HYPERBOLIC TYPE GEOMETRIES IN SUBDOMAINS

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This Licentiate thesis deals with hyperbolic type geometries in planar subdomains. It is known that hyperbolic type distance is always greater in a subdomain than in the original domain. In this work we obtain certain lower estimates for hyperbolic type distances in subdomains in terms of hyperbolic type distances of the original domains. In particular the domains that we consider are cyclic polygons and their circumcircles, sectors and supercircles.

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Contents

1 Introduction

The history of the hyperbolic type geometries began with the attempts to derive the Euclid's parallel postulate from other postulates of Euclid. These attemps failed but they produced results that gave birth to the non-Euclidean geometry. Some of the most significant contributors were E. Beltrami, J. Bolyai, C. F. Gauss and N. Lobatševski. The term *hyperbolic geometry* was first used by F. Klein in the year 1871. Klein's ideas became widely adopted and, in particular, to the use of Möbius invariant metrics in the geometric function theory (GFT). These ideas lead to the study of conformal invariants by L. Ahlfors and others. More information about the history of the hyperbolic geometry and about the major research done on this field can be found from [10], [11] and [16].

This thesis approaches the non-Euclidean geometry from the viewpoint of GFT. In GFT there are many different metrics which resemble the classical hyperbolic metric; for example the quasihyperbolic metric and the j-metric. These kinds of metrics have proven useful in GFT because many of them are invariant under certain classes of mappings, like under similarity mappings, Möbius mappings or conformal mappings. An overview of the recent research in this field can be found from [20].

In the beginning of this thesis we go over the fundamental concepts and results concerning hyperbolic type geometries. In the main part of the thesis we compare hyperbolic type distances in subdomains to hyperbolic type distances in the original domains. Specifically we study infinite strips, sectors, cyclic polygons and supercircles. In the final part of the thesis we obtain certain simple results concerning homeomorphisms and subdomain geometry.

1.1 The Metrics

First we introduce few important plane regions, notations and metrics.

Definition 1.1. The upper half-plane $\mathbb{H}$ is defined as

$$\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}\,(z) > 0\}$$

and the unit disk $\mathbb{D}$ is defined as

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

Definition 1.2. Let γ be a curve. Now the Euclidean length of γ is

$$l_{||}(\gamma) = \int_{\gamma} |dz|.$$

Definition 1.3. Let (D, k) be a metric space. Now a circle with respect to the metric k with center at c and radius r is defined as

$$C_k(c, r) = \{z \in D : k(c, z) = r\}.$$

Definition 1.4. Let (D, k) be a metric space. Now a ball with respect to the metric k with center at c and radius r is defined as

$$B_k(c, r) = \{z \in D : k(c, z) < r\}.$$

When dealing with a closed ball, we add an overline to B. If the metric is not mentioned, then it is assumed that we are dealing with the Euclidean metric.

Definition 1.5. Let $D \in \{\mathbb{H}, \mathbb{D}\}$ and let $d(z, \partial D)$ be the Euclidean distance between a point $z \in D$ and the boundary ∂D. Now we can define hyperbolic weight (or density) function in the following way:

$$w : D \to \mathbb{R}, w(z) = \frac{1}{d(z, \partial D)} \text{ for } D = \mathbb{H}$$

and

$$w : D \to \mathbb{R}, w(z) = \frac{2}{1 - |z|^2} \text{ for } D = \mathbb{D}.$$

Definition 1.6. If z_1 and z_2 are two points either in $\mathbb{H}$ or $\mathbb{D}$, then the hyperbolic distance between these two points is

$$\rho_D(z_1, z_2) = \inf_{\gamma \in \Gamma[z_1, z_2]} \int_{\gamma} w(z)|dz|$$

where $\Gamma[z_1, z_2]$ is the family of all rectifiable curves connecting z_1 and z_2 in the given region.

We can also construct hyperbolic metric in any simply connected proper subset of the complex plane $\mathbb{C}$.

Definition 1.7. Let D be holomorphically equivalent to $\mathbb{H}$ and $\xi : D \to \mathbb{H}$ be a conformal map. The hyperbolic distance between any two points $z_1, z_2 \in D$ is

$$\rho_D(z_1, z_2) = \inf_{\gamma \in \Gamma[z_1, z_2]} \int_{\xi \circ \gamma} \frac{|dz|}{d(z, \partial D)}$$

where $\Gamma[z_1, z_2]$ is the family of all rectifiable curves connecting z_1 and z_2 in D. This distance is independent of the choice of the map ξ so the hyperbolic distance in D is well-defined.

Theorem 1.8. *Hyperbolic distance is a metric in the domain it is generated in.*

Proof. See [1] and [2]. $\square$

There is also another way of generalizing the hyperbolic geometry first defined by F. W. Gehring and B. P. Palka in [4].

Definition 1.9. Let $D \subsetneq \mathbb{R}^n$ be a domain. The quasihyperbolic distance between any two points $z_1, z_2 \in D$ is

$$k_D(z_1, z_2) = \inf_{\gamma \in \Gamma[z_1, z_2]} \int_{\gamma} \frac{|dz|}{d(z, \partial D)}$$

where $\Gamma[z_1, z_2]$ is the family of all rectifiable curves connecting z_1 and z_2 in D.

Theorem 1.10. *Quasihyperbolic distance is a metric in $D \subsetneq \mathbb{R}^n$.*

Proof. See [4, Corollary 2.2.]. $\square$

Next we will consider the j-metric which is also known as the distance ratio metric. This metric was first introduced by F. W. Gehring and B. G. Osgood in the article [3] and later in the following modified form by M. Vuorinen in [19].

Definition 1.11. Let $D \subsetneq \mathbb{R}^n$. Now for points z_1 and z_2 in D, we define

$$j_D\left(z_1, z_2\right) = \log\left(1 + \frac{\left|z_1 - z_2\right|}{\min\left\{d\left(z_1, \partial D\right), d\left(z_2, \partial D\right)\right\}}\right).$$

Theorem 1.12. *The j-metric is a metric in the domain it is defined in.*

Proof. See [17, Lemma 2.2.]. □

The j-metric can be seen as a way to approximate the quasihyperbolic metric as the following theorem shows.

Theorem 1.13. *Let $D \subsetneq \mathbb{R}^n$ be a domain. Now $k_D\left(z_1, z_2\right) \geq j_D\left(z_1, z_2\right)$ for all points z_1 and z_2 in D.*

Proof. See [4, Lemma 2.1.]. □

The geometries induced by the aforementioned metrics are often referred in literature as hyperbolic type geometries as their behavior resembles that of the hyperbolic geometry. This is especially apparent in the way the distance between points in these geometries depends on the boundary of the domain.

1.2 Geodesics

Definition 1.14. Let $D \subsetneq \mathbb{R}^n$ be a domain and γ a curve in D. If k is a metric in the domain D and

$$k\left(z_1, z_2\right) + k\left(z_2, z_3\right) = k\left(z_1, z_3\right)$$

for all $z_1, z_3 \in \gamma$ and $z_2 \in \gamma^{'}$, where $\gamma^{'}$ is the subcurve of γ joining z_1 and z_3, then γ is a geodesic.

Geodesics are not always unique; for example quasihyperbolic geodesics in a punctured plane are not unique in a special case [14, p. 38]. We shall denote geodesic between points z_1 and z_2 in a metric k with $J_k\left[z_1, z_2\right]$.

We shall consider several metrics and geodesics in the cases they exist. In particular, we shall study the hyperbolic type metrics defined in the previous section.

Theorem 1.15. *In $\mathbb{H}$ and $\mathbb{D}$ hyperbolic geodesics are arcs of generalized circles which intersect the boundary of the region at a right angle.*

Proof. See [18]. $\square$

If one wants to find geodesics in a simply connected domain $D \subsetneq \mathbb{C}$, one has to map the geodesics of $\mathbb{H}$ to D with an appropriate conformal mapping. This follows from the definition of the general hyperbolic distance.

Quasihyperbolic geodesics are not as simple to find as hyperbolic ones but we do know some significant results regarding them.

Theorem 1.16. *a) If z_1 and z_2 are points in a domain $D \subsetneq \mathbb{R}^n$, there exists a quasihyperbolic geodesic that connects them.*

b) Quasihyperbolic geodesics are smooth curves.

c) If $D \subsetneq \mathbb{R}^n$ is convex, quasihyperbolic geodesics on D are unique.

d) If z_1 and z_2 are points in $\overline{B}(c,r)$ and $\overline{B}(c,r)$ is included in $D \subsetneq \mathbb{R}^n$, then $J_{k_D}[z_1,z_2] \subset \overline{B}(c,r)$.

