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STABILITY ANALYSIS IN MULTICRITERIA DISCRETE PORTFOLIO OPTIMIZATION

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Abstract

Almost every problem of design, planning and management in the technical and organizational systems has several conflicting goals or interests. Nowadays, multicriteria decision models represent a rapidly developing area of operation research.

While solving practical optimization problems, it is necessary to take into account various kinds of uncertainty due to lack of data, inadequacy of mathematical models to real-time processes, calculation errors, etc. In practice, this uncertainty usually leads to undesirable outcomes where the solutions are very sensitive to any changes in the input parameters. An example is the investment managing.

Stability analysis of multicriteria discrete optimization problems investigates how the found solutions behave in response to changes in the initial data (input parameters).

This thesis is devoted to the stability analysis in the problem of selecting investment project portfolios, which are optimized by considering different types of risk and efficiency of the investment projects. The stability analysis is carried out in two approaches: qualitative and quantitative. The qualitative approach describes the behavior of solutions in conditions with small perturbations in the initial data. The stability of solutions is defined in terms of existence a neighborhood in the initial data space. Any perturbed problem from this neighborhood has stability with respect to the set of efficient solutions of the initial problem. The other approach in the stability analysis studies quantitative measures such as stability radius. This approach gives information about the limits of perturbations in the input parameters, which do not lead to changes in the set of efficient solutions.

In present thesis several results were obtained including attainable bounds for the stability radii of Pareto optimal and lexicographically optimal portfolios of the investment problem with Savage's, Wald's criteria and criteria of extreme optimism. In addition, special classes of the problem when the stability radii are expressed by the formulae were indicated. Investigations were completed using different combinations of Chebyshev's, Manhattan and Hölder's metrics, which allowed monitoring input parameters perturbations differently.

Tiivistelmä

Lähes kaikki reaalimaailman optimointiongelmat sisältävät useita ristiriitaisia tavoitteita. Monitavoitteiset päätöksentekomallit ovatkin yksi operaatiotutkimuksen nopeimmin kehittyvistä osa-alueista. Reaalimaailman optimointiongelmia ratkaistaessa täytyy ottaa huomioon erilaisia epävarmuustekijöitä. Näitä ovat esimerkiksi puutteellinen data, epätarkkuus matemaattisessa mallissa sekä laskuvirheet. Nämä epävarmuustekijät johtavat usein ei-haluttuihin lopputuloksiin, joissa saadut ratkaisut ovat hyvin herkkiä syöttöparametrien suhteen. Hyvä esimerkki tällaisesta ongelmasta on sijoitustenhallintaongelma. Yleisesti diskreettien monitavoiteoptimointiongelmien herkkyysanalyysissä tutkitaan, miten löydetyt ratkaisut käyttäytyvät lähtötietojen (syöttöparametrit) muuttuessa.

Tämä opinnäytetyö käsittelee arvopaperisalkkujen hallinnointiongelman (investointiportfolio-ongelman) herkkyysanalyysiä. Portfolio-ongelmassa optimoidaan erityypisiä riskejä sekä erilaisten investointien tehokkuutta. Herkkyysanalyysin suorittamiseen voidaan valita kaksi eri lähestymistapaa: kvalitatiivinen tai kvantitatiivinen. Kvalitatiivisessa lähestymistavassa tutkitaan ratkaisun käyttäytymistä, kun lähtötiedoissa on pientä häiriötä. Ratkaisujen herkkyys/stabiilisuus määritellään alkuperäisen datan pienen ympäristön olemassaolona ongelman parametriavaruudessa. Mikä tahansa häiritty ongelma, jonka lähtötiedot löytyvät tästä ympäristöstä, on stabiili alkuperäisen ongelman optimaalisten ratkaisujen suhteen.

Herkkyysanalyysin kvantitatiivisessa lähestymistavassa puolestaan tutkitaan määrällisiä mittoja, kuten ongelman stabiilisuussädettä. Stabiilisuussäteen avulla voidaan ilmoittaa lähtötietojen häiriön suuruudelle rajat, joiden sisällä optimaalinen ratkaisu ei muutu.

Tässä työssä esitetään lukuisia tuloksia kuten esimerkiksi. saavutettavissa olevat rajat stabiilisuussäteelle sekä Pareto-optimaaliselle että leksikografisesti optimaaliselle portfoliolle käyttäen Savagen, Waldin ja äärimmäisen optimismin kriteerejä. Lisäksi työssä esitellään portfolio-ongelman erityistapaukset, joissa stabiilisuussäde voidaan esittää tietyllä kaavalla. Tutkimukset suoritetaan erilaisten metriikoiden yhdistelminä, jolloin syöttöparametrien häiriöitä voidaan seurata usealla eri tavalla.

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Contents

Chapter 1. Introduction

Every time we make a choice and try to implement this choice as well as possible, we are choosing an optimal choice from the alternatives available for us. Consequently, our life is a sequence of our decisions that we analyze during decision making.

Decision analysis is the science and art of designing or choosing the best alternatives depending on the decision maker's goals and preferences. We want to choose what best fits our goals, desires, lifestyle, values, etc. [104]. The difficulty lies when a conflict occurs between these various objectives and goals. Despite the fact that majority of decisions and compromises are made on the basis of intuition or chance, there are areas where mathematical modeling and programming are needed [75].

The decision making process uses such tools as different kind of indicators, ratios, weights, procedures, algorithms, etc. [77]. These tools tend to overload the decision maker with large amounts of information and the number of intervening variables. This makes the decision making process hard as regards making a rational decision. Therefore, some strategy is required to organize, classify, and evaluate this information, and also to analyze the results and benefit from them. The models and methodologies help to obtain a rational analysis. These models offer an outcome which could be analyzed by the decision maker, adding or deleting concepts, objectives, alternatives, etc.

Usually, a decision making problem contains three components: decision makers, who make the decisions; decision alternatives, from which the decision maker can choose; and criteria that evaluate the alternatives [81]. We might have only one decision maker who is responsible for deciding what to do, or several people, or organizations being involved in the decision making process. When there is more than one decision maker present, then they might all have different preferences, goals, objectives and criteria. Frequently, there is no decision which satisfies to every decision maker.

In the case of multiple decision makers, we might consider the problem as a multiobjective problem, where the objectives of the different decision makers are considered as the objectives of the problem [104]. The multiplicity of objectives is not the same as the plurality of decision makers. While one decision maker can perform the multiple objectives evaluation, plural decision makers may pursue only one objective in their decision problems [86]. Typical multiobjective problems arise when a single decision maker considers several objectives simultaneously. The decision maker faces different goals, objectives or criteria. It is necessary to consider conflicting goals in decision making processes in order to develop sustainable systems.

