ESSAYS ON OPTIMAL CONTROL OF SPECTRALLY NEGATIVE LÉVY DIFFUSIONS IN FINANCIAL APPLICATIONS

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To appear cultured, one should end with a quotation from someone wiser than oneself. Johann Wolfgang von Goethe is credited with this contribution to the field of aphorisms: *Grau ist alle Theorie, grün des Lebens goldner Baum.* Luckily, grey has always been one of my favourite colors.

Turku, Finland, 26.11.2008
Teppo Rakkolainen

TABLE OF CONTENTS

LIST OF ORIGINAL RESEARCH PAPERS

(1) Luis H. R. Alvarez – Teppo A. Rakkolainen: *A class of solvable optimal stopping problems of spectrally negative jump diffusions*.

(2) Luis H. R. Alvarez – Teppo A. Rakkolainen: *Optimal payout policy in presence of downside risk*, 2008, to appear in *Mathematical Methods of Operations Research*.

(3) Luis H. R. Alvarez – Teppo A. Rakkolainen: *On singular stochastic control and optimal stopping of spectrally negative jump diffusions*, 2008, to appear in *Stochastics: An International Journal of Probability and Stochastic Processes*.

(4) Luis H. R. Alvarez – Teppo A. Rakkolainen: *Investment timing in presence of downside risk: a certainty equivalent characterization*, 2008, to appear in *Annals of Finance*.

(5) Teppo A. Rakkolainen: *A class of solvable Dirichlet problems associated to spectrally negative jump diffusions*.

1 HISTORICAL BACKGROUND AND MOTIVATION

1.1 ON MATHEMATICAL MODELING OF RANDOM PHENOMENA

A fundamental feature of the real world phenomena occurring both in nature and in society is their *randomness* or – to say it in Greek – *stochasticity*. This means that these phenomena are, to a greater or lesser extent, governed by chance. In a *static* context, this implies that the value of a certain variable at a specified point in time is not a fixed quantity known in advance – instead, it is an unknown quantity, of which we may know the probabilities of it assuming certain potential values, but whose true value is not known until at the specified time point, at which we can observe the *realization* of the *random variable*. In a *dynamic* context, where the variable in question evolves in time, the presence of randomness implies that the values of the variable of interest at different moments of time constitute a *stochastic process*, a collection of random variables indexed by the time parameter.

A crucial decision in modeling is the choice of the time parameter set – discrete or continuous? The answer to this question depends naturally on the phenomenon considered. Even though one can in most cases argue that in reality, the phenomenon is discrete in nature (e. g. trades in a stock exchange do not happen continuously), the continuous-time model may be an acceptable approximation (e. g. if the aforementioned stock trading in an exchange is very frequent, assuming continuous trading may not introduce too much error). Continuous-time models are attractive because they tend to be mathematically more tractable than their discrete-time counterparts.

The aim of the present subsection is to give a short (and necessarily incomplete) overview of the history of mathematical modeling of randomness in continuous time. To enhance readability, the style is kept informal; however, the mathematical concepts mentioned here will be given precise definitions in later sections where the relevant theory is presented in detail, to the extent that these concepts are needed in our exposition.

The branch of mathematics concerned with modeling of random phenomena is called *stochastics*. The theoretical development leading to modern stochastics has its origins in the mathematical theory of probability, which was first given a precise axiomatic formulation in 1933 by A. Kolmogorov in his treatise *Grundbegriffe der Wahrscheinlichkeitsrechnung* (cf. Kolmogorov (1933)). Mathematicians had, of course, investigated probabilities and random events much earlier (popular applications being games of chance and gambling strategies), but no unified axiomatic theory had existed. As a discipline within mathematics, probability theory belongs into the domain of abstract measure theory, which was developed in the first decades of 20th century by mathematicians such as É. Borel, C. Carathéodory and H. Lebesgue, among others (for accounts on measure theory, see e. g. Dunford and Schwartz (1957) or Friedman (1982)).

While Kolmogorov's contribution was the essential foundation on which the theory of stochastic processes was built during the latter half of the 20th century, random processes had been considered by mathematicians earlier: In 1923, N. Wiener gave a precise mathematical definition of a continuous-time process known as the *Brownian motion* (and afterwards also as the *Wiener process*). This process was the starting point for the development of *stochastic integration theory* – in 1940s, K. Itô defined stochastic integrals (i.e. integrals in which we integrate over time with respect to a random process) for Brownian motion (cf. Itô and McKean (1974) and McKean (1969) for overviews and references to original research papers). An important insight of Itô was the representation of *diffusion processes* in terms of their stochastic dynamics – as *stochastic differential equations*. From this began the development of stochastic integration theory culminating in 1970s in the results of P.-A. Meyer and his school establishing processes known as *semimartingales* as the most general class of integrators possible, if one wishes to integrate *predictable* integrands (cf. Protter (2004) or Rogers and Williams (1994) for overviews of stochastic integration theory, and references to original papers).

From the point of view of this thesis, a crucially important class of stochastic processes was given a complete characterization (via infinitely divisible distributions) in 1934 by P. Lévy (cf. Lévy (1934))– namely, processes with stationary and independent increments, nowadays called Lévy processes (for an overview on the theory of Lévy processes, see Bertoin (1996) or Kyprianou (2006)).

There is a rich interplay between the theory of probability and stochastic processes and mathematical analysis. This is particularly striking for processes known as *Markov processes*; informally speaking, such processes have the property that given the present state of the process, the future of the process is independent of its past. In some sense, this property is a generalization of the independent increments property characteristic of Lévy processes. For a *(simple) Markov* process, this property holds for any deterministic moments of time; for a *strong Markov* process, it holds also for a certain class of random moments of time. It was, once again, Kolmogorov who established a connection between Markov processes and certain differential or integro-differential operators (cf. Kolmogorov (1931)) – integral terms appear if the process has path discontinuities, i. e. jumps. This line of research was carried further by W. Feller, who studied diffusion processes and *semigroups of operators* associated to the transition probabilities of these processes (cf. Feller (1936), Feller (1952) and Feller (1954)), and E. Dynkin, who gave the modern definition of a Markov process and introduced the characteristic operator of a strong Markov process (cf. Dynkin (1956), Dynkin and Yuskevich (1956), and Dynkin (1965) for an overview). Connections between stochastic processes and *harmonic* or *excessive* functions (that is, *potential theory*) were further investigated by G. Hunt, R. Blumenthal and R. Getoor (cf. Hunt (1958), Blumenthal (1957), Blumenthal and Getoor (1968)). When discussing the mathematical theory of diffusions, it is impossible to bypass the *martingale problem approach* pioneered around 1969 by D. Stroock and S. Varadhan (cf. Stroock and Varadhan (1979) for an overview and references). Martingale techniques have since then proved to be an immensely important and powerful tool in analyzing stochastic processes, especially in the field of mathematical finance, and particularly in option pricing (cf. Musiela and Rutkowski (2005) for a reasonably recent overview).

1.2 ON STOCHASTIC MODELS IN FINANCE AND ECONOMICS

Stochastic models have been applied in finance for a long time in the sense that already in 1900, L. Bachelier modeled the evolution of stock prices with a Brownian motion (cf. Bachelier (1900)). However, after this the interplay between stochastics on one hand and finance and economics on the other was relatively sparse for a long time: main areas of application which used methods from stochastics and also contributed to further developments in stochastics were statistics, physics and engineering sciences.

An important contribution by P. Samuelson in 1965 was to model stock prices with a *geometric Brownian motion* (cf. Samuelson (1965)), which does not allow negative stock prices and is also in some other respects a more realistic model for stock prices than a Brownian motion. Moreover, the geometric Brownian motion model is also eminently solvable in comparison with most alternatives. The importance of the geometric Brownian motion model (and its extensions and generalizations) in the theory of mathematical finance can hardly be overemphasized: in 1973 F. Black, M. Scholes and R. Merton developed their option pricing methodology based on the geometric Brownian motion (cf. Black and Scholes (1973) and Merton (1973)). Together with the development of information technology and several factors which increased the volatility of financial markets, such as the collapse of the Bretton Woods fixed exchange rate system and the oil crises in 1970s, and the decreasing of regulation in 1980s, the success of this model initiated the widespread application of increasingly sophisticated stochastic models in financial markets.

Despite the enormous success of the Black–Scholes model, with the passage of time its limitations have become increasingly apparent and the model assumptions have been criticized. These assumptions imply that the differences of logarithms of daily returns (*log-returns*) should form an independent and identically distributed sequence of normally (in particular, symmetrically) distributed random variables, with volatility constant over time. This is not consistent with empirical observations on the distribution of log-returns; from these observations are derived the so-called *stylized facts* of financial time series:

(i) Distribution is skewed

(ii) Distribution is leptokurtic (has heavier and longer tails than the normal distribution)

(iii) Volatility changes over time

(iv) Extreme values appear in clusters

(v) Series of squared values is autocorrelated

(cf. McNeil et al. (2005)). All of these are at odds with the normality and independence assumptions. In as much as theories and models are always approximations of reality, the inability to reproduce all aspects of the phenomenon modeled does not necessarily invalidate the model. However, it is important to be aware of such limitations of the model in order to be able to evaluate the suitability and reliability of the model for different uses. For example, in risk management the assumption of a Gaussian world (i. e. normal distributions) may be a very dangerous error indeed if the true distributions are leptokurtic and/or significantly skewed – and empirical findings do indicate that for the most part the financial real world is far from Gaussian (cf. McNeil et al. (2005), Mikosch (2004) and Hull (2003)). During the last two decades some financial catastrophes or near-catastrophes have been considered to be partly due to inadequate models (e. g. in Jorion (2000), the role played by (invalid) Gaussian assumptions in the downfall of hedge fund Long-Term Capital Management in 1998 is considered).

With regard to the choice between continuous- and discrete-time models in financial economics, it may be mentioned that traditionally in applications such as pricing of derivatives continuous-time models have been popular because of their *tractability*: full use of stochastic integration and change of measure techniques is possible. In discrete time models – time series analysis – the emphasis has usually been on obtaining *realistic* models capable of reproducing the observed behavior and suitable for predicting purposes (cf. Mikosch (2004)).

In financial economics, more advanced models have been developed during the last 25 years to address the problem of non-Gaussian distributions. These developments have fueled also the development of the mathematical discipline of stochastics, leading to a new fruitful interplay between financial economics and stochastics.

In time series analysis, perhaps the most successful developments have been the ARCH (*AutoRegressive Conditionally Heteroskedastic*) and GARCH (*Generalized ARCH*) families of nonconstant volatility models, due to R. Engle and T. Bollerslev, respectively (cf. Engle (1982) and Bollerslev (1986)).

In the continuous-time framework, a significant development beginning in 1990s has been the increasing interest in models based on *Lévy processes*, which has spawned a multitude of applied research during the last fifteen years. In these models one replaces Wiener processes and Itô stochastic differential equations driven by them with Lévy processes and corresponding Lévy stochastic differential equations. As solutions of (sufficiently regular) Itô stochastic differential equations are diffusions, solutions of Lévy stochastic differential equations are often called *jump diffusions*. These models are able to generate the nonconstant volatility and leptokurtic return distributions observed in real world time series (cf. Cont and Voltchkova (2005)). In addition, their dynamics allow instantaneous jumps, which open up possibilities for modeling catastrophic events, e. g. stock market crashes. The cost of this added flexibility is a more complicated and less tractable model. However, as several researchers have demonstrated with their work, much can be done and many problems solved in models based on Lévy processes (for a reasonably up-to-date list of references, see for example the bibliography in Kyprianou (2006)). Alternatives to Lévy models also capable of overcoming some of the limitations of the Gaussian models include *stochastic volatility models* and *regime-switching models*.

The research papers constituting the main part of this thesis belong to that multitude of applied research spawned by the growing interest in Lévy models, which was mentioned in the previous paragraph. The aim of the research was to analyze certain dynamic optimization problems relevant in finance and financial economics in the presence of jumps.

1.3 SOME PROBLEMS OF FINANCIAL ECONOMICS

1.3.1 PRICING OF REAL OPTIONS

A problem of considerable theoretical interest and practical relevance in financial economics is the pricing of *real options*. The problem arises in evaluating investments in real assets under uncertainty. Such investments can contain significant *embedded options* such as *abandonment*, *expansion*, *contraction*, *deferral* or *extension* options, as the investor typically is not obliged to make the investment "now-or-never", but has the option of waiting for new information before the investment decision. On the other hand, once made, the investment decision is typically at least partially irreversible – disinvestment is not possible, at least not without incurring some costs. Embedded options can also be found in financial products such as insurance contracts, e. g. the policyholder may have a surrender option, in which he/she can at any time during the contract terminate the policy and obtain its surrender value (cf. Møller and Steffensen (2007)).

In fact, almost any decision-making under uncertainty can be interpreted as pricing of a real option – essential features being the presence of *uncertainty*, the possibility of choosing the *timing* and the (partial) *irreversibility* of the decision. Classical methods of evaluating investment opportunities – such as net present value calculations – disregard the value of these options, which can be highly significant. The real options approach addresses this shortcoming by using option pricing methods originally developed for financial options to price correctly the options embedded in investment decisions. Usually these options are so-called American options in the sense that they can be exercised at any time during the lifetime of the investment opportunity. From the mathematical point of view, pricing of American options is equivalent to solving *optimal stopping problems*: our objective is to choose the optimal moment of time at which to stop a stochastic process. It should be emphasized that even though a natural way of formulating the problem of investing or disinvesting in real assets is to consider the stream of payments generated by the asset, this problem can often be transformed into a standard stopping problem by using the resolvent operator of the underlying process (cf. Paper IV). As the time horizon in real option problems is usually very long and in many cases there is no natural expiration date, a standard device for simplifying the problem is to assume that the time horizon is infinite, i. e. the American option in question is assumed to be perpetual (to have no finite expiration date). We will come back to this in a later section.

For the definitive detailed exposition of the real options approach, the reader should consult Dixit and Pindyck (1994), while some later extensions and recent developments can be found in Boyarchenko and Levendorskiĭ (2007).

1.3.2 DISTRIBUTION OF DIVIDENDS

Paying out dividends to shareholders is one way for a firm to distribute its profits. *A priori* a question of interest is, how to do this in an optimal way with respect to some chosen criterion? Actually, the famous Miller–Modigliani theorem states that in a rational market, in absence of transaction costs and other market imperfections, dividend policy is irrelevant for valuation purposes: the value of a firm is determined solely by its assets' earning power and investment policy (cf. Miller and Modigliani (1961)). However, these findings are somewhat controversial, as in real world a perfect market does not exist (see Ross (1977), Easterbrook (1984), Miller and Rock (1985) and Jensen (1986) for studies addressing the impact of different imperfections), and furthermore there is some empirical evidence on the importance of dividend policies in the strategic decision making process of corporations in some industries (e. g. insurance industry), see Akhigbe et al. (1993).

The usual criterion to be optimized in dividend distribution is the expected present value of the stream of future dividends over the lifetime of the corporation. Mathematically, this can be formulated as a problem of controlling a stochastic process – the firm's reservoir of assets, out of which the dividends are paid, evolves in time subject to uncertainty. In case there are no transaction costs, one ends up with a *singular stochastic control problem*. The term singular arises from the fact that the theoretical optimal policy is then typically of local time push-type variety: all value in excess of a certain level is instantaneously paid out. In presence of transaction costs, when each dividend payment incurs a fixed cost, such controls are no longer feasible as they would lead to infinite transaction costs: in these cases, optimal policy usually consists of discrete lump sum payments (impulses), and one speaks about *stochastic impulse control problems*. It is worth noticing that for these classes of control problems, optimal stopping rules are also feasible controls, being rather simple instances of impulse controls.

We shall return to the theory of optimal stochastic control in more detail in the third section of this Introduction; before this, however, it is necessary to give both some motivations for the relevance, and a precise account of the key mathematical properties of the class of jump diffusion models we intend to consider. The following last subsection of this section will take care of the motivational aspect, while the mathematical formulation of the model will be the subject of the next section.

1.4 DOWNSIDE RISK: A DISCUSSION

When one studies graphical representations of the development of stock indices from periods containing stock market crashes and/or bubbles, one is often tempted to characterize the behavior of the index at the time of the shock as resembling more an instantaneous jump than any continuous increase or decrease. From the point of view of mathematical modeling, this leads one to question the suitability of such stochastic models where the model assumptions imply that the variables of interest evolve with time in a continuous fashion.

It is also a well-documented empirical fact that in the financial markets, the responses to good news and to bad news are different: bad news tend to have a more dramatic impact on the markets – an observation in line with the *bad news principle* presented in Bernanke (1983). This observed behavior suggests that while assumption of continuity may be reasonable when one considers increases in the value of, say, a stock index, this may not be the case when one considers decreases in the value of the same index. Moreover, from the perspective of risk management, assumption of continuous decreases in the value of assets can be very dangerous indeed, as it means that the applied model is unable to incorporate properly the negative impact of potential catastrophic events. In the context of risk management it may not be so essential to capture the impacts of potential sudden unanticipated increases in value – hence assuming continuous increases may be acceptable from a prudent point of view.

In the research papers of this thesis, the term *downside risk* refers to the possibility of sudden unanticipated decreases in the value of interest: mathematically speaking, the risk that the stochastic process modeling the value *jumps* instantaneously to a lower level (as opposed to decreasing continuously).

While being able to accommodate both upward and downward jumps is something worth striving for, we found that particularly amenable to analysis is the case of *spectrally negative* processes, i. e. processes which have only downward jumps but increase continuously. In fact, we established that many techniques and solution methods applicable to a class of models with continuous paths can actually be extended to cover also spectrally negative models with path discontinuities.

2 JUMP DIFFUSION MODELS: BUILDING BLOCKS AND BASIC PROPERTIES

2.1 PRELIMINARIES

In this section familiarity with measure theory is assumed, and hence definitions of concepts such as σ-algebras are not given here. Note that even though many of the results presented in this section are valid in the multidimensional setting, we shall restrict our attention to the one-dimensional case only. The exposition in this subsection relies heavily on Protter (2004) and Rogers and Williams (1994), and we mostly refer the reader to these monographs for proofs.

A *filtration* $\mathbb{F} = \{\mathscr{F}_t\}_{t\in[0,\infty)}$ is an increasing family of σ-algebras. Let $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space satisfying the usual hypotheses

(i) $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets of $\mathscr{F}$;

(ii) $\mathbb{F}$ is right-continuous, i. e. $\mathscr{F}_t = \cap_{u>t}\mathscr{F}_u$.

Throughout this second section, we will assume that such a filtered probability space is given. Informally speaking, we can think of the σ-algebra $\mathscr{F}_t$ as representing information available at time t, and hence filtration $\mathbb{F}$ describes the flow of available information; as this is increasing, we are assuming that the amount of available information does not decrease with time ("we do not forget things").

An important class of random moments of time is defined in the following:

Definition 2.1. *A random variable $\tau : \Omega \to [0,\infty]$ is a* stopping time *(with respect to $\mathbb{F}$), if $\{\tau \leq t\} \in \mathscr{F}_t$ for every $t \in [0,\infty]$.*

Intuitively, a stopping time is a random time τ such that based on the information available at deterministic time t, we know whether we have stopped by time t (case $\tau \leq t$) or not (case $\tau > t$). The set of all ($\mathbb{F}$-) stopping times is denoted by $\mathscr{T}$. Stopping times can be classified into three types as follows.

Definition 2.2. *A stopping time τ is*

(a) predictable*, if there exists an increasing sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ such that $\tau_n \uparrow \tau$, almost surely;*

(b) accessible*, if there exists a sequence* $(\tau_n)_{n\in\mathbb{N}}$ *of predictable stopping times such that*

$$\mathbb{P}(\cup_{n=1}^{\infty}\{\omega : \tau_n(\omega) = \tau(\omega) < \infty\}) = \mathbb{P}(\tau < \infty);$$

(c) totally inaccessible*, if for every predictable stopping time* ζ*,*

$$\mathbb{P}(\{\omega : \tau(\omega) = \zeta(\omega) < \infty\}) = 0.$$

Naturally, any fixed time t is a predictable stopping time.

In analogue with σ-algebras $\mathscr{F}_t$ for deterministic times t, we want to have a mathematical representation of information available at a random stopping time.

Definition 2.3. *If* τ *is a stopping time, then the* stopping time σ-algebra

$$\mathscr{F}_\tau = \Big\{F \in \mathscr{F} \,|\, F \cap \{\tau \le t\} \in \mathscr{F}_t, \text{ for all } t \ge 0\Big\}.$$

Here is the general definition of a one-dimensional real-valued stochastic process:

Definition 2.4. *A one-dimensional* stochastic process *X on* $(\Omega, \mathscr{F}, \mathbb{P})$ *is a collection of* $\mathbb{R}$*-valued random variables* $\{X_t : \Omega \to \mathbb{R}\}_{t\in[0,\infty)}$.

In the general setup, one needs to be precise about the concept of equality between two stochastic processes.

Definition 2.5. *Two processes X and Y are* modifications*, if* $X_t = Y_t$ *almost surely, for each t. Two processes X and Y are* indistinguishable*, if almost surely, for all t,* $X_t = Y_t$.

Example 2.6 (Counting process)**.** *A special stochastic process can be constructed from a strictly increasing sequence of positive random variables* $\{T_n\}_{n\in\mathbb{N}}$ *with* $T_0 = 0$. *Recall that the indicator function*

$$\mathbf{1}_{\{t\ge T_n\}} = \begin{cases} 1, & \text{if } t \ge T_n(\omega) \\ 0, & \text{if } t < T_n(\omega). \end{cases}$$

Definition 2.7. *The* $\mathbb{N}\cup\{\infty\}$*-valued process* $N = \{N_t\}_{t\in[0,\infty)}$ *defined by*

$$N_t = \sum_{n=0}^{\infty} \mathbf{1}_{\{t\ge T_n\}}$$

is the counting process *associated to the sequence* $\{T_n\}_{n\in\mathbb{N}}$.

Random variable $T = \sup_n T_n$ *is the* explosion time *of* N*; in case* $T = \infty$ *almost surely (a.s.), we have a counting process* without explosions. ■

Definition 2.8. *A stochastic process* X *is* adapted *to the filtration* $\mathbb{F}$*, if* $X_t \in \mathscr{F}_t$ *for each* $t \in [0,\infty)$.

Example 2.9 (Counting process, cont'ed)**.** *A counting process is adapted, if and only if the associated sequence* $\{T_n\}_{n\in\mathbb{N}}$ *consists of stopping times: indeed, if* $\{T_n\}_{n\in\mathbb{N}}$ *are stopping times, then for each* $t \geq 0$ *we have* $\mathbf{1}_{\{t\geq T_n\}} \in \mathscr{F}_t$ *for every* $n \in \mathbb{N}$ *and hence* $N_t \in \mathscr{F}_t$ *for each* $t \geq 0$*, while if for some* $k \in \mathbb{N}$*,* T_k *is not a stopping time, then clearly* N_t *cannot be adapted and the second implication follows by contraposition.* ■

A stochastic process is always by definition adapted to its natural filtration, which can be interpreted as the information generated by the history of the process.

Definition 2.10. *The* natural filtration $\mathbb{G} = \{\mathscr{G}_t\}_{t\in[0,\infty)}$ *of a stochastic process* X *is the smallest filtration such that* $X_t \in \mathscr{G}_t$ *for each* $t \in [0,\infty)$.

Example 2.11 (Counting process, cont'ed)**.** *It is possible to show that the natural filtration of a counting process is right-continuous (cf. Protter (2004),Theorem I.25), and hence the completed natural filtration of a counting process satisfies the usual hypotheses.* ■

Example 2.12 (Galmarino's criterion)**.** *A nonnegative* $\mathscr{G}_\infty$*-measurable random variable* $\tau(X(\omega))$ *is a stopping time with respect to the natural filtration* $\mathbb{G}$*, if and only if for every fixed* t *and any scenarios* ω *and* $\tilde{\omega}$ *the conditions* $X_s(\omega) = X_s(\tilde{\omega})$ *for* $0 \leq s \leq t$ *and* $\tau(X(\omega)) \leq t$ *imply* $\tau(X(\omega)) = \tau(X(\tilde{\omega}))$*. In other words,* τ *is a* $\mathbb{G}$*-stopping time, if based on the behavior of* X *up to any fixed time* t *one can say whether moment* τ *occurs or not. Hence, for example, the* first passage time *of* b, $\min\{s \,|\, X_s > b\}$*, is a* $\mathbb{G}$*-stopping time, while the* last exit time *at* 0, $\max\{s \,|\, X_s = 0\}$ *is not.* ■

For each fixed scenario $\omega \in \Omega$, the function $t \mapsto X_t(\omega)$ is called a *sample path* or a *realization* of the stochastic process X. It is in terms of sample path behavior that the differences between the concepts of a modification and of indistinguishability (which were given in definition 2.5) are most easily understood: while the sample paths of two indistinguishable processes coincide for all $\omega \notin N$, where $\mathbb{P}(N) = 0$, for two modifications the set N on which the sample paths do not coincide need not even be measurable, since the interval $[0,\infty)$ is an uncountable set (cf. Protter (2004), Section I.1). Next we define a class of stochastic processes whose sample paths have certain nice regularity properties.

Definition 2.13. *A stochastic process X is called* càdlàg*, if it almost surely has sample paths which are right continuous with left limits.*

The acronym càdlàg is derived from French *continu à droite et pourvu de limites à gauche*. If processes X and Y are càdlàg, and Y is a modification of X, then X and Y are indistinguishable.

Example 2.14 (Counting process, cont'ed)**.** *A counting process without explosions has right-continuous paths with left limits, and hence is a càdlàg process.* ■

An extremely important class of càdlàg processes are *martingales*.

Definition 2.15. *An adapted process $M = \{M_t\}_{t\in[0,\infty)}$ is a* martingale *with respect to the filtration $\mathbb{F}$, if*

1. $\mathbb{E}\{|M_t|\} < \infty$;
2. *If $s \leq t$, then $\mathbb{E}\{M_t \,|\, \mathscr{F}_s\} = M_s$ almost surely.*

If an adapted process X satisfies the first condition of Definition 2.15 and, in addition, $\mathbb{E}\{X_t \,|\, \mathscr{F}_s\} \leq X_s$, almost surely, for $s \leq t$, then X is a *supermartingale*. For a *submartingale*, the reverse inequality holds.

Any martingale M has a unique modification which is càdlàg (cf. Protter (2004), Corollary of Theorem I.9). In some applications, the requirement that a process be a martingale is too stringent: it is sufficient that the martingale property holds *locally* in the sense of the following definition, where $X_{t\wedge T}$ is the process X *stopped at time T*, i.e. $X_{t\wedge T} = X_t$, if $t < T$ and $X_{t\wedge T} = X_T$, otherwise.

Definition 2.16. *An adapted càdlàg process X is a* local martingale *if there exists an increasing sequence of stopping times $\{T_i\}_{i\in\mathbb{N}}$ with $\lim_{n\to\infty} T_n = \infty$, almost surely, such that for each $n \in \mathbb{N}$, $X_{t\wedge T_n}\mathbf{1}_{\{T_n>0\}}$ is a uniformly integrable martingale.*

To introduce the next class of processes, we need the following auxiliary definition.

Definition 2.17. *A càdlàg process $N = \{N_t\}_{t \in [0,\infty)}$ is called a* finite variation process, *if almost all of the paths of N are of finite variation on each compact interval of $[0,\infty)$.*

Semimartingales are stochastic processes which can be represented as the sum of a local martingale and a finite variation process:

Definition 2.18. *An adapted càdlàg process X is a* semimartingale, *if and only if $X_t = X_0 + M_t + N_t$, where M is a local martingale and N is a finite variation process.*

Definition 2.19. *For semimartingales X and Y the* quadratic covariation process of X and Y, *$[X,Y] = \{[X,Y]_t\}_{t \in [0,\infty)}$, is defined by*

$$[X,Y]_t = X_t Y_t - \int_0^t X_{s-} dY_s - \int_0^t Y_{s-} dX_s.$$

In the special case $Y = X$ we obtain the quadratic variation process of X.

The quadratic variation of a semimartingale X is a càdlàg, increasing and adapted process, and for any sequence of random partitions tending to identity,

$$X_0^2 + \sum_i \left(X^{T_{i+1}^n} - X^{T_i^n}\right)^2 \to [X,X], \text{ as } n \to \infty,$$

where for each n the partition is given by $0 = T_0^n \leq T_1^n \leq \cdots \leq T_{k_n}^n$ with T_i^n stopping times (cf. Protter (2004) Theorem II.22).

Semimartingales are preserved by convex transformations: for a convex function f and a semimartingale X , $f(X)$ is also a semimartingale and, moreover,

$$f(X_t) - f(X_0) = \int_0^t f'(X_{s-}) dX_s + A_t, \tag{1}$$

where f' is the left derivative of f and $A = \{A_t\}_{t \in [0,\infty)}$ is adapted, right continuous and increasing (cf. Protter (2004), Theorem IV.66). Using the convexity of the function $x \mapsto |x|$, we can define the following concept, which plays an important role in many applications.

Definition 2.20. *Let X be a semimartingale, and define*

$$h_a(X_t) := |X_t - a| = |X_0 - a| + \int_0^t sign(X_{s-} - a) dX_s + A_t^a,$$

where A_t^a is the increasing process of equation (1). *Then the* local time of X at *a is defined by*

$$l_t^a(X) = A_t^a - \sum_{0<s\le t} \left\{h_a(X_s) - h_a(X_{s-}) - h_a'(X_{s-})\Delta X_s\right\}.$$

Local time process $\{l_t^a(X)\}_{t\ge 0}$ is continuous in t (cf. Protter (2004) pp. 212–213). It can be interpreted as an occupation density relative to $d[X,X]_s^c$, the path-by-path continuous part of the quadratic variation process: namely, for f bounded and Borel measurable,

$$\int_{-\infty}^{\infty} l_t^y(X) f(y)dy = \int_0^t f(X_{s-})d[X,X]_s^c.$$

Here is an important class of random times:

Definition 2.21. *Let X be a stochastic process and $B \subset \mathbb{R}$ a Borel set. The* hitting time of B for X *is the random variable $T = \inf\{t > 0 : X_t \in B\}$.*

We can now give more examples of stopping times: for càdlàg processes, hitting times of Borel sets are $\mathbb{G}$-stopping times (for nice and short proofs in the case of B open or closed, we refer to Protter (2004), Section I.1, Theorems 3 and 4 – for the general case, the statement of this deep result known as Début Theorem can be found in Rogers and Williams (1994) with a reference to a proof given).

We conclude this subsection with some useful continuity concepts for stochastic processes.

Definition 2.22. *Stochastic process X is* continuous in probability, *if for each $t \in [0,\infty)$ and each $\varepsilon > 0$ we have* $\lim_{s\to t} \mathbb{P}\left(|X_t - X_s| > \varepsilon\right) = 0$.

Definition 2.23. *Stochastic process X is* quasi-left continuous *or* left-continuous over stopping times, *if for any strictly increasing sequence of stopping times $\{\tau_n\}$ and a stopping time τ such that $\tau_n \uparrow \tau$ we have that $X_{\tau_n} \to X_\tau$, $\mathbb{P}$-almost surely.*

Definition 2.24. *Stochastic process X is* continuous, *if there exists a modification of X which has almost surely continuous sample paths.*

After these preliminaries, we are in a position to introduce the building blocks of a jump diffusion model in the following subsection.

2.2 DRIVING SOURCES OF RANDOMNESS

2.2.1 BROWNIAN MOTION

The best known example of a càdlàg process with continuous paths is the Brownian motion:

Definition 2.25. *An adapted process* $W = \{W_t\}_{t\in[0,\infty)}$ *is a* Brownian motion *or a* Wiener process, *if*

(i) $W_0 = 0$ *almost surely;*

(ii) for $0 \le s < t < \infty$*, the increment* $W_t - W_s$ *is independent of* $\mathscr{F}_s$*;*

(iii) for $0 < s < t$*, the increment* $W_t - W_s$ *is normally distributed with mean* 0 *and variance* $t - s$*.*

The existence of such a process can be proved by a combining a path-space construction of finite-dimensional distributions with the Daniell–Kolmogorov extension theorem (cf. Rogers and Williams (1994), Section I.6). It can be proved that a Brownian motion has (more precisely, can be chosen to have) almost surely continuous sample paths, hence it is *a fortiori* both a càdlàg process and continuous in probability. By virtue of the continuity of the sample paths, a Brownian motion can also be viewed as a probability measure (so-called *Wiener measure*) on the path space $\mathscr{C}([0,\infty);\mathbb{R})$, the space of continuous functions on $[0,\infty)$ (cf. Rogers and Williams (1994), Section II.71, and Stroock and Varadhan (1979), Chapters 1–2). The quadratic variation of a Brownian motion has exceedingly simple form:

Theorem 2.26. *For a Brownian motion* W *and* $\{\sigma_n\}_{n=1}^{\infty} = \{(T_i^n)_{i=1}^{\infty}\}_{n=1}^{\infty}$*, a sequence of random partitions of* $[0,t]$ *tending to identity,*

$$\lim_{n\to\infty} \sum_i \left(W_{T_{i+1}^n} - W_{T_i^n}\right)^2 = [W,W]_t = t.$$

In particular, $\mathbb{E}[W_t^2] = t$*.*

However, not all path properties of a Brownian motion are nice and regular. Using the fact that $[W,W] = t$, one can prove the following results.

Theorem 2.27. *For almost all* ω*, the sample paths* $t \mapsto W_t(\omega)$ *of a Brownian motion are of unbounded variation on any interval.*

Proof. The proof of this standard result can be found (for example) in Protter (2004), p. 19. □

Theorem 2.28. *For almost all ω, the sample path $t \mapsto W_t(\omega)$ is nondifferentiable for all $t \geq 0$.*

Proof. A remarkably simple and straightforward proof relying on the fact that the quadratic covariation $\mathbb{E}[W_s, W_t] = \min(s,t)$ is given in McKean (1969). □

It is precisely this “wild“ behavior of the sample paths which causes the difficulties in defining an integral with respect to a Brownian motion in the traditional way as a limit of sums.

2.2.2 POISSON RANDOM MEASURES

The most simple example of a discontinuous càdlàg stochastic process is a *Poisson process*.

Definition 2.29. *An adapted counting process $N = \{N_t\}_{t\in[0,\infty)}$ is a* Poisson process, *if*

(i) $N_0 = 0$ almost surely;

(ii) for $0 \leq s < t < \infty$, the increment $N_t - N_s$ is independent of $\mathscr{F}_s$;

(iii) for $0 \leq s < t < \infty$, $0 \leq u < v < \infty$, $t - s = v - u$ the increment $N_t - N_s$ has the same distribution as $N_v - N_u$.

It is a standard result that then the random variable N_t has Poisson distribution, i. e.

$$\mathbb{P}(N_t = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!},$$

for some $\lambda \geq 0$, and that the counting process N is continuous in probability and does not have explosions (cf. Protter (2004), Section I.3, Theorem 23). The parameter λ is called the *intensity* of the process: it describes the expected (average) number of jumps of deterministic size 1 that the process performs in a time unit. Poisson process N can also be constructed directly as $N_t = \sup\{n \in \mathbb{N} \,|\, S_n \leq t\}$, where $S_n = \tau_1 + \ldots + \tau_n$ with $\{\tau_i\}_{i\in\mathbb{N}}$ being a sequence of independent random variables distributed exponentially with parameter λ.

An immediate generalization is a stochastic process with jumps occurring randomly in time as a Poisson process, and – in addition – the size of an individual jump being a random variable. Such a process is called a *compound*

Poisson process: the individual jump sizes form a collection of independent and identically distributed random variables ξ_i, $i \in \mathbb{N}$, and the state of the process at time t is a sum of a random number N_t of random elements ξ_i, where the random number of terms is Poisson distributed with parameter λt:

$$R_t = \sum_{i=1}^{N_t} \xi_i.$$

A compound Poisson process can be thought of as a time-changed random walk $S_n = \sum_{i=1}^{n} \xi_i$, with time change $n \mapsto N_t$ – lengths of time periods between jumps are independent and exponentially distributed random variables instead of being equal to the deterministic constant 1.

The elementary instances of a Poisson process and a compound Poisson process can be subsumed into the more general framework of *Poisson point processes*. Let us now introduce this framework.

A *point process* is a random distribution of points in a space. Mathematically, this can be described by the following definitions.

Definition 2.30. *Let E be a locally compact space with a countable basis and let $\mathscr{E}$ be the Borel σ-algebra of subsets of E. A* point measure *on E is a measure m of form*

$$m = \sum_{i=1}^{\infty} \mathbf{1}_{x_i},$$

where the measures $\mathbf{1}_x$ are defined for $A \in \mathscr{E}$ via $\mathbf{1}_x(A) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$, and $\{x_i \,|\, i \in \mathbb{N}\} \subset E$ is a countable collection of points.

Denote now by $M_p(E)$ the space of all point measures defined on E and by $\mathscr{M}_p(E)$ the smallest σ-algebra of subsets of $M_p(E)$ making all evaluation maps $m \to m(F)$ from $M_p(E)$ to $[0,\infty]$ measurable for all $F \in \mathscr{E}$.

Definition 2.31. *A* point process *on E is a measurable map $N : (\Omega, \mathscr{F}, \mathbb{P}) \to (M_p(E), \mathscr{M}_p(E))$; that is, a point measure valued random variable – hence a point process is actually a* random measure.

For our purposes in this thesis the following class of random measures is sufficient.

Definition 2.32. *Let ν be a σ-finite measure on E. A random measure ϕ on E is called a* Poisson random measure with characteristic measure ν *or a* Poisson point process with intensity ν*, if the following are satisfied:*

(i) $\phi(B)$ *is Poisson distributed with parameter* $\nu(B)$*, for any* $B \in \mathscr{E}$ *such that* $\nu(B) < \infty$*;*

(ii) variables $\phi(B_1), \ldots, \phi(B_n)$ *are independent, if* $B_1, \ldots, B_n \in \mathscr{E}$ *are disjoint.*

To illustrate Poisson random measures, consider the case of a finite measure ν on E (i.e. $\nu(E) < \infty$) and set $\lambda := \nu(E)$. Let then ξ_i be a sequence of independent and identically distributed E-valued random variables with distribution $\lambda^{-1}\nu$ and suppose that the integer-valued random variable N is independent of all ξ_is and Poisson distributed with parameter λ. Then the random measure

$$\phi := \sum_{i=1}^{N} \mathbf{1}_{\xi_i},$$

is a Poisson random measure with intensity ν. In case of only a σ-finite ν, we can partition E into a countable number of Borel sets $\{B_n\}_{n\in\mathbb{N}}$ such that each B_n has a finite ν-measure, construct independent Poisson random measures ϕ_n with intensities $\mathbf{1}_{B_n}\nu$ and get a Poisson random measure with intensity ν by setting $\phi = \sum_{n=1}^{\infty} \phi_n$.

We can consider a Poisson measure on the product space $E \times [0,\infty)$ with intensity $\mu = \nu \otimes dt$. Because almost surely, $\phi(E \times \{t\}) \in \{0,1\}$, for all $t \geq 0$, we can represent ϕ in terms of a $E \cup \{\Gamma\}$-valued stochastic process $e(t)$, where Γ is an isolated additional point: if $\phi(E \times \{t\}) = 0$, then set $e(t) = \Gamma$; if $\phi(E \times \{t\}) = 1$, then the restriction of ϕ to $E \times \{t\}$ is a point mass at, say, (ε, t) and we set $e(t) = \varepsilon$. Then the Poisson random measure can be expressed as

$$\phi = \sum_{t\geq 0} \delta_{(e(t),t)}$$

and $e = \{e(t)\}_{t\in[0,\infty)}$ is a Poisson point process with characteristic measure ν.

In this thesis, our locally compact space with a countable basis is always a subset of $\mathbb{R} \setminus \{0\}$. For a more detailed account of point processes the reader should consult Chapter 3 of Resnick (1987).

2.2.3 LÉVY PROCESSES

For the results of this section, we refer to Bertoin (1996) unless otherwise indicated.

The class of Lévy processes consists of all stochastic processes which are continuous in probability and have independent and stationary increments; in particular, Brownian motion, Poisson process and compound Poisson process are examples of a Lévy process. Because of the independent and stationary increments, a Lévy process can be viewed as the continuous-time analogue of a random walk. Here is a formal definition:

Definition 2.33. *An adapted process* $L = \{L_t\}_{t\in[0,\infty)}$ *is a* Lévy process*, if*

(i) $L_0 = 0$ *almost surely;*

(ii) for $0 \leq s < t < \infty$*, the increment* $L_t - L_s$ *is independent of* $\mathscr{F}_s$*;*

(iii) for $0 < s < t$*, the increment* $L_t - L_s$ *has the same distribution as* L_{t-s}*;*

(iv) L is continuous in probability.

Any Lévy process has a unique càdlàg modification, which is also a Lévy process: hence a Lévy process is càdlàg (modulo indistinguishability), see Theorem I.30 in Protter (2004). The completed natural filtration of a Lévy process satisfies the usual hypotheses, see Theorem I.31 in Protter (2004). Lévy processes are intimately connected to *infinitely divisible laws*.

Definition 2.34. *A probability measure* μ *on* $\mathbb{R}$ *is called* infinitely divisible*, if for any* $n \in \mathbb{N}$ *there exists a probability measure* μ_n *such that the characteristic function*

$$\mathfrak{F}\mu(\xi) := \int_{\mathbb{R}} \exp\{i\xi u\}\mu(du)$$

satisfies $\mathfrak{F}\mu = (\mathfrak{F}\mu_n)^n$.

For any infinitely divisible law μ there exists a unique continuous function $\Psi : \mathbb{R} \to \mathbb{C}$ such that $\Psi(0) = 0$ and $\mathfrak{F}\mu(\xi) = \exp\{-\Psi(\xi)\}$ for all $\xi \in \mathbb{R}$. This function is called the *characteristic exponent* of μ. An fundamental result characterizing infinitely divisible laws in terms of characteristic exponents is the Lévy–Khintchine formula:

Theorem 2.35 (Lévy–Khintchine formula)**.** *A function* $\Psi : \mathbb{R} \to \mathbb{C}$ *is the characteristic exponent of an infinitely divisible probability measure on* $\mathbb{R}$*, if and only if there exist* $a \in \mathbb{R}$*,* $\sigma \geq 0$ *and a measure* ν *on* $\mathbb{R} \setminus \{0\}$ *with* $\int_{\mathbb{R}}(1 \wedge |x|^2)\nu(dx) < \infty$ *such that*

$$\Psi(y) = iay + \frac{1}{2}\sigma^2 y^2 + \int_{\mathbb{R}} \left(1 - e^{iyz} + iyz\mathbf{1}_{\{|z|<1\}}\right) \nu(dz)$$

for every $y \in \mathbb{R}$.

We can now give a more precise formulation of the connection between infinitely divisible laws and Lévy processes: any infinitely divisible law can be viewed as the distribution of the random variable L_1, the Lévy process L evaluated at time $t = 1$.

Theorem 2.36. *Consider* $a \in \mathbb{R}$, $\sigma \geq 0$ *and a measure* ν *on* $\mathbb{R} \setminus \{0\}$ *such that* $\int (1 \wedge |x|^2) \nu(dx) < \infty$*, and define for any* $y \in \mathbb{R}$

$$\Psi(y) = iay + \frac{1}{2}\sigma^2 y^2 + \int_{\mathbb{R}} \left(1 - e^{iyz} + iyz\mathbf{1}_{\{|z|<1\}}\right) \nu(dz).$$

Then there exists a unique probability measure $\mathbb{P}$ *on* Ω *under which L is a Lévy process with characteristic exponent* Ψ*, and the jump process of L is a Poisson point process with characteristic measure (or* Lévy measure*)* ν.

Proof. See Bertoin (1996), pp. 13–15. □ □

From the proof of the previous theorem, as it is given in Bertoin (1996), one can obtain the following version of the Itô–Lévy decomposition characterizing the structure of Lévy processes as superpositions of simpler processes.

Corollary 2.37. *A Lévy process* L_t *can be decomposed into a sum of four independent components*

$$L_t = at + \sigma W_t + \sum_{i=1}^{N_t} \xi_i + J_t,$$

where the first component is a deterministic drift, the second component is a scaled Brownian motion, the third component is a compound Poisson process with jumps of size at least 1 *and the fourth component is a pure jump martingale with jumps of size less than* 1.

As both the drift and the Brownian component in the decomposition of Corollary 2.37 are continuous, we see that the jump process ΔL of a Lévy process L consists of a compound Poisson process and a pure jump martingale, which may or may not be expressible as a (compensated) compound Poisson process. These two different cases can be characterized in terms of the total mass of the Lévy measure ν of the jump process:

Theorem 2.38. *A pure jump Lévy process has a finite Lévy measure, if and only if it is a compound Poisson process. In this case, we speak of a* finite activity *jump process. If the Lévy measure has infinite total mass, we speak of an* infinite activity *jump process.*

It is because of the possibility of infinite jump activity that we need to consider $\mathbb{R}\setminus\{0\}$ instead of $\mathbb{R}$ – the integrability properties of the Lévy measure ν ensure that $\int_{\{|z|>\varepsilon\}} \nu(dz) < \infty$ for any $\varepsilon > 0$, but do not exclude the possibility that $\int_{\{|z|<\varepsilon\}} \nu(dz) = \infty$ for any $\varepsilon > 0$. In this case ν could not be σ-finite on $\mathbb{R}$, since any open set containing 0 will have infinite ν-measure.

A particularly interesting class of Lévy processes are *spectrally negative Lévy processes*, that is, Lévy processes with no positive jumps. For these processes it is possible to define for $q \geq 0$ the *q-scale function* $W^q(x)$, which corresponds in some sense to the scale function of a regular diffusion (cf. the following subsection). The q-scale function has many applications in fluctuation theory and solving exit problems for these processes (cf. Bertoin (1996), Chapter VII, and Kyprianou (2006), Chapter 8).

2.3 DIFFUSIONS AND STOCHASTIC DIFFERENTIAL EQUATIONS

The superposition of a scaled Brownian motion σW, $\sigma \geq 0$, and a linear deterministic drift μt, $\mu \in \mathbb{R}$, is a process which evolves in time subject to uncertainty. Its dynamics can be represented in the form of a stochastic integral equation

$$X_t = X_0 + \int_0^t \mu ds + \int_0^t \sigma dW_s,$$

provided that sense can be made of the stochastic differential dW. Itô's stochastic integration theory tells us just how to make sense of such stochastic differentials, and we can then rewrite the above equation describing the stochastic dynamics of a Brownian motion with drift as as a *stochastic differential equation*

$$dX_t = \mu dt + \sigma dW_t,$$

$X_0 = x$. We can interpret this equation as follows: the dynamics of process X are *driven* by two processes, a deterministic drift process $t \mapsto t$ and a stochastic process $t \mapsto W_t$, and these processes have constant and deterministic coefficients μ and σ, respectively. The driving Brownian motion is the *source of randomness* in the model. This interpretation leads immediately to two very natural questions: first, what happens if we allow the coefficients to depend on the state of the process, and second, could we have a more general process as the source of randomness. Both questions have been addressed quite thoroughly, the first one by the theory of diffusions, the second one by the general stochastic integration theory as developed by P.-A. Meyer and his school. We will concentrate on the theory of diffusions in this subsection, while more general driving processes will be brought into picture in the next subsection.

For the considerations that are to follow, we require some additional definitions, since we want to characterize diffusions as continuous Markov processes.

Definition 2.39. *An ($\mathbb{F}$-)adapted process X is* (simple) Markov *with respect to $\mathbb{F}$, if for each $t \geq 0$, the σ-algebras $\mathscr{F}_t$ and $\sigma\{X_s \,|\, s \geq t\}$ are conditionally independent, given X_t.*

Loosely speaking, the (simple) Markov property means that given the present state of the process, the future of the process is independent of its past history. This property is a generalization of the independent increments property: hence, all Lévy processes are Markov processes. Simple Markov property is equivalent to the following: for $u \geq t$ and f a bounded and Borel measurable function

$$\mathbb{E}\{f(X_u) \,|\, \mathscr{F}_t\} = \mathbb{E}\{f(X_u) \,|\, \sigma\{X_t\}\}.$$

Consequently, for a Markov process X, $s < t$ and f bounded Borel, we can define a *transition function* via

$$\mathscr{P}_{s,t}(X_s, f) = \mathbb{E}\big\{f(X_t) \,|\, \mathscr{F}_s\big\}. \tag{2}$$

If the transition function satisfies the property $\mathscr{P}_{s,t} = \mathscr{P}_{t-s}$ for $t \geq s$, then the Markov process in question is *time homogeneous*, and the transition functions form a *transition semigroup* of operators $(\mathscr{P}_t)_{t\geq 0}$. For the purposes of this thesis, the class of time homogeneous Markov processes is sufficiently general.

Definition 2.40. *A time homogeneous simple Markov process is* strong Markov*, if for any stopping time T with $\mathbb{P}(T < \infty) = 1$ and $s \geq 0$,*

$$\mathbb{E}\Big\{f(X_{T+s}) \,|\, \mathscr{F}_T\Big\} = \mathscr{P}_s(X_T, f),$$

for f bounded and Borel measurable.

For detailed expositions of Markov processes, the reader is referred to Dynkin (1965), Chapter III of Rogers and Williams (1994) and Part II of Gikhman and Skorokhod (1975).

We are now in a position to give the general definition of a diffusion process (in the sense of Dynkin (1965)).

Definition 2.41. *A* diffusion *is a strong Markov process with almost surely continuous sample paths.*

The standard form of a time-homogeneous one-dimensional stochastic differential equation driven by a Brownian motion is

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \tag{3}$$

$X_0 = x \in I$, where $I \subseteq \mathbb{R}$ is the state space of the diffusion and $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to [0,\infty)$ are usually Lipschitz continuous functions satisfying some growth condition (cf. Øksendal (2003)). Typically, the conditions are needed to ensure the existence and uniqueness of an adapted and square integrable strong solution X. As usual, we will use symbols $\mathbb{P}_x$ and $\mathbb{E}_x$ for probabilities and expectations under the assumption $X_0 = x$. Solutions of stochastic differential equations such as (3) are diffusions, if a relatively mild continuity assumption is satisfied: if the coefficients are Lipschitz continuous, then the solution X is a strong Markov process. In fact, this last result remains true if the driving process is any Lévy process, see Protter (2004) Theorem V.32. However, not all diffusions can be represented via stochastic differential equations driven by a Brownian motion, and not all solutions of stochastic differential equations driven by a Brownian motion are diffusions. In Øksendal (2003) processes whose dynamics are characterized by (sufficiently regular) equations of form (3) are called *Itô diffusions*.

In this thesis we also restrict our attention to *regular* diffusions, that is, we assume that $\mathbb{P}_x(\tau_y < \infty) = 1$ for all $x, y \in I$, where $\tau_y = \inf\{t \,|\, X_t = y\}$. In other words, every point of the state space can be hit from every other point of the state space.

Any regular one-dimensional diffusion can be obtained by first time changing a Brownian motion and then making a transformation of the state space. These operations can be done with *speed measures* and *scale functions*, respectively. To be more specific, let X be a regular diffusion. Such a diffusion is said to be on *natural scale* if

$$\mathbb{P}_x(X_{\tau_{[a,b]}} = a) = \frac{b-x}{b-a} \text{ and } \mathbb{P}_x(X_{\tau_{[a,b]}} = b) = \frac{x-a}{b-a}$$

for every interval $[a,b] \subseteq I^o$, where $\tau_{[a,b]}$ is the first exit time of X from interval $[a,b]$.

Theorem 2.42. *There exists a strictly increasing continuous function $s(x)$ such that $s(X)$ is on natural scale on $s(I)$.*

Proof. See Bass (1997), pp. 79–80. □

The function $s(x)$ is the *scale function* of diffusion X. Given a regular diffusion on natural scale, a measure $m(dx)$ is the *speed measure* for the diffusion if

$$\mathbb{E}_x \tau_{[a,b]} = \int G_{a,b}(x,y) m(dy),$$

where $(a,b) \subseteq I$ and

$$G_{a,b}(x,y) = \begin{cases} 2(x-a)(b-y)/(b-a), & a < x \leq y < b \\ 2(y-a)(b-x)/(b-a), & a < y \leq x < b \\ 0, & \text{otherwise} \end{cases}$$

The speed measure governs how quickly the diffusion moves through intervals.

Theorem 2.43. *A regular diffusion on natural scale in an open interval I has a unique speed measure which characterizes the law of the diffusion.*

Proof. See Bass (1997), pp. 81–85. □

Both the scale function and the speed measure can be expressed in terms of the infinitesimal characteristics of the process:

$$s(x) = \int_c^x \exp\left\{ -\int_c^y \frac{2\mu(u)}{\sigma^2(u)} du \right\} dy,$$

where c is some constant, and

$$m(dx) = \frac{dx}{\sigma^2(x)}.$$

For more on diffusions and stochastic differential equations, we refer the reader to Øksendal (2003), Bass (1997) and Chapter V in Rogers and Williams (1994).

2.4 GENERATORS AND r-HARMONIC FUNCTIONS

Time-homogeneous stochastic differential equations driven by a Brownian motion have a strong connection to second order elliptic differential equations: we can associate with the stochastic differential equation (3) a second order linear differential operator $\mathscr{L}$ mapping functions $f \in \mathscr{C}_0^2(I;\mathbb{R})$ to continuous functions via

$$(\mathscr{L}f)(x) = \frac{\sigma^2(x)}{2} f''(x) + \mu(x) f'(x).$$

More precisely, for sufficiently regular functions this operator coincides with the *infinitesimal generator* of process X:

Definition 2.44. *If X is a time-homogeneous diffusion in $\mathbb{R}$, then the* infinitesimal generator $\mathscr{A}$ *of X is defined by*

$$\mathscr{A} f(x) := \lim_{t \downarrow 0} \frac{\mathbb{E}_x\{f(X_t)\} - f(x)}{t}$$

for $x \in \mathbb{R}$. The set of functions for which the limit exists for all $x \in \mathbb{R}$ is denoted by $\mathscr{D}_{\mathscr{A}}$.

For Itô diffusions, $\mathscr{C}_0^2(I;\mathbb{R}) \subset \mathscr{D}_{\mathscr{A}}$ and for $f \in \mathscr{C}_0^2(I;\mathbb{R})$, $\mathscr{A} f = \mathscr{L} f$ (cf. Øksendal (2003), Theorem 7.3.3). A closely associated concept is the *characteristic operator* of X.

Definition 2.45. *If X is a time-homogeneous diffusion in $\mathbb{R}$, then the* characteristic operator $\tilde{\mathscr{A}}$ *of X is defined by*

$$\tilde{\mathscr{A}} f(x) := \lim_{U \downarrow x} \frac{\mathbb{E}_x\{f(X_{\tau_U})\} - f(x)}{\mathbb{E}_x\{\tau_U\}}$$

for $x \in \mathbb{R}$, where the open sets U decrease to x in the sense that $U_k \supset U_{k+1}$ and $\cap_k U_k = \{x\}$. The set of functions for which the limit exists for all $x \in \mathbb{R}$ and all $\{U_k\}$ is denoted by $\mathscr{D}_{\tilde{\mathscr{A}}}$.

It is always the case that $\mathscr{D}_{\mathscr{A}} \subseteq \mathscr{D}_{\tilde{\mathscr{A}}}$ and $\mathscr{A} f = \tilde{\mathscr{A}} f$ for $f \in \mathscr{D}_{\mathscr{A}}$ (see Dynkin (1965), Part I, p. 143). Moreover, $\mathscr{C}^2(I;\mathbb{R}) \subset \mathscr{D}_{\tilde{\mathscr{A}}}$ and for $f \in \mathscr{C}^2(I;\mathbb{R})$, we have that $\tilde{\mathscr{A}} f = \mathscr{L} f$ (cf. Øksendal (2003), Section 7.5).

Classical one-dimensional *harmonic functions* are functions f such that

$$(\Delta f)(x) = f''(x) = 0,$$

where Δ is the second order differential operator known as the Laplacian. This concept of "Δ-harmonicity" generalizes naturally to $\mathscr{L}$-harmonicity, where $\mathscr{L}$ is any second order differential operator – or, more generally, an integro-differential operator. Of particular interest in the applications considered in this thesis is a special operator $\mathscr{L}_r$ derived from the operator $\mathscr{L}$ via

$$(\mathscr{L}_r f)(x) = (\mathscr{L} f)(x) - r f(x).$$

This operator corresponds to the process with infinitesimal generator $\mathscr{L}$ *killed* or *discounted* at rate r. The next definition spells out our use of terminology with respect to the harmonicity concepts (observe our notation: we use $\mathscr{G}$ and $\mathscr{G}_r$ to refer to (more general) integro-differential operators, while $\mathscr{L}$ and $\mathscr{L}_r$ are reserved for "pure" differential operators with no integral term present).