Proof. See [3, Lemma 1.], [13, Corollary 4.8.], [15, Theorem 2.11.] and [13, Theorem 2.2.] respectively. $\square$

Quasihyperbolic distances or good approximations of them are known at least in $\mathbb{H}$, $\mathbb{D}$, punctured plane [14], punctured disk, angular domains [12] and infinite strip [5], [6].

On the other hand the j-metric has geodesics only in few special cases as the following theorem shows.

Theorem 1.17. *Let z_1 and z_2 be points in a domain $D \subsetneq \mathbb{R}^n$ and $J_j[z_1,z_2]$ be a geodesic in j-metric. Now $J_j[z_1,z_2]$ is equal to the line segment $[z_1,z_2]$. Also there exists such a point $w \in \partial D$ that w is in the set of closest boundary points for every point in $J_j[z_1,z_2]$ and that w and $J_j[z_1,z_2]$ are collinear.*

Proof. See [7, Lemma 2.9.]. $\square$

2 Subdomain Geometry

In [8], R. Klén, Y. Li and M. Vuorinen introduced the following problem:

Problem 2.1. *Let D_1 and D_2 be two proper subdomains of $\mathbb{R}^n$ such that $D_1 \subset D_2$ and that $\partial D_1 \cap \partial D_2$ is either empty or discrete set. Does there exist a constant $c > 1$ such that*

$$m_{D_1}(z_1, z_2) \geq c m_{D_2}(z_1, z_2) \quad \forall z_1, z_2 \in D_1,$$

where $m_{D_n} \in \{j_{D_n}, k_{D_n}\}$ for $n \in \{1, 2\}$?

R. Klén, Y. Li and M. Vuorinen gave answer to this problem in three cases and stated a conjecture for a more general case. They are as follows:

Theorem 2.2. *Let*

$$D_1 = \{(x, y) \in \mathbb{C} : |x| + |y| < 1\} \text{ and } D_2 = \left\{(x, y) \in \mathbb{C} : |x|^2 + |y|^2 < 1\right\}.$$

Then $k_{D_1}(z_1, z_2) \geq \sqrt{2} k_{D_2}(z_1, z_2)$ for all points in D_1.

Regardless of this there is no appropriate constant $c > 1$ in these domains for the j-metric.

Proof. See [8, Theorem 4.1.]. □

Theorem 2.3. *For $s \in (0, 1)$, let $D_1 = \{(x, y) \in \mathbb{C} : |x|^s + |y|^s < 1\}$ and $D_2 = \{(x, y) \in \mathbb{C} : |x| + |y| < 1\}$. Then $k_{D_1}(z_1, z_2) \geq 2^{\frac{1}{s}-1} k_{D_2}(z_1, z_2)$ for all points in D_1.*

Proof. See [8, Theorem 4.3.]. □

Theorem 2.4. *If D_1 is a bounded proper subdomain of $D_2 \subsetneq \mathbb{R}^n$, then*

$$m_{D_1}(z_1, z_2) \geq \left(1 + \frac{2d(D_1, \partial D_2)}{\operatorname{diam}(D_1)}\right) m_{D_2}(z_1, z_2) \quad \forall z_1, z_2 \in D_1,$$

where $m_{D_n} \in \{j_{D_n}, k_{D_n}\}$ for $n \in \{1, 2\}$.

Proof. See [8, Theorem 4.6.]. □

Conjecture 2.5. *For $0 < s < t$, let $D_s = \{(x, y) \in \mathbb{C} : |x|^s + |y|^s < 1\}$ and $D_t = \{(x, y) \in \mathbb{C} : |x|^t + |y|^t < 1\}$. Then $k_{D_s}(z_1, z_2) \geq 2^{\frac{1}{s}-\frac{1}{t}} k_{D_t}(z_1, z_2)$ for all points in D_s.*

Sometimes the domains of the previous conjecture are referred as super-circles as they can be seen as generalizations of the unit circle.

2.1 Subdomain Geometry in Infinite Strips and in Sectors

Following theorems deal with the preceding subdomain Problem 2.1 in an infinite strip and in a sector. First we present a result concerning the quasihyperbolic metric of some infinite strips.

Proposition 2.6. *For $h > 0$ and $t \in (0, h)$, let $S_h = \{z \in \mathbb{C} : |\mathrm{Im}\,(z)\,| < h\}$ and $S_t = \{z \in \mathbb{C} : |\mathrm{Im}\,(z)\,| < t\}$. Now for all points in S_t:*

$$\frac{h}{t} k_{S_h}(z_1, z_2) \leq k_{S_t}(z_1, z_2).$$

Proof. Let us first prove that

$$d(z, \partial S_h) \geq \frac{h}{t} d(z, \partial S_t)$$

for each $z \in S_t$. Now

$$\frac{d(z, \partial S_h)}{d(z, \partial S_t)} = \frac{d(z, \partial S_t) + (h - t)}{d(z, \partial S_t)} = 1 + \frac{h - t}{d(z, \partial S_t)} \geq 1 + \frac{h - t}{t}$$

so

$$d(z, \partial S_h) \geq \frac{h}{t} d(z, \partial S_t).$$

Let us next suppose that z_1 and z_2 are points in S_t and that γ is a quasihyperbolic geodesic connecting z_1 and z_2 in S_t. Now $\gamma \subset S_h$ for $\gamma \subset S_t \subset S_h$, and so we get

$$k_{S_h}(z_1, z_2) \leq \int_\gamma \frac{|dz|}{d(z, \partial S_h)} \leq \int_\gamma \frac{|dz|}{\frac{h}{t} d(z, \partial S_t)} = \frac{t}{h} k_{S_t}(z_1, z_2).$$

This proves the inequality. □

Next theorem will generalize the case of the infinite strip to proper unbounded subdomains. But before the theorem, we must first introduce a new important definition.

Definition 2.7. Let D be a proper subdomain of $\mathbb{R}^n$. Now *medial axis* of D is the set of points in D that have two or more closest points on the boundary of D.

If it is not stated otherwise, medial axis is defined in relation to the Euclidean distance.

Theorem 2.8. *Let D_1 and D_2 be unbounded proper subdomains of $\mathbb{R}^n$. If $D_1 \subsetneq D_2$, $R = d(D_1, \partial D_2) \in \mathbb{R}_+$ and*

$$r = \sup \{d(z, \partial D_1) : z \in \textit{medial axis of } D_1\} \in \mathbb{R}_+,$$

then for all points in D_1

$$\left(1 + \frac{R}{r}\right) k_{D_2}(z_1, z_2) \leq k_{D_1}(z_1, z_2).$$

Proof. We shall proceed in the same fashion as in the proof of the preceding proposition. Clearly $d(z, \partial D_2) \geq d(z, \partial D_1) + R$. Now by evaluating the quotient of distance functions $d(z, \partial D_i)$, we get

$$\frac{d(z, \partial D_2)}{d(z, \partial D_1)} \geq \frac{d(z, \partial D_1) + R}{d(z, \partial D_1)} = 1 + \frac{R}{d(z, \partial D_1)}.$$

Next $r \geq d(z, \partial D_1)$, so we have

$$\frac{d(z, \partial D_2)}{d(z, \partial D_1)} \geq 1 + \frac{R}{r}.$$

Now the theorem follows by a similar argument as in the proof of the preceding proposition. □

The case of a sector and the quasihyperbolic distance will be divided into four lemmas.

Definition 2.9. If $\alpha \in (0, \pi]$, then the sector defined by α is

$$S_\alpha = \{z \in \mathbb{C} : |\arg(z)| < \alpha\}.$$ [1]

Lemma 2.10. *Let $\beta \in (0, \frac{\pi}{2}]$ and $\alpha \in (0, \beta)$. For all points in S_α we have*

$$\frac{1}{\sin\left(\frac{\pi}{2} + \alpha - \beta\right)} k_{S_\beta}(z_1, z_2) < k_{S_\alpha}(z_1, z_2),$$

where $\frac{1}{\sin\left(\frac{\pi}{2} + \alpha - \beta\right)} > 1$.

[1] An in-depth study of the quasihyperbolic distance and geodesics in sectors can be found from [12].

Proof. Let us first prove that

$$d(z, \partial S_\beta) > \frac{1}{\sin\left(\frac{\pi}{2} + \alpha - \beta\right)} d(z, \partial S_\alpha)$$

for each $z \in S_\alpha$.

