are informationally equivalent if and only if the support of the function* $\widehat{\rho}$ *is* $\widehat{\Omega}$*.*

Chapter 4

Position and momentum observables

4.1 Definition of position and momentum observables

Let us consider a particle moving in the real line $\mathbb{R}$. The physical description of the particle should be essentially the same in two reference frames differing by a constant motion or a translation. Therefore, we choose the symmetry group of the particle to be the collection of space translations and velocity boosts.

Let us describe the particle by a Hilbert space $\mathcal{H}$, and denote by σ the symmetry action corresponding to the transformations consisting of a velocity boost followed by a space translation. Hence, σ is a continuous group homomorphism from $\mathbb{R}^2$ to the automorphism group $\Sigma(\mathcal{H})$. For every $q, p \in \mathbb{R}$, there is an operator $W_{q,p} \in \mathcal{U}(\mathcal{H}) \cup \overline{\mathcal{U}}(\mathcal{H})$ such that

$$\sigma_{q,p}(T) = W_{q,p} T W_{q,p}^*, \qquad T \in \mathcal{S}(\mathcal{H}). \tag{4.1}$$

Since the group $\mathbb{R}^2$ is connected, each $W_{q,p}$ is a unitary operator. For any $q, p \in \mathbb{R}$, we denote $U_q = W_{q,0}$ and $V_p = W_{0,p}$. These unitary operators correspond to a space translation and a velocity boost, respectively. The automorphism $\sigma_{q,p}$ does not change if $W_{q,p}$ is multiplied by a complex number of unit modulus. As the group $\mathbb{R}$ has only exact multipliers, we can choose $W_{q,p}$ in such a way that the mappings $U : q \mapsto U_q$ and $V : p \mapsto V_p$ are unitary representations of $\mathbb{R}$.

It is natural to require that the measurement outcome distribution of the position of the particle should be shifted if the location of the measuring apparatus is changed. This can be thought of as a basic property of any position measurement. In addition, the measurement outcome distribution should be the same irrespective of the motion of the measuring apparatus. We take these two features as the defining properties of a position observable.

Definition 8. An observable E with the outcome space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a *position observable* if, for all $q, p \in \mathbb{R}$ and $X \in \mathcal{B}(\mathbb{R})$,

$$U_q E(X) U_q^* = E(X+q), \tag{4.2}$$

$$V_p E(X) V_p^* = E(X). \tag{4.3}$$

A momentum observable should behave differently in the symmetry transformations. Namely, a measurement of the momentum of the particle should give the same result if the location of the measuring apparatus is changed. However, if the measuring apparatus is put in a motion with a constant velocity, then the outcome probability distribution should be shifted. The shift depends on the mass m of the particle. Here we set $m = 1$, and therefore, we arrive to the following definition.

Definition 9. An observable F with the outcome space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a *momentum observable* if, for all $q, p \in \mathbb{R}$ and $X \in \mathcal{B}(\mathbb{R})$,

$$V_p F(X) V_p^* = F(X+p), \tag{4.4}$$

$$U_q F(X) U_q^* = F(X). \tag{4.5}$$

The concrete descriptions of the position and momentum observables depend on the form of the unitary representations U and V. We assume that the system is *elementary* with respect to the symmetry action σ, meaning that for any pure state T_0 of the particle, every other pure state is a superposition of some pure states of the form $\sigma_{q,p}(T_0)$, $q, p \in \mathbb{R}$. This assumption is equivalent to the requirement that the symmetry action σ is irreducible; see, for instance, [26]. For any irreducible symmetry action of $\mathbb{R}^2$, there is a corresponding irreducible unitary representation of the Heisenberg group. By the Stone-von Neumann Theorem, these representation are well known; see e.g. [38]. We assume that the particle can be in at least two different pure states, and thus, the Hilbert space $\mathcal{H}$ is not one-dimensional.

