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A CERTAINTY EQUIVALENT CHARACTERIZATION OF A CLASS OF PERPETUAL AMERICAN CONTINGENT CLAIMS

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ABSTRACT

This paper analyzes the certainty equivalent deterministic characterization of a class of stochastic valuations arising typically in studies considering irreversible investment in the presence of revenue and cost uncertainty. We demonstrate that certainty equivalence can be attained by adjusting either the growth rate of the underlying diffusions modelling the revenues and costs or by adjusting the interest rate at which future cash flows are discounted. We also consider the comparative static properties of these adjustments and demonstrate that the sensitivity of the optimal policy with respect to changes in volatility is a process-specific, and not a payoff-specific, property.

CONTENTS

1 INTRODUCTION

In their seminal study, McDonald and Siegel (1986) considered the optimal irreversible investment decision of a rationally managed firm facing both revenue and cost uncertainty. Given the linearity of the exercise payoff accrued from exercising the irreversible investment opportunity and the assumed flexibility and perpetuity of the timing of the actual irreversible investment opportunity, they modelled the investment problem as an optimal stopping problem of a two-dimensional geometric Brownian motion. They solved the problem explicitly and proved that the optimal investment rule can be characterized as a requirement that the benefit-cost ratio of the investment project has to exceed a critical threshold which is typically significantly greater than one and, therefore, potentially very different from the standard net present value rule suggesting that an investment opportunity should be exercised whenever benefits exceed costs. McDonald and Siegel (1986) also consider the comparative static properties of the optimal investment rule and established that increased volatility increases the value of the investment opportunity and decelerates rational investment by expanding the continuation region where investment is suboptimal.

Olsen and Stensdland (1992) extended the analysis of McDonald and Siegel (1986) to a more general setting where the exercise payoff was assumed to be linearly homogenous (but otherwise general) and the underlying dynamics modelling the stochastically fluctuating prices and costs are characterized by a n-dimensional geometric Brownian motion with potentially correlated driving Brownian motions. They demonstrated that, under the above mentioned conditions, the value of the optimal investment strategy maintains both the homogeneity and the convexity properties of the exercise payoff. As a consequence of these findings, they were able to establish that the continuation region where exercising the investment opportunity is suboptimal contains a half-space characterized by a linear boundary with known coefficients representing the exercise thresholds of the associated two-dimensional McDonald and Siegel-problems. Olsen and Stensland (1992) also demonstrated that for a linear exercise payoff and a twice continuously differentiable value function, increased volatility increases the value of the investment opportunity and postpones rational exercise by increasing the optimal boundary and, in this way, expanding the continuation region.

Hu and Øksendal (1998) reconsidered the problems addressed in McDonald and Siegel (1986) and Olsen and Stensdland (1992) and presented a rigorous proof of the findings of McDonald and Siegel (1986) by relying on variational inequalities. They also extended the findings of Olsen and Stensdland (1992) by proving that the stopping region is always contained in some half-space. As a consequence of this observation, Hu and Øksendal (1998) were able to establish a set of parametric conditions under which the optimal investment boundary can be actually characterized as the boundary of a half-space.

Along the lines of McDonald and Siegel (1986), we consider the valuation of an investment opportunity in the presence of two stochastically fluctuating and potentially correlated geometric Brownian motions. However, for the sake of generality and in order to extend the results by McDonald and Siegel (1986), Olsen and Stensdland (1992), and Hu and Øksendal (1998) we assume that the exercise payoff is homogenous of degree η and twice continuously differentiable. This generalization is of interest since it permits the analysis of the impact of risk aversion on the optimal irreversible investment policy in the case where the utility function of the risk averse investor is of the *HARA*-type (Merton, 1971). Instead of relying on variational inequalities, we tackle the considered valuation by relying on a combination of stochastic calculus and the classical theory of diffusions and present a certainty equivalent characterization of the considered class of valuations (Alvarez, 2004). More precisely, we prove that under a considerably weak set of conditions the homogeneity of the exercise payoff and the characterization of the underlying stochastically fluctuating dynamics results into a solvable one-dimensional valuation which, in turn, can be characterized in terms of the minimal excessive mappings for the associated one-dimensional quotient processes. Put somewhat differently, we state a set of conditions under which the two-dimensional optimal stopping problem can be transformed into a standard solvable one-dimensional stopping problem. Given this characterization, we are able to present a certainty equivalent formulation of the considered two-dimensional stopping problem in terms of an associated one-dimensional and deterministic timing problem, which is adjusted to the risk of the underlying diffusions. We also analyze the comparative static properties of both the value and exercise threshold of the optimal investment policy and state a set of conditions under which increased total volatility increases the value of the investment policy and decelerates investment by expanding the continuation region. In contrast to the findings by

Olsen and Stensdland (1992), we find that the sign of the relationship between increased volatility and the optimal policy does not generally depend on the convexity or concavity of the exercise payoff. Thus, our results indicate that *the sensitivity of the optimal policy with respect to changes in the volatility of the underlying diffusions is a process-specific, and not a payoff-specific, property* (Alvarez, 2003).

Our study proceeds as follows. In Section 2, we present the considered class of two-dimensional stopping problems and state our main results on the certainty equivalent characterization of the considered valuations and their sensitivity with respect to changes in the volatility of the underlying diffusions. In Section 3 we illustrate our general results explicitly in three examples. Finally, we present some concluding comments in Section 4.

2 THE UNDERLYING STOCHASTIC DYNAMICS

The main objective of this study is to consider how a class of valuations arising in the literature on irreversible decision making can be solved by relying on deterministic models adjusted to the risk of the underlying stochastic value processes. In order to accomplish this task, assume that the underlying two-dimensional value dynamics evolve on a complete filtered probability space $(\Omega, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F})$ according to the stochastic dynamics described by the stochastic differential equations

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \quad X_0 = x \in \mathbb{R}_+, \tag{2.1}$$

$$dY_t = \alpha Y_t dt + \beta Y_t dZ_t, \quad Y_0 = y \in \mathbb{R}_+, \tag{2.2}$$

where $\mu, \alpha \in \mathbb{R}, \sigma, \beta \in \mathbb{R}_+$ are exogenously given constants, and W_t and Z_t are potentially correlated Brownian motions satisfying the condition $dW_t dZ_t = \rho dt$, $\rho \in [-1, 1]$.