Let us fix z and let $A_m \in \partial S_m$ be such a point that $|z - A_m| = d(z, \partial S_m)$ for $m \in \{\alpha, \beta\}$. Also we define $B = [A_\beta, z] \cap \partial S_\alpha$. Now by the trigonometry of right triangle, we get $\angle A_\beta B O = \frac{\pi}{2} + \alpha - \beta$ where O is the origin. Next because they are vertical angles, $\angle A_\beta B O = \angle z B A_\alpha$. Then once again by the trigonometry of right triangle, we get that $|z - B| = \frac{d(z, \partial S_\alpha)}{\sin\left(\frac{\pi}{2} + \alpha - \beta\right)}$. Finally because $d(z, \partial S_\beta) > |z - B|$, we have

$$d(z, \partial S_\beta) > |z - B| = \frac{d(z, \partial S_\alpha)}{\sin\left(\frac{\pi}{2} + \alpha - \beta\right)}.$$

This concludes the first part of the proof.

Now using analogical method as in the preceding proofs, we get

$$\frac{1}{\sin\left(\frac{\pi}{2} + \alpha - \beta\right)} k_{S_\beta}(z_1, z_2) < k_{S_\alpha}(z_1, z_2).$$

Finally,

$$\frac{1}{\sin\left(\frac{\pi}{2} + \alpha - \beta\right)} > 1 \ \ \forall \alpha \in (0, \beta)$$

follows from the fact that $\alpha < \beta \leq \frac{\pi}{2}$. □

Lemma 2.11. *Let $\beta \in (\frac{\pi}{2}, \pi]$ and $\alpha \in (0, \beta - \frac{\pi}{2}] \setminus \left\{\frac{\pi}{2}\right\}$. For all points in S_α we have*

$$\frac{1}{\sin\alpha} k_{S_\beta}(z_1, z_2) \leq k_{S_\alpha}(z_1, z_2),$$

where $\frac{1}{\sin\alpha} > 1$.

Proof. Once again we start by evaluating the quotient of the two distance functions. Let $z = x + iy = re^{i\theta}$. Without loss of generality we can assume $y \geq 0$. Now because the origin is the closest point on ∂S_β for all points in S_α and because $1 + \tan^2\alpha = \frac{1}{\cos^2\alpha}$, it follows from the general equation of the distance between line and a point that

$$\frac{d(z, \partial S_\beta)}{d(z, \partial S_\alpha)} = \frac{\sqrt{x^2 + y^2}}{|x\tan\alpha - y||\cos\alpha|}.$$

By reducing with x and using the facts that $\tan\alpha > \tan\theta$ and that z is in the first quarter, we get

$$\frac{d\left(z,\partial S_\beta\right)}{d\left(z,\partial S_\alpha\right)} = \frac{\sqrt{1+\tan^2\theta}}{\left(\tan\alpha-\tan\theta\right)\cos\alpha}.$$

Finally, by finding a lower bound for the numerator and an upper bound for the denominator, we have

$$\frac{d\left(z,\partial S_\beta\right)}{d\left(z,\partial S_\alpha\right)} \geq \frac{1}{\sin\alpha}.$$

From this the lemma follows. $\square$

Lemma 2.12. *Let $\beta \in (\frac{\pi}{2},\pi]$ and $\alpha \in \left(\beta-\frac{\pi}{2},\frac{\pi}{2}\right)$. For all points in S_α we have*

$$\frac{1}{\sin\alpha} k_{S_\beta}\left(z_1,z_2\right) < k_{S_\alpha}\left(z_1,z_2\right),$$

where $\frac{1}{\sin\alpha} > 1$.

Proof. Let z_1 and z_2 be two points in S_α and let γ be the quasihyperbolic geodesic connecting these points in S_α. Now let us divide γ into two parts γ_1 and γ_2 so that $\gamma_1 \subset S_{\beta-\frac{\pi}{2}}$ and that $\gamma_2 \subset \left(S_\alpha \setminus S_{\beta-\frac{\pi}{2}}\right)$. Now because in the domain $S_{\beta-\frac{\pi}{2}}$ the distance functions can be evaluated in the same way as in Lemma 2.11 and because in the domain $\left(S_\alpha \setminus S_{\beta-\frac{\pi}{2}}\right)$ the distance functions can be evaluated in the same way as in Lemma 2.10, we get

$$\begin{aligned} k_{S_\beta}\left(z_1,z_2\right) &\leq \int_{\gamma_1} \frac{|dz|}{d\left(z,\partial S_\beta\right)} + \int_{\gamma_2} \frac{|dz|}{d\left(z,\partial S_\beta\right)} \\ &< \sin\alpha \int_{\gamma_1} \frac{|dz|}{d\left(z,\partial S_\alpha\right)} + \sin\left(\frac{\pi}{2}+\alpha-\beta\right) \int_{\gamma_2} \frac{|dz|}{d\left(z,\partial S_\alpha\right)}. \end{aligned}$$

Because $\alpha < \frac{\pi}{2}$ and $\frac{\pi}{2} < \beta \leq \pi$, clearly $\sin\alpha > \sin\left(\frac{\pi}{2}+\alpha-\beta\right)$. So we can further evaluate the preceding inequality:

$$k_{S_\beta}\left(z_1,z_2\right) < \sin\alpha \int_{\gamma_1} \frac{|dz|}{d\left(z,\partial S_\alpha\right)} + \sin\alpha \int_{\gamma_2} \frac{|dz|}{d\left(z,\partial S_\alpha\right)} = \sin\alpha \;\; k_{S_\alpha}\left(z_1,z_2\right).$$

This concludes the proof. $\square$

Lemma 2.13. *Let $\beta \in (\frac{\pi}{2},\pi]$ and $\alpha \in [\frac{\pi}{2},\beta)$. For the domains S_α and S_β the answer to Problem 2.1 is negative for the quasihyperbolic distance.*

Proof. Now for all points in the positive part of the real axis the closest point of the boundary of either domain is the origin. This means that $d(x, \partial S_\alpha) = d(x, \partial S_\beta)$ for all $x \in \mathbb{R}_+$. Then by [12, Theorem 4.10.] the quasihyperbolic geodesic between two points on the positive real axis is a line segment for both domains so $k_{S_\alpha}(x_1, x_2) = k_{S_\beta}(x_1, x_2)$ for all points on the positive real axis. So, in this case, there cannot be a constant $c > 1$ satisfying the assertion of Problem 2.1. □

The next theorem combines the results of the preceding lemmas.

Theorem 2.14. *a) If $\beta \in (0, \frac{\pi}{2}]$ and $\alpha \in (0, \beta)$, then*

$$\frac{1}{\sin\left(\frac{\pi}{2} + \alpha - \beta\right)} k_{S_\beta}(z_1, z_2) < k_{S_\alpha}(z_1, z_2) \quad \forall z_1, z_2 \in S_\alpha.$$

b) If $\beta \in (\frac{\pi}{2}, \pi]$ and $\alpha \in \left(0, \frac{\pi}{2}\right)$, then

$$\frac{1}{\sin\alpha} k_{S_\beta}(z_1, z_2) < k_{S_\alpha}(z_1, z_2) \quad \forall z_1, z_2 \in S_\alpha.$$

c) If $\beta \in (\frac{\pi}{2}, \pi]$ and $\alpha \in [\frac{\pi}{2}, \beta)$, then for the domains S_α and S_β the answer to Problem 2.1 is negative for the quasihyperbolic distance.

Proof. Theorem follows from the four preceding lemmas. □

Next we will prove that for the infinite strip and the sector the answer to Problem 2.1 will be negative in the case of the j-metric.

Theorem 2.15. *If the domains given in Problem 2.1 are two infinite strips or two sectors, there does not exist a suitable constant $c > 1$ for the j-metric.*

Proof. Let us first prove this theorem in the case of infinite strips. For $h > 0$ and $t \in (0, h)$, let $S_h = \{z \in \mathbb{C} : |\mathrm{Im}(z)| < h\}$ and $S_t = \{z \in \mathbb{C} : |\mathrm{Im}(z)| < t\}$. We shall examine a limit of quotient of j_{S_t} and j_{S_h}. Let $x > 1$. Now for fixed h and t

$$\lim_{x\to\infty} \frac{j_{S_t}(1, x)}{j_{S_h}(1, x)} = \lim_{x\to\infty} \frac{\log\left(1 + \frac{x-1}{t}\right)}{\log\left(1 + \frac{x-1}{h}\right)} = 1$$

which proves the theorem in the case of infinite strips.