The preceding assumptions fix, essentially, the form of U and V. We may choose $\mathcal{H} = L^2(\mathbb{R})$, and the unitary representations U and V act on $\psi \in \mathcal{H}$ as

$$\left[U_q \psi\right](x) = \psi(x-q), \tag{4.6}$$

$$\left[V_p \psi\right](x) = e^{ipx} \psi(x). \tag{4.7}$$

By Stone's theorem there are unique selfadjoint operators P and Q generating the unitary representations U and V, i.e., $U_q = e^{-iqP}$ and $V_p = e^{ipQ}$ for every $q, p \in \mathbb{R}$. We denote by Π_P and Π_Q the sharp observables corresponding to the operators P and Q, respectively. It follows from (4.6) that the sharp observable Π_Q has the form

$$[\Pi_Q(X)\psi](x) = \chi_X(x)\psi(x), \tag{4.8}$$

and it is directly verified that Π_Q satisfies the symmetry conditions (4.2) and (4.3). Therefore, Π_Q is a position observable, and, up to unitary equivalence, it is the only sharp position observable. We call it the *canonical position observable.*

There are also other position observables than the canonical position observable. Namely, as discussed in Section 3.4, a probability measure ρ on $\mathcal{B}(\mathbb{R})$ defines a covariant fuzzification E_ρ of the canonical position observable Π_Q by the formula

$$E_\rho(X) = \int \rho(X - q)\ d\Pi_Q(q), \qquad X \in \mathcal{B}(\mathbb{R}). \tag{4.9}$$

It is obvious from the structure of the observable E_ρ that it satisfies also the invariance requirement (4.3), and thus, E_ρ is a position observable. These kinds of observables correspond to imprecise position measurements and they have been discussed, for instance, in [1], [16], and [33].

Apart from the fuzzifications of Π_Q, there exist also other observables satisfying the covariance condition (4.2). For a classification of the translation covariant observables, see, for instance, [28]. There are even noncommutative observables, which cannot be fuzzifications of any sharp observable. However, since we require also the invariance in the definition of position observables, these cases are ruled out. The proof[1] of Proposition 6 is given in Appendix A of Article IV.

Proposition 6. *Any position observable E is of the form $E = E_\rho$ for a unique probability measure ρ.*

Let $\mathcal{F}$ denote the Fourier-Plancherel transformation on $\mathcal{H}$. Since $\mathcal{F}U(q) = V(-q)\mathcal{F}$ and $\mathcal{F}V(p) = U(p)\mathcal{F}$, an observable E is a position observable if and only if the mapping $X \mapsto \mathcal{F}^{-1}E(X)\mathcal{F}$ is a momentum observable. Therefore, the previous discussion on position observables is directly applicable to the case of momentum observables. The sharp observable Π_P is given by

$$\Pi_P(X) = \mathcal{F}^{-1}\Pi_Q(X)\mathcal{F}, \tag{4.10}$$

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[1] The uniqueness of ρ is proved in Section 3 of Article V.

and we call it the *canonical momentum observable.* A probability measure ν on $\mathcal{B}(\mathbb{R})$ defines a momentum observable F_ν by the formula

$$F_\nu(X) = \int \nu(Y - p)\ d\Pi_P(p), \qquad Y \in \mathcal{B}(\mathbb{R}), \tag{4.11}$$

and every momentum observable has this form for some ν.

4.2 Sharpness of a position observable

Besides translation covariance and velocity boost invariance, the canonical position observable Π_Q has still more symmetry properties. Indeed, let $\mathbb{R}_+$ be the set of positive real numbers regarded as a multiplicative group. If $a \in \mathbb{R}_+$ and $X \subseteq \mathbb{R}$, the set $aX = \{ax \mid x \in X\}$ is a dilation of X. With the multiplication, $\mathbb{R}_+$ is a transformation group on $\mathbb{R}$.

The group $\mathbb{R}_+$ has a unitary representation D on $\mathcal{H}$, given by

$$[D(a)\psi](x) = \frac{1}{\sqrt{a}}\psi\left(a^{-1}x\right), \qquad \psi \in \mathcal{H}. \tag{4.12}$$

It is directly verified that for every $a \in \mathbb{R}_+$ and $X \in \mathcal{B}(\mathbb{R})$, we have

$$D(a)\Pi_Q(X)D(a)^* = \Pi_Q(aX). \tag{4.13}$$

This shows that the canonical position observable Π_Q has no scale dependence: any dilation of the outcome space gives a unitarily equivalent observable. We take this property as the following definition.