A multiobjective approach helps the decision maker to organize and synthesize information in a way that leads them to feel comfortable and confident about making a decision, and minimizing the potential for post-decision regret by being satisfied that all the objectives or factors have properly been taken into account [4]. There are two major approaches to multiobjective decision making [86]. One is *multiobjective optimization* that has been developed for single objective problems and later integrated into dealing with multiobjective problems. This approach is useful for quantitative analysis of multiobjective models that can be applied in the analytical phase. The other approach is *multiobjective decision analysis*, which is useful for treating the judgmental phase with axiomatic and numerical representations. Multiobjective decision analysis concerns the subjective phase setting cooperation with rational procedures in decision processes. Each separate approach represents only one particular phase of multiobjective decision making process. Both multiobjective optimization and decision analysis should be combined with each other and as analytical and judgmental phases are included in the multiobjective decision analysis.

Later some tools for the decision maker are presented containing the multiobjective models, common methods for problem solving, and post-optimal analysis of the solutions. The focus is placed on the conditions and constraints in the behavior of found solutions of the multiobjective combinatorial problem under changes or uncertainty in the initial information.

Chapter 2 deals with multiobjective combinatorial optimization and widespread methods for the multiobjective problems with discrete variables. The methods divided into two categories. In the first section, we considered the multiobjective methods that were designed especially for problems with a multiobjective character and that basically combine the multiple objectives into one single objective. The second section describes the methods adapted from single objective combinatorial optimization for treating problems with multiple objectives.

Chapter 3 is devoted to combinatorial problem with many objectives such as a problem of investing in different kind of securities or investment projects. In addition to the classical Markowitz's investment portfolio problem, the chapter contains a modified version as the project portfolio problem. This version is presented as a project portfolio problem that takes into account the different kinds of risk and efficiencies of the projects.

In Chapter 4, the question of stability is considered. A survey of investigations in the field of post-optimal analysis is provided. The stability analysis shows which kind of perturbations may occur in the initial data in order to maintain the efficiency or inefficiency of the solutions. Quantitative and qualitative characteristics of stability are introduced for the multiobjective discrete problems.

The main topic of this thesis is the quantitative characteristic of the stability such as the stability radius. Additional attention is paid to this aspect in Chapter 5. This chapter shows how many deviations in the initial data may occur in order for some solutions to lose their preassigned properties. The bounds and formulae obtained for the stability radii of the project portfolio selection problem are shown. Additionally, Section 5.2 discusses approaches for calculating the stability radius.

Chapter 2. Multicriteria combinatorial problems

The methods developed for solving multiobjective optimization problems are mostly operated with continuous variables. However, it is well known that most of real-world processes are discrete. Moreover, multiobjective problems with discrete variables can have special difficulties which are different from those with continuous variables. The discrete problems cannot be solved by simply combining discrete programming methods and multiobjective programming methods [99]. Combinatorial optimization provides a powerful tool for modelling many real-world applications in operations research, engineering, biological sciences, and computer science. It allows optimization problems to be solved over discrete structures by combining techniques from combinatorics, mathematical programming, and the theory of algorithms [10]. Multiobjective combinatorial optimization provides a real challenge for the future, namely to introduce multiobjective approaches into the large class of combinatorial optimization applications. In this chapter the basics of multiobjective optimization are discussed and an introduction to some methods in multiobjective combinatorial optimization is presented, including the definitions of efficient (nondominated) solutions.

The general *multiobjective combinatorial optimization problem* is posed as follows:

$$\begin{aligned} \min \quad & f(x) = (f_1(x), f_2(x), \ldots, f_s(x)), \\ \text{subject to} \quad & x \in X, \end{aligned}$$

where $s \geq 2$ is the number of objective functions; $x = (x_1, x_2, \ldots, x_n)^T \in X \subset \mathbf{Z}^n$, $n \in \mathbf{N}$, is the vector representing the decision variables; n is the number of variables; X represents the feasible decision space; $f(x)$ is a vector of objective functions $f_k(x)$, $k = \overline{1, s}$, which are also called objectives, criteria, payoff functions, cost functions, or value functions.

Typically, in combinatorial optimization two types of objective functions (criteria) are considered, namely the *sum* and the *bottleneck objectives* [15]:

$$f_k(x) = \sum_{j=\overline{1,n}} c_{jk} x_j, \quad k = \overline{1, s}, \text{ or}$$

$$f_k(x) = \max_{j=\overline{1,n}} c_{jk} x_j, \quad k = \overline{1, s},$$

where c_{jk}, $j = \overline{1, n}$, are the elements of the k-th column of the matrix $C \in \mathbf{R}^{n \times s}$.

This definition includes multiobjective versions of the shortest path, minimum spanning tree, assignment, knapsack, travelling salesman problems [14].

In contrast to single objective optimization, a solution in a multiobjective problem is more of a concept than a definition. Typically, there is no single global solution because the objectives are usually in conflict. As a consequence, other concepts must be established to define what an efficient solution is. The predominant concept in defining an efficient solution is that of *Pareto optimality* [80], where it is impossible to make any objective of the solution better without making at least one objective of this solution worse. This is defined as follows:

Definition 1. *A solution $x^0 \in X$ is Pareto optimal if for any solution $x \in X$ $f_k(x) = f_k(x^0)$ for each $k = \overline{1,s}$, or there exists $k^0 = k^0(x)$ such that $f_{k^0}(x) > f_{k^0}(x^0)$.*

Another approach is lexicographic, when the decision maker determines the order in which the objectives have to be optimized according to their absolute importance.

Definition 2. *A solution $x^0 \in X$ is lexicographically optimal if $f_k(x) = f_k(x^0)$ for any $k = \overline{1,s}$, or for any solution $x \in X$, $f_{k^0}(x) > f_{k^0}(x^0)$ where $k^0 = \min\{k = \overline{1,s} : \ f_k(x) \neq f_k(x^0)\}$.*

This ordering means that the most important objective is infinitely more important than a less important objective. This implies that solutions are ordered by first evaluating them based on the foremost objective. If a set of solutions have comparable values in the foremost objective, the comparison continues onto lower level objectives until the solutions can be distinguished. If the most important objective has an alternative optima, it is then possible to use the next most important objective. If the most important objective function has a unique solution, the other objectives do not have any influence on the solution. Thus, the less important objective functions might not be taken into consideration at all. The main difficulty of this approach is to put the objective functions into an absolute order of importance.