The proof for two sectors is completely analogical; if we pick the same points as before, the minimum distance once again stays constant and the quotient approaches to 1. □

2.2 Subdomain Geometry and Cyclic Polygons

The next theorem generalizes Theorem 2.2 for a cyclic polygon and the corresponding circumcircle.

Theorem 2.16. *If P is a cyclic polygon and C is such that ∂C is the corresponding circumcircle, the answer to Problem 2.1 is positive in the case of the quasihyperbolic metric but negative in the case of the j-metric.*

Proof. Let $A_1, ..., A_k$ be the vertices of P and $a_1, ..., a_k$ be the edges of P. We also define r to be the radius of C and M to be the medial axis of P. Because P is a cyclic polygon, it is also a convex polygon so M is composed of line segments and $\partial M \cap \partial P = \{A_1, ..., A_k\}$. Because of this M divides P into such subregions $V_1, ..., V_k$ that each of them shares one edge with P; for example P and V_1 share a_1 as in Figure 1.

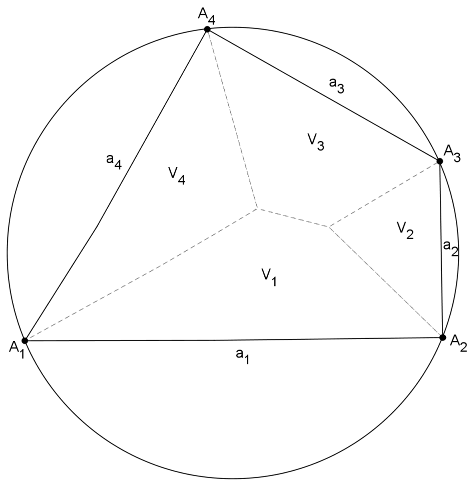

Figure 1: A cyclic polygon and the corresponding circumcircle with the used notations. The medial axis is drawn with dashed line segments.

First we prove that the quotient $\frac{d(z,\partial C)}{d(z,\partial P)}$ reaches the minimum value in M. We do this by examining each of the subregions beginning with V_1. For this first part we assume that the circumcenter c is the origin. We also rotate the complex plane so that a_1 acts as a basis for P and is collinear with the real axis with imaginary coordinate y_1. We may do this because rotations clearly preserve P and C, and so they also preserve the quasihyperbolic geometry of P and C. Next for $z = te^{i\theta} \in V_1$, we get

$$\frac{d(z, \partial C)}{d(z, \partial P)} = \frac{r - t}{t \sin \theta - y_1}.$$

Let us denote the right side of the above identity by $f_1(t)$. By differentiating with respect to t, we get

$$f_1'(t) = \frac{y_1 - r \sin \theta}{(t \sin \theta - y_1)^2}.$$

Now $f_1'(t) > 0$ when the line segment $\left[z, re^{i\theta}\right]$ does not intersect the medial axis, $f_1'(t) = 0$ when $re^{i\theta}$ is a vertex shared by V_1 and P and $f_1'(t) < 0$ when the line segment $\left[z, re^{i\theta}\right]$ intersects the medial axis. By repeating this process for all subregions V_m we see that corresponding $f_m(t)$ always decreases when it approaces M so the quotient $\frac{d(z,\partial C)}{d(z,\partial P)}$ reaches the minimum value in M.

Next we shall consider what happens to the quotient on the medial axis. We start by considering the line segment $(A_1, I_1] \subset M$ where I_1 is the closest junction point to A_1 on M. Next with suitable rotation and translation we map the vertex A_1 to the origin and side a_1 on the positive real axis so that the shape of P and C are preserved. Now for $z = te^{i\theta} \in (A_1, I_1]$ we have

$$\frac{d(z, \partial C)}{d(z, \partial P)} = \frac{r - \sqrt{r^2 + t^2 - 2rt \cos \beta_1}}{t \sin \theta}$$

where β_1 is either $\angle cA_1I_1$ or $\angle I_1A_1c$ depending on the position of $c \notin (A_1, I_1]$. We make a note that in the case $c \in (A_1, I_1]$ the above equation becomes $\frac{d(z,\partial C)}{d(z,\partial P)} = \frac{1}{\sin\theta} > 1$ when $z \in (A_1, c]$ and $\frac{d(z,\partial C)}{d(z,\partial P)} = \frac{2r-t}{t\sin\theta} > 1$ when $z \in (c, I_1]$ so in this case the quotient is either a constant or clearly decreasing. Then if $c \notin (A_1, I_1]$, we mark the right hand side of the above equation with $g_1(t)$ and differentiate with respect to t. This way we get

$$g_1'(t) = \frac{r\left(r - t\cos\beta_1 - \sqrt{r^2 + t^2 - 2rt\cos\beta_1}\right)}{t^2\sqrt{r^2 + t^2 - 2rt\cos\beta_1}\sin\theta}.$$

Whether this derivative is positive or not depends on the expression $r - t\cos\beta_1 - \sqrt{r^2 + t^2 - 2rt\cos\beta_1}$. By simple calculations we see that this expression is strictly smaller than zero. This means that the quotient of the distance functions decreases as the point z moves farther away from the vertex A_1.

By repeating the above process for each line segment $(A_m, I_m]$ we can conclude that $\frac{d(z,\partial C)}{d(z,\partial P)}$ reaches its minimum either on some point in the set $\{I_m : m \in \{1, ..., k\}\}$ or on some line segment $[I_u, I_v] \in M$ where $u, v \in \{1, ..., k\}$. Especially, this means that the minimum of the quotient is not reached when the point z on the medial axis approaches a vertex of P. Now there is clearly not a point in M where the quotient of the distance functions is equal to 1 so there must be a constant c such that $\frac{d(z,\partial C)}{d(z,\partial P)} \geq c$ for all $z \in P$ and especially

$$\min\left\{\frac{d\left(I_m, \partial C\right)}{d\left(I_m, \partial P\right)} : m \in \{1, ..., k\}\right\} \geq c > 1.$$

Finally, by the same process as in the proof of Proposition 2.6 we conclude that

$$k_P\left(z_1, z_2\right) \geq c k_C\left(z_1, z_2\right) \quad \forall z_1, z_2 \in P$$

where c is as above.

To prove the case of the j-metric we translate the circumcenter c back to the origin and let $I_1 + r_1 e^{i\theta_1} = A_1$ and $I_2 + r_2 e^{i\theta_2} = A_2$ where I_m and A_m are constructed as above. Also let $\epsilon \in (0, \min\{r_1, r_2\})$. If $w_1 = I_1 + (r_1 - \epsilon) e^{i\theta_1}$ and $w_2 = I_2 + (r_2 - \epsilon) e^{i\theta_2}$, then w_1 and w_2 are contained in the medial axis and $d(w_1, \partial C) = \epsilon = d(w_2, \partial C)$. We also may assume that $d(w_1, \partial P) \leq d(w_2, \partial P)$. Now $\frac{\epsilon}{c} \geq d(w_1, \partial P)$, so

$$\frac{j_C\left(w_1, w_2\right)}{j_P\left(w_1, w_2\right)} = \frac{\log\left(1 + \frac{|w_1 - w_2|}{\epsilon}\right)}{\log\left(1 + \frac{|w_1 - w_2|}{d(w_1, \partial P)}\right)} \leq \frac{\log\left(1 + \frac{|w_1 - w_2|}{\epsilon}\right)}{\log\left(1 + \frac{c|w_1 - w_2|}{\epsilon}\right)}.$$

The right side of the above inequality approaches 1 when ϵ approaches 0. This means that $\frac{j_C(w_1,w_2)}{j_P(w_1,w_2)}$ must also approach 1 because it is always equal to or greater than 1. In conclusion there cannot be a constant $c > 1$ for the j-metric. $\square$

Corollary 2.17. *If P is a regular polygon or a triangle and C is such that ∂C is the corresponding circumcircle, then*

$$k_P\left(z_1, z_2\right) \geq \frac{d\left(I, \partial C\right)}{d\left(I, \partial P\right)} k_C\left(z_1, z_2\right) \quad \forall z_1, z_2 \in P$$

where I is the intersection point of the bisectors of the internal angles of P.

There is no constant $c > 1$ as required in Problem 2.1 in the case of the j-metric.