Definition 10. An observable $E : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ is *covariant under dilations* if there exists a unitary representation $D : \mathbb{R}_+ \to \mathcal{U}(\mathcal{H})$ such that for all $a \in \mathbb{R}_+$ and $X \in \mathcal{B}(\mathbb{R})$,

$$D(a)E(X)D(a)^* = E(aX). \tag{4.14}$$

It is shown in Section II.B of Article IV that a position observable is covariant under dilations exactly when it is sharp. This fact and Proposition 3 are summarized in the following proposition.

Proposition 7. *Let E be a position observable. The following conditions are equivalent:*

(i) E is covariant under dilations;

(ii) E is an approximately sharp observable;

(iii) E is a sharp observable;

(iv) E is a translation of Π.

4.3 Functional coexistence of position and momentum observables

The problem of joint measurability of position and momentum observables in quantum mechanics has a long history, and different viewpoints have been presented; see, for instance, [17] and [18]. Naturally, an analysis of this problem depends on the definitions of a position observable, a momentum observable and a joint measurement. In this section we take the concept of functional coexistence as a formalization of the possibility of measuring two observables together. An operational analysis of this notion is presented in [52], while [50] contains the mathematical results needed here. For two sharp observables, the functional coexistence is equivalent to their commutativity and this case is treated in [68] and [69]. Generally, however, commutativity is not a necessary requirement for functional coexistence.

Intuitively, two observables E_1 and E_2 are jointly measurable if there is a single measurement which allows producing measurement outcome statistics of both E_1 and E_2. To formulate this idea explicitly, let us assume that the observables E_1 and E_2 have outcome spaces $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$, respectively. For E_1 and E_2 to be jointly measurable, there should be an observable G with the outcome space $(\Omega, \mathcal{A})$ giving measurement outcome statistics of E_1 and E_2. We take this to mean that there exist measurable functions $f_1 : \Omega \to \Omega_1$ and $f_2 : \Omega \to \Omega_2$ such that for any $X \in \mathcal{A}_1$ and $Y \in \mathcal{A}_2$,

$$E_1(X) = G(f_1^{-1}(X)), \quad E_2(Y) = G(f_2^{-1}(Y)). \tag{4.15}$$

In short, the observables E_1 and E_2 are functions of G.

Definition 11. Two observables E_1 and E_2 are *functionally coexistent* if there is an observable G such that E_1 and E_2 are functions of G.

Since we study the case of position and momentum observables, we give the following definition only for observables with the outcome space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, for simplicity.

Definition 12. Let E_1 and E_2 be observables with the outcome space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. An observable $G : \mathcal{B}(\mathbb{R}^2) \to \mathcal{L}(\mathcal{H})$ is their *joint observable* if for all $X, Y \in \mathcal{B}(\mathbb{R})$,

$$E_1(X) = G(X \times \mathbb{R}), \quad E_2(Y) = G(\mathbb{R} \times Y).$$

In this case E_1 and E_2 are the *margins* of G.

It is clear from the definitions that two observables having a joint observable are functionally coexistent. As the real line $\mathbb{R}$ is a locally compact, metrizable and separable topological space and we are considering observables with the outcome space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, the existence of a joint observable is actually equivalent to the functional coexistence; see, for instance, [50]. Therefore, the problem of functional coexistence of position and momentum observables can be approached by studying their joint observables.

Looking at the symmetry conditions (4.2), (4.3), (4.5) and (4.4), we notice that an observable $G : \mathcal{B}(\mathbb{R}^2) \to \mathcal{L}(\mathcal{H})$ has a position observable and a momentum observable as its margins if and only if, for every $q, p \in \mathbb{R}$ and $X, Y \in \mathcal{B}(\mathbb{R})$, the following conditions hold:

$$\begin{aligned} W_{q,p} G(X \times \mathbb{R}) W_{q,p}^* &= G(X \times \mathbb{R} + (q,p)), & (4.16) \\ W_{q,p} G(\mathbb{R} \times Y) W_{q,p}^* &= G(\mathbb{R} \times Y + (q,p)). & (4.17) \end{aligned}$$

A characterization of the observables satisfying (4.16) and (4.17) seems to be an open problem.

An important class of observables satisfying conditions (4.16) and (4.17) are observables which are covariant under phase space translations.