Definition 2.46. *Let $\mathscr{G}$ be an integro-differential operator. A function $f \in \mathscr{C}^2(I;\mathbb{R})$ with the necessary integrability properties is called* harmonic with respect to $\mathscr{G}$, *if $\mathscr{G}f = 0$. Analogously, a function f is called* r-harmonic with respect to $\mathscr{G}$, *if $\mathscr{G}_r f = 0$.*

If the equalities in the previous definition are replaced by the inequalities $\mathscr{G}f \leq 0$ and $\mathscr{G}_r f \leq 0$, respectively, them f is called *superharmonic* or *r-superharmonic with respect to $\mathscr{G}$*, respectively. If these inequalities are reversed, then f is called *subharmonic*. In cases when it is clear from the context which operator is being considered, we will drop the explicit reference "with respect to" and simply use the terms *harmonic* and *r-harmonic*. It is worth observing that the scale function $s(x)$ of a linear diffusion satisfies $\mathscr{L}s = 0$, i. e. is harmonic with respect to the infinitesimal generator $\mathscr{L}$ of the diffusion, while the q-scale function $W^q(x)$ of a spectrally negative Lévy process satisfies $\mathscr{G}_q W^q = 0$, i.e. is q-harmonic with respect to the infinitesimal generator $\mathscr{G}$ of the process.

A crucially important result for strong Markov processes and their generators is

Theorem 2.47 (Dynkin's formula)**.** *Let X be a strong Markov process with infinitesimal generator $\mathscr{G}$ and $f \in \mathscr{D}_{\mathscr{G}}$. Then*

$$f(X_t) - f(x) - \int_0^t (\mathscr{G}f)(X_{s-})ds$$

is a local martingale.

Proof. See Dynkin (1965), part I, p. 133. □

From Theorem 2.47 immediately follows the more usually encountered formulation of Dynkin's formula for Itô diffusions: for stopping times τ such that $\mathbb{E}\{\tau\} < \infty$,

$$\mathbb{E}_x\{f(X_\tau)\} = f(x) + \mathbb{E}_x\Big\{\int_0^\tau (\mathscr{G}f)(X_s)ds\Big\}$$

for $f \in \mathscr{C}_0^2(I;\mathbb{R})$, when X is an Itô diffusion. From Dynkin's formula we see that for a function $f \in \mathscr{C}^2(I;\mathbb{R})$ with the necessary integrability properties, we have the implication

$$(\mathscr{G}f)(x) \leq 0 \text{ for all } x \in I \ \Rightarrow\ \mathbb{E}_x\{f(X_\tau)\} \leq f(x) \text{ for all } \tau \in \mathscr{T}, x \in I. \quad (4)$$

Hence for such functions our definition of superharmonicity via the generator implies superharmonicity in the – in optimal stopping applications more usual – sense that the left side of (4) is satisfied. However, we need to be careful when dealing with functions that are not $\mathscr{C}^2$, as is the case in Paper I. A concept closely related to superharmonicity is *excessivity*.

Definition 2.48. *A nonnegative lower semicontinuous function $f : I \to \mathbb{R}$ is called r-*excessive with respect to *X, if $\mathbb{E}_x\{e^{-rt}f(X_t)\} \leq f(x)$ for all $t > 0$, $x \in I$.*

Quite obviously, nonnegative superharmonic functions in the sense of the left side of (4) are excessive. In Hunt (1958) it it shown that for quasi-left continuous strong Markov processes, nonnegative excessive functions are superharmonic in the sense of the left side of (4).

The research papers gathered together in this thesis utilize the intimate connection between the increasing solution $\psi(x)$ of the characteristic integro-differential equation $\mathscr{G}_r\psi = 0$ and the value functions of certain optimal control problems to solve these problems for a certain class of strong Markov processes – namely *spectrally negative jump diffusions*. For Itô diffusions this connection has been established in literature earlier – one significant difference in our framework is that while for linear diffusions the existence and uniqueness of both an increasing solution $\tilde{\psi}(x)$ and a decreasing solution $\tilde{\phi}(x)$ of the characteristic linear differential equation $\mathscr{L}_r f = 0$ are known and these solutions can be completely characterized (cf. Borodin and Salminen (2002), pp. 18–20), for integro-differential equations the situation is more complicated.

2.5 JUMP DIFFUSIONS AND LÉVY STOCHASTIC DIFFERENTIAL EQUATIONS

In this last subsection of the second section in our Introduction, we recall some basics about stochastic differential equations with more general driving processes than a Brownian motion. In particular, for purposes of more realistic modeling, we want the solutions to be such that discontinuous sample paths are allowed. However, we still want the solution to be a strong Markov process, and hence we do not consider the most general setup possible (with driving processes being only semimartingales). By virtue of Theorem V.32 in Protter (2004), we know that for the solution to be strong Markov, the driving process must be a Lévy process. Given the results concerning Lévy processes from subsection 2.2, the relevant stochastic differential equation can written as follows:

$$dX_t = \mu(X_{t-})dt + \sigma(X_{t-})dW_t + \int_{\mathscr{S}} \gamma(X_{t-}, z)\tilde{N}(dt, dz), \tag{5}$$

$X_0 = x \in I$, where $I \subseteq \mathbb{R}$, $\mathscr{S} \subset \mathbb{R}$, $\tilde{N}$ is a compensated Poisson random measure with characteristic Lévy measure ν, and coefficient functions μ, σ and γ are sufficiently regular (cf. Øksendal and Sulem (2005)).

As was previously mentioned, a jump diffusion (or a *Lévy diffusion*) is a strong Markov process. Moreover, as its jump times coincide with the jump times of the driving Lévy process, it is also left-continuous over stopping times (cf. Paper I, Section 2) and is hence a *Hunt process*. For $f \in \mathscr{C}_0^2(I;\mathbb{R})$, the infinitesimal generator of the process with stochastic dynamics given by (5) is

$$(\mathscr{G}f)(x) = \mu(x)f'(x) + \sigma(x)f''(x) + \int_{\mathscr{S}} \left\{ f(x+\gamma(x,z)) - f(x) - \gamma(x,z)f'(x) \right\} \nu(dz). \tag{6}$$

Dynkin's formula holds for jump diffusions as well (cf. Øksendal and Sulem (2005), Theorem 1.24). Using the change-of-variable formula for semimartingales from Peskir (2007), we can extend the implication (4) from $\mathscr{C}^2$ functions to continuous functions which are $\mathscr{C}^2$ outside a finite subset of I (see Paper I).

Since our research interest here has been directed towards downside risk as defined in subsection 1.4, we consider in Papers I–V only *spectrally negative* jump diffusions, that is, processes with dynamics given by (5) and $\gamma(x,z) \leq 0$ for all $(x,z) \in I \times \mathscr{S}$. With the sole exception of Paper I, we have furthermore found it convenient to restrict the form of γ even further by requiring the jump sizes to be proportional to the current state of the process and setting hence $\gamma(x,z) = -xz$. This restriction leads to significant gains on the tractability side while retaining much flexibility as we can choose the Poisson random measure $\tilde{N}$ fairly freely. In fact, it is possible to show that in the presence of proportional jumps there exists an exponential martingale $M = \{M_t\}_{t \in [0,\infty)}$

such that multiplication of the jump diffusion X by M yields a process whose dynamics are characterized by an ordinary Itô stochastic differential equation driven by a Brownian motion. Even though some of our results, especially the general theorems of Paper III, are applicable also in the case where the jumps have infinite activity, mostly we consider the finite jump activity case $\nu(\mathbb{R} \setminus \{0\}) < \infty$ – in this case, the requirements on the Lévy measure boil down to the existence of a continuously differentiable density for the jump size distribution $\mathfrak{m}$ (where $\nu = \lambda \mathfrak{m}$).

Moreover, a standing assumption (familiar from the theory of linear diffusions) is that the boundaries of the state space I are *natural boundaries* for X – that is, in absence of control the boundaries are unattainable in finite time. This implies that the lifetime of the uncontrolled process $\xi = \infty$.

The assumption of spectral negativity yields the following result, which is a cornerstone of our analysis. This theorem – a reproduction of Lemma 3.2 from Paper I – states that for a jump diffusion X started from x, the moment generating function of τ_y, the first hitting time of a level $y > x$, has a representation in terms of the minimal smooth and increasing r-harmonic function. We wish to point out here that in Kou and Wang (2003), the authors obtain similar representations for first passage times of a level in the case of a Brownian motion augmented with compound Poisson jumps when jump sizes have a double exponential distribution. For continuous Markov processes, a representation of this type was obtained at least as early as in 1953 (cf. Darling and Siegert (1953)).

Theorem 2.49. *For a jump diffusion X started from $x < y$, $x, y \in I$,*

$$\mathbb{E}_x \left\{ e^{-r\tau_y} \right\} = \psi(x)/\psi(y),$$

where $\psi \in \mathscr{C}^2(I;\mathbb{R})$ is increasing and solves $\mathscr{G}_r \psi = 0$.

Proof. See Paper I, proof of Lemma 3.2. □

By virtue of this representation, it is possible to express the value functions of several optimal control problems in terms of the solution of $\mathscr{G}_r \psi = 0$, and we turn our attention to this in the following last section of this Introduction. Of course, it is *a priori* not clear whether a smooth function ψ solving $\mathscr{G}_r \psi = 0$ exists – while some research papers give existence and uniqueness results for solutions of integro-differential equations even more general than the ones considered in this thesis (cf. Alvarez and Tourin (1996) and Jakobsen and Karlsen (2006)), typically the solutions then are not solutions in the classical sense (that is, elements of $\mathscr{C}^2(I;\mathbb{R})$), but *viscosity solutions* (that is, elements of some larger function space, such as the space of all continuous – or all semi-continuous – functions). The standard reference for viscosity solutions theory is Crandall et al. (1992), which deals with second order differential equations. Extensions of the theory to integro-differential equations are discussed in Alvarez and Tourin (1996), Jakobsen and Karlsen (2006) and Barles and Imbert (2008).

The difficulties in establishing the existence of an increasing smooth solution to $\mathscr{G}_r\psi = 0$ led to Paper V, where the existence result is obtained for a class of jump diffusion models whose drift and diffusion coefficients satisfy certain regularity conditions. Typical representatives of this class are jump diffusion versions of the logistic model.

3 CONSIDERED CONTROL PROBLEMS

3.1 GENERAL FORMULATION

In control theory, the usual setup is that we are given a dynamical system, which we are allowed to control subject to some restrictions. Allowed control actions are called *admissible*, and when solving an *optimal control problem*, we try to find the admissible control actions leading to the best possible result as measured by some specified criterion described mathematically by a *objective functional*. While classical control theory deals with deterministic systems described by ordinary or partial differential equations, in stochastic control theory the systems to be controlled are stochastic and consequently their dynamics are described with stochastic differential equations. A comprehensive and up-to-date overview on optimal control theory for Markov processes is Fleming and Soner (2006), to which we refer for the results of this section unless otherwise indicated.

Much of economics and finance deals with *optimization*, and most often the objective criterion is either maximization of profits or minimization of losses. As financial quantities such as the cash flows from an investment or the value of a firm's assets are both dynamical and stochastic in nature, stochastic control theory seems to be tailor-made for tackling these problems.

In all the research papers constituting this thesis the mathematical problem considered is of the following type: given a jump diffusion X with dynamics given by an equation of type (5), find the admissible control process $Z^* = \{Z_t^*\}_{t\in[0,\infty)}$ such that

$$\sup_{Z\in\mathscr{A}} \mathbb{E}_x \int_0^{\xi^Z} e^{-rs} f(X_{s-}^Z)dZ_s = \mathbb{E}_x \int_0^{\xi^Z} e^{-rs} f(X_{s-}^{Z^*})dZ_s^*, \tag{7}$$

where $r > 0$ is the applied constant discount rate, ξ^Z is the lifetime of the controlled process X^Z, $f : I \to \mathbb{R}$ is a functional describing the instantaneous marginal yields accrued from exerting the control Z, and the class of admissible control processes $\mathscr{A}$ is specified by certain restrictions. In particular, in line with optimal dividend distribution and optimal harvesting applications, we assume that control actions take the form of *displacements* (i. e. bring the state of the controlled process down from the current level). In other words, the admissible controls can be interpreted as *payout policies* or *harvesting strategies*.

Let us now see how different additional restrictions on the class of admissible controls lead to different control problems.

3.2 OPTIMAL STOPPING

If we are only allowed to control X by irreversibly liquidating all value, we can choose only the moment at which to stop the process once-and-for-all. In this case problem (7) transforms into an *optimal stopping problem*

$$V(x) = \sup_{\tau \in \mathscr{T}} \mathbb{E}_x \left\{ e^{-r\tau} f(X_\tau) \right\}, \tag{8}$$

where $\mathscr{T}$ is the set of all $\mathbb{F}$-stopping times. Optimal stopping problems originated in A. Wald's sequential analysis (cf. Wald (1947)) in a discrete time framework. Broadly speaking, there are two main approaches to solving optimal stopping problems: the *martingale approach* initiated by J. Snell (cf. Snell (1952)) with his characterization of the solution by what is nowadays called *Snell's envelope*, the smallest supermartingale dominating the reward; and the *Markovian approach* relying on the analytical tools provided by the theory of Markov processes. This second approach has its origins in Kolmogorov's studies (cf. Kolmogorov (1931)). The methods used in this thesis represent this second approach, more precisely, we rely strongly on the *superharmonic characterization* of the solution due to E. Dynkin (cf. Dynkin (1963)). This characterization states that the value function is the smallest superharmonic majorant of the reward.

The Markovian structure simplifies the optimal stopping problem quite a bit: it is intuitively clear that by virtue of the strong Markov property, we do not need to consider controls depending on the past history of the process. Instead, it suffices to consider *feedback controls* depending only on the current state $X_t \in I$. This implies that we can partition the state space I into a *continuation region* C and a *stopping region* $S = I \setminus C$. Since the value $V(x)$ dominates the reward $f(x)$, and furthermore in the stopping region $V(x) = f(x)$, we have

$$C = \{x \in I \,|\, V(x) > f(x)\} \text{ and } S = \{x \in I \,|\, V(x) = f(x)\}.$$

If $V(x)$ is lower semicontinuous and $f(x)$ is upper semicontinuous, then C is open and S is closed; hence the first entry time of X into S, $\tau_S = \inf\{t \geq 0 \,|\, X_t \in S\}$ is a stopping time because X is càdlàg. We have the following nice result concerning the existence of an optimal stopping time:

Theorem 3.1. *Suppose $f(x)$ is upper semicontinuous and satisfies*

$$\mathbb{E}_x\left(\sup_{t\geq 0}|f(X_t)|\right)<\infty.$$

If $V(x)$ is lower semicontinuous, then:

(a) $\mathbb{P}_x(\tau_S<\infty)=1$ for all $x\in I$ implies that τ_S is the optimal stopping time;

(b) $\mathbb{P}_x(\tau_S<\infty)<1$ for some $x\in I$ implies that, almost surely, there is no optimal stopping time.

Proof. See Peskir and Shiryaev (2006), pp. 34–48. □

It is well-known that via the superharmonic characterization, determination of $V(x)$ can be reduced to a *free-boundary problem*

$$\begin{cases}(\mathscr{G}_r V)(x)\leq 0, & x\in I\\ V(x)\geq f(x), & (V(x)>f(x), x\in C; V(x)=f(x), x\in S)\end{cases}\tag{9}$$

where so far both $V(x)$ and C are unknown (observe that determining C is equivalent to determining its boundary ∂C – hence the terminology). If $f(x)$ is continuous on I, then $\hat{V}(x)=\mathbb{E}_x[e^{-r\tau_S}f(X_{\tau_S})]$ solves the *Dirichlet problem*

$$\begin{cases}(\mathscr{G}_r\hat{V})(x)=0, & x\in C\\ \hat{V}(x)=f(x), & x\in\partial C.\end{cases}\tag{10}$$

By Theorem 3.1, if $\mathbb{P}_x(\tau_S<\infty)=1$ for all $x\in I$, then $V(x)=\hat{V}(x)$. Hence the value function of the stopping problem is a solution of the Dirichlet problem. However, solving the stopping problem still requires determination of the optimal free boundary ∂C.

As optimal stopping problems are dynamic programming problems, they can also be solved by applying *Bellman's optimality principle*, which in this context takes the well-known form of *Hamilton–Jacobi–Bellman integro-variational inequality*

$$\max\{(\mathscr{G}_r V)(x), f(x)-V(x)\}=0, \forall x\in I$$

(cf. Øksendal and Sulem (2005), Chapter 2). From this formulation one can derive the previous free-boundary problem characterization of the solution by assuming that we can partition I into a continuation region and a stopping region (cf. Lempa (2007)).

Our exposition in this subsection relied on Peskir and Shiryaev (2006), and we refer the reader to this monograph for additional information on the general theory of optimal stopping.

3.3 SINGULAR STOCHASTIC CONTROL

In classical control theory the displacement of the state due to control is differentiable in time. *Singular control models* first considered in Bather and Chernoff (1967a) and Bather and Chernoff (1967b) allow a displacement which is not absolutely continuous with respect to the time variable t.

If we are allowed to control X by bringing the state down by any amount at any time and these control actions incur no transaction costs, then our control problem (of maximizing the utility of the sum of all displacements) is transformed into a singular stochastic control problem

$$J(x) = \sup_{Z \in \mathscr{A}} \mathbb{E}_x \int_0^{\xi} e^{-rs} f(X_s^Z) dZ_s, \tag{11}$$

where the class of admissible control policies $\mathscr{A}$ consists of all adapted, nonnegative and nondecreasing càdlàg processes $Z = \{Z_t\}_{t \in [0,\infty)}$. These control policies can be singular with respect to the Lebesgue measure dt (whence the name). In fact, it turns out that the optimal control usually is of this type.

Similarly to the case of stopping problems, the Markovian structure is very helpful in simplifying the control problem: we can consider the state space I and split it into two regions, a *action region* A and a *inaction region* $N = I \setminus A$. From A the process is immediately moved into N, where its exit from N by creeping (continuous movement) is prevented by *reflection* at ∂N in an appropriate direction.

In our setup of spectrally negative processes, a further simplification is achieved if the set $N = (a,b)$ with $a = \inf I$. Then the only possible jump in the control process occurs at time 0, if $X_0 = x > b$. After the initial time the process can hit level b only by creeping and controlling via reflecting keeps it in N. The resulting control process $\hat{Z}$ is a combination of a potential initial impulse $\max\{x-b,0\}$ and the *local time* of X on the boundary $\partial N (= b)$. The local time process is the solution of the *Skorokhod problem*

$$\begin{cases} X_t^{\hat{Z}} = x + \int_0^t \left\{ \mu(X_{s-}^{\hat{Z}})ds + \sigma(X_{s-}^{\hat{Z}})dW_s - \int_{\mathscr{S}} \gamma(X_{s-}^{\hat{Z}}, z)\tilde{N}(ds,dz) - f(X_{s-}^{\hat{Z}})d\hat{Z}_s \right\}, \\ \hat{Z}_0 = 0,\ \int_0^t \mathbf{1}_{\{X_s^{\hat{Z}} \notin \partial N\}} d\hat{Z}_s = 0, \forall t \geq 0,\ \hat{Z} \text{ continuous and increasing,} \end{cases} \tag{12}$$

and the controlled process $X^{\hat{Z}}$ is X *with oblique reflection at* ∂N. The problem of determining the value function $J(x)$ can be reduced to a free-boundary problem of *Neumann* type

$$\begin{cases} (\mathscr{G}_r \hat{J})(x) = 0, & x \in N \\ \hat{J}'(x) = f(x), & x \in \partial N \end{cases} \tag{13}$$

by virtue of the following result.

Theorem 3.2. *A solution of the Neumann problem* (13) *satisfies*

$$\hat{J}(x) = \mathbb{E}_x \int_0^\infty e^{-rs} f(X_s^Z) dZ_s$$

for Z solving the Skorokhod problem (12).

Proof. Let $\{\tau_n^Z\}_{n\in\mathbb{N}}$ be a sequence of stopping times such that $\tau_n^Z \to \infty$, as $n \to \infty$, and $\tau_n^Z < \infty$ almost surely for each $n < \infty$ (for a, b finite, take exit times from intervals $(a+1/n, b-1/n)$; for infinite-valued boundaries, take exit times from intervals $(-n,n)$).

Apply Itô formula to obtain

$$\begin{aligned} e^{-r\tau_n^Z} \hat{J}(X_{\tau_n^Z}^Z) &= \hat{J}(x) + \int_{0+}^{\tau_n^Z} e^{-rs} \left(\mathscr{G}_r \hat{J}\right)(X_{s-}^Z)ds + \int_{0+}^{\tau_n^Z} e^{-rs} \hat{J}'(X_{s-}^Z)\sigma(X_{s-}^Z)dW_s - \\ &- \int_{0+}^{\tau_n^Z} e^{-rs} \hat{J}'(X_{s-}^Z) \int_{\mathscr{S}} \gamma(X_{s-}^Z, z)\tilde{N}(ds,dz) - \int_{0+}^{\tau_n^Z} e^{-rs} \hat{J}'(X_t^Z) dZ_s. \end{aligned}$$

Because $\hat{J}(x)$ solves $\mathscr{G}_r \hat{J} = 0$ on N and for the reflected process $X_t^Z \in \bar{N}$ for all $t > 0$, because Itô integrals of predictable and locally bounded integrands preserve local martingales, and because $\hat{J}'(x) = f(x)$ on $\partial N = \text{support}(Z)$, taking expectations yields

$$\mathbb{E}_x \left[e^{-r\tau_n^Z} \hat{J}(X_{\tau_n^Z}^Z) \right] = \hat{J}(x) - \mathbb{E}_x \int_{0+}^{\tau_n^Z} e^{-rs} f(X_{s-}^Z) dZ_s.$$

Letting now $n \to \infty$, we obtain the desired result. □

Again, to solve the control problem, one needs to determine the optimal reflection boundary b and thus fix a particular solution of the Neumann problem (13).

Being dynamic programming problems, also singular stochastic control problems can be approached via Hamilton–Jacobi–Bellman integro-variational inequalities, which in this case take the form

$$\max\{(\mathscr{G}_r J)(x), f(x) - J'(x)\} = 0, \forall x \in I$$

(cf. Øksendal and Sulem (2005), Chapter 5).

For more detailed expositions about singular stochastic control of Markov processes, we refer the reader to Chapter 5 in Øksendal and Sulem (2005), and Chapter VIII in Fleming and Soner (2006).

3.4 STOCHASTIC IMPULSE CONTROL

If we are allowed to control X by paying out any part of the value at any time but each action incurs a fixed intervention cost $c > 0$, we obtain a stochastic impulse control problem

$$J_I^c(x) = \sup_{\{(\tau_i,\zeta_i)\}_{i\in\mathscr{I}}} \mathbb{E}_x \left\{ \sum_{i\in\mathscr{I}} e^{-r\tau_i} \big(g(X_{\tau_i-}, \zeta_i) - c\big) \right\}, \tag{14}$$

where τ_i, $i \in \mathscr{I}$ are the intervention times, $0 < \zeta_i \leq X_{\tau_i-}$, $i \in \mathscr{I}$ are the interventions or impulses (displacements), which are assumed to be $\mathscr{F}_{\tau_i}$-measurable, $\mathscr{I}$ is an index set at most countably infinite, and function $g(x,z)$ describes the utility derived from an impulse of magnitude z when in state x. An impulse of magnitude z will cause the state of the process to jump immediately from X_{τ_i-} to $X_{\tau_i-} - z$. Thus, in general we need to choose both a sequence of intervention times τ_i and the sizes of interventions ζ_i. Intuitively, the optimal policy will consist of discrete impulses, as very frequent controlling – such as is implied by a reflection policy – would incur very high transaction costs offsetting the benefits of added flexibility in controlling.

Note that an optimal stopping problem is a special case of the impulse control problem, which can be obtained by restricting the sequence of intervention times to consist of exactly one element τ and the corresponding single impulse to be equal to $\zeta = X_\tau$. The result is

$$\sup_{\tau\in\mathscr{T}} \mathbb{E}_x \left(e^{-r\tau}(g(X_\tau) - c)\right);$$

setting $c = 0$ reproduces the optimal stopping problem in absence of transaction costs. On the other hand, an impulse control is an increasing and non-negative, càdlàg pure jump process and would hence qualify as an admissible control for a singular stochastic control problem. Hence we see that all three control problems can be viewed as the same basic problem with a varying degree of flexibility in controlling, as is pointed out in Alvarez and Virtanen (2006).

Impulse control problems can be solved by a version of Hamilton–Jacobi–Bellman approach using so-called *quasi-integrovariational inequalities*

$$\begin{cases} \max\{(\mathscr{G}_r J_I)(x), (\mathscr{M} J_I)(x) - J_I(x)\} = 0, & x \in I \\ J_I(x) = (\mathscr{M} J_I)(x), & x \in \partial N, \end{cases} \tag{15}$$

where the *intervention operator* $\mathscr{M}$ acts on measurable functions $J(x)$ via

$$(\mathscr{M} J)(x) = \sup_{z \in [0,x]} \{g(x,z) + J(x-z)\} \tag{16}$$

(cf. Øksendal and Sulem (2005), Chapter 6).

The use of quasi-variational inequalities in solving impulse control problems is due to A. Bensoussan and J. Lions (cf. Bensoussan and Lions (1980) for an overview).

3.5 BARRIER AND BAND STRATEGIES

In all three control problems presented in the previous subsections, the sets of admissible controls are *a priori* quite large and may contain very exotic elements. In contrast, for optimal stopping and singular control problems *barrier strategies* or *threshold rules* constitute a very simple subclass of admissible controls. In these strategies a control action is taken every time the controlled process hits a certain barrier level $y \in I$: in a stopping problem, we simply stop the process at the first hitting time of level y, while in a singular control problem, we reflect the process downward at level y. In terms of the push region A of the previous section, this corresponds to taking $A = [y, \infty)$. In an impulse control problem, the corresponding simple strategy is a *band strategy*: we make an intervention at each hitting time of level y to bring the state of the process to level $z < y$.

In the general case determination of the optimal control strategy entails optimization over a very complicated set, for which it is usually not possible to characterize the elements explicitly. But if we could show that the optimal strategy is in fact of barrier (or band) type, then the situation would simplify dramatically: we could simply optimize with respect to the barrier level y (band levels y and z) over the interval $I \subseteq \mathbb{R}$ ($I \times I$) of possible barrier (band) levels, as the optimal barrier (band) strategy would also be optimal overall. Hence the dynamic optimization problem would reduce to a static nonlinear programming problem. In light of this, obtaining sufficient conditions for optimality of barrier or band strategies is of interest.

This approach was applied in the general framework of linear diffusions in Alvarez (1999), Alvarez (2000), Alvarez (2001) and Alvarez (2004). In the research papers of this thesis, we show that this approach works also for a class of spectrally negative jump diffusions. This observation relies on Theorem 2.49, which combined with spectral negativity yields as an immediate corollary the following representation of the value of a barrier strategy in the context of optimal stopping.

Corollary 3.3. *Let $x, y \in I$, $x < y$, and let $g(x)$ be continuous. Then*

$$\mathbb{E}_x\{e^{-r\tau_y} g(X_{\tau_y})\} = g(y)\frac{\psi(x)}{\psi(y)},$$

where $\psi(x)$ is an increasing solution of $\mathscr{G}_r\psi = 0$.

In Paper IV, we apply this representation to establish a certainty equivalence relation between the value of an optimal stopping problem for a geometric Lévy process and the value of a deterministic optimal timing problem.

Of course, the optimal strategy is not always a barrier strategy. To give an example, in Loeffen (2008), dividend optimization for a spectrally negative Lévy process is considered and a sufficient condition for the optimality of a barrier strategy in terms of the q-scale function of the Lévy process is derived. Azcue and Muler (2005) in turn prove that optimal dividend policy in absence of transaction costs is a band strategy in the Cramér–Lundberg model when controlling risk via reinsurance is allowed.

4 SUMMARIES OF INCLUDED PAPERS

PAPER I: A CLASS OF SOLVABLE OPTIMAL STOPPING PROBLEMS OF SPECTRALLY NEGATIVE JUMP DIFFUSIONS

In this study we consider the optimal stopping problem

$$V(x) = \sup_{\tau \in \mathscr{T}} \mathbb{E}_x \left\{ e^{-r\tau} g(X_\tau) \right\}$$

for a class of spectrally negative jump diffusions with stochastic dynamics given by a stochastic differential equation of type (5), where the state space $I = (a,b) \subset \mathbb{R}$ is an open interval and the boundaries a and b are assumed to be unattainable in finite time. The coefficient functions are assumed to be such that a unique strong square integrable solution of (5) exists (cf. Øksendal and Sulem (2005), Theorem 1.19 for the most usual sufficient conditions for this).

We show that for a continuous reward function g which is $\mathscr{C}^2$ outside a finite set $\mathscr{N} \subset I$, the value of the optimal stopping problem has a representation in terms of the increasing smooth solution of the integro-differential equation $\mathscr{G}_r \psi = 0$, that is, that

$$V(x) = \psi(x) \sup_{y \geq x} \left\{ \frac{g(y)}{\psi(y)} \right\}.$$

This representation implies that the dynamic optimization problem can be reduced to a standard nonlinear static optimization problem, where we optimize over threshold levels $y \in I$. As we consider rewards that are only continuous, we need to consider solutions in viscosity sense. We also prove that a similar representation is valid for any $\mathscr{C}^2$ transformation of our jump diffusion, although for a different class of reward functions.

Moreover, we obtain useful inequalities by showing that the value of the problem for a jump diffusion can be sandwiched between the values of the corresponding problem for an associated continuous diffusion. We also consider comparative statics and establish some sufficient conditions under which increased volatility and increased jump intensity unambiguously decelerate the rational exercise.

PAPER II: OPTIMAL PAYOUT POLICY IN PRESENCE OF DOWNSIDE RISK

We consider the stochastic dividend optimization problem in absence of transaction costs, that is, the singular stochastic control problem

$$V_S(x) = \sup_{D\in\mathbb{A}} \mathbb{E}_x \int_0^{\tau_0^D} e^{-rs} dD_s,$$

where τ_0^D is the lifetime of the controlled cash reserve X^D and $\mathbb{A}$ is the class of admissible dividend policies consisting of all adapted, nonnegative and nondecreasing càdlàg processes. The cash reserve X^D, from which the dividends are paid out, evolves in accordance with the stochastic differential equation

$$dX_t^D = \mu(X_{t-}^D)dt + \sigma(X_{t-}^D)dW_t - X_{t-}^D \int_0^1 z\tilde{N}(dt,dz) - D_t,$$

$X_0^D = 0$, where D is the cumulative dividends process. We make the following assumptions concerning the uncontrolled cash reserve process X:

(i) The expected cumulative present value is finite, i.e.

$$\mathbb{E}_x \int_0^\infty e^{-rs} X_{s-} ds < \infty;$$

(ii) X is regular in the sense that for all $x,y \in (0,\infty)$, $\mathbb{P}_x(\tau_y < \infty) = 1$;

(iii) Upper boundary ∞ of the state space is natural (unattainable in finite time);

(iv) $\mu(x)$ and $\sigma(x)$ are analytic at $x = 0$ and satisfy $\mu(0) = \sigma(0) = \sigma'(0) = 0$, $\mu'(0) \geq 0$;

(v) cash flows $\{|\mu(X_{t-})|\}_{t\geq 0}$ and $\{\sigma^2(X_{t-})\}_{t\geq 0}$ have finite expected cumulative present values;

(vi) The characteristic measure ν associated to the compensated Poisson random measure $\tilde{N}(dt,dz)$ has finite total mass and can be represented as $\nu(dz) = \lambda \mathfrak{m}(dz) = \lambda f_{\mathfrak{m}}(z)dz$ with $f_{\mathfrak{m}}(z)$ continuously differentiable.

We also consider the associated optimal stopping problem

$$V_{OSP}(x) = \sup_{\tau\in\mathscr{T}} \mathbb{E}_x \left[e^{-r\tau} X_\tau\right],$$

where $\mathscr{T}$ is the set of all stopping times.

In addition, we consider stochastic dividend optimization in the presence of a fixed transaction cost $c > 0$, that is, the stochastic impulse control problem

$$V_I^c(x) = \sup_{(\tau,\xi)\in\mathscr{V}} \mathbb{E}_x \left[\sum_{i=1}^{N} e^{-r\tau(i)} \left(\xi(i) - c\right) \right],$$

where the class of admissible controls $\mathscr{V}$ consists of sequences of intervention times $\{\tau(i)\}_{i=1}^N$ and impulses $\{\xi(i)\}_{i=1}^N$. We also consider the associated optimal stopping problem

$$V_{OSP}^c(x) = \sup_{\tau\in\mathscr{T}} \mathbb{E}_x \left[e^{-r\tau}(X_\tau - c) \right].$$

We establish sufficient conditions on the *net appreciation rate* $\rho(x) = \mu(x) - rx$ for the solvability of the considered problems in terms of the minimal $\mathscr{G}_r$-harmonic map, that is, the unique increasing solution $\psi(x)$ of the integro-differential equation $\mathscr{G}_r f = \mathscr{G} f - rf = 0$, where $\mathscr{G}$ is the infinitesimal generator of the underlying reserve process X. Utilizing this representation of the value function of the control problem in terms of $\psi(x)$, we extend the results of Alvarez and Virtanen (2006) by showing that the values of the considered control problems are ordered in a particularly strong fashion, which is in accordance with our intuition. Namely, as impulse controls and stopping rules are admissible controls for the singular control problem, and stopping rules are impulse controls, the considered control problems can be ordered according to the degree of flexibility in controlling. We show that not only are the values themselves ordered as could be intuitively expected, but also their marginal values (derivatives) are similarly ordered: the value of the singular control problem is not only greater than that of the problems with less flexibility in controlling, but it also grows faster.

PAPER III: ON SINGULAR STOCHASTIC CONTROL AND OPTIMAL STOPPING OF SPECTRALLY NEGATIVE JUMP DIFFUSIONS

In this study we extend some of the results of Alvarez and Virtanen (2006), Alvarez and Lempa (2008) and Paper II, which relate to linear payoffs (constant yields) or deal with continuous linear diffusions, to nonlinear payoffs (state-dependent yields) in spectrally negative Lévy models. More precisely, we consider the stochastic optimal control problems

$$J(x) = \sup_{Z\in\Lambda} \mathbb{E}_x \int_0^\infty e^{-rs} g'(X_{s-}^Z) dZ_s$$

and

$$V(x) = \sup_{\tau\in\mathscr{T}} \mathbb{E}_x[e^{-r\tau} g(X_\tau)],$$

where the function $g(x) \in \mathscr{C}^2(\mathscr{I};\mathbb{R})$ is nondecreasing and satisfies inequalities $\lim_{x\downarrow 0} g(x) = 0 < \lim_{x\to\infty} g(x)$, Λ is the class of nonnegative, nondecreasing, right-continuous and adapted processes and $\mathscr{T}$ is the set of $\mathbb{F}$-stopping times.

The underlying jump diffusion is assumed to satisfy assumptions similar to assumptions (i)-(vi) of Paper II, with the exception that (iv) is replaced by an assumption about the existence of a smooth increasing $\psi(x)$; in this case also the lower boundary of the state space needs to be assumed unattainable (in the setup of Paper II, this follows directly from (iv)). Moreover, in (vi) we can dispense with the requirement that ν has finite total mass: our general representation result holds true also in the infinite jump activity case.

We give sufficient conditions for the values of the considered stochastic control problems to have a representation in terms of the minimal increasing r-harmonic map, both in the continuous diffusion and the jump diffusion case. In addition, we show that the values of these problems in the jump diffusion case can be bounded from above and below by the values of certain corresponding control problems of an associated continuous diffusion. Naturally, this sandwiching result is valid only in the case of a finite jump activity, as the associated continuous diffusion can be defined only if $\nu = \lambda\mathfrak{m}$ for a finite $\lambda > 0$.

We also show that the results on the impact of flexibility on valuation, obtained for a class of linear diffusions in Alvarez and Virtanen (2006) and for a class of jump diffusions in Paper II, generalize to the setup with nonlinear reward structures.

PAPER IV: INVESTMENT TIMING IN PRESENCE OF DOWNSIDE RISK: A CERTAINTY EQUIVALENT CHARACTERIZATION

We study the values of single threshold investment rules

$$J^y(x) = \mathbb{E}_x \left[e^{-r\tau_y} g(X_{\tau_y})\right]$$

for continuous and nondecreasing payoffs $g(x)$ in a geometric Lévy model

$$dX_t = X_{t-}\left\{\mu dt + \sigma dW_t - \int_0^1 z\tilde{N}(dt,dz)\right\},$$

$X_0 = x$, $\sigma > 0$, satisfying the following assumptions:

(i) the characteristic measure ν of the Poisson random measure $\tilde{N}(dt,dz)$ has a representation $\nu = \lambda\mathfrak{m}$ with $\mathfrak{m}(dz)$ absolutely continuous with respect to the Lebesgue measure;

(ii) $\mu + \lambda\,\mathbb{E}[z + \ln(1-z)] > \frac{\sigma^2}{2}$.

In this case the unique increasing solution of the integro-differential equation $\mathscr{G}_r f = 0$ is $\psi(x) = x^{\Psi}$, where ψ is the unique positive root of equation

$$P(\psi) = \frac{\sigma^2}{2}\psi(\psi-1) + \tilde{\mu}\psi - \tilde{r} + \lambda\int_0^1 (1-z)^{\Psi}\mathfrak{m}(dz) = 0,$$

where $\tilde{\mu} = \mu + \lambda\overline{m}$ and $\tilde{r} = r + \lambda$.

We show that the value of a single threshold investment policy (a barrier strategy) under stochastic dynamics of geometric Lévy type coincides with the value of a single threshold investment policy under risk-adjusted deterministic dynamics. To make this more precise, we define a new process Z with deterministic dynamics described by the ordinary differential equation

$$Z_t' = \alpha Z_t,$$

$Z_0 = x$. Denoting the value of a threshold policy for Z by

$$L^y_{\theta,\alpha}(x) = e^{-\theta t_y} g(Z_{t_y}),$$

where t_y is the deterministic first hitting time of level $y > x$ for Z, we prove that if $\theta = \alpha\psi$, then $L^y_{\theta,\alpha}(x) = J^y(x)$ for all $x \in \mathbb{R}_+$. Hence we obtain a certainty equivalence by considering Z and adjusting either the discount rate from r to $\theta = \mu\psi$ or the infinitesimal drift from μ to $\alpha = r/\psi$. We establish sufficient conditions for the overall optimal policy to be a barrier strategy, in which case the problem of the optimal timing of an irreversible investment in a geometric Lévy model – where two different sources of randomness are present, and one of these is discontinuous – is equivalent to an optimal timing problem associated with a *continuous and deterministic* underlying process. Thus our findings extend the results of Alvarez (2004b) to a jump diffusion setting.

PAPER V: A CLASS OF SOLVABLE DIRICHLET PROBLEMS ASSOCIATED TO SPECTRALLY NEGATIVE JUMP DIFFUSIONS

We consider a regular (cf. assumption (ii) of Paper II) spectrally negative jump diffusion X living on $(0,u) \subseteq \mathbb{R}_+$ with stochastic dynamics

$$dX_t = \mu(X_{t-})dt + \sigma(X_{t-})dW_t - X_{t-}\int_0^1 z\tilde{N}(dt,dz),$$

$X_0 = 0$, and unattainable boundaries. We derive sufficient conditions on $\mu(x)$ and $\sigma(x)$ for the existence and uniqueness of a smooth increasing solution of the characteristic integro-differential equation $\mathscr{G}_r\psi = 0$. The conditions are derived using an approach resembling the Frobenius method for constructing solutions of linear differential equations, and hence as a byproduct we obtain a semi-explicit analytical expression for the solution $\psi(x)$ of $\mathscr{G}_r f = 0$ in terms of a Frobenius series:

$$\psi(x) = x^{\Psi}\sum_{i=1}^{\infty} c_i x^i,$$

where Ψ solves the indicial equation

$$\tilde{p}(\Psi) = \frac{a''(0)}{2}\Psi^2 + \left(\mu'(0) + \lambda\overline{m} - \frac{a''(0)}{2}\right)\Psi - (r+\lambda) + \lambda\int_0^1 (1-z)^{\Psi}\mathfrak{m}(dz) = 0$$

with $a(x) = \frac{1}{2}\sigma^2(x)$, $c_0 \neq 0$ and the coefficients $\{c_i\}_{i=1}^{\infty}$ satisfy a certain recurrence relation and are such that the relevant power series is absolutely convergent on the state space $(0,u)$. Here are the sufficient conditions:

(i) $\mu(x)$ and $\sigma(x)$ are analytic at $x = 0$ and satisfy $\mu(0) = \sigma(0) = \sigma'(0) = 0$, $\mu'(0) \geq 0$ (this is assumption (iv) in Paper II);

(ii) there exists $m \in \mathbb{N}$ such that $a^{(n)}(0) = 0$ and $\mu^{(n)}(0) = 0$ for all $n > m$;

(iii) the unique positive root Ψ of the indicial equation satisfies

$$\mu'(0) + \lambda\overline{m} + \left(\Psi + \frac{1}{2}\right)a''(0) > -\lambda\int_0^1 \ln(1-z)(1-z)^{\Psi+1}\mathfrak{m}(dz).$$

Naturally, in general the integro-differential indicial equation can be solved only numerically, and for practical purposes we need to approximate the solution by truncating the power series.

The main reason why the Frobenius method can be modified to accommodate our setup is the assumption of jumps being proportional to the current state: this allows the incorporation of the jump term into the terms of the power series.

We illustrate the result with two explicit examples, which are both instances of what we here refer to as a logistic jump diffusion model. That is, a process with drift essentially of form $\mu(x) = x(1-x)$. More generally, from our sufficient conditions (i)-(ii) one can see that a typical representative of the class of jump diffusions whose characteristic equations are solvable via this method is such that $\mu(x)$ and $\sigma(x)$ are polynomial functions.

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PAPER I

Luis H. R. Alvarez – Teppo A. Rakkolainen: *A class of solvable optimal stopping problems of spectrally negative jump diffusions*

A CLASS OF SOLVABLE OPTIMAL STOPPING PROBLEMS OF SPECTRALLY NEGATIVE JUMP DIFFUSIONS

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ABSTRACT
We consider the optimal stopping of a class of spectrally negative jump diffusions. We state a set of conditions under which the value is shown to have a representation in terms of an ordinary nonlinear programming problem. We establish a connection between the considered problem and a stopping problem of an associated continuous diffusion process and demonstrate how this connection may be applied for characterizing the stopping policy and its value. We also establish a set of typically satisfied conditions under which increased volatility as well as higher jump-intensity decelerates rational exercise by increasing the value and expanding the continuation region.
AMS Subject Classification: Primary 60G40; secondary 60J60, 60J75;
Keywords: jump diffusions; optimal stopping; nonlinear programming; perpetual American options;

1 INTRODUCTION

It is a well-known result from literature on mathematical finance that the price of a perpetual American option on an underlying asset whose value can be characterized as a stochastic process coincides with the value of an optimal stopping problem for this process (see, for example, Karatzas and Shreve (1999) pp. 54–87 and Øksendal (2003), pp. 290–298). Such option prices, while naturally of interest in themselves, can also be used as upper bounds for prices of American options with finite expiration dates. Thus, their role is of importance from a risk management point of view as well. Perpetual optimal stopping problems arise quite naturally also in the real options literature on the valuation of irreversible investment opportunities (see Dixit and Pindyck (1994) for an extensive textbook treatment of this theory, and Boyarchenko and Levendorskiĭ (2007) for some more recent developments in this field). In that modeling framework the investment decision is usually interpreted as an opportunity but not an obligation to obtain a stochastically fluctuating return in exchange from a payment (sunk cost) which may or may not be stochastic as well. Given the considerable planning horizon of the valuation of real investment opportunities, the time horizon is typically assumed to be infinite, i.e. the considered optimal timing problem of the investment opportunity is assumed to be perpetual.

When the dynamics of the underlying process are characterizable via an Itô stochastic differential equation of form

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \tag{1}$$

with W a standard Wiener process, the stopping problem has been widely studied by relying on various techniques. Perhaps the most common approach is to rely on variational inequalities or the classical Hamilton-Jacobi-Bellman approach due to its applicability in a multidimensional setting as well (cf. Øksendal (2003) and Øksendal and Reikvam (1998)). In the one-dimensional setting there are, however, several different techniques for analyzing the perpetual stopping problem. The most general approach is probably provided by studies relying on the integral characterization of excessive functions for diffusion processes and the Martin boundary theory (cf. Salminen (1985) and Borodin and Salminen (2002), pp. 32–35). Alternatively, the considered problem can be analyzed by relying on the relationship between functional concavity and r-excessivity along the lines of the pioneering work by Dynkin (1965) (Chapters XV and XVI) and by Dynkin and Yuskevich (1969) which has been subsequently applied within a general optimal stopping framework in Dayanik and Karatzas (2003). A third technique for studying the perpetual optimal stopping problem in the linear diffusion setting is provided by the approaches relying on the well-known relationship between excessivity and superharmonicity with respect to first exit times from open sets with compact closure in the state space of the considered diffusion (cf. Dynkin (1965), Theorem 12.4). In this case, the optimal stopping problem is reduced to the optimization of arbitrary boundaries and, therefore, can be analyzed by relying on ordinary nonlinear programming techniques (cf. Alvarez (2001) and Alvarez (2004)).

More recently, the shortcomings of continuous path models driven by a Brownian motion have become apparent and, consequently, more general models allowing path discontinuities have been studied. In many ways the most simple generalizations of the traditional continuous path models are jump diffusion models, in which the driving noise is a Lévy process. Lévy processes can be used to construct more realistic models of financial quantities, as they are able to generate jump discontinuities and the leptokurtic feature of return distributions, unlike the Gaussian models based on a Brownian motion and the normal distribution. For a taste of the plentiful research done on pricing American options and optimal stopping in Lévy models, see (for example) Gerber and Landry (1998), Gerber and Shiu (1998), Duffie et al (2000), Mordecki (2002a), Mordecki (2002b), Boyarchenko and Levendorskiĭ (2002), Boyarchenko and Levendorskiĭ (2005), Alili and Kyprianou (2005) and Mordecki and Salminen (2007).

In risk management a criticism often leveled against the continuous models is their inability to model downside risk: the possibility of an instantaneous drop in the value of an asset. In real life markets phenomena closely resembling such instantaneous drops are often observed (for example, sudden unanticipated deterioration of stock market values, credit defaults, etc.). An empirically observed fact is that in the stock market reactions to negative shocks are usually significantly stronger than the reactions to positive ones (this is the celebrated ”bad news” principle originally introduced in the seminal study by Bernanke (1983)). In light of this asymmetric nature of the reaction to unanticipated shocks, a prudent approach is to disregard possibilities for positive surprises and to take fully into account the possibilities for disadvantageous future occurrences. Consequently, a one-sided model that allows instantaneous downward jumps can be seen as an acceptable model from a prudent risk management point of view.

Motivated by our previous arguments, it is our objective in this study to consider a spectrally negative one-dimensional jump diffusion, say X, with a state space $I = (a,b) \subseteq \mathbb{R}$ and natural boundaries a and b. Interestingly, we establish that given some extra conditions on X, the value of the optimal stopping problem has a representation in terms of an ordinary nonlinear programming problem (cf. Alvarez (2001) and Alvarez (2003) for an associated result in the continuous diffusion case). This representation is valid for continuous, almost everywhere twice continuously differentiable ($\mathscr{C}^2$) reward functions g satisfying the condition

$$g(x)/\psi(x) \text{ has a unique maximizer } x^* \in I \text{ and is non-increasing for } x > x^*,$$

where ψ is an increasing solution of the integro-differential equation $\mathscr{G}\psi = r\psi$ with $\mathscr{G}$ being the infinitesimal generator of X. The representation is proved using the viscosity solution approach and, thus, smooth pasting may not necessarily hold. We find that given a jump diffusion for which the representation is valid in a certain class of reward functions, any strictly increasing $\mathscr{C}^2$ transformation also has a similar representation, albeit for a different class of reward functions.

For the sake of comparison, we consider an optimal stopping problem of an associated continuous diffusion process which can be obtained by removing the pure jump part of the considered Lévy diffusion. We demonstrate that the value of the considered jump-diffusion stopping problem can be "sandwiched" between the values of two stopping problems which are defined with respect to the associated continuous diffusion. This finding is of interest since it can be applied for deriving bounds for the exercise threshold of the considered optimal stopping problem for the underlying jump-diffusion. Moreover, since the restricting values defined with respect to the continuous diffusion differ only by the rate at which they are discounted, our findings indicate that under some circumstances the downside jump-risk can be directly incorporated into the continuous diffusion case by adjusting the discount rate appropriately (for some results in this direction, see Alvarez and Rakkolainen (2008)). This characterization is also important in the analysis of the impact of downside risk on the optimal stopping policy since according to this representation the optimal exercise boundary is lower for the underlying jump-diffusion than for the associated dominating continuous diffusion process provided that both valuations are discounted at the same rate.

We also consider the comparative static properties of the optimal stopping policy and its value and present a set of relatively general conditions under which the value of the considered problem is monotone and convex. Along the lines of previous studies considering the optimal stopping of linear diffusions, we find that in such a case higher volatility increases the value of the optimal strategy and expands the continuation region where stopping is suboptimal by increasing the optimal exercise threshold. These observations are of interest since they indicate that higher volatility decelerates the rational exercise of investment opportunities by increasing the option value of waiting in the presence of jumps as well. We also analyze the impact of increased jump-intensity on the optimal policy and its value and find that if the value is convex, then higher jump-intensity increases the value of waiting and decelerates rational exercise by expanding the continuation region. These observations emphasize the potentially significant combined negative effect of jump-risk and continuous systematic risk on the timing of irreversible investment policies.

The contents of this study are as follows. In section 2, we present the model and the assumptions used throughout the study. The representation of the stopping problem in terms of an ordinary optimization problem is stated and proved in section 3, together with the result on the validity of the representation for increasing $\mathscr{C}^2$ transformations of Lévy diffusions admitting the representation. Some useful inequalities related to the associated continuous diffusion are presented in section 4, and Section 5 is devoted to comparative statics. Explicit illustrations are given in section 6, and section 7 concludes.

2 THE SETUP AND BASIC ASSUMPTIONS

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space carrying a standard Wiener process $W = \{W_t\}$ and a compound Poisson process $J = \{J_t\}$ with intensity λ and a jump size distribution on $\mathscr{S} \subseteq \mathbb{R}$ characterized by a probability distribution $\mathfrak{m}$. We define a Lévy process $L = \{L_t\}$ by

$$L_t = t + W_t + J_t. \tag{2}$$

We equip $(\Omega, \mathscr{F}, \mathbb{P})$ with the completed natural filtration $\mathbb{F}$ generated by this process. The natural filtration of a Lévy process is right-continuous, and thus the completed filtration satisfies the usual hypotheses (see Protter (2004) Theorem I.31). We consider the optimal stopping problem

$$V(x) = \sup_{\tau \in \mathscr{T}} \mathbb{E}_x \left\{ e^{-r\tau} g(X_\tau) \right\}, \tag{3}$$

where $X = \{X_t\}$ is the jump diffusion driven by L with initial value $X_0 = x \in I$ and dynamics given by the stochastic differential equation

$$dX_t = \alpha(X_{t-})dt + \sigma(X_{t-})dW_t + \int_{\mathscr{S}} \gamma(X_{t-}, z)\tilde{N}(dz, dt). \tag{4}$$

In the above equations $\tilde{N}(U,t)$ is a compensated Poisson random measure with characteristic (Lévy) measure $\nu(dz) = \lambda\mathfrak{m}(dz)$, where $\mathfrak{m}$ is a probability measure on $\mathscr{S}$, and $\mathscr{T}$ is the set of all $\mathbb{F}$ -stopping times. Note that the driving jump process is, as a compensated process, a martingale – this is no restriction, as non-martingale jump dynamics can be reduced to the form 4 by adding and subtracting a correction term on the left side of the stochastic differential equation. We denote the expectation of the jump size by $\overline{m}$. The state space of the Lévy diffusion is an open interval $I := (a,b) \subseteq \mathbb{R}$ where a and b are natural boundaries (not attainable in finite time). We assume that the coefficient functions in 4 satisfy some sufficient conditions for the existence of a unique adapted càdlàg solution $X \in L^2(\mathbb{P})$ without explosions; in the case of an infinite interval I, the usual sufficient conditions are at most linear growth and Lipschitz continuity, see Øksendal and Sulem (2005) Theorem 1.19. The global Lipschitz condition guarantees that the explosion time of the process is a.s. infinite (see Protter (2004) Theorem V.40) and that X is strong Markov (cf. Protter (2004) Theorem V.32). Observe also that the jump times of X coincide with the jump times of the driving Lévy process, which are totally inaccessible stopping times (cf. Protter (2004) Theorem III.4): this implies that X is quasi-left continuous (or left-continuous over stopping times) and hence is a Hunt process. To avoid degeneracies, we finally assume that $\sigma(x) > 0$ on I.

The solution of the optimal stopping problem is known to be closely related to the integro-differential equation defined for $f \in C_0^2(\mathbb{R})$ by

$$\mathscr{G}f = rf, \tag{5}$$

where $(\mathscr{G}f)(x)$ is the generator of X given by

$$\begin{aligned}&\frac{1}{2}\sigma^2(x)f''(x) + \alpha(x)f'(x) + \\ &+ \lambda \int_{\mathscr{S}} \left\{f(x+\gamma(x,z)) - f(x) - f'(x)\gamma(x,z)\right\}\mathfrak{m}(dz).\end{aligned} \tag{6}$$

Integrating the last two terms of the integrand in (6) and using the notation $(\mathscr{G}_r) := (\mathscr{G} - r)$ we can write (5) equivalently as

$$(\mathscr{G}_r f)(x) = \frac{1}{2}\sigma^2(x)f''(x) + \tilde{\alpha}(x)f'(x) - \tilde{r}f(x) + \lambda \int_{\mathscr{S}} f(x+\gamma(x,z))\mathfrak{m}(dz) = 0,$$

where $\tilde{\alpha}(x) = \alpha(x) - \lambda \int_{\mathscr{S}} \gamma(x,z)\mathfrak{m}(dz)$ and $\tilde{r} = r + \lambda$.

Next the assumptions used throughout the rest of this study are stated. The following additional assumptions concerning the dynamics of X are made:

X1. $\tau_{(a,\bar{x})} = \inf\{t \geq 0 : X_t \geq \bar{x}\} < \infty$ $\mathbb{P}_x$-a.s. for all $a < x < \bar{x} < b$;

X2. $a - x < \gamma(x,z) \leq 0$ for all $(x,z) \in I \times \mathscr{S}$.

Assumption *X2* implies that X has only negative jumps and that X cannot reach the lower boundary a by jumping. Thus $X_t \in I$ for all $t \geq 0$.

The reward function g is assumed to satisfy

g1. $g(x) = \max(\tilde{g}(x), 0)$ with $\tilde{g}$ increasing, continuous and $\mathscr{C}^2$ on $I \setminus \mathscr{N}$ for some finite set $\mathscr{N} \subset I$ with finite limits $\tilde{g}'(x\pm)$, $\tilde{g}''(x\pm)$ for $x \in \mathscr{N}$, and such that $\tilde{g}(a+) \leq 0$;

g2. $\mathbb{E}_x\left\{\sup_{t\geq 0} e^{-rt} g(X_t)\right\} < \infty$ for all $x \in I$.

Observe that assumption *g1* is satisfied by the reward of a standard American call option, in which case $\tilde{g}(x) = x - K$ for some strike price K. In fact, the imposed reward structure is natural for an option type contract, where we can always avoid losses by not exercising our option if the reward is negative.

We make the following assumption on the operator $\mathscr{G}_r$:

A1. $\mathscr{G}_r \psi = 0$ has a nonnegative solution $\psi \in \mathscr{C}^2(I)$ such that $\psi(a) = 0$.

It should be mentioned here that it is not at all clear whether a given integro-differential equation has such a smooth solution – the validity of assumption *A1* needs to be checked in each case. Finally, we need to make two assumptions on the behavior of the quotient g/ψ, namely,

Ag1. there exists a unique maximizer $x^* \in I$ of $g(x)/\psi(x)$ and $g(x)/\psi(x)$ is non-increasing for $x > x^*$.

Ag2. there exists $\hat{x} < x^*$ such that, for all $x \geq \hat{x}$ such that g is $\mathscr{C}^2$ at x,

$$(\mathscr{G}_r g)(x) \leq -\int_C \left\{ \frac{g(x^*)}{\psi(x^*)} \psi(x + \gamma(x,z)) - g(x + \gamma(x,z)) \right\} \nu(dz),$$

where $C = \left\{z \in \mathscr{S} : x + \gamma(x,z) < x^*\right\}$.