Given the above characterization of the underlying stochastic dynamics, assume now that the mapping $F : \mathbb{R}^2_+ \mapsto \mathbb{R}$ is twice continuously differentiable on $\mathbb{R}^2_+$, homogenous of degree η, and satisfies for all $(x, y) \in \mathbb{R}^2_+$ the condition

$$\lim_{t\to\infty} \mathbb{E}\left[e^{-rt} F(X_t, Y_t)\right] = 0.$$

Given these assumptions, we now plan to consider the valuation

$$V(x, y) = \sup_{\tau} \mathbb{E}\left[e^{-r\tau} F(X_\tau, Y_\tau)\right] \tag{2.3}$$

and to determine the stopping date τ^* at which this maximum is attained. Before proceeding in the analysis of the valuation, we first prove the following result extending the previous findings arising in studies considering exchange options and other contingent claims subject to linearly homogenous exercise payoffs (Björk, 1996, 284–285).

Lemma 2.1. *The optimal stopping problem* (2.3) *can be re-expressed as*

$$V(x, y) = y^\eta \sup_{\tau} \mathbb{E}_{x/y}\left[e^{-(r-\delta(\eta))\tau} F(P_\tau, 1)\right], \tag{2.4}$$

where $\delta(\eta) = \eta\alpha + \frac{1}{2}\beta^2\eta(\eta - 1)$ *and the underlying process* P_t *evolves according to the dynamics characterized by the stochastic differential equation*

$$dP_t = \xi P_t dt + \sigma P_t dW_t - \beta P_t dZ_t, \quad P_0 = x/y, \tag{2.5}$$

where $\xi = \mu - \alpha + (\beta^2 - \sigma\beta\rho)(1 - \eta)$. *Alternatively, the optimal stopping problem* (2.3) *can also be re-expressed as*

$$V(x, y) = x^\eta \sup_\tau \mathbb{E}_{y/x}\left[e^{-(r-\gamma(\eta))\tau} F(1, Q_\tau)\right], \tag{2.6}$$

where $\gamma(\eta) = \eta\mu + \frac{1}{2}\sigma^2\eta(\eta - 1)$ *and the underlying process* Q_t *evolves according to the dynamics characterized by the stochastic differential equation*

$$dQ_t = \chi Q_t dt + \beta Q_t dZ_t - \sigma Q_t dW_t, \quad Q_0 = y/x, \tag{2.7}$$

where $\chi = \alpha - \mu + (\sigma^2 - \sigma\beta\rho)(1 - \eta)$.

Proof. See Appendix A. □

In accordance with the valuation of European contingent contracts with linearly homogeneous exercise payoffs, Lemma 2.1 demonstrates that the original two-dimensional optimal stopping problem can be transformed into a standard one-dimensional optimal stopping problem by invoking Girsanovs theorem and the fact that the ratio between two geometric Brownian motions is itself a geometric Brownian motion. An important implication of Lemma 2.1 is that if $L : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ is twice continuously differentiable, homogenous of degree one, and r-excessive for the underlying diffusion (X_t, Y_t) then $L(P_t, 1)$ is $(r - \delta(\eta))$-excessive for the diffusion P_t and $L(1, Q_t)$ is $(r - \gamma(\eta))$-excessive for the diffusion Q_t.

Given the observations of Lemma 2.1, we now analyze the determination of the value of the functionals

$$\begin{aligned} G_b(x, y) &= y^\eta \mathbb{E}_{x/y}\left[e^{-(r-\delta(\eta))\tau_b} F(P_{\tau_b}, 1)\right] \\ &= y^\eta F(b, 1)\mathbb{E}_{x/y}\left[e^{-(r-\delta(\eta))\tau_b}\right], \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} H_a(x, y) &= x^\eta \mathbb{E}_{y/x}\left[e^{-(r-\gamma(\eta))\hat{\tau}_a} F(1, Q_{\hat{\tau}_a})\right] \\ &= x^\eta F(1, a)\mathbb{E}_{y/x}\left[e^{-(r-\gamma(\eta))\hat{\tau}_a}\right], \end{aligned} \tag{2.9}$$

where $\tau_b = \inf\{t \geq 0 : P_t = b\}$ denotes the first hitting time of the diffusion P_t to the state $b \in \mathbb{R}_+$ and $\hat{\tau}_a = \inf\{t \geq 0 : Q_t = a\}$ denotes the first hitting time of the diffusion Q_t to the state $a \in \mathbb{R}_+$. Before presenting our general characterization of the functionals, we assume that $r > \max(\delta(\eta), \gamma(\eta))$ and define the constants

$$\psi = \frac{1}{2} - \frac{\xi}{\theta^2} + \sqrt{\left(\frac{1}{2} - \frac{\xi}{\theta^2}\right)^2 + \frac{2(r - \delta(\eta))}{\theta^2}},$$

$$\varphi = \frac{1}{2} - \frac{\xi}{\theta^2} - \sqrt{\left(\frac{1}{2} - \frac{\xi}{\theta^2}\right)^2 + \frac{2(r - \delta(\eta))}{\theta^2}},$$

$\hat{\varphi} = \eta - \psi$, and $\hat{\psi} = \eta - \varphi$, where $\xi = \mu - \alpha + (\beta^2 - \sigma\beta\rho)(1 - \eta)$ and $\theta^2 = \sigma^2 + \beta^2 - 2\sigma\beta\rho$. As intuitively is clear, ψ denotes the positive and φ denotes the negative root of the quadratic characteristic equation $\frac{1}{2}\theta^2 a(a-1) + \xi a - (r - \delta(\eta)) = 0$ and, in turn, $\hat{\psi}$ denotes the positive and $\hat{\varphi}$ denotes the negative root of the quadratic characteristic equation $\frac{1}{2}\theta^2 a(a-1) + \chi a - (r - \gamma(\eta)) = 0$, where $\chi = \alpha - \mu + (\sigma^2 - \sigma\beta\rho)(1 - \eta)$. We can now establish the following.

Lemma 2.2. *Assume that* $r > \max(\delta(\eta), \gamma(\eta))$. *Then,*

$$G_b(x, y) = H_{1/b}(x, y) = y^{\eta - \psi} x^{\psi} b^{-\psi} F(b, 1)$$

whenever $x < yb$ *and*

$$G_b(x, y) = H_{1/b}(x, y) = y^{\eta - \varphi} x^{\varphi} b^{-\varphi} F(b, 1)$$

whenever $x > yb$.