Proof. This follows directly from the proof of the preceding theorem because in a regular polygon and in a triangle the medial axis has only one junction point: the unique intersection point of the angle bisectors. □

Corollary 2.18. *If P is a cyclic polygon and C is such that ∂C is the corresponding circumcircle, the point where the quotient $\frac{d(z,\partial C)}{d(z,\partial P)}$ reaches its minimum is not generally unique.*

Proof. We prove this corollary by giving an example. If P is a rectangle that is not a square, the medial axis M has two junction points, I_1 and I_2, and the circumcenter c is the midpoint of the line segment $[I_1, I_2]$. Now by the proof of Theorem 2.16 the quotient $\frac{d(z,\partial C)}{d(z,\partial P)}$ decreases as z approaches a junction point from vertex. On the other hand when z approaches c on $[I_1, I_2]$, distance $d\,(z, \partial P)$ stays constant but $d\,(z, \partial C)$ increases so the quotient of the distance functions increases. In conclusion the quotient reaches its minimum in both I_1 and I_2. □

In the case of an incircle and the corresponding polygon the answer to Problem 2.1 is negative.

Proposition 2.19. *There is no constant $c > 1$ when the regions in Problem 2.1 are a polygon P and C is such that ∂C is the corresponding incircle.*

Proof. Let O be the center of C and $A \in \partial P \cap \partial C$. Now clearly $d\,(z, \partial P) = d\,(z, \partial C)$ for all points on the line segment $[A, O]$. Then for points z_1 and z_2 on the line segment $[A, O]$, $J_{k_P}\,[z_1, z_2] = [z_1, z_2]$. Also by [12, Lemma 3.7.] $J_{k_C}\,[z_1, z_2] = [z_1, z_2]$. From these facts it follows that

$$k_P\,(z_1, z_2) = k_C\,(z_1, z_2) \quad \forall z_1, z_2 \in [A, O]$$

so there cannot be a constant $c > 1$. □

Next we will present a lemma concerning cases when the quotient of the distance functions approaches 1 and then consider Problem 2.1 in the intersection of two disks.

Lemma 2.20. *Let D_1 and D_2 be as in Problem 2.1. If the function $\frac{d(z,\partial D_2)}{d(z,\partial D_1)}$ equals to 1 for all $z \in D_1$ or approaches 1 when $z \in D_1$, then there is no constant $c > 1$ as required in Problem 2.1.*

Proof. If $\frac{d(z,\partial D_2)}{d(z,\partial D_1)} = 1$ for all $z \in D_1$, then $D_1 = D_2$, and the result is trivial. Let us then assume that for some $z \in D_1$ the quotient $\frac{d(z,\partial D_2)}{d(z,\partial D_1)} \neq 1$. This means that there is a curve $\gamma \subset D_1$ such that $\frac{d(z,\partial D_2)}{d(z,\partial D_1)}$ approaches 1 on the curve γ. Now for every $\epsilon > 0$ we can pick a suitable part N of the curve γ such that

$$\frac{d(z, \partial D_2)}{d(z, \partial D_1)} - 1 < \epsilon \;\; \forall z \in N.$$

We shall now finish the proof by contradiction. Let us assume that there exists $c > 1$ so that $k_{D_1}(z_1, z_2) \geq c k_{D_2}(z_1, z_2)$ for all points in D_1. Now we pick such $\epsilon > 0$ that $\frac{c}{\epsilon+1} > 1$. Next by similar process as in the proof of Proposition 2.6 we get that $k_{D_2}(z_1, z_2) > \frac{1}{\epsilon+1} k_{D_1}(z_1, z_2)$ for some points on the curve γ. By combining the two previous inequalities we have $k_{D_1}(z_1, z_2) > \frac{c}{\epsilon+1} k_{D_1}(z_1, z_2)$ but this is impossible for $\frac{c}{\epsilon+1} > 1$. $\square$

Theorem 2.21. *Let ∂C_1 and ∂C_2 be such circles that $\partial C_1 \cap \partial C_2 = \{A, B\}$. Now for domains $D_1 = C_1 \cap C_2$ and $D_2 = C_1 \cup C_2$ there is no constants $c > 1$ satisfying the assertion of Problem 2.1 for either of the two metrics.*

Proof. Let us first define few notations. Let r_m be the radius of the circle ∂C_m. Without loss of generality we may assume that the center of C_1, c_1, is the origin, that the center of the other circle, c_2, is on the positive real axis and that $\mathrm{Im}(B) > 0$. We shall begin by proving the case of the quasihyperbolic metric.

First we let $z = te^{i\theta} \in D_1$ and solve what is the equation for t when z is on the medial axis of D_1, M. Now $d(z, \partial C_1) = r_1 - t$ and $d(z, \partial C_2) = r_2 - |c_2 - z| = r_2 - \sqrt{c_2^2 + t^2 - 2c_2 t \cos\theta}$. Clearly z is on M when $d(z, \partial C_1) = d(z, \partial C_2)$. By solving this equation for t we get $t = \frac{(c_2+r_1-r_2)(c_2-r_1+r_2)}{2(-r_1+r_2+c_2\cos\theta)}$.

Second let $z = te^{i\theta}$ be on the medial axis. Because of symmetry and because we are only interested what happens close to a border point, we may assume $\mathrm{Im}(z) > 0$. We shall prove that the quotient of the distance functions $\frac{d(z,\partial D_2)}{d(z,\partial D_1)}$ approaches 1 when θ approaches $\arg(B) > 0$. Clearly

$|z-B| = d(z,\partial D_2)$. Now by the law of cosines we have

$$\frac{d(z,\partial D_2)}{d(z,\partial D_1)} = \frac{\sqrt{r_1^2+t^2-2tr_1\cos(\arg(B)-\theta)}}{r_1-t}.$$

By making the substitution $t = \frac{(c_2+r_1-r_2)(c_2-r_1+r_2)}{2(-r_1+r_2+c_2\cos\theta)}$ we reach

$$\frac{d(z,\partial D_2)}{d(z,\partial D_1)} = \frac{2(r_1-r_2-c_2\cos\theta)\sqrt{f(\theta)}}{c_2^2+r_1^2-r_2^2-2c_2r_1\cos\theta}$$

where

$$f(\theta) = r_1^2 + \frac{(c_2+r_1-r_2)^2(c_2-r_1+r_2)^2}{4(-r_1+r_2+c_2\cos\theta)^2} - \frac{r_1(c_2+r_1-r_2)(c_2-r_1+r_2)\cos(\arg(B)-\theta)}{-r_1+r_2+c_2\cos\theta}.$$

Next by straightforward calculation

$$\lim_{\theta\to\arg(B)} \frac{d(z,\partial D_2)}{d(z,\partial D_1)} = 1$$

so the claim for quasihyperbolic metric follows from Lemma 2.20.

The case of the j-metric is proved fundamentally in the same way as in the proof of Theorem 2.16. □

2.3 Subdomain Geometry and Supercircles

In this section we shall answer to Conjecture 2.5 in certain cases. First we must prove few preliminary results concerning subdomain geometry.

The following problem was presented by M. Vuorinen in an informal conversation at the University of Turku sometime during the summer of 2013.

Problem 2.22. *Let $G \subsetneq \mathbb{R}^n$ and $w \in G$. Point $b \in \partial G$ is such that $d(w,\partial G) = |w-b|$. Does there exist a domain $D \subsetneq G$ that fills conditions*

1. $[w,b) \subsetneq D$

and

2. $\exists c > 1 \ : \ k_D(z_1,z_2) \geq c k_G(z_1,z_2) \ \ \forall z_1,z_2 \in D$?

The following theorem shows that the answer to this problem is positive in the case of the complex plane.

Theorem 2.23. *Let $G \subsetneq \mathbb{C}$. Then there exists a domain $D \subsetneq G$ as required for Problem 2.22.*

Proof. First let

$$S_\alpha = \left\{ z \in \mathbb{H} : \frac{\pi}{2} - \alpha < \arg(z) < \frac{\pi}{2} + \alpha \right\}$$

for a fixed $\alpha \in \left(0, \arcsin\left(\frac{1}{16}\right)\right)$. Now by Theorem 2.14

$$\frac{1}{\sin\alpha} k_{\mathbb{H}}(z_1, z_2) < k_{S_\alpha}(z_1, z_2) \quad \forall z_1, z_2 \in S_\alpha$$

where $\frac{1}{\sin\alpha} > 16$.

