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ON METHODS FOR CONSTRAINING $F(R)$ THEORIES OF GRAVITY

by

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Contents

Abstract

Einstein's theory of general relativity is without doubt one of greatest achievements in the history of Mankind. Even so, there are some ways, in which it leaves room for improvement. The last one hundred years and especially the last fifteen have seen many possibilities to remedy the small cracks in general relativity. Since the 1990s it has been known that the Universe is experiencing accelerating expansion. Explaining this with general relativity alone is not without problems. For this reason we need to find the viable alternatives to general relativity.

While general relativity is based on certain assumptions, the various alternatives discard one or more of these assumptions for greater generality. One path leads to $f(R)$ theories of gravity, which let the gravitational action be a function of the Ricci curvature scalar instead of the plain linear term in general relativity. Thus, there is an infinite number of possible $f(R)$ gravity models.

Many of these possible $f(R)$ models can be ruled out as unphysical from the start. However, it is possible to construct models, which seem to fit observations even better than the highly successful general relativity with the cosmological constant. Even for these models, there might still be lurking some dynamics or other characteristics, which render them unphysical.

Further constraining the class of viable $f(R)$ theories provides us with better understanding of gravity itself and the characteristics required of a new gravitational theory. As such it paves way for understanding the needs of a working quantum gravity theory.

In this thesis I develop methods to better constrain viable $f(R)$ models and apply these methods to select models. I use both theoretical tools to examine the mathematical background of $f(R)$ for instabilities and link results to observational data. Even a mathematically sound candidate for a physical theory must stand trial to observations.

The methods I develop in this thesis can be applied to a wide range of $f(R)$ models for tests of viability. As the body of available data grows and

the observations become ever more precise, these methods will provide even more stringent bounds and rule out more models. Many of the methods can also be used other modified gravity theories besides $f(R)$ gravity.

Tiivistelmä

Einsteinin suhteellisuusteoria on epäilemättä yksi historian suurimmista saavutuksista. Kaikesta huolimatta se jättää hieman toivomisenvaraa. Viimeisen sadan - ja erityisesti viimeisen viidentoista vuoden aikana - on kehitelty monia vaihtoehtoja yleisen suhteellisuusteorian parannuksiksi. 1990-luvulta lähtien on ollut selvää, että maailmankaikkeus laajenee kiihtyvällä tahdilla. Tämän selittäminen yleisen suhteellisuusteorian avulla ei ole ongelmatonta. Tämän vuoksi on tärkeää etsiä vaihtoehtoja yleiselle suhteellisuusteorialle.

Yleinen suhteellisuusteoria perustuu tiettyihin oletuksiin. Vaihtoehtoiset gravitaatioteoriat poistavat tai lieventävät näitä oletuksia suuremman yleisyyden saavuttamiseksi. Yksi vaihtoehdoista ovat $f(R)$ teoriat, jotka antavat graviataatiovaikutuksen riippua Riccin kaarevuusskalaarista muutenkin kuin lineaarisena funktiona. Näin mahdollisia $f(R)$ malleja on ääretön määrä.

Monet mahdollisista $f(R)$ malleista voidaan sulkea sulkea pois epäfysikaalisina. On kuitenkin mahdollista rakentaa malleja, jotka sopivat havaintoihin jopa paremmin, kuin menestyksekäs suhteellisuusteoria kosmologisella vakiolla. Jopa näissä näennäisesti toimivisssa malleissa saattaa kuitenkin piillä ongelmia dynamiikassa tai muissa piirteissä, joiden vuoksi ne ovat lopulta epäfysikaalisia.

Fysikaalisesti mielekkäiden $f(R)$ teorioiden joukon rajaaminen tarjoaa parempaa ymmärrystä gravitaatiosta. Vaikka $f(R)$ teorioiden parista ei löytyisikään lopullista ratkaisua gravitaation ongelmaan, niiden avulla voidaan saada arvokasta tietoa, millainen mahdollisen kvanttigravitaatioteorian pitäisi olla.

Tässä väitöskirjassa kehitän menetelmiä, joiden avulla voidaan entistä tehokkaammin rajata mielekkäitä $f(R)$ malleja sekä sovellan näitä menetelmiä tiettyihin malleihin. Käytän matemaattisia työkaluja epästabiiliuden etsimiseen sekä testaan tuloksia havaintoaineistoon. Matemaattisesti kelvollinen malli saattaa kaatua havaintojen edessä ja toisaalta monet havainnot

täyttävä malli saattaa olla matemaattisesti huteralla pohjalla.

Menetelmiä, jotka esittelen tässä väitöskirjassa, voidaan soveltaa kaikkiin mahdollisiin $f(R)$ malleihin ja näiden mielekkyyttä voidaan testata. Sitä mukaa, kun havaintoaineistoa kertyy lisää, näillä menetelmillä päästään aina vain suurempaa tarkkuuteen ja sitä myötä tiukempiin rajoituksiin kelvollisille malleille. Monia näistä menetelmistä voidaan käyttää myös muihin yleistettyihin gravitaatioteorioihin kuin $f(R)$ teorioihin.

List of papers

This thesis consists of a review of the subject and the following original research articles:

I **Maximal symmetry and metric-affine $f(R)$ gravity**,
T. Multamäki, J. Vainio, and I. Vilja, Class.Quant.Grav. **26**, 075005 (2009) [`arXiv:0805.1834`].

II **Hamiltonian perturbation theory in $f(R)$ gravity**,
T. Multamäki, J. Vainio, and I. Vilja, Phys.Rev. D **81**, 064025 (2010) [`arXiv:0910.5659`].

III **Jeans analysis of Bok globules in $f(R)$ gravity**,
J. Vainio, and I. Vilja, GERG **48**, 10 (2016) [`arXiv:1512.04220`]

IV **$f(R)$ gravity constraints from gravitational waves**,
J. Vainio, and I. Vilja, GERG *Under review*, [`arXiv:1603.09551`]

Luku 1

On notation

Greek letters in indices refer to the four space-time coordinates. These four coordinates are $(t, x, y, z) = (t, \mathbf{x})$ or in polar coordinates (t, r, θ, ϕ). Latin letters in indices refer to the spatial coordinates, *e.g.* on a common time hypersurface Σ_t. Capital Latin letters refer to three-dimensional coordinates on a boundary ∂V of a space-time volume V.

The signature of the metric is $-, +, +, +$, the chosen sign for the Riemann curvature tensor is

$$R^{\alpha}_{\beta\mu\nu} = \partial_\mu \Gamma^{\alpha}_{\beta\nu} - \partial_\nu \Gamma^{\alpha}_{\beta\mu} + \Gamma^{\alpha}_{\kappa\mu}\Gamma^{\kappa}_{\beta\nu} - \Gamma^{\alpha}_{\kappa\nu}\Gamma^{\kappa}_{\beta\mu} \tag{1.1}$$

and the contraction of the Ricci tensor is done by contracting the first and the third index

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}. \tag{1.2}$$

The Einstein tensor is the combination of Ricci tensor and scalar

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}. \tag{1.3}$$

Symmetric parts of a tensor are marked with brackets

$$A_{(\mu\nu)} \equiv \frac{1}{2}\Big(A_{\mu\nu} + A_{\nu\mu}\Big) \tag{1.4}$$

and the anti-symmetric parts are marked

$$A_{[\mu\nu]} \equiv \frac{1}{2}\Big(A_{\mu\nu} - A_{\nu\mu}\Big). \tag{1.5}$$

The connection coefficients of the Levi-Civita connection, the Christoffel symbols, are written as

$$\left\{ {\lambda \atop \mu\nu} \right\} \equiv \frac{1}{2} g^{\lambda\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}) \tag{1.6}$$

The Levi-Civita tensor $\epsilon_{\alpha\beta\gamma\delta}$ is defined to be 1 for even permutations of 0123 and -1 for uneven permutations and zero otherwise.

The energy-momentum tensor is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}. \tag{1.7}$$

The coupling constant used in Lagrangians is

$$\chi \equiv \frac{8\pi G}{c^4}, \tag{1.8}$$

where G is the gravitational constant and c is the speed of light.

The Planck constant is denoted h and the reduced Planck constant is $\hbar = h/2\pi$. As the symbol h used for other purposes in this thesis, only the reduced Planck constant $\hbar$ is used in the following.

The constants related to the Planck scale are the Planck mass M_P and the Planck scale ℓ_P. In the context of this thesis I assume the natural units ($c = \hbar = 1$) unless stated otherwise.

Luku 2

Introduction

The discovery of the accelerating expansion of the Universe served as the launch of what could be called the golden age of cosmology. On one hand hand this observation called for new ideas and re-examining old ideas while on the other hand it lead the way to a growing body of observational data. What could be the cause of this phenomenon? What force could account for the acceleration of the entire Universe?

The answer to the riddle of accelerating expansion of the Universe is dark energy. However, this is just a name for the answer. We do not know, what dark energy is, at least not yet. There are several possible answers for this question. One rough way to characterize the possibilities is to examine the Einstein equation.

On one side of the equation is the matter content of the Universe. It is only natural to ask, if by adding some new kind of matter to the Universe would produce the accelerating expansion. Indeed, if there was some exotic matter with negative pressure abundantly available, the problem would be solved. The problem is, there is no observational evidence to back this kind of explanation.

The other side of the Einstein equation deals with how the theory itself affects gravitation. General Relativity (GR) is based on certain postulates (namely general covariance, equivalence principle and the rule of second derivatives). When one these postulates is removed (or at least weakened), modifications to gravitation become possible. These modified gravity theories can produce the observed accelerating expansion without the addition of new types of matter.

Modified gravity theories were first examined near the advent of general relativity. However, as Einstein's theory was more simple and explained all the observations in those days, there was little reason to explore more complicated theories. For the best part of the 20th century the observations did not encourage further study of modified gravity.