Proof. See Appendix B. □

Our main objective is to derive certainty equivalent characterizations for the stochastic valuation problem (2.3) in terms of associated deterministic valuations. To this end, define the deterministic processes $\bar{X}_t$ and $\bar{Y}_t$ by the ordinary differential equations

$$\bar{X}_t' = \tilde{\mu}\bar{X}_t, \quad \bar{X}_0 = x$$

and

$$\bar{Y}_t' = \tilde{\alpha}\bar{Y}_t, \quad \bar{Y}_0 = y$$

where $\tilde{\mu}, \tilde{\alpha}$ are exogenously given constants. Given this characterization, consider now the associated deterministic valuation

$$\hat{V}(x, y) = \sup_{t \geq 0} e^{-\tilde{r}t} F(\bar{X}_t, \bar{Y}_t), \tag{2.10}$$

where $\tilde{r} \in \mathbb{R}_+$ is an exogenously given discount rate. It is now a simple exercise in ordinary calculus to demonstrate that

$$\hat{V}(x,y) = y^\eta \sup_{t\geq 0} e^{-(\tilde{r}-\tilde{\alpha}\eta)t} F(\bar{P}_t, 1) = x^\eta \sup_{t\geq 0} e^{-(\tilde{r}-\tilde{\mu}\eta)t} F(1, \bar{Q}_t),$$

where $\bar{P}_t = (x/y)e^{(\tilde{\mu}-\tilde{\alpha})t}$ satisfies the ordinary differential equation $\bar{P}_t' = (\tilde{\mu} - \tilde{\alpha})\bar{P}_t$ and $\bar{Q}_t = (y/x)e^{(\tilde{\alpha}-\tilde{\mu})t}$ satisfies the ordinary differential equation $\bar{Q}_t' = (\tilde{\alpha} - \tilde{\mu})\bar{Q}_t$. Given this observation, we now plan to analyze the determination of the value of the functionals

$$\hat{G}_b(x,y) = y^\eta e^{-(\tilde{r}-\tilde{\alpha}\eta)T_b} F(\bar{P}_{T_b}, 1), \tag{2.11}$$

$$\hat{H}_a(x,y) = x^\eta e^{-(\tilde{r}-\tilde{\mu}\eta)\hat{T}_a} F(1, \bar{Q}_{\hat{T}_a}), \tag{2.12}$$

where $T_b = \inf\{t \geq 0 : \bar{P}_t = b\}$ denotes the first hitting time of the process $\bar{P}_t$ to the state $b \in \mathbb{R}_+$ and $\hat{T}_a = \inf\{t \geq 0 : \bar{Q}_t = a\}$ denotes the first hitting time of the process $\bar{Q}_t$ to the state $a \in \mathbb{R}_+$. We can now establish the following auxiliary lemma.

Lemma 2.3. *(A) Assume that $\tilde{\mu} > \tilde{\alpha}$, that $\tilde{r} > \eta \max(\tilde{\alpha}, \tilde{\mu})$, and that $x < yb$. Then,*

$$\hat{G}_b(x,y) = \hat{H}_{1/b}(x,y) = y^{\eta-\zeta} x^\zeta b^{-\zeta} F(b, 1),$$

where $\zeta = (\tilde{r} - \tilde{\alpha}\eta)/(\tilde{\mu} - \tilde{\alpha})$.
(B) Assume that $\tilde{\alpha} > \tilde{\mu}$, that $\tilde{r} > \eta \max(\tilde{\alpha}, \tilde{\mu})$, and that $x > yb$. Then,

$$\hat{G}_b(x,y) = \hat{H}_{1/b}(x,y) = y^{\eta-\zeta} x^\zeta b^{-\zeta} F(b, 1),$$

where $\zeta = (\tilde{r} - \tilde{\alpha}\eta)/(\tilde{\mu} - \tilde{\alpha})$.

Proof. See Appendix C. □

Along the lines of our previous findings on the functionals (2.8) and (2.9), Lemma 2.3 presents an explicit characterization of the functionals (2.11) and (2.12). A set of conditions under which the considered functionals coincide and certainty equivalence holds is now summarized in the following.

Corollary 2.4. *(A) Assume that $r > \max(\delta(\eta), \gamma(\eta))$, that $\tilde{r} > \eta \max(\tilde{\alpha}, \tilde{\mu})$, that $\tilde{\mu} > \tilde{\alpha}$, that $(\tilde{r} - \eta\tilde{\alpha}) = \psi(\tilde{\mu} - \tilde{\alpha})$, and that $x < yb$. Then, $\hat{G}_b(x,y) = G_b(x,y) = H_{1/b}(x,y) = \hat{H}_{1/b}(x,y)$.*
(B) Assume that $r > \max(\delta(\eta), \gamma(\eta))$, that $\tilde{r} > \eta \max(\tilde{\alpha}, \tilde{\mu})$, that $\tilde{\alpha} > \tilde{\mu}$, that $(\tilde{r} - \eta\tilde{\alpha}) = \varphi(\tilde{\mu} - \tilde{\alpha})$, and that $x > yb$. Then, $\hat{G}_b(x,y) = G_b(x,y) = H_{1/b}(x,y) = \hat{H}_{1/b}(x,y)$.

Proof. The alleged result is a direct implication of Lemma 2.2 and Lemma 2.3. □

Corollary 2.4 states a set of conditions under which the stochastic valuations (2.8) can be expressed in terms of an associated deterministic valuation (2.11). As we will later observe, this result plays a key role in the derivation and subsequent analysis of the certainty equivalent formulation of the optimal stopping problem (2.3). Before stating our main conclusions on that subject, we first state a set of sufficient conditions under which the considered valuation can be explicitly solved.

Theorem 2.5. *(A) Assume that* $r > \max(\delta(\eta), \gamma(\eta))$*, that the mapping* $F(p, 1)$ *is non-decreasing as a function of* p*, that* $p^{-\psi}F(p, 1)$ *attains a unique global maximum at* $b^* = \operatorname{argmax}\{p^{-\psi}F(p, 1)\}$ *and that* $L_\psi(p) = p^{1-\varphi}F_x(p, 1) - \psi p^{-\varphi}F(p, 1)$ *is non-increasing on* (b^*, ∞)*. Then*

$$V(x, y) = y^{\eta-\psi}x^{\psi} \sup_{py\geq x} \left[p^{-\psi}F(p, 1)\right] = y^{\hat{\varphi}}x^{\eta-\hat{\varphi}} \sup_{y\geq qx} \left[q^{-\hat{\varphi}}F(1, q)\right] \quad (2.13)$$

which can be re-expressed as

$$V(x, y) = \begin{cases} F(x, y) & x \geq b^*y \\ G_{b^*}(x, y) & x < b^*y \end{cases} = \begin{cases} F(x, y) & x \geq b^*y \\ H_{1/b^*}(x, y) & x < b^*y. \end{cases}$$