In a sense, the last assumption is needed to guarantee the r-superharmonicity of the value function V in the stopping region $x \geq x^*$, as will be seen in the proof of Theorem 3.5 later on. In most cases, this assumption is rather difficult to verify otherwise than numerically on a case by case basis. Observe that assumptions *Ag1* and *Ag2* have implications for the form of the reward function g: the set of allowable reward functions will depend on the behavior of function ψ.

It was pointed out to us by *Erik Baurdoux* that checking condition *Ag2* is not necessary, if one can show that $V(x) - g(x)$ is decreasing, as in that case the stopping set is necessarily of form $[l, b)$, and then by spectral negativity $l = x^*$. We can make use of this nice result in the first two of our illustrative examples in Section 6.

3 THE REPRESENTATION THEOREM

In Alvarez (2001), it is shown that (modulo some conditions) if the process X is a continuous linear diffusion the value function of the stopping problem (3) can be expressed as

$$V(x) = \psi(x) \sup_{y \geq x} \left\{ \frac{g(y)}{\psi(y)} \right\},$$

where $\psi(x)$ is the increasing fundamental solution of the differential equation $A\psi - r\psi = 0$, where A is the second order differential operator coinciding with the infinitesimal generator of X. Our main theorem states that this representation is also valid for a jump diffusion satisfying the assumptions of section 2. Before stating the main result, we present some auxiliary results necessary for the proof of the theorem. We begin with the following definition.

Definition 3.1. *A lower semicontinuous function f on I is* r-superharmonic, *if* $\mathbb{E}_x\{e^{-r\tau} f(X_\tau)\} \leq f(x)$ *for all* $\tau \in \mathscr{T}$, $x \in I$.

From Dynkin's formula it follows that if f is a $\mathscr{C}^2$ function, then r-superharmonicity of Definition 3.1 is implied by the stronger property

$$(\mathscr{G}_r f)(x) \leq 0 \text{ for all } x \in I.$$

For jump diffusions this implication is valid for any continuous nonnegative function which is $\mathscr{C}^2$ outside a finite set $\mathscr{N} \subset I$.

Lemma 3.2. *Let g be nonnegative and continuous on I and twice continuously differentiable outside a finite set of points $\mathscr{N} \subset I$. Then $(\mathscr{G}_r f)(x) \leq 0$ for all $x \in I \setminus \mathscr{N}$ implies that g is r-superharmonic.*

Proof. W.l.o.g. we consider a continuous function g which has one point of nondifferentiability at x_0 but is $\mathscr{C}^2$ on $I \setminus \{x_0\}$. Since X is a semimartingale, we can apply the change-of-variable formula from Peskir (2007) to get

$$\begin{aligned} g(X_t) &= g(x) + \int_0^t g'(X_{s-})dX_s^c + \tfrac{1}{2}\int_0^t g''(X_{s-})d[X,X]_s + \\ &+ \textstyle\sum_{0<s\leq t} \Big(g(X_s) - g(X_{s-}) - g'(X_{s-})\Delta X_s \Big) + \\ &+ \tfrac{1}{2}\int_0^t \left(g'(X_s) - g'(X_{s-})\right) \mathbf{1}_{\{X_{s-}=x_0\}}(X_{s-}) dl_s^{x_0}(X), \end{aligned}$$

where $l_s^{x_0}(X)$ is the local time of X at x_0, that is,

$$l_t^{x_0}(X) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{X_s \in (x_0-\varepsilon, x_0]\}} d[X,X]_s,$$

where the limit is taken in probability. Writing out $dX_s^c = \mu(X_{s-})ds + \sigma(X_{s-})dW_s$ and $d[X,X]_s = \frac{1}{2}\sigma^2(X_{s-})ds$ and rearranging terms (note that the individual jump magnitude $\Delta X_s = \gamma(X_{s-},z)$) yields

$$\begin{aligned} g(X_t) &= g(x) + \\ &+ \int_0^t \left\{ \left[\mu(X_{s-}) - \int_{\mathscr{S}} \gamma(X_{s-},z)\nu(dz)\right] g'(X_{s-}) + \tfrac{1}{2}\sigma^2(X_{s-})g''(X_{s-}) \right\} ds + \\ &+ \int_0^t \int_{\mathscr{S}} \{g(X_s) - g(X_{s-})\} N(ds,dz) + \int_0^t g'(X_{s-})\sigma(X_{s-})dW_s + \\ &+ \tfrac{1}{2}\int_0^t \left(g'(X_s) - g'(X_{s-})\right) \mathbf{1}_{\{X_{s-}=x_0\}}(X_{s-})\sigma^2(X_{s-})ds. \end{aligned}$$

Adding and subtracting $\int_0^t \int_{\mathscr{S}} \{g(X_s) - g(X_{s-})\} \nu(dz)$ and rearranging yields

$$\begin{aligned} g(X_t) &= g(x) + \int_0^t (\mathscr{G}g)(X_{s-})ds + \int_0^t g'(X_{s-})\sigma(X_{s-})dW_s + \\ &+ \int_0^t \int_{\mathscr{S}} \{g(X_s) - g(X_{s-})\} \tilde{N}(ds,dz) + \qquad (7) \\ &+ \tfrac{1}{2}\int_0^t \left(g'(X_s) - g'(X_{s-})\right) \mathbf{1}_{\{X_{s-}=x_0\}}(X_{s-})\sigma^2(X_{s-})ds. \end{aligned}$$

On the right side the third and fourth terms are stochastic integrals with respect to martingales. Now, with respect to the last term we observe that since all jump times of a Lévy process are totally inaccessible, $\mathbb{P}_x\left(\{\Delta X_{\tau_{x_0}} > 0\}\right) = 0$. Hence,

$$\begin{aligned}&\mathbb{E}_x\Big\{ \textstyle\int_0^t (g'(X_s) - g'(X_{s-}))\, \mathbf{1}_{\{X_{s-}=x_0\}}(X_{s-})\sigma^2(X_{s-})ds \Big\} = \\ &= (g(x_0+) - g(x_0-))\, \sigma^2(x_0)\mathbb{E}_x\Big\{ \textstyle\int_0^t \mathbf{1}_{\{X_{s-}=x_0\}}(X_{s-})ds \Big\}.\end{aligned}$$

Since $\mathbb{P}_x$ is a diffuse measure, by Fubini's theorem

$$\mathbb{E}_x \int_0^t \mathbf{1}_{\{X_{s-}=x_0\}}(X_{s-})ds = \int_0^t \mathbb{E}_x \left\{ \mathbf{1}_{\{X_{s-}=x_0\}}(X_{s-}) \right\} ds = 0.$$

Thus taking expectations in (7) yields

$$\mathbb{E}_x \{ g(X_t) \} = g(x) + \mathbb{E}_x \int_0^t (\mathscr{G} g)(X_{s-})ds.$$

Hence, finally, considering the discounted value $e^{-rt}g(X_t)$, we see that

$$\mathbb{E}_x \left\{ e^{-rt} g(X_t) \right\} = g(x) + \mathbb{E}_x \int_0^t e^{-rs}(\mathscr{G}_r g)(X_{s-})ds,$$

which means that $(\mathscr{G}_r g)(x) \le 0$ for all $x \in I \setminus \mathscr{N}$ implies that g is r-excessive for X. But by the results of Hunt (1958), for nonnegative functions r-excessivity for X is equivalent to r-superharmonicity for X, when X is a Hunt process. □

At this point, we introduce the notation $v(x) := \psi(x) \sup_{y \ge x} \left\{ \frac{g(y)}{\psi(y)} \right\}$ and consider the properties of this function.

Lemma 3.3. *Assume that g/ψ is continuous on I with a unique maximum point $x^* \in I$ and non-increasing for $x > x^*$. Then $v(x)$ is a continuous function of x and we have the representation*

$$v(x) = \begin{cases} g(x), & x \ge x^* \\ \psi(x)\frac{g(x^*)}{\psi(x^*)}, & x < x^*. \end{cases}$$

Proof. For a function $f := g/\psi$ satisfying our assumptions, it holds that

$$\sup_{y \ge x} f(y) = \begin{cases} f(x), & x \ge x^* \\ f(x^*), & x < x^*, \end{cases}$$

which is continuous if f is. The representation is immediate (multiply the above equation with $\psi(x)$). □

Lemma 3.3 demonstrates, that under our assumptions the value of the associated nonlinear programming problem is continuous. Interestingly, as in studies based on continuous diffusion models, lemma 3.3 characterizes the value in terms of the exercise payoff received at the exercise boundary and the ratio $\psi(x)/\psi(x^*)$ measuring the expected present value of a contract which pays the holder one dollar at the first date the underlying jump diffusion exceeds a beforehand fixed threshold level. This observation is expressed in more precise terms in the following lemma.

Lemma 3.4. *Suppose $\psi : I \mapsto \mathbb{R}_+$ is a nonnegative solution of $\mathscr{G}_r u = 0$ and $a < x < y < b$. Then*

$$\mathbb{E}_x[e^{-r\tau_{(a,y)}}] = \frac{\psi(x)}{\psi(y)}$$

and ψ is increasing. Moreover, in case $\psi(x)$ exists any other nonnegative and increasing solution of $\mathscr{G}_r u = 0$ is a constant multiple of $\psi(x)$ (i.e. $\psi(x)$ is unique up to a multiplicative constant).

Proof. Under assumption *X1* and the assumed boundary behavior of X_t at the boundary a, we can apply the Dynkin formula to ψ:

$$\mathbb{E}_x[e^{-r\tau_{(a,y)}}\psi(X_{\tau_{(a,y)}})] = \psi(x) + \mathbb{E}_x \int_0^{\tau_{(a,y)}} e^{-rt}(\mathscr{G}_r\psi)(X_t)dt.$$

Since ψ solves $\mathscr{G}_r\psi = 0$ and $X_{\tau_{(a,y)}} = y$ a.s. (because X has no positive jumps and it never attains a), this implies that

$$\psi(y)\mathbb{E}_x[e^{-r\tau_{(a,y)}}] = \psi(x),$$

from which the first two of the claimed results follow (for the latter one, note that $\mathbb{E}_x[e^{-r\tau_{(a,y)}}] \in (0,1]$). To establish uniqueness, assume that $\varsigma : I \mapsto \mathbb{R}_+$ is another increasing and nonnegative solution of equation $\mathscr{G}_r u = 0$. By applying a similar argument as above, we find that

$$\varsigma(x) = \frac{\varsigma(y)}{\psi(y)}\psi(x)$$

which completes the proof of our lemma. □

We wish to point out here that in Kou and Wang (2003), representation results similar to our lemma 3.4 are obtained for a Brownian motion augmented with a compound Poisson process with double exponentially distributed jumps.

It is worth emphasizing that the strong Markov property of the jump diffusion and the fact that it cannot jump upwards and, therefore, that it can increase only continuously imply that the function $\mathbb{E}_x[e^{-r\tau_{(a,y)}}]$ can always be expressed as a ratio of the form (8). However, it is not beforehand clear whether this ratio is always (i.e. for any jump diffusion model) twice continuously differentiable with respect to the current state or not. Hence, lemma 3.4 essentially demonstrates that in those cases where the integro-differential equation $\mathcal{G}_r u = 0$ has an increasing solution, the expected value $\mathbb{E}_x[e^{-r\tau_{(a,y)}}]$ can be expressed in terms of this solution and identity (8) holds. The key implication of this finding and our main result on the characterization of the value of the considered optimal stopping problem as an ordinary nonlinear programming problem is now summarized in the following.

Theorem 3.5. *Suppose X and g are such that* X1–X2, g1, g2, A1, Ag1 *and* Ag2 *are satisfied. Then the value function of problem* (3) *has the representation*

$$V(x) = \psi(x) \sup_{y \geq x} \left\{ \frac{g(y)}{\psi(y)} \right\}, \tag{8}$$

where ψ is an increasing solution of $\mathcal{G}_r \psi = 0$.

Proof. We use again the notation $v(x) := \psi(x) \sup_{y \geq x} \left\{ \frac{g(y)}{\psi(y)} \right\}$. Now, from the two previous lemmas it is obvious that $v(x)$ is the value of stopping at the first time the process X hits $x^* := \operatorname{argmax}\{g(y)/\psi(y)\}$, i.e.

$$v(x) = \mathbb{E}_x \left\{ e^{-r\tau_{x^*}} g(X_{\tau_{x^*}}) \right\}.$$

But then $v(x) \leq V(x)$, since $v(x)$ is the value of an admissible stopping rule.

To establish the opposite inequality, we show that v is a r-superharmonic majorant of g – since for a càdlàg Hunt process $V(x)$ is the smallest superharmonic majorant of the reward (cf. Peskir and Shiryaev (2006) Section IV.9, and also Mordecki and Salminen (2007)), this then implies the desired inequality. We achieve this by showing that v is a viscosity solution of the variational inequalities

$$\max\big((\mathcal{G}_r w)(x), g(x) - w(x)\big) = 0, \; x \in (a,b), \tag{9}$$

satisfying the boundary condition

$$w(a) = 0. \tag{10}$$

(for a detailed exposition of the original viscosity solutions theory for second order differential equations, see Crandall et al. (1992); extensions to integro-differential equations are considered in Alvarez and Tourin (1996), Jakobsen and Karlsen (2006) and Barles and Imbert (2008) – do note that in the present study we are dealing with the relatively simple case of a bounded Lévy measure). Observe that from the definition of $v(x)$ it is clear that $(\mathscr{G}_r v)(x) = 0$ for $x < x^*$, since $(\mathscr{G}_r \psi)(x) = 0$ and $\psi \in \mathscr{C}^2(I)$.

Since v is continuous by Lemma 3.3, it remains to show that v is a solution of the variational inequalities in the viscosity sense. First we establish the *subsolution property*. So let us take $x_0 \in I$ and suppose that $h \in \mathscr{C}^2(I)$ is such that $h(x) \geq v(x)$ for $x \in I$ and $h(x_0) = v(x_0)$. We have two possibilities:

(i) if $a < x_0 < x^*$, then $h(x) - v(x)$ is a smooth function at $x = x_0$ and has a local minimum there. First and second order conditions for a local minimum imply then that $v'(x_0) = h'(x_0)$ and $v''(x_0) \leq h''(x_0)$. Furthermore, $h(x_0 + \gamma(x_0,z)) \geq v(x_0 + \gamma(x_0,z))$. But then $(\mathscr{G}_r h)(x_0) \geq (\mathscr{G}_r v)(x_0) = 0$, and the variational inequality

$$\max\left((\mathscr{G}_r h)(x_0), g(x_0) - v(x_0)\right) \geq 0, \tag{11}$$

holds.

(ii) if $b > x_0 \geq x^*$, then $v(x_0) = g(x_0)$ and (11) is satisfied.

Thus, for all $h \in \mathscr{C}^2(I)$ and $x_0 \in I$ such that $h(x) \geq v(x)$ for all $x \in I$, and $h(x_0) = v(x_0)$, equation (11) is satisfied, so v is a viscosity subsolution of the variational inequality.

To show the *supersolution property* of v we take $x_0 \in I$ and $h \in \mathscr{C}^2(I)$ such that $h(x) \leq v(x)$ for all $x \in I$ and $h(x_0) = v(x_0)$. Now we have three possibilities:

(i) if $a < x_0 < x^*$, then $h(x) - v(x)$ is a smooth function at $x = x_0$ and has a local maximum there. First and second order conditions for a local maximum imply then that $v'(x_0) = h'(x_0)$ and $v''(x_0) \geq h''(x_0)$. Furthermore, $h(x_0 + \gamma(x_0,z)) \leq v(x_0 + \gamma(x_0,z))$. But then $(\mathscr{G}_r h)(x_0) \leq (\mathscr{G}_r v)(x_0) = 0$, and since $v(x_0) \geq g(x_0)$, the inequality

$$\max\left((\mathscr{G}_r h)(x_0), g(x_0) - v(x_0)\right) \leq 0, \tag{12}$$

is satisfied.

(ii) if $b > x_0 \geq x^*$ and g is $\mathscr{C}^2$ at x_0, then $v(x_0) = g(x_0)$ and so the second half of the left side of (12) equals 0. To obtain $(\mathscr{G}_r h)(x_0) \leq 0$, observe that by arguments similar to previous ones, $(\mathscr{G}_r h)(x_0) \leq (\mathscr{G}_r v)(x_0)$, and from the definition of v we get then, using assumption *Ag2*,

$$(\mathscr{G}_r v)(x_0) = (\mathscr{G}_r g)(x_0) + \int_C \{v(x_0 + \gamma(x_0,z)) - g(x_0 + \gamma(x_0,z))\}\nu(dz) \leq 0$$

(see section 2 for the definition of the set C). This implies that (12) holds.

(iii) if $b > x_0 \geq x^*$ and $x_0 \in \mathscr{N}$ (i.e. g is not $\mathscr{C}^2$ at x_0), we still have $g(x_0) = v(x_0)$. By (ii), under our assumptions

$$(\mathscr{G}_r h)(y) \leq (\mathscr{G}_r g)(y) + \int_C \{v(y + \gamma(y,z)) - g(y + \gamma(y,z))\}\nu(dz) \leq 0,$$

for all $x_0 < y < \min\{b, x_k\}$, where x_k is the point of $\mathscr{N} \cap (x_0, b)$ closest to x_0 . Since g is $\mathscr{C}^2$ on (x_0, x_k) and v is continuous, letting $y \downarrow x_0$, we get

$$(\mathscr{G}_r h)(x_0) \leq \lim_{y \downarrow x_0} \Big\{ (\mathscr{G}_r g)(y) + \int_C \{v(y + \gamma(y,z)) - \\ - g(y + \gamma(y,z))\}\nu(dz) \Big\} \leq 0.$$

So (12) holds.

We have established that for all $h \in \mathscr{C}^2(I)$ and $x_0 \in I$ such that $h(x) \geq v(x)$, for $x \in I$, and $h(x_0) = v(x_0)$ equation (12) holds, i.e. v is a viscosity supersolution of the variational inequality.

We have now proved that being continuous and both a viscosity sub- and supersolution, v is a continuous viscosity solution of the variational inequalities. This implies that $v(x) \geq g(x)$ and that v satisfies $(\mathscr{G}_r v)(x) \leq 0$ in viscosity solution sense: that is, for all $x \in I \setminus \mathscr{N}$; thus by lemma 3.2 v is a r-superharmonic majorant of g. Hence $v(x) \geq V(x)$. But this implies that $v(x) = V(x)$.

□

The representation of theorem 3.5 implies that if g is continuously differentiable at the point x^* maximizing $g(x)/\psi(x)$, then x^* can be solved from the first order condition for an extremum $g'(x)\psi(x) - g(x)\psi'(x) = 0$, which is equivalent to the logarithmic derivative condition $D_x[\ln g(x)] = D_x[\ln \psi(x)]$. In this case the well-known *smooth fit condition* is satisfied, i.e. the value is continuously differentiable. However, even if x^* happens to be a point of non-differentiability of g, the representation result holds – necessary conditions for a maximum of g/ψ are then

$$\lim_{y\to x^*-}\{g'(y)\psi(y)-g(y)\psi'(y)\}\geq 0 \text{ and } \lim_{y\to x^*+}\{g'(y)\psi(y)-g(y)\psi'(y)\}\leq 0,$$

which imply only that

$$g'(x^*-)\geq V'(x^*-)=\psi'(x^*)\frac{g(x^*)}{\psi(x^*)}\geq g'(x^*+)=V'(x^*+).$$

It is possible to prove that given a process X such that theorem 3.5 holds (for a certain class of reward functions) and any sufficiently regular transformation $f(\cdot)$, the representation is valid for the process Y defined by $Y_t=f(X_t)$ (although the class of allowable reward functions will be different). This is the content of the next theorem.

Theorem 3.6. *Let* $\{X_t\}$ *be a stochastic process such that assumptions* X1, X2 *and* A1 *are satisfied, and let* f *be a strictly increasing function in* $\mathscr{C}^2(I)$*. Denote the increasing solution in* A1 *for* X *by* ψ_1*. Define a new process* Y *by setting* $Y_t:=f(X_t)$*. Then* Y *satisfies assumptions* X1, X2 *and* A1*. Furthermore, the corresponding increasing solution in* A1 *for* Y *is given by* $\psi_1(f^{-1}(y))$.

Proof. A $\mathscr{C}^2$ transform of a jump diffusion is a jump diffusion. Being an increasing function, f maps the state space $I=(a,b)$ of X onto $J=(f(a),f(b))$, the state space of Y. Since

$$Y_t=f(X_t)>\bar{x}\Leftrightarrow X_t>f^{-1}(\bar{x}),$$

it follows that Y satisfies *X1*, and as X is spectrally negative and f is increasing, we have

$$\begin{aligned}|\Delta Y_t| &= |f(X_{t-})-f(X_{t-}+\Delta X_t)|\\ &= f(X_{t-})-f(X_{t-}+\Delta X_t)<f(X_{t-})-f(a)\end{aligned}$$

and thus *X2* is satisfied. Because X is assumed to satisfy *A1*, there exists an increasing solution ψ_1 of the integro-differential equation

$$\tilde{\mu}(x)\psi'(x)+\frac{1}{2}\sigma^2(x)\psi''(x)+\int_{\mathscr{S}}\psi(x+\gamma(x,z))\nu(dz)=\tilde{r}\psi(x).$$

Let $x := x(y) = f^{-1}(y)$. The transformation $\tilde{\psi}(y) = \psi(x) = (\psi \circ f^{-1})(y)$ leads to the integro-differential equation

$$\begin{aligned}&\tilde{\mu}(x(y))[x'(y)]^{-1}\tilde{\psi}'(y) + \frac{1}{2}\sigma^2(x(y))\Big\{[x'(y)]^{-2}\tilde{\psi}''(y)\\&- \tilde{\psi}'(y)x''(y)[x'(y)]^{-3}\Big\}+\\&+\int_{\mathscr{S}} \tilde{\psi}(x(y)+\gamma(x(y),z))\nu(dz) = \tilde{r}\tilde{\psi}(y),\end{aligned} \tag{13}$$

which is well-defined since under the assumptions on f, the inverse mapping theorem guarantees the existence and continuity of $x'(y)$ and $x''(y)$. Defining

$$\phi(y) := \psi_1(x(y))$$

we can by substituting ϕ into (13) establish that $\phi(y)$ is a solution of (13). As ψ_1 is increasing on I by assumption and $x'(y) = (f^{-1})'(y) = (f'(x))^{-1} > 0$ on J by the inverse mapping theorem, we have that

$$\phi'(y) = \psi_1(x(y))x'(y) > 0.$$

So $\phi(y)$ is an increasing function, and $\phi(f(a)) = \psi_1(a) = 0$, since X satisfies *A1*. Thus Y satisfies *A1*. □

4 USEFUL INEQUALITIES: SANDWICHING THE SOLUTION

In this section we plan to analyze how the considered stopping problem is related to two optimal stopping problems of an associated continuous diffusion model. To accomplish this task, consider now the associated diffusion

$$d\tilde{X}_t := \left(\mu(\tilde{X}_t) - \int_{\mathscr{S}} \gamma(\tilde{X}_t, z)\nu(dz)\right)dt + \sigma(\tilde{X}_t)dW_t. \tag{14}$$

It is worth mentioning that this associated diffusion is very useful in assessing the impact of downside risk on the optimal policy, as the Lévy diffusion X is, in fact, a superposition of $\tilde{X}$ and a spectrally negative, non-martingale jump process. As usually, we denote as $\tilde{\mathscr{A}}_\theta$ the differential operator

$$\tilde{\mathscr{A}}_\theta = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \left(\mu(x) - \int_{\mathscr{S}} \gamma(x,z)\nu(dz)\right)\frac{d}{dx} - \theta$$

associated with the continuous diffusion $\tilde{X}_t$ killed at the constant rate $\theta > 0$. Along the lines of the notation in our previous analysis, we denote as $\tilde{\psi}_\theta(x)$ the increasing fundamental solution (i.e. the minimal increasing θ-harmonic mapping for the diffusion $\{\tilde{X}_t; t \geq 0\}$; for a thorough characterization of these mappings, see Borodin and Salminen (2002), p. 33) of the ordinary linear second order differential equation $(\tilde{\mathscr{A}}_\theta u)(x) = 0$. As is well-known from the classical theory of diffusions, given this increasing fundamental solution we have for all $x \leq y$ (cf. Borodin and Salminen (2002), p. 18)

$$\mathbb{E}_x\left[e^{-\theta\tilde{\tau}_{(a,y)}}\right] = \frac{\tilde{\psi}_\theta(x)}{\tilde{\psi}_\theta(y)},$$

where $\tilde{\tau}_{(a,y)} = \inf\{t \geq 0 : \tilde{X}_t = y\}$ denotes the first hitting time of the diffusion $\tilde{X}_t$ to the state y. Therefore, the continuity of the exercise payoff yields that for all $x \leq y$ we have

$$\mathbb{E}_x\left[e^{-\theta\tilde{\tau}_{(a,y)}} g(\tilde{X}_{\tilde{\tau}_{(a,y)}})\right] = g(y)\frac{\tilde{\psi}_\theta(x)}{\tilde{\psi}_\theta(y)}$$

implying that

$$\sup_{y\geq x}\mathbb{E}_x\left[e^{-\theta\tilde{\tau}_{(a,y)}} g(\tilde{X}_{\tilde{\tau}_{(a,y)}})\right] = \tilde{\psi}_\theta(x)\sup_{y\geq x}\left[\frac{g(y)}{\tilde{\psi}_\theta(y)}\right]$$

provided that the supremum exists. In light of this observation it is naturally of interest to ask whether the discount rate θ can be chosen so as to yield representations which either dominate or are smaller that the value of the optimal stopping problem (3). Interestingly, the answer to this question turns out to be positive as is illustrated by our following theorem characterizing the relationship of the value of the optimal stopping problem with the values of two associated stopping problems defined with respect to the continuous diffusion (14).

Theorem 4.1. *For all $x \leq y$ we have*

$$\frac{\tilde{\psi}_{r+\lambda}(x)}{\tilde{\psi}_{r+\lambda}(y)} \leq \mathbb{E}_x\left[e^{-r\tau_{(a,y)}}\right] \leq \frac{\tilde{\psi}_r(x)}{\tilde{\psi}_r(y)}.$$

Consequently,

$$\tilde{\psi}_{r+\lambda}(x)\sup_{y\geq x}\left[\frac{g(y)}{\tilde{\psi}_{r+\lambda}(y)}\right] \leq \sup_{y\geq x}\mathbb{E}_x\left[e^{-r\tau_{(a,y)}} g(X_{\tau_{(a,y)}})\right] \leq \tilde{\psi}_r(x)\sup_{y\geq x}\left[\frac{g(y)}{\tilde{\psi}_r(y)}\right]$$

provided that the suprema exist. Therefore, if condition A1 is satisfied, then

$$\frac{\tilde{\psi}_{r+\lambda}(x)}{\tilde{\psi}_{r+\lambda}(y)} \leq \frac{\psi(x)}{\psi(y)} \leq \frac{\tilde{\psi}_r(x)}{\tilde{\psi}_r(y)}$$

for all $x \leq y$ and

$$\tilde{\psi}_{r+\lambda}(x) \sup_{y \geq x} \left[\frac{g(y)}{\tilde{\psi}_{r+\lambda}(y)}\right] \leq \psi(x) \sup_{y \geq x} \left[\frac{g(y)}{\psi(y)}\right] \leq \tilde{\psi}_r(x) \sup_{y \geq x} \left[\frac{g(y)}{\tilde{\psi}_r(y)}\right],$$

provided that the suprema exist.

Proof. Applying the Dynkin formula to $\tilde{\psi}_r(x)$ yields

$$\mathbb{E}_x[e^{-r\tau_{(a,y)}} \tilde{\psi}_r(X_{\tau_{(a,y)}})] = \tilde{\psi}_r(x) + \mathbb{E}_x \int_0^{\tau_{(a,y)}} e^{-rt} (\mathscr{G}_r \tilde{\psi}_r)(X_t) dt \leq \tilde{\psi}_r(x)$$

since

$$(\mathscr{G}_r \tilde{\psi}_r)(x) = \lambda \int_{\mathscr{S}} \{\tilde{\psi}_r(x + \gamma(x,z)) - \tilde{\psi}_r(x)\} \mathfrak{m}(dz) < 0$$

by the monotonicity of $\tilde{\psi}_r(x)$. Since $X_{\tau_{(a,y)}} = y$ a.s. (because X has no positive jumps and it never attains a) and $\tilde{\psi}_r(x)$ is continuous, this inequality implies that $\mathbb{E}_x[e^{-r\tau_{(a,y)}}] \leq \tilde{\psi}_r(x)/\tilde{\psi}_r(y)$ for all $x \leq y$. Analogously, applying the Dynkin formula to $\tilde{\psi}_{r+\lambda}(x)$ yields

$$\mathbb{E}_x[e^{-r\tau_{(a,y)}} \tilde{\psi}_{r+\lambda}(X_{\tau_{(a,y)}})] = \tilde{\psi}_{r+\lambda}(x) + \mathbb{E}_x \int_0^{\tau_{(a,y)}} e^{-rt} (\mathscr{G}_r \tilde{\psi}_{r+\lambda})(X_t) dt \geq \tilde{\psi}_{r+\lambda}(x)$$

since

$$(\mathscr{G}_r \tilde{\psi}_{r+\lambda})(x) = \lambda \int_{\mathscr{S}} \tilde{\psi}_{r+\lambda}(x + \gamma(x,z)) \mathfrak{m}(dz) > 0$$

by the positivity of $\tilde{\psi}_{r+\lambda}(x)$. Thus, $\mathbb{E}_x[e^{-r\tau_{(a,y)}}] \geq \tilde{\psi}_{r+\lambda}(x)/\tilde{\psi}_{r+\lambda}(y)$ for all $x \leq y$. The rest of the alleged results then follow from the nonnegativity of $g(x)$ and condition A1. □

Theorem 4.1 essentially establishes that in the present setting the value of the optimal stopping problem (3) satisfies the inequality $\tilde{V}_{r+\lambda}(x) \leq V(x) \leq \tilde{V}_r(x)$, where

$$\tilde{V}_\theta(x) = \sup_\tau \mathbb{E}_x \left[e^{-\theta\tau} g(\tilde{X}_\tau)\right],$$

provided that the sufficiency conditions guaranteeing the optimality of the stopping rule characterized by a single threshold are satisfied. For that class of problems, Theorem 4.1 also clearly indicates that $C_{r+\lambda} \subseteq C \subseteq C_r$ where $C_\theta = \{x \in I : \tilde{V}_\theta(x) > g(x)\}$ and $C = \{x \in I : V(x) > g(x)\}$. This observation is important since it demonstrates that the optimal exercise threshold x^*

is dominated by the exercise threshold of the dominating value $\tilde{V}_r(x)$ and is greater than the exercise threshold of the smaller value $\tilde{V}_{r+\lambda}(x)$. In this way, the findings of our Theorem 4.1 provide valuable information on the impact of pure (uncompensated) downside risk on the optimal decision. Moreover, since $\tilde{V}_{r+\lambda}(x) \leq \tilde{V}_\theta(x) \leq \tilde{V}_r(x)$ for all $x \in I$, we immediately observe that if the conditions of our Theorem 4.1 and Theorem 3.5 are satisfied, then there is a critical discount rate for which the stopping rule coincides in the continuous and in the jump-diffusion case. That is, there is a $\theta^* \in (r, r+\lambda)$ such that $x^* = \min\{x \in I : \tilde{V}_{\theta^*}(x) = g(x)\}$.

It is worth emphasizing that the proof of Theorem 4.1 essentially relies on the fact that if $u : I \mapsto \mathbb{R}_+$ is a sufficiently smooth and monotonically increasing function, then

$$(\tilde{\mathscr{A}}_{r+\lambda} u)(x) \leq (\mathscr{G}_r u)(x) \leq (\tilde{\mathscr{A}}_r u)(x).$$

Hence, our results clearly indicate that the class of sufficiently smooth monotonically increasing r-excessive mappings for the diffusion $\tilde{X}_t$ is larger than the class of sufficiently smooth monotonically increasing r-excessive mappings for the jump-diffusion X_t which, in turn, is larger than the class of sufficiently smooth monotonically increasing $(r+\lambda)$-excessive mappings for the diffusion $\tilde{X}_t$. This observation is interesting since it directly generates a natural ordering for the monotone (viscosity) solutions of the variational inequalities $\max\{(\mathscr{G}_r u)(x), g(x) - u(x)\} = 0$ and $\max\{(\tilde{\mathscr{A}}_\theta u)(x), g(x) - u(x)\} = 0$ with $\theta = r, r+\lambda$.

5 COMPARATIVE STATICS

In this section our main objective is to consider comparative static properties of the value function and the optimal policy and, especially, to analyze the impact of increased volatility on these factors. To this end, we consider two jump diffusions of the form (4), X and $\hat{X}$, which are otherwise identical but have different volatilities, $\sigma(x) > \hat{\sigma}(x)$. In accordance with this notation, we denote the values of the associated optimal stopping problems by V and $\hat{V}$, the associated differential operators as $\mathscr{G}_r$ and $\hat{\mathscr{G}}_r$, and the associated increasing fundamental solutions (given that assumption A1 is satisfied) as ψ and $\hat{\psi}$, respectively. Our first result emphasizing the role of these fundamental solutions is now summarized in the next theorem.

Theorem 5.1. *Assume that the increasing fundamental solution $\psi(x)$ is convex. Then*

$$\frac{\hat{\psi}(x)}{\hat{\psi}(y)} \leq \frac{\psi(x)}{\psi(y)}$$

for all $x \leq y$. Hence,

$$\hat{\psi}(x) \sup_{y \geq x} \left[\frac{g(y)}{\hat{\psi}(y)} \right] \leq \psi(x) \sup_{y \geq x} \left[\frac{g(y)}{\psi(y)} \right]$$

provided that the suprema exist. Moreover, if the conditions of Theorem 3.5 are satisfied, then $V(x) \geq \hat{V}(x)$ and, therefore,

$$\hat{C} = \{x \in I : \hat{V}(x) > g(x)\} \subseteq \{x \in I : V(x) > g(x)\} = C.$$

If the increasing fundamental solution $\hat{\psi}(x)$ is concave, then the inequalities and inclusions stated above are reversed.

Proof. Applying the Dynkin formula to $\psi(x)$ yields

$$\mathbb{E}_x[e^{-r\hat{\tau}_{(a,y)}} \psi(\hat{X}_{\hat{\tau}_{(a,y)}})] = \psi(x) + \mathbb{E}_x \int_0^{\hat{\tau}_{(a,y)}} e^{-rt} (\hat{\mathscr{G}}_r \psi)(\hat{X}_t) dt,$$

where $\hat{\tau}_{(a,y)} = \inf\{t \geq 0 : \hat{X}_t \geq y\}$. Since $\hat{X}_{\hat{\tau}_{(a,y)}} = y$ a.s. and $(\hat{\mathscr{G}}_r \psi)(x) = \left((\hat{\mathscr{G}}_r - \mathscr{G}_r + \mathscr{G}_r)\psi\right)(x) = \left((\hat{\mathscr{G}}_r - \mathscr{G}_r)\psi\right)(x) = \frac{1}{2}(\hat{\sigma}^2(x) - \sigma^2(x))\psi''(x) \leq 0$ by the X-harmonicity and the convexity of $\psi(x)$, we find that

$$\mathbb{E}_x[e^{-r\hat{\tau}_{(a,y)}}]\psi(y) = \frac{\hat{\psi}(x)}{\hat{\psi}(y)} \psi(y) \leq \psi(x)$$

from which the alleged results follow by the nonnegativity of the payoff $g(x)$. Establishing the reverse conclusions in case the fundamental solution $\psi(x)$ is concave is completely analogous. □

Theorem 5.1 extends previous findings based on continuous diffusions to the present setting as well and states a set of conditions in terms of the convexity (concavity) of the fundamental solution $\psi(x)$ under which increased volatility unambiguously decelerates (accelerates) rational exercise by expanding (shrinking) the continuation region where waiting is optimal. As is clear from this observation, the sign of the relationship between increased volatility and the optimal stopping policy is a process-specific property that as such does not depend on the precise form of the exercise payoff as long as the supremum at which the expected present value of the payoff is maximized exists and constitutes the optimal stopping rule.

It is worth noticing that the proof of our Theorem 5.1 indicates that the analysis of the impact of increased volatility on the optimal policy and its value reduces to the comparison of the r-superharmonic mappings characterized by the integro-differential operators $\mathscr{G}_r$ and $\hat{\mathscr{G}}_r$. Since $(\hat{\mathscr{G}}_r u)(x) \leq (\mathscr{G}_r u)(x)$ for any sufficiently smooth convex function $u : I \mapsto \mathbb{R}_+$ and $(\hat{\mathscr{G}}_r v)(x) \geq (\mathscr{G}_r v)(x)$ for any sufficiently smooth concave function $v : I \mapsto \mathbb{R}_+$, we find that the findings of our Theorem 5.1 generate a natural ordering for the convex (concave) solutions of the variational inequalities $\max\{(\hat{\mathscr{G}}_r u)(x), g(x) - u(x)\} = 0$ and $\max\{(\mathscr{G}_r u)(x), g(x) - u(x)\} = 0$.

We state next sufficient conditions for convexity of the value when the underlying process is the slightly less general

$$X_t = \int_0^t \mu(X_s)ds + \int_0^t \sigma X_s dW_s + \int_0^t \int_{\mathscr{S}} \gamma(z) X_s \tilde{N}(dz, ds),$$

where (1) the diffusion and jump components are assumed to be linear in the state variable and (2) the function $\mu(x)$ is assumed to have a locally Lipschitz derivative. In this setting we can state sufficient conditions for the convexity of the value function.

Theorem 5.2. *Suppose that g and μ are convex functions, and that*

$$rx - \mu(x)$$

is increasing. Then the value function of the stopping problem is convex.

Proof. We denote $Y_t^1 := \frac{\partial X_t}{\partial x}$. By virtue of Theorem V.40 of Protter (2004), we can differentiate the flow $X_t = X_t^x$ with respect to the initial state x to obtain

$$Y_t^1 = \int_0^t \mu'(X_s^x) Y_s^1 ds + \int_0^t \sigma Y_s^1 dW_s + \int_0^t \int_{\mathscr{S}} \gamma(z) Y_s^1 \tilde{N}(dz, ds),$$

which implies that

$$Y_t^1 = \exp\left(\int_0^t \mu'(X_s^x) ds\right) M_t \geq 0,$$

where

$$M_t = \exp\left(\sigma \int_0^t dW_s - \frac{1}{2}\sigma^2 t + \int_0^t \int_{\mathscr{S}} \ln(1 + \gamma(z)) N(ds, dz) - \lambda \overline{\gamma} t\right)$$

is an exponential martingale independent of x and

$$\overline{\gamma} := \int_{\mathscr{S}} \gamma(z) \mathfrak{m}(dz).$$

Thus differentiating the mapping

$$Q(t,x) := \mathbb{E}\left[e^{-rt}g(X_t^x)\right]$$

with respect to x yields

$$Q_x(t,x) = \mathbb{E}\left[\exp\left(-\int_0^t (r-\mu'(X_s^x))ds\right) g'(X_t^x)M_t\right] \geq 0,$$

which as a function of x is increasing, being under our assumptions the product of two non-negative and monotonically increasing functions. Thus $Q(t,x)$ is an increasing and convex function of x. Consequently, all elements of the increasing sequence $\{V_k(x)\}_{k\in\mathbb{N}}$ defined by

$$V_0(x) = \sup_{t\geq 0}\mathbb{E}\left[e^{-rt}g(X_t^x)\right]$$

$$V_{k+1}(x) = \sup_{t\geq 0}\mathbb{E}\left[e^{-rt}V_k(X_t^x)\right]$$

are increasing and convex. Furthermore, $V_k(x) \uparrow V(x)$. If $\alpha \in [0,1]$ and $x,y \in I$, then

$$\begin{aligned} \alpha V(x) + (1-\alpha)V(y) &\geq \alpha V_k(x) + (1-\alpha)V_k(y) \\ &\geq V_k(\alpha x + (1-\alpha)y) \end{aligned}$$

for all k. By monotone convergence

$$\alpha V(x) + (1-\alpha)V(y) \geq \lim_{k\to\infty} V_k(\alpha x + (1-\alpha)y) = V(\alpha x + (1-\alpha)y),$$

which implies the convexity of the value V. □

Theorem 5.2 states a set of conditions under which the sign of the relationship between increased volatility and the value of the considered optimal stopping problem is unambiguously positive. It is worth noticing that along the lines of the findings by Alvarez (2003) the monotonicity of the net appreciation rate $\mu(x) - rx$ is the key factor determining how higher volatility affects the optimal policy. The reason for this observation is naturally the fact that our evaluations are based on the compensated compound Poisson process (which is a martingale). If this were not the case, then the local expected behavior of the underlying jump process would naturally have a constant effect on the monotonicity requirement stated in Theorem 5.2.

Having characterized the impact of increased volatility on the optimal policy and its value, it is naturally of interest to analyze how the jump-intensity λ measuring the rate at which the downside risk is realized affects these factors. Along the lines of our previous notation, we now consider two jump diffusions of the form (4), X and $\hat{X}$, which are otherwise identical but are subject to different jump intensities, $\lambda > \hat{\lambda}$. In line with this notation, we denote the values of the associated optimal stopping problems again by V_λ and $V_{\hat{\lambda}}$, the associated differential operators by $\mathscr{G}_r$ and $\hat{\mathscr{G}}_r$, and the associated increasing fundamental solutions (given that assumption A1 is satisfied) by ψ_λ and $\psi_{\hat{\lambda}}$, respectively. Our main characterization on the impact of increased jump intensity on the value and the optimal policy is now summarized in our next theorem.

Theorem 5.3. *Assume that the increasing fundamental solution $\psi_\lambda(x)$ is convex. Then*

$$\frac{\psi_{\hat{\lambda}}(x)}{\psi_{\hat{\lambda}}(y)} \leq \frac{\psi_\lambda(x)}{\psi_\lambda(y)}$$

for all $x \leq y$. Hence,

$$\psi_{\hat{\lambda}}(x) \sup_{y \geq x} \left[\frac{g(y)}{\psi_{\hat{\lambda}}(y)}\right] \leq \psi_\lambda(x) \sup_{y \geq x} \left[\frac{g(y)}{\psi_\lambda(y)}\right]$$

provided that the supremum exists. Moreover, if the conditions of Theorem 3.5 are satisfied, then $V_\lambda(x) \geq V_{\hat{\lambda}}(x)$ and, therefore,

$$C_{\hat{\lambda}} = \{x \in I : V_{\hat{\lambda}}(x) > g(x)\} \subseteq \{x \in I : V_\lambda(x) > g(x)\} = C_\lambda.$$

If the increasing fundamental solution $\psi_{\hat{\lambda}}(x)$ is concave, then the inequalities and inclusions stated above are reversed.

Proof. The assumed convexity of the increasing fundamental solution $\psi_\lambda(x)$ implies that $\psi_\lambda(x+\gamma(x,z)) \geq \psi_\lambda(x) + \psi_\lambda'(x)\gamma(x,z)$ for any $z \in \mathscr{S}$ and, therefore, that

$$\int_{\mathscr{S}} \{\psi_\lambda(x+\gamma(x,z)) - \psi_\lambda(x) - \psi_\lambda'(x)\gamma(x,z)\}\mathfrak{m}(dz) > 0.$$

Consequently, we observe that

$$(\hat{\mathscr{G}}_r \psi_\lambda)(x) = (\hat{\lambda} - \lambda) \int_{\mathscr{S}} \{\psi_\lambda(x+\gamma(x,z)) - \psi_\lambda(x) - \psi_\lambda'(x)\gamma(x,z)\}\mathfrak{m}(dz) < 0$$

for all $x \in I$. Applying now Dynkin's theorem to $\psi_\lambda(x)$ then finally proves that $\psi_{\hat{\lambda}}(x)/\psi_{\hat{\lambda}}(y) \leq \psi_\lambda(x)/\psi_\lambda(y)$ for $x \leq y$. The rest of the alleged results then follow from the nonnegativity of the payoff $g(x)$ and Theorem 3.5. Establishing the reverse conclusions in the case where $\psi_{\hat{\lambda}}(x)$ is concave is completely analogous. □

Theorem 5.3 characterizes how the direction of the impact of increased jump-intensity λ on the optimal stopping policy and its value can be unambiguously determined when the fundamental solution is convex (concave). Along the lines of our findings on the impact of increased volatility, we observe that higher jump-intensity also slows down (speeds up) rational exercise by expanding (shrinking) the continuation region when $\psi(x)$ is convex (concave). This result is economically important, since it essentially states that if the value is convex on the continuation region where exercising is suboptimal, then the combined impact of downside risk and systematic market risk on the exercise incentives of rational investors is unambiguously negative.

6 EXPLICIT ILLUSTRATIONS

In this section our objective is to illustrate our main findings within explicitly parametrized examples based on different descriptions for the underlying stochastic dynamics. As usually, we first illustrate our findings for the arithmetic Lévy process and the geometric Lévy process since in those cases the representation obtained in the analysis of our previous sections is valid. We then extend our findings to cover also two other solvable cases: namely the constant elasticity of variance case and a logistic jump-diffusion case.

6.1 ARITHMETIC STOCHASTIC DYNAMICS

In the arithmetic case

$$dL_t = \mu dt + \sigma dW_t + \int_{(0,\infty)} \gamma z \tilde{N}(dt, dz)$$

(with $\gamma < 0$ so that *X2* holds) $I = \mathbb{R}$ and a sufficient condition for assumption *X1* to hold is $\mu + \gamma\lambda\overline{m} > 0$. The associated integro-differential equation

$$\frac{1}{2}\sigma^2\psi''(x) + (\mu + \gamma\lambda\overline{m})\psi'(x) - (r+\lambda)\psi(x) + \lambda\int_{(0,\infty)} \psi(x+\gamma z)\mathfrak{m}(dz) = 0$$

has an increasing solution $e^{k_1 x}$ where $k_1 > 0$ solves

$$\frac{1}{2}\sigma^2 k^2 + (\mu + \gamma\lambda\overline{m})k + \lambda\int_{(0,\infty)} e^{\gamma z k}\mathfrak{m}(dz) - (r+\lambda) = 0.$$

In light of our general observations we find that for any reward function g satisfying conditions *g1–g2*, and *Ag2*, and such that $e^{-k_1 x}g(x)$ has a unique maximizer $x^* \in \mathbb{R}$ and is non-increasing for $x > x^*$, the value of the optimal stopping policy can be represented as

$$V(x) = e^{k_1 x}\sup_{y\geq x}\left\{e^{-k_1 y}g(y)\right\} = \begin{cases} g(x), & x \geq x^*, \\ g(x^*)e^{k_1(x-x^*)}, & x < x^*, \end{cases} \tag{15}$$

where x^* is the unique maximizer of g/ψ, i.e. for a differentiable g the solution of $D_x[\ln g(x)] = k_1$. It is also worth pointing out that in accordance with the findings of our Theorem 4.1 we now find that the root $k_1 \in (\tilde{k}_1, \hat{k}_1)$, where

$$\hat{k}_1 = -\frac{\mu+\gamma\lambda\bar{m}}{\sigma^2} + \sqrt{\left(\frac{\mu+\gamma\lambda\bar{m}}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}}$$

denotes the positive root of the characteristic equation $\sigma^2k^2 + 2(\mu+\gamma\lambda\overline{m})k = 2(r+\lambda)$, and

$$\tilde{k}_1 = -\frac{\mu+\gamma\lambda\bar{m}}{\sigma^2} + \sqrt{\left(\frac{\mu+\gamma\lambda\bar{m}}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$$

denotes the positive root of the characteristic equation $\sigma^2k^2 + 2(\mu+\gamma\lambda\overline{m})k = 2r$. Consequently, we observe that in the present setting

$$e^{\tilde{k}_1 x}\sup_{y\geq x}\left\{e^{-\tilde{k}_1 y}g(y)\right\} \leq e^{k_1 x}\sup_{y\geq x}\left\{e^{-k_1 y}g(y)\right\} \leq e^{\hat{k}_1 x}\sup_{y\geq x}\left\{e^{-\hat{k}_1 y}g(y)\right\}$$

provided that the maximum exists. Moreover, given that in the present case $\tilde{\psi}_\theta(x) = e^{K_\theta x}$, where

$$K_\theta = -\frac{\mu+\gamma\lambda\bar{m}}{\sigma^2} + \sqrt{\left(\frac{\mu+\gamma\lambda\bar{m}}{\sigma^2}\right)^2 + \frac{2\theta}{\sigma^2}}$$

we observe that $\operatorname{argmax}\{e^{-k_1 y}g(y)\} = \operatorname{argmax}\{e^{-K_\theta y}g(y)\}$ whenever the identity

$$\theta = r + \lambda\int_{(0,\infty)}(1-e^{\gamma z k})\mathfrak{m}(dz) \tag{16}$$

holds. This observation is important since it demonstrates that in the present case both the value as well as the optimal stopping rule of the optimal stopping problem (3) of the underlying jump diffusion coincides with the value and stopping rule of the associated stopping problem of a continuous diffusion by properly adjusting the discount rate. Hence, our results indicate that whenever the value of the optimal policy admits the representation (15) the jump-risk can be viewed as a discount rate effect as characterized by the identity (16). More precisely, whenever the value of the optimal policy admits the representation (15) we have that $\tilde{V}_\theta(x) = V(x)$ by choosing the discount rate according to the identity (16).

It is worth noticing that according to our general results the strict convexity of the increasing fundamental solution $e^{k_1 x}$ implies that increased volatility σ as well as higher jump-intensity λ increases the value of the optimal stopping policy and raises the optimal boundary at which the underlying jump-diffusion should be stopped. To see that this is indeed the case consider the mapping

$$\bar{P}(\lambda,\sigma,k) = (\mu+\gamma\lambda\overline{m})k + \frac{1}{2}\sigma^2 k^2 + \lambda\int_{(0,\infty)} e^{\gamma z k}\mathfrak{m}(dz) - (r+\lambda).$$

Standard differentiation yields that $\bar{P}_\sigma(\lambda,\sigma,k) = \sigma k^2 > 0$ and

$$\bar{P}_\lambda(\lambda,\sigma,k) = \int_{(0,\infty)} \{e^{\gamma z k} - 1 + \gamma k z\}\mathfrak{m}(dz) > 0.$$

Therefore, the inequality $\bar{P}(\lambda,\sigma,0) = -r < 0$, the limiting condition $\bar{P}(\lambda,\sigma,k) \uparrow +\infty$ as $k \to \infty$, and the strict convexity of the function $\bar{P}(\lambda,\sigma,k)$ imply that $\partial k_1/\partial\sigma < 0$ and $\partial k_1/\partial\lambda < 0$ and, therefore, that $\partial e^{k_1(x-y)}/\partial\sigma > 0$ and $\partial e^{k_1(x-y)}/\partial\lambda > 0$ for all $x \le y$.

As a numerical illustration, consider the *capped option* reward function

$$g(x) = \max\{0, p\min(K,x) - qK\},$$

where we assume $p > q > 0$ and $K \le \frac{p}{p-q}\frac{1}{k_1} =: K_0$ to guarantee that g/ψ is maximized at K (if $K > K_0$, the maximizer is an interior point of $(0,K)$, see Alvarez (1996) for a detailed analysis and interpretation in the continuous setting). As stated, in this case g/ψ attains a unique maximum value at $x^* = K$, which is a point of nondifferentiability for g. Assumption *g1* is now satisfied. We do not need to worry about *Ag2* as we can show that for this particular reward function $V(x) - g(x)$ is decreasing: if $\tau^*(x)$ is the optimal stopping time for X_t^x, $x \ge y$ and X_t^z denotes the process started from z, then by suboptimality of $\tau^*(x)$ for X_t^y,

$$V(x) - V(y) \le \mathbb{E}\left\{e^{-r\tau^*(x)}\left(g(X^x_{\tau_*(x)}) - g(X^y_{\tau^*(x)})\right)\right\}.$$

Since the option will not be exercised below qK/p or above K, we can consider the linear reward $px - qK$. Because $X^z_\tau = z + X^0_\tau$, we get then

$$\mathbb{E}\left\{e^{-r\tau^*(x)}\left(p(x + X^0_{\tau_*(x)}) - p(y + X^0_{\tau^*(x)})\right)\right\} \leq px - qK - (py - qK),$$

which gives $V(x) - (px - qK) \leq V(y) - (py - qK)$ for $x \geq y$, i.e. we see that $V(x) - g(x)$ is decreasing. Hence the value of the optimal stopping problem is

$$V(x) = e^{k_1 x} \sup_{y \geq x}\left\{e^{-k_1 y} g(y)\right\} = \begin{cases} (p-q)K, & x \geq K \\ e^{k_1(x-K)}(p-q)K, & x < K \end{cases} \tag{17}$$

This is a continuous function, but its derivative has a discontinuity at x^*:

$$\lim_{x \to x^*-} V'(x) = k_1(p-q)K > 0 = \lim_{x \to x^*+} g'(x) = \lim_{x \to x^*+} V'(x),$$

and there is no smooth pasting. The graphs of the reward function, the function g/ψ and the value and its derivative for $p = 1$, $q = 0.5$, $r = 0.04$, $\mu = 0.075$, $\sigma = 0.1$, $\lambda = 0.1$, $\gamma = -0.5$, and $K = 0.75 \cdot K_0$ (implying that $k_1 = 1.1463$ and $K_0 = 1.7447$) are shown in Figure 1.

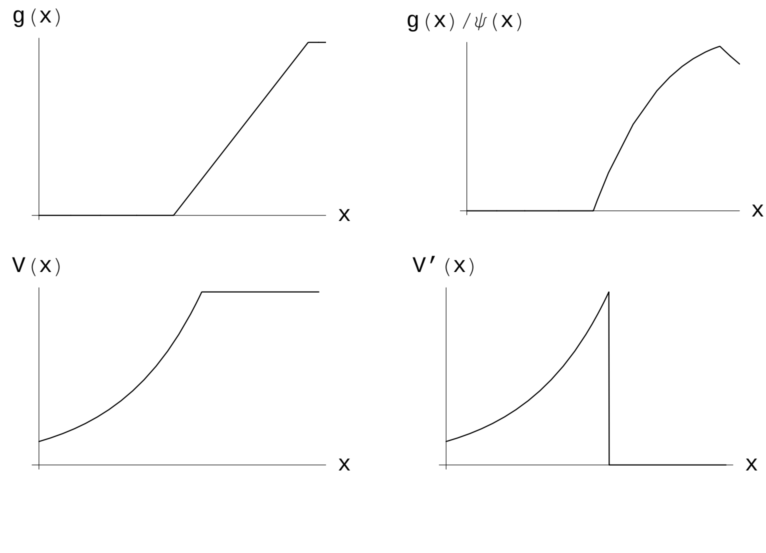

Figure 1: *The reward function g, the function g/ψ, the value function V and the derivative of the value V' for the capped option case*

6.2 GEOMETRIC STOCHASTIC DYNAMICS

Geometric processes have been of paramount importance in mathematical finance for several decades, with the most extensively used and well-known instance being the geometric Brownian motion $S_t = s_0 \exp\{\mu t + \sigma W_t\}$, where $\sigma > 0$ and W is a standard Wiener process. During the last decade, a considerable amount of research has been done on geometric Lévy models

$$Y_t = y_0 \exp\{\alpha t + \sigma W_t + J_t\}, \tag{18}$$

where in addition to the deterministic drift and the Gaussian component there is a jump process J_t in the exponent.