While the observations did not necessitate alternative theories, there was still some theoretical interest, especially in the 1960s. The Jordan-Fierz-Brans-Dicke theory of gravity (perhaps more commonly known as Brans-Dicke) had considerable impact on the development of modern models and is in itself considered viable. Also, the development of the parametrized post-Newtonian formalism paved way for comparing modified gravity models with observations.

All the other interactions in nature, besides gravity, can be explained through a single paradigm. It is now known that general relativity is incompatible with quantum mechanics, it cannot be quantized. This is another important reason to look into alternatives for general relativity. A working quantum gravity theory would pave the way for a so called Theory of Everything which would unify gravity with the other three interactions - electromagnetic, weak and strong interactions.

One way to generalize general relativity is to abandon the restriction on higher order derivatives. The naive motivation for this restriction is that all the other interactions follow this scheme. These commonly found second derivatives produce the simplest equations of motion. The somewhat more elaborate explanation for the second derivative rule is having a well-posed initial value problem and avoiding negative energy states. If higher order derivatives are allowed, the form of gravitational action is no longer unambiguous. The gravitational action can now include *e.g.* a function of the curvature scalar instead of the plain linear term.

These modified theories, which alter the dependence of the gravitational action on the curvature scalar are called $f(R)$ theories. There is an infinite number of theories in this class. This thesis focuses on finding new methods for constraining the possible ones and finding the characteristics needed for a viable theory.

In chapter 3 I discuss the reasons and motivation for developing modified theories of gravity. The mathematical foundations of both GR and modified gravity are elaborate and discussed in many textbooks. In chapter 4 I present some definitions and results of differential geometry, which are crucial in understanding the work in the attached papers. Chapter 5 offers a brief introduction into $f(R)$ theories of gravity, which are of the main interest in this thesis. As there has been considerable effort in the literature to constrain $f(R)$ theories, chapter 6 offers a short review on possible sources for constraints. Chapter 7 discusses perturbation theory and Hamiltonian formalism for the purposes of the latter three of the attached papers. The summary of my work in the attached papers is found in chapter 8.

Luku 3

Motivation for modified theories of gravity

The equations of motion for GR are found through variation of the Einstein-Hilbert Langrangian. The choice of this Lagrangian is natural as it is the most simple one exhibiting the desired dynamics. Provided some common assumptions are taken, which were introduced by Einstein [1, 2], the Einstein-Hilbert Lagrangian proves to be the unique choice, up to two constants. These are the cosmological constant and the gravitational constant. Relaxing these assumptions releases more degrees of freedom and provides for modified gravity. While the assumptions were originally presented by Einstein, the Lagrangian formulation was first introduced by Hilbert.

The core assumptions behind the Einstein-Hilbert Langrangian can be stated in several forms (see *e.g.* [3, 1, 2]). The axiomatic foundation (and lack thereof) of GR is discussed in [4]. The work of Brans and Dicke [5] has both been influential on the axiomatic base as well for modern modified gravity theories. Besides the elegant mathematical formulation of an axiom based derivation, these could provide better understanding for the properties necessary for classical gravity and show which conditions must be broken to achieve quantum gravity. One way to state the assumptions [2] is that any theory, which

1. is a metric theory
2. has second order field equations and the equations are linear in second order derivatives of the metric
3. has the correct Newtonian limit in the weak field approximation
4. has no fixed background metric

reduces to the Einstein-Hilbert action. According to [3], if the well-tested Einstein equivalence principle is taken as an assumption the remaining conditions leading to Einstein-Hilbert Lagrangian are

1. diffeomorphism-invariance of the action
2. field equations of second order for the metric
3. no more than 4 dimensions
4. the metric is the only field in the gravitational action

However, it is worth mentioning, that this set does not fix the gravitational constant, leaving two free constants.

Clearly, relaxing the second condition of either list of assumptions leads to higher order gravity and $f(R)$ theories. As we shall see in the following chapters, there will be numerous changes as there are more degrees of freedom entering the field. One interesting consequence of relaxing these conditions is a massive graviton, the mediating particle of the force of gravitation. The link between $f(R)$ gravity and graviton mass is further studied in the attached paper [6].

In standard GR the graviton has a zero mass. In order to give graviton a mass some generalization is needed, usually fixing the background metric [2], as this is the path of least resistance. However, the graviton receives a non-zero mass also through other generalizations, *e.g.* permitting higher order field equations.

The first higher order theories of gravity appeared shortly after Einstein's introduction of GR. Within ten years, both Eddington and Weyl proposed alternate variants for the gravity action. However, as at the time there was no imminent reason for more complicated theories, the interest waned. Later on, both theoretical reasons and observational reasons would turn the tables. The history of fourth order gravity has been briefly reviewed in [7].

After the early excitement, there was almost a half a decade long pause in the interest in higher-order theories. One of the notably exceptions are the works of Buchdahl [8, 9, 10]. In those days much of the interest was related to the Palatini variational principle which provides an alternative to the standard metric approach and the more recently (fully) developed metric-affine approach.

With the advent of unified theories, it become apparent that GR would have to be modified in order to bind it under a single theory with all the other interactions. Renormalization of GR does not seem to be possible due to the behaviour in the ultraviolet regime [11]. Therefore, it cannot be quantized like the other interactions. In the 1960s and 1970s it was shown that higher order gravitational actions are renormalizable (see *e.g.* [12]).

3.1 Observational motivations

During the 90s, several important observational programs were launched, which enabled cosmology to become a precision science. One could also argue, that at this time the Golden Age of cosmology began. These lead to the discovery that changed the picture of the Universe almost entirely. The Universe is expanding at an accelerating rate.

The studies of supernovae type SNIa revealed the accelerating expansion [13, 14]. Since then, observations from different sources and based on different physical phenomena have confirmed this to be likeliest interpretation. Standard GR needed the addition of the long-discarded, troublesome cosmological constant with a new sign to cover this new finding. However, many alternate explanations have (re)surfaced as well.

To create accelerating expansion of the entire Universe, something powerful must be involved. Ordinary matter and dark matter can only count for a small piece of the cosmic energy budget and for the acceleration, matter would have to have negative pressure. The cause for the acceleration is dubbed *dark energy*, but the sad truth is, no-one knows for certain, what dark energy really is. This is called the dark energy problem.

Dark energy fills up 68.5% of the cosmic energy budget [15]. The ordinary baryonic matter, such as planets and stars, takes up only about 5% and radiation covering far less than a percent. The remaining quarter is dark matter. In this sense, much of the energy content of the Universe is still unknown to us.

One way to categorize the possible answers to the dark energy problem is by looking at the equations of motion in GR, the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \chi T_{\mu\nu}. \tag{3.1}$$

On the right hand side we find the matter content of the Universe. The constant $\chi \equiv \frac{8\pi G}{c^4}$ presents the coupling strength between matter and gravitation. By adding some matter with negative pressure the accelerating expansion could be explained. The alternative would be to change the theory itself, *i.e.* make changes on the left hand side. These candidates for explaining the acceleration are called modified gravity.

The simplest modification would be the inclusion of the cosmological constant. It is arguable, whether this a modification at all, since it is allowed by the Einstein-Hilbert Lagrangian. Indeed, the ΛCDM model (or the Concordance model) which entails GR with cosmological constant and cold dark matter, explains the observed phenomena rather well. The problem arises with the explanation of the constant. Some authors use the term dark energy only to refer to the solutions arising from the matter sector. In this thesis I use the term dark energy for all the possible causes for the acceleration.

The cosmic microwave background (CMB) has provided another useful and rich source of data for modern cosmology. Especially the anisotropies in the CMB have received much attention. While the first satellite mission, COBE, provided only a crude outline of the CMB [16], later programs have provided ever more detail. The WMAP [17] and Planck [15] have provided for more stringent bounds for cosmological parameters, such as Ω_Λ, the portion of the dark energy of the cosmic energy budget.

Shortly after the supernova results, large-scale structures of the Universe were studied to back the finding and to find further constraints on the newly found dark energy [18, 19]. These dynamical models usually examine an effective equation of state parameter $w = p/\rho$. For the cosmological constant the equation of state parameter $w = -1$. The best fit alue is slightly $w < -1$ but the value -1 is not ruled out [15].

Several observations and experiments have shown GR to hold to a very high degree. Therefore, a viable modified gravity theory should have GR as a limit and pass the same tests as GR with Λ. There are several ways, in which this could be achieved, *e.g.* one possible way for this to happen is the so-called chameleon mechanism [20, 21] which effectively hides the modifications in low curvature regimes, such as the Solar System.

The tests of GR at Solar System level [22] place stringent limits on possible modifications to GR. However, these have not been able to rule out the possibility of modifications to the Einstein-Hilbert Lagrangian, while not exactly encouraging them either. Since the discovery of cosmic acceleration, the supernova data has been a valuable source of data for constraining possible $f(R)$ models and other modified gravity theories as well.

While not exactly as big a problem as the dark energy, dark matter is problematic. While being around for a long time in debates, there have been no observations and no conclusive explanations for dark matter. The galactic rotation curves of spiral galaxies do not obey the predictions of GR (or Newtonian gravity, for that matter) but require considerable extra mass. See *e.g.* [23] and [24]. With modified gravity and especially $f(R)$ theories of gravity it is possible to explain away at least part of the troublesome dark matter [25], [26].

With all the observations since the discovery of the accelerating expansion, the need for an explanation has not diminished. Rather, there is more proof for the acceleration and ever more demand for an explanation. While the cosmological constant could still be the answer to the puzzle, it is rather problematic as will be discussed next.

3.2 Theoretical motivations

There are a number of theory-based reasons to pursue alternatives to GR. While the simplicity and, as many would say, beauty of GR with Λ is certainly appealing, there is a demand for something more. In the following I briefly review some of the reasons.