Definition 13. An observable $G : \mathcal{B}(\mathbb{R}^2) \to \mathcal{L}(\mathcal{H})$ is a *covariant phase space observable* if for all $q, p \in \mathbb{R}$ and $Z \in \mathcal{B}(\mathbb{R}^2)$,

$$W_{q,p} G(Z) W_{q,p}^* = G(Z + (q,p)). \tag{4.18}$$

It is obvious that (4.18) implies (4.16) and (4.17), and hence, every covariant phase space observable is a joint observable of some position and momentum observables.

For any $T \in \mathcal{S}(\mathcal{H})$, we define an observable $G_T : \mathcal{B}(\mathbb{R}^2) \to \mathcal{L}(\mathcal{H})$ by

$$G_T(Z) = \frac{1}{2\pi} \int_Z W_{q,p} T W_{q,p}^* \, dq dp, \qquad Z \in \mathcal{B}(\mathbb{R}^2). \tag{4.19}$$

The observable G_T is a covariant phase space observable. Moreover, if G is a covariant phase space observable, then there is a unique operator $T \in \mathcal{S}(\mathcal{H})$ such that $G = G_T$; see, for instance, [27], [46], [71].

Let G_T be a covariant phase space observable and let $\sum_i \lambda_i |\varphi_i\rangle\langle\varphi_i|$ be the spectral decomposition of the operator T. The margins of G_T are a position observable E_ρ and a momentum observable F_ν, where

$$d\rho(q) = e(q)dq, \quad e(q) = \sum_i \lambda_i |\varphi_i(-q)|^2, \tag{4.20}$$

$$d\nu(p) = f(p)dp, \quad f(p) = \sum_i \lambda_i |\widehat{\varphi}_i(-p)|^2. \tag{4.21}$$

Therefore, a position observable E_ρ and a momentum observable F_ν are functionally coexistent if there is an operator $T \in \mathcal{S}(\mathcal{H})$ such that ρ and ν are given by (4.20) and (4.21). A measurement theoretical model of the corresponding joint position-momentum measurement is analyzed in [12].

Proposition 8. *Let* E_ρ *be a position observable and* F_ν *a momentum observable. If* E_ρ *and* F_ν *have a joint observable, then they also have a joint observable which is a covariant phase space observable.*

The core idea of the proof of Proposition 8 was presented in [72], and a detailed proof is given in the appendix of Article V. As a direct consequence of Proposition 8 we get the following characterizations of functionally coexistent position and momentum observables.

Corollary 1. *A position observable* E_ρ *[a momentum observable* F_ν*] is functionally coexistent with some momentum observable [position observable] if and only if the probability measure* ρ *[prob. measure* ν*] is absolutely continuous with respect to the Lebesgue measure.*

Corollary 2. *A position observable* E_ρ *and a momentum observable* F_ν *are functionally coexistent if and only if there exists an operator* $T \in \mathcal{S}(\mathcal{H})$ *such that* ρ *and* ν *are given by equations (4.20) and (4.21).*

4.4 The uncertainty principle

In 1927 W. Heisenberg presented a totally new aspect of a measurement imprecision. In his fundamental article [45] Heisenberg sketched an idea which became known as the uncertainty principle. One version of the uncertainty principle can be informally stated as follows:

(UP) A joint measurement of position and momentum must be imprecise, with the measurement inaccuracies satisfying an uncertainty relation.

This is not the only version of the uncertainty principle. There are other formulations referring, for instance, to state preparation, retrieved information, and measurement disturbance. It is beyond the scope of this thesis to try to summarize even the main parts of the issues related to the uncertainty principle. We merely show how the concepts and results of Section 4.3 can be used to give some natural formulations for (UP).

Let E_ρ and F_ν be functionally coexistent position and momentum observables. As discussed in the last section, there exists a state T such that the probability measures ρ and ν have the forms (4.20) and (4.21). It is a basic fact in harmonic analysis that a function and its Fourier transform cannot both be sharply localized. This inaccuracy has a multitude of possible mathematical characterizations[2], and therefore, a bunch of different uncertainty relations can be derived for the observables E_ρ and F_ν. Here we discuss three physically meaningful uncertainty relations manifesting (UP).