A geometric Lévy process $Y = \{Y_t\}$ with a finite Lévy measure $\nu = \lambda m$ is a jump diffusion whose dynamics are given by

$$dY_t = Y_{t-}\left\{\alpha dt + \sigma dW_t + \lambda \int_{\mathscr{S}} \gamma(z)(N(dt,dz) - \nu(dz)dt)\right\}, \tag{19}$$

where both the drift α and the diffusion coefficient σ are assumed to be positive. Note that in this case $I = \mathbb{R}_+$ and the explicit solution Y_t equals

$$y_0 \exp\left\{\tilde{\alpha} t + \sigma W_t + \int_0^t \int_{\mathscr{S}} \ln(1+\gamma(z))\tilde{N}(ds,dz)\right\}. \tag{20}$$

where $\tilde{\alpha} = \alpha - \frac{1}{2}\sigma^2$. To ascertain that *X1* holds, we require that $\tilde{\alpha} > 0$. For simplicity of exposition, we take $\gamma(z) = -z$ and to guarantee that *X2* is satisfied, we assume $\mathscr{S} \subseteq (0,1)$. Furthermore, to ensure the finiteness of the value of the optimal stopping problem, we need to impose the integrability condition $\alpha - r < 0$ (which is known in the literature on financial economics as the absence of speculative bubbles condition). The integro-differential operator $\mathscr{G}_r$ takes now the form

$$\frac{1}{2}\sigma^2 x^2 \psi''(x) + \hat{\alpha} x \psi'(x) - (r+\lambda)\psi(x) + \lambda \int_0^1 \psi(x - xz)\mathfrak{m}(dz) = 0, \tag{21}$$

where $\hat{\alpha} = \alpha + \lambda \bar{m}$. By guessing now the solution to be of form x^k, we obtain the characteristic equation for k:

$$\frac{1}{2}\sigma^2 k(k-1) + (\alpha + \lambda \bar{m})k - (r+\lambda) + \lambda \int_0^1 (1-z)^k \mathfrak{m}(dz) = 0, \tag{22}$$

If the integrability condition is satisfied, it is easy to show that this equation has a solution $k_1 > 1$, and thus $\psi(x) = x^{k_1}$ is an increasing smooth solution of (21) which vanishes at $x = 0$. Hence assumption *A1* is satisfied. In light of our representation of the value of the optimal policy in terms of an associated nonlinear programming problem, we find that for any reward function g satisfying conditions *g1* and *Ag2*, and such that $x^{-k_1}g(x)$ has a unique maximizer $x^* \in \mathbb{R}$ and is non-increasing for $x > x^*$, the value of the optimal stopping policy can be represented as

$$V(x) = x^{k_1} \sup_{y \geq x} \left\{ y^{-k_1} g(y) \right\} = \begin{cases} g(x), & x \geq x^* \\ g(x^*)(x/x^*)^{k_1}, & x < x^*, \end{cases} \tag{23}$$

where x^* is the unique maximizer of g/ψ, i.e. for a differentiable g the solution of $g'(x^*)x^*/g(x^*) = k_1$.

As in the arithmetic case, we observe that our Theorem 4.1 implies that in the present case the root of the equation (22) k_1 satisfies the condition $k_1 \in (\tilde{k}_1, \hat{k}_1)$, where

$$\hat{k}_1 = \frac{1}{2} - \frac{\alpha + \lambda\bar{m}}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\alpha + \lambda\bar{m}}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}}$$

denotes the positive root of the characteristic equation $\sigma^2 k(k-1) + 2(\alpha + \lambda\bar{m})k - 2(r+\lambda) = 0$ and

$$\tilde{k}_1 = \frac{1}{2} - \frac{\alpha + \lambda\bar{m}}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\alpha + \lambda\bar{m}}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$$

denotes the positive root of the characteristic equation $\sigma^2 k(k-1) + 2(\alpha + \lambda\bar{m})k - 2r = 0$. To demonstrate this observation, consider the behavior of the function

$$P(\lambda, \sigma, k) = \frac{1}{2}\sigma^2 k(k-1) + (\alpha + \lambda\bar{m})k - (r+\lambda) + \lambda \int_0^1 (1-z)^k \mathfrak{m}(dz).$$

We first observe that

$$P(\lambda, \sigma, \hat{k}_1) = \lambda \int_0^1 (1-z)^{\hat{k}_1} \mathfrak{m}(dz) > 0$$

and

$$P(\lambda, \sigma, \tilde{k}_1) = \lambda \int_0^1 ((1-z)^{\tilde{k}_1} - 1)\mathfrak{m}(dz) < 0.$$

However, since $P(\lambda, \sigma, 1) = \alpha - r < 0$ and $P(\lambda, \sigma, k)$ is strictly convex on $k > 1$ we observe that $\tilde{k}_1 < k_1 < \hat{k}_1$ and, therefore, that

$$x^{\hat{k}_1} \sup_{y \geq x} \left[g(y) y^{-\hat{k}_1} \right] \leq x^{k_1} \sup_{y \geq x} \left[g(y) y^{-k_1} \right] \leq x^{\tilde{k}_1} \sup_{y \geq x} \left[g(y) y^{-\tilde{k}_1} \right]$$

provided that the maximum exists. Moreover, since in this case $\tilde{\psi}_\theta(x) = x^{l_\theta}$, where

$$l_\theta = \frac{1}{2} - \frac{\alpha+\lambda\bar{m}}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\alpha+\lambda\bar{m}}{\sigma^2}\right)^2 + \frac{2\theta}{\sigma^2}}$$

we observe that $\operatorname{argmax}\{y^{-k_1}g(y)\} = \operatorname{argmax}\{y^{-l_\theta}g(y)\}$ provided that the identity

$$\theta = r + \lambda \int_0^1 (1-(1-z)^k)\mathfrak{m}(dz) \tag{24}$$

is satisfied. Along the lines indicated by our observations in the arithmetic case, we again observe that the effect of jump-risk on valuation can be captured by making an appropriate adjustment in the discount rate of the stopping problem of the associated continuous diffusion as is characterized by (24). Hence, whenever the value of the optimal stopping problem (3) admits the representation (23) we have $\tilde{V}_\theta(x) = V(x)$ by choosing the discount rate according to the identity (24).

It is also clear from our analysis that the increasing fundamental solution is strictly convex in this case as well. Thus, as our results in Theorem 5.1 and in Theorem 5.3 indicated, increased volatility and higher jump-intensity should increase the value and decelerate exercise by increasing the optimal stopping boundary. To see that this is indeed the case in the present example, we first observe that $P_\sigma(\lambda,\sigma,k) = \sigma k(k-1) > 0$ for $k > 1$ and

$$P_\lambda(\lambda,\sigma,k) = \bar{m}k - 1 + \int_0^1 (1-z)^k\mathfrak{m}(dz) = \mathbb{E}[zk - 1 + (1-z)^k].$$

Since the function $z \mapsto zk - 1 + (1-z)^k$ is strictly convex for $k > 1$ and attains a minimum at $z = 0$, we observe that $\mathbb{E}[zk-1+(1-z)^k] > 0$ and, therefore, that $P_\lambda(\lambda,\sigma,k) > 0$ for all $k > 1$. Since the positive root k_1 is attained on this set and $P(\lambda,\sigma,1) = \alpha - r < 0$, we find that $\partial k_1/\partial\lambda < 0$ and $\partial k_1/\partial\sigma < 0$. An interesting implication of this observation is that

$$\frac{\partial}{\partial\sigma}\left(\frac{x}{y}\right)^{k_1} > 0 \quad \text{and} \quad \frac{\partial}{\partial\lambda}\left(\frac{x}{y}\right)^{k_1} > 0$$

for all $x \le y$. Consequently, we observe that both increased volatility as well as higher jump-intensity increases the value of the problem and postpones exercise by raising the threshold at which the underlying process should be optimally stopped. Moreover, if the exercise payoff is continuously differentiable at the exercise boundary x^*, then

$$\frac{\partial}{\partial\sigma}\left[\frac{g'(x^*)x^*}{g(x^*)}\right]=\frac{\partial k_1}{\partial\sigma}<0 \quad \text{and} \quad \frac{\partial}{\partial\lambda}\left[\frac{g'(x^*)x^*}{g(x^*)}\right]=\frac{\partial k_1}{\partial\lambda}<0.$$

In other words, both increased volatility and higher jump intensity decreases the elasticity of the exercise payoff at the optimal exercise threshold x^*.

For the sake of explicit illustration, we now consider the case of a linear reward function in the geometric Lévy model. Let $g(x)=\max(ax-b,0)$ with $a,b>0$. This case contains the standard American call option (take $a=1$, $b=K$), and also the rewards of optimal stopping problems associated with irreversible investment decisions (see Boyarchenko (2004) for a very readable account on the relationship between perpetual American options and irreversible investment decisions). Clearly, the increasing function g satisfies *g1*. The function g/ψ has a unique maximum in $\mathbb{R}_+$ and is non-increasing for argument values larger than the maximizer, since the sign of

$$D_x\left[\frac{g(x)}{\psi(x)}\right]=\frac{x^{k_1-1}(ax-(ax-b)k_1)}{x^{2k_1}}$$

depends only on the linear decreasing function $a(1-k_1)x+k_1b$, so *Ag1* is satisfied. As in the previous example, we can show that $V(x)-g(x)$ is decreasing. Considering the linear function $g(x)=ax-b$, we get for $x\geq y$

$$V(x)-V(y)\leq a(x-y)\mathbb{E}_1\left\{e^{-(r-\Psi(1))\tau_*(x)}\right\},$$

where $\Psi(1)$ is the Laplace exponent of the spectrally negative Lévy process L_t in $Y_t=y_0\exp\{L_t\}$ and $\mathbb{E}_1$ denotes expectation under the equivalent measure defined via $\frac{d\mathbb{P}^1}{\mathbb{P}}|_{\mathscr{F}_t}=e^{-\Psi(1)t+L_t}$. Under our assumption $\alpha-r<0$ it is the case that $r>\Psi(1)$ and hence $V(x)-V(y)\leq a(x-y)$, implying that $V(x)-g(x)$ is decreasing. By theorem 3.5, the value of the optimal stopping problem can now be represented as

$$V(x)=x^{k_1}\sup_{y\geq x}\left\{y^{-k_1}(ay-b)\right\}=\begin{cases}ax-b, & x\geq x^*\\ (ax^*-b)(x/x^*)^{k_1}, & x<x^*\end{cases} \tag{25}$$

where $x^*=\frac{k_1b}{(k_1-1)a}$ is the unique maximizer of the function $g(x)/\psi(x)$. As usually in the real options literature on irreversible investment, we notice that the option multiplier $M=k_1/(k_1-1)$ determines the comparative static properties of the optimal exercise threshold x^*. In light of our findings this multiplier reads $\tilde{M} = \tilde{k}_1/(\tilde{k}_1 - 1)$ and $M = \hat{k}_1/(\hat{k}_1 - 1)$ for the stopping problems of the associated continuous diffusion. We illustrate these option multipliers in Figure 2 for Beta(c,d)-distributed jumps under the assumption that $\alpha = 0.02, r = 0.035, \lambda = 0.01, a = b = 1, c = 1.25, d = 2$. As Figure 2 in-

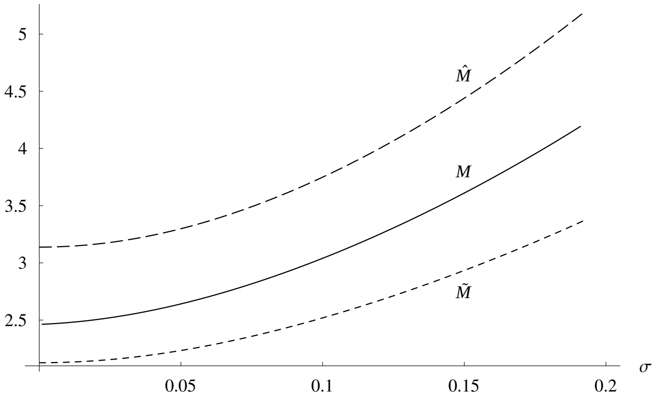

Figure 2: The impact of volatility on the option multipliers M, $\tilde{M}$, and $\hat{M}$

dicates, the option multipliers are increasing as functions of the underlying volatility coefficient. Moreover, the option multipliers satisfies the condition $M \in (\tilde{M}, \hat{M})$ as was established in our Theorem 4.1. The values of the optimal stopping problems are graphically illustrated for Beta(c,d)-distributed jumps in Figure 3 under the assumption that $\alpha = 0.02, r = 0.035, \lambda = 0.01, a = b = 1, c = 1.25, d = 2, \sigma = 0.1$ (which implies that $x^* = M = 2.95, \hat{x}^* = \hat{M} = 3.75$, and $\tilde{x}^* = \tilde{M} = 2.52$) Figure 3 illustrates explicitly the results of our Theorem 4.1 for the values of the stopping problems. It is of interest to notice that as was predicted by Theorem 4.1, the value $V(x)$ of the considered stopping problem is sandwiched between the two values $\tilde{V}_{r+\lambda}(x)$ and $\tilde{V}_r(x)$.

Consider next the concave reward function $g(x) = \max(\ln x, 0)$. Then function g satisfies *g1* and g/ψ has a unique maximum in $\mathbb{R}_+$ and is non-increasing for argument values larger than the maximizer, since the sign of

$$D_x\left[\frac{g(x)}{\psi(x)}\right] = \frac{1 - k_1 \ln x}{x^{k_1+1}}$$

depends only on the decreasing function $1 - k_1 \ln x$, so *Ag1* is satisfied. For geometric dynamics and a loglinear reward function, we can show that $V(x) - g(x)$ is decreasing, by reasoning similar to the case of arithmetic dynamics and

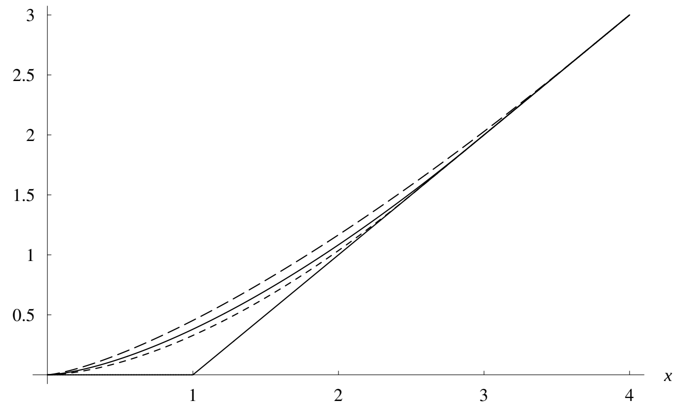

Figure 3: The exercise payoff $(x-1)^+$ and the values $V(x), \tilde{V}_r(x)$, and $\tilde{V}_{r+\lambda}(x)$

a linear reward considered in Example 6.1. By theorem 3.5, the value of the optimal stopping problem can now be represented as

$$V(x) = x^{k_1} \sup_{y \geq x} \left\{ y^{-k_1} \ln y \right\} = \begin{cases} \ln x, & x \geq \exp\{1/k_1\} \\ x^{k_1} \left[\frac{1}{ek_1} \right], & x < \exp\{1/k_1\} \end{cases} \tag{26}$$

where $x^* = \exp\{1/k_1\}$ is the unique maximizer of the function $g(x)/\psi(x)$. As was predicted by our Theorem 5.1 we find that under the assumption $r > \alpha$ we have $\partial x^*/\partial \sigma = -(x^*/k_1^2)\partial k_1/\partial \sigma > 0$ and $\partial x^*/\partial \lambda = -(x^*/k_1^2)\partial k_1/\partial \lambda > 0$. Hence, both increased volatility as well as higher jump-intensity decelerate optimal exercise by raising the optimal exercise boundary in this case as well. It is, however, worth noticing that in the present example the maximizing threshold x^* exists even when $k_1 < 1$, that is, even when the fundamental solution is not convex as a function of the state. Hence, for the exercise payoff $g(x) = \max(\ln x, 0)$ the condition $r > \alpha$ can be relaxed. If $0 < r \leq \alpha$ then $k_1 \in (0,1]$ since in that case $P(\lambda, \sigma, 0) = -r < 0$ and $P(\lambda, \sigma, 1) = \alpha - r \geq 0$. Under those circumstances the sign of the relationship between increased volatility and the optimal exercise strategy is reversed as the root k_1 becomes an increasing function of volatility. More precisely, if $r \leq \alpha$ then $\partial x^*/\partial \sigma = -(x^*/k_1^2)\partial k_1/\partial \sigma < 0$ and $\partial x^*/\partial \lambda = -(x^*/k_1^2)\partial k_1/\partial \lambda < 0$. We illustrate this observation graphically for Beta(c,d)-distributed jumps in Figure 4 under the assumption that $\alpha = 0.04, r = 0.02, \lambda = 0.01, a = b = 1, c = 1.25$, and $d = 2$. Figure 4 illustrates how the sign of the relationship between increased volatility and the optimal exercise threshold is reversed as the increas-

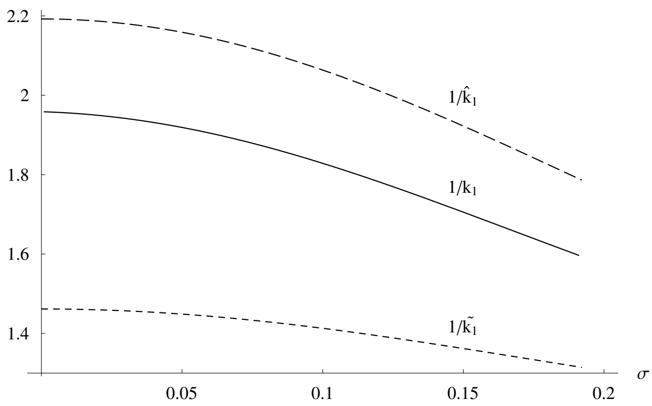

Figure 4: The impact of volatility on the exercise thresholds $1/k_1$, $1/\tilde{k}_1$, and $1/\hat{k}_1$

ing fundamental solution becomes concave. It is worth noticing that even in this case the order of the exercise thresholds remain naturally unchanged since the ordering of the values $V(x)$, $\tilde{V}_{r+\lambda}(x)$, and $\tilde{V}_r(x)$ is based only on nonnegativity and monotonicity.

6.3 CONSTANT ELASTICITY OF VARIANCE

Consider the *Constant Elasticity of Variance (CEV)* model with an added jump component

$$dS_t = S_t\Big\{rdt + \sigma S_t^{-\alpha} dW_t - \int_0^1 z\tilde{N}(dz,dt)\Big\}, \tag{27}$$

where $\alpha \in (0,1)$ is a known exogenously determined constant, and suppose that the first exit time $\tau_{(0,x)} < \infty$ for any $x \in \mathbb{R}_+$. Note that the deterministic drift is now set equal to the discount rate r. We assume the reward to be of logarithmic utility type: $g(x) = \max(\ln x, 0)$. The associated integro-differential equation is now

$$\begin{aligned}&\frac{1}{2}\sigma^2 x^{2(1-\alpha)}\psi''(x) + (r+\lambda\overline{m})x\psi'(x)\\ &+\lambda\int_0^1 \psi(x-xz)\mathfrak{m}(dz) = (r+\lambda)\psi(x),\end{aligned} \tag{28}$$

which has an increasing solution $\psi(x) = x$, as can easily be verified. Now assumptions *X1*, *X2* and *A1* are clearly satisfied. Since the reward is of log utility type, we also have for $x > 1$

$$D_x[g(x)/\psi(x)] = \frac{1-\ln x}{x^2},$$

whose sign depends on the decreasing function $1-\ln x$; the unique maximizer is $x^* = e$ and assumption *Ag1* is satisfied. If *Ag2* holds, by theorem 3.5, the value of the optimal stopping problem has the representation

$$V(x) = x \sup_{y \geq x} \left\{ \frac{\ln y}{y} \right\} = \begin{cases} \ln x, & x \geq e, \\ \frac{x}{e}, & x < e, \end{cases} \tag{29}$$

as the unique maximum of $\ln x/x$ is $x = e$. It is worth noting that in this case we could not have taken a linear $g(x)$ as then *Ag1* is no longer satisfied. Due to our specific choice of parameters, the optimal value and the threshold are independent of the parameter values. For the associated diffusion, the increasing fundamental solution $\tilde{\psi}_\theta(x)$ of the characteristic differential equation can be expressed as

$$\tilde{\psi}_\theta(x) = xF_1\left(\frac{1}{2\alpha}\left(1-\frac{\theta}{r+\lambda\overline{m}}\right), 1+\frac{1}{2\alpha}, \frac{-r-\lambda\overline{m}}{\alpha\sigma^2}x^{2\alpha}\right),$$

where $F_1(a,b,z)$ is the Kummer confluent hypergeometric function. Notice that the solution depends on the values of parameters. As a numerical illustration, consider the case $(r,\lambda,\overline{m},\sigma,\alpha) = (0.035, 0.1, 0.2, 0.1, 0.75)$. In this case $\tilde{x}^* \approx 1.54$ and $\hat{x}^* \approx 4.79$.

6.4 LOGISTIC JUMP DIFFUSION

In order to illustrate how Theorem 3.6 can be applied in the analysis of the stopping problem consider a stochastic process X such that the ratio $X/(1-X)$ evolves as a geometric Lévy process, i.e.

$$G_t := \frac{X_t}{1-X_t} = \frac{x}{1-x}\exp\left\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t + \int_0^t \int_{(0,1)} z\tilde{N}(ds,dz)\right\}.$$

Note that X lives on $I = (0,1)$. Being a $\mathscr{C}^2$ function of a jump diffusion, $X_t = f(G_t) = G_t/(1+G_t)$ is also a jump diffusion and application of Itô formula yields the dynamics of X:

$$\begin{aligned} dX_t &= (1-X_t)X_t\{\mu + \lambda\overline{m} - X_t\sigma^2 + \tfrac{C(X_t)}{1-X_t}\}dt \\ &+\sigma(1-X_t)X_t dW_t + +X_t \textstyle\int_{(0,1)} \left\{\frac{1-z}{1-zX_t} - 1\right\}\tilde{N}(dt,dz), \end{aligned} \tag{30}$$

where $C(x) = \int_{(0,1)} \{\frac{1-z}{1-zx} - 1\} \nu(dz)$ arises from compensating the driving jump process in (30). Thus the integro-differential equation associated with the optimal stopping problem for X is

$$\begin{aligned}(1-x)x\alpha(x)\psi'(x) + \frac{1}{2}\sigma^2(1-x)^2x^2\psi''(x) \\ + \int_{(0,1)} \psi(x + xc(x,z))\nu(dz) = \tilde{r}\psi(x),\end{aligned} \tag{31}$$

where $\alpha(x) = \mu + \lambda\overline{m} - x\sigma^2$ and $c(x,z) = \frac{1-z}{1-zx} - 1$. Via transformation $\tilde{\psi}(y) := \psi(\frac{y}{y+1})$ equation (31) transforms into the following integro-differential equation:

$$\frac{1}{2}\sigma^2 y^2 \tilde{\psi}''(y) + (\mu + \lambda\overline{m})y\tilde{\psi}'(y) + \int_{(0,1)} \tilde{\psi}(y - yz)\nu(dz) = \tilde{r}\tilde{\psi}(y),$$

which has an increasing solution $\tilde{\psi}(y) = y^{k_1}$, where k_1 is the positive root of equation

$$\frac{1}{2}\sigma^2 k(k-1) + (\mu + \lambda\overline{m})k + \int_{(0,1)} (1-z)^k \nu(dz) = \tilde{r}.$$

Then by theorem 3.6 $\phi(x) := (\frac{x}{1-x})^{k_1}$ is an increasing solution of equation (31), and furthermore $\phi(0) = 0$. Since the process X by virtue of theorem 3.6 satisfies assumptions *X1* and *X2*, theorem 3.5 implies that for any reward function g satisfying assumptions *g1* and *Ag1–Ag2*

$$\begin{aligned}V(x) &= \left(\frac{x}{1-x}\right)^{k_1} \sup_{y \geq x} \left\{\frac{g(y)(1-y)^{k_1}}{y^{k_1}}\right\} \\ &= \begin{cases} g(x), & x \geq x^*, \\ g(x^*)\left(\frac{x(1-x^*)}{x^*(1-x)}\right)^{k_1}, & x < x^*, \end{cases}\end{aligned} \tag{32}$$

where x^* again satisfies the logarithmic derivative condition.

As a numerical illustration, consider a *bull spread* type reward (where $0 < K_1 < K_2 < 1$)

$$g(x) = \max\{0, x - K_1\} - \max\{0, x - K_2\}$$

in a logistic jump diffusion model with parameters

$$(\mu, \sigma, \gamma, \lambda, r, a, b, c, d, K_1, K_2) = (0.1, 0.3, -1, 0.1, 0.15, 1, 1, 1.5, 1, 0.4, 0.6)$$

and Beta(c,d) distributed jumps ($\mathfrak{m}(dz) = (1/\beta(c,d))z^{c-1}(1-z)^{d-1}dz$). These values lead to $k_1 = 1.2591$, and an increasing solution of the characteristic equation for the logistic process is given by $\psi(x) = \left(\frac{x}{1-x}\right)^{k_1}$. Thus

$$\frac{g(x)}{\psi(x)} = \begin{cases} \left(\frac{1-x}{x}\right)^{k_1}(K_2 - K_1), & 1 > x > K_2 \\ \left(\frac{1-x}{x}\right)^{k_1}(x - K_1), & K_1 < x \leq K_2 \\ 0, & 0 < x \leq K_1, \end{cases}$$

and this is decreasing for $x > K_2$ and has a unique maximum at

$$x_0^* = \frac{1-k_1}{2} + \frac{1}{2}\sqrt{(1-k_1)^2 + 4k_1K_1} > K_1,$$

provided that $x_0^* < K_2$; in the case $x_0^* \geq K_2$ it is optimal to stop at K_2 and the value will not exhibit smooth pasting. By Theorems 3.5 and 3.6, the value function is given by

$$V(x) = \begin{cases} \max\{0, x - K_1\} - \max\{0, x - K_2\}, & x \geq x^* \\ (x^* - K_1)\left(\frac{x(1-x^*)}{x^*(1-x)}\right)^{k_1}, & x < x^*, \end{cases}$$

where $x^* = \min\{x_0^*, K_2\}$ – provided that *Ag2* is satisfied. For the given parameters, we have $x^* = 0.59185$ and we can establish numerically that *Ag2* holds. The value function is presented graphically in Figure 5 (unbroken curve).

To illustrate Theorem 5.1, consider the process $\hat{X}$ with parameters identical to those of X except that the volatility coefficient $\hat{\sigma} = 0.4 > 0.3 = \sigma$. We can compute $\hat{k}_1 = 1.21464$ and $\hat{x}^* = 0.59793 > 0.59185 = x^*$. This is in line with the theorem, and the value $\hat{V}(x)$ is depicted in Figure 5 (dashed curve).

7 CONCLUSIONS

In this study we generalized a representation result known to hold for continuous linear diffusions to include a class of spectrally one-sided Lévy diffusions: given some conditions, the optimal stopping problem for a one-dimensional spectrally negative Lévy diffusion can be reduced to an ordinary nonlinear programming problem. As the proof of our representation relied on the viscosity solution approach, differentiability is not required, and we are able to deal with nonsmooth reward functions as well. The class of processes for which the representation holds, contains the standard arithmetic and geometric

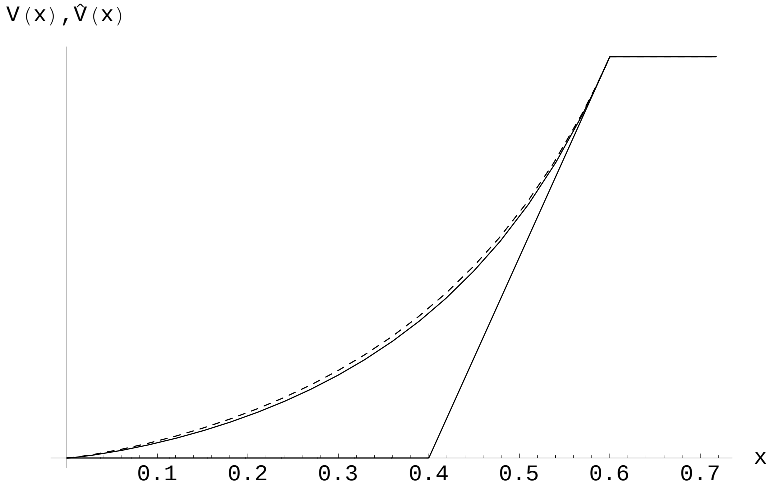

Figure 5: The impact of volatility on the value

Lévy processes. We established that the class is closed with respect to strictly increasing $\mathscr{C}^2$ transforms, although the transform naturally changes the set of allowable reward functions.

Considering the fact that optimal stopping problems feature prominently in pricing of American options and in real options theory, reducing the stopping problem of a Lévy diffusion into a standard programming problem can significantly facilitate the ongoing research on these areas of mathematical finance. We demonstrated this by deriving several interesting comparative static properties of spectrally negative Lévy diffusions using our representation, and found out that a useful tool in obtaining bounds for the value of the optimal stopping of a Lévy diffusion is the corresponding stopping problem for an associated continuous diffusion. By choosing the discount rates appropriately (namely, as r and $r+\lambda$), we were able to sandwich the value of the jump diffusion problem between the values of two optimal stopping problems of this continuous diffusion. In fact, our findings indicate the existence of a critical discount rate θ^* such that the value and the threshold of the stopping problem of the jump diffusion with discount rate r coincide with the value and the threshold for the stopping problem of the associated diffusion with discount rate θ^*. Furthermore, it turned out that the impact of volatility on the optimal policy and its value in our setting is similar to the continuous case: for values convex (concave) below the optimal threshold, increased risk decelerates (accelerates) rational investment by expanding or leaving unchanged (shrinking or leaving unchanged) the continuation region and increasing or leaving unchanged (decreasing or leaving unchanged) the optimal threshold and the value of waiting. The impact of downside risk as measured by the intensity of the compound Poisson jump process on the optimal value was found out to be similar to the impact of the diffusion risk (as measured by the volatility). We also established that the key factor determining the relevant convexity/concavity properties of the value is (provided that it exists) the increasing fundamental solution of the associated integro-differential equation, which is process-specific. Thus we saw that the impact of volatility or downside risk is not dependent on the precise form of the exercise payoff, as long as the conditions for the optimality of the stopping rule characterized by a single threshold are met.

Motivated by our views on the importance of taking into account the downside risk, we concentrated our attention on the spectrally negative case with an increasing reward. However, the corresponding results can (with obvious modifications) be shown to hold for spectrally positive Lévy diffusions and decreasing reward functions.

In addition to their usefulness in obtaining information about the comparative static properties of Lévy diffusions and their relations (similarities and differences) to the continuous diffusion case, our results raise a few interesting questions. Firstly, it would be of interest to obtain precise knowledge on the scope of applicability of our representation. This boils largely down to the question: when is the assumption on the existence of an increasing smooth solution to the characteristic integro-differential equation true, and can conveniently verifiable sufficient conditions for this be found? Secondly, could a more convenient (i.e. analytically verifiable) substitute for our condition *Ag2* be derived for cases where $V(x) - g(x)$ cannot be shown to be decreasing, and if so, how generally applicable this substitute would be? The answers to these rather difficult questions, however, are outside the scope of the present study, and are therefore left for future research – with regard to the first question, sufficient conditions for the existence assumption to hold in a certain class of spectrally negative jump diffusions are obtained in Rakkolainen (2008).

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PAPER II

Luis H. R. Alvarez – Teppo A. Rakkolainen: *Optimal payout policy in presence of downside risk*, 2008, to appear in *Mathematical Methods of Operations Research*.

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ORIGINAL ARTICLE

Optimal payout policy in presence of downside risk

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Abstract We analyze the determination of a value maximizing dividend payout policy for a broad class of cash reserve processes modeled as spectrally negative jump diffusions. We extend previous results based on continuous diffusion models and characterize the value of the optimal dividend distribution strategy explicitly. We also characterize explicitly the values as well as the optimal dividend thresholds for a class of associated optimal liquidation and sequential lump sum dividend control problems. Our results indicate that both the value as well as the marginal value of the optimal policies are increasing functions of policy flexibility in the discontinuous setting as well.

Keywords Dividend optimization · Downside risk · Impulse control · Jump diffusion · Optimal stopping · Singular stochastic control

JEL Classification C61 · G35

1 Introduction

Dividends are one way in which firms distribute their retained earnings. The *dividend payout policy* of a firm should specify the rules according to which dividends are paid out—most importantly, the size of a dividend payment and its timing. As far as the impact of the dividend policy on the shareholder value is considered, a crucial question is the presence or absence of transaction costs and other similar market imperfections.

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The celebrated Miller–Modigliani theorem states that in a frictionless market, dividend policy is irrelevant (cf. Miller and Modigliani 1961). However, in addition to the most obvious example of a firm paying dividends to its shareholders, also bonuses given to customers by an insurance undertaking can be viewed as dividend distribution. If the insurance entity is not a mutual company, then the customers and the shareholders of the company do not coincide. Given the importance of *bonus policies* as competitive elements, it is clear that the determination of a value-maximizing bonus policy is of interest.

Mathematically, the problem of determining the optimal dividend distribution policy in absence of transaction costs can be formulated as a singular stochastic control problem. The singular controls can usually be expressed in terms of the local times of the underlying reserve process, i.e. they correspond to so-called barrier strategies, in which all retained earnings exceeding a given level are distributed to shareholders. If there is a fixed cost associated with a transaction, the optimal dividend policy takes the form of an impulse control consisting of lump sum dividends distributed at discrete moments of time. In Alvarez and Virtanen (2006) it is shown that in a diffusion model, under relatively general conditions, the value of the impulse control problem is always dominated by the value of the singular control problem. This is quite intuitive, as the singular case is the one allowing the most flexible dividend policies and an impulse control is an admissible dividend policy for the singular control problem as well. It could be argued that an impulse control corresponds more closely to actual reality. However, even if this argument would be accepted, we can still extract much useful information from the solution of the singular problem—besides, despite the dearth of closed form expressions for the local time process itself, the decision rule implied by a local time control is intuitive and casts light on the required rate of return in the associated discrete setting as well.

In modeling the stochastic dynamics of the cash flow, continuous processes have been more popular than processes with discontinuities—largely due to their mathematically more convenient properties. The dividend problem has been considered, among many others, in Taksar (2000) and Gerber and Shiu (2004) (in an insurance context utilizing a diffusion approximation of the surplus process). However, from a risk management point of view the assumption of path continuity neglects the *downside risk*, the possibility of an instantaneous deterioration in the value of the reservoir of assets. This risk can be significant, as is evidenced, for example, by the effects of unanticipated stock market crashes which may cause large instantaneous drops in the asset values. It is also well known that there is an asymmetry in the response of the market to new information: reactions to bad news are considerably stronger than reactions to good news [this is the celebrated *bad news principle* originally introduced in Bernanke (1983)]. Moreover, in insurance applications most quantities of interest are naturally jump processes due to the discontinuous nature of the underlying claims process. These considerations have led to a growing interest in models with stochastic dynamics allowing jumps, and in recent years, several results have been obtained. The most popular choice of dynamics appears to be the Lévy process in one form or another (the reason being again, of course, the relative tractability of this setting in comparison with more general Markovian dynamics; the standard reference for the theory of Lévy processes is Bertoin (1996)). We mention particularly

Perry and Stadje (2000) and Bar-Ilan et al. (2004), where a stochastic cash management model with dynamics characterized by a finite activity Lévy process is considered, and the recent papers by Avram et al. (2007) and Kyprianou and Palmowski (2007), where the authors investigate the optimal dividend policies under dynamics given by a spectrally negative Lévy process. Boyarchenko (2004), in turn, considers the optimal timing of capital investment both in the single investment opportunity case as well as in the sequential incremental investment case when the underlying price process follows an exponential Lévy process. To mention studies dealing with the dividend payout problem in the context of insurance and risk theory applications, Azcue and Muler (2005) consider the problem of maximizing discounted expected cumulative dividends in the classical Cramér–Lundberg model when the insurer can choose both dividend and reinsurance strategies, establishing the optimality of a band strategy in this model, while Schmidli (2006) solves the dividend payout problem using the HJB equation and in Schmidli (2008) an overview of optimal dividend strategies both in the classical risk model and its diffusion approximation are given. Earlier, Dassios and Embrechts (1989) discussed barrier type strategies in the context of piecewise deterministic Markov processes, while Shreve et al. (1984) considered the dividend distribution problem for a large class of continuous diffusion processes. *Optimal stopping* and *option pricing applications* in the context of (general and one-sided) Lévy processes have been by now studied extensively in literature; for a taste, see Boyarchenko and Levendorskiĭ (2000, 2002, 2005, 2006, 2007a,b,c) Gerber and Landry (1998), Gerber and Shiu (1998), Alili and Kyprianou (2005), Mordecki (2002a,b) and Mordecki and Salminen (2007). Transforms applicable to solving many econometric and valuation problems for *affine* jump diffusions have been considered in Duffie et al. (2000), while Bayraktar and Egami (2008) optimize venture capital or R&D investments in a jump diffusion model. A good overview on stochastic control of jump diffusions is given in Øksendal and Sulem (2005).

In light of this increased interest on Lévy models and recognition of the importance of downside risk, it is to some extent surprising that the possibilities suggested by the classical theory of diffusions and minimal superharmonic maps seem to have largely been neglected in the studies based on one-dimensional jump diffusion models [for an exception, see Mordecki and Salminen (2007)]. It is namely the case that several of the results derived for continuous diffusions via the classical theory (cf. Alvarez (2004)) can be shown to hold true for a relatively broad class of spectrally negative jump diffusions as well, as has been demonstrated for optimal stopping problems in Alvarez and Rakkolainen (2006). The main advantage of these results is naturally the reduction of the considered dynamic stochastic control problem to static optimization.

Motivated by the previous considerations, our objective in this study is to consider the determination of the optimal dividend payout strategy of a competitive corporation when the retained earnings from which dividends are paid out evolves as a spectrally negative jump diffusion with geometric (i.e. proportional) jumps. The jumps of the process reflect the unanticipated potentially significant downside risk faced by the corporation. We extend the findings of Alvarez and Virtanen (2006), focusing on the dividend policy within a continuous diffusion setting, and delineate those circumstances under which their findings hold in the discontinuous setting as well. We state a set of relatively general sufficient conditions under which the optimal dividend payout strategy constitutes a threshold policy requiring that dividends should be paid out as soon as the cash reservoirs exceed a critical threshold above which the option value of retaining a further unit of capital vanishes. Extending the results of Alvarez (2001), we prove that under these conditions the value of the optimal singular dividend policy has a representation in terms of the minimal increasing r-superharmonic mapping with respect to the underlying reserve process, that is, in terms of the increasing fundamental solution of an associated integro-differential equation characterizing the smooth minimal r-harmonic maps. This representation is important since it allows the reduction of the original dynamic programming problem into a static minimization problem which can be analyzed by relying on ordinary static optimization techniques. Given the close connection of singular control with optimal stopping and impulse control, we establish that under our assumptions both the associated optimal liquidation problem as well as the associated discrete lump-sum dividend optimization (i.e. impulse control) problem are solvable in terms of the minimal increasing r-superharmonic map. We also extend the sandwiching result originally established in Alvarez and Rakkolainen (2006) and demonstrate that the value of the considered stochastic control problems of the underlying discontinuous cash flow dynamics can be sandwiched between the values of two associated stochastic control problems based on a continuous cash flow process in the present setting as well. This finding is useful especially in cases where deriving the value of the optimal policy is very difficult, since it presents two typically explicitly solvable boundaries for the value.

To the authors' best knowledge the extension of the representation results for continuous diffusions to the discontinuous setting is completely novel in the singular stochastic control and stochastic impulse control framework. Furthermore, the current study significantly refines and clarifies the results on optimal stopping obtained in Alvarez and Rakkolainen (2006). In this way our findings contribute to a larger class of problems than just dividend optimization. The reason for this is that the approach developed in our paper is applicable in other economically interesting management problems of stochastically fluctuating flows as well. A closely related problem is that of *optimal reserve management* arising in studies considering the management of foreign exchange reserves by central banks. A second closely related problem arising in the literature on natural resource economics is the determination of the *harvesting policy which maximizes the expected cumulative yield accrued from a stochastically fluctuating renewable resource stock*. This class of decision making problems is again, from the mathematical point of view, similar to the optimal dividend distribution problem considered in this study.

Our study proceeds as follows. In Sect. 2 we specify the stochastic dynamics of our jump diffusion, state our main assumptions on the parameters of the process and the associated integro-differential equation and present the mathematical formulation of the dividend control problem in absence of transaction costs. Section 3 gathers some auxiliary results, including a crucial uniqueness and existence theorem. These results are then used in the next section where the representation of the value of the singular dividend control problem in terms of the minimal r-superharmonic map is stated and proved. In particular, a representation theorem for the associated optimal stopping problem is proved. In Sect. 5 we turn our attention to the determination of the optimal dividend control in presence of a fixed transaction cost, give the definition of the ensuing impulse control problem and obtain the result that both the impulse control problem and its associated optimal stopping problem are solvable in terms of the minimal increasing r-superharmonic map as well. In the next section we illustrate our general results with an explicit mean-reverting model, the logistic Lévy diffusion. In particular, we demonstrate how our results allow us to evaluate the impact that the shape of the jump size distribution has on the optimal policies. Finally, concluding comments are presented in Sect. 7.

2 Basic setup and assumptions

Our main objective in this study is to investigate the combined impact of continuous risk as well as potentially discontinuous downside risk on the rational dividend policy and on the value of a risk neutral firm. In order to accomplish this task, we assume that the reservoir of retained earnings from which dividends are paid out evolves in the absence of interventions according to a Lévy diffusion whose dynamics are governed by the stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t - \int_{(0,1)} X_t z\tilde{N}(dt, dz), \tag{1}$$

$X_0 = x > 0$, where $\tilde{N}(dt, dz)$ is a compensated compound Poisson process (and thus a martingale) with the associated Lévy measure $\nu = \lambda\mathfrak{m}$, and $\mathfrak{m}$ is the jump size distribution, which is assumed to have a density $f_{\mathfrak{m}} \in C((0,1))$. It is worth noting that if $\tilde{N}$ is just a compound Poisson process (hence not a martingale), we can add to the drift and subtract from the jump component a suitable compensator to obtain a stochastic differential equation of form (1). We assume that the standard absence of speculative bubbles condition is met and consider only cash flow processes with finite expected cumulative present values. That is, we analyze processes X satisfying the inequality

$$\mathbb{E}_x \int_0^{\zeta} e^{-rs} X_s ds < \infty, \tag{2}$$

where $\zeta \in (0, \infty]$ denotes the lifetime of the process and $r > 0$ denotes the constant discount rate. For notational convenience, we denote the class of cash flows with finite expected cumulative present value by $\mathcal{L}^1$. The drift coefficient $\mu(x)$ and the volatility coefficient $\sigma(x) > 0$ in (1) are assumed to be such that a unique adapted, cádlág semimartingale solution X of (1) exists [sufficient is Lipschitz continuity, see Protter (2004) Theorem V.7]. In addition, we make the following assumptions:

(i) functions $\mu(x)$ and $\sigma(x)$ are analytic at $x = 0$ and satisfy the boundary conditions;
(ii) $\mu(0) = 0$, $\mu'(0) \geq 0$, $\sigma(0) = 0$ and $\sigma'(0) = 0$;
(iii) cash flows $|\mu(X)| := \{|\mu(X_t)|\}_{t\in[0,\infty)} \in \mathcal{L}^1$ and $\sigma(X) := \{\sigma(X_t)\}_{t\in[0,\infty)} \in \mathcal{L}^2$, where $\mathcal{L}^2 = \{Y : \mathbb{E}_x \int_0^{\zeta} e^{-rs} Y_s^2 ds < \infty\}$.

The state space of the underlying is $I = (0, \infty)$, where the upper boundary ∞ is assumed to be unattainable. Note that assumption (ii) combined with our assumptions on the jump structure implies that also the lower boundary 0 is unattainable. The negative coefficient of the jump part in (1) implies that the process is spectrally negative: it can decrease discontinuously but increases only continuously. This spectral negativity will play a crucial role in our analysis. The following assumption is made:

A1. X is *regular* in the sense that for all $x, y \in I$ it holds that $\mathbb{P}_x(\tau_y < \infty) = 1$, where $\tau_y = \inf\{t > 0 : X_t \geq y\}$.

Assumption *A1* ascertains the a.s. finiteness of the first exit time τ_u of X from any interval of form $(0, u)$ with $u < \infty$. The underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is equipped with the natural filtration $\mathbb{F} = \{\sigma(X_s : s \leq t)\}_{t \in \mathbb{R}_+}$.

The integro-differential operator coinciding with the infinitesimal generator of X is defined for sufficiently smooth mappings $f(x)$ by

$$(\mathcal{G} f)(x) = \frac{1}{2}\sigma^2(x) f''(x) + \mu(x) f'(x) + \lambda \int_{(0,1)} \{f(x - xz) - f(x) + xzf'(x)\}\mathfrak{m}(dz). \tag{3}$$

We will make use of the notation $\mathcal{G}_r u = \mathcal{G}u - ru$ and require

A2. There exists an increasing solution $\psi \in C^2(I)$ of $\mathcal{G}_r \psi = 0$ such that $\psi(0) = 0$.

Note our use of terminology: by *increasing* (resp. *decreasing*), we mean *strictly* increasing (resp. decreasing); correspondingly, a *non-decreasing* (resp. *non-increasing*) function is increasing (resp. decreasing) but not necessarily strictly so. A function $f \in C^2(I)$ is called *r-(super)harmonic*, if $\mathcal{G}_r \psi = (\leq) 0$. Under our assumptions (i)–(iii) on μ and σ, condition *A2* is automatically satisfied (cf. Rakkolainen 2007).

With regard to assumption *A2*, following things are worth pointing out. First, by virtue of Lemma 3.2 in Alvarez and Rakkolainen (2006) a smooth solution of $\mathcal{G}_r \psi = 0$ is monotone and unique up to a multiplicative constant—hence we can always get an increasing solution by choosing the constant suitably. Smoothness of the solution may present some problems in the general setting. However, in Chan and Kyprianou (2006) it is shown that in the case of a Lévy process with a nonzero Gaussian coefficient, the solution (which in this particular case is called *r-scale function*) of the corresponding integro-differential equation belongs to $C^2(I)$. This might indicate that in our jump diffusion setting, as long as the volatility coefficient is nonzero, the increasing solution should be smooth. In Rakkolainen (2007), sufficient conditions (i)–(iii) for assumption *A2* to hold are derived. This case fits in a natural fashion to situations where the solution can be constructed via a Frobenius type method. More generally, if either the (continuous) supremum process $\bar{X} = \{\bar{X}_t\}_{t \in [0,\infty)}$ of the jump diffusion X or the entrance times to closed sets from below have densities smooth with respect to the initial state x, then assumption *A2* will be satisfied. Since $\mathbb{P}_x[\tau_y < t] = \mathbb{P}_x[\bar{X}_t \geq y]$

for $x < y$, our statement follows from the identity

$$\psi(x)/\psi(y) = \mathbb{E}_x\left\{e^{-r\tau_y}\right\} = \int_0^\infty re^{-rt}\mathbb{P}_x[\bar{X}_t \geq y]dt$$

for $x < y$.

We define a differential operator associated with $\mathcal{G}_r$ for $f \in C^2(I)$ by

$$(\tilde{\mathcal{A}}_\theta f)(x) = \frac{1}{2}\sigma^2(x)f''(x) + \tilde{\mu}(x)f'(x) - \theta f(x), \tag{4}$$

where $\theta \in (0, \infty)$ and

$$\tilde{\mu}(x) = \mu(x) + \lambda x \cdot \int_{(0,1)} z\mathfrak{m}(dz) = \mu(x) + \lambda\bar{z}x. \tag{5}$$

This operator is related to the continuous diffusion $\tilde{X}$ given by

$$d\tilde{X}_t = \tilde{\mu}(\tilde{X}_t)dt + \sigma(\tilde{X}_t)dW_t. \tag{6}$$

Along the lines of our previous notation, we denote as $\tilde{\psi}_\theta(x)$ the increasing fundamental solution of the ordinary linear second order differential equation $(\tilde{\mathcal{A}}_\theta u)(x) = 0$ [for a comprehensive characterization of these mappings, see Borodin and Salminen (2002), p. 33]. As we will later demonstrate, the mappings $\tilde{\psi}_r(x)$ and $\tilde{\psi}_{r+\lambda}(x)$ can be applied for providing useful inequalities concerning the considered stochastic control problems.

Having characterized the underlying stochastic cash flow dynamics (1) in the absence of interventions we now denote the controlled cash flow dynamics as X_t^D and assume that it is characterized by the stochastic differential equation

$$X_t^D = x + \int_0^t \mu(X_{s-}^D)ds + \int_0^t \sigma(X_{s-}^D)dW_s - \int_0^t \int_{(0,1)} X_{s-}^D z\tilde{N}(ds, dz) - D_t, \tag{7}$$

$X_{0-}^D = x$, where D_t denotes the cumulative dividends paid up to time t. As usually, we call a dividend payout strategy *admissible* if it is predictable and the resulting adapted, cádlág cumulative dividends process $D = \{D_t\}_{t\in[0,\infty)}$ is non-negative and non-decreasing. We denote the class of admissible policies by $\mathcal{A}$. Under our assumptions X^D is a semimartingale [being a Markov process generated by a pseudodifferential operator, see Jacob and Schilling (2001)]. In light of this characterization, our objective is to consider the determination of an admissible payout policy maximizing the expected cumulative present value of the dividend flow. Formally, our objective is to solve the cash flow management problem

$$V_S(x) = \sup_{D\in\mathcal{A}} \mathbb{E}_x \int_0^{\tau_0^D} e^{-rs} dD_s, \tag{8}$$

where $\tau_0^D = \inf\{t > 0 : X_t^D \leq 0\}$ denotes the lifetime of the controlled reserve process X^D. It is worth emphasizing that in our model liquidation is always the result of a control action (and, thus, *endogenous*), as the assumed boundary behavior of X implies that exogenous liquidation in finite time is not possible.

As was pointed out in Alvarez and Virtanen (2006), the singular control setting is the one allowing the greatest flexibility in dividend policies, as single optimal stopping rules and discrete impulse policies (sequential stopping) are in fact admissible controls (belong to $\mathcal{A}$). In light of this observation, we define the optimal stopping problem associated to the singular stochastic control problem (8) as

$$V_{\text{OSP}}(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x \left[e^{-r\tau} X_\tau\right], \tag{9}$$

where $\mathcal{T}$ is the set of all $\mathbb{F}$-stopping times. Note that the valuation in (9) is perpetual since as was mentioned above, the underlying reserves cannot vanish nor explode in finite time.

3 Some auxiliary results

Before proceeding in our analysis of the considered dividend optimization problem in a general setting, we first define the net appreciation rate $\rho : I \to \mathbb{R}$ of the stock X as $\rho(x) = \mu(x) - rx$. As will turn out later in our analysis, this mapping plays a key role in the determination of the optimal payout policy and its value. Under our assumptions on μ and X, the corresponding cash flow $\rho(X) := \{\rho(X_t)\}_{t\in[0,\infty)}$ has a finite expected cumulative present value, that is, $\rho(X) \in \mathcal{L}^1$. An interesting result based on this mapping is now summarized in the following.

Lemma 3.1 *For all $x \in I$ it holds that*

$$V_S(x) \leq x + \sup_{D\in\mathcal{A}} \mathbb{E}_x \int_0^{\tau_0^D} e^{-rs} \rho(X_s^D) ds. \tag{10}$$

Especially, if $\rho(x) \leq 0$ for all $x \in I$ then the optimal strategy is to liquidate the corporation immediately and pay out the entire reserve instantaneously. In that case the value of the optimal dividend policy reads as $V_S(x) = x$ for all $x \in I$. Moreover, $V_{\text{OSP}}(x) = x$ for all $x \in I$ as well.

Proof Applying the generalized Itô theorem to the identity mapping $x \mapsto x$ yields

$$\mathbb{E}_x \left[e^{-r\tau_N} X_{\tau_N}^D\right] = x + \mathbb{E}_x \int_0^{\tau_N} e^{-rs} \rho(X_s^D) ds - \mathbb{E}_x \int_0^{\tau_N} e^{-rs} dD_s,$$

where $\tau_N = N \wedge \tau_0^D \wedge \inf\{t \geq 0 : X_t^D > N\}$ is an increasing sequence of almost surely finite stopping times tending towards τ_0^D. Reordering terms, invoking the nonnegativity of the controlled jump-diffusion, and letting $N \to \infty$ yields by dominated convergence inequality (10). The optimality of instantaneous liquidation is then clear in light of (10).

Lemma 3.1 characterizes the circumstances under which the so-called *take the money and run* policy (i.e. immediate liquidation of the company) is optimal. As intuitively is clear, waiting is suboptimal whenever the value of the reserves depreciates at all states and subsequently no intertemporal gains may be accrued by postponing the payout decision into the future. An interesting implication of the findings of Lemma 3.1 is that if the net appreciation rate has a global maximum at $\hat{x} = \operatorname{argmax}\{\rho(x)\}$, then

$$x \leq V_S(x) \leq x + \frac{\rho(\hat{x})}{r}$$

for all $x \in I$. Thus, as long as the net appreciation rate is bounded, the value of the optimal policy can grow at most at a linear rate for large reservoirs.

Lemma 3.1 characterizes the optimal policy only in the extreme case of instantaneous liquidation. However, in order to characterize the optimal dividend payout policy in a more general setting more analysis is naturally needed. Before proceeding in our analysis we first define the continuously differentiable mappings $H : I^2 \mapsto \mathbb{R}$ and $\tilde{H}_\theta : I^2 \mapsto \mathbb{R}$ as

$$H(x, y) = \begin{cases} x - y + \frac{\psi(y)}{\psi'(y)} & x \geq y \\ \frac{\psi(x)}{\psi'(y)} & x < y \end{cases} \tag{11}$$

and

$$\tilde{H}_\theta(x, y) = \begin{cases} x - y + \frac{\tilde{\psi}_\theta(y)}{\tilde{\psi}'_\theta(y)} & x \geq y \\ \frac{\tilde{\psi}_\theta(x)}{\tilde{\psi}'_\theta(y)} & x < y. \end{cases} \tag{12}$$

It is worth noticing that for a given fixed $y \in I$ the function $x \mapsto H(x, y)$ satisfies the variational equalities

$$\begin{aligned} (\mathcal{G}_r H)(x, y) &= 0, \quad x < y \\ \partial_x H(x, y) &= 1, \quad x \geq y. \end{aligned}$$

Analogously, for a given fixed $y \in I$ the function $x \mapsto \tilde{H}_\theta(x, y)$ satisfies the variational equalities

$$\begin{aligned} (\tilde{\mathcal{A}}_\theta \tilde{H}_\theta)(x, y) &= 0, \quad x < y \\ \partial_x \tilde{H}_\theta(x, y) &= 1, \quad x \geq y. \end{aligned}$$

As we will later observe, these functions can be applied for solving the variational inequalities $\max\{(\mathcal{G}_r v)(x), 1 - v'(x)\} = 0$, $\max\{(\mathcal{G}_r v)(x), (x - c) - v(x)\} = 0$, $\max\{(\tilde{\mathcal{A}}_\theta u)(x), 1 - u'(x)\} = 0$, and $\max\{(\tilde{\mathcal{A}}_\theta u)(x), (x - c) - u(x)\} = 0$ associated to the considered singular control and optimal stopping problems. We can now establish the following result characterizing how the values of the mapping $\tilde{H}_\theta(x, y)$ defined with respect to the minimal increasing r-harmonic function for the continuous diffusion $\tilde{X}$ can be applied for bounding the values of the mapping $H(x, y)$ defined with respect to the jump-diffusion X.

Lemma 3.2 *For all $x, y \in I$ it holds that $\tilde{H}_{r+\lambda}(x, y) \leq H(x, y) \leq \tilde{H}_r(x, y)$. Consequently, $\sup_{y\in I} \tilde{H}_{r+\lambda}(x, y) \leq \sup_{y\in I} H(x, y) \leq \sup_{y\in I} \tilde{H}_r(x, y)$ provided that the supremum exists.*

Proof As was established in Theorem 4.1 of Alvarez and Rakkolainen (2006) we have that

$$\frac{\tilde{\psi}_{r+\lambda}(x)}{\tilde{\psi}_{r+\lambda}(y)} \leq \frac{\psi(x)}{\psi(y)} \leq \frac{\tilde{\psi}_r(x)}{\tilde{\psi}_r(y)} \tag{13}$$

for all $0 < x \leq y < \infty$. This inequality and the fundamental theorem of integral calculus in turn implies that

$$\int_x^y \frac{\tilde{\psi}'_{r+\lambda}(t)}{\tilde{\psi}_{r+\lambda}(y)} dt \geq \int_x^y \frac{\psi'(t)}{\psi(y)} dt \geq \int_x^y \frac{\tilde{\psi}'_r(t)}{\tilde{\psi}_r(y)} dt.$$

Applying now the mean value theorem and letting $x \uparrow y$ then shows that

$$\frac{\tilde{\psi}'_{r+\lambda}(y)}{\tilde{\psi}_{r+\lambda}(y)} \geq \frac{\psi'(y)}{\psi(y)} \geq \frac{\tilde{\psi}'_r(y)}{\tilde{\psi}_r(y)} \tag{14}$$

for all $y \in I$. Noticing now that

$$\frac{\psi(x)}{\psi'(y)} = \frac{\psi(x)}{\psi(y)} \frac{\psi(y)}{\psi'(y)} \leq \frac{\tilde{\psi}_r(x)}{\tilde{\psi}_r(y)} \frac{\tilde{\psi}_r(y)}{\tilde{\psi}'_r(y)} = \frac{\tilde{\psi}_r(x)}{\tilde{\psi}'_r(y)}$$

and

$$\frac{\psi(x)}{\psi'(y)} = \frac{\psi(x)}{\psi(y)} \frac{\psi(y)}{\psi'(y)} \geq \frac{\tilde{\psi}_{r+\lambda}(x)}{\tilde{\psi}_{r+\lambda}(y)} \frac{\tilde{\psi}_{r+\lambda}(y)}{\tilde{\psi}'_{r+\lambda}(y)} = \frac{\tilde{\psi}_{r+\lambda}(x)}{\tilde{\psi}'_{r+\lambda}(y)}$$

then demonstrates that $\tilde{H}_{r+\lambda}(x, y) \leq H(x, y) \leq \tilde{H}_r(x, y)$ for all $x, y \in I$. Since a continuously differentiable mapping is bounded on the interior of its domain and attains its extreme values either at the points where it derivative vanishes or at the boundaries of its domain, the proposed ordering of the values follows from the inequality above.