3.2.1 Quantization, quantum gravity

The standard model (see *e.g.* [27]) does a very good job describing electromagnetism, the weak and the strong interactions, especially after the discovery of the Higgs boson [28]. However, it does not include gravitation. To simplify, QFT does a good job describing small scales and high energies, while GR works in the large scale.

The search for a single theory describing all the interactions in nature has been one of the main goals of modern physics for a hundred years. Sometimes this hypothetical theory is a called the *Theory of Everything* (ToE). There have been a number of candidates in this direction, but none have been satisfactory so far. While this is an interesting avenue, it is not the only one explored in the field of quantum gravity (*e.g.* see [29] and references therein for loop quantum gravity).

One of the fundamental problems in combining quantum field theory and GR, the best current theories describing each, is the concept of time. The nature of time in quantum mechanics is entirely different to that in

GR, in which it is far from absolute [30]. It is not possible to incorporate GR and QFT into a single theory, as they stand now.

String theories are one of the main candidates for ToE (see *e.g.* [31]). As this thesis is about modified gravity and especially, $f(R)$ theories of gravity, string theories worth mentioning as it has been shown that certain $f(R)$ theories are found as the limit of string theories [32].

On the low-energy field theory limit the spin-two particle can be identified as the graviton. However, the corrections to GR are expected to be with very small couplings, so that they manifest mainly at close to Planck scale, $\ell_P \sim 10^{-35}$m. On the other hand, on the large scale remedies to cosmological problems are expected.

Due to this connection, studying $f(R)$ gravity provides a way to probe into the viability of candidates of quantum gravity. This is important as practically all the candidates are notoriously difficult [31] to prove (or rather, disapprove) with experiments and observations. If the extra dimensions are of Planck scale, they maybe forever out of reach for experiments. In this context, comparing $f(R)$ theories with observations is far easier. If a considered string theory would have a $f(R)$ limit, which is proved non-viable, that string theory would be non-viable as well.

Another advantage of $f(R)$ theories over GR, when it comes to quantization of gravity, is the better possibility for renormalization. This was originally found for other higher order theories [33, 12] but has later extended to general $f(R)$ theories, see *e.g.* [34].

In this sense, deeper understanding of $f(R)$ gravity and the viability criteria involved pave way for discovering a viable ToE. For example the $f(R)$ actions would cause differences in regularization [35]. The correct unification theory would properly describe the accelerating expansion. As such, the limit behaviour would probably incorporate the expansion as $f(R)$ gravity does.

3.2.2 Cosmological constant problem

While the GR with the cosmological constant, Λ, fits very well with the observations described in the last section, it is far from problem-free. The Λ can arise from quantum effects, but the magnitude of the theoretical vacuum energy density (caused by the cosmological constant) and the observed one do not meet [36]. In fact they are off by a factor of 10^{-120}. As densities are related to mass scales through a power-law, the discrepancy in mass does not look as problematic, since $M_{vac}^{theory} \sim M_P \sim 10^{18}$GeV and $M_{vac}^{obs} \sim 10^{-3}$eV, which is still not good, though, in the words of Sean Carroll [37].

With proper regularization and renormalization this classic cosmological constant problem can be mitigated [38, 39]. Correct definition of the vacuum in Quantum Field Theory (QFT) would be one remedy for the issue, but this has not been achieved for yet. For the moment, the cosmological constant problem endures, yet in a more elaborate form [40].

The value of the cosmological constant is also problematic to some candidates for ToE, as these require a strictly zero cosmological constant (see *e.g.* [41, 42] and references therein).

Besides the value of the cosmological constant, there is another problem, the coincidence problem. While the current date cosmological constant is comparable to the energy density of matter, this should happen for only a short period of time, so why now?

3.2.3 Other modified gravity theories

The dark energy problem still reigns strong and there is a host of modified gravity theories, including $f(R)$ gravity to offer alternatives to ΛCDM. Even if these do not provide an answer to cosmic acceleration, one is interested in understanding why not. While there are numerous other works discussing these, for the purposes of this thesis, I only mention some alternatives and references for the interested reader. This is by no means an exhaustive list of theories, rather it briefly presents a subset which I have come across my studies. An useful review of many alternatives is found in [43] and in [44].

One advantage over many other possible generalizations of gravity is avoiding the Ostrogradski instability [45], which is related to having higher order field equations. A contemporary English discussion on the subject can be found in [46]. I return to the Ostrogradski instability in chapter 6.

As a close relative of $f(R)$, it is natural to mention first the $f(R,T)$ theories which have raised interest rather recently [47]. In these models the Einstein-Hilbert Lagrangian is replaced by a function depending on the scalar curvature and the trace of the energy-momentum tensor.

In so-called $f(\mathcal{G})$ gravity the Ricci scalar in the action is supplemented by a function of the topological Gauss-Bonnet invariant, $\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$. One of the benefits (or some might say, disadvantages) of these theories is the close relation to string theories [48]. See *e.g.* [49], [50] and references therein.

In the DGP brane gravity [51] the 3-dimensional Universe we observe is embedded into a higher dimension space-time. While there are string theories and brane-world gravity models [52] requiring several more extra dimensions, the DGP models make do with one extra dimension. These models are able to produce the accelerated expansion without any additional dark energy fields [53].

Luku 4

On differential geometry

This chapter presents concepts of differential geometry needed as background mathematics for some of the papers included. While most of the following could be generalized to more dimensions, for the context of this thesis, it is enough to focus on the case of four-dimensional space-times. The material presented in this chapter is found in many text books, but as much of it seldom used in the context of $f(R)$ gravity I present the most crucial concepts for understanding the included papers. Also, unifying the notation used in the papers and literature should prove helpful. This chapter is largely based on the books [54], [55], [56], [57] and [58].

4.1 Metric-affine space-time

A *metric-affine* space-time has its geometry defined by the *metric* and the *affine connection*[1]. Metric-affine spaces have several special cases. The terms in what follows are those of [59] and [60]. While they are not too common in contemporary literature, I believe they illustrate the differences in connections and their significance.

By assuming *metricity* of the metric, *i.e.* $\nabla_\lambda g_{\mu\nu} = 0$, the result is called the *Riemann-Cartan space*. If *torsion* is assumed to be zero, but not metricity, the space is called *(pseudo-)Riemann space*. If even *curvature* is zero, the space is a flat *Minkowski space*. There is also the possibility of zero curvature with non-zero torsion, which is called *Weitzenböck space*. A more detailed classification can be found in [59]. In the case of Einstein-Hilbert Langrangian, the Riemann space case is general relativity and the

[1] In the context of this thesis there is no ambiguity and I use both the terms affine connection and connection. In literature one might also encounter the spelling connexion.

Minkowski one is special relativity.

For a general manifold M without a connection, it meaningless to discuss dynamics. For this purpose *parallel transport* of vectors must be specified and for this the affine connection is needed. A differential manifold equipped with an affine connection is sometimes called a *linearly connected space.* If a vector u^μ is parallely transported from x^μ to $x^\mu + dx^\mu$, the change can now be written

$$du^\mu = -\Gamma^\mu_{\alpha\beta} u^\beta dx^\alpha. \tag{4.1}$$

In the most general case there are 4^3 independent connection coefficients. The connection Γ is not a tensor. However, the anti-symmetric part does transform as a tensor

$$S_{\mu\nu}{}^\lambda \equiv \frac{1}{2}(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}) \tag{4.2}$$

and is called *(Cartan's) torsion tensor*. The covariant derivative can now be defined in terms of the connection. In order to have a constant quantity in space-time, its covariant derivative, defined by the connection, should vanish. To put it in another way, a tensor field is parallel transported along a curve σ if its covariant derivative along the curve is zero.

As the connection defines the parallel transport, it characterizes the geometry of the manifold M. It helps the intuitive picture to consider the parallel transport of a vector. Let u^λ and v^λ be two infinitesimal vectors. Parallel transporting u^λ along v^λ results in a new vector $u^\lambda - \Gamma^\lambda_{\mu\nu} u^\mu v^\nu$. Correspondingly, parallel transporting v^λ along u^λ yields $v^\lambda - \Gamma^\lambda_{\mu\nu} v^\mu u^\nu$. Intuitively this should form a parallelogram and indeed, this can be the case, but only with the condition

$$u^\lambda + v^\lambda - \Gamma^\lambda_{\mu\nu} u^\mu v^\nu = v^\lambda + u^\lambda - \Gamma^\lambda_{\mu\nu} u^\mu v^\nu. \tag{4.3}$$

The condition of closing parallelograms is now found to be

$$(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}) u^\mu v^\nu = 2 S_{\mu\nu}{}^\lambda u^\mu v^\nu = 0. \tag{4.4}$$

Another geometric quantity defined by the connection is curvature. As the example above shows, in a general case parallel transfer is heavily path-dependendent. If a vector is parallel transported from point P around an infinitesimal area, back to P, the change is proportional to the Riemann curvature tensor [60]

$$R^{\alpha}_{\beta\mu\nu} = \partial_\mu \Gamma^{\alpha}_{\beta\nu} - \partial_\nu \Gamma^{\alpha}_{\beta\mu} + \Gamma^{\alpha}_{\kappa\mu}\Gamma^{\kappa}_{\beta\nu} - \Gamma^{\alpha}_{\kappa\nu}\Gamma^{\kappa}_{\beta\mu}. \tag{4.5}$$

A priori, in metric-affine space-times the curvature depends only on the connection and is independent of the metric. The Riemann curvature tensor could be also be defined in terms of the torsion tensor [61].