It is well known that for any state T, the variances of the probability distributions of the canonical observables Π_Q and Π_P satisfy the inequality

$$\mathrm{Var}(p_T^{\Pi_Q}) \cdot \mathrm{Var}(p_T^{\Pi_P}) \geq \frac{1}{4}. \tag{4.22}$$

Let E_ρ be a position observable and F_ν a momentum observable. The variances of the probability distributions of E_ρ and F_ν in a state T are given by

$$\mathrm{Var}(p_T^{E_\rho}) = \mathrm{Var}(p_T^{\Pi_Q}) + \mathrm{Var}(\rho), \tag{4.23}$$

and

$$\mathrm{Var}(p_T^{F_\nu}) = \mathrm{Var}(p_T^{\Pi_P}) + \mathrm{Var}(\nu). \tag{4.24}$$

If the observables E_ρ and F_ν are not sharp, then $\mathrm{Var}(\rho)$ and $\mathrm{Var}(\nu)$ are nonzero and they satisfy the strict inequality

$$\mathrm{Var}(p_T^{E_\rho}) \cdot \mathrm{Var}(p_T^{F_\nu}) > \frac{1}{4}. \tag{4.25}$$

However, as there are no requirements for the probability measures ρ and ν, the product of the variances in (4.25) can be arbitrarily close to the lower bound $\frac{1}{4}$. We emphasize that the inequalities (4.22) and (4.25) refer to separate measurements of position and momentum and therefore, they are not related to (UP).

Assume then that ρ and ν have the forms (4.20) and (4.21). As shown, for instance, [16, Section III.2.4] and [65, Section 5.4], we then have

$$\mathrm{Var}(p_T^{E_\rho}) \cdot \mathrm{Var}(p_T^{F_\nu}) \geq 1 \tag{4.26}$$

for any state T. This together with Corollary 2 leads to the following conclusion:

Proposition 9. *A necessary condition for the functional coexistence of a position observable E_ρ and a momentum observable F_ν is that for every state T,*

$$\mathit{Var}(p_T^{E_\rho}) \cdot \mathit{Var}(p_T^{F_\nu}) \geq 1. \tag{UR1}$$

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[2] An extensive mathematical survey of various uncertainty relations is presented, for instance, in the review article [40] of G. Folland and A. Sitaram.

The next uncertainty relation is based on the idea that the accuracy of an observable is associated with the resolution of the measurement. We need the following concept, which has been discussed in [13].

Definition 14. Let E be an observable with the outcome space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $\frac{1}{2} \leq \delta < 1$. An effect $E(X)$ is *δ-realizable* if there exists a state T such that

$$p^E_T(X) > \delta. \tag{4.27}$$

The inequality (4.27) means that the measurement outcome belongs to the set X with a probability greater than δ. The reason for the restriction $\delta \geq \frac{1}{2}$ is to avoid the situation where an effect and its complement are both δ-realizable in the same state.

Definition 14 has a close connection with the concepts discussed in Section 2.2. Namely, an observable E is approximately sharp exactly when every nonzero effect $E(X)$ is δ-realizable for every $\frac{1}{2} \leq \delta < 1$. Similarly, E is regular exactly when every nonzero effect $E(X)$ is $\frac{1}{2}$-realizable. It is a direct implication of Proposition 4 and Corollary 1 that if a position observable and a momentum observable are functionally coexistent, neither of them is regular. In the following we formulate an uncertainty relation, which can be seen as a refinement of this fact.

For any $x \in \mathbb{R}$ and $\alpha \in \mathbb{R}_+$, we denote the interval $[x - \frac{\alpha}{2}, x + \frac{\alpha}{2}]$ by $I_{x;\alpha}$. Given an observable E and a number $\frac{1}{2} \leq \delta < 1$, we define

$$\gamma(E; \delta) := \inf\{\alpha > 0 \mid E(I_{x;\alpha}) \text{ is } \delta\text{-realizable for every } x \in \mathbb{R}\}. \tag{4.28}$$

This number has the following interpretation: given any interval with a length greater than $\gamma(E; \delta)$, the corresponding effects are δ-realizable.

The essential idea of Proposition 10 and its proof were presented in Article IV. Since the formulation contains a mistake, we give the proof here. A more extensive analysis is given in [24].