Lemma 3.2 states two interesting inequalities characterizing how the value of the function $H(x, y)$ can be sandwiched between the values $\tilde{H}_{r+\lambda}(x, y)$ and $\tilde{H}_r(x, y)$. This observation is of interest since it demonstrates that the solutions of the associated variational inequalities are ordered. As we will later observe, these functions are closely related to the values of the optimal dividend policies in the considered three different cases. Two interesting implications of Lemma 3.2 needed later in the analysis of the associated dividend optimization problems are now summarized in the following.

Corollary 3.3 (A) *Assume that* $y > \eta > 0$. *Then*

$$\frac{\tilde{\psi}_{r+\lambda}(x)}{\tilde{\psi}_{r+\lambda}(y) - \tilde{\psi}_{r+\lambda}(y-\eta)} \leq \frac{\psi(x)}{\psi(y) - \psi(y-\eta)} \leq \frac{\tilde{\psi}_r(x)}{\tilde{\psi}_r(y) - \tilde{\psi}_r(y-\eta)}$$

for all $x \leq y$.
(B) *For all* $x \in I$ *it holds*

$$\frac{\tilde{\psi}_{r+\lambda}(x)}{\tilde{\psi}'_{r+\lambda}(x)} - x \leq \frac{\psi(x)}{\psi'(x)} - x \leq \frac{\tilde{\psi}_r(x)}{\tilde{\psi}'_r(x)} - x.$$

Proof Noticing that

$$\frac{\psi(x)}{\psi(y) - \psi(y-\eta)} = \frac{\psi(x)/\psi(y)}{1 - \psi(y-\eta)/\psi(y)}$$

and applying the inequality (13) proves part (A). Part (B) is a direct consequence of (14).

Before stating our main result on the general convexity properties of the increasing solution $\psi(x)$, we now present the next lemma.

Lemma 3.4 *Assume that* $\phi(x) \in C^2(\mathbb{R}_+)$ *is non-decreasing and that there exists* $x_1 \in \mathbb{R}_+$ *such that* $\phi(x)$ *is strictly concave on* $(0, x_1)$ *and strictly convex on* (x_1, x_2)*, where* $x_2 > x_1$. *Define* $u : (x_1, x_2) \to [0, x_1)$ *via* $u(z) = \inf\{y \in [0, x_1) : \phi'(y) \leq \phi'(z)\}$. *Then*

$$u(z) = \begin{cases} 0, & x_2 > z > x_1, \quad z \geq \tilde{\Phi}(\phi'(0)) \\ \Phi(\phi'(z)), & x_2 > z > x_1, \quad z < \tilde{\Phi}(\phi'(0)), \end{cases} \tag{15}$$

where the function $\Phi : \big(\phi'(x_1), \phi'(x_2)\big) \to (0, x_1)$ *is defined as* $\Phi = (\phi' \mid_{(0,x_1)})^{-1}$ *and* $\tilde{\Phi} : \big(\phi'(x_1), \phi'(x_2)\big) \to (x_1, x_2)$ *is defined as* $\tilde{\Phi} = (\phi' \mid_{(x_1,x_2)})^{-1}$. *Moreover,* $u(z)$ *is continuously differentiable for* $z < \tilde{\Phi}(\phi'(0))$.

Proof Assumptions imply that $\phi'(x)$ is a unimodal continuously differentiable function with a unique minimum at x_1. Since it is decreasing on $[0, x_1)$, $\phi'(0) \leq \phi'(z)$ implies that the inequality in the definition of $u(z)$ is satisfied for all $y \in [0, x_1)$ and hence $u(z) = 0$. On the other hand, $\phi'(0) \leq \phi'(z)$ is in the present case equivalent to $z \geq \tilde{\Phi}(\phi'(0))$. For $z < \tilde{\Phi}(\phi'(0))$, the continuous differentiability of $u(z)$ on

$(0, \tilde{\Phi}(\phi'(0))) \cap (x_1, x_2)$ follow from the inverse function theorem, as $\phi'(x)$ is continuously differentiable and $\phi''(x) \neq 0$ on $(0, x_1)$, implying that $\Phi(y)$ is continuously differentiable on $(\phi'(x_1), \phi'(0))$, being the inverse function of the restriction $\phi' \mid_{(0,x_1)}$.

Given this auxiliary result, we are now in position to prove the following theorem stating a set of sufficient conditions under which the monotonicity properties of $\psi'(x)$ can be unambiguously characterized. In the statement of the theorem, the convention $(a, a) = \emptyset$ is used.

Theorem 3.5 *Assume that the net appreciation rate $\rho(x)$ satisfies the limiting inequalities $\lim_{x\to\infty} \rho(x) < 0 \leq \lim_{x\downarrow 0} \rho(x)$, that there exists a unique threshold $\hat{x} \in I \cup \{0\}$ such that $\rho(x)$ is increasing on $(0, \hat{x})$ and decreasing on $(\hat{x}, \infty)$, and that $\rho(x)$ is concave on $(\hat{x}, \infty)$. Then equation $\psi''(x) = 0$ has a unique root $x^* \in (\hat{x}, \infty)$ so that $\psi''(x) \lesseqqgtr 0$ for $x \lesseqqgtr x^*$ and $x^* = \operatorname{argmin}\{\psi'(x)\}$.*

Proof We first establish that under our assumptions the increasing solution is locally concave on a neighborhood of the origin. To accomplish this task, we first notice that the integro-differential equation $(\mathcal{G}_r\psi)(x) = 0$ can be re-expressed as

$$I(x) = r(\psi(x) - x\psi'(x)) - \rho(x)\psi'(x) - J(x, \psi(x)), \tag{16}$$

where $I(x) = \frac{1}{2}\sigma^2(x)\psi''(x)$, and

$$J(x, \psi(x)) = \int_{(0,1)} \{\psi(x - xz) - \psi(x) + xz\psi'(x)\}\nu(dz). \tag{17}$$

Assume now that there is a set $(0, \varepsilon)$, $\varepsilon < x_0 = \rho^{-1}(0)$, where the increasing fundamental solution is convex. Since a convex mapping satisfying the boundary condition $\psi(0) = 0$ satisfies the inequalities $\psi'(x)x \geq \psi(x)$ and $\psi(x - xz) \geq \psi(x) - xz\psi'(x)$ for all $x \in (0, \varepsilon)$ and $z \in (0, 1)$, we find from (16) that $I(x) \leq -\rho(x)\psi'(x)$. The monotonicity of $\psi(x)$ and the positivity of $\rho(x)$ on $(0, x_0)$ then imply that $I(x) < 0$ which is a contradiction due to the assumed convexity of $\psi(x)$ on $(0, \varepsilon)$. This proves that $\psi(x)$ is locally concave on a set $(0, \varepsilon)$. We now show that $\psi(x)$ cannot become convex on $(0, \hat{x})$ and, therefore, that if equation $\psi''(x) = 0$ has a root, it has to be on $(\hat{x}, \infty)$ (in the case $\hat{x} = 0$, this is immediate and the considerations of the next two sentences are unnecessary). To see that this is indeed the case, we observe that if $x_1 < \hat{x}$ is the smallest root of $\psi''(x) = 0$, then

$$I'(x_1) = -\rho'(x_1)\psi'(x_1) - \int_{(0,1)} \{\psi'(x_1(1 - z)) - \psi'(x_1)\}(1 - z)\nu(dz) < 0$$

due to the monotonicity of $\psi(x)$, $\psi'(x)$, and $\rho(x)$. Hence, if equation $\psi''(x) = 0$ has a root, it has to be on $(\hat{x}, \infty)$. In order to establish that $\psi(x)$ has to become convex at some $x_2 \in (\hat{x}, x_0)$, where $x_0 = \rho^{-1}(0)$, assume that $\psi(x)$ is concave on the entire interval $(0, x_0)$. In that case we would have the inequalities $\psi'(x)x \leq \psi(x)$

and $\psi(x - xz) \leq \psi(x) - xz\psi'(x)$ for all $x \in (0, x_0)$ and $z \in (0, 1)$. Consequently, $I(x) \geq -\rho(x)\psi'(x)$ for all $x \in (0, x_0)$. Letting $x \uparrow x_0$ then yields that $I(x_0) \geq 0$ which is a contradiction due to the assumed concavity of $\psi(x)$. Combining this observation with our previous findings shows that equation $\psi''(x) = 0$ has at least one root $x^* \in (\hat{x}, x_0)$.

Given these findings, our objective is now to establish that the root x^* is unique whenever $\rho(x)$ is concave on $(\hat{x}, \infty)$. To observe that this is the case, we notice that (16) can be re-expressed as

$$\tilde{I}(x) = (r + \lambda)\left(\frac{\psi(x)}{S'(x)} - x\frac{\psi'(x)}{S'(x)}\right) - \tilde{\rho}(x)\frac{\psi'(x)}{S'(x)} - \tilde{J}(x), \tag{18}$$

where $\tilde{I}(x) = \frac{\sigma^2(x)\psi''(x)}{2S'(x)}$, $\tilde{\rho}(x) = \rho(x) - \lambda x(1 - \bar{z})$, $S'(x) = \exp\left(-\int \frac{2\tilde{\mu}(x)dx}{\sigma^2(x)}\right)$ denotes the scale density of the associated diffusion $\tilde{X}$, and

$$\tilde{J}(x) = \int_{(0,1)} \frac{\psi(x(1-z))}{S'(x)}\nu(dz).$$

Standard differentiation yields that

$$\frac{d}{dx}\left[\frac{\psi'(x)}{S'(x)}\right] = \left((r+\lambda)\psi(x) - \int_{(0,1)} \psi(x(1-z))\nu(dz)\right)m'(x)$$

$$\frac{d}{dx}\left[\frac{\psi(x)}{S'(x)} - x\frac{\psi'(x)}{S'(x)}\right] = \left(\tilde{\rho}(x)\psi(x) + x\int_{(0,1)} \psi(x(1-z))\nu(dz)\right)m'(x)$$

and

$$\tilde{J}'(x) = \int_{(0,1)} \frac{\psi'(x(1-z))}{S'(x)}(1-z)\nu(dz) + \tilde{\mu}(x)m'(x)\int_{(0,1)} \psi(x(1-z))\nu(dz)$$

where $m'(x) = 2/(\sigma^2(x)S'(x))$ denotes the speed measure of the associated diffusion $\tilde{X}$. Hence, we find that

$$\tilde{I}'(x) = \frac{\psi'(x)}{S'(x)}\left[-\rho'(x) + \int_{(0,1)} \left(1 - \frac{\psi'(x(1-z))}{\psi'(x)}\right)(1-z)\nu(dz)\right]. \tag{19}$$

In light of the definition of $\tilde{I}(x)$ and our findings on the local concavity of $\psi(x)$ on $(0, x^*)$, it is clear that $\tilde{I}'(x^*) \geq 0$, which implies that the expression within square brackets in (19) is non-negative at $x = x^*$. Assume now that equation $\psi''(x) = 0$ has another root $y^* > x^*$ at which the increasing fundamental solution becomes locally concave again. To establish that this is impossible, we first observe that the integral term in (19) can be re-expressed as

$$\int_0^{1-\frac{u(x)}{x}} \left(1 - \frac{\psi'(x(1-z))}{\psi'(x)}\right)(1-z)\nu(dz) + \int_{1-\frac{u(x)}{x}}^{1} \left(1 - \frac{\psi'(x(1-z))}{\psi'(x)}\right)(1-z)\nu(dz),$$

where $u(x) = \inf\{y \in (0, x^*] : \psi'(y) \le \psi'(x)\} \in C^1((x^*, y^*))$ by Lemma 3.4. It is now clear that the first term of this expression is positive due to the local convexity of $\psi(x)$ on (x^*, y^*). On the other hand, a direct application of Leibniz' rule to the second term proves

$$\begin{aligned} &\frac{d}{dx}\int_{1-\frac{u(x)}{x}}^{1} \left(1 - \frac{\psi'(x(1-z))}{\psi'(x)}\right)(1-z)\nu(dz) \\ &= \int_{1-\frac{u(x)}{x}}^{1} \frac{\psi''(x)\psi'(x(1-z)) - \psi'(x)\psi''(x(1-z))(1-z)}{\psi'(x)}(1-z)\nu(dz) \\ &\quad + \left(\frac{u'(x)x - u(x)}{x^2}\right)\left(1 - \frac{\psi'(u(x))}{\psi'(x)}\right) \cdot \frac{u(x)}{x} \cdot \lambda f_{\mathfrak{m}}\left(1 - \frac{u(x)}{x}\right) > 0, \end{aligned}$$

since $\psi''(x) > 0$ on (x^*, y^*), $\psi''(x(1-z)) < 0$ when $z > 1 - \frac{u(x)}{x}$, and $\psi'(x) = \psi'(u(x))$. Combining this observation with the non-negativity of the bracketed expression in (19) for $x = x^*$, the identity $u(x^*) = x^*$, and the assumed concavity of $\rho(x)$ on $(\hat{x}, \infty)$ then proves that

$$\tilde{I}'(x) > \frac{\psi'(x)}{S'(x)}\left[-\rho'(x^*) + \int_0^1 \left(1 - \frac{\psi'(x^*(1-z))}{\psi'(x^*)}\right)(1-z)\nu(dz)\right] \ge 0$$

for all $x \in (x^*, y^*)$. Letting $x \uparrow y^*$ now implies that $\tilde{I}'(y^*) > 0$ which is a contradiction since $\tilde{I}(x)$ should be non-increasing at y^*. Hence, we find that the root x^* is unique and constitutes the global minimum of $\psi'(x)$.

Theorem 3.5 states a set of conditions under which $\psi'(x)$ attains a unique global minimum so that $\psi(x)$ is concave below and convex above this critical threshold. It is worth mentioning that the conditions on the net appreciation rate ρ imply that $\rho(x) \to -\infty$, as $x \to \infty$. We conclude this section with the following useful observation.

Theorem 3.6 *Suppose that the assumptions of Theorem 3.5 are satisfied and define the function $F : I \mapsto \mathbb{R}_+$ as $F(x) = H(x, x^*)$. Then,*

(A) $F \in C^2(I)$, $(\mathcal{G}_r F)(x) \le 0$, $F'(x) \ge 1$, *and* $F''(x) \le 0$ *for all* $x \in I$, *and*
(B) $F(x) \ge H(x, y)$ *and* $F'(x) \ge H_x(x, y)$ *for all* $x, y \in I^2$ *and* $H_y(x, y) < 0$ *for all* $(x, y) \in \mathbb{R}_+ \times (x^*, \infty)$.
Moreover,
(C) *if* $\rho(x) - (1 - \bar{z})\lambda x$ *is increasing on a neighborhood of* 0 *then* $F(x) \ge \tilde{H}_{r+\lambda}(x, \tilde{x}^*(r + \lambda))$, *where* $\tilde{x}^*(r + \lambda) = \operatorname{argmin}\{\tilde{\psi}'_{r+\lambda}(x)\}$ *is the unique root of equation* $\tilde{\psi}''_{r+\lambda}(x) = 0$, *and*
(D) *if* $\lim_{x\to\infty}(\rho(x) + \lambda\bar{z}x) < 0$ *then* $F(x) \le \tilde{H}_r(x, \tilde{x}^*(r))$, *where* $\tilde{x}^*(r) = \operatorname{argmin}\{\tilde{\psi}'_r(x)\}$ *is the unique root of equation* $\tilde{\psi}''_r(x) = 0$.
(E) *if* $\rho(x) - (1 - \bar{z})\lambda x$ *is increasing on a neighborhood of* 0 *and* $\lim_{x\to\infty}(\rho(x) + \lambda\bar{z}x) < 0$ *then* $\tilde{H}_{r+\lambda}(x, \tilde{x}^*(r + \lambda)) \le H(x, x^*) \le \tilde{H}_r(x, \tilde{x}^*(r))$ *for all* $x \in I$.

Proof (A) Clearly $F \in C^2(I)$. Since $F(x)$ is a linear function with derivative equal to 1 on $[x^*, \infty)$, it is straightforward to compute that

$$(\mathcal{G}_r F)'(x) = \rho'(x) + \int_0^1 \left[F'(x - xz) - 1\right](1 - z)\nu(dz) \tag{20}$$

for $x \ge x^*$. We have assumed that $\rho'(x)$ is negative and decreasing on $[x^*, \infty)$. Furthermore, the integral in (20) can be written as

$$\int_{1-x^*/x}^1 \left[\frac{\psi'(x - xz)}{\psi'(x^*)} - 1\right](1 - z)\nu(dz),$$

since for $x - xz \ge x^*$ the integrand vanishes. This is a decreasing function of x since $\psi(x)$ is concave on $(0, x^*)$ and $x - xz < x^*$ in the region over which we integrate here. Hence, $(\mathcal{G}_r F)'(x)$ is decreasing on (x^*, ∞) and consequently if $(\mathcal{G}_r F)'(x^*) \le 0$, then $(\mathcal{G}_r F)(x)$ is non-increasing on (x^*, ∞). But

$$(\mathcal{G}_r F)'(x^*) = \rho'(x^*) + \int_{(0,1)} \left(\frac{\psi'(x^*(1 - z))}{\psi'(x^*)} - 1\right)(1 - z)\nu(dz),$$

and this quantity was shown to be negative in the proof of Theorem 3.5. As $(\mathcal{G}_r F)(x)$ is continuous and equal to 0 for all $x < x^*$, we necessarily have $(\mathcal{G}_r F)(x^*) = 0$. By continuity and monotonicity of $(\mathcal{G}_r F)(x)$ on (x^*, ∞), it follows that $(\mathcal{G}_r F)(x) \le 0$ for all $x \ge x^*$. The strict concavity of $\psi(x)$ on $(0, x^*)$ then proves that $F'(x) \ge 1$ and $F''(x) \le 0$ for all $x \in I$. Part (B) now follows directly from Theorem 3.2 in Alvarez and Virtanen (2006) since $\psi''(x) \lesseqqgtr 0$ for all $x \lesseqqgtr x^*$ and $x^* = \operatorname{argmin}\{\psi'(x)\}$. Part

(C) follows from Lemma 3.2 after noticing that if $\rho(x) - (1-\bar{z})\lambda x$ is increasing on a neighborhood of 0 then according to Lemma 3.1 in Alvarez and Virtanen (2006) equation $\tilde{\psi}''_{r+\lambda}(x) = 0$ has a unique root $\tilde{x}^*(r+\lambda) = \operatorname{argmin}\{\tilde{\psi}'_{r+\lambda}(x)\}$ so that $\tilde{\psi}''_{r+\lambda}(x) \lesseqqgtr 0$ for $x \lesseqqgtr \tilde{x}^*(r+\lambda)$. Establishing part (D) is entirely analogous. Part (E) finally follows from (C) and (D).

Theorem 3.6 demonstrates that if the conditions of Theorem 3.5 are satisfied then the value $H(x, y)$ attains a unique global maximum as a function of the threshold y. Interestingly, Theorem 3.6 proves that this maximal value $H(x, x^*)$ does not only dominate the values $H(x, y)$ for all $y \in I$, it also grows faster than any other of these values. This observation is important since it characterizes the circumstances under which the findings by Alvarez and Virtanen (2006) focusing on continuous cash flow models can be extended to the discontinuous setting as well. Theorem 3.6 also establishes a set of sufficient conditions under which the two associated values $\tilde{H}_{r+\lambda}(x, y)$ and $\tilde{H}_r(x, y)$ attain a unique global maximum as functions of the arbitrary threshold y. Whenever these optimal thresholds exist the value $H(x, x^*)$ belongs into the region bounded by the resulting values.

4 Optimal singular control of dividends

We are now in a position to state our main result on the optimal singular dividend payout policy for the considered class of jump diffusions modeling the underlying stochastically fluctuating reserve dynamics.

Theorem 4.1 *Assume that the assumptions of Theorem 3.5 are satisfied. Then the value of the singular control problem is given by $V_S(x) = H(x, x^*)$. The value is twice continuously differentiable, monotonically increasing and concave. Moreover, the marginal value of the optimal policy reads as*

$$V_S'(x) = \psi'(x) \sup_{y \geq x} \left\{ \frac{1}{\psi'(y)} \right\} = \begin{cases} 1 & x \geq x^* \\ \frac{\psi'(x)}{\psi'(x^*)} & x < x^*. \end{cases} \tag{21}$$

The corresponding optimal singular control consists of an initial impulse (lump sum dividend) $\xi_{0-} = (x - x^*)^+$ and a barrier strategy where all retained earnings in excess of x^* are instantaneously paid out as dividends.

Proof For notational convenience, we shall denote the proposed value function as $\tilde{V}(x)$ and the value function of the singular control problem as $V(x)$. Let $D \in \mathcal{A}$ be an arbitrary admissible policy and denote $J^D(x) = \mathbb{E}_x \int_0^{\tau_0^D} e^{-rs} dD_s$. Applying the generalized Itô formula [see Protter (2004), Theorem II.32] to the mapping $(t, x) \mapsto e^{-rt}\tilde{V}(X_t^D)$ yields

$$\mathbb{E}_x\left[e^{-r\tau_N}\tilde{V}\left(X^D_{\tau_N}\right)\right] = \tilde{V}(x) + \mathbb{E}_x\int_{0+}^{\tau_N} e^{-rs}\left(\mathcal{G}_r\tilde{V}\right)\left(X^D_s\right)ds$$
$$+\mathbb{E}_x\int_{0+}^{\tau_N} e^{-rs}\left\{\tilde{V}\left(X^D_{s-}+(\Delta D)_s\right)-\tilde{V}\left(X^D_{s-}\right)\right\}ds$$
$$-\mathbb{E}_x\int_{0+}^{\tau_N} e^{-rs}\tilde{V}'\left(X^D_{s-}\right)dD_s, \tag{22}$$

where $\tau_N = N \wedge \tau_0^D \wedge \inf\{t \geq 0 : X_t^D \geq N\}$ is an increasing sequence of almost surely finite stopping times converging to τ_0^D as $N \to \infty$. It is now clear from our Theorem 3.6 that the proposed value function is nonnegative and twice continuously differentiable and that it satisfies the inequalities $\tilde{V}'(x) \geq 1$ and $(\mathcal{G}_r F)(x) \leq 0$ for all $x \in I$. Combining these observations with (22) implies

$$0 \leq \mathbb{E}_x\left[e^{-r\tau_N}\tilde{V}(X^D_{\tau_N})\right] \leq \tilde{V}(x) - \mathbb{E}_x\int_{0+}^{\tau_N} e^{-rs}dD_s.$$

This inequality and the monotone convergence theorem then imply that

$$\tilde{V}(x) \geq \mathbb{E}_x\int_{0+}^{\tau_N} e^{-rs}dD_s \to \mathbb{E}_x\int_{0+}^{\tau_0^D} e^{-rs}dD_s$$

as $N \to \infty$. Thus $\tilde{V}(x) \geq J^D(x)$ for any $D \in \mathcal{A}$ and so $\tilde{V}(x) \geq V(x)$.

Denote now the proposed dividend strategy described in the theorem by $\hat{D}$. Under the proposed policy we have $X_t^D \in (0, x^*]$ t-almost everywhere, implying thus that$(\mathcal{G}_r V)(X_t^D) = 0$ t-almost everywhere. Hence, (22) takes now the form

$$\mathbb{E}_x\left[e^{-r\tau_N}\tilde{V}\left(X^{\hat{D}}_{\tau_N}\right)\right] = \tilde{V}(x) - \mathbb{E}_x\int_{0+}^{\tau_N} e^{-rs}\tilde{V}'\left(X^{\hat{D}}_{s-}\right)d\hat{D}_s. \tag{23}$$

However, since the proposed dividend policy increases only when the underlying process hits the threshold x^* and, therefore, when $V'(X_s) = 1$ we find that (23) can be re-expressed as

$$\tilde{V}(x) = \mathbb{E}_x\left[e^{-r\tau_N}\tilde{V}(X^{\hat{D}}_{\tau_N})\right] + \mathbb{E}_x\int_{0}^{\tau_N} e^{-rs}d\hat{D}_s.$$

Recalling that either $\tau_N \to \infty$ or $X^{\hat{D}}_{\tau_N} = 0$ for N large enough, letting $N \to \infty$ then gives $\tilde{V}(x) = J^{\hat{D}}(x)$ and consequently $\tilde{V}(x) \le V(x)$. But then $\tilde{V}(x) = V(x)$.

The capital theoretic implications of Theorem 4.1 are in line with the ones stated in Alvarez and Virtanen (2006): firstly, the optimal dividend threshold is attained on the set where net appreciation rate $\rho(x)$ of the underlying reserve is positive and thus dividends are paid out on the set where the expected per capita rate at which the reserves are increasing dominate the opportunity cost of investment; second, since the optimal dividend threshold is attained on the set where the net appreciation rate of the underlying reserve is decreasing, at the optimum the marginal yield accrued from retaining yet another marginal unit of stock undistributed is smaller than the interest rate r. Thus, the optimal dividend policy diverges from the deterministic golden rule of capital accumulation in the present jump-diffusion case as well. An important implication of Theorems 4.1 and 3.6 is now summarized in the following.

Corollary 4.2 *Assume that the assumptions of Theorem 3.5 are satisfied, that $\rho(x) - (1-\bar{z})\lambda x$ is increasing on a neighborhood of* 0 *and that* $\lim_{x\to\infty}(\rho(x) + \lambda\bar{z}x) < 0$. *Then, for all $x \in I$ it holds*

$$\tilde{V}^{r+\lambda}_S(x) \le V_S(x) \le \tilde{V}^r_S(x),$$

where

$$\tilde{V}^{\theta}_S(x) = \sup_{D\in\mathcal{A}} \mathbb{E}_x \int_0^\infty e^{-\theta s} d\tilde{D}_s,$$

and

$$d\tilde{X}^{\tilde{D}}_t = \tilde{\mu}\big(\tilde{X}^{\tilde{D}}_t\big)dt - \sigma\big(\tilde{X}^{\tilde{D}}_t\big)dW_t - d\tilde{D}_t,\ \tilde{X}^{\tilde{D}}_0 = x.$$

Proof Since $V_S(x) = H(x, x^*)$ and $\tilde{V}^{\theta}_S(x) = \tilde{H}_\theta(x, \tilde{x}^*(\theta))$ the result follows from Theorem 3.6.

Corollary 4.2 states a set of conditions under which the value of the optimal singular dividend policy is bounded by the value of an associated singular stochastic control problem of the continuous diffusion $\tilde{X}$. It is worth noticing that even though the jump intensity λ does not affect the existence of the optimal threshold x^* it affects the existence of an optimal policy for the associated problems. As the jump intensity λ increases the local growth rate of $\rho(x) - (1-\bar{z})\lambda x$ decreases and eventually vanishes (provided that $\mu'(0+) < \infty$). At the critical level $(1-\bar{z})\lambda = \mu'(0+)$ the optimal policy associated to the smallest value becomes trivial (instantaneous liquidation) and $\tilde{V}^{\lambda+r}_S(x) = x$. Analogous conclusions can be naturally drawn for the highest value as well.

Having analyzed the considered singular control problem, we now proceed in our analysis and study the associated optimal liquidation problem. An important implication of Theorems 3.6 and 4.1 characterizing the relationship between the optimal singular dividend policy and the value of the optimal liquidation policy is now summarized in the following representation theorem for the associated optimal stopping problem.

Theorem 4.3 *Suppose assumptions of Theorem 3.5 are satisfied and that* $\lim_{x\to\infty}(\rho(x)+\lambda\bar{z}x) < 0$. *Then, the value of the associated optimal stopping problem* (9) *reads as*

$$V_{OSP}(x) = \psi(x) \sup_{y\geq x} \left\{ \frac{y}{\psi(y)} \right\} = \begin{cases} x, & x \geq x_0^*, \\ \psi(x)\frac{x_0^*}{\psi(x_0^*)}, & x < x_0^*, \end{cases} \tag{24}$$

where the optimal stopping boundary $x_0^* \geq x^*$ *is the unique root of* $\psi(x) = x\psi'(x)$. *In particular,* $V_{OSP}(x) = H(x, x_0^*)$. *Moreover,* $V_{OSP}(x) \leq V_S(x)$ *and* $V'_{OSP}(x) \leq V'_S(x)$ *for all* $x \in I$.

Proof First we need to establish existence and uniqueness of x_0^*. For this, note that by Theorem 3.5, under our assumptions $\psi''(x) < 0$ for $x < x^*$ and $\psi''(x) > 0$ for $x > x^*$, which implies that $\psi(x)$ is strictly concave for $x < x^*$ and strictly convex for $x > x^*$. The strict concavity of $\psi(x)$ on $(0, x^*)$ implies that $0 = \psi(0) < \psi(x) - \psi'(x)x$ for all $x \in (0, x^*)$. Hence,

$$D_x\left[\frac{x}{\psi(x)}\right] = \frac{\psi(x) - x\psi'(x)}{\psi^2(x)} > 0, \tag{25}$$

for all $x < x^*$. On the other hand, the monotonicity of the function $\psi(x)$ and the spectral negativity of the jumps imply

$$\begin{aligned} \frac{d}{dx}\left[\frac{\psi(x)}{S'(x)} - x\frac{\psi'(x)}{S'(x)}\right] &= \left(\tilde{\rho}(x)\psi(x) + x\int_{(0,1)} \psi(x(1-z))\nu(dz)\right) m'(x) \\ &\leq (\rho(x) + \lambda\bar{z}x)\psi(x)m'(x). \end{aligned}$$

Combining this finding with the assumption $\lim_{x\to\infty}(\rho(x)+\lambda\bar{z}x) < 0$ and the positivity of $S'(x)$ shows that there must exist $x \in I$ such that $\psi(x) < x\psi'(x)$. By continuity, this implies the existence of $x_0^* \geq x^*$ such that $\psi(x_0^*) = x_0^*\psi'(x_0^*)$. Moreover, the convexity of $\psi(x)$ on (x^*, ∞) implies that

$$D_x\left[\psi(x) - x\psi'(x)\right] = -x\psi''(x) < 0$$

for all $x > x^*$. Thus $x/\psi(x)$ is decreasing for all $x > x^*$ and so x_0^* is unique. Note, in particular, that $x \mapsto x/\psi(x)$ is now a unimodal function with a unique maximum at x_0^*.

We will use the notation $v(x) = \psi(x)\sup_{y\geq x}\{y/\psi(y)\}$. It is immediate from the definition that $v(x) \geq x$ for all $x \in I$, that $v \in C^1(I) \cap C^2(I\setminus\{x_0^*\})$, and that $|v''(x_0^*\pm)| < \infty$. We will now prove that $v(x)$ is r-superharmonic with respect to X.

It is clear that since $v(x) = \psi(x)/\psi'(x_0^*)$ on $(0, x_0^*)$ and $(\mathcal{G}_r\psi)(x) = 0$, we have $(\mathcal{G}_r v)(x) = 0$ for all $x \in (0, x_0^*)$. To establish the r-superharmonicity of $v(x)$ on $[x_0^*, \infty)$ we first note that since $\psi(x_0^*) = \psi'(x_0^*)x_0^*$ and $\psi''(x_0^*) \geq 0$ the identity $(\mathcal{G}_r\psi)(x) = 0$ implies that

$$0 \geq -\frac{1}{2}\sigma^2(x_0^*)\psi''(x_0^*) = \rho(x_0^*)\psi'(x_0^*) + \int_0^1 \left(\psi(x_0^*(1-z)) - (1-z)x_0^*\psi'(x_0^*)\right)\nu(dz).$$

Dividing this inequality with $\psi'(x_0^*)$ then shows that

$$\lim_{x \to x_0^*+} (\mathcal{G}_r v)(x) = \rho(x_0^*) + \int_0^1 \left(\frac{\psi(x_0^*(1-z))}{\psi'(x_0^*)} - (1-z)x_0^*\right)\nu(dz) \leq 0.$$

We will now prove that $(\mathcal{G}_r v)(x)$ is non-increasing on (x_0^*, ∞) and, therefore, that $(\mathcal{G}_r v)(x) \leq 0$ for all $x \in (x_0^*, \infty)$. Differentiating the functional $(\mathcal{G}_r v)(x)$ and applying the inequalities $(\mathcal{G}_r V_s)'(x) \leq 0$ and $v'(x) \leq V_S'(x)$ established in Theorem 3.6 demonstrates that for all $x \in (x_0^*, \infty)$ we have

$$\begin{aligned}(\mathcal{G}_r v)'(x) &= \rho'(x) + \int_{(0,1)} (v'(x(1-z)) - 1)(1-z)\nu(dz) \\ &\leq (\mathcal{G}_r V_S)'(x) + \int_{1-x_0^*/x}^{1-x^*/x} (v'(x(1-z)) - 1)(1-z)\nu(dz) \leq 0\end{aligned}$$

since $\psi(x)$ is convex on (x^*, x_0^*). Hence $(\mathcal{G}_r v)(x) \leq 0$ for all $x \in I$. Consequently, $v(x)$ constitutes a nonnegative r-superharmonic majorant of x and, therefore, $v(x) \geq V_{\mathrm{OSP}}(x)$, as the latter is by the general theory the *least* r-superharmonic majorant of x [cf. Peskir and Shiryaev (2006), especially Sect. IV.9 therein].

In order to establish the opposite inequality we first observe that for $y > x$

$$\mathbb{E}_x\left[e^{-r\tau_y}X_{\tau_y}\right] = y\mathbb{E}_x\left[e^{-r\tau_y}\right] = y\frac{\psi(x)}{\psi(y)}$$

and for $y \leq x$, $\mathbb{E}_x\left[e^{-r\tau_y}X_{\tau_y}\right] = x$. Hence the choice $y = \operatorname{argmax}[z/\psi(z)] = x_0^*$ yields

$$v(x) = \mathbb{E}_x\left[e^{-r\tau_{x_0^*}}X_{\tau_{x_0^*}}\right] \leq V_{\mathrm{OSP}}(x).$$

Thus $V_{\mathrm{OSP}}(x) \leq v(x) \leq V_{\mathrm{OSP}}(x)$, and the claimed representation is proved. In particular, by definition of x_0^* we have $\frac{1}{\psi'(x_0^*)} = \frac{x_0^*}{\psi(x_0^*)}$ and $\frac{\psi(x_0^*)}{\psi'(x_0^*)} - x_0^* = 0$, and hence, recalling the definition of $H(x, y)$ from (11), we get that

$$H(x, x_0^*) = \begin{cases} x - x_0^* + \frac{\psi(x_0^*)}{\psi'(x_0^*)} = x, & x \geq x_0^*, \\ \frac{\psi(x)}{\psi'(x_0^*)} = \psi(x) \cdot \frac{x_0^*}{\psi(x_0^*)}, & x < x_0^*. \end{cases}$$

The last two claims follow in a straightforward fashion from Theorem 3.6, since $V_{\mathrm{OSP}}(x) = H(x, x_0^*)$.

We wish to point out that the representation of the value of the stopping problem given in the previous theorem holds also for more general jump diffusions and reward functions under some additional conditions, as has been shown in Alvarez and Rakkolainen (2006). An interesting implication of Theorem 4.3 and Lemma 3.2 extending the observation of Corollary 4.2 to the optimal liquidation case as well is now summarized in the following.

Corollary 4.4 *Assume that the assumptions of Theorem 3.5 are satisfied, that $\rho(x) - (1-\bar{z})\lambda x$ is increasing on a neighborhood of 0 and that $\lim_{x\to\infty}(\rho(x) + \lambda \bar{z} x) < 0$. Then, for all $x \in I$ it holds*

$$\tilde{V}_{\mathrm{OSP}}^{r+\lambda}(x) \leq V_{\mathrm{OSP}}(x) \leq \tilde{V}_{\mathrm{OSP}}^{r}(x), \tag{26}$$

where

$$\tilde{V}_{\mathrm{OSP}}^{\theta}(x) = \sup_{\tau} \mathbb{E}_x \left[e^{-\theta\tau} \tilde{X}_\tau \right] = \tilde{\psi}_\theta(x) \sup_{y \geq x} \left[\frac{y}{\tilde{\psi}_\theta(y)} \right].$$

Moreover, $\tilde{x}_0^*(r+\lambda) < x_0^* < \tilde{x}_0^*(r)$, where $\tilde{x}_0^*(\theta)$ denotes the unique root of the optimality condition $\tilde{\psi}_\theta(\tilde{x}_0^*(\theta)) = \tilde{\psi}'_\theta(\tilde{x}_0^*(\theta))\tilde{x}_0^*(\theta)$.

Proof Since $V_{\mathrm{OSP}}(x) = H(x, x_0^*)$ and $\tilde{V}_{\mathrm{OSP}}^{\theta}(x) = \tilde{H}_\theta(x, \tilde{x}^*(\theta))$ inequality (26) follows from Theorem 3.6 and Lemma 3.2. The ordering $\tilde{x}_0^*(r+\lambda) < x_0^* < \tilde{x}_0^*(r)$ is a direct implication of Corollary 3.3, the proof of Theorem 4.3, and the inequality $\tilde{\psi}''_\theta(x) \lesseqgtr 0$ for $x \lesseqgtr \tilde{x}_0^*(\theta)$.

5 Optimal impulse control of dividends

Let us next consider the problem of determining the *optimal impulse control* in our Lévy diffusion model in case where the corporation incurs a fixed cost $c > 0$ each time it distributes dividends. An impulse type dividend control consists of an increasing sequence of $\mathbb{F}$-stopping times $\tau = (\tau(i)), i \leq N \leq \infty$, (intervention times) and a corresponding sequence of non-negative impulses $\xi = (\xi(i)), i \leq N \leq \infty$, (interventions). The standard approach is to seek an optimal impulse control $\hat{\nu} = (\hat{\tau}, \hat{\xi})$ in the whole class of admissible impulse controls

$$\mathcal{V} = \left\{ (\tau, \xi) \,:\, \tau(i) \in \mathcal{T},\ 0 \leq \xi(i) \leq X_{\tau(i)},\ 1 \leq i \leq N \right\}$$

such that the expected cumulative present value of the policy,

$$J^{\tau,\xi}(x) = \mathbb{E}_x\left[\sum_{i=1}^{N} e^{-r\tau(i)}(\xi(i) - c)\right],$$

is maximized, that is, $(\hat{\tau}, \hat{\xi})$ should satisfy

$$V_{\mathrm{I}}^{c}(x) = \sup_{(\tau,\xi)\in\mathcal{V}} J^{\tau,\xi}(x) = \mathbb{E}_x\left[\sum_{i=1}^{N} e^{-r\hat{\tau}(i)}(\hat{\xi}(i) - c)\right].$$

The associated optimal stopping problem is defined as

$$V_{\mathrm{OSP}}^{c}(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x\left[e^{-r\tau}(X_\tau - c)\right]. \tag{27}$$

The following analogue of Theorem 4.3 and its Corollary 4.4 holds for this stopping problem.

Lemma 5.1 (A) *Suppose assumptions of Theorem 3.5 are satisfied. Then the value of the associated optimal stopping problem (27) reads as*

$$V_{OSP}^{c}(x) = \psi(x)\sup_{y\geq x}\left\{\frac{y-c}{\psi(y)}\right\} = H(x, x_c^*) \tag{28}$$

where the optimal stopping boundary $x_c^* \geq x_0^* \geq x^*$ is the unique root of $\psi(x) = (x-c)\psi'(x)$. Moreover, $V_{OSP}^{c}(x) \leq V_S(x)$ and $V_{OSP}^{c\prime}(x) \leq V_S'(x)$ for all $x \in I$.

(B) *If also $\rho(x) - (1-\bar{z})\lambda x$ is increasing on a neighborhood of 0 and $\lim_{x\to\infty}(\rho(x) + \lambda\bar{z}x) < 0$ then, for all $x \in I$ it holds*

$$\tilde{V}_{\mathrm{OSP}}^{r+\lambda,c}(x) \leq V_{OSP}^{c}(x) \leq \tilde{V}_{\mathrm{OSP}}^{r,c}(x), \tag{29}$$

where

$$\tilde{V}_{\mathrm{OSP}}^{\theta,c}(x) = \sup_{\tau}\mathbb{E}_x\left[e^{-\theta\tau}(\tilde{X}_\tau - c)\right] = \tilde{\psi}_\theta(x)\sup_{y\geq x}\left[\frac{(y-c)}{\tilde{\psi}_\theta(y)}\right].$$

Moreover, $\tilde{x}_c^*(r+\lambda) < x_c^* < \tilde{x}_c^*(r)$, where $\tilde{x}_c^*(\theta)$ denotes the unique root of the optimality condition $\tilde{\psi}_\theta(\tilde{x}_c^*(\theta)) = \tilde{\psi}_\theta'(\tilde{x}_c^*(\theta))\tilde{x}_c^*(\theta)$.

Proof This is simply a straightforward replication of the proof of Theorem 4.3 and its Corollary 4.4, *mutatis mutandis*.

Lemma 5.1 states a set of conditions under which the solution of the associated liquidation problem (27) is an ordinary threshold policy. As Lemma 5.1 proves, the standard balance equation, requiring that the value of the optimal policy is equal to its total costs (the sum of the direct cost and the lost option value), is satisfied at the optimal exercise threshold in the discontinuous setting as well. Moreover, according to the findings of Lemma 5.1 the value of the liquidation problem (27) can be sandwiched between the values of two associated stopping problems defined with respect to a continuous diffusion process.

We shall follow an approach similar to the one adopted in Alvarez and Virtanen (2006) and determine the optimal choice within a more restricted class of impulse dividend controls based on a single threshold level and a fixed dividend size. We will then proceed to give reasonably general conditions under which this optimal choice in the restricted class is, in fact, optimal also in the larger class $\mathcal{V}$. To avoid unnecessary duplication, we will mostly refer to Alvarez and Virtanen (2006) for detailed arguments when the presence of jumps does not affect the analysis.

Consider a dividend policy (τ_y, η) such that a constant dividend η is paid out when the underlying reaches a specified threshold level y, and in case $x > y$ an exceptional, state-dependent initial dividend $x - y + \eta$ is paid out to bring the state below level y. By relying on a similar reasoning as in Sect.4 of Alvarez and Virtanen (2006) we find that the value of such a policy has the following representation in terms of the minimal r-superharmonic map $\psi(x)$:

$$J^{\tau_y,\eta}(x) = F_c(x) = \begin{cases} x - y + \frac{(\eta-c)\psi(y)}{\psi(y)-\psi(y-\eta)}, & x \geq y \\ \frac{(\eta-c)\psi(x)}{\psi(y)-\psi(y-\eta)}, & x < y. \end{cases} \tag{30}$$

Consider now the inequality constrained nonlinear programming problem

$$\sup_{\eta\in[0,y],\, y\in I} h(\eta, y) = \sup_{\eta\in[0,y],\, y\in I} \frac{(\eta - c)}{\psi(y) - \psi(y-\eta)}. \tag{31}$$

If there exists a unique optimal pair (η_c^*, y_c^*) maximizing $h(\eta, y)$, we can define

$$F_c^*(x) = \begin{cases} x - y_c^* + h(\eta_c^*, y_c^*)\psi(y_c^*), & x \geq y_c^* \\ h(\eta_c^*, y_c^*)\psi(x), & x < y_c^*. \end{cases} \tag{32}$$

It is an immediate consequence of the necessary first order conditions for optimality in (31) that

$$F_c^*(x) = H(x, y_c^*) = \begin{cases} x - y_c^* + \frac{\psi(y_c^*)}{\psi'(y_c^*)}, & x \geq y_c^* \\ \frac{\psi(x)}{\psi'(y_c^*)}, & x < y_c^* \end{cases} \tag{33}$$

(to see this, consider the function $h : (0, y) \times I \to \mathbb{R}$ in (31), which under our assumptions is a smooth function of two variables, and set the gradient equal to zero—the resulting equations imply that $h(\eta_c^*, y_c^*) = \frac{1}{\psi'(y_c^*)}$, from which the claimed representation of $F_c^*(x)$ follows). It is now clear that $F_c^*(x)$ belongs to the class of mappings considered in Theorem 3.6 and hence $F_c^*(x) \leq V_S(x)$ and $F_c^{*\prime}(x) \leq V_S'(x)$ (given existence and uniqueness of (η_c^*, y_c^*)). We will now establish a set of sufficient conditions for the existence of a unique optimal pair (η_c^*, y_c^*) solving (31).

Lemma 5.2 *Suppose, in addition to the assumptions of Theorem 3.5, that* $\lim_{x\downarrow 0} \psi'(x) = \infty$. *Then there exists a unique pair* $(\eta_c^*, y_c^*) \in (c, y_c^*) \times (x^*, x_c^*)$, *which satisfies the necessary first order conditions* $\psi'(y_c^*) = \psi'(y_c^* - \eta_c^*)$ *and* $\psi(y_c^*) - \psi(y_c^* - \eta_c^*) = \psi'(y_c^* - \eta_c^*)(\eta_c^* - c)$.

Proof Under assumptions of Theorem 3.5 there exists a unique x^* such that $\psi'(x)$ is decreasing (increasing) on $(0, x^*)$ $((x^*, \infty))$. If $\lim_{x\downarrow 0} \psi'(x) = \infty$, then this implies that for any $y \in (x^*, \infty)$ there exists a unique $\hat{y} \in (0, x^*)$ such that $\psi'(y) = \psi'(\hat{y})$. Moreover, we can define $\hat{x}^* = x^*$. Hence the function $y \mapsto \hat{y}$ from $[x^*, \infty)$ onto $(\hat{y}_{\min}, x^*]$ is well-defined, where $\hat{y}_{\min} = (\psi')^{-1}(\lim_{x\to\infty} \psi'(x))$. It is decreasing and continuous (even C^1, see the proof of Lemma 3.4). Consider then the continuous function

$$L(y) = \psi(y) - \psi(\hat{y}) - \psi'(\hat{y})(y - \hat{y}) + c \cdot \psi'(\hat{y})$$

defined for $y \in [x^*, \infty)$. Now $L(x^*) = c \cdot \psi'(x^*) > 0$ and

$$\begin{aligned}
L(x_c^*) &= \psi(x_c^*) - \psi(\hat{x}_c^*) - \psi'(\hat{x}_c^*)(x_c^* - \hat{x}_c^*) + c \cdot \psi'(\hat{x}_c^*) \\
&= \left(\psi(x_c^*) - \psi'(\hat{x}_c^*)x_c^*\right) - \psi(\hat{x}_c^*) + \psi'(\hat{x}_c^*)\hat{x}_c^* + c \cdot \psi'(\hat{x}_c^*) \\
&= \left(\psi(x_c^*) - \psi'(x_c^*)x_c^*\right) - \psi(\hat{x}_c^*) + \psi'(\hat{x}_c^*)\hat{x}_c^* + c \cdot \psi'(\hat{x}_c^*) \\
&= -c \cdot \psi'(x_c^*) - \left(\psi(\hat{x}_c^*) - \psi'(\hat{x}_c^*)\hat{x}_c^*\right) + c \cdot \psi'(\hat{x}_c^*) < 0,
\end{aligned}$$

since $\psi(x) - \psi'(x)x > 0$ for all $x < x^*$. Thus there exists $y_c^* \in (x^*, x_c^*)$ such that $L(y_c^*) = 0$, in other words, the choice $(\eta, y) = (y_c^* - \hat{y}_c^*, y_c^*)$ satisfies the first order conditions for optimality in (31). To establish uniqueness of this solution, note that

$$L'(y) = \psi''(\hat{y}) \cdot \hat{y}'(y) \cdot (c - y + \hat{y}) + \left(\psi'(y) - \psi'(\hat{y})\right),$$

whose sign is determined by the last factor of the first term on the right hand side, both other factors of the term being always negative for $y \in (x^*, \infty)$—observe that the second term equals zero, by definition of $\hat{y}$. As $c - y + \hat{y} = c > 0$ for $y = x^*$, $\lim_{y\to\infty}(c - y + \hat{y}) = -\infty$ and furthermore $c - y + \hat{y}$ is decreasing in y, we see that $L(y)$ is a unimodal function with a unique maximum. Since $L(x^*) = c > 0$, the root of $L(y) = 0$ is necessarily unique.

Having established sufficient conditions for existence and uniqueness of the pair (η_c^*, y_c^*) satisfying the necessary optimality conditions of (31), we now proceed to state our second main theorem, whose proof requires a verification lemma.

Lemma 5.3 *Assume that the mapping $g : I \to I$ is increasing and satisfies the conditions $g \in C^1(I) \cap C^2(I \setminus \mathcal{D})$, where $\mathcal{D}$ is a set of zero measure, and $|g''(x\pm)| < \infty$ for all $x \in \mathcal{D}$. Suppose further that g satisfies the quasi-variational inequality*

$$\sup_{\eta \in [0,x]} \{\eta - c + g(x - \eta)\} \leq g(x)$$

for all $x \in I$ and the variational inequality $(\mathcal{G}_r g)(x) \leq 0$ for all $x \in I \setminus \mathcal{D}$. Then $g(x) \geq V_I^c(x)$ for all $x \in I$.

Proof The estimations in the proof of Lemma 2.1 in Alvarez and Virtanen (2006) go through also in our setting, when one notices two things. First, an application of the dominated convergence theorem yields

$$\int_0^1 \psi_k(x - xz)\mathfrak{m}(dz) \to \int_0^1 \psi(x - xz)\mathfrak{m}(dz)$$

as $k \to \infty$, for any $x \in I$, and thus uniformly on compact subsets of I. Hence the approximation result from Appendix D in Øksendal (2003) is valid also in our jump diffusion model. Second, for a spectrally negative jump diffusion X and an increasing non-negative function g

$$\begin{aligned} g(X^\nu_{\tau_j -}) - g(X^\nu_{\tau_j}) &= g(X^\nu_{\tau_j -}) - g(X^\nu_{\tau_j -} - \eta_{\tau_j} - |\Delta X^\nu_{\tau_j}|) \\ &\geq g(X^\nu_{\tau_j -}) - g(X^\nu_{\tau_j -} - \eta_{\tau j}), \end{aligned}$$

which ensures that the inequalities derived in Alvarez and Virtanen (2006) remain valid in our model.

Having proved the auxiliary Lemma 5.3, we are now in position to present our main finding on the optimal stochastic lump-sum dividend policy and its value.

Theorem 5.4 *Suppose the assumptions of Lemma 5.2 are satisfied. Then*

$$V_I^c(x) = J^{\tau_{y_c^*}, \eta_c^*}(x) = F_c^*(x) = H(x, y_c^*),$$

where $y_c^* \in (x^*, x_c^*)$ and $\eta_c^* = y_c^* - \hat{y}_c^*$ solve (31). In other words, the value of the optimal single threshold dividend policy coincides with the value of the optimal impulse control problem.

Proof As the single threshold dividend policy $\nu = \nu(\eta_c^*, y_c^*)$ is clearly an admissible impulse control, we have $F_c^*(x) \leq V_\text{I}^c(x)$. To establish the converse inequality, by Lemma 5.3 it is enough to show that the increasing function $F_c^*(x)$ is sufficiently smooth and satisfies the relevant quasi-variational inequalities. It is easy to see by standard differentiations that $F_c^*(x) \in C^1(I) \cap C^2(I \setminus \{y_c^*\})$ and that $\lim_{x \downarrow y_c^*} |F_c^{*\prime\prime}(x)| = 0$ and $\lim_{x \uparrow y_c^*} |F_c^{*\prime\prime}(x)| < \infty$. By boundedness on compacts of continuous maps and the fact that $X_t^\nu \leq y_c^*$, t-almost everywhere, we furthermore have $\lim_{t\to\infty} \mathbb{E}_x \left[e^{-rt} F_c^*(X_t^\nu)\right] = 0$ for all $x \in I$. To see that $F_c^*(x)$ satisfies the variational inequality, note that for $x < y_c^*$

$$\left(\mathcal{G}_r F_c^*\right)(x) = \left(\psi'(y_c^*)\right)^{-1} \left(\mathcal{G}_r \psi\right)(x) = 0,$$

and for $x \geq y_c^*$, by Theorem 3.6,

$$\begin{aligned}\left(\mathcal{G}_r F_c^*\right)'(x) &= \rho'(x) + \int_0^1 \left\{F_c^{*\prime}(x - xz) - 1\right\}(1 - z)\nu(dz) \\ &\leq (\mathcal{G}_r V_s)'(x) + \int_{1-y_c^*/x}^{1-x^*/x} (F_c^{*\prime}(x(1-z)) - 1)(1-z)\nu(dz) \leq 0,\end{aligned}$$

since $\psi(x)$ is convex on (x^*, y_c^*). This implies that $\left(\mathcal{G}_r F_c^*\right)(x)$ is decreasing for $x \geq y_c^*$ and hence $\left(\mathcal{G}_r F_c^*\right)(x) \leq 0$ for all $x \in I$. Finally, to establish that the quasi-variational inequality $F_c^*(x) \geq \sup_{\eta\in[0,x]} \left[\eta - c + F_c^*(x - \eta)\right]$ holds, we may proceed exactly as in Appendix E in Alvarez and Virtanen (2006). Thus, since $F_c^*(x)$ satisfies the quasi-variational inequalities, by Lemma 5.3, $F_c^*(x) \geq V_{\mathrm{I}}^c(x)$ and hence $F_c^*(x) = V_{\mathrm{I}}^c(x)$.

Results obtained in this section are similar to the ones obtained for continuous linear diffusions in Alvarez and Virtanen (2006) and highlight the similarities in behavior of continuous diffusions and spectrally negative jump diffusions with natural boundaries and geometric jumps. Along the lines of our previous analysis, we are now in position to establish the following interesting comparison result extending our sandwiching results to the present setting as well.

Theorem 5.5 *For all $\eta \in [c, y]$ and $x \in I$ we have $\tilde{K}_{r+\lambda}(x) \leq F_c(x) \leq \tilde{K}_r(x)$, where the function $\tilde{K}_\theta : I \mapsto \mathbb{R}_+$ is defined as*

$$\tilde{K}_\theta(x) = \begin{cases} x - y + \frac{(\eta-c)\tilde{\psi}_\theta(y)}{\tilde{\psi}_\theta(y)-\tilde{\psi}_\theta(y-\eta)}, & x \geq y \\ \frac{(\eta-c)\tilde{\psi}_\theta(x)}{\tilde{\psi}_\theta(y)-\tilde{\psi}_\theta(y-\eta)}, & x < y. \end{cases} \tag{34}$$

Consequently, if the conditions of Lemma 5.2 are satisfied, $\rho(x) - (1-\bar{z})\lambda x$ is increasing on a neighborhood of 0, $\lim_{x\to\infty}(\rho(x) + \lambda\bar{z}x) < 0$, and $\lim_{x\downarrow 0} \tilde{\psi}_{r+\lambda}'(x) = \lim_{x\downarrow 0} \tilde{\psi}_r'(x) = \infty$, then $\tilde{H}_{r+\lambda}(x, \tilde{y}_c^*(r+\lambda)) \leq V_I^c(x) \leq \tilde{H}_r(x, \tilde{y}_c^*(r))$, where $(\tilde{y}_c^*(\theta), \tilde{\eta}_c^*(\theta))$ denotes the unique pair maximizing the function

$$\frac{(\eta - c)}{\tilde{\psi}_\theta(y) - \tilde{\psi}_\theta(y - \eta)}.$$

Proof The inequality $\tilde{K}_{r+\lambda}(x) \leq F_c(x) \leq \tilde{K}_r(x)$ is a direct consequence of part (A) of our Corollary 3.3. As was established in Lemma 4.1 of Alvarez and Virtanen

(2006), our assumptions guarantee the existence and uniqueness of the optimal pairs $(\tilde{y}_c^*(r+\lambda), \tilde{\eta}_c^*(r+\lambda))$ and $(\tilde{y}_c^*(r), \tilde{\eta}_c^*(r))$. Combining this observation with the result of Theorem 5.4 completes our proof.