Second let $G \subsetneq \mathbb{C}$, $w \in G$ and $b \in \partial G$ be as in Problem 2.22. Now we define $G' \subset G$ to be a simply connected domain that includes the line segment $[w, b)$. This kind of domain must exist, because $d(w, \partial G) = |w - b|$ implies that $B(w, |w-b|) \subset G$. By Riemann mapping theorem there exists a biholomorphic and conformal map f_{rmt} from G' onto $\mathbb{D}$. Then Möbius transformation

$$m_1(z) = e^{i\theta} \frac{z - f_{rmt}(w)}{1 - \overline{f_{rmt}(w)} z}$$

with suitable θ is an automorphism of the unit disk that maps $f_{rmt}(w)$ to the origin and $f_{rmt}(b)$ to -1. Another Möbius transformation

$$m_2(z) = \frac{iz + i}{-z + 1}$$

maps the unit disk $\mathbb{D}$ onto the upper half-plane $\mathbb{H}$ so that $m_2(-1) = 0$ and $m_2(0) = i$. Now, the composite function $f_1 = m_2 \circ m_1 \circ f_{rmt}$ maps G' bijectively and conformally onto $\mathbb{H}$ and especially b to the origin and w to i so that curve $f_1([w, b))$ meets the real axis at right angle. Finally, if we choose appropriate coefficient $0 < a \le 1$ and define $f = af_1$, then $f([w, b)) \subsetneq S_\alpha$.

Third let us define $D = f^{-1}(S_\alpha)$. Now, by [9, Proposition 1.6.], we get

$$\frac{1}{4} k_D(z_1, z_2) \le k_{S_\alpha}(f(z_1), f(z_2)) \le 4 k_D(z_1, z_2) \quad \forall z_1, z_2 \in D$$

and

$$\frac{1}{4} k_{G'}\left(z_1, z_2\right) \leq k_{\mathbb{H}}\left(f\left(z_1\right), f\left(z_2\right)\right) \leq 4 k_{G'}\left(z_1, z_2\right) \quad \forall z_1, z_2 \in G'.$$

By combining these two two-sided inequalities and the one from the first part of the proof, we have

$$\begin{aligned} k_D\left(z_1, z_2\right) &\geq \frac{1}{4} k_{S_\alpha}\left(f\left(z_1\right), f\left(z_2\right)\right) \\ &> \frac{1}{4 \sin \alpha} k_{\mathbb{H}}\left(f\left(z_1\right), f\left(z_2\right)\right) \\ &\geq \frac{1}{16 \sin \alpha} k_{G'}\left(z_1, z_2\right) \end{aligned}$$

for all z_1, z_2 in D. Also $G' \subset G$, so $k_{G'}\left(z_1, z_2\right) \geq k_G\left(z_1, z_2\right)$ which finally gives us

$$k_D\left(z_1, z_2\right) > \frac{1}{16 \sin \alpha} k_G\left(z_1, z_2\right) \quad \forall z_1, z_2 \in D.$$

Clearly, $\frac{1}{16 \sin \alpha} > 1$. This means that D is as required in the theorem, and the proof is complete. $\square$

We can also give a positive answer to Problem 2.22 in the general case of $\mathbb{R}^n$ as the following proof, suggested by T. Sugawa during an informal conversation at the Second Chinese-Finnish Seminar and Workshop on Modern Trends in Classical Analysis and Applications in August 2013, shows.

Theorem 2.24. *Let $G \subsetneq \mathbb{R}^n$. Then there exists a domain $D \subsetneq G$ as required for Problem 2.22.*

Proof. Let $G \subsetneq \mathbb{R}^n$, $w \in G$ and $b \in \partial G$ be as in Problem 2.22. Also let B be such an Euclidean ball that $b \in \partial B \cap \partial G$ and that it includes the line segment $[w, b)$. Next we define C_β to be such a right circular cone that $C_\beta \subset G$, $[w, b) \subset C_\beta$ and $\partial C_\beta \cap \partial G = b$, and C_α to be an other right circular cone such that $C_\alpha \subsetneq C_\beta$, $[w, b) \subset C_\alpha$ and $\partial C_\alpha \cap \partial C_\beta = b$.

Now by a similar argument as in the proof of Lemma 2.10 we get

$$\frac{1}{\sin\left(\frac{\pi}{2} + \alpha - \beta\right)} k_{C_\beta}\left(z_1, z_2\right) < k_{C_\alpha}\left(z_1, z_2\right) \quad \forall z_1, z_2 \in C_\alpha$$

and because $C_\beta \subset B \subset G$, we finally get

$$\frac{1}{\sin\left(\frac{\pi}{2}+\alpha-\beta\right)} k_G\left(z_1, z_2\right) < k_{C_\alpha}\left(z_1, z_2\right) \quad \forall z_1, z_2 \in C_\alpha$$

so C_α is suitable for the region D in Problem 2.22. $\square$

Based upon the proof of Theorem 2.23 M. Vuorinen presented the following problem in an informal discussion at the University of Turku sometime during the summer of 2013.

Problem 2.25. *Does there exist such a domain $D \subset \mathbb{H}$ that ∂D is differentiable at the origin O, that $\partial D \cap \partial \mathbb{H} = \{O\}$ and that the answer to Problem 2.1 is positive for the domains D and $\mathbb{H}$?*

We shall show that answer to this problem is negative.

Theorem 2.26. *There does not exist any such domain D as described in Problem 2.25.*

Proof. We shall prove this claim by considering the quotient of the distance functions $\frac{d(z,\partial\mathbb{H})}{d(z,\partial D)}$ when z is on the imaginary axis. First from the definition of D it follows that the derivative of ∂D at the origin O is 0 for otherwise ∂D would continue below the real axis which is impossible. From this it follows that there exist $b > 0$ such that the line segment $(O, ib] \subset D$ for otherwise the derivative would not be 0 at the origin.

If $\frac{d(z,\partial\mathbb{H})}{d(z,\partial D)} = 1$ for some points on the line segment $(O, ib]$, then the claim follows from Lemma 2.20. Otherwise let $\partial D_+ = \{z \in \partial D : \mathrm{Re}\,(z) \geq 0\}$ and $\partial D_- = \{z \in \partial D : \mathrm{Re}\,(z) \leq 0\}$. We may assume without losing generality that $d\left(z, \partial D_+\right) \leq d\left(z, \partial D_-\right)$ when $z \in (O, ic]$ where $c \leq b$.

Next we shall prove that the quotient $\frac{d(z,\partial\mathbb{H})}{d(z,\partial D_+)}$ approaches 1 as z approaches the origin on the line segment $(O, ic]$. Let $P = (x, if\left(x\right))$ be such a part of the curve ∂D_+ that f is strictly increasing. P must exist for otherwise D would not be as defined. Let $K \in P$ be such a point that $|z - K| = d\left(z, P\right)$ for $z \in (O, ic]$. Let $z = iy$ and let us solve y for $K = (x, f\left(x\right))$. By making use of the fact that the line segment $[z, K]$ is normal to the point K we get

$$y - f\left(x\right) = -\frac{1}{f'\left(x\right)}\left(0 - x\right)$$

so we get

$$y = \frac{x}{f'(x)} + f(x).$$

Then by using the definition of the Euclidean distance we get

$$|z - K| = \sqrt{x^2 + \left(\frac{x}{f'(x)}\right)^2}.$$

Finally, we are ready to evaluate the quotient $\frac{d(z,\partial\mathbb{H})}{d(z,\partial D_+)}$ on the line segment $(O, ic]$. It is clear that the origin O is the closest point to z on $\partial\mathbb{H}$, so now

$$\begin{aligned}\frac{d(z,\partial\mathbb{H})}{d(z,\partial D_+)} &= \frac{\frac{x}{f'(x)} + f(x)}{\sqrt{x^2 + \left(\frac{x}{f'(x)}\right)^2}} \\ &= \frac{1}{\sqrt{1 + (f'(x))^2}} + \left(\frac{f(x)}{x}\right)\left(\frac{1}{\sqrt{1 + (f'(x))^{-2}}}\right).\end{aligned}$$

Now z approaches the origin on $(O, ic]$ as x approaches 0. We make use of the algebraic limit theorem and the fact that $f'(0) = 0$ together with $f(0) = 0$ implies $\lim_{x\to 0} \frac{f(x)}{x} = 0$ and calculate

$$\lim_{x\to 0} \frac{d(z,\partial\mathbb{H})}{d(z,\partial D_+)} = 1 + 0 \cdot 0 = 1.$$

The claim follows from this and by Lemma 2.20. □

Corollary 2.27. *If D_1 and D_2 are as in Problem 2.25 and $D_1 \subset D_2$, then there is no constant $c > 1$ as required in Problem 2.1.*

Proof. Now for all $z \in D_1$ we have

$$\frac{d(z,\partial D_2)}{d(z,\partial D_1)} = \frac{d(z,\partial D_2)}{d(z,\partial\mathbb{H})}\frac{d(z,\partial\mathbb{H})}{d(z,\partial D_1)}.$$

Both the quotients on the right hand side of the above relation approach 1 when z approaches the origin on the imaginary axis, so the claim follows by Lemma 2.20. □

With the help of Theorem 2.26 and Corollary 2.27, we may now go over some cases of Conjecture 2.5. We shall begin with a general lemma and then proceed to results concerning the conjecture itself.