(B) Assume that $r > \max(\delta(\eta), \gamma(\eta))$*, that the mapping* $F(p, 1)$ *is non-increasing as a function of* p*, that* $p^{-\varphi}F(p, 1)$ *attains a unique global maximum at* $b^* = \operatorname{argmax}\{p^{-\varphi}F(p, 1)\}$ *and that* $L_\varphi(p) = p^{1-\psi}F_x(p, 1) - \varphi p^{-\psi}F(p, 1)$ *is non-increasing on* $(0, \tilde{b})$*. Then*

$$V(x, y) = y^{\eta-\varphi}x^{\varphi} \sup_{py\leq x} \left[p^{-\varphi}F(p, 1)\right] = y^{\hat{\psi}}x^{\eta-\hat{\psi}} \sup_{y\leq qx} \left[q^{-\hat{\psi}}F(1, q)\right] \quad (2.14)$$

which can be re-expressed as

$$V(x, y) = \begin{cases} G_{b^*}(x, y) & x > b^*y \\ F(x, y) & x \leq b^*y \end{cases} = \begin{cases} H_{1/b^*}(x, y) & x > b^*y \\ F(x, y) & x \leq b^*y. \end{cases}$$

Proof. See Appendix D. □

Theorem 2.5 states a set of general conditions under which an optimal exercise policy exists and is unique. Along the lines of the seminal study by McDonald and Siegel (1986), Theorem 2.5 demonstrates that the optimal stopping policy can be characterized in terms of first exit times of the associated one-dimensional diffusions P_t and Q_t from open intervals. It is worth noticing that the characterization of the optimal exercise boundary can be interpreted in terms of the general properties of homogenous mappings and Euler's theorem for homogenous functions. More precisely, since $F(x,y) = y^\eta F(x/y,1) = x^\eta F(1,y/x)$ and $\eta F(x,y) = F_x(x,y)x + F_y(x,y)y$ the optimality condition $F_x(p^*,1)p^* = \psi F(p^*,1)$ characterizing the optimal boundary in case (A) of Theorem 2.5 can be alternatively expressed as $F_y(1,q^*)q^* = (\eta-\psi)F(1,q^*) = \hat{\varphi}F(1,q^*)$. Similarly, the optimality condition $F_x(p^*,1)p^* = \varphi F(p^*,1)$ characterizing the optimal boundary in case (B) of Theorem 2.5 can be alternatively expressed as $F_y(1,q^*)q^* = (\eta-\varphi)F(1,q^*) = \hat{\psi}F(1,q^*)$.

A set of important implications of Theorem 2.5 characterizing the impact of increased volatility on the optimal exercise thresholds is now summarized in the following.

Corollary 2.6. *The mappings $K_\theta(p) = pF_x(p,1)/\psi - F(p,1)$ and $\hat{K}_\theta(p) = pF_x(p,1)/\varphi - F(p,1)$ satisfy the condition*

$$\begin{aligned}\frac{\partial K_\theta}{\partial \theta}(p) &= \frac{pF_x(p,1)}{\psi}\frac{2(\psi-1)}{\theta(\psi-\varphi)}\\ \frac{\partial \hat{K}_\theta}{\partial \theta}(p) &= \frac{pF_x(p,1)}{\varphi}\frac{2(1-\varphi)}{\theta(\psi-\varphi)}.\end{aligned}$$

Hence, if the conditions of part (A) of Theorem 2.5 are satisfied and

$$r > (\eta-1)(\alpha + \frac{1}{2}\beta^2\eta - \beta^2 + \sigma\beta\rho) + \mu \tag{2.15}$$

then increased total volatility θ postpones rational exercise by increasing the value of $K_\theta(p)$ and, therefore, by increasing the exercise threshold b^*. If the conditions of part (B) of Theorem 2.5 are satisfied then increased total volatility θ postpones rational exercise by increasing the value of $\hat{K}_\theta(p)$ and, therefore, by decreasing the exercise threshold b^*.

Proof. Since $\theta^2(\psi-1)(1-\varphi) = 2(r-\delta(\eta)-\xi)$ the alleged result is a straightforward implication of Theorem 2.5 and the inequalities

$$\frac{\partial \psi}{\partial \theta} = \frac{2\psi(1-\psi)}{\theta(\psi-\varphi)} < 0$$

and

$$\frac{\partial\varphi}{\partial\theta} = \frac{2\varphi(\varphi-1)}{\theta(\psi-\varphi)} > 0.$$

□

Corollary 2.6 states that under the conditions of part (A) of Theorem 2.5 increased total volatility will decelerate rational investment by expanding the continuation region where exercising the investment opportunity is suboptimal whenever the condition (2.15) is satisfied (and, therefore, under which $\psi > 1$). Interestingly, Corollary 2.6 states that under the conditions of part (B) of Theorem 2.5 increased total volatility will always decelerate rational investment independently of the relative size of the negative root φ. It is, however, worth noticing that this observation does not necessarily imply that an increase in the volatility coefficients σ or β would have a similar effect since the impact of an increase in these factors naturally depends on the correlation structure of the driving Brownian motions. More precisely, since $\partial\theta/\partial\sigma = (\sigma - \beta\rho)/\theta$, $\partial\theta/\partial\beta = (\beta - \sigma\rho)/\theta$, and $\partial\theta/\partial\rho = -\sigma\beta/\theta$ we find that the impact of increased volatility coefficients σ and β is ambiguous while increased correlation will unambiguously accelerate investment. Moreover, it is also worth pointing out that the results of Corollary 2.6 clearly indicate that the sign of the relationship between increased volatility and the optimal investment policy is typically a process-specific, and not a payoff-specific, property (Alvarez, 2003).

Our main result on the certainty equivalent formulation of the considered class of infinitely-lived valuations is now summarized in the following.

Theorem 2.7. *(A) Assume that $\tilde{\mu} > \tilde{\alpha}$, that $\tilde{r} > \eta \max(\tilde{\alpha}, \tilde{\mu})$, and that $(\tilde{r} - \eta\tilde{\alpha}) = \psi(\tilde{\mu} - \tilde{\alpha})$. Assume also that the conditions of part (A) of Theorem 2.5 are satisfied. Then $V(x, y) = \hat{V}(x, y)$.*
(B) Assume that $\tilde{\alpha} > \tilde{\mu}$, that $\tilde{r} > \eta \max(\tilde{\alpha}, \tilde{\mu})$, and that $(\tilde{r} - \eta\tilde{\alpha}) = \varphi(\tilde{\mu} - \tilde{\alpha})$. Assume also that the conditions of part (B) of Theorem 2.5 are satisfied. Then $V(x, y) = \hat{V}(x, y)$.