The general Riemann curvature tensor does not have all the usual symmetries found in the text books as most books are based on the Levi-Civita connection, which I will introduce later in this chapter. Actually, the general curvature tensor could be contracted into two separate Ricci tensors [62]. However, as one the possibilities results in a null Ricci scalar, the choice is unambiguous. The Ricci tensor is the familiar

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}. \tag{4.6}$$

Even though the contraction looks familiar, it should be noted, that the general Ricci tensor is not symmetric in the indices. Another note is, that this contraction does not require a metric.

As the connection defines transport and change, something else is needed for measuring distances and angles. This is naturally the metric $g_{\mu\nu}$. In terms of the metric the square of the space-like interval ds (or time-like $d\tau$) can be written as

$$ds^2 = -g_{\mu\nu}(x)dx^\mu dx^\nu. \tag{4.7}$$

It is possible to find a coordinate system, in which the metric is of Minkowski form. While there is an intuitive idea about measuring distances, like with the connection, in a general case the intuitive idea breaks down. For the metric, the troublesome possibility is the *non-metricity tensor*

$$Q_{\lambda\mu\nu} \equiv -\nabla_\lambda g_{\mu\nu}, \tag{4.8}$$

which describes how distances change in parallel transport. Non-zero $Q_{\lambda\mu\nu}$ would mean that the unit length would not be preserved, which would make dynamic considerations rather troublesome, as (4.7) is not invariant. In the most general case the metric would have 4^2 degrees of freedom, but in the context of this thesis symmetry of the metric is assumed at all times.[2] There

[2] See *e.g.* [63], [64] for non-symmetric gravity considerations and [65] on some of the

are therefore 10 degrees of freedom left in the metric. The symmetry of the metric leads to symmetry in the last indices of the non-metricity tensor but is not enough to make the connection symmetric.

Indices of tensors can be lowered and raised with the metric. Especially, the *Ricci scalar* is found to be

$$g^{\mu\nu}R_{\mu\nu} = R^{\mu}{}_{\mu} \equiv R. \tag{4.9}$$

An often used combination of Ricci tensor and scalar is the *Einstein tensor*

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}. \tag{4.10}$$

The main usefulness of the Einstein tensor is the fact that in GR the equations of motion can be written in a compact manner with it. It is also useful in comparing a $f(R)$ model with GR.

In a general metric-affine space the connection can be decomposed into parts with geometric interpretations. The resulting decomposition is best written with the introduction of the permutation tensor

$$P^{\alpha\beta\gamma}_{\nu\mu\lambda} \equiv \delta^{\alpha}_{\nu}\delta^{\beta}_{\mu}\delta^{\gamma}_{\lambda} + \delta^{\alpha}_{\mu}\delta^{\beta}_{\lambda}\delta^{\gamma}_{\nu} - \delta^{\alpha}_{\lambda}\delta^{\beta}_{\nu}\delta^{\gamma}_{\mu}. \tag{4.11}$$

Now the decomposition can be written as

$$\Gamma^{\lambda}_{\mu\nu} = g^{\lambda\kappa}P^{\alpha\beta\gamma}_{\nu\mu\kappa}\Big(\frac{1}{2}\partial_{\alpha}g_{\beta\gamma} - g_{\gamma\delta}S_{\alpha\beta}{}^{\delta} + \frac{1}{2}Q_{\alpha\beta\gamma}\Big). \tag{4.12}$$

The first term is the *Christoffel symbol* part, familiar from all the text books of GR. It should be stressed that Christoffel symbols are not tensors, but their variations are. The second term represents torsion and the last non-metricity. Geometrically these terms represent what happens to, say, a vector during parallel transport. The combination of torsion terms is called the *contortion tensor*

$$K_{\mu\nu}{}^{\lambda} \equiv S_{\nu}{}^{\lambda}{}_{\mu} - S_{\mu\nu}{}^{\lambda} - S^{\lambda}{}_{\mu\nu}. \tag{4.13}$$

Due to the antisymmetry in the torsion tensor, also the contortion tensor is antisymmetric in the first two indices.

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stability issues.

In standard GR the metric is the sole geometry defining quantity and the space is a Riemann space. The connection is the *Levi-Civita connection* and the coefficients are simply the Christoffel symbols, *i.e.* the metric defines the connection. Considering the general quantities of metric-affine spaces, this means that non-metricity and torsion must vanish. This is caused by the Einstein equivalence principle in GR. In this case, the connection can be simply written as

$$\Gamma^{\lambda}_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\} \equiv \frac{1}{2} g^{\lambda\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}). \tag{4.14}$$

Therefore, it is uniquely defined by the metric.

4.2 On symmetries

Let $(M, g_{\mu\nu})$ be a globally hyperbolic space-time, where the hyperbolicity is required for the initial value problem considerations mentioned in chapter 5. The space-time metric $g_{\mu\nu}$ is related to the position vector x^μ defining the curve. The integrals in the gravitational action (*e.g.* (5.2)) are over a volume V of the space-time $(M, g_{\mu\nu})$. This volume is foliated with space-like hypersurfaces Σ_t. These are Cauchy surfaces parametrized by a global time $t(x^\mu)$ and let n^μ be the unit normal vector field to the hypersurfaces. The global time remains arbitrary and unphysical until one knows the metric, as they are related through $d\tau = \sqrt{g_{00}} dt$. At this point it is only set that t must be a single-valued function of coordinates x^μ, to ensure non-intersection of the hypersurfaces.

Let σ be a curve on the manifold M and let ξ^μ be vector tangent to the curve σ and let A^μ be a vector defined in the neighborhood of σ. The *Lie derivative* of vector A^μ along the curve can be written as

$$\mathcal{L}_\xi A^\mu \equiv \partial_\nu A^\mu \xi^\nu - \partial_\nu \xi^\mu A^\nu. \tag{4.15}$$

This generalizes to higher rank tensors. The Lie derivative of a tensor is a tensor. In the torsionless case, the partial derivatives in (4.15) can be replaced with covariant ones. As this thesis examines also metric-affine gravity, which includes torsion, the partial derivatives are kept. For the purposes of this thesis the most important is the Lie derivative of second rank covariant tensor, (*e.g.* the metric tensor) for which the formula reads

$$\mathcal{L}_\xi A_{\mu\nu} = \xi^\lambda \partial_\lambda A_{\mu\nu} + A_{\lambda\nu}\partial_\mu \xi^\lambda + A_{\mu\lambda}\partial_\nu \xi^\lambda. \tag{4.16}$$

If the Lie derivative of a tensor field along the curve σ is zero, then it is said to be *Lie transported* along the curve. A Lie transported tensor field is also *form invariant* [66]. A space $(M, g_{\mu\nu})$ is form invariant, if the associated metric tensor is form invariant. For infinitesimal transformations this is equivalent with $\mathcal{L}_\xi g_{\mu\nu} = 0$ [58]. The coordinates for the position vector x^μ can be freely chosen and by choosing a coordinate system in which the spatial x^a are constant, the curve is parametrized by the only non-constant coordinate, t.

It should be noted, that any of the other coordinates could be used as well and the t need not be tied to the physical time. At this point it is a general parameter. In this coordinate system the tangent vectors are simply $\dot{x}^\mu = \delta^\mu_0$. As the partial derivatives vanish, in this coordinate system the Lie derivative effectively reduces to ∂_t. This means that if a tensor field is Lie transported, its components are independent of parameter t.

Symmetries of the space-time $(M, g_{\mu\nu})$ are smooth local diffeomorphisms $\phi_t : U \to V$ associated with the vector field ξ^μ, where U and V are open submanifolds of M, which preserve some feature of M (see *e.g.* [57]). The ϕ_t is the local flow generated by ξ^μ [58]. Therefore, for an infinitesimal t it can be approximated that

$$\big(\phi_t(x)\big)^\mu = x^\mu + t\xi^\mu. \tag{4.17}$$

A smooth vector field ξ *conserves* a tensor T on the space-time $(M, g_{\mu\nu})$, if for all smooth local diffeomorpishms ϕ_t, $T = \phi_t * T$ in U. This is equivalent with the vanishing of the Lie derivative *i.e.*

$$\mathcal{L}_\xi T = \lim_{t\to 0} \frac{1}{t}\Big(\phi_t^* T_{\phi_t(x)} - T_x\Big) = 0. \tag{4.18}$$

The vector field ξ is called *affine* if it is a smooth global vector field of M and its every local diffeomorphism ϕ_t is an *affine* map. A map ϕ_t is affine if it preserves geodesics and their affine parameters. It can be shown (see [57] and references therein), that ξ is affine iff $\mathcal{L}_\xi \nabla = 0$ or

$$\nabla_\mu \nabla_\nu \xi^\lambda = R^\lambda_{\mu\nu\sigma}\xi^\sigma. \tag{4.19}$$

The covariant derivative of the vector field ξ can be split into symmetric and antisymmetric parts. For the resulting symmetric tensor $f_{\mu\nu}$ and antisymmetric tensor $F_{\mu\nu}$

$$\nabla_\nu \xi_\mu = \frac{1}{2} f_{\mu\nu} + F_{\mu\nu}. \tag{4.20}$$

One may notice, that the symmetric part, $f_{\mu\nu}$ is equal to the Lie derivative of the metric in the case of a symmetric connection

$$f_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = \mathcal{L}_\xi g_{\mu\nu}. \tag{4.21}$$

By inserting the decomposition (4.20) to the affine condition (4.19) and considering the symmetries of the curvature tensor, $f_{\mu\nu}$ and $F_{\mu\nu}$, two conditions can be separated

$$\nabla_\lambda f_{\mu\nu} = 0, \tag{4.22}$$

$$\nabla_\lambda F_{\mu\nu} = R_{\mu\nu\lambda\sigma} \xi^\sigma. \tag{4.23}$$

The first can be shown to be equivalent with ξ being an affine vector field [57]. *Killing vector fields* are a special case of this vector field, for which $f_{\mu\nu} = 0$. The affine condition now is