Lexicographic methods are not so commonly used by themselfs in engineering design, but used jointly with other techniques, such as in goal programming or as a part of a selection mechanism in genetic algorithms [75].

The following discusses the complexity of solving multiobjective combinatorial optimization problems. The book [37] has an in-depth introduction to computational complexity.

Computational complexity theory provides information about the difficulty of solving a decision problem, with the answers “yes“ or “no“, and also provides the number of operations in an algorithm for obtaining these answers in the worst cases [13].

Definition 3. *A decision problem belongs to the class P of problems, if there exists a deterministic algorithm that answers the decision problem and performs $O(p(n))$ operations, where p is a polynomial of n, and n is the length* *of the string representing the input.*

Definition 4. *An algorithm is called polynomial time algorithm if there is a polynomial* p*, such that the running time of the algorithm is* $O(p(n))$.

It is worth noting, that algorithms for solving multiobjective combinatorial optimization problems are often non-polynomial.

A nondeterministic algorithm is an algorithm that, even for the same input, can exhibit different behaviors on different runs, as opposed to a deterministic algorithm.

Definition 5. *A decision problem belongs to the class* NP *if there is a nondeterministic polynomial time algorithm that solves the decision problem.*

An algorithm that solves a problem in nondeterministic polynomial time can run in polynomial time or exponential time depending on the choices it makes during execution.

With each decision problem there is also the counting problem, which counts "yes" answers.

Definition 6. *A counting problem belongs to the class* $\#P$ *if there exists a nondeterministic algorithm that find the answer to the counting problem and such that the longest computation that confirms a "yes" answer is bounded by a polynomial in the size of the instance.*

Definition 7. *A decision problem is* $NP(\#P)$*-complete if this problem belongs to the class* NP $(\#P)$ *and every problem from the class* NP $(\#P)$ *can be reduced to this decision problem in polynomial time.*

It transpires that multiobjective combinatorial optimization problems in the sense of finding or counting efficient solutions, in general, are NP- and $\#P$-complete, respectively. Moreover, the number of efficient solutions can be exponential as regards the size of the problem. This makes it difficult to find algorithms that run in a reasonable (polynomial) time, even when in the single objective case the problem has an efficient algorithm [12]. Consequently, such problems are called *intractable.*

For lexicographic optimization, it is known that a lexicographically optimal solution is always efficient. Lexicographic optimization can also be viewed as a special case of algebraic optimization [105].

Solving a multiobjective problem means finding the set of Pareto optimal solutions, also known as the Pareto set, the nondominated set, and the set of efficient or nondominated solutions. Generating Pareto optimal solutions is an important part of multiobjective optimization, but it is only the first phase in the decision making process. After finding the Pareto set the decision maker depending on his or her preferences chooses the right solution. The Pareto optimal solution must satisfy the decision maker's preference structure. This choice requires knowledge about the problem being treated and the factors related to it [3].

It is not necessary that the optimization phase and the analytical phase be consecutive. Depending on the situation in the decision process, there might be three forms of the cooperation between the problem solver and the (final) decision maker, as was mentioned in [14].

In a *priori form*, at the beginning of the decision making process all the preferences are known and used as parameters in the model. For example, goal-programming methods are based on this technique.

In a *posteriori form*, the set of all efficient solutions is first generated and then analyzed according to the decision maker's preferences. At the end of the process, one solution is chosen from among the set of solutions provided by the solver.

In an *interactive form* the decision maker and the solver cooperate all the time throughout the process. In addition, the decision maker introduces the preferences during the resolution process and uses the knowledge, obtained during the problem resolution, in the next preferences. Practical problems are often solved according to the interactive mode.

Further, several examples are presented of the exact methods for solving multiobjective combinatorial optimization. These methods are divided into two groups. The first group contains methods such as the scalarization, the compromise solution method, goal programming that was designed especially for solving problems with multiobjective character, and were effective for small sized problems. The second group consists of the dynamic programming, the branch and bound and the two phases methods. They were originally designed for single objective combinatorial problems and then adapted for multiobjective problems.

2.1. Multicriteria methods

One of the most popular exact methods to solve multiobjective combinatorial optimization problems is *scalarization*. In scalarization, all the objectives are combined into one single objective. In this way, the new problem has a real-valued objective function, possibly depending on some parameters. After the multiobjective optimization problem has been scalarized, the widely developed theory and methods for single objective optimization can be used [75].

Different multiobjective optimization methods apply various scalarizing functions differently. In [76] several widely-known scalarizing functions were discussed and their modifications together with some variations introduced more recently. The scalarization may be performed once or repeatedly as a part of an iterative process. When methods are introduced in the literature, the optimality of the results produced is usually proved. On the other hand, it is not so common to justify why some specific form of scalarization is used.

Another method in multiobjective optimization is the *compromise solution method* which minimizes the distance between the potential Pareto optimal solution and an ideal solution f^I or utopian solution f^U which are defined according to the individual minima of each objective as follows:

$$f_k^I = \min_{x \in X} f_k(x), \quad k = \overline{1, s},$$

$$f^U = f^I - \epsilon u, \quad u = (1, 1, \ldots, 1) \in \mathbf{R}^s, \; \epsilon > 0.$$

In general, ideal or utopia solution is unattainable. The next best thing is a solution that is as close as possible to the utopia solution. Such a solution is called a *compromise solution* and is Pareto optimal. A difficulty with the idea of a compromise solution is the definition of closeness. The term close usually implies that one minimizes the Chebyshev or Euclidean distances, which are respectively defined as follows:

$$\max_{k=\overline{1,s}} |f_k(x) - f_k^I(x)|,$$

$$\left(\sum_{k=\overline{1,s}} (f_k(x) - f_k^I(x))^2 \right)^{1/2}.$$

The 3rd alternative, goal programming, is one of the first methods expressly created for multiobjective optimization. In goal programming the decision maker specifies a goal for the objective functions and any deviations from these goals are minimized. The goal of the objective function $f_k(x)$ will be denoted by b_k, $k = \overline{1, s}$. So, for minimization problems, goals are of the form $f_k(x) \leq b_k$, $k = \overline{1, s}$. Then, the total deviation from the goals $\sum_{k=\overline{1,s}} |d_k|$ is minimized, where d_k is the deviation from the goal b_k for the k-th objective. To model the absolute values, d_k is split into positive and negative parts such that $d_k = d_k^+ - d_k^-$, with $d_k^+ \geq 0$, $d_k^- \geq 0$, and $d_k^+ d_k^- = 0$. Consequently, $\|d_k\| = |d_k^+ + d_k^-|$, d_k^+ and d_k^- represent underachievement and overachievement, respectively, where achievement implies that a goal has been reached. The optimization problem is formulated as follows:

$$\begin{aligned}
\min \quad & \sum_{k=\overline{1,s}} (d_k^+ + d_k^-), \\
\text{subject to} \quad & f_k(x) + d_k^+ - d_k^- = b_k, \quad k = \overline{1,s}, \\
& d_k^+,\ d_k^- \geq 0, \quad k = \overline{1,s}, \\
& d_k^+ d_k^- = 0, \quad k = \overline{1,s}, \\
& x \in X.
\end{aligned}$$

The constraints of the problem can be regarded as a subset of the goals that have the same form.