Proof. The alleged result is a direct implication of Theorem 2.5 and Corollary 2.4. □

Theorem 2.7 states a certainty equivalent formulation of the optimal stopping problem (2.3) in terms of the associated deterministic valuation (2.10). One of the most important implications of Theorem 2.7 is that if $\tilde{\mu} = \mu$ and

$\tilde{\alpha} = \alpha$ then certainty equivalence can be attained by adjusting the discount rate for risk according to the characterization

$$\tilde{r} = \eta\alpha + \psi(\mu - \alpha)$$

in case (A) and according to the characterization

$$\tilde{r} = \eta\alpha + \varphi(\mu - \alpha)$$

in case (B). Moreover, as was already indicated by our Corollary 2.6, we find that the impact of an increase in σ, β, or the correlation coefficient ρ on the risk adjusted discount rate is ambiguous in case (A) while an increase in the correlation coefficient ρ unambiguously increases the risk adjusted discount rate in case (B). More precisely,

$$\frac{\partial \tilde{r}}{\partial \sigma} = \frac{2\psi(1-\psi)(\mu-\alpha)}{\theta^2(\psi-\varphi)}(\sigma - \beta\rho), \qquad \frac{\partial \tilde{r}}{\partial \beta} = \frac{2\psi(1-\psi)(\mu-\alpha)}{\theta^2(\psi-\varphi)}(\beta - \sigma\rho)$$

in case (A) and

$$\frac{\partial \tilde{r}}{\partial \sigma} = \frac{2\varphi(\varphi-1)(\mu-\alpha)}{\theta^2(\psi-\varphi)}(\sigma - \beta\rho), \qquad \frac{\partial \tilde{r}}{\partial \beta} = \frac{2\varphi(\varphi-1)(\mu-\alpha)}{\theta^2(\psi-\varphi)}(\beta - \sigma\rho)$$

in case (B). The risk adjusted discount rate is explicitly illustrated as a function of the volatility coefficient σ for various values of the correlation coefficient in Figure 1 under the assumptions that $\eta = 1$, $r = 0.045, \alpha = 0.02, \mu = 0.025$, and $\beta = 0.05$. In accordance with our findings, we find that when $\rho = 50\%$ the risk-adjusted discount rate attains a global maximum at the point $\sigma = \rho\beta = 0.025$.

Alternatively, if the discount rate is not adjusted to the volatility of the underlying processes, then according to our Theorem 2.7 certainty equivalence can be attained only by adjusting either one (or both) of the growth rates of the underlying diffusions. Consequently, the sensitivity of the required adjustments satisfy the comparative statics

$$(\psi - \eta)\frac{\partial \tilde{\alpha}}{\partial \theta} - \psi\frac{\partial \tilde{\mu}}{\partial \theta} = (\tilde{\mu} - \tilde{\alpha})\frac{\partial \psi}{\partial \theta}$$

in case (A) of Theorem 2.7 and

$$(\varphi - \eta)\frac{\partial \tilde{\alpha}}{\partial \theta} - \varphi\frac{\partial \tilde{\mu}}{\partial \theta} = (\tilde{\mu} - \tilde{\alpha})\frac{\partial \varphi}{\partial \theta}$$

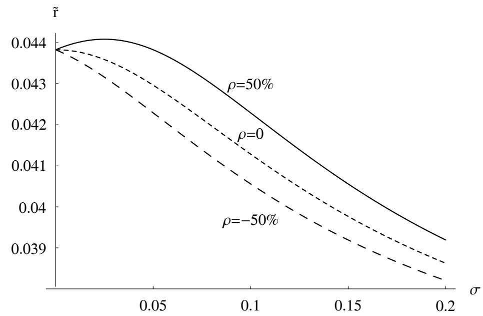

Figure 1: The risk-adjusted discount rate as a function of σ

in case (B) of Theorem 2.7. The risk-adjusted growth rate $\tilde{\mu}$ is illustrated as a function of the volatility coefficient σ for various values of the correlation coefficient in Figure 2 under the assumptions that $\eta = 1$, $r = 0.045, \alpha = \tilde{\alpha} = 0.02, \mu = 0.025$, and $\beta = 0.05$.

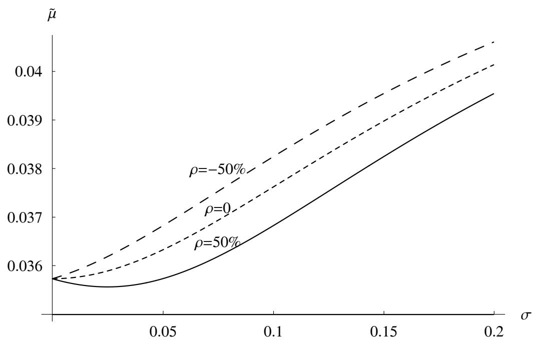

Figure 2: The risk-adjusted growth rate as a function of σ

3 EXPLICIT ILLUSTRATIONS

3.1 THE EXCHANGE OPTION

In order to illustrate our general results assume now that $F(x,y) = x - y$. It is now a well known result that in this case if the condition $r > \max(\mu, \alpha)$ is met, then (McDonald and Siegel, 1986; Olsen and Stensland, 1992; Hu and Øksendal, 1998)

$$V(x,y) = y^{1-\psi}x^{\psi} \sup_{py\geq x}\left[p^{-\psi}(p-1)\right] = \begin{cases} x - y & x \geq b^* y \\ y^{1-\psi}x^{\psi}b^{*-\psi}(b^*-1) & x < b^* y, \end{cases}$$

where

$$b^* = \frac{\psi}{\psi - 1} > 1$$

denotes the optimal boundary at which the process X_t/Y_t should be optimally stopped. Similarly, if $\tilde{r} > \tilde{\mu} > \tilde{\alpha}$ then

$$\hat{V}(x,y) = y^{1-\zeta}x^{\zeta} \sup_{py\geq x}\left[p^{-\zeta}(p-1)\right] = \begin{cases} x - y & x \geq \tilde{b} y \\ y^{1-\zeta}x^{\zeta}\tilde{b}^{-\zeta}(\tilde{b}-1) & x < \tilde{b} y, \end{cases}$$

where

$$\tilde{b} = \frac{\zeta}{\zeta - 1} = \frac{\tilde{r} - \tilde{\alpha}}{\tilde{r} - \tilde{\mu}} > 1$$

denotes the optimal boundary at which the deterministic process $\bar{X}_t/\bar{Y}_t$ should be optimally stopped. Hence, as was indicated by our main theorem, we find that in the present example $V(x,y) = \hat{V}(x,y)$ as long as the conditions $\tilde{r} > \tilde{\mu} > \tilde{\alpha}$, $r > \max(\mu, \alpha)$, and $\tilde{r} - \tilde{\alpha} = \psi(\tilde{\mu} - \tilde{\alpha})$ are satisfied.