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0, \tag{4.24}$$

$$\mathcal{L}_\xi g_{\mu\nu} = 0. \tag{4.25}$$

The Killing vectors describe the infinitesimal isometries of a space. The first of the two equations is known as the *Killing equation.* One should notice this equation includes covariant derivatives instead of partial derivatives. As stated above in (4.21), under certain conditions, these two are equivalent. This can be seen by rewriting the partial derivatives in (4.16) as

$$g_{\lambda\nu} \partial_\mu \xi^\lambda = \partial_\mu (g_{\lambda\nu} \xi^\lambda) - \xi^\lambda \partial_\mu g_{\lambda\nu} = \partial_\mu \xi_\nu - \xi^\lambda \partial_\mu g_{\lambda\nu}. \tag{4.26}$$

The Lie derivative of the metric can now be written as

$$\begin{aligned} \mathcal{L}_\xi g_{\mu\nu} &= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + (\xi^\lambda \partial_\lambda g_{\mu\nu} - \xi^\lambda \partial_\mu g_{\lambda\nu} - \xi^\lambda \partial_\nu g_{\mu\lambda}) && (4.27) \\ &= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - 2\xi_\lambda \begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix}. && (4.28) \end{aligned}$$

The last term is the Christoffel symbol, the connection coefficients of the Levi-Civita connection. On the other hand, the Killing equation can be written as

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \Gamma^\lambda_{\nu\mu}\xi_\lambda - \Gamma^\lambda_{\mu\nu}\xi_\lambda. \tag{4.29}$$

On comparing these two equations the condition for the equivalence is revealed to be

$$2\xi_\lambda \begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix} = \Gamma^\lambda_{\nu\mu}\xi_\lambda + \Gamma^\lambda_{\mu\nu}\xi_\lambda. \tag{4.30}$$

The vector field ξ is arbitrary, so the connection must by of a form

$$\Gamma^\lambda_{\mu\nu} = \begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix} + C_{\mu\nu}{}^\lambda, \tag{4.31}$$

where $C_{\mu\nu}{}^\lambda$ is a tensor anti-symmetric in the first two indices [61, 67]. It was shown above, that the connection can be decomposed into three parts (4.12): the Christoffel symbol, torsion and non-metricity parts. As can be seen from the condition above, the Killing equation equivalence can be reached with non-zero torsion. However, the non-metricity must vanish.

When the equivalence holds, the metric remains unchanged along the Killing vector. Therefore, a test particle would not have gravitational forces acting on it in this direction [68], which leads to conservation of its momentum in this direction.

For all Killing vectors, the general equation (4.19) naturally holds. If ξ_μ and $\nabla_\nu \xi_\mu$ are known at some point P, the higher order derivatives of the Killing vector can be found using (4.19) and will be a linear combination of $\xi_\mu(P)$ and $\nabla_\nu \xi_\mu(P)$. Then in the neighborhood of P the function $\xi_\mu(x)$ can be constructed as a Taylor series. Any one of the Killing vectors can be written as

$$\xi_\mu(x) = A^\alpha_\mu(x;P)\xi_\alpha(P) + B^{\alpha\beta}_\mu(x;P)\nabla_\alpha K_\beta(P), \tag{4.32}$$

where A^α_μ and $B^{\alpha\beta}_\mu$ are functions that depend on the metric and the point P, but they do not depend on the initial values $\xi_\mu(P)$ and $\nabla_\nu \xi_\mu(P)$ and are therefore common for all the Killing vectors of the space $(M, g_{\mu\nu})$. It should be stressed that the Killing vectors depend on the metric.

The first term of (4.32) is related to the initial values $\xi_\mu(P)$ of which there can be as many linearly independent ones as there are dimensions, n. As for the second term and its initial values $\nabla_\nu \xi_\mu(P)$, there are $n(n-1)/2$ linearly independent ones due to the symmetry (4.24). This totals up to $n(n+1)/2$ independent Killing vectors.

Let $\hat{\Omega}$ be a smooth strictly positive function. The metric transformed via

$$\hat{g}_{\mu\nu} = \hat{\Omega}^2 g_{\mu\nu} \tag{4.33}$$

is found through a *conformal transformation*. As discussed later, $f(R)$ theories and scalar tensor theories are in some sense a conformal transformation away from GR. Also, an equation for a field ψ may be or may not be *conformally invariant*. It is invariant if $\exists s \in \mathbb{R}$, for which $\hat{\psi} = \hat{\Omega}^s \psi$ is a solution with the metric $\hat{g}_{\mu\nu} = \hat{\Omega}^2 g_{\mu\nu}$ iff ψ is a solution with the metric $g_{\mu\nu}$. Using the transformed metric $\hat{g}$ a covariant derivative, of the Levi-Civita connection, can be constructed and denoted $\hat{\nabla}$.

For example the conservation laws can be examined through the vanishing of the energy-momentum tensor

$$\nabla_\mu T^{\mu\nu} = 0. \tag{4.34}$$

The energy-momentum tensor is properly introduced later chapter 5, but in this case any symmetric two-tensor would do. This equation is not conformally invariant as is evident from

$$\hat{\nabla}(\hat{\Omega}^s T^{\mu\nu}) = \nabla_\mu(\hat{\Omega}^s T^{\mu\nu}) + \Gamma^\mu_{\mu\lambda}\hat{\Omega}^s T^{\mu\nu} + \Gamma^\nu_{\mu\lambda}\hat{\Omega}^s T^{\mu\lambda} \tag{4.35}$$

$$= \hat{\Omega}^s \nabla_\mu T^{\mu\nu} + \hat{\Omega}^{s-1}\nabla_\mu \hat{\Omega}\Big[(s+n+2)T^{\mu\nu} - g^{\mu\nu}T\Big]. \tag{4.36}$$

Clearly, conformal invariance is achieved only in the case of $T \equiv g_{\mu\nu}T^{\mu\nu} = 0$ and $s = -n-2$.

4.3 Maximal symmetry

A *maximally symmetric space* is defined as having maximum number of Killing vectors, which is $n(n+1)/2$ for a n dimensional manifold. As the number of Killing vectors is maximal, the Lie derivative of a general tensor must vanish (generalization of equation (4.16))

$$0 = \partial_\mu \xi^\lambda T_{\lambda\nu\ldots}(x) + \partial_\nu \xi^\lambda T_{\mu\lambda\ldots}(x) + \cdots + \xi^\lambda(x)\partial_\lambda T_{\mu\nu\ldots}(x). \tag{4.37}$$

As these equations restrict invariant tensors, maximal symmetry (which is equivalent with isotropy and homogeneity) effectively cut the degrees of freedom involved. The following treatment is largely similar to [66] and is important as context for the attached paper [67].

According to the cosmological principle (or the generalized Copernican principle) the Universe is homogeneous and isotropic in large scales. As these mean that the Universe looks the same in all directions and at all points, the consequences on symmetry are vast. Homogeneity and isotropy severely restrict the possible form of tensors. As this means the vanishing of Lie derivative and form invariance under an infinitesimal isometric transformation $x \to x'$, the following must hold

$$T_{\mu\nu\ldots}(x) = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu}\cdots T_{\alpha\beta\ldots}(x'). \tag{4.38}$$

For the transformation $x'^\mu = x^\mu + \epsilon\xi^\mu(x)$ this is equivalent with (4.37). For a scalar $S(x)$ this becomes

$$\xi^\mu(x)\partial_\mu S(x) = 0. \tag{4.39}$$

As the Killing vectors ξ^μ are form invariant, an arbitrary value can be chosen for $\xi^\mu(x)$ forcing $\partial_\mu S(x) = 0$, *i.e.* a scalar tensor must always be constant in a maximally symmetric space. A similar treatment can be achieved for higher order tensors as well. Using the form invariance further, the Killing vector can be chosen so, that for an arbitrary point P

$$\xi^\mu(P) = 0, \tag{4.40}$$

$$\nabla_\nu \xi_\mu(P) = g_{\alpha\beta}(P)\Big(\frac{\partial \xi^\alpha(x)}{\partial x^\mu}\Big)_{x=P}. \tag{4.41}$$

In the latter equation anti-symmetry is enforced by the Killing equation (4.24). Using these in (4.37) and keeping in mind that $\nabla_\nu\xi_\mu$ is an arbitrary matrix, the equation can be written as

$$\delta^\alpha_\mu T^\beta_{\nu\ldots} + \delta^\alpha_\nu T^\beta{}_\mu{}^\beta{}_{\ldots} + \cdots = \delta^\beta_\mu T^\alpha_{\nu\ldots} + \delta^\beta_\nu T^\alpha{}_\mu{}^\beta{}_{\ldots} + \cdots, \tag{4.42}$$

which holds for all points. In the context of this thesis, tensors up to rank 3 are of interest. This calculation of the form for rank 3 tensors is carried out explicitly in [67]. For tensors of rank 1 and 2 the calculations can be found in the following.

For a vector $A_\mu(x)$, the condition (4.42) reads as

$$\delta^\alpha_\mu A^\beta = \delta^\beta_\mu A^\alpha, \tag{4.43}$$

and contracting α and μ reveals that $\delta^\alpha_\alpha = n \neq 1$. Therefore, for a maximally symmetric space, form-invariant vectors must vanish. Similarly, for a rank 2 tensor $B_{\mu\nu}$ in (4.42), with the α and μ contraction

$$nB^\beta{}_\nu + B_\nu{}^\beta = B^\beta{}_\nu + \delta^\beta_\nu B^\mu_\mu. \tag{4.44}$$

Lowering the indices and as the indices are arbitrary, the same equation can be subtracted with interchanged indices β and ν to obtain

$$(n-2)(B_{\beta\nu} - B_{\nu\beta}) = 0. \tag{4.45}$$

As the number of dimensions in this context is not 2, the tensor $B_{\mu\nu}$ must be symmetric. The lowering of indices in (4.44) and using the symmetry reveals, that all rank 2 tensors must be of the form

$$B_{\mu\nu} = f g_{\mu\nu}, \tag{4.46}$$

where the function $f \equiv B^\mu_\mu/n$ does not depend on the coordinates of the maximally symmetric space. This is due to the fact the Killing vectors can at any point be chosen to have an arbitrary value.