Let us present widespread approaches for goal programming. In the *weighted approach* a subclass of goal programming is composed, in which weights are assigned to the deviation of each objective from its perspective goal [9]. This approach is closely related to the *method of weighted metrics* or *compromise programming*. In the lexicographic approach, the decision maker must specify a lexicographic order for the goals, so that the deviations $|d_k| = d_k^+ + d_k^-$ for the objectives are ordered in terms of priority and minimized lexicographically. The weighted and lexicographic approaches provide Pareto optimal solutions if the goals form a Pareto optimal point or if all deviation variables, d_k^+ for functions being increased and d_k^- for functions being reduced, have positive values at the optimum [75]. The latter condition suggests that all of the goals must be unattainable.

2.2. Combinatorial methods

Most of the multiobjective combinatorial methods are based on scalarizing the general multiobjective problem to a single objective optimization problem. Scalarization of the problem provides a possibility to solve multiobjective problems using the algorithms for single objective optimization problems. When solving a scalarized problem the decision maker obtains one Pareto optimal solution that is preferred for him or her in a single run. However, there exist algorithms that can solve multiobjective problem directly without scalarizing the problem. Now the algorithms adapted from single objective combinatorial optimization are briefly reviewed.

Multiobjective dynamic programming is an extension of the scalar dynamic programming to handle multiple objectives. The survey of developed algorithms for dynamic programming is given in [38].

The purpose of multiobjective dynamic programming is to identify the nondominated set of variables that lead to a set of nondominated solutions to the multiobjective optimization problem [5]. From all of the objectives, one is chosen as the objective function to be optimized while each of the others is taken as a state variable. Decisions are being ranked made in stages. Thus, the standard dynamic programming algorithm can be adopted and the entire nondominated set can be obtained from one solution of the problem.

Another method, adapted for multiobjective approach, is the *branch and bound method* [103], modified in order to handle multiple objectives and to yield the whole set of efficient solutions. Unlike the single objective case, the multiobjective branch and bound algorithm is not directly applicable with general integer variables. These have to be transformed into a sum of 0-1 variables (see, e.g., [36, 73]).

As it was described in [74], the combinatorial tree in the branch and bound method is traversed as in the single objective case, setting a priority list for the binary variables and using depth first search. The procedure is adapted for the multiple objective character of the problem. The optimum is calculated for each objective function at each intermediate node for formatting the ideal vector. At the final or terminating nodes the multiobjective problems are being solved for generating temporal efficient points which stored in the list. At each final node the list is updated after comparing new temporal efficient points with those already stored in the list. This list is called incumbent in accordance with the incumbent solution in the single objective case. The incumbent list after the completion of searching of the combinatorial tree is the set of efficient solutions.

The multiobjective case of the *branch and bound method* is much more complicated than the single objective case. It requires more computationally intensive and multiple optimizations at each intermediate node.

The *two-phase method* is applied mostly for bi-objective problems [100]. The decision making process is divided into two phases. The first phase generates the set of efficient solutions using scalarization technique and/or solving a sequence of single objective problems. In the second phase the decision maker searches for the most preferred efficient solution among the efficient solutions using a branch and bound [101] or ranking algorithms [50, 51].

The two-phase decision making approach allows to reduce the load of information to the decision maker reducing the number of candidates for being an efficient solution gradually focus on the most preferred solution [73, 93].

Chapter 3. Investment problems

A problem of allocation investment capital for maximizing profit is a common problem with multiple approaches. In this chapter, we describe a problem concerning of investment portfolio selection that requires a technique of multiobjective combinatorial optimization.

The decision maker, who is an investor in this case, is faced with a choice from among a large number of projects or assets. Usually, the funding is not sufficient to choose all the available securities. The problem is how to select securities to acquire the maximum benefit. The problem is to find securities for investments with high expected return and low risk. Generally, securities have low risk with a low rate of return. However, if a high rate of return is required, then it is necessary to tolerate a high risk [87].

Funding can be switched from one investment to another which has the same expected return but less risk, or one, which has the same risk but a higher return, or one that has both higher return and less risk. Each time the investors want to move as far as possible in the direction of increasing return and as far as possible in the direction of decreasing risk. The investor does not hold single assets [49]. The investor holds groups or portfolios of assets, because any separate investment asset has a higher risk than the portfolio of those assets. The investor tries to choose the most profitable subset of assets, one which does not exceed the budget and has the leas risk for the portfolio [11]. Creating the portfolio by diversification and mixing of a variety of investments the investor reduces the riskiness of the portfolio.

First model of the investment portfolio selection was proposed by Markowitz in [71]. Following Markowitz's investment theory, the investor must increase the expected profit for a given level of risk, or choose a portfolio with the lowest level of risk for a given expected profit [72]. In this model risk is measured by the variance of its return and the efficient set of portfolios is chosen using optimization methods.

There are some difference between investing in securities and projects. We can buy and sell securities in any quantities. In the case of investment projects we invest or do not invest in a project and the projects' decision variables are binary.

In this thesis, we formulated the model of the project portfolio selection using Savage's criteria, taking into account different types of risk such as scope risk, schedule risk, resource risk, technology risk, etc. [62]. In the case where the level of risk is fixed, the model uses Wald's criteria to consider the various types of portfolio efficiency [84].

3.1. Markowitz's portfolio problem

Markowitz provided investors with a model to optimize the risk and return of their portfolios of assets. In the model, the risk and return of single assets were not viewed separately, but they were considered only in their portfolio context. In addition, the investor needs to consider both expected return and risk, which means a variability of return. Following Markowitz's theory, the main goal for an investor is to find a nondominated portfolio with an optimal trade-off between the expected return and the risk of the investment. Markowitz's model can be formulated in bi-criterion format [95] as

$$\begin{aligned} \min \quad & x^T V x \quad \text{variance}, \\ \max \quad & e^T x \quad \text{expected return}, \\ \text{subject to} \quad & x \in X, \end{aligned}$$

where V is a covariance matrix, e is a vector of mean returns of the assets, x is a vector, which contains the proportions or the weight of each asset, and X is the set of feasible portfolios x.