Proposition 10. *A necessary condition for the functional coexistence of a position observable E_ρ and a momentum observable F_ν is that for any $\delta_1, \delta_2 \in [\frac{1}{2}, 1)$ satisfying*

$$\sqrt{1 - \delta_1} + \sqrt{1 - \delta_2} < 1, \tag{4.29}$$

we have

$$\gamma(E_\rho; \delta_1) \cdot \gamma(F_\nu; \delta_2) \geq \left(1 - \sqrt{1 - \delta_1} - \sqrt{1 - \delta_2}\right)^2. \tag{UR2}$$

Proof. By Corollary 2, there exists a vector valued function $\theta \in L^2(\mathbb{R}, \mathcal{H})$ such that $d\rho(q) = \|\theta(q)\|^2_{\mathcal{H}}\, dq$ and $d\nu(p) = \left\|\widehat{\theta}(p)\right\|^2_{\mathcal{H}} dp$.

Let $\alpha > \gamma(E_\rho; \delta_1)$ and $\beta > \gamma(F_\nu; \delta_2)$. For any $X \in \mathcal{B}(\mathbb{R})$, the norm of the multiplicative operator $E_\rho(X)$ is given by the formula

$$\|E_\rho(X)\| = \operatorname{ess\,sup}_{x\in\mathbb{R}} \rho(X - x). \tag{4.30}$$

Since the map $x \mapsto \rho(I_{x;\alpha})$ is continuous, this implies that

$$\sup_{x\in\mathbb{R}} \int_{I_{x;\alpha}} \|\theta(x)\|^2_{\mathcal{H}}\; dx > \delta_1. \tag{4.31}$$

Similarly, we have

$$\sup_{\xi\in\mathbb{R}} \int_{I_{\xi;\beta}} \left\|\widehat{\theta}(\xi)\right\|^2_{\mathcal{H}}\; d\xi > \delta_2. \tag{4.32}$$

It now follows from [35, Theorem 2] (extended to the case of vector valued functions) that

$$\alpha \cdot \beta \geq \left(1 - \sqrt{1-\delta_1} - \sqrt{1-\delta_2}\right)^2. \tag{4.33}$$

Therefore, (UR2) holds. □

Our third uncertainty relation is stated for a joint observable of position and momentum observables. The proof of Proposition 11 is given in Section 2 of Article V.

Proposition 11. *Let G be a joint observable of a position observable E_ρ and a momentum observable F_ν. For any bounded set $Z \in \mathcal{B}(\mathbb{R}^2)$, there exists a number $k_Z < 1$ such that for every state T,*

$$p^G_T(Z) \leq k_Z. \tag{UR3}$$

This result means that the localization of the particle in any bounded set Z in the phase space contains unavoidable inaccuracy. There is no state such that the measurement outcome is in Z with probability greater than k_Z.

4.5 Coexistence of position and momentum observables

The concept of functional coexistence is well motivated and it seems to capture the idea of joint measurability of two observables. However, there is also a (potentially) more general approach to this subject. The concept of coexistence goes back to the works of G. Ludwig. For analysis of this notion, we refer to [50] and [52]. A convenient survey is given in [49].

Definition 15. Two observables E_1 and E_2 are *coexistent* if there is an outcome space $(\Omega, \mathcal{A})$ and an observable $G : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ such that

$$\mathrm{ran}(E_1) \cup \mathrm{ran}(E_2) \subseteq \mathrm{ran}(G).$$

It is clear from the definitions that functionally coexistent observables are coexistent. For sharp observables these concepts are the same. It is not known, in general, if coexistent observables are functionally coexistent. To see why these concepts might be different, assume that E and F are observables and F is a function of E. It follows that $\mathrm{ran}(F) \subseteq \mathrm{ran}(E)$. On the other hand, the condition $\mathrm{ran}(F) \subseteq \mathrm{ran}(E)$ need not imply that F is a function of E; a simple example is given in [37]. Evidently, this observation does not prove that the concepts are different as there can still exist a third observable G such that both E and F are functions of G.

The characterization of coexistent pairs of position and momentum observables is an open problem. Here we state a few partial results. The proofs can be found in Section 5 of Article V.

Proposition 12. *Let E_ρ be a position observable and F_ν a momentum observable. If ran$(E_\rho) \cup$ ran(F_ν) contains a nontrivial projection (not equal to O or I), then E_ρ and F_ν are not coexistent.*

Corollary 3. *Let E_ρ be a position observable which is a convex combination of two sharp position observables. Then ran(E_ρ) contains a nontrivial projection and therefore, it is not coexistent with any momentum observable F_ν.*

Corollary 3 has an obvious dual statement with the roles of position and momentum observables reversed.

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