For rank three tensors the results are found in the paper attached to the thesis [67]. As above, when the number of dimensions is equal to the rank of the tensor, there is a special case. Due to symmetry and anti-symmetry conditions found in the paper, in a maximally symmetric space form-invariant tensors must vanish. As the torsion and non-metricity tensors are of rank three, in a maximally symmetric space the connection is always the Levi-Civita connection. For $n = 3$, the only restriction is the invariance under cyclic permutations

$$C_{kjn} = C_{nkj}. \tag{4.47}$$

It is now clear, that demanding maximal symmetry causes severe restrictions, of which all physical system cannot pass. However, in many physical systems it is possible to find a maximally symmetric subspace. Depending on the context, I use both the terms subspace and hypersurface on the three-dimensional subspace. The subspaces still pose restrictions on the general metric of the whole space(-time). While the global space-time has n dimensions and the maximally symmetric subspace has m dimensions, coordinates $u^i,\ i \in [1,\ldots,m]$ can be used on the submanifold and $v^I,\ i \in [1,\ldots,n-m]$ keep track of the submanifolds. Now the space-time $(M, g_{\mu\nu})$ is foliated by the maximally symmetric subspaces and the combination of u^i and v^I can be used as the coordinates on $(M, g_{\mu\nu})$. It is now possible (see [66]) to write the metric as

$$g_{\mu\nu}dx^\mu dx^\nu = g_{IJ}(v)dv^I dv^J + f(v)\gamma_{ij}(u)du^i du^j, \tag{4.48}$$

where the γ_{ij} is the induced metric on the submanifold. This offers much simplification to the general case as there are no temporal-spatial cross-terms, plus g_{IJ} and f are independent of the coordinates of the submanifold.

For the four-dimensional space-time this would mean 10 Killing vectors for the considerations in this thesis. However, this is often undesired as it can be shown that maximal symmetry is equivalent with a space being homogeneous and isotropic [66]. As according to current paradigm the Universe is experiencing expansion, the space-time cannot be homogeneous and isotropic, but the spatial three-space can be. Therefore, there is three-dimensional subspace and one time coordinate. The metric is now of the familiar form

$$g_{\mu\nu}dx^\mu dx^\nu = g_{00}(t)dt^2 + a(t)\gamma_{ab}(x)dx^a dx^b. \tag{4.49}$$

The time-dependent function $a(t)$ basically measures the dynamics of the scale of the subspace. In spherical coordinates the induced metric is

$$\gamma_{ab} = \frac{1}{1-k^2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{4.50}$$

where the parameter k refers to the curvature of the three-dimensional subspace. Especially, $k = 0$ is the case of flat three-dimensional Σ, $k = 1$ refers to closed, positive curvature subspaces Σ and $k = -1$ refers to negative curvature subspaces.

This type of metric is called the *Friedmann-Robertson-Walker metric* (FRW). There is also the concept of *Friedmann-Robertson-Walker universe*, which refers to a space-time which has the FRW metric and obeys the Friedmann equations [66]. In the context of this thesis, the interest lies in the flat $k = 1$. This is not only due to simplicity but several experiments, *e.g.* Boomerang [69] and MAXIMA [70] back this claim.

In summation, enforcing maximal symmetry also leads to a constant Ricci scalar, or to put it otherwise, the metric is the Friedmann-Robertson-Walker metric. Especially for empty space, the resulting de Sitter behaviour in context of the trace equation will be of use in the following chapter.

In order to use the Killing equations, $Q_{\lambda\mu\nu} = 0$ must be assumed. This assumption is backed up by all experiments supporting special relativity as the metricity establishes the local Minkowskian structure. This is not to be confused with the fact that the space-time is always locally Minkowskian, regardless of the connection. This is an inherent characteristic of manifolds and related to general covariance expected of physical theories.

The general affine connection has now reduced to

$$\Gamma^{\lambda}_{\mu\nu} = \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} - K_{\mu\nu}{}^{\lambda}, \tag{4.51}$$

where the contortion tensor (4.13) has been used. Due to the (anti-) symmetry of the torsion and the contortion tensor, the degrees of freedom in the connection have reduced from 64 to 24. The Christoffel part is determined by the symmetric metric with 10 degrees of freedom.

With symmetries it is possible to further reduce the degrees of freedom. In [67] maximal symmetry is used to reduce the degrees of freedom effectively.

Luku 5

$f(R)$ theories of gravity

The equations of motion for GR, the so-called Einstein equations, are found through variation of the Einstein-Hilbert action. In the absence of matter it reads simply (see *e.g.* [66, 71, 1, 58])

$$S_{EH} = \frac{1}{2\chi} \int d^4x \sqrt{-g} R. \tag{5.1}$$

Here $\chi \equiv \frac{8\pi G}{c^4}$ and the square root of the determinant g is required to make the integrand a density, *i.e* it is the volume element. It is worth noting that the action is dimensionless. As mentioned in chapter 3, the Einstein-Hilbert action is the essentially unambiguos choice with the postulates of GR as discussed in chapter 3.

In $f(R)$ theories the dependence of the gravitational action on the curvature scalar is allowed to be a general function of R. This is not the only possible, and certainly not the most general way to generalize the action to allow higher order terms. Rather, it is a simple way to do it or as some might argue, the most simple way. Following the same reasoning as choosing the Einstein-Hilbert action, it provides generalization possibilities without unnecessary complications. The action with an included matter part is

$$S = \frac{1}{2\chi} \int d^4x \sqrt{-g} f(R) + S_M(g_{\mu\nu}, \Gamma, \psi), \tag{5.2}$$

where the latter term is the general matter term [72, 20] and the ψ refer to all matter fields. It is apparent that the choice of $f(R) = R$ yields standard Einstein equations and as a special case the Newtonian limit can be found [73]. Variation of the gravitational action produces the equations of motion. There are several different choices for the variational principle, which yield to a degree different equations. Most of this thesis concentrates on the metric variational principle (or formalism), which is also the most widely used one.

5.1 On variational principles

As discussed in chapter 4 a general metric-affine space-time has its geometry defined by two quantities: the metric and the connection. These are *a priori* the dynamic variables of a geometric theory and independent. Besides these two, there are usually independent matter fields as well. If no dependence between the geometric quantities is assumed, the *metric-affine variational principle* is chosen. If a torsionless, metric connection is assumed (*i.e.* the Levi-Civita connection), the *metric variational principle* is chosen. The third choice on variational principle is the *Palatini variational principle*, which could be said to be halfway between the two. In the following the different variational principles are briefly discussed to the extent needed in the context of this thesis. For a more in-depth discussion, see [72] and [74].

5.1.1 Metric variational principle

The metric formalism is by far the most common choice and arguably the most simple one. It can be found in all textbooks on GR. For the purposes of this thesis the metric formalism takes the most important position as it is used in the associated papers [75] and [6].

In the metric formalism the space-time is a Riemann space and the connection coefficients are Christoffel symbols (4.14), which are dependent on the metric and its derivatives. As the curvature scalar appearing in the action depends on the derivatives of connection, the action is in total second order in the derivatives. The action can now be written with arguments as

$$S = \frac{1}{2\chi}\int d^4x\sqrt{-g}f\big(R(g,\partial g,\partial^2 g)\big) + S_M(g,\psi). \tag{5.3}$$

For this reason the metric formalism is sometimes called the *second order formalism*. The associated question of surface terms in the action is discussed later in the next section 5.2.

The standard variation practices (*e.g.* [1, 71, 66] net us with the generalized Einstein equations and the trace equation

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu f'(R) + g_{\mu\nu}\Box f'(R) = \chi T_{\mu\nu}, \tag{5.4}$$

$$3\Box f'(R) + f'(R)R - 2f(R) = \chi T. \tag{5.5}$$

Here $T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta\sqrt{-g}\mathcal{L}_m}{\delta g^{\mu\nu}}$ [68] is the energy-momentum tensor and $T \equiv T^\alpha_\alpha$. The prime denotes derivatives with respect to R, ∇_μ is the covariant derivative and $\Box \equiv \nabla^\mu\nabla_\mu$. The latter equation is the trace equation which is found by contracting the indices of the first equation. As for the matter term, the metric and the matter are minimally coupled. It can be shown that the energy-momentum is conserved in metric $f(R)$ gravity as is the case in GR [76].

One should notice that in the definition of the energy momentum tensor, the derivative in question is the functional derivative (also known as the variational derivative) instead of a partial derivative. The definition of the variational derivative is[77]

$$\left.\frac{dF[f+\epsilon\phi]}{d\epsilon}\right|_{\epsilon=0} = \int \frac{\delta F[f]}{\delta f(x)}\phi(x)dx, \tag{5.6}$$

where $\epsilon > 0$ and $F[f]$ is a functional ϕ a test function.

One major difference in the equations of motion of the general $f(R)$ case as opposed to the Einstein-Hilbert action can be seen from the trace equation. The relation of the Ricci scalar to matter is now dependent not only on the metric but also on its derivatives. It can also be seen, that while for GR the empty space solution is always zero curvature, in $f(R)$ theories the curvature is not necessarily zero.