A portfolio is nondominated if it is either risk minimal for a given return level or has the maximum return for a given level of risk. In order to find portfolio with minimal risk we have quadratic optimization with linear constraints. In the case of the portfolio with maximal return, the objective function is linear and the constraints are quadratic [49].

Markowitz's model is generally solved by obtaining the nondominated set using ε-constraint method [47, 96]. One of the objectives is converted to constraints with ε right-hand sides. Mostly it is an expected return. Now Markowitz's model transforms into single objective problem that minimize risk for q different levels of the expected returns:

$$\begin{aligned} \min \quad & x^T V x \quad \text{variance}, \\ \text{subject to} \quad & e^T x = \varepsilon, \quad \varepsilon \in \{a_{\min}, a_2, \ldots, a_{q-1}, a_{\max}\}, \\ & x \in X, \end{aligned}$$

where $a_{\min}$ is the minimum expected return value, $a_{\max}$ is the maximum expected return value, which were calculated previously. The other a_l, $l = \overline{2, q-1}$, are constructed at equally spaced dotted representation of the nondominated frontier within the range $[a_{\min}, a_{\max}]$. It solves for every $l = \overline{1, q}$, and then computes its criterion vector.

The ε-constraint method approach computes for each level of expected return the portfolio that minimizes risk. A further advantage of the approach is that single objective problem is quadratic that requires a standard quadratic programming solver.

After solving the investor selects from the nondominated set an efficient portfolio which fits best the preferences about the trade-off between risk and return to the investor's preferences.

Markowitz's model as well as all the most theoretical models has been criticized. The main criticism relates to the lack of possibility to adapt the model for the additional criteria beyond risk and return. Based on Markowitz's mean-variance formulation many approaches using additional criteria have been proposed [94]. However, despite such efforts, all of them have their drawbacks.

Proposed approaches for solving multicriteria problems of portfolio selection can be divided into *priori*, *interactive* and *posteriori methods*. In a priori methods preferences of the investor is sought before and included into the model. A required information can be unknown so early. Such methods produce only single solutions and do not give a chance to the investor for alternatives to choose more preferable form the nondominated set. The doubts arise that some suitable solution have been missed.

For interactive methods computation and decision making process are going at the same time. But the process requires a lot of iterations and the investor does not have a look at the whole nondominated set of portfolios.

Methods in the posteriori category generate the whole nondominated set and then the investor has possibility to select a most preferable portfolio and to be sure that his or her desires were fully satisfied. However the posteriori methods can overload the investor providing the huge volume of information and raise some problems for computing the nondominated set.

3.2. Project portfolio problem

Markowitz's investment model for portfolio selection deals with mean and variance. In this context, risk is understood in terms of the standard deviation of return. However, the investor may face different kinds of risk, e.g. risk due to different market price fluctuations, changing trends and fashions, error in sales forecasting, default on different types of financial obligation etc. [6]. Risk is an immense subject covering many sources and types with different potential impact. Thus risk may take place in different forms. A manager's responsibility is to identify the types of risk that involve the investment portfolio, measure their impact and finally decide how to reduce them.

Inspired by Markowitz's theory, we formulated the multicriteria model for project portfolio selection. In this model we assumed that the investor prefers some investment portfolios, which do not exit the budget. Each project is rated by a number of the types of risk. The problem is to choose a portfolio with a minimal acceptable level of risk. The partial criterion for choosing portfolios was the well-known Savage's criterion [85]:

$$f_k(x) = \max_{i=\overline{1,m}} R_{ik}x, \qquad k = \overline{1,s},$$

where m is a number of expected states in the market, $R_{ik} = (r_{i1k}, r_{i2k}, \ldots, r_{ink})$, $i = \overline{1,m}$, is a row of the k-th cut $R_k \in \mathbf{R}^{m\times n}$, $k = \overline{1,s}$, of the risk matrix $R = [r_{ijk}] \in \mathbf{R}^{m\times n\times s}$; $R_{ik}x$ is a total value of k-th type of risk for the portfolio $x = (x_1, x_2, \ldots, x_n)^T \in \mathbf{E}^n$, $\mathbf{E} = \{0, 1\}$, which is a Boolean vector, composed of components x_j, $j = \overline{1,n}$, equal to 1 if the investor chooses j-th projects and equal to 0 otherwise. In [7] there was proposed the survey of different techniques for risk measurement.

Using Savage's criteria, the investor chooses a portfolio with a minimum level of risk in the worst scenario of a market situation. This approach is inherent for the pessimistic expectation. It aims at achieving the guaranteed result.

When the level of risk for the preferable portfolios is fixed and appropriated, it is worth considering the efficiency of the portfolio that can be represented by different indicators, e.g. net present value (NPV), net future value (NFV), net uniform series (NUS) etc. [84]. When choosing the most preferable portfolio in view of k-th indicator of project efficiency, the investor uses Wald's criterion [102]:

$$f_k(x) = \min_{i=\overline{1,m}} E_{ik}x, \qquad k = \overline{1,s}.$$

Here $E_{ik}x$ is the total value of efficiency of the portfolio x, where $E_{ik} = (e_{i1k}, e_{i2k}, \ldots, e_{ink})$, $i = \overline{1,m}$, is a row of the k-th cut $E_k \in \mathbf{R}^{m\times n}$, $k = \overline{1,s}$, of the efficiency matrix $E = [e_{ijk}] \in \mathbf{R}^{m\times n\times s}$. Using this criterion, the investor chooses the most efficient portfolio in the worst scenario at the market state.

The criterion of extreme optimism

$$f_k(x) = \max_{i=\overline{1,m}} E_{ik}x, \qquad k = \overline{1,s},$$

is used, when the investor chooses a portfolio, which would be the most efficient in the most favorable market state.

Chapter 4. Stability analysis

Stability analysis in multicriteria optimization is of great importance both from the theoretical and practical point of view. A primary concern in stability analysis is to verify how efficient solution values change when the data contain errors or changes. This is preceded by some reasons such as the decision maker having a lack of information in the initial data or deviations from the theoretical model in the real world [75].

Multicriteria models under any changes in the initial data can behave unpredictably. Even small changes in the initial data can have a huge influence on the efficient solutions [98]. Stability analysis attempts to determine the situation and conditions when perturbations in the problem parameters produce change in the efficient solutions to the problem. Any deviations in the initial data, which are connected with implementation in the real world, are not neglected. The decision maker is supported by stability analysis information, in order to quantify the solution stability of the application problem.