6 Explicit illustration: logistic jump diffusion

To illustrate our general results with a particular example, we consider a *logistic jump diffusion* given by

$$dX_t = X_t \left\{ a(b - X_t)dt + \sigma dW_t - \int_0^1 zN(dt, dz) \right\}, \tag{35}$$

where parameters $a > 0$, $b > 0$, $\sigma > 0$ and the associated Lévy measure of the compensated compound Poisson process N is $\nu = \lambda\mathfrak{m}(dz)$, where $\mathfrak{m}$ is the relative jump size distribution defined on $(0, 1)$. The downside risk is thus characterized by the jump intensity λ and the form of the jump size distribution. If $ab > r$, the considered jump diffusion satisfies the conditions of Theorem 4.1, as then the net appreciation rate $\rho(x) = ax(b-x) - rx = -ax^2 + (ab-r)x$. In this chapter we will take the relative jump size to be Beta(α, β) distributed. This allows us to consider both symmetric and skewed distributions by varying the parameters α and β. We assume that the discount rate $r = 0.025$ and the fixed transaction cost $c = 0.05$.

With regard to analyzing the effect of λ, we wish to point out that care may be needed when dealing with large values of λ, since in the limit $\lambda \to \infty$ we get a spectrally negative compound Poisson process with drift $\lambda\bar{z} > 0$, which (being a martingale) does *not* satisfy the classical Cramer–Lundberg net profit condition and hence oscillates and will hit 0 in finite time almost surely, violating our assumptions on the boundary behavior of the jump diffusion.

We are interested in the effect of introducing jumps—downside risk—on the optimal thresholds. The benchmark case is now the absence of downside risk, $\lambda = 0$, in which case the associated integro-differential equation $\mathcal{G}_r u = 0$ reduces to a linear ordinary second order differential equation, whose increasing fundamental solution $\psi(x)$ can be expressed in terms of the Kummer confluent hypergeometric function. Optimal boundaries for the singular control, impulse control and stopping problems (respectively) can then be solved from equations

$$\begin{aligned} &\psi''(x) = 0, \\ &\begin{cases} \psi(y) - \psi(y-\eta) = \psi'(y-\eta)(\eta - c) \\ \psi'(y) = \psi'(y-\eta) \quad \text{and} \end{cases} \\ &\psi(x) = \psi'(x)(x-c). \end{aligned} \tag{36}$$

This yields with the assumed parameter values the first row of Table 1. For non-zero intensities λ, the integro-differential equation is not (semi-)explicitly solvable except in the case $\alpha = \beta = 1$, i.e. when relative jump sizes are uniformly distributed. In this special case the integro-differential equation can be reduced to a third

Table 1 Optimal boundaries for the jump diffusion (35) and the associated continuous diffusion, when intensity $\lambda \in \{0.1, 1, 10\}$ and relative jump size distribution is Beta(α, α), $\alpha \in \{1, 6, 10\}$

λ	α		x^*	y_c^*	η_c^*	x_c^*
0	–	(X, r)	1.003	1.423	0.781	2.378
0.1	–	$(\tilde{X}, r)$	1.263	1.734	0.887	3.011
	–	$(\tilde{X}, \tilde{r})$	0.684	1.134	0.782	1.472
	1	(X, r)	1.053	1.503	0.830	2.469
	6	(X, r)	1.054	1.500	0.824	2.509
	10	(X, r)	1.054	1.500	0.824	2.511
1	–	$(\tilde{X}, r)$	3.546	4.383	1.639	8.594
	1	(X, r)	1.169	1.833	1.166	2.514
	6	(X, r)	1.296	1.931	1.133	2.901
	10	(X, r)	1.307	1.939	1.129	2.947
10	–	$(\tilde{X}, r)$	26.07	29.05	5.932	69.35
	1	(X, r)	1.168	2.699	2.340	2.921
	6	(X, r)	1.356	2.795	2.278	3.164
	10	(X, r)	1.381	2.808	2.270	3.199

order linear differential equation by considering $\Phi(x) := \int_0^x \psi(y)dy$; the obtained differential equation is then solvable in terms of generalized hypergeometric functions. For the general case, $\lambda \neq 0$ and either $\alpha \neq \beta$ or $\alpha \neq 1$, we can apply the Frobenius method. That is, we will assume that the solution ψ of $\mathcal{G}_r u = 0$ is of form $\psi(x) = x^\varsigma \sum_{n=0}^{\infty} \gamma_n x^n$ with $\varsigma > 0$ (which is clearly in $C^2(I)$), plug this into $\mathcal{G}_r u = 0$ and solve the resulting indicial (integral) equation

$$\left(ab+\lambda\frac{\alpha}{\alpha+\beta}\right)\varsigma+\frac{1}{2}\sigma^2\varsigma(\varsigma-1)-\tilde{r}+\frac{\lambda}{B(\alpha,\beta)}\int_0^1(1-z)^\varsigma z^{\alpha-1}(1-z)^{\beta-1}dz=0$$

for ς and the recursion relation

$$\left(ab+\lambda\frac{\alpha}{\alpha+\beta}\right)\gamma_n(\varsigma+n)-a\gamma_{n-1}(\varsigma+n-1)+\frac{\sigma^2}{2}\gamma_n(\varsigma+n)(\varsigma+n-1)$$
$$-\left(\tilde{r}-\frac{\lambda}{B(\alpha,\beta)}\int_0^1(1-z)^{\varsigma+n}z^{\alpha-1}(1-z)^{\beta-1}dz\right)\gamma_n=0.$$

for $\{\gamma_n\}$. If $\varsigma > 0$ solves the indicial equation and the obtained sequence of coefficients $\{\gamma_n\}$ is such that the power series converges, we have found a smooth solution of the considered integro-differential equation satisfying the required boundary condition. In Rakkolainen (2007) it is shown that in the logistic jump diffusion case such a solution is necessarily monotone and unique up to a multiplicative constant. A numerical approximation (to desired accuracy) for ψ can be obtained by truncating the infinite series in $\psi(x) = x^\varsigma \sum_{n=0}^{\infty} \gamma_n x^n$ at some $n_0 \in \mathbb{N}$. In the present case, the recursion

Table 2 Optimal boundaries for the jump diffusion (35) and the associated continuous diffusion, when intensity $\lambda \in \{0.1, 1, 10\}$ and relative jump size distribution is Beta distributed with variance 0.01 and mean $\bar{z} \in \{0.25, 0.5, 0.75\}$

λ	$\bar{z}$		x^*	y_c^*	η_c^*	x_c^*
0.1	0.25	$(\tilde{X}, r)$	1.133	1.579	0.835	2.696
		(X, r)	1.019	1.448	0.795	2.430
		$(\tilde{X}, \tilde{r})$	0.551	0.971	0.715	1.204
	0.5	$(\tilde{X}, r)$	1.263	1.734	0.887	3.011
		(X, r)	1.054	1.499	0.823	2.511
		$(\tilde{X}, \tilde{r})$	0.684	1.134	0.782	1.472
	0.75	$(\tilde{X}, r)$	1.392	1.887	0.937	3.324
		(X, r)	1.086	1.555	0.862	2.529
		$(\tilde{X}, \tilde{r})$	0.817	1.294	0.844	1.744
1	0.25	$(\tilde{X}, r)$	2.284	2.932	1.253	5.495
		(X, r)	1.151	1.647	0.906	2.806
	0.5	$(\tilde{X}, r)$	3.546	4.383	1.639	8.594
		(X, r)	1.309	1.941	1.128	2.959
	0.75	$(\tilde{X}, r)$	4.803	5.809	1.982	11.73
		(X, r)	1.192	1.977	1.348	2.557
10	0.25	$(\tilde{X}, r)$	13.57	15.51	3.864	34.68
		(X, r)	1.501	2.401	1.549	3.293
	0.5	$(\tilde{X}, r)$	26.07	29.05	5.932	69.35
		(X, r)	1.387	2.811	2.268	3.208
	0.75	$(\tilde{X}, r)$	38.58	42.42	7.684	105.3
		(X, r)	1.184	3.106	2.824	3.256

relation can be manipulated to the form $\gamma_{n+1} = [c_1(n)/c_2(n)]\gamma_n$, where essentially (since the integral term is in any case bounded from above by λ) c_1 is linear and c_2 quadratic in n. This implies that $\gamma_n \sim (1/n^n)\gamma_0$ and thus the coefficients converge to zero quite rapidly as n increases.

It is worth noting that in principle, the outlined approximation approach is always applicable if the jump component has the geometric form assumed throughout our study and the coefficient functions of the compensated diffusion part are polynomials $\tilde{\mu}(x) = \sum_{i=0}^{N} \tilde{p}_i x^i$ and $(1/2)\sigma^2(x) = \sum_{j=0}^{M} q_i x^i$ such that $q_0 = q_1 = \tilde{p}_0 = 0$. Naturally in more general cases the rate of convergence for the coefficient sequence is not necessarily as rapid as in the logistic case.

We apply the outlined procedure to solve the (approximative) optimal thresholds for intensities $\lambda \in \{0.1, 1, 100\}$ and two different sets of parameters α and β:

(i) symmetric jump size distribution with constant mean 1/2 ($\alpha = \beta$), for $\alpha \in \{1, 5, 10\}$; as parameter value increases, the distribution becomes more concentrated around its mean; and
(ii) skewed jump size distribution with constant variance 0.01 and variable mean $\bar{z} \in \{0.25, 0.5, 0.75\}$;

This gives us an illustration of the impact of variable uncertainty in the jump risk with constant “average jump risk” (i), and of skewness of the jump size distribution (ii). Note that with variance fixed, skewness and mean have opposing effects: for a small mean (which is “good” in the sense that downward jumps are small on average) the distribution is skewed to the right, i.e. towards larger jump sizes, and vice versa. In addition, we will compute the optimal policies in the associated optimization problems for the corresponding continuous (drift-corrected) diffusion $\tilde{X}$. It should be noted that for the associated diffusion instantaneous liquidation is optimal in the problem with discount rate $r + \lambda$ if $\tilde{r} := r + \lambda \geq ab$). The results are given in Tables 1 (symmetric distributions) and 2 (skewed distributions). In both tables, instantaneous liquidation is optimal for $\lambda \in \{1, 10\}$ and hence rows corresponding to $(\tilde{X}, \tilde{r})$ have been omitted in these cases.

Inspection of the results shows that the numerical results are in line with our findings: the exercise boundaries for the jump diffusion X are in all cases between the corresponding boundaries for the associated diffusion $\tilde{X}$, provided that the lower boundaries in question exist (i.e. that *take the money and run* policy is not optimal). From Table 1 one sees that increasingly concentrated jump size distribution seems to lead to higher exercise thresholds for all problems and to a lower dividend size for the impulse control problem. This effect is similar for all sample values of λ, though naturally almost negligible for the smallest sample value and more pronounced for the larger values. It appears from Table 2 that such monotonicity does not hold for the case (ii).

7 Concluding comments

In this study we considered the determination of the optimal dividend policy of a risk-neutral firm when the stochastic dynamics of the underlying cash flow are characterizable as a spectrally negative jump diffusion with natural boundaries and geometric jumps. We established a relatively broad set of conditions typically satisfied in most mean-reverting models under which the optimal singular dividend policy is characterizable via the minimal r-superharmonic map with respect to the underlying jump diffusion. A significant consequence of this representation is that the dynamic dividend optimization problem can be reduced to an equivalent static nonlinear minimization problem. As corollaries of this result we then showed that the associated sequential impulse dividend problem as well as the associated optimal liquidation problem are also solvable in terms of the minimal r-superharmonic map. In line with previous observations based on continuous cash flow dynamics, the values of these problems were shown to be ordered in an exceptionally strong way: the value of the singular stochastic control problem dominates the value of the associated impulse control problem which, in turn, dominates the value of the associated optimal stopping problem. However, we also demonstrated that the marginal values are ordered in an analogous way. Hence our results unambiguously indicate that increased policy flexibility has a positive effect on both the value as well as on the marginal value of the optimal policy in the jump diffusion case as well. We also stated a set of typically satisfied conditions under which the values of the considered dividend optimization problems can be sandwiched between the values of two associated dividend optimization problems based on a continuous cash flow dynamics.

Our results generalize the results obtained previously in literature for linear diffusions and demonstrate the strong similarities between the behavior of linear diffusions and spectrally negative jump diffusions with geometric jumps and natural boundaries. From an applied point of view, spectrally negative processes are a very relevant generalization of processes with continuous paths, as they allow the incorporation of discontinuous unanticipated negative shocks into the modeling of the underlying cash flow dynamics. Taking this downside risk into account can be viewed as essential for any model meant to be used in prudent risk management.

While our model allows fairly rich jump structures, as we are reasonably free to choose the distribution of the relative jump sizes, it assumes that the jump component enters the defining stochastic differential equation in geometric form and that the boundaries are natural. It might be of interest to know whether, and to what extent, our results could be extended to encompass more general forms of the jump component and different boundary behaviors [a partial answer is given by Loeffen (2008), who derives sufficient conditions for optimality of barrier strategies in the case of a spectrally negative Lévy process]. Such extensions are out of the scope of the present study and are, therefore, left for future research.

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PAPER III

Luis H. R. Alvarez – Teppo A. Rakkolainen: *On singular stochastic control and optimal stopping of spectrally negative jump diffusions*, 2008, to appear in *Stochastics: An International Journal of Probability and Stochastic Processes*.

ON SINGULAR STOCHASTIC CONTROL AND OPTIMAL STOPPING OF SPECTRALLY NEGATIVE JUMP DIFFUSIONS

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ABSTRACT
We consider a broad class of singular stochastic control problems of spectrally negative jump diffusions in the presence of potentially nonlinear state-dependent exercise payoffs. We analyze these problems by relying on associated variational inequalities and state a set of sufficient conditions under which the value of the considered problems can be explicitly derived in terms of the increasing minimal r-harmonic map. We also present a set of inequalities bounding the value of the optimal policy and prove that increased policy flexibility increases both the value of the optimal strategy as well as the rate at which this value grows.

JEL classification: C61; G35;

MSC: 93E20; 60G51; 60J60;

Keywords: jump diffusion; optimal stopping; singular control; variational inequalities

1 INTRODUCTION

One of the principal lessons in the literature on real options is that policy flexibility creates value (for example, investment timing or operational flexibility in general, cf. [15]). This result is intuitively clear, since increased policy flexibility can typically be interpreted as an enlargement of the class of admissible policies and, therefore, as a larger opportunity set from which the applied policy can be chosen. However, even though the positivity of the sign of the relationship between increased policy flexibility and value is intuitive, it is not beforehand obvious how increased policy flexibility affects the rate at which the value of the optimal policy grows as a function of the current state of the controlled stochastic dynamics. This question was recently addressed in [5] within a dividend optimization framework based on continuous diffusion processes modeling the underlying cash flow dynamics. They established that increased policy flexibility unambiguously raises both the value of the optimal policy as well as the rate at which it grows in the typically applied case where the admissible controls are assumed to be non-negative, non-decreasing, right-continuous, and adapted. The analysis of [5] was subsequently extended to the discontinuous jump diffusion setting in [4]. They reconsidered the class of cash flow management problems addressed in [5] in the case where the underlying cash flow is modeled as a spectrally negative jump diffusion and established that increased flexibility has a positive effect both on the value and on the marginal value of the optimal policy in that setting as well. Even though the findings of [5] and [4] are interesting from the point of view of cash flow management applications (especially under risk neutrality) their results are based on linear payoffs. Hence, their analysis overlooks the potentially significant effect of nonlinear payoffs arising, for example, in studies considering cash flow management in the presence of risk aversion or in studies analyzing optimal harvesting of renewable resources in the presence of variable stock dependent prices.

For reasons of mathematical convenience, processes with continuous paths have been more popular in modeling the underlying stochastic dynamics of cash flow management problems than processes with discontinuities. The problem of optimal dividend distribution policy in the continuous path setting has been considered, for example, in [27] and [19] (among many others). From a risk management point of view, the models with continuous sample paths fail to capture the downside risk: the possibility of a large instantaneous drop in the asset value. The effects of stock market crashes demonstrate the potential significance of this discontinuous risk and indicate that in prudent risk management this risk should not be overlooked. Moreover, especially in insurance applications many quantities of interest are jump processes. Furthermore, the celebrated *bad news principle* introduced in [10] tells us that the market reacts to new information in an asymmetrical way: the impact of bad news is considerably more dramatic than that of good news. In recent years these considerations have led to an increased interest and activity in studying models with stochastic dynamics allowing jumps, and several results have been obtained. Because of the relative tractability of the ensuing models, by far the most popular choice of dynamics has been the Lévy process or its exponential. Our main object of interest in this study is the *optimal dividend problem*, and of the many fairly recent results in this area, we wish to mention particularly [24] and [8], where the underlying dynamics are characterized by a finite activity Lévy process, and the recent papers by [7] and [20], where the dynamics are assumed to be characterizable via a spectrally negative Lévy process and the fluctuation theory of such Lévy processes is then used to solve the optimal policies. For a taste of the by now extensively studied *optimal stopping* and *option pricing* applications in the context of general and one-sided arithmetic and geometric Lévy processes, see [17], [18], [21], [22], [6], [13] and [23]. In a recent preprint [9] venture capital investments in a jump diffusion model are optimized. In [16], transforms applicable to solving many econometric and valuation problems for affine jump diffusions have been considered.

It is somewhat surprising that despite the increased interest in models based on Lévy processes and recognition of the importance of downside risk in prudent risk management, the tools provided by the classical theory of diffusions and minimal excessive maps have largely been neglected in the analysis of one-dimensional jump diffusion models (an exception is [23]). However, the results allowing the reduction of the dynamic problem to static optimization, which have been derived for diffusions via the classical theory in [1], also hold true for spectrally negative jump diffusions (modulo some conditions). This has already been demonstrated for optimal stopping problems in [3] and for singular control problems with a linear payoff in [4].

Motivated by the arguments stated above, our objective in this study is to consider, within a relatively general spectrally negative jump-diffusion setting, the impact of increased policy flexibility on both the value as well as on the marginal value of the optimal policy in the presence of a nonlinear and state-dependent payoff. To be precise, we consider the value of the *singular stochastic control problem*

$$J(x) = \sup_{Z \in \Lambda} \mathbb{E}_x \int_0^\infty e^{-rs} g'(X_s^Z) dZ_s, \tag{1}$$

(where Λ denotes the class of non-negative, non-decreasing, right-continuous, and adapted processes), and the value of the *optimal stopping problem*

$$V(x) = \sup_{\tau \in \mathscr{T}} \mathbb{E}_x \left[e^{-r\tau} g(X_\tau) \right], \tag{2}$$

(where $\mathscr{T}$ is the appropriate set of stopping times), when the underlying process X is a jump diffusion with dynamics described by

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t - \int_{(0,1)} X_t z \tilde{N}(dt, dz), \tag{3}$$

$X_0 = x > 0$, where $\tilde{N}(dt, dz)$ is a compensated Poisson random measure. Along the lines of the studies [5] and [4] we first present a class of functionals constituting the solutions of two associated boundary value problems and establish an ordering between these values. Given that these functionals depend on the arbitrary boundary, we then investigate those circumstances under which these functionals attain a global maximum as functions of the boundary by relying on ordinary static optimization techniques. We then state both in a continuous diffusion setting as well as in the general discontinuous jump-diffusion setting a set of sufficient conditions under which these extreme values of the associated functionals constitute the values of the considered stochastic control problems. In accordance with the findings based on linear payoffs, we find that increased policy flexibility increases both the value and the marginal value of the optimal policy in the nonlinear setting as well. Moreover, in accordance with the observations of [3] on optimal stopping of spectrally negative jump-diffusions, our results indicate within a relatively broad setting that the value of the optimal singular stochastic policy of the underlying discontinuous jump-process can be typically sandwiched between the values of two associated singular control problems of a continuous diffusion. This result is of interest since it characterizes circumstances under which the value of the optimal policy can be attained with the value of the optimal policy defined with respect to the associated continuous diffusion by appropriately adjusting the discount rate (i.e. makes possible the interpretation of downside jump risk as a discount rate effect).

The content of this study is as follows. In section two we present in detail the assumptions on the considered underlying jump diffusion and the associated continuous diffusion process. Furthermore, we define some classes of auxiliary functionals related to the variational inequalities associated to the considered problems and state a technical lemma characterizing how the functionals associated to the underlying jump diffusion are related to the functionals associated to continuous diffusion models. The proof of this lemma is relegated to Appendix. In section three we then begin to analyze the considered functionals in the continuous diffusion setting and establish a set of conditions under which these functionals can be interpreted as the values of associated singular stochastic control problems. The results of this section are finally extended to the jump diffusion case in section four. In section five we illustrate numerically our findings by relying on a logistic jump-diffusion model. Some concluding comments are stated in section six.

2 BASIC SETUP AND ASSUMPTIONS

The main objective of this study is to analyze the impact of a state dependent yield on the value maximizing singular stochastic control policy of a spectrally negative jump diffusion and in this way extend previous findings based on a constant yield. In order to accomplish this task, we assume that the controlled jump diffusion evolves in the absence of interventions according to a Lévy diffusion defined on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ equipped with the natural filtration $\mathbb{F} = \{\sigma(X_s : s \leq t)\}_{t \in \mathbb{R}_+}$. The stochastically fluctuating dynamics of the controlled process are assumed to be governed in the absence of interventions on the state space $\mathscr{I} = (0, \infty)$ by the stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t - \int_{(0,1)} X_t z \tilde{N}(dt, dz), \tag{4}$$

$X_0 = x > 0$, where $\tilde{N}(dt, dz)$ is a compensated Poisson random measure with the associated Lévy measure $\nu = \lambda \mathfrak{m}$, and $\mathfrak{m}$ is the jump size distribution, which is assumed to have a density $f_{\mathfrak{m}} \in C((0,1))$. Notice that we are here assuming that the total mass of ν is finite, or, equivalently, that the discontinuous component of the driving Lévy process can be described as a (compensated) compound Poisson process – however, some of the results we shall derive hold also in the case when the total mass of ν is infinite, provided that ν possesses a density in $C((0,1))$, and we will return to this matter at the end of section four. The drift coefficient $\mu(x)$ and the volatility coefficient $\sigma(x) > 0$ are, in turn, assumed to satisfy the usual conditions for the existence of a unique adapted cádlág solution $X \in L^2(\mathbb{P})$ of (4) (Lipschitz continuity and at most linear growth, see [29] Theorem 1.19). In addition, we assume that $\mu(x)$ is continuously differentiable.

For the sake of simplicity, we assume that the boundaries 0 and ∞ of the state space of X_t are unattainable. In light of this assumption it is clear that in the present setting even though the underlying jump diffusion may be expected to tend towards its boundaries, it is never expected to hit them in finite time (i.e. $\tau_0 = \infty$ a.s.). Hence, the considered jump-diffusion cannot vanish nor become infinitely large exogenously. As usually, we define the class $\mathscr{L}^1$ as the space of measurable mappings $h : \mathscr{I} \mapsto \mathbb{R}$ satisfying the inequality

$$\mathbb{E}_x \int_0^\infty e^{-rs} |h(X_s)| ds < \infty, \tag{5}$$

where $r > 0$ denotes the constant discount rate.

The negative coefficient of the jump part in (4) implies that the process is spectrally negative: even though the process can decrease discontinuously, it can increase only continuously. This spectral negativity will play a crucial role in our analysis. The following assumption is made:

A1. X is *regular* in the sense that for all $x, y \in \mathscr{I}$ it holds that $\mathbb{P}_x(\tau_y < \infty) = 1$, where $\tau_y = \inf\{t > 0 : X_t \geq y\}$.

Assumption *A1* ascertains the a.s. finiteness of the first exit time τ_u of X from any interval of form $(0,u)$ with $u < \infty$.

The integro-differential operator associated to the jump-diffusion X is defined for sufficiently smooth mappings $f(x)$ by

$$\begin{aligned}(\mathscr{G}f)(x) = &\frac{1}{2}\sigma^2(x) f''(x) + \mu(x) f'(x) \\ &+ \lambda \int_{(0,1)} \{f(x - xz) - f(x) + xzf'(x)\} \mathfrak{m}(dz).\end{aligned} \tag{6}$$

We will make use of the notation $\mathscr{G}_r u = \mathscr{G}u - ru$ and assume

A2. There exists an increasing solution $\psi \in C^2(\mathscr{I})$ of $\mathscr{G}_r \psi = 0$ such that $\psi(0) = 0$.

By virtue of Lemma 3.2 in [3] such increasing solution whenever it exists is unique up to a multiplicative constant. With regard to the existence of such a solution, in [14] it is shown that in the case of an (arithmetic) Lévy process with a nonzero Gaussian coefficient the solution (which in this particular case is called *r-scale function*) belongs to $C^2(\mathscr{I})$. Moreover, in [26] such a solution is constructed by a Frobenius type method for a (from an applied point of view) reasonably extensive class of jump diffusions. However, it is worth mentioning that the smoothness of the solution may present some problems in a completely general setting.

We define a differential operator associated with $\mathscr{G}_r$ for $f \in C^2(\mathscr{I})$ by

$$(\tilde{\mathscr{A}}_\theta f)(x) = \frac{1}{2}\sigma^2(x) f''(x) + \tilde{\mu}(x) f'(x) - \theta f(x), \tag{7}$$

where $\theta \in (0,\infty)$ and

$$\tilde{\mu}(x) = \mu(x) + \lambda x \cdot \int_{(0,1)} z\mathfrak{m}(dz) = \mu(x) + \lambda \bar{z} x. \tag{8}$$

This operator is related to the continuous diffusion $\tilde{X}$ given by

$$d\tilde{X}_t = \tilde{\mu}(\tilde{X}_t)dt + \sigma(\tilde{X}_t)dW_t. \tag{9}$$

Along the lines of our previous notation, we denote as $\tilde{\psi}_\theta(x)$ the increasing fundamental solution of the ordinary linear second order differential equation $(\tilde{\mathscr{A}}_\theta u)(x) = 0$ (for a comprehensive characterization of these mappings, see [12], p. 33). As we will later demonstrate, the mappings $\tilde{\psi}_r(x)$ and $\tilde{\psi}_{r+\lambda}(x)$ can be applied for providing useful inequalities concerning the considered stochastic control problems. We also denote as

$$\tilde{S}'(x) = \exp\left(-\int \frac{2\tilde{\mu}(x)}{\sigma^2(x)} dx\right)$$

the density of the scale function and as $\tilde{m}'(x) = 2/(\sigma^2(x)\tilde{S}'(x))$ the density of the speed measure of the diffusion $\tilde{X}$.

Recall that we were interested in problems (1) and (2). We assume that the mapping $g : \mathscr{I} \mapsto \mathbb{R}_+$ is twice continuously differentiable, non-decreasing, and satisfies the limiting inequalities $\lim_{x\downarrow 0} g(x) = 0 < \lim_{x\to\infty} g(x)$. Now we can present two classes of functionals associated to the considered stochastic control problems. Given our assumptions on the function $g(x)$ we define the continuously differentiable mappings $H : \mathscr{I}^2 \mapsto \mathbb{R}$ and $\tilde{H}_\theta : \mathscr{I}^2 \mapsto \mathbb{R}$ as

$$H(x,y) = \begin{cases} g(x) - g(y) + \psi(y)\frac{g'(y)}{\psi'(y)} & x \geq y \\ \psi(x)\frac{g'(y)}{\psi'(y)} & x < y \end{cases} \tag{10}$$

and

$$\tilde{H}_\theta(x,y) = \begin{cases} g(x) - g(y) + \tilde{\psi}_\theta(y)\frac{g'(y)}{\tilde{\psi}'_\theta(y)} & x \geq y \\ \tilde{\psi}_\theta(x)\frac{g'(y)}{\tilde{\psi}'_\theta(y)} & x < y. \end{cases} \tag{11}$$

Analogously, we define the continuous mappings $K : \mathscr{I}^2 \mapsto \mathbb{R}$ and $\tilde{K}_\theta : \mathscr{I}^2 \mapsto \mathbb{R}$ as

$$K(x,y) = \begin{cases} g(x) & x \geq y \\ g(y)\frac{\psi(x)}{\psi(y)} & x < y \end{cases} \tag{12}$$

and

$$\tilde{K}(x,y) = \begin{cases} g(x) & x \geq y \\ g(y)\frac{\tilde{\psi}_\theta(x)}{\tilde{\psi}_\theta(y)} & x < y. \end{cases} \tag{13}$$

It is now worth pointing out that for a given fixed $y \in \mathscr{I}$ the functions $x \mapsto H(x,y)$ and $x \mapsto \tilde{H}_\theta(x,y)$ satisfy the variational equalities

$$\begin{aligned} (\mathscr{G}_r H)(x,y) &= 0, \quad x < y \\ \partial_x H(x,y) &= g'(x), \quad x \geq y \end{aligned}$$

and

$$\begin{aligned} (\tilde{\mathscr{A}}_\theta \tilde{H}_\theta)(x,y) &= 0, \quad x < y \\ \partial_x \tilde{H}_\theta(x,y) &= g'(x), \quad x \geq y, \end{aligned}$$

respectively. Analogously, we also notice that for a given fixed $y \in \mathscr{I}$ the functions $x \mapsto K(x,y)$ and $x \mapsto \tilde{K}_\theta(x,y)$ satisfy the variational equalities

$$\begin{aligned} (\mathscr{G}_r K)(x,y) &= 0, \quad x < y \\ K(x,y) &= g(x), \quad x \geq y \end{aligned}$$

and

$$
\begin{aligned}
(\tilde{\mathscr{A}_\theta}\tilde{K}_\theta)(x,y) &= 0, \quad x<y \\
\tilde{K}_\theta(x,y) &= g(x), \quad x \geq y,
\end{aligned}
$$

respectively. As we will later observe in our analysis, these functions can be applied for solving the variational inequalities $\max\{(\mathscr{G}_r v)(x), g(x)-v(x)\}=0$, $\max\{(\mathscr{G}_r v)(x), g'(x)-v'(x)\}=0$, $\max\{(\tilde{\mathscr{A}_\theta} v)(x), g(x)-v(x)\}=0$, and $\max\{(\tilde{\mathscr{A}_\theta} v)(x), g'(x)-v'(x)\}=0$ associated to a relatively broad class of singular stochastic control and optimal stopping problems of the underlying stochastic processes.

We end this section by stating the following technical result, which extends previous observations in [4] on linear payoffs to our more complex setting and characterizes how the values of the mappings $\tilde{H}_\theta(x,y)$ and $\tilde{K}_\theta(x,y)$ can be applied for bounding the values of the mappings $H(x,y)$ and $K(x,y)$.

Lemma 2.1. *(A) For all $x \in \mathscr{I}$ it holds that $\tilde{H}_{r+\lambda}(x,y) \leq H(x,y) \leq \tilde{H}_r(x,y)$. Thus, $\sup_{y\in\mathscr{I}} \tilde{H}_{r+\lambda}(x,y) \leq \sup_{y\in\mathscr{I}} H(x,y) \leq \sup_{y\in\mathscr{I}} \tilde{H}_r(x,y)$ provided that the suprema exist.*
(B) For all $x \in \mathscr{I}$, $\tilde{K}_{r+\lambda}(x,y) \leq K(x,y) \leq \tilde{K}_r(x,y)$. Thus, $\sup_{y\in\mathscr{I}} \tilde{K}_{r+\lambda}(x,y) \leq \sup_{y\in\mathscr{I}} K(x,y) \leq \sup_{y\in\mathscr{I}} \tilde{K}_r(x,y)$ provided the suprema exist. Moreover,

$$
g'(x)\frac{\tilde{\psi}_{r+\lambda}(x)}{\tilde{\psi}'_{r+\lambda}(x)} - g(x) \leq g'(x)\frac{\psi(x)}{\psi'(x)} - g(x) \leq g'(x)\frac{\tilde{\psi}_r(x)}{\tilde{\psi}'_r(x)} - g(x)
$$

for all $x \in \mathscr{I}$.

Proof. See Appendix. □

Lemma 2.1 states two interesting inequalities characterizing how the value of the function $H(x,y)$ can be sandwiched between the values $\tilde{H}_{r+\lambda}(x,y)$ and $\tilde{H}_r(x,y)$.

3 THE CONTINUOUS DIFFUSION CASE

We begin our analysis by focusing on the case where the underlying controlled dynamics evolve according to the continuous diffusion $\tilde{X}$ described by the stochastic differential equation (9). Before proceeding in our analysis, we assume that the following assumptions are satisfied throughout this section.

(A1) The boundaries of the state-space are natural for $\tilde{X}$.

(A2) $(\tilde{\mathscr{A}}_\theta g) \in \mathscr{L}^1$.

(A3) There is a unique threshold $\hat{x}_\theta \in \mathscr{I}$ so that $(\tilde{\mathscr{A}}_\theta g)(x)$ is increasing on $(0, \hat{x}_\theta)$ and decreasing on $(\hat{x}_\theta, \infty)$.

(A4) The mapping $(\tilde{\mathscr{A}}_\theta g)(x)$ satisfies the limiting inequalities $\lim_{x \downarrow 0}(\tilde{\mathscr{A}}_\theta g)(x) \geq 0 > \lim_{x \to \infty}(\tilde{\mathscr{A}}_\theta g)(x)$.

In light of these assumptions, we are now in position to prove our first result on the considered class of control problems.

Theorem 3.1. *(A) There is a unique threshold* $\tilde{x}_\theta^* = \operatorname{argmax}\{g'(x)/\tilde{\psi}_\theta'(x)\}$ *satisfying the ordinary first order condition* $g''(\tilde{x}_\theta^*)\tilde{\psi}_\theta'(\tilde{x}_\theta^*) = g'(\tilde{x}_\theta^*)\tilde{\psi}_\theta''(\tilde{x}_\theta^*)$ *so that* $\tilde{H}_\theta(x, \tilde{x}_\theta^*) \geq \tilde{H}_\theta(x, y)$ *and* $\partial_x \tilde{H}_\theta(x, \tilde{x}_\theta^*) \geq \partial_x \tilde{H}_\theta(x, y)$ *for all* $(x, y) \in \mathscr{I}^2$*. Moreover,*

$$\partial_x \tilde{H}_\theta(x, \tilde{x}_\theta^*) = \tilde{\psi}_\theta'(x) \sup_{y \geq x} \left\{ \frac{g'(y)}{\tilde{\psi}_\theta'(y)} \right\}.$$

(B) There is a unique threshold $\tilde{z}_\theta^* = \operatorname{argmax}\{g(x)/\tilde{\psi}_\theta(x)\}$ *satisfying the ordinary first order condition* $g'(\tilde{z}_\theta^*)\tilde{\psi}_\theta(\tilde{z}_\theta^*) = g(\tilde{z}_\theta^*)\tilde{\psi}_\theta'(\tilde{z}_\theta^*)$ *so that* $\tilde{K}_\theta(x, \tilde{z}_\theta^*) \geq \tilde{K}_\theta(x, y)$ *for all* $(x, y) \in \mathscr{I}^2$*. Moreover,*

$$\tilde{K}_\theta(x, \tilde{z}_\theta^*) = \tilde{\psi}_\theta(x) \sup_{y \geq x} \left\{ \frac{g(y)}{\tilde{\psi}_\theta(y)} \right\}.$$

(C) $\tilde{H}_\theta(x, \tilde{z}_\theta^*) = \tilde{K}_\theta(x, \tilde{z}_\theta^*)$*,* $\partial_x \tilde{H}_\theta(x, \tilde{z}_\theta^*) = \partial_x \tilde{K}_\theta(x, \tilde{z}_\theta^*)$*,* $\tilde{H}_\theta(x, \tilde{x}_\theta^*) \geq \tilde{K}_\theta(x, \tilde{z}_\theta^*)$*, and* $\partial_x \tilde{H}_\theta(x, \tilde{x}_\theta^*) \geq \partial_x \tilde{K}_\theta(x, \tilde{z}_\theta^*)$ *for all* $x \in \mathscr{I}$*.*
(D) For all $x \in \mathscr{I}$ *it holds that* $\tilde{H}_{r+\lambda}(x, \tilde{x}_{r+\lambda}^*) \leq \sup_{y \in \mathscr{I}} H(x, y) \leq \tilde{H}_r(x, \tilde{x}_r^*)$ *and* $\tilde{K}_{r+\lambda}(x, \tilde{z}_{r+\lambda}^*) \leq \sup_{y \in \mathscr{I}} K(x, y) \leq \tilde{K}_r(x, \tilde{z}_r^*)$*.*

Proof. (A) As was established in Corollary 3.2 of [2] our assumptions imply that for all $x \in \mathscr{I}$ we have

$$\frac{g''(x)}{\tilde{S}'(x)}\tilde{\psi}_\theta'(x) - \frac{\tilde{\psi}_\theta''(x)}{\tilde{S}'(x)}g'(x) = \frac{2r\tilde{I}(x)}{\sigma^2(x)} \tag{14}$$

where the functional $\tilde{I} : \mathscr{I} \mapsto \mathbb{R}$ is defined as

$$\tilde{I}(x) = \int_0^x \tilde{\psi}_\theta(y)[(\tilde{\mathscr{A}}_\theta g)(x) - (\tilde{\mathscr{A}}_\theta g)(y)]\tilde{m}'(y)dy. \tag{15}$$

It is clear from our assumptions on $(\tilde{\mathscr{A}}_\theta g)(x)$ that $\tilde{I}(x)$ is increasing on $(0,\hat{x}_\theta)$ and decreasing on $(\hat{x}_\theta,\infty)$. Moreover, since the integrand in (15) is positive as long as $x\in(0,\hat{x}_\theta)$ we find that $\tilde{I}(x)>0$ for all $x\in(0,\hat{x}_\theta)$. However, since

$$\tilde{I}(\tilde{x}_0) = -\int_0^{\tilde{x}_0} \tilde{\psi}_\theta(y)(\tilde{\mathscr{A}}_\theta g)(y)\tilde{m}'(y)dy < 0,$$

where $\tilde{x}_0$ is the unique root of equation $(\tilde{\mathscr{A}}_\theta g)(x)=0$, we find by invoking the continuity and monotonicity of $\tilde{I}(x)$ that equation $\tilde{I}(x)=0$ has a unique root $\tilde{x}^*_\theta\in(\hat{x}_\theta,x_0)$ so that $\tilde{I}(x)\gtreqless 0$ for all $x\gtreqless\tilde{x}^*_\theta$. Noticing from (15) that

$$\frac{d}{dx}\left[\frac{g'(x)}{\tilde{\psi}'_\theta(x)}\right] = \frac{2r\tilde{S}'(x)}{\tilde{\psi}'^2_\theta(x)\sigma^2(x)}\tilde{I}(x)$$

finally proves that $\tilde{x}^*_\theta = \text{argmax}\{g'(x)/\tilde{\psi}'_\theta(x)\}$. Since

$$\frac{\partial\tilde{H}_\theta}{\partial y}(x,y) = \min(\tilde{\psi}_\theta(x),\tilde{\psi}_\theta(y))\frac{d}{dy}\left[\frac{g'(y)}{\tilde{\psi}'_\theta(y)}\right]$$

we observe that $\tilde{H}_\theta(x,\tilde{x}^*_\theta)\geq\tilde{H}_\theta(x,y)$ for all $(x,y)\in\mathscr{I}^2$. Establishing that $\partial_x\tilde{H}_\theta(x,\tilde{x}^*_\theta)\geq\partial_x\tilde{H}_\theta(x,y)$ for all $(x,y)\in\mathscr{I}^2$ is entirely analogous. Finally, the representation

$$\partial_x\tilde{H}_\theta(x,\tilde{x}^*_\theta) = \tilde{\psi}'_\theta(x)\sup_{y\geq x}\left\{\frac{g'(y)}{\tilde{\psi}'_\theta(y)}\right\}$$

follows directly from the proven monotonicity properties of the mapping $g'(x)/\tilde{\psi}'_\theta(x)$. Part (B), in turn, follows directly form Lemma 4.1 in [2]. Finally, part (C) and (D) follow from our Lemma 2.1. □

In light of our results in Theorem 3.1 it is naturally of interest to investigate the properties of the extremal values associated to the critical thresholds. This is accomplished in the following.

Theorem 3.2. *(A) The function $\tilde{J}_\theta(x)=\tilde{H}_\theta(x,\tilde{x}^*_\theta)$ satisfies the variational inequalities*

$$\max\{(\tilde{\mathscr{A}}_\theta\tilde{J}_\theta)(x), g'(x)-\tilde{J}'_\theta(x)\}=0.$$

Especially,

$$\tilde{J}_\theta(x) = \sup_{\tilde{Z}\in\Lambda}\mathbb{E}_x\int_0^\infty e^{-\theta s}g'(\tilde{X}^{\tilde{Z}}_s)d\tilde{Z}_s, \tag{16}$$

where

$$d\tilde{X}^{\tilde{Z}}_t = \tilde{\mu}(\tilde{X}^{\tilde{Z}}_t)dt+\sigma(\tilde{X}^{\tilde{Z}}_t)dW_t - d\tilde{Z}_t, \tilde{X}^{\tilde{Z}}_{0-}=x, \tag{17}$$

and Λ denotes the class of non-negative, non-decreasing, right-continuous, and adapted processes.
(B) The function $\tilde{V}_\theta(x) = \tilde{K}_\theta(x, \tilde{z}^*_\theta)$ satisfies the variational inequalities

$$\max\{(\tilde{\mathscr{A}}_\theta \tilde{V}_\theta)(x), g(x) - \tilde{V}_\theta(x)\} = 0.$$

Especially,

$$\tilde{V}_\theta(x) = \sup_{\tau \in \mathscr{T}} \mathbb{E}_x \left[e^{-\theta\tau} g(\tilde{X}_\tau) \right], \tag{18}$$

where $\mathscr{T}$ is the set of all $\mathbb{F}$-stopping times.

Proof. (A) It is clear that $(\tilde{\mathscr{A}}_\theta \tilde{J}_\theta)(x) = 0$ for all $x \in (0, \tilde{x}^*_\theta)$. On the other hand, since $(\tilde{\mathscr{A}}_\theta \tilde{J}_\theta)(\tilde{x}^*_\theta) = 0$ and $\tilde{x}^*_\theta \in (\hat{x}_\theta, \infty)$, we find that $(\tilde{\mathscr{A}}_\theta \tilde{J}_\theta)(x) < 0$ on $(\tilde{x}^*_\theta, \infty)$. Noticing that

$$\tilde{J}'_\theta(x) - g'(x) = \begin{cases} 0 & x \geq \tilde{x}^*_\theta \\ \psi'(x) \left[\frac{g'(\tilde{x}^*_\theta)}{\psi'(\tilde{x}^*_\theta)} - \frac{g'(x)}{\psi'(x)} \right] & x < \tilde{x}^*_\theta \end{cases}$$

and invoking the maximality of the value $g'(\tilde{x}^*_\theta)/\psi'(\tilde{x}^*_\theta)$ then proves that $\tilde{J}'_\theta(x) \geq g'(x)$ for all $x \in \mathscr{I}$ and that the value $\tilde{J}_\theta(x)$ satisfies the alleged variational inequalities. The interpretation of the function $\tilde{J}_\theta(x)$ as the value of the associated singular stochastic control problem (16) follows from Theorem 1 in [1]. Establishing part (B) is entirely analogous. □

Theorem 3.2 demonstrates that under our assumptions the considered functionals $\tilde{H}_\theta(x, \tilde{x}^*_\theta)$ and $\tilde{K}_\theta(x, \tilde{z}^*_\theta)$ can be interpreted as the values of associated stochastic control problems. It is worth emphasizing that part (C) of our Theorem 3.1 demonstrates that $\tilde{J}_\theta(x) \geq \tilde{V}_\theta(x)$ and $\tilde{J}'_\theta(x) \geq \tilde{V}'_\theta(x)$ for all $x \in \mathscr{I}$. Consequently, given the interpretation of these values in terms of the values of two associated singular stochastic control problems (16) and (18) we find that increased policy flexibility has a positive impact both on the value and on its growth rate in the present setting as well. The second order properties and the resulting comparative static properties of the value of the optimal policy are now characterized in the following.

Theorem 3.3. *Assume that the payoff $g(x)$ is concave. Then the value function $\tilde{J}_\theta(x)$ is concave. Moreover, increased volatility decreases the value $\tilde{J}_\theta(x)$ of the optimal policy.*

Proof. Noticing that $\psi''(x)g'(x) < g''(x)\psi'(x) \leq 0$ on $(0, \tilde{x}^*_\theta)$ proves the alleged concavity of the value function $\tilde{J}_\theta(x)$. Assume that $\sigma(x) \leq \hat{\sigma}(x)$ for all $x \in \mathscr{I}$ and denote the value of the optimal policy associated to the more volatile dynamics as $\tilde{J}^{\hat{\sigma}}_\theta(x)$. Since $\tilde{J}'_\theta(x) \geq g'(x)$ for all $x \in \mathscr{I}$ and

$$\frac{1}{2}\hat{\sigma}^2(x)\tilde{J}''_\theta(x) + \tilde{\mu}(x)\tilde{J}'_\theta(x) - \theta\tilde{J}_\theta(x) \leq \frac{1}{2}(\hat{\sigma}^2(x) - \sigma^2(x))\tilde{J}''_\theta(x) \leq 0$$

we find that $\tilde{J}_\theta(x)$ dominates the value $\tilde{J}^{\hat{\sigma}}_\theta(x)$ of the optimal policy associated to the more volatile dynamics and, therefore, that $\tilde{J}_\theta(x) \geq \tilde{J}^{\hat{\sigma}}_\theta(x)$ for all $x \in \mathscr{I}$. □

4 THE JUMP-DIFFUSION CASE

The results of the previous sections indicate that the values of the functionals $H(x,y)$ and $K(x,y)$ associated to the underlying jump-diffusion can be bounded by the corresponding functionals defined with respect to the continuous diffusion $\tilde{X}$. It is not, however, directly clear when the conclusions of our Theorem 3.1 and Theorem 3.2 can be extended to the jump-diffusion case and more analysis is needed. A set of sufficient conditions under which the considered functionals constitute the value of the associated stochastic control problems are now summarized in the following.

Theorem 4.1. *(A) Assume that the mapping $g'(x)/\psi'(x)$ attains a unique global maximum at $x^* = \text{argmax}\{g'(x)/\psi'(x)\}$ and that the mapping $J(x) = H(x,x^*)$ satisfies the inequality $(\mathscr{G}_r J)(x) \leq 0$ for all $x \in \mathscr{I}$. Then,*

$$J(x) = \sup_{Z\in\Lambda} \mathbb{E}_x \int_0^\infty e^{-rs} g'(X^Z_s)dZ_s, \tag{19}$$

where

$$dX^Z_t = \mu(X^Z_t)dt + \sigma(X^Z_t)dW_t - X^Z_t \int_{(0,1)} z\tilde{N}(dt,dz) - dZ_t, X^Z_{0-} = x, \tag{20}$$

and Λ denotes the class of non-negative, non-decreasing, right-continuous, and adapted processes. The optimal policy consists of reflecting the controlled process downwards at level x^* (see [27] pp. 39–40 for a characterization of the associated Skorohod problem). Furthermore, $J(x) \geq H(x,y)$ and $J'(x) \geq H_x(x,y)$ for all $(x,y) \in \mathscr{I}^2$, and if $g'(x)/\psi'(x)$ is non-increasing on (x^*,∞) then

$$J'(x) = \psi'(x) \sup_{y\geq x}\left[\frac{g'(y)}{\psi'(y)}\right] \tag{21}$$

(B) Assume that the mapping $g(x)/\psi(x)$ attains a unique global maximum at $z^* = \operatorname{argmax}\{g(x)/\psi(x)\}$ and that the mapping $V(x) = K(x,z^*)$ satisfies the inequality $(\mathscr{G}_r V)(x) \leq 0$ for all $x \in \mathscr{I}$. Then,

$$V(x) = \sup_{\tau \in \mathscr{T}} \mathbb{E}_x \left[e^{-r\tau} g(X_\tau) \right], \tag{22}$$

where $\mathscr{T}$ is the set of all $\mathbb{F}$-stopping times. Moreover, if $g(x)/\psi(x)$ is non-increasing on (z^*,∞) then

$$V(x) = \psi(x) \sup_{y \geq x} \left[\frac{g(y)}{\psi(y)} \right] \tag{23}$$

(C) If the conditions of part (A) and (B) are satisfied, then $J(x) \geq V(x)$ and $J'(x) \geq V'(x)$ for all $x \in \mathscr{I}$.

Proof. (A) Differentiating the function $J(x) = H(x,x^*)$ and dividing the resulting derivative with $g'(x)$ yields

$$\frac{J'(x)}{g'(x)} = \begin{cases} 1 & x \geq x^* \\ \frac{\psi'(x) g'(x^*)}{g'(x) \psi'(x^*)} & x < x^*. \end{cases}$$

The assumed maximality of x^* and the monotonicity of $g(x)$ then imply that $J'(x) \geq g'(x)$ for all $x \in \mathscr{I}$. Applying now the generalized Itô theorem (see [25] Theorem II.32) to the mapping $(t,x) \mapsto e^{-rt} J(x)$ yields

$$\begin{aligned} \mathbb{E}_x \left[e^{-rT_n} J(X^Z_{T_n}) \right] &= J(x) + \mathbb{E}_x \int_0^{T_n} e^{-rs} (\mathscr{G}_r J)(X^Z_s) ds - \mathbb{E}_x \int_0^{T_n} e^{-rs} J'(X^Z_s) dZ_s \\ &+ \mathbb{E}_x \sum_{s<T_n} e^{-rs} \left[J(X^Z_s) - J(X^Z_{s-}) - J'(X^Z_{s-}) \Delta X^Z_s \right], \end{aligned}$$

where the summation goes over the control-driven intervention dates and $T_n = n \wedge \inf\{t \geq 0 : X^Z_t \notin (1/n, n)\}$ is a sequence of almost surely finite stopping times converging to $\tau^Z_0 = \inf\{t \geq 0 : X^Z_t = 0\}$ (given the assumed unattainability of ∞). Since $(\mathscr{G}_r J)(X^Z_t) = 0$ almost surely for all t, the proposed policy increases only when $J'(X^Z_t) = g'(X^Z_t)$, and $J(X^Z_s) - J(X^Z_{s-}) = g(X^Z_s) - g(X^Z_{s-})$ at the optimal boundary, we find that

$$\begin{aligned} \mathbb{E}_x \left[e^{-rT_n} J(X^Z_{T_n}) \right] &= J(x) - \mathbb{E}_x \int_0^{T_n} e^{-rs} g'(X^Z_s) dZ_s \\ &+ \mathbb{E}_x \sum_{s<T_n} e^{-rs} \left[g(X^Z_s) - g(X^Z_{s-}) - g'(X^Z_{s-}) \Delta X^Z_s \right]. \end{aligned}$$

Noticing that the proposed singular control policy is continuous t-almost everywhere since the underlying jump-diffusion increases continuously to the boundary x^* (due to the spectrally negative jumps) finally shows that

$$\mathbb{E}_x\left[e^{-rT_n}J(X^Z_{T_n})\right] = J(x) - \mathbb{E}_x\int_0^{T_n} e^{-rs}g'(X^Z_s)dZ_s.$$

Letting $n \to \infty$ and invoking the finiteness of the value $J(x)$ on compacts then implies

$$J(x) = \mathbb{E}_x\int_0^{\tau_0^Z} e^{-rs}g'(X^Z_s)dZ_s.$$

proving the first part of our claim. Noticing that

$$\frac{\partial H}{\partial y}(x,y) = \min(\psi(x),\psi(y))\frac{d}{dy}\left[\frac{g'(y)}{\psi'(y)}\right]$$

then proves that $J(x) \geq H(x,y)$ and $J'(x) \geq H_x(x,y)$ for all $(x,y) \in \mathscr{I}^2$ and that $J'(x)$ can be expressed as in (21). Establishing part (B) is entirely analogous. Part (C) finally follows from the proof of part (A). □

Theorem 4.1 states a set of sufficient conditions under which $H(x,x^*)$ constitutes the value of an associated singular stochastic control problem and $K(x,z^*)$ constitutes the value of an associated optimal stopping problem of the underlying jump diffusion. Along the lines of our previous observations on the relationship between these values, Theorem 4.1 proves that increased policy flexibility does not only increase the value of the optimal policy, it also increases the rate at which this value grows in the presence of a state dependent yield as well.

A set of useful inequalities characterizing how the values $H(x,x^*)$ and $K(x,z^*)$ are related to the values $\tilde{J}_\theta(x)$ and $\tilde{V}_\theta(x)$ are now summarized in the following.

Lemma 4.2. *Assume that the conditions (A1)-(A4) are satisfied for $\theta = r$ and $\theta = r+\lambda$. Then, for all $x \in \mathscr{I}$ it holds*

(i) $\tilde{J}_{r+\lambda}(x) \leq \sup_{y\in\mathscr{I}} H(x,y) \leq \tilde{J}_r(x)$, *and*

(ii) $\tilde{V}_{r+\lambda}(x) \leq \sup_{y\in\mathscr{I}} K(x,y) \leq \tilde{V}_r(x)$.

Moreover,

(a) *If the conditions of part (A) of Theorem 4.1 are also satisfied, then* $\tilde{J}_{r+\lambda}(x) \leq J(x) \leq \tilde{J}_r(x)$ *for all* $x \in \mathscr{I}$.

(b) *If the conditions of part (B) of Theorem 4.1 are also satisfied, then* $\tilde{V}_{r+\lambda}(x) \leq V(x) \leq \tilde{V}_r(x)$ *for all* $x \in \mathscr{I}$ *and* $\tilde{z}^*_{r+\lambda} \leq z^* \leq \tilde{z}^*_r$.

Proof. The alleged inequalities are direct implications of Lemma 2.1, Theorem 3.2, and Theorem 4.1. □

Having stated a set of sufficient conditions under which the optimal policy constitutes a standard threshold policy, we now plan to proceed in our analysis and state a relatively general set of sufficient conditions, in terms of the infinitesimal characteristics of the underlying jump diffusion and the payoff, under which the existence and uniqueness of an optimal threshold is guaranteed. Along the lines of our previous section, we now make the following assumptions.

(B1) There is a unique threshold $\hat{x} = \text{argmax}\{(\mathscr{G}_r g)(x)\} \in \mathscr{I}$ so that $(\mathscr{G}_r g)(x)$ is increasing on $(0,\hat{x})$ and decreasing on $(\hat{x},\infty)$,

(B2) The mapping $(\mathscr{G}_r g)(x)$ satisfies the limiting inequalities $\lim_{x\downarrow 0}(\mathscr{G}_r g)(x) \geq 0 > \lim_{x\to\infty}(\mathscr{G}_r g)(x)$,

(B3) $(\mathscr{G}_r g)(x)$ is locally concave on $(\hat{x},\infty)$, and

(B4) $(\mathscr{G}_r g)(x)$ is continuously differentiable on $\mathscr{I}$.

Given these assumptions, we can now establish the following theorem, which characterizes unambiguously the monotonicity properties of the mapping $g'(x)/\psi'(x)$ on $\mathscr{I}$.

Theorem 4.3. *Assume that conditions (B1) - (B4) are satisfied, that the payoff* $g(x)$ *is concave, that* $\lim_{x\downarrow 0} g(x) = 0$, *and that* $\lim_{x\downarrow 0} g'(x)/\psi'(x) < \infty$. *Then there exists a unique threshold*

$$x^* = \text{argmax}\left\{\frac{g'(x)}{\psi'(x)}\right\} \in (\hat{x}, x_0),$$

where $x_0 := \arg\{(\mathscr{G}_r g)(x) = 0\}$, *such that*

$$\frac{d}{dx}\left[\frac{g'(x)}{\psi'(x)}\right] \gtreqqless 0, \quad x \lesseqqgtr x^*.$$

Proof. Applying the identity $(\mathscr{G}_r\psi)(x)=0$ shows that

$$\frac{d}{dx}\left[\frac{g'(x)}{\psi'(x)}\right]=\frac{2}{\sigma^2(x)\psi'(x)}\left[\frac{1}{2}\sigma^2(x)g''(x)-\frac{1}{2}\sigma^2(x)\frac{\psi''(x)}{\psi'(x)}g'(x)\right]=\frac{2I(x)}{\sigma^2(x)\psi'(x)}$$

where

$$I(x)=(\mathscr{G}_r g)(x)+rR(x)+\int_0^1\eta(x,z)\nu(dz), \tag{24}$$

$R(x)=g(x)-g'(x)\psi(x)/\psi'(x)$, and

$$\eta(x,z)=g(x)-g(x-xz)-\frac{g'(x)}{\psi'(x)}(\psi(x)-\psi(x-xz)).$$

We begin by showing that $I(x)>0$ necessarily on a neighborhood of 0. Assume that this is not the case and that there exists some $\varepsilon\in(0,\hat{x})$ such that $I(x)\leq 0$ and, therefore, such that $g'(x)/\psi'(x)$ is non-increasing for all $x\in(0,\varepsilon)$. Combining assumptions (B1) and (B2) with the observations $\lim_{x\downarrow 0}R(x)=0$,

$$R'(x)=-\psi(x)\frac{d}{dx}\left[\frac{g'(x)}{\psi'(x)}\right]\geq 0,\quad x\in(0,\varepsilon)$$

$\eta(x,0)=0$, and

$$\eta_z(x,z)=\left(\frac{g'(x-xz)}{\psi'(x-xz)}-\frac{g'(x)}{\psi'(x)}\right)x\psi'(x-xz)\geq 0,\quad x\in(0,\varepsilon)$$

shows that $I(x)>0$ for all $x\in(0,\varepsilon)$ which is a contradiction with our supposition $I(x)\leq 0$ on $(0,\varepsilon)$. Thus, we necessarily have $I(x)>0$ on a neighborhood of 0.