In a maximally symmetric space (*i.e.* homogenous and isotropic) the Ricci scalar is constant. As this case mostly refers to the large scale considerations of trace equation (5.5), the study of the de Sitter solutions often provides a useful tool. For constant curvature and empty space the trace equation becomes

$$f'(R)R - 2f(R) = 0, \tag{5.7}$$

from which the curvature R can be solved algebraically. The constant curvature solution must be found in order for a $f(R)$ candidate to have the correct late-time acceleration behaviour, *i.e.* resembling ΛCDM. Moreover, the solution is necessary of the weak-field limit.

5.1.2 Metric-affine variational principle

In the metric affine formalism the basis is a more general metric-affine spacetime with *a priori* no dependence between the metric and the connection. Both torsion and non-metricity are allowed. The curvature scalar can be written in terms of the connection, its derivatives and the metric (but not its derivatives). Therefore, no second derivatives enter the action. With the arguments, it can now be written

$$S = \frac{1}{2\chi}\int d^4x\sqrt{-g}f\big(g, R(\Gamma, \partial\Gamma)\big) + S_M(g, \Gamma, \psi). \tag{5.8}$$

As the matter part is *a priori* coupled to the connection, the matter is not solely characterized by the energy-momentum tensor $T_{\mu\nu}$. For this purpose the *hypermomentum tensor* must be introduced [60, 72]

$$\Delta_\lambda{}^{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta\Gamma^\lambda_{\mu\nu}}. \tag{5.9}$$

The physical meaning of the hypermomentum has not been studied nearly as extensively as its better-known cousin, the energy-momentum tensor. However, it has been proposed that through hypermomentum the coupling between some matter fields and gravitation is more natural [59]. As the hypermomentum carries some of the characteristics of matter, this changes the role of the energy-momentum tensor $T_{\mu\nu}$ as well. *A priori* the divergence of the energy-momentum tensor in metric-affine $f(R)$ gravity does not vanish.

In the last section, the contribution of the derivatives of the metric to the energy-momentum tensor was discussed in the metric formalism. Now, covariant derivatives do not produce derivatives of the metric, only the connection. In this sense, this contribution is broken off the energy-momentum tensor and into hypermomentum. Therefore, the new energy-momentum tensor is denoted as $\hat{T}_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\partial\sqrt{-g}\mathcal{L}_m}{\partial g^{\mu\nu}}$. An example of this kind of matter is a Dirac field.

While the generality of the spacetime and the connection is desirable in a theoretical, and perhaps aesthetical sense, the total generality of the connection poses problem. The field equations set the connection, but due to transformation invariances of the gravity section of the action this is problematic. Namely, projective transformations of the connection

$$\Gamma^{\lambda}_{\mu\nu} \to \Gamma^{\lambda}_{\mu\nu} + \delta^{\lambda}_{\mu}\xi_{\nu}, \tag{5.10}$$

where ξ_ν is an arbitrary covariant vector field, are unconstrained. As the curvature scalar is constructed with the connection, it turns out the Ricci curvature scalar and therefore the gravitational action is invariant under projective transformations

$$R_{\mu\nu} \to R_{\mu\nu} - \partial_\mu\xi_\nu + \partial_\nu\xi_\mu. \tag{5.11}$$

The same cannot be said about the matter section which would lead to inconsistent field equations. This invariance can be broken with non-symmetric metrics and some other ways, but these are outside the scope of this thesis [62]. The path of least resistance is to set conditions on the connection in a way, that breaks the invariance while retaining as much generality as possible. It can be seen from the transformation (5.10) that four degrees of freedom need to be fixed as there are that many components of ξ_μ. The situation is much alike the gauge invariance of electromagnetism, in which the invariance is broken by fixing the gauge. To preserve as much generality as possible, and looking at (5.11) for hints on where to begin, the natural choice is to constrain the non-symmetric part of the connection, *i.e* set a constraint on the torsion tensor. Therefore, the contraction of the torsion tensor is set [78]

$${S_{\lambda\mu}}^{\lambda} \equiv S_\mu = 0. \tag{5.12}$$

Adding a Lagrange multiplier term in the action (5.8) includes this constraint

$$S = \frac{1}{2\chi}\int d^4x\sqrt{-g}f\big(g, R(\Gamma,\partial\Gamma)\big) + \int d^4x\sqrt{-g}B^\mu S_\mu + S_M(g,\Gamma,\psi), \tag{5.13}$$

where B^μ is the Lagrange multiplier field. The field equations can now be found to be

$$
\begin{aligned}
f'(R)R_{(\mu\nu)} - \frac{1}{2}f(R)g_{\mu\nu} =& \chi\hat{T}_{\mu\nu}, \qquad (5.14)\\
\chi\sqrt{-g}\Big(\Delta_\lambda{}^{\mu\nu} - \frac{2}{3}\Delta_\sigma{}^{\sigma[\nu}\delta_\lambda^{\mu]}\Big) =& 2\sqrt{-g}f'(R)g^{\mu\sigma}S_{\sigma\lambda}{}^{\nu} -\\
&-\nabla_\lambda\Big(\sqrt{-g}f'(R)g^{\mu\nu}\Big) +\\
&+\nabla_\sigma\Big(\sqrt{-g}f'(R)g^{\mu\sigma}\Big)\delta_\lambda^\nu, \qquad (5.15)\\
S_\mu =& 0. \qquad (5.16)
\end{aligned}
$$

Manipulating these equations leads to two findings, the symmetric part of the hypermomentum is related to the non-metricity and the anti-symmetric part is related to the torsion tensor [62]. This translates to torsion being present in the Universe only in the presence of matter fields coupled to the connection. An example of a matter field with a non-zero hypermomentum would be Dirac fields related to fermions.

In the case of vanishing hypermomentum, the field equations simplify a great deal as the second set of equations enforce the Levi-Civita connection. The details can be found in [62], but eventually manipulations reveal that

$$
\nabla_\lambda\Big(\sqrt{-g}f'(R)g^{\mu\nu}\Big) = 0 \qquad (5.17)
$$

$$
S_{\sigma\lambda}{}^{\nu} = 0, \qquad (5.18)
$$

which will be revisited in the next section. The first line of equations (5.14) then becomes the familiar equations of the metric case (5.4). For an empty space the resulting equations of motion would be the same as for GR with Λ.

These results are in line with the attached paper [67] where the connection components are found using maximal symmetry. In absence of matter the degrees of freedom reduce to two and enforcing (5.12) or non-metricity in the spatial parts of the connection reduces the the connection to a Levi-Civita one, just like for the Einstein Hilbert-Lagrangian. There remains only a spurious degree of freedom.

The matter action includes derivatives of the included matter fields. As the general covariance of the field equations is desired, the derivatives in the action should be covariant ones. This leads to the connection explicitly coupling to most matter fields. There are a few examples of matter fields for which the hypermomentum would naturally vanish. For scalar fields the covariant derivative reduces to a partial one and is therefore not coupled to the connection. Also, for the electromagnetic field a covariant action can be written without coupling to the connection.

5.1.3 Palatini variational principle

The Palatini formalism is a slight generalization of the metric formalism. In a way, the Palatini formalism could be said to be half-way between these two formalisms (see *e.g.* [79] and references therein). Like in the metric-affine formalism, the connection is assumed *a priori* independent of the metric. Despite the name, it was originally developed by Einstein [80] and was at first thought to be an equivalent way to derive the Einstein field equations. At the advent of $f(R)$ gravity in the works of Buchdahl, the Palatini formalism was also used [8, 9, 10]. Sometimes the approach is called the Palatini device.

In the Palatini formalism the matter part of the action is assumed to be independent of the connection, unlike in the metric-affine formalism. As the matter is uncoupled to the connection, the covariant derivative would be defined by another connection, the Levi-Civita one. Therefore, there is only a minimal coupling to the metric and nothing else. The action is of the form

$$S = \frac{1}{2\chi} \int d^4x \sqrt{-g} f\big(g, R(\Gamma, \partial\Gamma)\big) + S_M(g, \partial g, \psi). \tag{5.19}$$

Since the dynamics of test particles in Palatini formalism is ultimately described only by the metric, it can be argued that the approach is basically metric formalism with and added field. Since the covariant derivative is defined by the Levi-Civita connection, the standard results like the conservation of the energy momentum tensor hold in Palatini formalism, unlike in the metric-affine approach.

The name Palatini has also stuck to a formula useful in deriving the field equations with the independent connection. This is the Palatini formula for variation of the Ricci tensor

$$\delta R_{\mu\nu} = \nabla_\lambda \delta\Gamma^\lambda_{\mu\nu} - \nabla_\nu \delta\Gamma^\lambda_{\mu\lambda}. \tag{5.20}$$

The equations of motion are found in a similar manner to the metric-affine formalism, though without the need for the Langrange multiplier. After some manipulation the equations are found to be

$$f'(R)R_{(\mu\nu)} - \frac{1}{2}f(R)g_{\mu\nu} = \chi T_{\mu\nu}, \tag{5.21}$$

$$\nabla_\lambda\left(\sqrt{-g}f'(R)g^{\mu\nu}\right) = 0. \tag{5.22}$$

One should notice, that the energy-momentum tensor here is the same as in the metric formalism. While the first set of equations has stayed the mostly same in the last two introduced formalisms, the second one merits more thought. In the absence of hypermomentum, this set of equations was reached also in the metric-affine formalism. For the Einstein-Hilbert action this tells the covariant derivative of the metric density vanishes,

$$\nabla_\lambda\left(\sqrt{-g}g^{\mu\nu}\right) = 0. \tag{5.23}$$

As can be found in most textbooks (*e.g.* [68]), this is one the possible definitions of the Levi-Civita connection. After the brief study of the metric-affine formalism, this comes as no surprise as the hypermomentum was not included. Even so, for more general forms of $f(R)$ the equations found through metric and Palatini formalisms are not equal.