3.2 RISK AVERSION AND INVESTMENT

In this subsection we reconsider the problem of the previous section by assuming that the decision maker is risk averse. In order to describe risk aversion, we assume that the utility function of the decision maker is of the standard *HARA*-form $U(x) = \frac{1}{\kappa}x^{\kappa}$, where $\kappa \in (0,1)$ is a known exogenously given constant and $1-\kappa$ measures the relative risk aversion of the decision maker. Given these assumption, we now plan to investigate the problem (2.3) in the case where $F(x,y) = \frac{1}{\kappa}\max(x-y,0)^{\kappa}$. It is now clear that the exercise payoff is now homogenous of degree κ and, therefore, that our general findings apply. Especially, we observe that

$$F(x,y) = y^{\kappa}\frac{1}{\kappa}\max(x/y-1,0)^{\kappa} = x^{\kappa}\frac{1}{\kappa}\max(1-y/x,0)^{\kappa}.$$

Now, assume that $r > \max(\delta(\kappa), \eta(\kappa))$. Then

$$\begin{aligned} V(x,y) &= \frac{y^{\kappa-\psi}x^{\psi}}{\kappa}\sup_{py\geq x}\left[p^{-\psi}\max(p-1,0)^{\kappa}\right] \\ &= \begin{cases} \frac{1}{\kappa}(x-y)^{\kappa} & x \geq b^* y \\ \frac{y^{\kappa-\psi}x^{\psi}}{\kappa}b^{*-\psi}(b^*-1)^{\kappa} & x < b^* y, \end{cases} \end{aligned}$$

where

$$b^* = \frac{\psi}{\psi-\kappa} > 1$$

denotes the optimal threshold for the process X_t/Y_t at which the value is optimal. It is worth noticing that

$$\frac{\partial b^*}{\partial \theta} = \frac{2\kappa\psi(\psi-1)}{\theta(\psi-\varphi)(\psi-\kappa)^2} \gtreqqless 0, \quad \psi \gtreqqless 1$$

which demonstrates that the impact of increased volatility on the optimal exercise threshold is negative on the set $\kappa < \psi < 1$ and, therefore, that increased total volatility needs not to decelerate investment under uncertainty and risk aversion. Similarly, we observe that

$$\frac{\partial b^*}{\partial \kappa} = \frac{\psi}{(\psi-\kappa)^2} > 0$$

proving that increased relative risk aversion accelerates rational investment by decreasing the optimal exercise threshold at which the investment opportunity should be exercised (see Alvarez and Koskela (2004) for a similar result in the one-dimensional setting).

When $\tilde{r} > \kappa\tilde{\mu} > \kappa\tilde{\alpha}$, we observe that

$$\begin{aligned} \hat{V}(x,y) &= \frac{y^{\kappa-\zeta}x^{\zeta}}{\kappa}\sup_{py\geq x}\left[p^{-\zeta}\max(p-1,0)^{\kappa}\right] \\ &= \begin{cases} \frac{1}{\kappa}(x-y)^{\kappa} & x \geq \tilde{b} y \\ \frac{y^{\kappa-\zeta}x^{\zeta}}{\kappa}\tilde{b}^{-\zeta}(\tilde{b}-1)^{\kappa} & x < \tilde{b} y, \end{cases} \end{aligned}$$

where

$$\tilde{b} = \frac{\zeta}{\zeta - \kappa} = \frac{\tilde{r} - \kappa\tilde{\alpha}}{\tilde{r} - \kappa\tilde{\mu}} > 1$$

denotes the optimal threshold for the process $\bar{X}_t/\bar{Y}_t$ at which the value is optimal. Thus, as was indicated by our main theorem, $V(x, y) = \hat{V}(x, y)$ as long as the conditions $\tilde{r} > \kappa\tilde{\mu} > \kappa\tilde{\alpha}$, $r > \max(\delta(\kappa), \eta(\kappa))$, and $\tilde{r} - \kappa\tilde{\alpha} = \psi(\tilde{\mu} - \tilde{\alpha})$ are satisfied.

3.3 AN EXOTIC OPTION

In order to illustrate our results in a more general case, consider now a case where the exercise payoff of the investment opportunity reads as

$$F(x, y) = px^{\kappa}y^{1-\kappa} - qx,$$

where $\kappa \in (0, 1)$, $p, q \in \mathbb{R}_+$ are exogenously given known constants. It is clear that $F(x, y)$ is twice continuously differentiable and homogenous of degree one. Moreover

$$F(x, y) = x\left(p(y/x)^{1-\kappa} - q\right).$$

Thus, whenever $r > \max(\mu, \alpha)$, we observe that

$$\begin{aligned} V(x, y) &= x^{1-\hat{\psi}}y^{\hat{\psi}} \sup_{sx \geq y} \left[s^{-\hat{\psi}}(ps^{1-\kappa} - q)\right] \\ &= \begin{cases} x\left(p(y/x)^{1-\kappa} - q\right) & y > s^*x \\ x^{1-\hat{\psi}}y^{\hat{\psi}}s^{*-\hat{\psi}}(ps^{*1-\kappa} - q) & y \leq s^*x, \end{cases} \end{aligned} \tag{3.1}$$

where

$$s^* = \left(\frac{q\hat{\psi}}{p(\hat{\psi} + \kappa - 1)}\right)^{\frac{1}{1-\kappa}} > \left(\frac{q}{p}\right)^{\frac{1}{1-\kappa}}$$

denotes the optimal stopping threshold for the process Y_t/X_t at which the value is optimal. On the other hand, we can write as well that

$$\begin{aligned} V(x, y) &= y^{1-\varphi}x^{\varphi} \sup_{uy \leq x} \left[u^{-\varphi}(pu^{\kappa} - qu)\right] \\ &= \begin{cases} y^{1-\varphi}x^{\varphi}u^{*-\varphi}(pu^{*\kappa} - qu^*) & y \leq (1/u^*)x \\ x\left(p(y/x)^{1-\kappa} - q\right) & y > (1/u^*)x \end{cases} \end{aligned} \tag{3.2}$$

where

$$u^* = \left(\frac{q(\varphi - 1)}{p(\varphi - \kappa)}\right)^{\frac{1}{\kappa - 1}}$$

denotes the optimal stopping threshold for the process X_t/Y_t at which the value is optimal. Note that the condition $\hat{\psi} = 1 - \varphi$ implies immediately that $s^* = 1/u^*$ and that the expressions (3.1) and (3.2) are the same. Thus, as was indicated by our main theorem, it makes no difference whether one solves the problem with respect to the quotient process X_t/Y_t or Y_t/X_t, the value of the problem is the same in both cases.