Second, we will show that $I(x)>0$ for all $x\in(0,\hat{x})$. Suppose to the contrary that $I(\bar{x})=0$ at some $\bar{x}<\hat{x}$. Then

$$\begin{aligned}I'(\bar{x}) &= (\mathscr{G}_r g)'(\bar{x})+rR'(\bar{x})+\int_0^1\eta_x(\bar{x},z)\nu(dz)\\ &= (\mathscr{G}_r g)'(\bar{x})+\int_0^1(1-z)\psi'(\bar{x}-\bar{x}z)\left(\frac{g'(\bar{x})}{\psi'(\bar{x})}-\frac{g'(\bar{x}-\bar{x}z)}{\psi'(\bar{x}-\bar{x}z)}\right)\nu(dz)\end{aligned}$$

The term $(\mathscr{G}_r g)'(\bar{x})$ in the above expression is positive by the assumed monotonicity of $(\mathscr{G}_r g)(x)$ on $(0,\hat{x})$. Moreover, since $\frac{g'(x)}{\psi'(x)}$ is increasing on $(0,\bar{x})$, we find that the integrand is positive as well. Hence, $I'(\bar{x})\geq 0$, which is a contradiction as $I(x)$ must be decreasing at $\bar{x}$. Thus, $I(x)>0$ for all $x\in(0,\hat{x})$.

Third, we show that there exists $y\in(\hat{x},x_0)$ such that $I(y)<0$. Suppose that this is not the case and that $I(x)\geq 0$ for all $x\leq x_0$. Then $\eta(x,z)\leq 0$ and

$R(x) < 0$ for all $x \in (0, x_0)$ since $\eta(x,0) = 0$, $\eta_z(x,z) < 0$, and $R'(x) < 0$ on $(0, x_0)$ by the monotonicity of $g'(x)/\psi'(x)$. This implies that

$$I(x_0) = rR(x_0) + \int_0^1 \eta(x_0, z)\nu(dz) < 0 \tag{25}$$

which is a contradiction and shows that there exists $x^* \in (\hat{x}, x_0)$ such that $I(x^*) = 0$ and $I(x) < 0$ on some interval $(x^*, x^* + \varepsilon)$.

Finally, to establish uniqueness of x^*, we show that once negative, $I(x)$ cannot become positive at a higher state. We proceed again by contradiction. Suppose there exists $y^* \in (x^*, \infty)$ such that $I(y^*) = 0$. The desired contradiction is obtained, if we can show that we necessarily have $I'(y^*) < 0$, as a negative quantity cannot *decrease* to 0. Consider

$$\begin{aligned} I'(y^*) &= (\mathcal{G}_r g)'(y^*) + \\ &+ \int_0^1 \left\{ (1-z)\left(\tfrac{g'(y^*)}{\psi'(y^*)} \psi'(y^* - y^* z) - g'(y^* - y^* z) \right) \right\} \nu(dz). \end{aligned} \tag{26}$$

We can split the second term above into two parts as follows. Denote

$$W(x,y) := \frac{g'(x)}{\psi'(x)} \psi'(y) - g'(y)$$

and define for all $x \in (x^*, y^*]$

$$\tilde{x} := \sup \left\{ y < x^* \, : \, \frac{g'(y)}{\psi'(y)} \le \frac{g'(x)}{\psi'(x)} \right\} \tag{27}$$

and consider

$$\begin{aligned} &\int_0^{1-\tilde{x}/x} (1-z) W(x, x - xz)\nu(dz) + \\ &+ \int_{1-\tilde{x}/x}^1 (1-z) W(x, x - xz)\nu(dz). \end{aligned} \tag{28}$$

The first integral corresponds to "small" jumps and is always negative since for $y \in (\tilde{x}, x)$ we have $\frac{g'(x)}{\psi'(x)} \le \frac{g'(y)}{\psi'(y)}$. The second integral, corresponding to "large" jumps is always positive. To obtain the desired contradiction, it is enough to show that the second integral in (28) is a decreasing function of x, because then by local concavity of $(\mathcal{G}_r g)'(x)$ on $(\hat{x}, \infty)$,

$$I'(y^*) < (\mathcal{G}_r g)'(x^*) + \int_{1-\tilde{x}^*/x^*}^1 (1-z) W(x^*, x^* - x^* z)\nu(dz) = I(x^*) < 0$$

(note here that $\tilde{x} \uparrow x^*$ as $x \downarrow x^*$). To show the required monotonicity, apply the Leibniz rule to the second term of (28) to obtain its derivative as

$$\int_{1-\tilde{x}/x}^1 \left\{ (1-z) \left(D_x \left[\frac{g'(x)}{\psi'(x)} \right] \psi'(x - xz) + (1-z) W_y(x, x - xz) \right) \right\} \nu(dz)$$

by the definition of $\tilde{x}$. The first part of the integrand is negative, because $x \in (x^*, y^*)$ and on this interval $I(x) < 0$. By concavity of $g(x)$, for all $x < x^*$

$$\frac{g'(x)}{\psi'(x)}\psi''(x) \leq g''(x) \leq 0,$$

and hence $\psi''(x) \leq 0$ on $(0, x^*)$. Then, for all $y \in (0, \tilde{x})$

$$W_y(x,y) \leq W_y(y,y) \leq 0.$$

Thus the second term of (28) is decreasing on (x^*, y^*) and $I'(y^*) < 0$, which completes the proof of our theorem. □

Theorem 4.3 presents a set of relatively general conditions under which there is a unique interior threshold x^* at which the ratio $g'(x)/\psi'(x)$ is maximized. Along the lines of the findings of our sufficiency Theorem 4.1 we are now in position to establish our key result on the considered class of singular stochastic control problems of jump diffusions.

Theorem 4.4. *Assume that the conditions of Theorem 4.3 are satisfied. Then $H(x,x^*)$ is r-superharmonic for the underlying jump diffusion X_t and, therefore, constitutes the value of the associated singular stochastic control problem* (19). *Moreover, $H(x,x^*)$ is strictly concave and increased volatility decreases its value.*

Proof. Since $v(x) = H(x,x^*)$ is r-harmonic for the underlying jump diffusion X_t on $(0,x^*)$ it is sufficient to consider the functional $(\mathscr{G}_r v)(x)$ on (x^*, ∞). Noticing that $v(x) = g(x) - R(x^*)$, where $R(x) = g(x) - \frac{g'(x)}{\psi'(x)}\psi(x)$ yields

$$(\mathscr{G}_r v)'(x) = (\mathscr{G}_r g)'(x) + \int_{1-\frac{x^*}{x}}^{1} \left(\frac{g'(x^*)}{\psi'(x^*)}\psi'(x-xz) - g'(x-xz) \right) (1-z)\nu(dz).$$

In this expression $(\mathscr{G}_r g)'(x) < 0$ since x^* is attained on the set $(\hat{x}, \infty)$ where $(\mathscr{G}_r g)(x)$ is decreasing. On the other hand, since

$$g''(t) > \psi''(t)\frac{g'(t)}{\psi'(t)} > \psi''(t)\frac{g'(x^*)}{\psi'(x^*)}$$

for all $t \in (0,x^*)$, we find that the integrand is decreasing as a function of x as well and, therefore, that $(\mathscr{G}_r v)'(x) < (\mathscr{G}_r v)'(x^*) = 0$ for all $x \in (x^*, \infty)$. Hence, $(\mathscr{G}_r v)'(x) < 0$ for all $x \in (x^*, \infty)$. Combining this observation with the identity $(\mathscr{G}_r v)(x) = 0$ on $(0, x^*)$ then proves that $(\mathscr{G}_r v)(x) \leq 0$ for all $x \in \mathscr{I}$

and, therefore, that the proposed value function is r-superharmonic for the underlying jump diffusion. $H(x,x^*)$ now constitutes the value of the associated singular stochastic control problem (19) by part (A) of Theorem 4.1.

It is clear from the proof of our Theorem 4.3 that the proposed value function is concave on $\mathscr{I}$ by the assumed concavity of the payoff $g(x)$. Establishing now the negativity of the sign of the relationship between increased volatility and the value of the optimal policy is analogous with the proof of Theorem 3.3. □

An interesting direct consequence of Theorem 4.4 presenting a natural ordering between the values of the considered control problems is now summarized in our next corollary.

Corollary 4.5. *Assume that the conditions of Theorem 4.3 are satisfied. Then* $J(x) \geq V(x)$ *for all* $x \in \mathscr{I}$.

Proof. It is clear from the proof of Theorem 4.4 that $(\mathscr{G}_r J)(x) \leq 0$ for all $x \in \mathscr{I}$. Moreover, since $J'(x) \geq g'(x)$ for all $x \in \mathscr{I}$ we find by integrating over the set $(0,x)$ and invoking the limits $\lim_{x\downarrow 0} J(x) = \lim_{x\downarrow 0} g(x) = 0$ that $J(x) \geq g(x)$ for all $x \in \mathscr{I}$. Hence, $J(x)$ constitutes a r-excessive majorant of the payoff $g(x)$. Since $V(x)$ is the least of these majorants, we find that $J(x) \geq V(x)$. □

Having characterized circumstances under which the considered singular stochastic control problem is solved by a standard threshold policy and under which the value of the optimal policy dominates the value of the associated optimal stopping problem, it is natural of interest to investigate when the optimal stopping problem admits a similar solution in the general setting as well. This is accomplished in our next theorem.

Theorem 4.6. *Assume that the conditions of Theorem 4.3 are satisfied and that* $\lim_{x\to\infty} g'(x)/\psi'(x) = 0$. *Then, the conditions of part (B) of Theorem 4.1 are satisfied and there is a unique* $z^* = \operatorname{argmax}\{g(x)/\psi(x)\} \in (x^*,\infty)$ *satisfying the ordinary first order condition* $g'(z^*)\psi(z^*) = g(z^*)\psi'(z^*)$. *In that case the value of the optimal stopping policy* $\tau_{z^*} = \inf\{t \geq 0 : X_t \geq z^*\}$ *reads as* (23).

Proof. Consider the behavior of the function $u(x) = g(x)/\psi(x)$. Standard differentiation shows that

$$u'(x) = -\frac{\psi'(x)}{\psi^2(x)} R(x).$$

It is clear from the proof of Theorem 4.3 that

$$R'(x) = -\psi(x)\frac{d}{dx}\left[\frac{g'(x)}{\psi'(x)}\right] \lesseqgtr 0, \quad x \lesseqgtr x^*$$

and $\lim_{x\downarrow 0} R(x) = 0$. Applying these results show that $u'(x) > 0$ for all $x \in (0,x^*)$. We will now demonstrate that the function $u(x)$ cannot be increasing on the entire $\mathscr{I}$. To see that this is indeed the case, assume the opposite and, therefore, that $u(x)$ is increasing on the entire $\mathscr{I}$. Then $u(x) > u(y)$ for all $x > y$ and, therefore,

$$\frac{g'(x)}{\psi'(x)} - \frac{g(x)}{\psi(x)} < \frac{g'(x)}{\psi'(x)} - \frac{g(y)}{\psi(y)}$$

for all $x > y$. Letting $x \to \infty$ and invoking the assumption $\lim_{x\to\infty} g'(x)/\psi'(x) = 0$ shows that

$$\lim_{x\to\infty}\left[\frac{g'(x)}{\psi'(x)} - \frac{g(x)}{\psi(x)}\right] < -\frac{g(y)}{\psi(y)} < 0$$

which is a contradiction, since

$$u'(x) = \frac{\psi'(x)}{\psi(x)}\left[\frac{g'(x)}{\psi'(x)} - \frac{g(x)}{\psi(x)}\right].$$

Hence, equation $u'(x) = 0$ has at least one root $z^* \in (x^*, \infty)$. The uniqueness of this root follows from the monotonicity of $R(x)$.

Having established the existence and uniqueness of the threshold z^*, we now denote the proposed value function as $P(x)$ and notice that since

$$P(x) = \mathbb{E}_x\left[e^{-r\tau_{z^*}} g(X_{\tau_{z^*}})\right]$$

we necessarily have that $V(x) \geq P(x)$. In order to prove the opposite inequality, we firs notice that $P \in C^1(\mathscr{I}) \cap C^2(\mathscr{I}\setminus\{z^*\})$, that $|P''(z^*\pm)| < \infty$, that $P(x) \geq g(x)$ for all $x \in \mathscr{I}$, and that $(\mathscr{G}_r P)(x) = 0$ on $(0,z^*)$. We now establish that $(\mathscr{G}_r P)(x) < 0$ on (z^*,∞). To see that this is indeed the case, we first consider the functional

$$(L_\psi g)(x) = \frac{g'(x)}{S'(x)}\psi(x) - \frac{\psi'(x)}{S'(x)}g(x),$$

where

$$S'(x) = \exp\left(-\int \frac{2\mu(x)}{\sigma^2(x)}dx\right).$$

It is clear from our analysis above that $(L_\psi g)(x) \gtreqless 0$ for all $x \lesseqgtr z^*$ and $(L_\psi g)'(z^*) < 0$. Applying the identity $(\mathscr{G}_r \psi)(x) = 0$ shows that

$$(L_\psi g)'(z^*) = \left[(\mathscr{G}_r g)(z^*) + \int_0^1 \left(\frac{\psi(z^*(1-z))g(z^*)}{\psi(z^*)} - g(z^*(1-z))\right)\nu(dz)\right]M(z^*),$$

where $M(x) = \psi(x)m'(x)$ and $m'(x) = 2/(\sigma^2(x)S'(x))$. Since

$$\lim_{x\downarrow z^*}(\mathcal{G}_r P)(x) = \frac{(L_\psi g)'(z^*)}{\psi(z^*)m'(z^*)} < 0$$

we notice that the proposed value function is locally r-superharmonic at the optimal threshold z^*. Moreover, we also observe that for all $x \in (z^*, \infty)$

$$(\mathcal{G}_r P)(x) = (\mathcal{G}_r g)(x) + \int_{1-\frac{z^*}{x}}^{1} \left(\psi(x-xz)\frac{g'(z^*)}{\psi'(z^*)} - g(x-xz) \right) \nu(dz)$$

implying that

$$(\mathcal{G}_r P)'(x) = (\mathcal{G}_r g)'(x) + \int_{1-\frac{z^*}{x}}^{1} \left(\psi'(x-xz)\frac{g'(z^*)}{\psi'(z^*)} - g'(x-xz) \right) (1-z)\nu(dz).$$

Applying the proof of Theorem 4.4 yields

$$\begin{aligned}(\mathcal{G}_r P)'(x) &= (\mathcal{G}_r J)'(x) + \int_{1-\frac{x^*}{x}}^{1} \psi'(x-xz)\left[\frac{g'(z^*)}{\psi'(z^*)} - \frac{g'(x^*)}{\psi'(x^*)}\right](1-z)\nu(dz) \\ &+ \int_{1-\frac{z^*}{x}}^{1-\frac{x^*}{x}} \left[\psi'(x-xz)\frac{g'(z^*)}{\psi'(z^*)} - g'(x-xz)\right](1-z)\nu(dz) < 0\end{aligned}$$

since z^* is attained on the set where $(\mathcal{G}_r J)'(x)$ is decreasing, $x^* = \operatorname{argmax}\{g'(x)/\psi'(x)\}$, and $g'(x)/\psi'(x)$ is strictly decreasing on (x^*, z^*). Hence, $(\mathcal{G}_r P)'(x) < 0$ on (z^*, ∞) which guarantees that $(\mathcal{G}_r P)(x) < 0$ for all $x \in (z^*, \infty)$. Consequently, $P(x)$ constitutes a r-excessive majorant of the payoff $g(x)$. Since $V(x)$ is the least of these majorants, we find that $P(x) \geq V(x)$ which completes the proof of our theorem. □

It is worth noticing that Theorems 4.1, 4.3, 4.4 and 4.6 hold true even in the case when the Lévy measure of the driving process has infinite total mass (this is the so-called *infinite activity* (of jumps) case) – provided that the measure has a continuously differentiable density. This can been seen by examining the proofs carefully and noticing that the only potential problems arising from $\int_0^1 \nu(dz) = \infty$ are related to the ν-integrability of several functions defined in terms of $\psi(x)$ and $g(x)$. However, the assumptions made on $\psi(x)$ and $(\mathcal{G}_r g)(x)$ clearly imply that

$$\left| \int_0^1 \left\{ f(x-xz) - f(x) + xzf'(x) \right\} \nu(dz) \right| < \infty$$

for $f = \psi$ and $f = g$. This, in turn, implies the ν-integrability of $\eta(x,z)$ defined in the proof of Theorem 4.3. With regard to other functions we need to integrate with respect to ν, we note that on any set $(\varepsilon,1) \subset (0,1)$, $\varepsilon > 0$, the measure ν has finite total mass, and all integrands considered in the proofs are at least continuous on these sets. Hence the relevant integrals are finite-valued, and since we assume $\nu(dz) = h(z)dz$ for $h \in C((0,1))$, also Leibniz rule can be applied where necessary. Naturally, the results concerning the associated diffusion $\tilde{X}$ are not generalizable to the case of a jump diffusion with infinite activity of jumps, since $\tilde{X}$ can not be defined as we have done here if the total mass of ν is infinite. It is perhaps worth mentioning here that models driven by Lévy processes encountered in the financial literature tend to have either a nonzero Brownian part and finite jump activity or no Brownian component and infinite jump activity.

5 ILLUSTRATION

To illustrate our general results we consider the *logistic jump diffusion* model, where the stochastic dynamics of the underlying cash flow are characterized by the stochastic differential equation

$$dX_t = aX_t(b - X_t)dt + \sigma X_t dW_t - X_t \int_0^1 z\tilde{N}(dz, dt), \tag{29}$$

where $a, b, \sigma > 0$, $ab > r$ and the Lévy measure associated with $\tilde{N}$ is $\lambda b(\alpha;\beta;z)dz$ with λ being the jump intensity and $b(\alpha;\beta;z)$ being the density of a Beta(α,β) distribution. As we are interested in the impacts of volatility σ and the jump structure (characterized in this setting by parameters λ, α and β)) on the optimal control, we shall fix the values $a = 0.5$, $b = 2$ and assume the constant risk-free interest rate r to be 5 %. In [4] it was demonstrated that in this setting the increasing fundamental solution of $\mathcal{G}_r u = 0$ has a Frobenius series representation with coefficients that converge to zero fairly quickly – we capitalize on this fact in solving the control problems.

We assume that the reward function $g(x) = \sqrt{x}$, which is a special case of the popular power utility function form x^γ, $\gamma \in (0,1)$ – in terms of the investor's risk preferences, such utility functions imply that the investor is risk averse. We investigate the impacts of jump intensity, shape and location of the relative jump size distribution and diffusion volatility on the optimal singular control and the optimal stopping rule. With regard to the relative jump size distribution, we shall here consider two aspects. First, the impact of the degree of concentration of the distribution by fixing $\alpha = \beta$ (and hence the mean to $1/2$); increasing the common value of these parameters leads to a distribution more and more concentrated around its mean. Second, the impact of the location of the distribution by fixing $\alpha = 1$; then as β increases from a value close to 0 to a very large value, the mean of the distribution decreases from nearly 1 to nearly 0. To study the effect of risk aversion on the optimal policy, we compare the obtained numerical results with the corresponding results in the risk neutral situation, i.e. in the linear case $g(x) = x$.

Using Theorem 3.2, we can determine the optimal thresholds for the optimal control problems for the associated continuous diffusion $\tilde{X}$. The results of doing this for various values of intensity λ, volatility σ and different combinations of α and β are given in Tables 1–4. From Table 2 and Figure 1 we can see that increased volatility increases the optimal threshold and decreases the optimal value, which is in accordance with Theorem 3.3.

Similarly, since the reward satisfies the assumptions of Theorems 4.4 and 4.6, we can apply these results to determine the optimal thresholds for the optimal control problems for the jump diffusion X. The results of doing this for various values of intensity λ, volatility σ and different combinations of α and β are also given in Tables 1–4. From Table 2 and Figure 2 we see that in accordance with the last part of Theorem 4.4, increased volatility increases the optimal threshold and decreases the optimal value.

		λ				
		0.2	0.4	0.6	0.8	1
$\tilde{X}$	x^*	0.676981	0.743883	0.81075	0.87759	0.944409
	z^*	2.244790	2.472510	2.70115	2.93073	3.161240
X	x^*	0.618723	0.625722	0.630787	0.633582	0.633675
	z^*	2.079780	2.129780	2.166390	2.187830	2.192210

Table 1: *The impact of jump intensity λ on the optimal thresholds x^* and z^* for the diffusion $\tilde{X}$ and the jump diffusion X.*

We can compare the obtained numerical results with the risk neutral situation. Tables 5–8 reproduce the optimal thresholds expressed now as a proportion of the optimal threshold in the risk neutral case. It is seen that in comparison with a risk-neutral investor, the risk averse investor has lower optimal thresholds. The difference is more pronounced in the singular control

		σ			
		0.2	0.4	0.6	0.8
$\tilde{X}$	x^*	0.944409	0.974522	1.01746	1.06725
	z^*	3.161240	3.593280	4.12726	4.66522
X	x^*	0.633675	0.646766	0.659945	0.660685
	z^*	2.192210	2.347260	2.477770	2.489330

Table 2: *The impact of volatility σ on the optimal thresholds x^* and z^* for the diffusion $\tilde{X}$ and the jump diffusion X.*

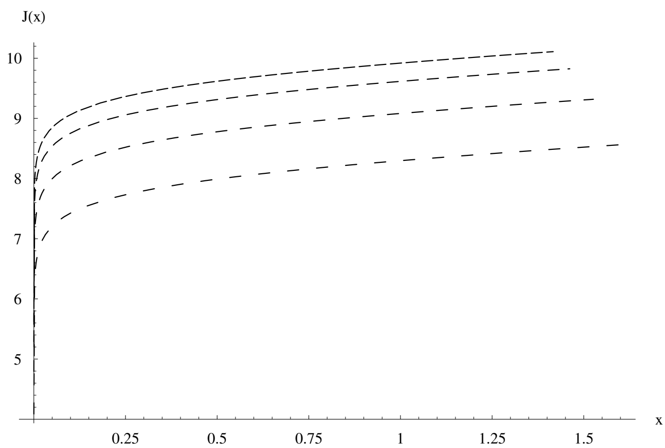

Figure 1: *The value $J(x)$ of the singular control problem for different volatilities σ when the underlying process is the diffusion $\tilde{X}$. The highest value corresponds to $\sigma = 0.2$ and the lowest value to $\sigma = 0.8$.*

		α			
		0.1	0.5	1.0	10
X	x^*	0.375831	0.588971	0.633675	0.649268
	z^*	1.145130	1.943590	2.192210	2.331930

Table 3: *The impact of varying $\alpha = \beta$ on the optimal thresholds x^* and z^* for the jump diffusion X. The optimal thresholds for the diffusion $\tilde{X}$ do not depend on α and are $x^* = 0.944409$ and $z^* = 3.16124$.*

problems, where the threshold values for the risk averse investor are between $37 - 64$ % of the corresponding values for the risk neutral investor, with the most typical values being around $50 - 60$ %. For the stopping problems the

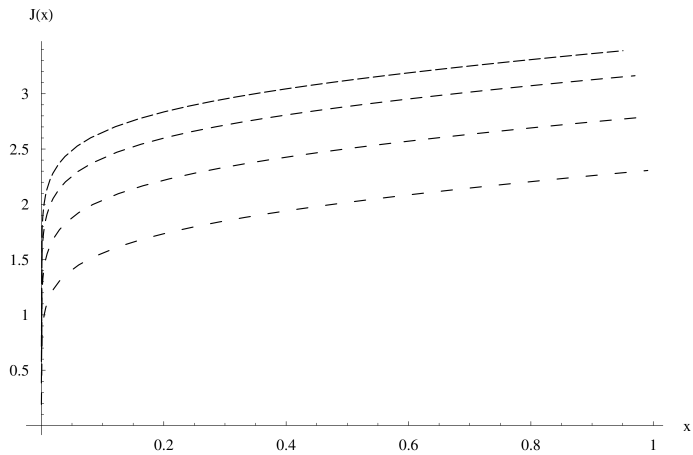

Figure 2: *The value $J(x)$ of the singular control problem for different volatilities σ when the underlying process is the jump diffusion X. The highest value corresponds to $\sigma = 0.2$ and the lowest value to $\sigma = 0.8$.*

		β		
		0.5	1.0	10
$\tilde{X}$	x^*	1.14477	1.05573	0.72159
	z^*	3.85820	3.54745	2.39650
X	x^*	0.517487	0.633675	0.613266
	z^*	1.653030	2.192210	2.052190

Table 4: *The impact of varying parameter β with $\alpha = 1$ on the optimal thresholds x^* and z^* for the diffusion $\tilde{X}$ and the jump diffusion X.*

corresponding range is $54 - 91$ %, with most values being around $75 - 91$ %.

6 CONCLUDING COMMENTS

In this study we considered the optimal singular control and optimal stopping problems with a nonlinear and state-dependent payoff for a class of spectrally negative jump diffusions. In the language of economics, these problems have meaningful interpretations both in financial economics and in natural resource economics. In the first case we speak about an investor's optimal dividend policy and optimal timing of an irreversible investment, in the second about optimal harvesting and optimal depletion of a renewable resource. We showed

		λ				
		0.2	0.4	0.6	0.8	1
$\tilde{X}$	x^*	0.624	0.628	0.631	0.633	0.636
	z^*	0.894	0.898	0.902	0.905	0.908
X	x^*	0.609	0.599	0.588	0.575	0.564
	z^*	0.881	0.871	0.860	0.847	0.831

Table 5: *The optimal thresholds x^* and z^* for the diffusion $\tilde{X}$ and the jump diffusion X with different values of λ, as a proportion of the optimal threshold in the risk-neutral case.*

		σ			
		0.2	0.4	0.6	0.8
$\tilde{X}$	x^*	0.636	0.617	0.593	0.569
	z^*	0.908	0.888	0.862	0.834
X	x^*	0.564	0.542	0.504	0.471
	z^*	0.831	0.793	0.743	0.680

Table 6: *The optimal thresholds x^* and z^* for the diffusion $\tilde{X}$ and the jump diffusion X with different values of σ, as a proportion of the optimal threshold in the risk-neutral case.*

		α			
		0.1	0.5	1.0	10
X	x^*	0.370	0.531	0.564	0.581
	z^*	0.546	0.782	0.831	0.857

Table 7: *The optimal thresholds x^* and z^* for the jump diffusion X with different values of α, as a proportion of the optimal threshold in the risk-neutral case.*

that under some relatively general conditions typically satisfied in most mean-reverting models with a risk averse investor or diminishing marginal utility (captured by a decreasing demand function), the problems are solvable in terms of the increasing fundamental solution of the associated integro-differential equation. Moreover, for continuous diffusions, this result was shown to hold without any assumption on the investor's risk preferences or behavior of marginal utility as described by the convexity properties of the payoff.

		β		
		0.5	1.0	10
$\tilde{X}$	x^*	0.641	0.639	0.627
	z^*	0.914	0.911	0.897
X	x^*	0.461	0.564	0.617
	z^*	0.679	0.831	0.886

Table 8: *The optimal thresholds x^* and z^* for the diffusion $\tilde{X}$ and the jump diffusion X with different values of β, as a proportion of the optimal threshold in the risk-neutral case.*

We focused on studying the impact of increased policy flexibility on both the value as well as on the marginal value of the optimal singular control. Along the lines of previous studies based on linear models, we established that the impact of increased flexibility is positive in the nonlinear setting as well by relying on a combination of variational inequalities and ordinary static optimization techniques. We also demonstrated that the value of the optimal control in the jump diffusion setting can be sandwiched between the values of the optimal controls for two associated diffusions and established that under our assumptions the sign of the relationship between volatility of the underlying and the optimal value is negative.

Restricting the attention to one-sided jumps can be justified by two arguments. First, there is the time-honored principle of prudence: A risk manager should take into account all uncertain future losses, while it may be prudent to disregard some uncertain future profits. Second, there is the asymmetrical response of the market to unexpected shocks: upward movements in asset values caused by a positive shock tend to be smaller and less steep than downward movements caused by a negative shock. While certainly interesting if it is possible, a generalization of our results to Lévy diffusions with two-sided jumps may prove to be a very challenging problem, as we crucially relied on the fact that by virtue of spectral negativity, the underlying process can increase only continuously. Such a highly nontrivial generalization is out of the scope of the present study and is left for future research.

An interesting (and perhaps less difficult to address) question is whether our sufficient conditions in the jump diffusion case could be restated solely in terms of the functional obtained by applying the associated integro-differential operator to the payoff, and whether this would lead to weaker sufficient conditions than the ones stated here – in particular, could the results be made independent of the convexity properties of the payoff, as our results for the associated diffusion seem to suggest. However, given that our aim in this study was to obtain results applicable in models relevant from an applied point of view, and in this respect risk aversion and diminishing marginal utility are widely accepted as reasonable assumptions, such a generalization is, despite its aesthetic appeal from a mathematician's point of view, out of the scope of the present study and we leave it for future research.

A third potentially interesting extension of our approach would be to analyze the considered stochastic control problems in the presence of a potentially finite exogenous liquidation time. Such an extension would cast light on the question of how liquidation risk affects the optimal dividend policy and in this way the minimal reserves.

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A PROOF OF LEMMA 2.1

Proof. As was established in Theorem 4.1 of [3] we have that

$$\frac{\tilde{\psi}_{r+\lambda}(x)}{\tilde{\psi}_{r+\lambda}(y)} \leq \frac{\psi(x)}{\psi(y)} \leq \frac{\tilde{\psi}_r(x)}{\tilde{\psi}_r(y)} \tag{30}$$

for all $0 < x \leq y < \infty$. Thus, we find by multiplying inequality (30) with $g(y)$, where $y \in \mathscr{I}$, that

$$g(y)\frac{\tilde{\psi}_{r+\lambda}(x)}{\tilde{\psi}_{r+\lambda}(y)} \leq g(y)\frac{\psi(x)}{\psi(y)} \leq g(y)\frac{\tilde{\psi}_r(x)}{\tilde{\psi}_r(y)}$$

proving the two first claims of part (B) of our lemma. As was established in Theorem 4.1 of [3] inequality (30) and the fundamental theorem of integral calculus in turn imply that

$$\frac{\tilde{\psi}'_{r+\lambda}(y)}{\tilde{\psi}_{r+\lambda}(y)} \geq \frac{\psi'(y)}{\psi(y)} \geq \frac{\tilde{\psi}'_r(y)}{\tilde{\psi}_r(y)} \tag{31}$$

for all $y \in \mathscr{I}$. Noticing now that

$$\frac{\psi(x)}{\psi'(y)} = \frac{\psi(x)}{\psi(y)}\frac{\psi(y)}{\psi'(y)} \leq \frac{\tilde{\psi}_r(x)}{\tilde{\psi}_r(y)}\frac{\tilde{\psi}_r(y)}{\tilde{\psi}_r'(y)} = \frac{\tilde{\psi}_r(x)}{\tilde{\psi}_r'(y)}$$

and

$$\frac{\psi(x)}{\psi'(y)} = \frac{\psi(x)}{\psi(y)}\frac{\psi(y)}{\psi'(y)} \geq \frac{\tilde{\psi}_{r+\lambda}(x)}{\tilde{\psi}_{r+\lambda}(y)}\frac{\tilde{\psi}_{r+\lambda}(y)}{\tilde{\psi}_{r+\lambda}'(y)} = \frac{\tilde{\psi}_{r+\lambda}(x)}{\tilde{\psi}_{r+\lambda}'(y)}$$

and invoking the monotonicity of the function $g(x)$ then shows that

$$g'(y)\frac{\tilde{\psi}_{r+\lambda}(x)}{\tilde{\psi}_{r+\lambda}'(y)} \leq g'(y)\frac{\psi(x)}{\psi'(y)} \leq g'(y)\frac{\tilde{\psi}_r(x)}{\tilde{\psi}_r'(y)}$$

which completes the proof of the remaining alleged results. □

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PAPER IV

Luis H. R. Alvarez – Teppo A. Rakkolainen: *Investment timing in presence of downside risk: a certainty equivalent characterization*, 2008, to appear in *Annals of Finance*.

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RESEARCH ARTICLE

Investment timing in presence of downside risk: a certainty equivalent characterization

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Abstract We demonstrate that the value of a single threshold investment strategy under stochastic dynamics allowing both continuous fluctuations and instantaneous downward jumps has a certainty equivalent representation in terms of the value of this strategy under risk-adjusted deterministic dynamics, and that this risk adjustment can be made either to the discount rate or to the expected infinitesimal growth rate of the underlying. In this way our analysis characterizes a class of optimal timing problems of irreversible investments for which the solution of the stochastic problem coincides with the solutions of certain risk-adjusted deterministic optimal timing problems.

Keywords Downside risk · Certainty equivalence · Exponential Lévy process · Optimal stopping · Risk adjustment · Threshold policy

JEL Classification C61 · G31

1 Introduction

Most major investment projects share three common features. First, the industry specificity of capital and other similar factors imply that most investments are irreversible and, therefore, that the costs associated with undertaking the project are sunk. Second, investment returns and costs are typically subject to considerable uncertainty. Third, most investment projects can be delayed in time resulting into valuable timing flexibility. In light of these arguments it is clear that the determination of the optimal

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exercise strategy of deferrable irreversible investment opportunities in the presence of uncertainty constitutes one of the key problems in the modern literature on real options and its application in modeling irreversible investment decisions (cf. Dixit and Pindyck 1994, chaps. 5 and 6). In standard approaches to this subject the underlying stochastic dynamics characterizing the uncertainty faced by the risk neutral investor are modeled as a geometric process, which can be either a geometric Brownian motion with its continuous paths, or, in the more recent literature, an exponential Lévy process which allows path discontinuities. As the planning horizon in real options problems is typically very long, the usual approach is to model the valuation and rational exercise of the investment opportunity as the determination of the price and exercise policy of a perpetual American option written on dividend paying stock. This pricing problem was originally solved within a continuous setting by Samuelson (1965) and McKean (1965) (see also McDonald and Siegel 1986; Merton 1973) for extensions and Chap. 5 in Dixit and Pindyck (1994) and Chap. 28 of Hull (2003) for accessible textbook treatments). Subsequently, the investment timing problem was solved in full generality by modeling the underlying return dynamics as a Lévy process by Mordecki (2002) (see also Boyarchenko and Levendorskiǐ 2002; Boyarchenko 2004).

The reason for the popularity of the geometric model in the financial literature is of course its delightful tractability given its close connection to first order autoregressive processes. This tractability allows a detailed analysis of the dependencies and interrelations of model parameters, thus enhancing the possibilities of deriving statistically testable implications, economic interpretations and comparative static properties of the model. For an exemplary economic interpretation of the results in Mordecki (2002), see Boyarchenko (2004) and the *record-setting news principles* formulated therein. Especially tractable is the situation where only downward jumps are allowed, that is, when the driving discontinuous Lévy process is spectrally negative. There are two practical justifications for relying on such a one-sided model capturing the potentially catastrophic downside risk. The first justification is the *bad news priciple* introduced in Bernanke (1983) (see also Boyarchenko and Levendorskiǐ 2005a) for the *record setting bad news principle* extending Bernanke's result). According to this principle the market response to new information is asymmetric in the sense that only bad news affect investment. Competitive factors like entry or patent races and innovation processes are also natural justifications for downward jumps in revenue dynamics (cf. Dixit and Pindyck 1994, p. 65). A third natural justification is loss aversion since it also results into an asymmetric response to increased risk of potential losses.

Despite the wealth of first class research done on the optimal stopping of spectrally negative exponential models, studies dealing with the *certainty equivalent formulation of optimal investment timing problems in the presence of downside risk* are scarce (for a comprehensive characterization and economic motivation of certainty equivalence principles, see Chap. 3 in Laffont 1989); see, however, Boyarchenko and Levendorskiǐ (2005b), where a certainty equivalence in terms of deterministic functions and with a view to explaining discounted utility anomalies is considered. This scarcity is somewhat surprising, since in spite of the presence of two very different underlying driving processes, the continuous risk characterized by the Wiener process on one hand and the discontinuous downside risk described by the jump process on the other, our study demonstrates that both the value and optimal exercise boundary of the optimal timing of an irreversible investment opportunity under uncertainty coincide for a large class of investment problems with the value and optimal exercise boundary of an associated *risk adjusted continuous and deterministic* optimal investment timing problem. Along the lines indicated by the capital asset pricing model, our findings show that this risk adjustment can be made either to the rate at which the exercise payoff is discounted or to the percentage growth rate of the deterministic process modeling the underlying state variable in the absence of uncertainty. By extending the analysis of the study (Alvarez 2004) focusing on continuous dynamics to the present discontinuous setting, we derive the risk adjusted discount rate and percentage growth rate of the associated deterministic dynamics explicitly and analyze the sensitivity of the adjustments with respect to changes in volatility and in the intensity at which jumps are expected to occur. Interestingly, our findings indicate that the sign of the impact of increased volatility or increased jump intensity depends on the convexity/concavity of the value on the continuation region where the investment opportunity is left unexercised which, in turn, is dictated by the sign of the difference between the risk free interest rate and the expected percentage growth rate of the underlying. According to our findings, increased volatility or increased jump intensity decreases the risk adjusted discount rate and, therefore, the opportunity cost of investment whenever the risk free interest rate dominates the percentage growth rate of the underlying. The reason for this observation is naturally the convexity of the expected present value of a unit of money received at exercise as a function of the current state of the underlying. Under such circumstances the positive impact of increased uncertainty on the value of waiting always dominates its impact on the expected present value of the exercise payoff. This increases the required rate of return which, in turn, results into a lower risk adjusted discount rate. In contrast, if the percentage growth rate of the underlying dominates the risk free interest rate then the opposite conclusion is valid. Analogously, increased volatility or increased jump intensity increases (decreases) the risk adjusted growth rate in the convex (concave) case. We also state a set of sufficient conditions under which both the values and the optimal investment thresholds of the original discontinuous optimal stopping problem and the risk adjusted continuous deterministic timing problem coincide.

It is worth emphasizing that Boyarchenko and Levendorskiǐ (see Boyarchenko and Levendorskiǐ 2002, 2005a,b, 2006, 2007a,b,c) have also applied single threshold policies as a starting point for their analysis. However, instead of focusing on the minimal functions associated to the underlying state variable as we do, they rely on general Wiener–Hopf factorization techniques (see Boyarchenko and Levendorskiǐ 2007a,b) for a careful explanation of that approach in a discrete time framework and an intuitive interpretation in terms of so-called expected present value operators). Due to the chosen factorization approach, they have to rely on the infimum and supremum processes of the underlying. Naturally, operating with these processes in a general setting is a demanding task given their complex behavior in the jump diffusion setting. However, in light of the close connection between the probability distribution of first passage times and the distribution of the supremum and infimum process, it is clear that our approaches are related.

The content of this study is as follows. In Sect. 2 we present the considered class of exponential Lévy models. Our certainty equivalent representations of the values of threshold policies are stated and proved in the following section, followed by the sufficient conditions for the stochastic optimal timing problem to have a similar representation. An explicit illustration based on beta-distributed jumps is then presented in Sect. 4, and finally concluding comments are made in Sect. 5.

2 Basic setup and assumptions

The main objective of our study is to state an explicit certainty equivalent formulation for a class of investment timing problems subject to discontinuous downside risk. To accomplish this task, let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a filtered probability space equipped with the natural filtration $\mathbb{F}$ of a Lévy process $L = \{L_t : t \in [0, \infty)\}$, which is a superposition of two martingales: a standard Wiener process $W = \{W_t : t \in [0, \infty)\}$ and $J = \{J_t : t \in [0, \infty)\}$, a compensated jump process whose Poisson random measure and Lévy measure are $\tilde{N}(dt, dz)$ and $\lambda \mathfrak{m}(dz)$, respectively. The distribution of the size of individual jumps, $\mathfrak{m}$, is assumed to be absolutely continuous with support contained in (0, 1) and expectation equal to $\overline{m}$. The evolution of the underlying jump diffusion X is governed on the state space $(0, \infty)$ by the stochastic differential equation

$$dX_t = X_{t-} \left\{ \mu dt + \sigma dW_t - \int_0^1 z \tilde{N}(dt, dz) \right\}, \quad X_0 = x. \tag{1}$$

The minus sign before the last term on the right hand side of (1) means that the process is spectrally negative: it can jump downwards discontinuously but increases only continuously. These downward jumps represent the downside risk: the possibility of an unanticipated instantaneous potentially significant deterioration in the value of the underlying. As usually, the integro-differential operator $\mathcal{L}_r$ associated to the underlying jump-diffusion X in the presence of discounting is defined for sufficiently smooth mappings u by

$$\begin{aligned}(\mathcal{L}_r u)(x) = &\frac{1}{2}\sigma^2 x^2 u''(x) + \mu x u'(x) - r u(x) \\ &+ \lambda \int_0^1 \left\{ u(x - xz) - u(x) + xzu'(x) \right\} \mathfrak{m}(dz),\end{aligned} \tag{2}$$

where $r > 0$ denotes the prevailing discount rate. The increasing fundamental solution of the integro-differential equation $(\mathcal{L}_r u)(x) = 0$ is x^{ψ}, where $\psi > 0$ constitutes the unique positive root of the characteristic equation

$$P(\psi) = \frac{1}{2}\sigma^2 \psi(\psi - 1) + (\mu + \lambda \overline{m})\psi - (r + \lambda) + \lambda \int_0^1 (1 - z)^{\psi} \mathfrak{m}(dz) = 0. \tag{3}$$

In particular, we observe that $\psi \lesseqqgtr 1$ when $r \lesseqqgtr \mu$ (since $P(0) = -r < 0$, $P(1) = \mu - r$, $\lim_{\psi\to\infty} P(\psi) = \infty$, and $P(\psi)$ is convex).

Given our characterization of the underlying stochastically fluctuating return dynamics, define the first entrance time of the underlying to the set $[y, \infty)$ as $\tau_y = \inf\{t \geq 0 : X_t \geq y\}$ and assume that the payoff $g : \mathbb{R}_+ \mapsto \mathbb{R}$ received whenever the investment opportunity is exercised is continuous and non-decreasing. The expected present value of the payoff received by following the timing rule based on the single exercise threshold y now reads as

$$J^y(x) = \mathbb{E}_x\left[e^{-r\tau_y} g(X_{\tau_y})\right]. \tag{4}$$

It is at this point worth pointing out that the first entrance time of the underlying to the set $[y, \infty)$ may be infinite. Therefore, more conditions are required in order to guarantee the almost sure attainability of the state y in finite time. Our next auxiliary lemma states a sufficient condition under which $\tau_y < \infty$ almost surely.

Lemma 2.1 *Assume that*

$$\mu + \lambda\mathbb{E}[z + \ln(1 - z)] > \frac{1}{2}\sigma^2. \tag{5}$$

Then, $\mathbb{P}_x[\tau_y < \infty] = 1$ *for all* $x \leq y$.

Proof See Appendix A. □

In light of the considered class of threshold strategies, it is naturally also of interest to investigate the optimal stopping problem

$$V(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x\left[e^{-r\tau} g(X_\tau)\right], \tag{6}$$

where $\mathcal{T}$ is the set of all $\mathbb{F}$-stopping times, and to establish conditions under which the optimal policy constitutes a threshold policy. It is worth noticing that the results we obtain in this case where an instantaneous lump-sum payoff is obtained by exercising at an admissible stopping time, are (under a certain integrability condition) also valid for more general problems subject to accumulated cash flows arising typically in studies focusing on either optimal entry or exit. Namely, if the cash flow $f(x)$ accrued from continuing operation has a finite expected cumulative present value and

$$r > \mu + \lambda\overline{m} + \lambda\left(\mathbb{E}\left[\frac{1}{1-z}\right] - 1\right), \tag{7}$$

then the optimal entry problem

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}_x\left\{\int_\tau^\infty e^{-rs} f(X_{s-})ds + e^{-r\tau} g(X_\tau)\right\} \tag{8}$$

and the optimal exit problem

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left\{ \int_0^\tau e^{-rs} f(X_{s-}) ds + e^{-r\tau} g(X_\tau) \right\} \tag{9}$$

can be reduced to an optimal stopping problem of form (6) by using the expected cumulative present value

$$(R_r f)(x) = \mathbb{E}_x \left\{ \int_0^\infty e^{-rs} f(X_{s-}) ds \right\}. \tag{10}$$

To be precise, we have the following.

Lemma 2.2 *Suppose that* (7) *holds. If the cash flow $f(x)$ is Lipschitz continuous and non-decreasing, then $(R_r f)(x)$ inherits these properties. Furthermore, problems* (8) *and* (9) *are equivalent to optimal stopping problems*

$$V_{entry}(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left\{ e^{-r\tau} \left((R_r f)(X_\tau) + g(X_\tau) \right) \right\} \tag{11}$$

and

$$V_{exit}(x) = (R_r f)(x) + \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left\{ e^{-r\tau} \left(g(X_\tau) - (R_r f)(X_\tau) \right) \right\}, \tag{12}$$

respectively.

Proof See Appendix B. □

Lemma 2.2 characterizes how standard entry or exit problems involving cumulative cash flows can be reduced into a standard stopping problem based on a lump-sum exercise payoff. It is worth noticing that even though the payoff is lump sum, it naturally depends on the characteristics of the underlying jump-diffusion.

Having characterized the considered class of valuations, we now introduce an associated *continuous and deterministic* process Z whose evolution on $(0, \infty)$ is described by the ordinary first order differential equation

$$Z_t' = \alpha Z_t, \quad Z_0 = x \in (0, \infty), \tag{13}$$

where the constant percentage growth rate $\alpha \in \mathbb{R}_+$ will be defined explicitly in our subsequent analysis. It is worth noticing that if $\alpha = \mu$ then the process Z can be viewed as the deterministic limiting case in the dynamics (1) which is attained as the volatility coefficient σ and the jump intensity λ vanish. For notational convenience, we denote the (deterministic) first entrance time of Z to the set $[y, \infty)$ by t_y and by

$$L^y_{\theta,\alpha}(x) = e^{-\theta t_y} g(Z_{t_y})$$

the discounted present value of a policy based on the single threshold y (cf. (4)) when the discount rate is $\theta > 0$. Accordingly, we denote by

$$V_\theta^\alpha(x) = \sup_{t \geq 0} e^{-\theta t} g(Z_t)$$

the value of the associated deterministic optimal timing problem. The corresponding optimal threshold (if it exists) is denoted by $y^*_{\theta,\alpha}$.

3 Certainty equivalence and investment timing

The considered stochastic model for the underlying dynamics is subject to two different sources of uncertainty: the continuous risk represented by the Wiener process and the discontinuous downside risk captured by the jump process. However, despite the presence of these two fundamentally different sources of stochasticity, it can be shown that there exists a continuous deterministic process for which the values of a threshold policy coincide in the deterministic and stochastic case. This is the content of our first theorem.

Theorem 3.1 *Assume that $\theta > 0$, that $y \in (0, \infty)$ is an arbitrary exercise threshold, and that condition (5) is satisfied. If $\theta/\alpha = \psi$, then $L^y_{\theta,\alpha}(x) = J^y(x)$ for all $x \in \mathbb{R}_+$.*

Proof See Appendix C. □

Theorem 3.1 is a certainty equivalence relation: *by making an appropriate risk adjustment, we can characterize the value of a threshold policy in presence of uncertainty as the value of the same policy in absence of uncertainty*. According to Theorem 3.1 this appropriate adjustment can be made in such a way that the growth rate of the deterministic process equals the infinitesimal drift μ of the stochastic process by changing the applied discount rate from r to $\psi\mu$. In that case, $L^y_{\psi\mu,\mu}(x) = J^y(x)$. Alternatively, to keep the discount rate invariant with respect to the risk adjustment, we can set the growth rate of the deterministic process equal to r/ψ instead of μ. In that case, we naturally have that $L^y_{r,r/\psi}(x) = J^y(x)$.

An important consequence of Theorem 3.1 is that if the overall optimal investment rule is a threshold policy, then the optimal timing problem (6) is equivalent to the optimal stopping problem of the deterministic process Z subject to $\alpha = \mu$ and a risk adjusted discount rate. Furthermore, it is also equivalent to the optimal stopping problem of Z subject to the discount rate r and the risk adjusted percentage growth rate $\alpha = r/\psi$. These results are now summarized in the following theorem, which gives sufficient conditions for the optimal investment rule to be a standard threshold policy.

Theorem 3.2 *Suppose that $x^{-\psi}g(x)$ attains a unique global maximum at $x^* \in (0, \infty)$, that the functional $(\mathcal{L}_r J^{x^*})(x)$ is non-increasing for all $x > x^*$, and that condition (5) is satisfied.*

(A) Let $\rho = \mu\psi$. Then $y^*_{\rho,\mu} = x^*$ and $V^\mu_\rho(x) = V(x)$.
(B) Let $\alpha = r/\psi$. Then $y^*_{r,\alpha} = x^*$ and $V^\alpha_r(x) = V(x)$.

Proof See Appendix D. □

Theorem 3.2 states a set of sufficient conditions under which the problem of determining the optimal timing of an irreversible investment in a market subject to both continuous risk and discontinuous downside risk can be reduced to a certainty equivalent deterministic stopping problem of a continuous underlying state variable by making a suitable risk-adjustment either to the discount rate applied in discounting the future cash flows or to the infinitesimal growth rate at which the underlying stochastic process is expected to grow. Essentially, this means that the impact of both types of risk on the optimal decision can be characterized as a *discount rate effect*.

We feel that Theorem 3.2 offers a relatively straightforward and simple complementary alternative to other solution methods obtained in the literature for optimal investment timing problems of the type considered here. In our method one needs to first solve the increasing solution of an integro-differential equation. After that the optimal timing problem can be reduced into a standard non-linear programming problem which can be handled by relying on ordinary optimization techniques. In this way the original problem is taken into a framework which permits the application of standard marginalistic arguments familiar from basic microeconomic theory. In our opinion, the use of such arguments makes the chain of reasoning more transparent than does invoking highly sophisticated mathematical results such as the Wiener–Hopf factorization.

In particular, the independence of the risk-adjusted discount rate on the exact form of the reward structure allows a quick assessment of the riskiness of several alternative cash flows with equal infinitesimal growth rates modeled as spectrally negative geometric Lévy processes: simply solve the positive root of the characteristic equation, ψ, and calculate the adjusted discount rate $\rho = \mu\psi$ for each process. The differences $r - \rho$ are then measures of the riskiness of the different cash flows. The relationship between infinitesimal growth rate and risk adjustment can be illustrated *à la Markowitz* in the $(\mu, r - \rho)$-plane: different volatilities and jump structures generate different functions $\rho(\mu) = \mu\psi(\mu)$. In contrast, such comparisons cannot be done via solving the optimal thresholds, as these will, in addition to the characteristics of the process, depend also on the reward function $g(x)$.

Having presented our main findings on the certainty equivalent formulation of the considered class of investment timing problems, we are now in a position to establish some comparative static results on the behavior of the risk adjustment as a function of diffusion volatility σ and jump intensity λ. This is accomplished in our next theorem.

Theorem 3.3 *The risk adjusted discount rate $\rho = \mu\psi$ resulting into certainty equivalence when $\alpha = \mu$ satisfies the inequalities*

$$\rho = \mu\psi \lesseqqgtr r, \quad \frac{\partial \rho}{\partial \sigma} \lesseqqgtr 0 \quad \text{and} \quad \frac{\partial \rho}{\partial \lambda} \lesseqqgtr 0 \quad \text{when} \quad r \gtreqqless \mu.$$

Analogously, the risk adjusted percentage growth rate $\alpha = r/\psi$ resulting into certainty equivalence satisfies the inequalities

$$\alpha = r/\psi \gtreqqless \mu, \quad \frac{\partial \alpha}{\partial \sigma} \gtreqqless 0 \quad \text{and} \quad \frac{\partial \alpha}{\partial \lambda} \gtreqqless 0 \quad \text{when} \quad r \gtreqqless \mu.$$

Proof See Appendix E. □

Theorem 3.3 characterizes the sensitivity of the risk adjustment resulting into certainty equivalence with respect to changes in the volatility coefficient and in the jump intensity of the underlying dynamics. An interesting implication of our observations is that the impact of discontinuous downside risk on the risk adjustments depends on the convexity or concavity of the fundamental solution x^{ψ}. As a consequence, it is clear from Theorem 3.3

$$\psi \gtreqless \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}, \quad \text{whenever} \quad r \gtreqless \mu.$$

Therefore, our findings indicate that discontinuous downside risk decreases the risk adjusted discount rate and increases the risk adjusted growth rate in comparison with the continuous diffusion case where jump risk is absent whenever the condition $r > \mu$ is satisfied. Naturally, if $r < \mu$ then the opposite conclusion is true. As our numerical illustrations indicate, the impact of jump risk may be significant and therefore approaches overlooking it may either under- or overestimate significantly the required rate of return of a rational investor.

4 Illustration: beta-distributed jumps

In order to illustrate our general findings explicitly, we now assume that the jump-size is beta-distributed with density

$$\mathfrak{m}'(z) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} z^{a-1}(1-z)^{b-1}.$$

In this case ψ constitutes the positive root of the equation

$$\frac{1}{2}\sigma^2\psi(\psi - 1) + \left(\mu + \frac{\lambda a}{a+b}\right)\psi - (r+\lambda) + \lambda\frac{\Gamma(a+b)\Gamma(b+\psi)}{\Gamma(b)\Gamma(a+b+\psi)} = 0.$$

The risk adjusted discount rate $\rho = \mu\psi$ is explicitly illustrated as a function of volatility in Fig. 1 both in the absence of downside risk (dashed curve) and in the

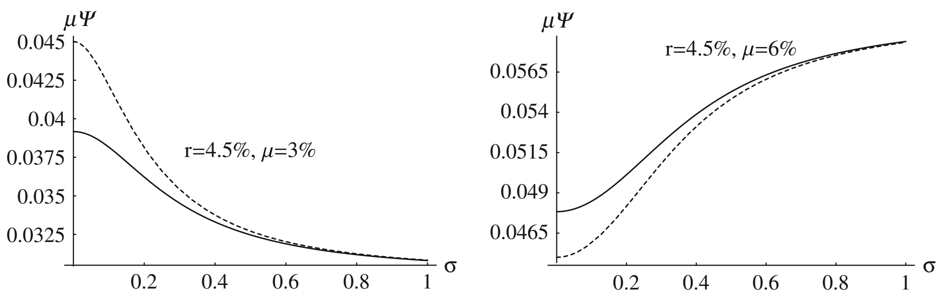

Fig. 1 The risk adjusted discount rate $\rho = \mu\psi$ as a function of σ

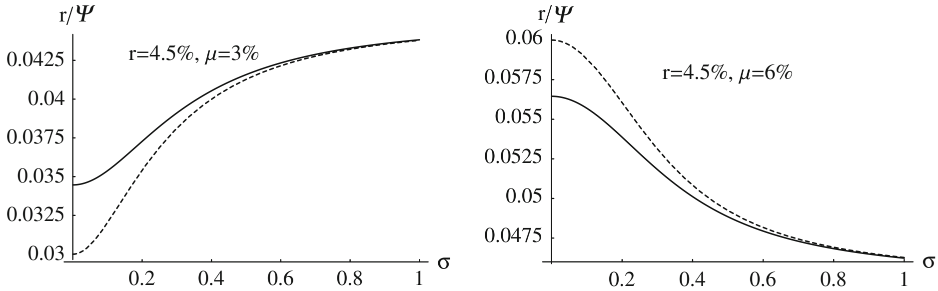

Fig. 2 The risk adjusted growth rate $\alpha = r/\psi$ as a function of σ

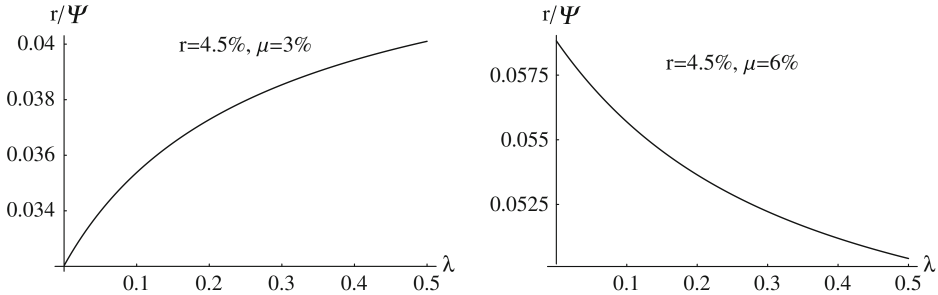

Fig. 3 The risk adjusted growth rate $\alpha = r/\psi$ as a function of λ

presence of downside risk (uniform curve) under the parameter specifications $a = 1.5, b = 2$, and $\lambda = 0.1$ (implying that $\bar{m} = 3/7$). The risk adjusted growth rate $\alpha = r/\psi$ is in turn explicitly illustrated as a function of volatility in Fig. 2 both in the absence of downside risk (dashed curve) and in the presence of downside risk (uniform curve) under the parameter specifications $a = 1.5$, $b = 2$, and $\lambda = 0.1$.