Lemma 2.28. *Let D_1 and D_2 be two domains as in Problem 2.1 and let $c = \inf\left\{\frac{d(z,\partial D_2)}{d(z,\partial D_1)} : z \in D_1\right\} > 1$. If $J_{k_{D_1}}[w_1, w_2] = J_{k_{D_2}}[w_1, w_2]$ for points in a nonempty $W \subset D_1$ and if the infimum c is reached in the closure of W, then c is the best possible constant that fills the requirements of Problem 2.1.*

Proof. We shall prove by contradiction. Let us suppose $h > c$ and that $k_{D_1}(z_1, z_2) \geq hk_{D_2}(z_1, z_2)$ for all points in the domain D_1. Especially for points in W we get

$$\int_J \frac{c|dz|}{d(z, \partial D_2)} < \int_J \frac{h|dz|}{d(z, \partial D_2)} \leq \int_J \frac{|dz|}{d(z, \partial D_1)}$$

where J is a geodesic joining the points z_1 and z_2. Because the path is the same in each integral and because the integrand is now always positive, we get

$$\frac{c}{d(z, \partial D_2)} < \frac{h}{d(z, \partial D_2)} \leq \frac{1}{d(z, \partial D_1)}$$

and

$$c < h \leq \frac{d(z, \partial D_2)}{d(z, \partial D_1)} \quad \forall z \in W.$$

This implies that $h = \inf\left\{\frac{d(z,\partial D_2)}{d(z,\partial D_1)} : z \in D_1\right\}$ which is a contradiction, and so the claim follows. $\square$

Proposition 2.29. *If $s = 1$ and $t > 2$, then Conjecture 2.5 is false but there is a constant $c > 1$ as required in Problem 2.1.*

Proof. For all $z \in D_1$ we have

$$\frac{d(z, \partial D_t)}{d(z, \partial D_1)} = \frac{d(z, \partial D_t)}{d(z, \partial D_2)} \frac{d(z, \partial D_2)}{d(z, \partial D_1)}.$$

We know by the proof of Theorem 2.16 and by Corollary 2.27 that each of the quotients on the right hand side of the above relation reaches its infimum on the medial axis M, of D_1. Next by the proof of Theorem 2.16 we also know that $\frac{d(w,\partial D_2)}{d(w,\partial D_1)} = \sqrt{2}$ for all points on M. Then the quotient $\frac{d(w,\partial D_t)}{d(w,\partial D_2)} = 1$ on M as $t > 2$. In conclusion we have

$$\frac{d(z, \partial D_t)}{d(z, \partial D_1)} \geq \sqrt{2} \quad \forall z \in D_1.$$

From this it follows that $k_{D_1}(z_1,z_2) \geq \sqrt{2} k_{D_t}(z_1,z_2)$ for all points in D_1.

Finally we prove that the conjecture is false in this case. First $2^{1-\frac{1}{2}} \geq 2^{1-\frac{1}{t}}$ if and only if $t \leq 2$. Then by [12, Lemma 3.7.] $J_{k_{D_1}}[z_1,z_2] = J_{k_{D_t}}[z_1,z_2]$ for all points on the line segment $(i,-i)$. Now by Lemma 2.28 the constant $\sqrt{2}$ is the best possible and so the conjecture is false. □

Proposition 2.30. *If $1 < s < t$ for the domains in Conjecture 2.5, there is no constant $c > 1$ as required in Problem 2.1.*

Proof. If we make a translation $t(z) = z + i$, the situation becomes as in Corollary 2.27 and the claim follows. □

Proposition 2.31. *If $0 < s < 1$ and $t = 2$, then Conjecture 2.5 is true.*

Proof. Let D_1 be a domain such that $D_s \subset D_1 \subset D_t$. Then for all points $z \in D_s$, we have

$$\frac{d(z,\partial D_t)}{d(z,\partial D_s)} = \frac{d(z,\partial D_t)}{d(z,\partial D_1)} \frac{d(z,\partial D_1)}{d(z,\partial D_s)}.$$

First by the proof of Theorem 2.16 the quotient $\frac{d(z,\partial D_t)}{d(z,\partial D_1)}$ reaches its infimum on the medial axis M, of D_1, where it is the constant $\sqrt{2}$. Then by the proof of Theorem 2.3 the quotient $\frac{d(z,\partial D_1)}{d(z,\partial D_s)}$ reaches its infimum at the origin, O and the minimum is $2^{\frac{1}{s}-1}$. Because $O \in M$, we get

$$\frac{d(z,\partial D_t)}{d(z,\partial D_s)} \geq \sqrt{2} \cdot 2^{\frac{1}{s}-1} = 2^{\frac{1}{s}-\frac{1}{2}} \quad \forall z \in D_s.$$

Now the claim follows by a similar argument as in the proof of Proposition 2.6. □

Proposition 2.32. *If $0 < s < 1$ and $t > 2$, then Conjecture 2.5 is false but there is a constant $c > 1$ as required in Problem 2.1.*

Proof. Let D_2 be a domain such that $D_s \subset D_2 \subset D_t$. Then for all points $z \in D_s$, we have

$$\frac{d(z,\partial D_t)}{d(z,\partial D_s)} = \frac{d(z,\partial D_t)}{d(z,\partial D_2)} \frac{d(z,\partial D_2)}{d(z,\partial D_s)}.$$

First as in the proof of Lemma 2.29 the quotient $\frac{d(z,\partial D_t)}{d(z,\partial D_2)}$ reaches its infimum on the medial axis M, of D_2, where it is the constant 1. Second, by Proposition 2.31, the quotient $\frac{d(z,\partial D_2)}{d(z,\partial D_s)}$ reaches its infimum value $2^{\frac{1}{s}-\frac{1}{2}}$ at the origin $O \in M$. From these facts we conclude that

$$\frac{d\left(z,\partial D_t\right)}{d\left(z,\partial D_s\right)} \geq 1 \cdot 2^{\frac{1}{s}-\frac{1}{2}} = 2^{\frac{1}{s}-\frac{1}{2}} \ \ \forall z \in D_s.$$

From this it follows that $k_{D_s}\left(z_1, z_2\right) \geq 2^{\frac{1}{s}-\frac{1}{2}} k_{D_t}\left(z_1, z_2\right)$ for all points in D_s.

Finally $2^{\frac{1}{s}-\frac{1}{2}} < 2^{\frac{1}{s}-\frac{1}{t}}$ and $2^{\frac{1}{s}-\frac{1}{2}}$ is the best possible coefficient in this case by a similar argument as in the proof of Lemma 2.29 so Conjecture 2.5 is false. $\square$

2.4 Subdomain Geometry and Homeomorphisms

We can also ask whether homeomorphisms preserve the condition of Problem 2.1. To deal with this question we first introduce a new definition.

Definition 2.33. If $f : D \to D'$ and $\varphi : [0,\infty) \to [0,\infty)$ are homeomorphisms and if

$$\varphi^{-1}\left(k_D\left(z_1, z_2\right)\right) \leq k_{D'}\left(f\left(z_1\right), f\left(z_2\right)\right) \leq \varphi\left(k_D\left(z_1, z_2\right)\right)$$

for all points in D, then f is φ-solid. If above condition is true for all subsets of D then f is fully φ-solid.

If the above inequality is missing the left hand side, f is φ-semisolid.