Moreover, assume that $r > \tilde{\mu} > \tilde{\alpha}$. Then

$$\begin{aligned}\hat{V}(x, y) &= x^{1-\zeta} y^{\zeta} \sup_{sx \geq y} \left[s^{-\zeta}(p s^{1-\kappa} - q) \right] \\ &= \begin{cases} x \left(p(y/x)^{1-\kappa} - q\right) & y > \tilde{s}x \\ x^{1-\zeta} y^{\zeta} \tilde{s}^{-\zeta}(p\tilde{s}^{1-\kappa} - q) & y \leq \tilde{s}x, \end{cases}\end{aligned}$$

where

$$\tilde{s} = \left(\frac{q\zeta}{p(\zeta + \kappa - 1)}\right)^{\frac{1}{1-\kappa}} = \left(\frac{q(r - \tilde{\alpha})}{p(r - (k\tilde{\alpha} + \tilde{\mu}(1 - \kappa)))}\right)^{\frac{1}{1-\kappa}} > \left(\frac{q}{p}\right)^{\frac{1}{1-\kappa}}$$

denotes the optimal stopping threshold for the process $\bar{Y}_t/\bar{X}_t$ at which the value is optimal. Hence we discover that in the present example $V(x, y) = \hat{V}(x, y)$ as long as the conditions $r > \tilde{\mu} > \tilde{\alpha}$ and $r - \tilde{\alpha} = \psi(\tilde{\mu} - \tilde{\alpha})$ are satisfied. This discovery is consistent with our main theorem.

4 CONCLUSIONS

In this paper we considered the certainty equivalent characterization of a class of valuations arising typically in studies considering irreversible investment in the presence of both cost and revenue uncertainty. Assuming that the underlying diffusions evolve according to a pair of potentially correlated geometric Brownian motions and that the exercise payoff is homogenous of degree η was shown to result under a set of conditions to a solvable one-dimensional valuation which can be characterized in terms of the minimal excessive mappings for the associated one-dimensional quotient processes. Given this characterization, we presented a certainty equivalent formulation of the considered functionals and studied their comparative static properties and, especially, their sensitivity with respect to changes in the volatilities of the underlying processes.

Although assuming that the underlying processes evolve according to a pair of potentially correlated geometric is not restrictive (at least from the point of view of financial applications) the assumed homogeneity of the exercise payoff rules out valuations subject to constant sunk costs and other similar factors resulting into non-homogeneous payoff structures. Thus, a natural extension of our analysis would be to assume a more general functional formulation of the exercise payoff and consider certainty equivalence within such a framework. Unfortunately, such analysis is out of the scope of this study and is, therefore, left for future analysis.

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A PROOF OF LEMMA 2.1

Proof. The assumed homogeneity of the exercise payoff $F(x,y)$ implies that $F(x,y) = y^\eta F(x/y,1)$ and, therefore, that

$$V(x,y) = \sup_\tau \mathbb{E}^{\mathbb{P}}\left[e^{-r\tau} Y_\tau^\eta F(X_\tau/Y_\tau,1)\right].$$

On the other hand, since $Y_t^\eta = ye^{\delta(\eta)t}M_t$, where $\delta(\eta) = \eta\alpha + \frac{1}{2}\beta^2\eta(\eta-1)$ and $M_t = e^{\beta\eta Z_t - \beta^2\eta^2 t/2}$ is a positive exponential martingale, we can now define the equivalent measure $\mathbb{Q}$ by the likelihood-ratio $d\mathbb{Q}/d\mathbb{P} = M_t$. This implies that

$$V(x,y) = y^\eta \sup_\tau \mathbb{E}^{\mathbb{Q}}\left[e^{-(r-\delta(\eta))\tau} F(X_\tau/Y_\tau,1)\right],$$

where the processes X_t and Y_t are characterized under the measure $\mathbb{Q}$ by the stochastic differential equations

$$dX_t = (\mu + \sigma\beta\eta\rho)X_t dt + \sigma X_t d\tilde{W}_t, \quad X_0 = x \in \mathbb{R}_+ \tag{A.1}$$
$$dY_t = (\alpha + \beta^2\eta)Y_t dt + \beta Y_t d\tilde{Z}_t, \quad Y_0 = y \in \mathbb{R}_+, \tag{A.2}$$

where $\tilde{W}_t$ and $\tilde{Z}_t$ are Brownian motions defined under the equivalent measure $\mathbb{Q}$. The strong uniqueness of the solutions of the stochastic differential equations (A.1) and (A.2) now, in turn, imply that

$$V(x,y) = y^\eta \sup_\tau \mathbb{E}^{\mathbb{P}}\left[e^{-(r-\delta(\eta))\tau} F(\tilde{X}_\tau/\tilde{Y}_\tau,1)\right],$$

where

$$d\tilde{X}_t = (\mu + \sigma\beta\eta\rho)\tilde{X}_t dt + \sigma \tilde{X}_t dW_t, \quad \tilde{X}_0 = x \in \mathbb{R}_+ \tag{A.3}$$
$$d\tilde{Y}_t = (\alpha + \beta^2\eta)\tilde{Y}_t dt + \beta \tilde{Y}_t dZ_t, \quad \tilde{Y}_0 = y \in \mathbb{R}_+. \tag{A.4}$$

Finally, since the process $P_t = \tilde{X}_t/\tilde{Y}_t$ satisfies the stochastic differential equation

$$dP_t = \xi P_t dt + \sigma P_t dW_t - \beta P_t dZ_t, \quad P_0 = x/y,$$

where $\xi = \mu - \alpha + (\beta^2 - \sigma\beta\rho)(1-\eta)$, we find that the optimal stopping problem can be re-expressed as

$$V(x,y) = y^\eta \sup_\tau \mathbb{E}_{x/y}\left[e^{-(r-\delta(\eta))\tau} F(P_\tau,1)\right].$$

Proving the alternative representation (2.6) of the considered stochastic valuation (2.3) is entirely analogous. □

B PROOF OF LEMMA 2.2

Proof. It is well-known from the literature on linear diffusions that the functional

$$u_b(p) = \mathbb{E}_p\left[e^{-(r-\delta(\eta))\tau_b}\right]$$

where $\tau_b = \inf\{t \geq 0 : P_t = b\}$ denotes the first hitting time of the diffusion P_t to the state $b \in \mathbb{R}_+$, can be expressed as (Borodin and Salminen, 2002, 132)

$$u_b(p) = \begin{cases} (p/b)^{\psi} & p \leq b \\ (p/b)^{\varphi} & p \geq b, \end{cases}$$

where ψ denotes the positive and φ the negative root of the quadratic characteristic equation of the ordinary second order (Euler's) differential equation

$$\frac{1}{2}\theta^2 p^2 u_b''(p) + \xi p u_b'(p) - (r - \delta(\eta))u_b(p) = 0.$$