The sensitivity of the risk adjusted growth rate $\alpha = r/\psi$ with respect to changes in the jump intensity λ is illustrated in Fig. 3 under the parameter specifications $a = 1.5, b = 2$, and $\sigma = 0.1$.

Finally, in Fig. 4 we have plotted the riskiness—as measured by the difference $r - \rho$—of the underlying cash flow as a function of the infinitesimal growth rate μ for some different values of volatility σ, jump intensity λ and jump size distribution parameter a.

5 Concluding comments

In this study we presented a certainty equivalent characterization of the value of a single threshold investment policy in a spectrally negative geometric Lévy market. We also stated a set of sufficient conditions under which such a threshold timing policy is optimal and in that way derived a certainty equivalent formulation for a relatively broad class of irreversible investment problems in the presence of downside risk. By transforming the original problem subject to discontinuous stochastic dynamics to a

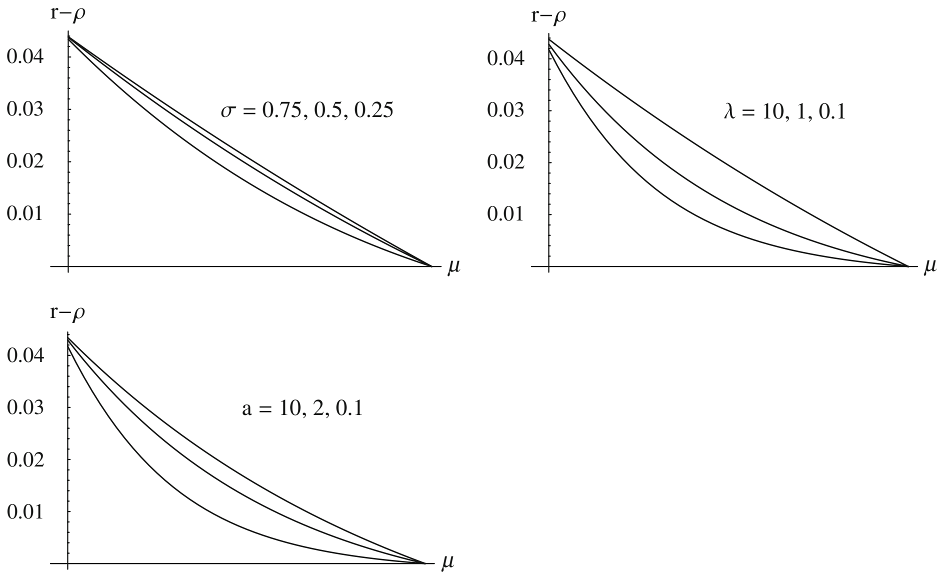

Fig. 4 The risk adjustment $r - \rho$ as a function of μ for some different values of risk parameters

problem subject to risk-adjusted continuous deterministic dynamics, we were able to use the tools of standard differential calculus to obtain comparative static results on the dependency of the risk adjustment on the model parameters such as volatility and jump intensity. According to our findings, the impact of increased volatility or increased jump intensity on the risk-adjustments depend on the convexity or concavity of the value on the region where the investment opportunity is left unexercised. We showed that increased volatility or increased jump intensity decreases (increases) the risk adjusted discount rate in the convex (concave) case. Analogously, increased volatility or increased jump intensity increases (decreases) the risk adjusted growth rate in the convex (concave) case.

Generalizing our analysis to situations allowing upward jumps would naturally constitute an interesting extension of our analysis. However, in that case establishing a representation of the values of the optimal threshold policies in terms of the minimal excessive maps is not so straightforward as in the present setting since the supremum process of the underlying is no longer continuous. Consequently, such a generalization would require different techniques than the one we have applied here, and is hence out of the scope of the present study. A more promising direction in many ways is to consider a wider class of spectrally negative jump diffusions. However, as such a general setting nonetheless is far more complex to analyze, we leave it for future research.

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A Proof of Lemma 2.1

Proof Since

$$\mathbb{E}_x[e^{-r\tau_y}; \tau_y < \infty] = \left(\frac{x}{y}\right)^{\psi}$$

for all $x \leq y$ we find that

$$\mathbb{P}_x[\tau_y < \infty] = \lim_{r\downarrow 0}\left(\frac{x}{y}\right)^{\psi}.$$

Consider now the convex function

$$P_0(\psi) = \frac{1}{2}\sigma^2\psi(\psi-1) + (\mu + \lambda\overline{m})\psi - \lambda + \lambda\int_0^1 e^{\psi\ln(1-z)}\mathfrak{m}(dz).$$

Since $P_0(0) = 0$, $P_0(1) = \mu > 0$, and

$$P_0'(0) = \mu - \frac{1}{2}\sigma^2 + \lambda\int_0^1 (z + \ln(1-z))\mathfrak{m}(dz)$$

we notice that condition (5) guarantees that $\lim_{r\downarrow 0}\psi = 0$ and, therefore, that $\mathbb{P}_x[\tau_y < \infty] = 1$ for all $x \in \mathbb{R}_+$. □

B Proof of Lemma 2.2

Proof Denote the Lipschitz constant of $f(x)$ by K. Since

$$(R_r f)(x) = \mathbb{E}_0\int_0^{\infty} e^{-rs} f(xe^{Y_{s-}})ds,$$

where Y is a Lévy process with dynamics

$$Y_t = \left(\mu + \lambda\overline{m} - \frac{1}{2}\sigma^2\right)t + \sigma W_t + \int_0^t\int_0^1 \ln(1-z)N(ds, dz),$$

$Y_0 = 0$, it follows that for $x, y \in (0, \infty)$

$$|(R_r f)(x) - (R_r f)(y)| \leq K|x-y|\mathbb{E}_0\int_0^{\infty} e^{-rs}e^{Y_{s-}}ds,$$

which shows that $(R_r f)(x)$ is continuous, if the expectation in the last expression is finite. But this expectation is the resolvent operator of Y applied to the function $y \mapsto e^y$. For $\tau \sim \exp(r)$ independent of Y, this resolvent, R_r^Y, satisfies equation

$$\mathbb{E}[f(Y_\tau)] = r(R_r^Y f)(0)$$

(see Bertoin 1996, I.2). Using conditioning on τ we get

$$(R_r^Y f)(0) = \frac{1}{r}\mathbb{E}\left\{\mathbb{E}\left\{e^{Y_\tau} \mid \tau\right\}\right\},$$

and by the mutual independence of the components of Y, the inner expectation has the following explicit expression as a function of τ:

$$\exp\left\{(\mu + \lambda\overline{m} + \lambda(\mathbb{E}[1/(1-z)] - 1))\tau\right\} =: \exp(\gamma\tau).$$

As $\tau \sim \exp(r)$, the outer expectation is computed by integrating $\exp((\gamma - r)t)$ with respect to t from 0 to ∞. The integral is finite if $\gamma - r < 0$, which is equivalent to (7).

Denote the transition density of Y by $p(s, y)$. Then for all $0 < x_1 \le x_2 < \infty$ and a non-decreasing function $f(x)$ we have

$$\begin{aligned}(R_r f)(x_1) - (R_r f)(x_2) &= \mathbb{E}_0 \int_0^\infty e^{-rs}\left(f(x_1 e^{Y_{s-}}) - f(x_2 e^{Y_{s-}})\right) ds \\ &= \int_0^\infty \int_0^\infty e^{-rs}\left(f(x_1 e^y) - f(x_2 e^y)\right) p(s, y) dy ds \le 0,\end{aligned}$$

so $(R_r f)(x)$ is non-decreasing.

Finally, to obtain the equivalence of problems (9) and (12), we note that

$$\mathbb{E}_x \int_0^\tau e^{-rs} f(X_{s-}) ds = (R_r f)(x) - \mathbb{E}_x \int_\tau^\infty e^{-rs} f(X_{s-}) ds$$

and do the change of variables $s = \tau + u$ in the last term. Then, conditioning on X_τ and observing that $X_{\tau+u} = X_\tau \hat{X}_u$, where $\hat{X} \sim (X \mid X_0 = 1)$, we get

$$\mathbb{E}_{X_\tau} \int_0^\infty e^{-ru} f(X_{\tau+u}) du = (R_r f)(X_\tau),$$

which establishes the required result. The proof for the entry problem is completely analogous. □

C Proof of Theorem 3.1

Proof By virtue of the spectral negativity of X, $g(X_{\tau_y}) = g(y)$ a.s., and by Theorem 3.2 in Alvarez and Rakkolainen (2006) $\mathbb{E}_x[e^{-r\tau_y}] = \left(\frac{x}{y}\right)^{\psi}$ for $x < y$. Hence

$$J^y(x) = \mathbb{E}_x\left[e^{-r\tau_y} g(X_{\tau_y})\right] = \begin{cases} g(x), & x \geq y, \\ g(y)\left(\frac{x}{y}\right)^{\psi}, & x < y. \end{cases} \tag{14}$$

Thus in the stochastic case the expected present value of a stopping policy based on a single threshold has a simple representation in terms of the payoff g and the increasing fundamental solution of $(\mathcal{L}_r u)(x) = 0$. On the other hand, for the deterministic process Z it is easy to compute $t_y = (1/\alpha)(\ln y - \ln x)$ and consequently

$$L^y_{\theta,\alpha}(x) = \begin{cases} g(x), & x \geq y, \\ g(y)\left(\frac{x}{y}\right)^{\theta/\alpha}, & x < y. \end{cases} \tag{15}$$

We see that if $\psi = \theta/\alpha$, then (14) and (15) coincide and we are done. □

D Proof of Theorem 3.2

Proof Using the notation of (4), the value of the single threshold policy is $J^{x^*}(x) \leq V(x)$ by maximality of $V(x)$. To establish the opposite inequality, note that in the current setup $J^{x^*}(x) \in C^1((0,\infty)) \cap C^2((0,\infty) \setminus \{x^*\})$ with the limits $\lim_{x \to x^* \pm} |D^2_{xx} J^{x^*}(x)| < \infty$, $J^{x^*}(x) \geq g(x)$ for all $x \in (0,\infty)$, $J^{x^*}(x)$ is r-harmonic for all $x < x^*$ and the assumptions of the theorem guarantee the superharmonicity of $J^{x^*}(x)$ for $x \geq x^*$. Hence $J^{x^*}(x)$ is a superharmonic majorant of $g(x)$ and consequently $J^{x^*}(x) \geq V(x)$ since $V(x)$ is the least superharmonic majorant of $g(x)$. Thus

$$V(x) = J^{x^*}(x) = \begin{cases} g(x), & x \geq x^* \\ g(x^*)\left(\frac{x}{x^*}\right)^{\psi}, & x < x^*, \end{cases} \tag{16}$$

where $x^* = \operatorname{argmax}\{x^{-\psi} g(x)\}$. On the other hand, the fundamental solution of $\alpha x \psi'(x) - \theta\psi(x) = 0$ is $x^{\theta/\alpha}$ and hence we also have

$$V^{\alpha}_{\theta}(x) = \begin{cases} g(x), & x \geq y^*_{\theta,\alpha} \\ g(y^*_{\theta,\alpha})\left(\frac{x}{y^*_{\theta,\alpha}}\right)^{\theta/\alpha}, & x < y^*_{\theta,\alpha}, \end{cases} \tag{17}$$

where $y^*_{\theta,\alpha} = \operatorname{argmax}\{x^{-\theta/\alpha} g(x)\}$. Thus we see that if $\psi = \theta/\alpha$, then $y^*_{\theta,\alpha} = x^*$ and $V^{\alpha}_{\theta}(x) = V(x)$. But since ψ solves (3), taking $\theta = \mu\psi$ and $\alpha = \mu$ yields (A), while taking $\theta = r$ and $\alpha = r/\psi$ yields (B). □

E Proof of Theorem 3.3

Proof It is clear from the definition of the root ψ that

$$\mu\psi = r - \lambda \int_0^1 [(1-z)^{\psi} + \psi z - 1]\mathfrak{m}(dz) - \frac{1}{2}\sigma^2\psi(\psi - 1).$$

Since the twice continuously differentiable (on (0, 1)) mapping $\pi(z) = (1-z)^{\psi} + \psi z - 1$ satisfies the conditions $\pi(0) = 0, \pi(1) = \psi - 1, \pi'(z) = \psi(1-(1-z)^{\psi-1})$, and $\pi''(z) = \psi(\psi-1)(1-z)^{\psi-2}$ we observe that $\mu\psi \lesseqqgtr r$ when $r \gtreqqless \mu$. Analogously, noticing that

$$\frac{r}{\psi} = \mu + \lambda \int_0^1 \psi^{-1}[(1-z)^{\psi} + \psi z - 1]\mathfrak{m}(dz) + \frac{1}{2}\sigma^2(\psi - 1)$$

demonstrates that $r/\psi \gtreqqless \mu$ when $r \gtreqqless \mu$.

Assume that $\hat{\sigma} \geq \sigma$ and $r > \mu$ and denote the increasing solution of the integro-differential equation $(\mathcal{L}_r^{\sigma} u)(x) = 0$ as x^{ψ_σ}, where

$$\left(\mathcal{L}_r^{\sigma} u\right)(x) = \frac{1}{2}\sigma^2 x^2 u''(x) + (\mu + \lambda\bar{m})xu'(x) - (r+\lambda)u(x) + \lambda \int_0^1 u(x - xz)\mathfrak{m}(dz).$$

It is now clear that

$$(\mathcal{L}_r^{\sigma} x^{\psi_{\hat{\sigma}}}) = \frac{1}{2}(\sigma^2 - \hat{\sigma}^2)\psi_{\hat{\sigma}}(\psi_{\hat{\sigma}} - 1)x^{\psi_{\hat{\sigma}}} < 0$$

since $\psi_{\hat{\sigma}} > 1$ whenever $r > \mu$. Dynkin's theorem now implies that for all $x < y$ we have

$$x^{\psi_{\hat{\sigma}}} > \mathbb{E}_x\left[e^{-r\tau_y} X_{\tau_y}^{\psi_{\hat{\sigma}}}\right] = y^{\psi_{\hat{\sigma}}}\mathbb{E}_x\left[e^{-r\tau_y}\right] = y^{\psi_{\hat{\sigma}}}\left(\frac{x}{y}\right)^{\psi_\sigma}$$

demonstrating that $\psi_\sigma > \psi_{\hat{\sigma}}$ whenever $r > \mu$. Establishing the opposite inequality when $r < \mu$ is entirely analogous. If $r = \mu$ then $\psi = 1$. Hence, we have established that

$$\frac{\partial\psi}{\partial\sigma} \lesseqqgtr 0 \quad \text{when} \quad r \gtreqqless \mu.$$

Assume now that $\hat{\lambda} \geq \lambda$ and $r > \mu$ and denote the increasing solution of the integro-differential equation $(\mathcal{L}_r^{\lambda} u)(x) = 0$ as x^{ψ_λ}, where

$$\left(\mathcal{L}_r^{\lambda}u\right)(x) = \frac{1}{2}\sigma^2x^2u''(x) + \mu xu'(x) - ru(x) + \lambda\int_0^1 [u(x-xz) - u(x) + u'(x)xz]\mathfrak{m}(dz).$$

Now

$$(\mathcal{L}_r^{\lambda}x^{\psi_{\hat{\lambda}}}) = (\lambda - \hat{\lambda})x^{\psi_{\hat{\lambda}}}\int_0^1 [(1-z)^{\psi_{\hat{\lambda}}} - 1 + \psi_{\hat{\lambda}}z]\mathfrak{m}(dz) < 0$$

by the convexity of the increasing solution $x^{\psi_{\hat{\lambda}}}$ when $r > \mu$. Applying analogous arguments as in the analysis of the sensitivity with respect to changes in the volatility coefficient now proves that

$$\frac{\partial\psi}{\partial\lambda} \lesseqqgtr 0 \quad \text{when} \quad r \gtreqqless \mu.$$

The alleged inequalities now follow directly from the definition of the risk adjustments by standard differentiation. □

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PAPER V

Teppo A. Rakkolainen: *A class of solvable Dirichlet problems associated to spectrally negative jump diffusions*

A CLASS OF SOLVABLE DIRICHLET PROBLEMS ASSOCIATED TO SPECTRALLY NEGATIVE JUMP DIFFUSIONS

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ABSTRACT
We consider the characteristic integro-differential equation associated to a spectrally negative Lévy diffusion with proportional jumps and natural boundaries, discounted at rate r, and present sufficient conditions for the unique increasing solution of this equation to be expressible in terms of a Frobenius series. This solution corresponds in some sense to the scale function (or r-scale function) of a linear diffusion (or a Lévy process) and can be utilized to solve several optimal control problems of such a Lévy diffusion. To illustrate the general result, we consider as specific examples two mean-reverting Lévy diffusions.

Keywords: Frobenius method; infinitesimal generator; integro-differential equation; Lévy diffusion; optimal stochastic control;

AMS Subject Classification: 45J05; 34B18; 93E20;

1 INTRODUCTION

Stochastic control theory plays a major role in modern financial and insurance mathematics, as several problems relevant in financial theory can conveniently be posed and (often) solved in the framework of optimal control of stochastic processes. In the optimal control problem quantities of interest are the *value function*, giving the value of the optimal action as a function of the state of the underlying process, and the corresponding optimal values of the *decision variables*.

To name the two applications which were the primary motivations for the author of this study, *real option models of the optimal timing of irreversible investment opportunities* lead to *optimal stopping* problems (cf. [8], [14], [16], [17], [18], [23], [24], [25] and [4]), and optimal *dividend payout* policies of a firm can be determined in the framework of *singular stochastic control* or *stochastic impulse control*, the choice depending on the absence or presence of transaction costs (cf. [1], [2], [3], [5], [7], [9], [10], [11], [19], [21], [22], [26] and [28]). It is well known (cf. [2], [3] and [7]) that if X is a continuous diffusion, then the value functions of these problems under a discount (or killing) rate $r \geq 0$ are characterizable in terms of the minimal r-harmonic maps, that is, the fundamental solutions of the second order linear differential equation $(\mathscr{A}u)(x) = ru(x)$, where $\mathscr{A}$ is the infinitesimal generator of X (see [13] for a thorough characterization of these mappings). The scale function of a continuous diffusion satisfies an equation of this type with $r = 0$ and the q-scale function of a Lévy process with $r = q$ (cf. [13] and [12] or [20]).

The representation of values in terms of minimal r-harmonic maps has been (partially) extended to spectrally negative jump diffusions and their infinitesimal generators $\mathscr{G}$ in [4] and [5] – however, subject to some restrictive assumptions. In particular, in these studies it is *assumed* that a smooth (that is, $\mathscr{C}^2$) increasing solution of the integro-differential equation defined by the infinitesimal generator of the jump diffusion satisfying the boundary condition $u(0) = 0$ exists. In general, questions of existence and uniqueness of a smooth solution to an integro-differential equation are not easy to resolve, and actually constructing a solution is a difficult task. Hence there is room for improvement, and this is what we are going to do in this study. We will derive a set of sufficient conditions under which the Dirichlet problem defined by the characteristic integro-differential equation

$$(\mathscr{G}u)(x) = ru(x) \tag{1}$$

together with the boundary condition $u(0) = 0$ has a unique (up to a multiplicative constant) increasing and non-negative solution expressible in terms of a Frobenius series.

2 BASIC SETUP AND ASSUMPTIONS

Let $(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F})$ be a filtered probability space equipped with the completed natural filtration $\mathbb{F} := \{\mathscr{F}_t^X\}_{t\in[0,\infty)}$ of a Lévy diffusion $X := \{X_t\}_{t\in[0,\infty)}$ with state space $S := (0,u)$, $u \in (0,\infty]$, and dynamics governed by the stochastic differential equation

$$dX_{t-} = \mu(X_{t-})dt + \sigma(X_{t-})dW_t - X_{t-}\int_0^1 z\tilde{N}(dt,dz), \tag{2}$$

$X_0 = x$. In equation (2) $W := \{W_t\}_{t\in[0,\infty)}$ is a standard Wiener process, $N(t,U)$ is a Poisson random measure with Lévy measure $\lambda\mathfrak{m}$ so that for each Borel set $U \subset \mathbb{R}$, $\bar{U} \cap \{0\} = \emptyset$, the process $\tilde{N}(t,U) := N(t,U) - \lambda\mathfrak{m}(U)t$ is a martingale. From equation (2) we see that process X has *proportional jumps* and is *spectrally negative*, i.e. can jump downwards but increases only continuously. We assume a constant discount rate $r > 0$. Coefficient functions μ and σ, discount rate r, intensity λ and distribution $\mathfrak{m}$ constitute what we shall refer to as the *given data*. We make the following assumptions concerning the behavior of X:

> *X1.* X is *regular*, i.e. $\mathbb{P}_x(\tau_y < \infty) = 1$ for all $x, y \in S$, where $\tau_y = \inf\{t > 0 : X_t \geq y\}$;
>
> *X2.* The boundaries 0 and u are *natural* for X, i.e. unattainable in finite time.

Furthermore, we assume that the functions $\mu : S \to \mathbb{R}$ and $a : S \to \mathbb{R}_+$, $a := \frac{1}{2}\sigma^2$, are *analytic at* 0, i.e. we assume

> *A1.* Maclaurin series $\mu(x) = \sum_{i=0}^{\infty} \frac{\mu^{(i)}(0)}{i!}x^i$ and $a(x) = \sum_{i=0}^{\infty} \frac{a^{(i)}(0)}{i!}x^i$ are absolutely convergent.

In addition, $\mu(x)$ and $a(x)$ are required to satisfy the boundary conditions

B1. $\mu(0)=0$, $\mu'(0)\geq 0$, $a(0)=0$ and $a^{'}(0)=0$

and the condition

R1. There exists $n\in\mathbb{N}$ such that $a^{(m)}(0)=0$ and $\mu^{(m)}(0)=0$ for all $m\geq n$.

Note, in particular, that the boundary conditions *B1* imply that $a^{''}(0)\geq 0$. For $f\in\mathscr{C}_0^2$, we define the integro-differential operator

$$\begin{aligned}(\mathscr{G}_r f)(x) \;&:= (\mu(x)+\lambda\overline{m}x)f^{'}(x)+\tfrac{\sigma^2(x)}{2}f^{''}(x)-(r+\lambda)f(x)+\\ &+\lambda\textstyle\int_0^1 f(x-xz)\mathfrak{m}(dz),\end{aligned} \tag{3}$$

where the Lévy measure $\nu:=\lambda\mathfrak{m}$ and $\overline{m}:=\int_0^1 z\mathfrak{m}(dz)$. This operator is related to the infinitesimal generator $\mathscr{G}$ of X via the identity $\mathscr{G}_r u=\mathscr{G}u-ru$.

To summarize, our standing assumptions on X and the coefficients of the stochastic differential equation (2) are *X1*, *X2*, *A1*, *B1* and *R1*.

3 EXISTENCE AND UNIQUENESS OF THE INCREASING SOLUTION

Some auxiliary notation (to be used throughout the rest of this study) and two simple lemmas are needed before we can state our main theorem. Define

$$\tilde{p}(\psi):=\frac{a^{''}(0)}{2}\psi^2+\left(\mu^{'}(0)+\lambda\overline{m}-\frac{a^{''}(0)}{2}\right)\psi-(r+\lambda)+\lambda\int_0^1(1-z)^{\psi}\mathfrak{m}(dz), \tag{4}$$

$e(\psi):=\lambda\int_0^1(1-z)^{\psi}\mathfrak{m}(dz)$ and $p(\psi):=\tilde{p}(\psi)-e(\psi)$. Moreover, denote

$$\bar{\psi}:=\arg\{\tilde{p}(\psi)=0\,:\,\psi>0\} \tag{5}$$

(if the right hand side exists) and denote by $\{\bar{c}_i\}_{i=0}^{\infty}$ the sequence such that the power series $\sum_{i=0}^{\infty}\bar{c}_i x^i$ is absolutely convergent and the $\bar{c}_i$ solve the recurrence

$$\begin{aligned}&\textstyle\sum_{j=1}^{n+1}c_{n+1-j}(\bar{\psi}+n+1-j)\frac{\mu^{(j)}(0)}{j!}+\\ &+\textstyle\sum_{j=2}^{n+2}c_{n+2-j}\left(\bar{\psi}^2+(2n-2j+3)\bar{\psi}+(n-j+2)(n-j+1)\right)\frac{a^{(j)}(0)}{j!}+\\ &+c_n\left(\lambda\overline{m}(\bar{\psi}+n)+\lambda\textstyle\int_0^1(1-z)^{\bar{\psi}+n}\mathfrak{m}(dz)-(r+\lambda)\right)=0\end{aligned} \tag{6}$$

with initial values $\bar{c}_i = c_i$, $i = 0, 1, \ldots, d$, in case such a sequence exists. Here the required number of initial values

$$d = \max_{n \in \mathbb{N}} \left\{ \sharp \{ i : A_{n,i} \neq 0 \} \right\},$$

where the coefficients $A_{n,i}$ are obtained from the representation

$$\sum_{i=0}^{n} A_{n,i} c_i = 0 \tag{7}$$

of the recurrence (6). These coefficients depend on the derivatives of $\mu(x)$ and $a(x)$ at $x = 0$, and by virtue of assumption *R1*, d is finite. Armed with these notations, we are in a position to state the following lemma giving sufficient conditions for the existence of a unique positive root $\bar{\psi}$ of $\tilde{p}(\psi) = 0$.

Lemma 3.1. *Under our standing assumptions there exists a unique $\bar{\psi} > 0$ such that $\tilde{p}(\bar{\psi}) = 0$. Furthermore, $\bar{\psi} \lesseqqgtr 1$ if $\mu^{'}(0) \gtreqqless r$.*

Proof. Using the notations introduced in the beginning of this section, we consider $\tilde{p}(\psi) = p(\psi) + e(\psi)$ as the sum of a second degree polynomial $p(\psi)$ and a "perturbation" $e(\psi)$. For $\psi > 0$ we have

$$e(\psi) = \lambda \int_0^1 (1-z)^{\psi} \mathfrak{m}(dz) \in (0, \lambda)$$

and thus $p(\psi) \leq \tilde{p}(\psi) \leq p(\psi) + \lambda$ for $\psi \in (0, \infty)$. Hence $\tilde{p}(0) \in (-(r + \lambda), -r)$. If $a^{''}(0) > 0$, then $p(\psi) \to \infty$, as $\psi \to \infty$, and hence also $\tilde{p}(\psi) \to \infty$, as $\psi \to \infty$, which by continuity of $\tilde{p}(\psi)$ establishes the existence of a positive root to $\tilde{p}(\psi) = 0$. Since the sum of convex functions is convex and $\tilde{p}(0) < 0$, this positive root is necessarily unique. Similar arguments apply, if $a^{''}(0) = 0$ and $\mu^{'}(0) > -\lambda \overline{m}$. Moreover, $\tilde{p}(1) = \mu^{'}(0) - r$, which establishes the last claim. □

The following lemma requires one additional assumption.

Lemma 3.2. *Let ψ be the unique positive root of $\tilde{p}(\psi) = 0$ and assume that*

$$\mu'(0) + \lambda \overline{m} + \left(\psi + \frac{1}{2} \right) a''(0) > -\lambda \int_0^1 \ln(1-z)(1-z)^{\psi+1} \mathfrak{m}(dz) \tag{8}$$

Then the function

$$\begin{aligned} A_{k,k} \;\; &:= (\psi + k)(\mu'(0) + \lambda \overline{m}) + \left(\psi^2 + (2k-1)\psi + k(k-1) \right) \tfrac{a''(0)}{2} + \\ &+ \lambda \textstyle\int_0^1 (1-z)^{\psi+k} \mathfrak{m}(dz) - r - \lambda \end{aligned}$$

(which corresponds to $A_{n,n}$ in equation (7)) is strictly positive for $k \in [1, \infty)$.

Proof. Because ψ is the root of $\tilde{p}(\psi)=0$, we have

$$A_{1,1} = \tilde{p}(\psi)+a''(0)\psi+\mu'(0)+\lambda\overline{m}+\lambda\int_0^1(1-z)^{\psi+1}\mathfrak{m}(dz)-\lambda\int_0^1(1-z)^{\psi}\mathfrak{m}(dz)$$

$$= a''(0)\psi+\mu'(0)+\lambda\left(\int_0^1 z\mathfrak{m}(dz)-\int_0^1 z(1-z)^{\psi}\mathfrak{m}(dz)\right),$$

and this is strictly positive under our assumptions on $a''(0)$ and $\mu'(0)$. Furthermore,

$$\frac{\partial}{\partial k}A_{k,k}=\mu'(0)+\lambda\overline{m}+\psi a''(0)+(2k-1)\frac{a''(0)}{2}+\lambda\int_0^1\ln(1-z)(1-z)^{\psi+k}\mathfrak{m}(dz)$$

is nondecreasing in k for $k\geq 1$ and at $k=1$ equals

$$\mu'(0)+\lambda\overline{m}+\psi a''(0)+\frac{a''(0)}{2}+\lambda\int_0^1\ln(1-z)(1-z)^{\psi+1}\mathfrak{m}(dz),$$

which is positive by assumption (8). □

Note that the additional assumption (8) is of different type than our standing assumptions listed in Section 2, as it involves the root of $\tilde{p}(\psi)=0$, which is typically solvable only numerically.

With the help of lemmas 3.1 and 3.2, we can prove the following theorem establishing the existence of a smooth solution to the Dirichlet problem given by (1) together with the boundary condition $u(0)=0$. The theorem relies on the Frobenius method for obtaining series solutions to linear differential equations, as presented in [31]. Due to the form in which the jump component enters the stochastic differential equation (2), this method can be applied to our integro-differential equation.

Theorem 3.3. *If the series $\sum_{i=0}^{\infty}\bar{c}_i x^i$ is absolutely convergent and condition* (8) *is satisfied, then $\psi(x):=x^{\bar{\psi}}\sum_{i=0}^{\infty}\bar{c}_i x^i$ is a smooth solution of* (1) *such that $\psi(0)=0$. Moreover, this solution is unique (up to a multiplicative constant) in the class of solutions expressible in terms of a Frobenius series.*

Proof. Consider the function $\phi(x):=x^{\psi}\sum_{k=0}^{\infty}c_k x^k$, where $\psi>0$ and $\{c_k\}_{k=0}^{\infty}$ is a sequence of real numbers such that the above power series is absolutely convergent. Then it is straightforward to calculate

$$\phi'(x)=x^{\psi-1}\sum_{k=0}^{\infty}c_k(\psi+k)x^k,$$

$$\phi''(x)=x^{\psi-2}\sum_{k=0}^{\infty}c_k\left(\psi(\psi-1)+2\psi k+k(k-1)\right)x^k \text{ and}$$

$$\phi(x-xz)=x^{\psi}(1-z)^{\psi}\sum_{k=0}^{\infty}c_k x^k(1-z)^k.$$

By plugging these expressions into equation (1) we get the equation

$$\begin{aligned}&\mu(x)\sum_{k=0}^{\infty}c_k(\psi+k)x^{k-1}+a(x)\sum_{k=0}^{\infty}c_k\left(\psi(\psi-1)+2\psi k+k(k-1)\right)x^{k-2}-\\&-(r+\lambda)\sum_{k=0}^{\infty}c_kx^k+\lambda\overline{m}\sum_{k=0}^{\infty}c_k(\psi+k)x^k+\\&+\lambda\sum_{k=0}^{\infty}c_k\int_0^1(1-z)^{\psi+k}\mathfrak{m}(dz)x^k=0.\end{aligned} \tag{9}$$

Replacing now $\mu(x)$ and $a(x)$ by their Maclaurin series and applying the Cauchy rule for products of absolutely convergent infinite sums this can be written in the form

$$\sum_{k=0}^{\infty}d_kx^k=0, \tag{10}$$

where the coefficient d_k depends on $\{\mu^{(i)}(0)\}_{i=0}^{k+2}$, $\{a^{(i)}(0)\}_{i=0}^{k+2}$, $\mathfrak{m}$, λ, r, ψ and $\{c_i\}_{i=0}^{k+2}$. This follows from assumption *B1*, under which the coefficients of negative powers of x in (10) equal zero: to illustrate, we can represent the computations of the second product in equation (9) in the form of a table as

$i \mid k$	0	1	2	$\ldots$
0	$a(0)c_0\psi(\psi-1)x^{-2}$	$a(0)c_1\psi(\psi+1)x^{-1}$	$a(0)c_2(\psi^2+3\psi+2)x^0$	$\ldots$
1	$a'(0)c_0\psi(\psi-1)x^{-1}$	$a'(0)c_1\psi(\psi+1)x^0$	$a'(0)c_2(\psi^2+3\psi+2)x^1$	$\ldots$
2	$\frac{a''(0)}{2}c_0\psi(\psi-1)x^0$	$\frac{a''(0)}{2}c_1\psi(\psi+1)x^1$	$\frac{a''(0)}{2}c_2(\psi^2+3\psi+2)x^2$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

from which is easy to see that under our assumptions the first two lines will contain only zeroes. By general properties of power series, equation (10) holds, if and only if $d_k=0$ for all $k\in\mathbb{N}\cup\{0\}$. These relations can be represented as an infinite system of linear equations

$$\begin{array}{llllll}A_{0,0}c_0 & +A_{0,1}c_1 & +A_{0,2}c_2 & & & =0\\ A_{1,0}c_0 & +A_{1,1}c_1 & +A_{1,2}c_2 & +A_{1,3}c_3 & & =0\\ \vdots & \vdots & \vdots & & & \vdots\\ A_{k,0}c_0 & +A_{k,1}c_1 & +A_{k,2}c_2 & +\ldots & +A_{k,k+2}c_{k+2} & =0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\end{array} \tag{11}$$

Writing out $d_0=0$ (the first equation of system (11)) explicitly we get

$$\begin{aligned}&c_0\left[\frac{a''(0)}{2}\psi^2+\left(\mu'(0)+\lambda\overline{m}-\frac{a''(0)}{2}\right)\psi-(r+\lambda)+\lambda\int_0^1(1-z)^{\psi}\mathfrak{m}(dz)\right]+\\&+c_1\left[(\psi+1)\mu(0)+\psi^2a'(0)\right]+c_2(\psi^2+3\psi+2)a(0)=0.\end{aligned} \tag{12}$$

The coefficient of c_0 in (12) can be written, using the notations introduced in the beginning of this section, as $\tilde{p}(\psi)$. Assumption *B1* implies that the coefficients of c_1 and c_2 in (12) vanish, and hence under our assumptions $d_0 = 0$, if and only if ψ solves $\tilde{p}(\psi) = 0$. By lemma 3.1, under the assumptions of the theorem the positive solution of $\tilde{p}(\psi) = 0$ exists and is unique. Generally, the explicit expression for $d_k = 0$ in (10) is

$$\begin{aligned} &\textstyle\sum_{j=0}^{k+1} c_{k+1-j}(\psi+k+1-j)\frac{\mu^{(j)}(0)}{j!}+ \\ &+\textstyle\sum_{j=0}^{k+2} c_{k+2-j}\left(\psi^2+(2k-2j+3)\psi+(k-j+2)(k-j+1)\right)\frac{a^{(j)}(0)}{j!}+ \\ &+c_k\left[\lambda\overline{m}(\psi+k)+\lambda\textstyle\int_0^1(1-z)^{\psi+k}\mathfrak{m}(dz)-(r+\lambda)\right]=0. \end{aligned} \tag{13}$$

Again by assumption *B1*, terms containing c_{k+1} and c_{k+2} in equation (13) will vanish, yielding the linear recursion relation (6), which is of form $A_{k,k}c_k = \sum_{j=0}^{k-1} A_{k,j}c_j$. Moreover, by assumption *R1* only finitely many $A_{n,i}$ differ from zero and consequently the system of linear equations (11) is under our assumptions a finite system of d equations:

$$\begin{array}{llllll} A_{0,0}c_0 & & & & & =0 \\ A_{1,0}c_0 & +A_{1,1}c_1 & & & & =0 \\ \vdots & \vdots & \vdots & & & \vdots \\ A_{d,0}c_0 & +A_{d,1}c_1 & +A_{d,2}c_2 & +\ldots & +A_{d,d}c_d & =0, \end{array} \tag{14}$$

from which it is easy to see that if $A_{k,k} \neq 0$, for all $k = 1,2,\ldots,d$, then we can choose any $c_0 \neq 0$ and successively solve c_i for $i = 1,2,\ldots,d$ from (14). By lemma 3.2, the map $k \mapsto A_{k,k}$ is nondecreasing on $[1,\infty)$ and $A_{1,1} > 0$, if we choose $\psi = \bar{\psi}$, where $\bar{\psi}$ is defined in (5). Hence, with this choice of ψ we necessarily have $A_{k,k} > 0$, for all $k = 1,2,\ldots,d$ and we can solve the values $c_1,\ldots,c_d$ required for initiating the recursion.

If the sequence $\{\bar{c}_i\}_{i=0}^{\infty}$ is such that $\sum_{k=0}^{\infty}\bar{c}_k x^k$ converges, then the function

$$x^{\bar{\psi}}\sum_{i=0}^{\infty}\bar{c}_i x^i = \psi(x)$$

is a solution of (1) which is obviously smooth and vanishes at $x = 0$. Since we can freely choose $\bar{c}_0$, any scalar multiple of $\psi(x)$ is also a solution, so that this solution is unique up to a multiplicative constant in the class of solutions of form $x^{\psi}\sum_{i=0}^{\infty}c_i x^i$. □

A question left unanswered by theorem 3.3 is whether there are smooth solutions $u(x)$ to (1) which satisfy the boundary condition $u(0) = 0$ and are not of the form $x^{\psi} \sum_{i=0}^{\infty} c_i x^i$. The next theorem states that under our assumptions such solutions cannot exist.

Theorem 3.4. *Under our assumptions a smooth solution $\psi(x)$ of the Dirichlet problem*

$$\begin{cases} (\mathscr{G}_r \psi)(x) = 0, & x \in S \\ \psi(0) = 0 \end{cases} \tag{15}$$

is unique up to a multiplicative constant. Moreover, it is necessarily monotone.

Proof. Let $\psi(x)$ be a smooth solution of $(\mathscr{G}_r \psi)(x) = 0$ such that $\psi(0) = 0$. By regularity of X, $\tau_y := \inf\{t \geq 0 : X_t \in [y,u)\} < \infty$ (a.s.) for any $y \in S$. Hence by the Dynkin formula

$$\mathbb{E}_x \left\{ e^{-r\tau_y} \psi(X_{\tau_y}) \right\} = \psi(x) + \mathbb{E}_x \int_0^{\tau_y} e^{-rt} (\mathscr{G}_r \psi)(X_t) dt = \psi(x).$$

By spectral negativity of X, if $x < y$, then $\psi(X_{\tau_y}) = \psi(y)$ (a.s.). Thus we get

$$(0,1] \ni \mathbb{E}_x \left\{ e^{-r\tau_y} \right\} = \psi(x)/\psi(y) \tag{16}$$

for any $0 < x < y < u$. In particular, this representation implies that $\psi(x)$ is monotone on S.

Suppose that $\eta(x)$ is also a smooth solution of the considered Dirichlet problem. But then $\mathbb{E}_x\{e^{-r\tau_y}\}$ has a similar representation in terms of $\eta(x)$ as the one given in (16) in terms of $\psi(x)$, and this implies that $\eta(x) = [\eta(y)/\psi(y)]\psi(x)$. Hence $\eta(x)$ is a scalar multiple of $\psi(x)$. □

Now we can state our solvability result as a corollary of theorems 3.3 and 3.4.

Corollary 3.5. *Under the assumptions of theorems 3.3 and 3.4, the function*

$$\psi(x) = x^{\bar{\psi}} \sum_{k=0}^{\infty} \bar{c}_i x^k \tag{17}$$

is the unique (up to a multiplicative constant) monotone and smooth solution of Dirichlet problem (15).

Note in particular that we can always choose the multiplicative constant in such a way that we obtain an increasing solution, which then by virtue of the boundary condition $\psi(0) = 0$ will be non-negative.

4 ILLUSTRATIONS

In this last section we illustrate our general findings with two explicit examples, both being instances of a stochastic version of the logistic growth model. We shall compare the obtained solutions with the known solutions of the corresponding (linear differential) equations for a continuous diffusion (cf. [1] and [7]). In the first example the state space is the positive half-line.

Example 4.1. *(Logistic jump diffusion on* $(0,\infty)$*) Consider a regular jump diffusion living on* $(0,\infty)$ *with dynamics characterized by*

$$dX_{t-} = X_{t-}\left\{(a - bX_{t-})dt + \sigma dW_t - \int_0^1 z\tilde{N}(dt,dz)\right\}, \tag{18}$$

$X_0 = x > 0$, *where* a, b *and* σ *are strictly positive real numbers and the Lévy measure of Poisson point process* N *is* $[\lambda/B(\alpha,\beta)]z^{\alpha-1}(1-z)^{\beta-1}dz$ *with* $B(\alpha,\beta)$ *being the Beta function (i.e. relative jump sizes follow a Beta distribution). In this case,* $\mu(x) = ax - bx^2$ *and* $a(x) = \frac{1}{2}\sigma^2 x^2$ *are analytic and satisfy boundary conditions* B1. *Furthermore, now*

$$p(\psi) = \frac{\sigma^2}{2}\psi^2 + (a + \lambda\overline{m} - \frac{\sigma^2}{2})\psi - (r+\lambda)$$

and $e(\psi) = (\lambda/B(\alpha,\beta))\int_0^1 z^{\alpha-1}(1-z)^{\psi+\beta-1}dz = \lambda\,(B(\alpha,\psi+\beta)/B(\alpha,\beta))$. *By theorem 3.3 the unique smooth increasing solution to the integro-differential equation associated with jump diffusion* (18) *is given by*

$$\psi(x) = x^{\bar{\psi}}\sum_{i=0}^{\infty}\bar{c}_i x^i, \tag{19}$$

where $\bar{\psi}$ *is as in theorem 3.3 and the sequence* $\{\bar{c}_i\}$ *solves recurrence*

$$\begin{aligned} & c_n(\bar{\psi}+n)\tfrac{\mu^{(1)}(0)}{1!} + c_{n-1}(\bar{\psi}+n-1)\tfrac{\mu^{(2)}(0)}{2!} \\ & +c_n\left(\bar{\psi}^2 + (2n-3)\bar{\psi} + n(n-1)\right)\tfrac{a^{(2)}(0)}{2!} + \\ & +c_n\left(\lambda\overline{m}(\bar{\psi}+n) + \lambda\textstyle\int_0^1(1-z)^{\bar{\psi}+n}\mathfrak{m}(dz) - (r+\lambda)\right) = 0. \end{aligned} \tag{20}$$

In this particular example the convergence of the power series in (19) *presents no problems, as recursion* (20) *can easily be rearranged to form* $c_n = [s(n)/t(n)]c_{n-1}$ *with* $s(n)$ *a first degree polynomial in* n *and* $t(n)$ *essentially a second degree* *polynomial in n since the integral term containing n is bounded from above and below. Hence* $|\bar{c}_n/\bar{c}_{n-1}| \to 0$ *as* $n \to \infty$ *and thus the power series in* (19) *has radius of convergence* ∞.

If there is no jump component $\tilde{N}$, *process X is a continuous diffusion and the solution* $\varphi(x)$ *of the corresponding linear differential equation can be expressed in terms of confluent hypergeometric functions as*

$$\varphi(x) = x^{\eta} M(\eta, 1+\eta-\alpha, 2abx/\sigma^2), \tag{21}$$

where M is the Kummer confluent hypergeometric function and

$$\begin{aligned} \eta &= \tfrac{1}{2} - \tfrac{a}{\sigma^2} + \sqrt{\left(\tfrac{1}{2} - \tfrac{a}{\sigma^2}\right)^2 + \tfrac{2r}{\sigma}^2} \\ \alpha &= \tfrac{1}{2} - \tfrac{a}{\sigma^2} - \sqrt{\left(\tfrac{1}{2} - \tfrac{a}{\sigma^2}\right)^2 + \tfrac{2r}{\sigma}^2} \end{aligned} \tag{22}$$

(cf. [7]).

As a numerical illustration, consider the case $a = 0.1$, $b = 0.05$, $\sigma = 0.3$, $\lambda = 0.1$, $\alpha = \beta = 1$ *and* $r = 0.05$. *Then the positive root of* $\tilde{p}(\psi) = 0$ *is* $\bar{\psi} \approx 0.66667$ *and we can check numerically that condition* (8) *is satisfied. Taking the first* 250 *terms* ($\bar{c}_{249}$ *is here of order* 10^{-488}) *of the power series, we obtain the numerical approximation of* $\psi(x)$ *presented graphically on the left in Figure 1 (unbroken curve). For comparison, we have also plotted the increasing fundamental solution of the linear differential equation associated with the diffusion* $X_t + X_t \int_0^1 z\tilde{N}(dt,dz)$ *(dashed curve). Note that the magnitudes of the function values are meaningless, since the solutions are unique only up to a multiplicative constant. It appears to be the case that although the solutions have broadly similar convexity properties (first concave, then after a certain threshold convex), the impact of jumps on the value of this threshold is quite dramatic: in presence of jumps the threshold value* ($x^* \approx 0.694$) *is only a fraction of the corresponding value in absence of jumps* ($\tilde{x}^* \approx 15.3$). *As it is precisely this threshold value which is of essential importance in determining the optimal dividend distribution or optimal harvesting strategies (cf. [7] and [5]), this result illustrates the fact that adding martingale jumps to the underlying dynamics does have serious implications from the point of view of optimal control.*

In table 1, we have solved the critical value x^* *such that* $\psi''(x^*) = 0$ *for different values of* σ. *For comparison, also critical values* $\tilde{x}^*$ *such that* $\varphi''(\tilde{x}^*) = 0$

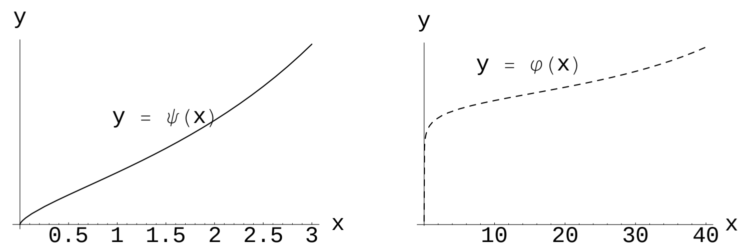

Figure 1: The increasing solutions $\psi(x)$ and $\varphi(x)$ for Example 1.

σ	0.1	0.25	0.5	0.75	0.9
x^*	0.568	0.659	0.814	0.899	0.926
$\tilde{x}^*$	10.8	14.0	19.4	19.5	19.5

Table 1: *Critical values x^* for the jump diffusion of Example 1 for different values of parameter σ.*

are presented in the table.

Our second example illustrates the case of a state space with a finite upper boundary. This example has been considered in the context of optimal harvesting of a population in [22] and [1] in absence of jumps.

Example 4.2. *(Logistic jump diffusion on $(0,1/c)$) Consider a regular jump diffusion living on $(0,1/c)$ with dynamics characterized by*

$$dX_{t-} = X_{t-}\left\{a(1-cX_{t-})dt + b(1-cX_{t-})dW_t - \int_0^1 z\tilde{N}(dt,dz)\right\}, \qquad (23)$$

$X_0 = x > 0$, where $a,b,c \in \mathbb{R}_+$ and the characteristic measure of Poisson point process N is again $[\lambda/B(\alpha,\beta)]z^{\alpha-1}(1-z)^{\beta-1}dz$ with $B(\alpha,\beta)$ being the Beta function. In this case $\mu(x) = ax - acx^2$ and $a(x) = \frac{1}{2}b^2x^2(1-cx)^2$. These are analytic at $x = 0$ and differentiation yields

$$a^{'}(x) = b^2x(1-cx)(1-2cx), \quad a^{''}(x) = b^2(1-6cx+6c^2x^2),$$

$$a^{(3)}(x) = b^2(-6c+12c^2x), \qquad a^{(4)}(x) = 12b^2c^2,$$

and $a^{(i)}(x) = 0$ for $i \geq 5$. By theorem 3.3 the unique increasing solution to the integro-differential equation associated with jump diffusion (23) is given by

$$\psi(x) = x^{\bar{\psi}} \sum_{i=0}^{\infty} \bar{c}_i x^i, \qquad (24)$$

where $\bar{\psi}$ is as in theorem 3.3 and the sequence $\{\bar{c}_i\}$ solves recurrence

$$\begin{aligned}
&c_n\big[(a+\lambda\overline{m})(\bar{\psi}+n)+\left(\bar{\psi}^2+(2n-1)\bar{\psi}+n(n-1)\right)\tfrac{b^2}{2}+\\
&+\lambda\textstyle\int_0^1(1-z)^{\bar{\psi}+n}\mathfrak{m}(dz)-(r+\lambda)\big]-\\
&-c_{n-1}\left[ac(\bar{\psi}+n-1)+b^2c\left(\bar{\psi}^2+(2n-3)\bar{\psi}+(n-1)(n-2)\right)\right]+\\
&+c_{n-2}\left[\tfrac{1}{2}b^2c^2\left(\bar{\psi}^2+(2n-5)\bar{\psi}+(n-2)(n-3)\right)\right]=0,
\end{aligned}\tag{25}$$

which is of form $r(n)c_n = s(n)c_{n-1} - t(n)c_{n-2}$. In this case the question of convergence of the power series in representation (24) *is slightly more problematic than in the previous example. We would, of course, like to have a radius of convergence $R \geq 1/c$ in order for* (24) *to be well-defined in the state space* $(0,1/c)$*. Using notations*

$$\begin{aligned}
&r(n) := (a+\lambda\overline{m})(\bar{\psi}+n)+\left(\bar{\psi}^2+(2n-1)\bar{\psi}+n(n-1)\right)\tfrac{b^2}{2}+\\
&+\lambda\textstyle\int_0^1(1-z)^{\bar{\psi}+n}\mathfrak{m}(dz)-(r+\lambda),\\
&s(n) := ac(\bar{\psi}+n-1)+b^2c\left(\bar{\psi}^2+(2n-3)\bar{\psi}+(n-1)(n-2)\right),\\
&t(n) := \tfrac{1}{2}b^2c^2\left(\bar{\psi}^2+(2n-5)\bar{\psi}+(n-2)(n-3)\right)
\end{aligned}$$

and dividing recurrence (25) *by $r(n)c_{n-1}$, we obtain*

$$\frac{c_n}{c_{n-1}} = \frac{s(n)}{r(n)} - \frac{t(n)}{r(n)}\frac{c_{n-2}}{c_{n-1}}. \tag{26}$$

Denoting now $\tilde{R} := \lim_{n\to\infty}\frac{c_{n-1}}{c_n} \neq 0$ (if $\tilde{R}=0$, then the radius of convergence would equal ∞ and there would be nothing to prove) and letting $n\to\infty$ in (26) *we obtain the equation*

$$\frac{1}{\tilde{R}} = \lim_{n\to\infty}\frac{s(n)}{r(n)} - \tilde{R}\lim_{n\to\infty}\frac{t(n)}{r(n)}.$$

It is an easy computation to check that $\lim_{n\to\infty}\frac{s(n)}{r(n)} = 2c$ and $\lim_{n\to\infty}\frac{t(n)}{r(n)} = c^2$. Then multiplication with $\tilde{R}$ leads to equation

$$c^2\tilde{R}^2 - 2c\tilde{R} + 1 = 0,$$

whose unique solution is $\tilde{R} = \frac{1}{c}$. *But then the radius of convergence* $R = |\tilde{R}| = \frac{1}{c}$ *and hence the power series in* (24) *always converges in the state space* $(0, 1/c)$.

If there is no jump component $\tilde{N}$, *process* X *is a continuous diffusion and the solution* $\varphi(x)$ *of the corresponding linear differential equation can be represented in terms of standard hypergeometric functions as*

$$\varphi(x) = \left(\frac{cx}{1-cx}\right)^{\alpha_1} F(\gamma_1, \gamma_2, \gamma_3, -\left(\frac{cx}{1-cx}\right)), \tag{27}$$

where F *is the hypergeometric function and*

$$\begin{aligned}
\gamma_1 &= 1 - \tfrac{\alpha_2}{2} + \tfrac{\alpha_1}{2} - \tfrac{1}{2}\sqrt{\alpha_2^2 - 2\alpha_2(2+\alpha_1) + (2-\alpha_1)^2} \\
\gamma_2 &= 1 - \tfrac{\alpha_2}{2} + \tfrac{\alpha_1}{2} + \tfrac{1}{2}\sqrt{\alpha_2^2 - 2\alpha_2(2+\alpha_1) + (2-\alpha_1)^2} \\
\gamma_3 &= 1 - \alpha_2 + \alpha_1 \\
\alpha_1 &= \tfrac{1}{2} - \tfrac{a}{b^2} + \sqrt{\left(\tfrac{1}{2} - \tfrac{a}{b^2}\right)^2 + \tfrac{2r}{b}^2} \\
\alpha_2 &= \tfrac{1}{2} - \tfrac{a}{b^2} + \sqrt{\left(\tfrac{1}{2} - \tfrac{a}{b^2}\right)^2 - \tfrac{2r}{b}^2}
\end{aligned} \tag{28}$$

(cf. [1]).

As a numerical illustration, consider the case $a = 0.1$, $b = 0.3$, $c = 1$, $\lambda = 0.1$, $\alpha = \beta = 1$ *and* $r = 0.05$. *Then the positive root of* $\tilde{p}(\psi) = 0$ *is* $\bar{\psi} \approx 0.66667$, *condition* (8) *can be checked numerically, and taking the first* 250 *terms of the power series, we obtain the numerical approximation of* $\psi(x)$ *presented graphically on the left in Figure 2 (unbroken curve). For comparison, the increasing fundamental solution* $\varphi(x)$ *given by* (27) *is also plotted on the right in Figure 2 (dashed curve). We can make the same observations as in the previous example: while convexity properties of the solutions are broadly similar, the presence of jumps has a considerable impact on the inflection point where the second derivative of the solution vanishes. Now* $x^* \approx 0.146$, *while* $\tilde{x}^* \approx 0.924$.

In table 2, we have solved the critical value x^* *such that* $\psi''(x^*) = 0$ *for different values of* b. *For comparison, also critical values* $\tilde{x}^*$ *such that* $\varphi''(\tilde{x}^*) = 0$ *are presented in the table. We see that while increased volatility increases the*

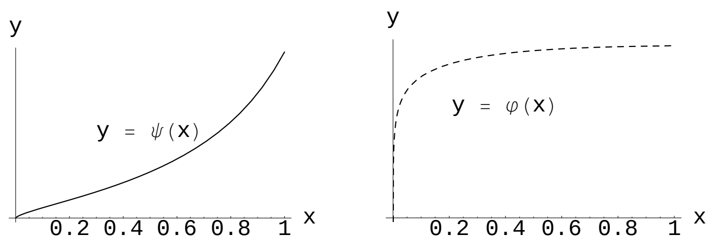

Figure 2: The increasing solutions $\psi(x)$ and $\varphi(x)$ for Example 2.

b	0.1	0.25	0.5	0.75	0.9
x^*	0.169	0.154	0.109	0.077	0.054
$\tilde{x}^*$	0.511	0.577	0.886	0.951	0.957

Table 2: *Critical value x^* for the jump diffusion of Example 2 for different values of parameter b.*

critical values in absence of jumps, the effect is reversed in presence of jumps (it should be noted, however, that in both cases the value functions *of the corresponding singular control problems decrease as volatility increases, in accordance with the general results on the impact of volatility on the value in [6]).*

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