Stability analysis is theoretically and practically important for multicriteria optimization problems. It helps to find relationships between the initial data and the efficient solution and to improve the judgmental process that formulated the initial data.

The stability notions are inextricably linked with the methods for solving multicriteria optimization problems. The procedures of stability analysis remain insufficiently developed and have been considered mostly for continuous problems (see, e.g., [2, 97]). Direct transformation of methods and results, obtained for the stability analysis in linear or nonlinear programming (for example, the results from [35, 82]) to some discrete optimization problems provides only very simple conclusions [91]. Such transformation does not consider the combinatorial specific of the problem.

There exist several approaches for providing the stability analysis in the multicriteria problems [45]. In this thesis, two main approaches for the post-optimal analysis that provide quantitative and qualitative characteristics of the stability of the multiobjective combinatorial problems are considered.

4.1. Qualitative approach

During post-optimal analysis of the efficient solution set a qualitative approach was used concentrating on obtaining specific analytical conditions. These conditions should guarantee a certain pre-specified behavior of the set, characterizing the stability of the problem with small perturbations of the initial data. The stability is considered as one of the classical properties of continuous or semicontinuous (e.g., Hausdorff or Berger) point-to-set mapping, which associates a set of efficient solutions with each set of parameters of the problem [55].

In [58] five of the most common types of stability were distinguished: strong quasistability, strong stability, quasistability, stability and superstability of the problem. Each type of stability describes a behavioral aspect of one or the whole set of efficient solutions in conditions of small changes in the initial data. The types of stability are defined in terms of the existence of a neighborhood of the initial data, in the problem parameters space. In this space any perturbed problem with the initial data from this neighborhood has a stability with respect to the set of efficient solutions of the initial problem.

Let $Opt^s(A)$ be the set of efficient solutions in the initial problem; $Opt^s(A+B)$ be the set of efficient solutions in the perturbed problem, where $B \in \Omega(\varepsilon) = \{\text{matrix } B \text{ has the same dimention as matirx } A : \; \|B\| < \varepsilon\}$ and $\|B\|$ is a norm of the matrix B in the problem parameters space; and s be a number of criteria in the problem.

The first type of stability reports the situation when at least one efficient solution remains efficient in both initial and perturbed problems simultaneously. This type of stability is called the *strong quasistability of the problem* and it is given by

$$\exists \varepsilon > 0 \quad \exists x^0 \in Opt^s(A) \quad \forall B \in \Omega(\varepsilon) \qquad (x^0 \in Opt^s(A+B)).$$

When the set of efficient solutions of the initial problem and the set of efficient solutions of the problem with perturbed parameters have similar points. We have so-called the *strong stability of the problem.* That is,

$$\exists \varepsilon > 0 \quad \forall B \in \Omega(\varepsilon) \qquad (Opt^s(A) \cap Opt^s(A+B) \neq \emptyset).$$

The type of stability is called *quasistability of the problem* when the set of all efficient solutions remains efficient for small changes in the initial data. Formally,

$$\exists \varepsilon > 0 \quad \forall B \in \Omega(\varepsilon) \qquad (Opt^s(A) \subseteq Opt^s(A+B)).$$

The *stability of the problem* characterizes the case when small perturbations of the initial data do not lead to appearance of new efficient solutions

$$\exists \varepsilon > 0 \quad \forall B \in \Omega(\varepsilon) \qquad (Opt^s(A+B) \subseteq Opt^s(A)).$$

The *superstability* of the problem happens when any perturbations of the initial data do not affect the set of efficient solutions

$$\exists \varepsilon > 0 \quad \forall B \in \Omega(\varepsilon) \qquad (Opt^s(A) = Opt^s(A+B)).$$

One of the first study of the qualitative approach have been conducted in [57] for the problem of finding the Pareto set among integer points of a convex polyhedron. Various types of stability of this multicriteria problem to changes of the initial data under constraints were analyzed. Necessary and sufficient conditions were obtained for different types of stability with respect to constraints.

Several types of stability against perturbations of vector criterion coefficients were analyzed from the same point of view for a multicriteria integer optimization problem with quadratic criterion functions in [58]. Necessary and sufficient conditions were formulated and analyzed for each type of stability. The topological structure of the sets of the initial data, on which some solution remains efficient, was described.

The paper [60] deals with searching for a unified approach to studying different types of stability of multicriteria integer optimization problems. Namely, searching for concepts that could compose a general basis for the description of various types of stability, and using these concepts for formulating necessary and sufficient stability conditions.

In [54] it was studied the behavioural of the Pareto set under perturbations of the parameters. Some conditions for stability by vector criterion were derived for the mixed integer problem. Similar results were derived for multicriteria quadratic optimization problems [59].

The necessary and sufficient conditions of the stability for multicriteria combinatorial problems with nonlinear partial bottleneck criteria were presented in [30] using terms of several types of binary relations given on a system of subsets of a finite set. Similarly, five types of stability for the set of lexicographically optimal solutions under small changes in the parameters of the vector criterion were presented in [18] for the lexicographic integer optimization problem with criteria, represented by absolute values of linear functions. In [18] it was shown that the structure of the lexicographic set and the image of this set in the criterion space are closely connected with the solvability of some corresponding system of integer linear equations. In other words, each element of such lexicographic set can be considered as an approximation in the case of insolvability of such system or a solution otherwise. The research was related to the behavior analysis of this kind of approximations (solutions) under small changes of parameters.

For the multicriteria combinatorial problem with MINMIN criteria [25] and for the multicriteria combinatorial median location problem [24] with Pareto and lexicographic principles of optimality, necessary and at the same time sufficient condition for each type of stability were found.

The necessary and sufficient conditions for stability and quasistability of the multicriteria Boolean lexicographic problem were obtained in [46].

4.2. Quantitative approach

Another approach to the stability analysis of the discrete optimization problems is associated with obtaining quantitative measures of the stability of the problem under admissible variations of the parameters. One of the quantitative characteristics of the stability is the *stability radius*. Along with the conditions of existence, the above mentioned neighborhood of the stability in the initial data, an approach was developed to find the bounds or formulae and also the algorithms for calculating the stability radius of a discrete optimization problem, which refers to the maximum radius of the stability sphere [91].

The need for such studies is primarily caused by the two following reasons. First, to validate correctness of a specific optimization model: it is important to know the limits of changes in the input parameters, which do not lead to changes of the efficient solutions. Second, the design of algorithms for solving discrete optimization problems can be based on the procedure of searching for the stability radius. Such procedures may be useful, for example, in the construction of algorithms that solve some groups of adjacent problems in which input data vary slightly [44].