On the other hand, the functional

$$\hat{u}_a(q) = \mathbb{E}_q\left[e^{-(r-\gamma(\eta))\hat{\tau}_a}\right]$$

where $\hat{\tau}_a = \inf\{t \geq 0 : Q_t = a\}$ denotes the first hitting time of the diffusion Q_t to the state $a \in \mathbb{R}_+$, can be expressed as (cf. Borodin and Salminen, 2002, 132)

$$\hat{u}_a(q) = \begin{cases} (q/a)^{\hat{\psi}} & q \leq a \\ (q/a)^{\hat{\varphi}} & q \geq a, \end{cases}$$

where $\hat{\psi}$ denotes the positive and $\hat{\varphi}$ the negative root of the quadratic characteristic equation of the ordinary second order (Euler's) differential equation

$$\frac{1}{2}\theta^2 q^2 \hat{u}_a''(q) + \chi q \hat{u}_a'(q) - (r - \gamma(\eta))\hat{u}_a(q) = 0.$$

Since $\hat{\varphi} = \eta - \psi$ and $\hat{\psi} = \eta - \varphi$, we notice that the homogeneity of the exercise payoff implies that $G_b(x, y) = y^{\eta}F(b, 1)u_b(x/y) = y^{\eta}b^{\eta}F(1, 1/b)u_b(x/y) = x^{\eta}F(1, 1/b)\hat{u}_{1/b}(y/x) = H_{1/b}(x, y)$. This completes the proof of the lemma. □

C PROOF OF LEMMA 2.3

Proof. (A) Since $\bar{P}_t = pe^{(\tilde{\mu}-\tilde{\alpha})t}$ we find that if $p < b$ then the condition $\tilde{\mu} > \tilde{\alpha}$ implies that

$$T_b = \frac{1}{(\tilde{\mu} - \tilde{\alpha})} \ln\left(\frac{b}{p}\right) < \infty.$$

Similarly, since $\bar{Q}_t = qe^{(\tilde{\alpha}-\tilde{\mu})t}$ we find that if $q > 1/b$ then the condition $\tilde{\mu} > \tilde{\alpha}$ implies that

$$\hat{T}_{1/b} = \frac{\ln(bq)}{(\tilde{\mu} - \tilde{\alpha})} < \infty.$$

Hence, we find that that if $p < b$ then

$$v_b(p) = e^{-(\tilde{r}-\tilde{\alpha}\eta)T_b} = \left(\frac{p}{b}\right)^{\frac{\tilde{r}-\tilde{\alpha}\eta}{\tilde{\mu}-\tilde{\alpha}}}.$$

and if $q > 1/b$ then

$$\hat{v}_{1/b}(q) = e^{-(\tilde{r}-\tilde{\mu}\eta)\hat{T}_{1/b}} = \left(\frac{1}{qb}\right)^{\frac{\tilde{r}-\tilde{\mu}\eta}{\tilde{\mu}-\tilde{\alpha}}}$$

Noticing that $y^\eta v_b(x/y) = y^\eta(x/(yb))^\zeta$ and $x^\eta \hat{v}_{1/b}(y/x) = x^\eta(x/(yb))^{\zeta-\eta}$, where $\zeta = (\tilde{r} - \tilde{\alpha}\eta)/(\tilde{\mu} - \tilde{\alpha})$, then proves the alleged result. Establishing part (B) is entirely analogous. □

D PROOF OF THEOREM 2.5

Proof. (A) Define the value function $J : \mathbb{R}_+ \mapsto \mathbb{R}_+$ as

$$J(p) = \sup_\tau \mathbb{E}_p\left[e^{-(r-\delta(\eta))\tau} F(P_\tau, 1)\right] \tag{D.1}$$

and denote the proposed value function as $J_{b^*}(p)$. Since

$$J_{b^*}(p) = \mathbb{E}_p\left[e^{-(r-\delta(\eta))\tau^*} F(P_{\tau^*}, 1)\right],$$

where $\tau^* = \inf\{t \geq 0 : P_t \geq b^*\}$ denotes the first exit time of the underlying diffusion P_t from the set $(0, b^*)$, we immediately find that $J(p) \geq J_{b^*}(p)$.

To prove the opposite inequality, we first observe that the proposed value function $J_{b^*}(p)$ is continuously differentiable on $\mathbb{R}_+$, twice continuously differentiable on $\mathbb{R}_+\backslash\{b^*\}$. Moreover, it satisfies the conditions $J''_{b^*}(b^*-) =$

$\psi(\psi - 1)b^{*-2}F(b^*, 1) < \infty, J''_{b^*}(b^*+) = F_{xx}(b^*+, 1) < \infty$. The maximality of $b^* = \operatorname{argmax}\{p^{-\psi}F(p, 1)\}$ implies that $J_{b^*}(p) \geq F(p, 1)$ for all $p \in \mathbb{R}_+$. Moreover, since $\theta^2 p^2 J''_{b^*}(p)/2 + (\mu - \alpha + (\beta^2 - \sigma\beta\rho)(1-\eta))pJ'_{b^*}(p) - (r - \delta(\eta))J_{b^*}(p) = 0$ on $(0, b^*)$, the assumed monotonicity of the mapping $L_\psi(p) = p^{1-\varphi}F_x(p, 1) - \psi p^{-\varphi}F(p, 1)$ (Salminen, 1985) on (b^*, ∞) implies that

$$L'_\psi(p) = \frac{2}{\theta^2}p^{-\varphi-1}\left[\frac{1}{2}\theta^2 p^2 F_{xx}(p, 1) + \xi p F_x(p, 1) - (r - \delta(\eta))F(p, 1)\right] \leq 0$$

for all $p \in (b^*, \infty)$. Hence $\frac{1}{2}\theta^2 p^2 J''_{b^*}(p) + \xi p J'_{b^*}(p) - (r - \delta(\eta))J_{b^*}(p) \leq 0$ for all $p \in \mathbb{R}_+ \backslash \{b^*\}$. Consequently, $J_{b^*}(p)$ constitutes a $(r - \delta(\eta))$-excessive majorant of the payoff $F(p, 1)$ for the process P_t. Since $J(p)$ is the least of these majorants, we find that $J_{b^*}(p) \geq J(p)$ and, therefore, that $J_{b^*}(p) = J(p)$. The rest of the alleged result follows from Lemma 2.1 and Lemma 2.2. Establishing part (B) is entirely analogous. □

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