Proposition 2.34. *Let D_1 and D_2 be such subdomains of $\mathbb{R}^n$ that they fulfill the conditions of Problem 2.1 for the quasihyperbolic metric with a constant $c > 1$. If $f : D_2 \to D_2'$ is fully φ-solid, D_1' is the image of D_1 under f and $\varphi(t) \leq t$ for all t, then $k_{D_1'}\left(f\left(z_1\right), f\left(z_2\right)\right) \geq c k_{D_2'}\left(f\left(z_1\right), f\left(z_2\right)\right)$ for all points in D_1.*

Proof. First because φ is a homeomorphism from $[0,\infty)$ to $[0,\infty)$, φ and its inverse are strictly increasing. From this and the fact that $\varphi(t) \leq t$, it follows that $\varphi^{-1}(t) \geq t$ for all t.

Now because f is fully φ-solid, $k_{D_1'}\left(f\left(z_1\right), f\left(z_2\right)\right) \geq \varphi^{-1}\left(k_{D_1}\left(z_1, z_2\right)\right)$.

Then because the domains D_1 and D_2 fill the criteria of Problem 2.1 and because of the inequalities mentioned in the previous paragraph, we get

$$\begin{aligned} k_{D_1'}\left(f\left(z_1\right), f\left(z_2\right)\right) &\geq \varphi^{-1}\left(c k_{D_2}\left(z_1, z_2\right)\right) \\ &\geq c k_{D_2}\left(z_1, z_2\right) \geq c \varphi\left(k_{D_2}\left(z_1, z_2\right)\right). \end{aligned}$$

Finally we again use the fact that f is fully φ-solid and get that

$$k_{D_1'}\left(f\left(z_1\right), f\left(z_2\right)\right) \geq c k_{D_2'}\left(f\left(z_1\right), f\left(z_2\right)\right)$$

for all points in D_1' which proves the claim. □

Proposition 2.35. *Let D_1 and D_2 be such subdomains of $\mathbb{R}^n$ that they fulfill the conditions of Problem 2.1 for the quasihyperbolic metric with a constant $c > 1$. If $f : D_2 \to D_2'$ is fully φ-solid, D_1' is the image of D_1 under f and $\varphi(t) = Mt^p$ with $M \geq 1$ and $p > 1$, then*

$$k_{D_1'}\left(f\left(z_1\right), f\left(z_2\right)\right) \geq \left(\frac{c}{M^2}\right)^{\frac{1}{p}} \left(k_{D_2'}\left(f\left(z_1\right), f\left(z_2\right)\right)\right)^{\frac{1}{p^2}}$$

for all points in D_1.

Proof. Let z_1 and z_2 be points in D_1. First because f is fully φ-solid, we get

$$k_{D_1'}\left(f\left(z_1\right), f\left(z_2\right)\right) \geq \varphi^{-1}\left(k_{D_1}\left(z_1, z_2\right)\right) = \left(\frac{1}{M} k_{D_1}\left(z_1, z_2\right)\right)^{\frac{1}{p}}.$$

Then because φ is stricly increasing, φ^{-1} is also stricly increasing, and so we get

$$k_{D_1'}\left(f\left(z_1\right), f\left(z_2\right)\right) \geq \left(\frac{c}{M} k_{D_2}\left(z_1, z_2\right)\right)^{\frac{1}{p}} = \left(\frac{c}{M^2}\right)^{\frac{1}{p}} \left(M k_{D_2}\left(z_1, z_2\right)\right)^{\frac{1}{p}}.$$

But $M^{\frac{1}{p}} \geq M^{\frac{1}{p^2}}$ and $\left(k_{D_2}\left(z_1, z_2\right)\right)^{\frac{1}{p}} = \left(\left(k_{D_2}\left(z_1, z_2\right)\right)^p\right)^{\frac{1}{p^2}}$, so we have

$$k_{D_1'}\left(f\left(z_1\right), f\left(z_2\right)\right) \geq \left(\frac{c}{M^2}\right)^{\frac{1}{p}} \left(M\left(k_{D_2}\left(z_1, z_2\right)\right)^p\right)^{\frac{1}{p^2}}.$$

Finally $M\left(k_{D_2}\left(z_1, z_2\right)\right)^p = \varphi\left(k_{D_2}\left(z_1, z_2\right)\right)$ and f is φ-solid, so

$$k_{D_1'}\left(f\left(z_1\right), f\left(z_2\right)\right) \geq \left(\frac{c}{M^2}\right)^{\frac{1}{p}} \left(k_{D_2'}\left(f\left(z_1\right), f\left(z_2\right)\right)\right)^{\frac{1}{p^2}}.$$ □

3 Concluding Remarks

The most important results of this thesis deal with Problem 2.1 that concerns comparing hyperbolic type distances in subdomains to hyperbolic type distances in the original domains. This problem was first introduced and answered to in certain cases in the article [8] by R. Klén, Y. Li and M. Vuorinen. In this thesis the previous research was expanded which is most apparent in Theorem 2.16 that generalizes an erstwhile theorem concerning a square and the corresponding circumcircle to all cyclic polygons. For future research, this problem could be studied in the case of other metrics discussed in [20] that were not mentioned in this thesis.

Also noteworthy is the advancement in the study of Conjecture 2.5 that deals with Problem 2.1 in the case of supercircles. Propositions 2.29, 2.30, 2.31 and 2.32 cover some cases of this conjecture and hopefully they will in the future help turning the conjecture into proven theorems.

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References

[1] J. W. Anderson: *Hyperbolic Geometry, 2. edition*, Springer-Verlag, New York, 2005.

[2] A. F. Beardon: *The Geometry of Discrete Groups*, Springer-Verlag, New York, 1983.

[3] F. W. Gehring and B. G. Osgood: *Uniform Domains and the Quasihyperbolic Metric*, J. Anal. Math., Vol. 36, p. 50–74, 1979.

[4] F. W. Gehring and B. P. Palka: *Quasiconformally Homogeneous Domains*, J. Anal. Math., Vol. 30, p. 172–199, 1976.

[5] M. Glader: *Hyperbolinen ja Kvasihyperbolinen Geometria*, Master's Thesis, University of Turku, 2011.

[6] R. Klén: *Local Convexity Properties of Metric Balls*, Licentiate's Thesis, University of Turku, 2007.

[7] R. Klén: *Local Convexity Properties of j-metric Balls*, Ann. Acad. Sci. Fenn. Math., Vol. 33, p. 281–293, 2008.

[8] R. Klén, Y. Li and M. Vuorinen: *Subdomain Geometry of Hyperbolic Type Metrics*, Pr. Inst. Mat. Nats. Akad. Nauk Ukr. Mat. Zastos., Vol. 10, p. 177–194, 2013.

[9] R. Klén, M. Vuorinen and X. Zhang: *Quasihyperbolic Metric and Möbius Transformations*, Proc. Amer. Math. Soc., Vol. 142, p. 311–322, 2014.

[10] R. Kühnau: *Handbook of Complex Analysis: Geometric Function Theory. Vol. 1*, North-Holland, Amsterdam, 2002.

[11] R. Kühnau: *Handbook of Complex Analysis: Geometric Function Theory. Vol. 2*, Elsevier Science B. V., Amsterdam, 2005.

[12] H. Lindén: *Quasihyperbolic Geodesics and Uniformity in Elementary Domains*, Doctoral Thesis, Yliopistopaino, Helsinki, 2005.

[13] G. J. Martin: *Quasiconformal and Bi-Lipschitz Homeomorphisms, Uniform Domains and the Quasihyperbolic Metric*, Trans. Amer. Math. Soc., Vol. 292, p. 169–191, 1985.

[14] G. J. Martin and B. G. Osgood: *The Quasihyperbolic Metric and Associated Estimates on the Hyperbolic Metric*, J. Anal. Math., Vol. 47, p. 37–53, 1986.

[15] O. Martio and J. Väisälä: *Quasihyperbolic Geodesics in Convex Domains II*, Pure Appl. Math. Q., Vol. 7, p. 379–393, 2011.

[16] J. Milnor: *Hyperbolic Geometry: The First 150 Years*, Bull. Amer. Math. Soc., Vol. 6, p. 9–24, 1982.

[17] P. Seittenranta: *Möbius-Invariant Metrics*, Math. Proc. Cambridge Philos. Soc., Vol. 125, p. 511–533, 1999.

[18] S. Stahl: *The Poincaré Half-Plane: A Gateway to Modern Geometry*, Jones and Bartlett Publishers, Boston, 1993.

[19] M. Vuorinen: *Conformal Invariants and Quasiregular Mappings*, J. Anal. Math., Vol. 45, p. 69–115, 1985.

[20] M. Vuorinen: *Geometry of Metrics*, J. Analysis, Vol. 18, p. 399–424, 2010.