The concept of the stability radius was first introduced in [61] for the scalar combinatorial problem. The particular definition of the stability radius concept depends on chosen principles of optimality (if the problem is multicriteria), uncertain data, and a type of distance metric used to measure the closeness in problem parameters spaces. Various types of metrics allow us to consider a specific of problem parameters perturbation. Therefore, in the case of Chebyshev's metric l_∞ only the maximal changes are taken into account in the initial data that allow independence of the perturbations. In the case of Manhattan metric l_1 every change of the initial data can be monitored in total. Hölder's metric l_p, $1 \leq p \leq \infty$, is the metric with the parameter and includes such extreme cases as Chebyshev's metric l_∞, Manhattan metric l_1 and also Euclidean metric l_2. Thus, using Hölder's metric l_p in order to obtain the stability radius that depends on the properties of the initial data – the control of perturbations can be varied. However, using Hölder's metric is not always justified. This metric overloud investigations. Obtained qualitative characteristics are often unable to provide information for the analytical use.

Therefore, it is of great interest to study the stability radius of the problems using different kinds of metrics, which allow the specifics of perturbation in the parameters of the problem to be taken into account [41]. Using different metrics during the investigations the formula for the stability radius of a scalar linear combinatorial problem was obtained in the case of Chebyshev's metric [61] and for special classes of this problem in the case of Manhattan metric [42]. With the same metric in [8,89] the stability was studied of an approximated solution of the scalar linear Boolean combinatorial problems.

Stability of scalar scheduling problems were studied in [56, 88, 90, 92].

Note that in [43] it was proposed a general approach to obtain the formula for the stability radius of the scalar combinatorial problem, based on reducing the problem of finding the stability radius to the mathematical programming problem of the special type.

Following is a brief overview of the multicriteria discrete problems for which were obtained formulae or bounds of the stability radii.

In the paper [67] formulae for radii of three types of stability (stability, quasistability and strong quasistability) were derived for the problem, which criteria are functions of Σ-MINMAX and Σ-MINMIN.

The bounds for stability and quasistability radii of multicriteria combinatorial problem with the transformed bottleneck criteria were found in [21]. Attainable bounds and formulae of various types of stability of multicriteria integer linear programming problem [16, 26–29, 31, 33], a multicriteria Boolean problem [19, 20, 22, 34] and multicriteria quadratic Boolean programming problem [17] were also obtained.

For finite cooperative game of several players with parametric principle of optimality such that the relations between players in a coalition are based on the Pareto maximum in [23] or lexicographic dominance relationship in [29] a quantitative analysis of the stability was carried out of the game situation which is optimal for the given partition method with respect to perturbations of the parameters of the payoff functions in the space with different metrics. The formulae were obtained for the stability radius for such situation.

The formulae for the stability radius of an efficient solution of the Boolean optimization problem which partial criteria are the positive cuts of linear functions to the non-negative semi-axis was presented in [32] in case of Manhattan metric l_1 in the problem parameters space.

It is also important to note that sometimes the stability radius does not provide us with complete information about the quality of a given solution in the case when problem data are outside the stability region. Some attempts to study a quality of the problem solution in this case are connected with concepts of stability and accuracy functions. These functions were first introduced in [63, 64] for scalar combinatorial optimization problem. Later, in [68] the results were extended to the multicriteria linear discrete optimization problem with Pareto and lexicographic optimality principles. Similar results were obtained for Boolean linear programming [70] and game theory problem formulations [78]. Moreover, as it was shown recently (see, e.g., [65, 66]), calculating stability and accuracy functions is closely related to analyzing problem robustness. Robust optimization in that context is understood as a process aiming to produce solutions that optimize an additionally constructed objective. The objective must assure that the efficient solution will remain feasible under worst case realization of uncertain problem input parameters. Robust optimization is also known as worst-case or minimax regret optimization, and efficient solutions of worst case optimization are often referred to as robust solutions (see, e.g., [1, 53, 79]).

Chapter 5. Stability radius

In this chapter, the quantitative characteristic of the stability of multicriteria problems, such as the stability radius is considered in more detail. For the stability radius, it is possible to use the same definitions for the types of the stability (stability, quasistability, strong stability, strong quasistability and superstability) as in the qualitative approach. However, mostly investigations only search for two types: the stability of one efficient solution and the stability of the problem. Commonly the researchers investigate the stability radius of one efficient solution x^0:

$$\rho^s(x^0, A) = \begin{cases} \sup\ \Xi_1 & \text{if } \ \Xi_1 \neq \emptyset, \\ 0 & \text{if } \ \Xi_1 = \emptyset, \end{cases}$$

where

$$\Xi_1 = \{\varepsilon > 0 : \ \forall B \in \Omega(\varepsilon) \quad (x^0 \in Opt^s(A + B))\}.$$

This characteristic shows that in the perturbed problem with perturbations less than $\rho^s(x^0, A)$ the chosen efficient solution remains efficient. The stability of one efficient solution can be considered as an analogue of the quasistability of the problem.

Another way to find the quantitative characteristic of the problem stability is to find the stability radius of the problem:

$$\rho^s(A) = \begin{cases} \sup\ \Xi_2 & \text{if } \ \Xi_2 \neq \emptyset, \\ 0 & \text{if } \ \Xi_2 = \emptyset, \end{cases}$$

where

$$\Xi_2 = \{\varepsilon > 0 : \ \forall B \in \Omega(\varepsilon) \quad (Opt^s(A + B) \subseteq Opt^s(A))\}.$$

During investigation the decision maker obtains such characteristic, which shows that during perturbations not exceeding the stability radius $\rho^s(A)$, no new efficient solution appears in the perturbed problem.

5.1. Formulae and bounds

Most of the results, related to obtaining the formulae or bounds for the stability radius, were devoted to the multicriteria problems with linear criteria. In this thesis we started the investigation of the stability radii for the problem with the nonlinear criteria such as Savage's, Wald's criteria and criteria of extreme optimism (see Section 3.2).

Nonlinear criteria have another structure. They require another approaches and methods [75]. In the most cases of linear criteria the formulae were derived for the stability radii. These formulae were represented as the ratio of two norms. There were the distance between the values of the functions for two solutions in the numerator and the norm of the difference between these two solutions in the denominator. The selection of solutions was depended from the principle of optimality (lexicographical or Pareto).

In the case of nonlinear criteria the topology of solutions does not allow deriving similar results. The formulae in the nonlinear case are cumbrous, contain numerous cases and inconvenient to use. For these reasons, the bounds for the stability radii of an efficient solution and the stability radii of the problem were obtained. It was proved that the stability radii are not less the lower bounds and not exceed the upper bounds. In some cased the stability radii are equal to one of the bounds. In papers [I]–[VIII] there were introduced the particular classes of the problem when the stability radii are expressed by the formulae equal to lower or upper bounds. These facts allow to say the obtained bounds for the stability radii are attainable. The investigations were also provided with some different combinations of Chebyshev's, Manhattan and Hölder's metrics in the problem parameters spaces. It gave possibility to monitor perturbations in different ways, described in Chapter 4.

In papers [II] and [III] the attainable bounds for the stability radii of one Pareto optimal portfolio (see Definition 1) of the project portfolio problem with Savage's criteria (see Section 3.2) were obtained. The problem parameters space was endowed by Manhattan metric in [II]. In paper [III] for the investigation of the stability radius, the combination of Chebyshev's and Manhattan metrics was used. The results were obtained in the case of Chebyshev's metric in the risk criteria space and Manhattan metric in the market state space and in the portfolio space. There was also showed that, when the problem had only one supposed market state and Savage's criterion transformed into the linear criterion, the lower and upper bounds were equal to each other. In this case, the stability radius is expressed by the formula, obtained earlier in [26]. Manhattan metric was also used in paper [I] in every space of the problem parameters for obtaining the stability radius of a lexicographically optimal portfolio (see Definition 2) of the project portfolio selection problem with Savage's criteria. In [VI] attainable bounds for the stability radius of one Pareto optimal portfolio of the bi-criteria problem were obtained. The portfolios were evaluated using Savage's criterion for risk and Wald's criterion for the efficiency of the portfolios. The market state space and the portfolio space were endowed by Hölder's metric.

For the multicriteria problem with Savage's criteria in [V], Wald's criteria in [VIII] and for the bi-criteria problem with Savage's criterion and criterion of extreme optimism in [VII], there were obtained attainable bounds for the stability radii of the problems. For the stability analysis there was used Chebyshev's metric in every problem parameters space for the problem with Savage's criteria. For the problem with Wald's criteria the quantitative characteristic of the stability investigated in the case of Manhattan metric in the project and market state spaces and Chebyshev's metric in the efficiency criteria space. In the bi-criteria case of the problem used Hölder's metric. Stability analysis of a Pareto optimal portfolio of the project portfolio selection problem, using three different combination Hölder's and Chebyshev's metrics, was presented in [IV]. The quantitative characteristics in the form of attainable bounds of the stability of a Pareto optimal portfolio were derived.

5.2. Calculation of stability radius

When considering the formulae or bounds of the stability radius the normal question raised is how to calculate that stability radius. Calculation of the stability radius of efficient solutions can be harder than finding these efficient solutions. For example, the results of the investigations in [39] claim that the problem of finding the stability radius of the shortest path problem is NP-complete. That is why many researchers were interested in finding regularity between solving the problem and calculating the stability radius. In [40] the complexity of solving the problem and the complexity of finding the stability radius of found solutions were compared. The results of the experiments in [40] showed that efforts involved in the algorithm for computing the stability radius of the travelling salesman problem are comparable in complexity with efforts involved in the algorithm for finding the efficient solution. It was also claimed that the algorithm used for finding the stability radius could be used as a component in devising an algorithm using a library of previously solved problems. The survey and new investigations in the field of comparing the complexity of the problems and its stability radius were released in [8, 48]. It was proposed that an algorithm runs in polynomial time if the optimization problem itself is polynomially solvable. Some results in reducing complexity of the stability radius computation were detected in [69].

An adaptation of the multiobjective evolutionary algorithm (NSGA-II) was applied for the calculation of the stability radius of an efficient solution to the shortest path problem in [52]. The behavior of the derived algorithm with the known exact method in terms of solutions diversity and computational complexity was also compared.

The paper [83] was addressed to the problem of computing the stability radius of an efficient solution in the context of multiobjective combinatorial optimization. The stability radius of an efficient solution was modeled as a particular inverse optimization problem. The model was solved by an algorithm which required only solving a logarithmic number of mixed integer programs. It contains a linear number of constraints and variables compared with the instance of the combinatorial optimization problem if its feasible set can be defined by linear constraints.

Chapter 6. Conclusion

Most of the decision making problems have a multiple criteria structure with conflicting objectives. It is necessary to consider conflicting goals in the decision making processes in order to make the right decision.

Multicriteria combinatorial optimization models real-world processes over discrete structures. The solving methods of multiobjective optimization can be classified in many ways according to different criteria, and can be categorized into two relatively distinct subsets: generating methods and preference-based methods. In general, most of the methods for solving multiobjective combinatorial optimization problems are based on scalarization. Some methods were designed for problems of the multiobjective type. Other methods were adapted from single objective combinatorial optimization. In this thesis, we discussed the basics of multiobjective combinatorial optimization, including the definitions of efficient solutions, and introduced some methods for solving such problems. One of the multiobjective problems particularly considered here was the investment problem which was considered in two formulations: Markowitz's portfolio model and the project portfolio model.

Multicriteria models that undergo changes in the initial data can behave unpredictably. Even small changes in the initial data can have an enormous influence on the efficient solution. Stability analysis attempts to determine the situation and conditions when perturbations in the problem parameters produce a light change in its efficient solution.

In Chapter 2, the definitions of the multiobjective combinatorial problem were given and the concepts of its efficient solutions represented. In this chapter several methods for solving the problems were introduced. Some of these methods were designed especially for problems with multiple objectives and generally based on the scalarizing technique. Other methods were constructed for problems with a combinatorial structure, which were adapted for the multicriteria case.

Chapter 3 discussed problems such as investment problems arising during selecting different kind of securities or investment projects for funding which have a multicriteria character. In the first section the formulation of the well-known Markowitz's portfolio investment problem was introduced and solving it by the ε-constraint method was described. An account was also given of some of the drawbacks of these models.

In the second section, a new approach was proposed for the formulation of the investment problems in the multicriteria case when the different types of risk and efficiency of the investment projects are taken into account.

Stability analysis of the multiobjective combinatorial problems was considered in Chapter 4. The two major directions of investigation were described in this chapter: the quantitative and qualitative approaches. Five types of stability were described in detail and a survey of recent results was also proposed.

Chapter 5 was dedicated to the stability radius of the multicriteria problem, which represents the supreme level of perturbations in of the initial data, where the solutions to the problems preserve their preassigned properties. In this chapter the results obtained in this thesis were introduced. The attainable bounds for the stability radii of the efficient solutions of the project portfolio selection problem were described. The second section was devoted to approaches for calculating the stability radius.

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