However, the form of the equations hints that it is only a conformal transform away from the definition of the Levi-Civita connection [81]. By creating a conformally transformed metric

$$\bar{g}_{\mu\nu} \equiv f'(R)g_{\mu\nu}, \tag{5.24}$$

and while the same holds for tensor densities

$$\sqrt{-\bar{g}}\bar{g}_{\mu\nu} = \sqrt{-g}f'(R)g_{\mu\nu}, \tag{5.25}$$

the connection of the Palatini formalism turns out to be a Levi-Civita connection of the transformed metric $\bar{g}$. This further implies that connection can be eliminated and written in terms of $g_{\mu\nu}$ and R.

The equations (5.21) which are shared by the Palatini and metric-affine formalisms differ from the metric equations (5.4) in the lack of the derivative terms. Therefore, for empty space the result would be similar to GR with

Λ. This is revealed by the trace equation

$$f'(R)R - 2f(R) = \chi T. \tag{5.26}$$

This forces the curvature to be an algebraic function of T. Especially, for empty space, the curvature is constant R_0 and allows for rearranging the equations of motion into the standard Einstein equations plus $\Lambda = R_0/4$. An interesting special case is $f(R) = R^2$ for which the trace equation is fulfilled trivially. This case is revisited later in this chapter.

In [81] the different solutions of the trace equation are analyzed. The most important result for the purposes of this thesis, is the statement that the equation must have solutions or the equations of motion are not consistent.

5.2 On surface terms

An overlooked fact about the derivation of Einstein equations (3.1) from (5.1) is not entirely straightforward. The problematic part is the often omitted surface term [82, 83]

$$\delta S_{EH} = \frac{1}{2\chi}\Big(\int_V d^4x\sqrt{-g}G_{\mu\nu}\delta g^{\mu\nu} - 2\int_{\partial V} d^3x\sqrt{|h|}\delta K\Big), \tag{5.27}$$

where K is the extrinsic curvature $K = K^a_a = h^{ab}h^c_a\nabla_b n_c$ of the surface ∂V and $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ is the induced metric on the surface [84, 55]. The form of the surface term depends on the foliation, which is further discussed in chapters 4 and 7. Variations of the field do vanish on the surface, so the metric $g_{\mu\nu}$ can be fixed to the boundary. This does not lead to the vanishing of the second term, however. This would require fixing the first derivatives of the metric as well. The existence of this problematic surface term is often overlooked as the term does not affect the differential order of the field equations [85].

The problem can be rectified by modifying the original actions. While the modifications are to remove the surface term, the covariance of the original action must be maintained. As can be seen from (5.27), the surface term is the total variation of a surface action. Therefore, the boundary problem can be bypassed with the action [83]

$$S_{EHB} = \frac{1}{2\chi}\int d^4x\sqrt{-g}R + \frac{1}{\chi}\int_{\partial V} d^3x\sqrt{|h|}K. \tag{5.28}$$

Even though this modification does not have an impact on the Einstein equations, and therefore, the classical dynamics, there are consequences [82, 83, 86]. Its effects are visible in the black hole entropy and the Hamiltonian formulation. The surface terms in the Hamiltonian formalism are examined in the attached paper [87] and in chapter 7.

The boundary problem is not absent in $f(R)$ gravity. Like the Einstein-Hilbert action, equation (5.2) yields a surface term during the variation [84, 72, 85] and the term does not vanish by simply fixing the metric on the boundary. The same remedy as in the Einstein-Hilbert case is not available as the surface terms do not necessarily sum up to a total variation in this case.

There is a major difference however, $f(R)$ theories are by nature and middle name, higher order theories. Therefore, there are more degrees of freedom that can be fixed on the surface, thus eliminating the surface term. The resulting equations of motion are unaffected by this fixing and the classical formulation remains the same. If one would consider canonical quantization, this fixing would have to be more rigorously treated.

The equations of motion (5.4) reveal also that for $f(R)$ gravity empty space with $T = 0$ does not necessarily lead to $R = 0$. As the degrees of freedom increase, the admitted solutions are more numerous. This in turn disqualifies some tools that were available in GR.

5.3 On Birkhoff theorem

The *Birkhoff theorem* states that for GR the Schwarzschild solution is the unique solution with spherical symmetry, to the equations of motion in absence of matter. In a more mathematical way this it can be stated that a spherically symmetric vacuum space-time admits the fourth Killing vector, which is orthogonal to the hypersurfaces of the spheres [88] and time-like at infinity. A spherically symmetric manifold has three Killing vectors and can be foliated into spheres [68]. In this case the metric is called *static*.

As discussed in [75] the Birkhoff theorem does not generally hold for higher order theories as there may be other spherically symmetric vacuum solutions than the Schwarzschild solution [89]. However, in some cases the Birkhoff theorem does hold even for $f(R)$ theories.

In [89] the $f(R)$ theories are studied using scalar-tensor theory equivalence, which is covered later in section 5.5. The treatment includes writing the $f(R)$ contribution as an effective energy-momentum tensor, which is discussed later in section 6.1. It is found that the Birkhoff theorem is cannot be used if R is not constant. Of course, in this case the solutions would reduce to those of GR with cosmological constant.

In order to have the de Sitter solution for empty space with $T = 0$, the trace equation must satisfy

$$f'(R)R - 2f(R) = 0, \tag{5.29}$$

which further leads to a constant curvature solution. When the model in question is chosen by setting the function f, this turns into an algebraic equation for R. The constant curvature scalar translates to a curvature tensor $R_{\mu\nu} = Cg_{\mu\nu}$, with C a real constant. Clearly, this resembles closely the situation of GR with Λ as mentioned above. The sign of the constant differentiates between de Sitter and anti de Sitter solutions. In this scenario the Birkhoff theorem is valid.

Even though generally the Ricci scalar would have to be time-independent for the Birkhoff theorem, it is valid in the weak field limit with small velocities [73]. This is the case of Solar System examinations and systems of similar mass, like small dust clouds. The theorem is found to hold for approximations up to the second order in the background metric.

5.4 Viability considerations

In order to be considered a viable $f(R)$ theory must pass several theoretical criteria besides the experimental constraints. The constraints on $f(R)$ are more rigorously analyzed in chapter 6, but here we look into the basic requirements to be considered on constructing a viable model. As stated before, GR with Λ agrees extremely well with observations, so the viable models must retain many of the characteristics of ΛCDM. Here we examine the three minimal criteria for a viable $f(R)$ stated by Faraoni [90].

One of the main motivations of $f(R)$ theories is reconciling the dark energy problem, *e.g.* producing the late time acceleration. Besides the acceleration a viable theory should reproduce the other widely agreed phases and dynamics of the Universe. Namely it should reproduce the correct dynamics of inflation, radiation dominated era and matter dominated era. The second is constrained by baryogenesis [91, 92, 93] and nucleosynthesis [94, 95, 96]. The last is in turn constrained by the formation of astrophysical structures [97, 91]. The late time acceleration is described by the CMB, supernova and large-scale structure observations as stated previously in section 3.1. See [98] on reconstruction of modified gravity based on the observed dynamics and the expansion history.

As the Newton mechanics have proven themselves time and again, a $f(R)$ candidate must have the correct Newtonian and post-Newtonian limits [99, 100, 101, 102, 103, 104, 105]. A general formalism for treating all modified gravity in the weak-field limit is still lacking, which is problematic in comparing the constraints [106, 107]. The effects of $f(R)$ in this scale are not straightforward, but there are some general requirements. Another way to put this criterion is to demand the perturbations of the metric to have the correct behaviour.

Formally, much of these conditions can be summarized as the limits [20]

$$\lim_{R\to 0} f(R) - R = 0, \tag{5.30a}$$

$$\lim_{R\to \infty} f(R) - R = \Lambda, \tag{5.30b}$$

and the conditions for Newtonian limits for curvatures exceeding the present day cosmological background value, $R > \tilde{R}$

$$|f(R) - R| \ll R, \tag{5.31}$$

$$|f'(R) - 1| \ll 1, \tag{5.32}$$

$$Rf''(R) \ll 1. \tag{5.33}$$

A viable theory candidate must be stable both at the classical and quantum level. Perhaps the most notable stability criterion is the Dolgov-Kawasaki stability criterion [108, 90, 109], which concerns the stability of the matter. The instabilities of de Sitter space in the gravity sector have been studied in [110, 111, 112, 113, 114]. There are other stability criteria and other sources of instabilities as well, such as the ghosts [115, 116, 117, 118]. The one-loop quantization of $f(R)$ and the stability issues related to black holes are studied in [35].

Any candidates for viable $f(R)$ theories must admit a well-posed Cauchy problem in order to make physical predictions. As for GR this condition is fulfilled for most forms of matter [55] but the situation changes for $f(R)$ theories. This is due to the extra degrees of freedom which create auxiliary scalar fields, leading to a changed initial value problem. Most studies into this matter have been through the equivalence with scalar tensor theories (see [119, 120] and references therein). The chosen frame (Einstein or Jordan) does not affect the outcome whether the Cauchy problem is well-posed or not.

It should be reminded that there are two related but different concepts here, whether a problem is *well-posed* and/or *well-formulated* [121]. If the initial data on a Cauchy surface [55] produces uniquely determined dynamical evolution, the problem is well-formulated. Of the two conditions, being well-formulated is more easily met. In order to be well-posed, further properties are needed. Related to the perturbations discussed above, initial small perturbations must remain small and the causal structure must be preserved. With these further properties a problem is well-posed.

The scalar-tensor theory considerations have revealed the Cauchy problem to be well-posed [119, 120], except for the cases of $\omega = 0$ and $\omega = -3/2$. Sadly, these are exactly the cases of $f(R)$ in metric ($\omega = 0$) and in Palatini ($\omega = -3/2$) formalism. This issue is addressed in [122, 123] and it is found that metric $f(R)$ theories do have well-posed Cauchy problem. This is generally not the case for Palatini $f(R)$ theories or metric-affine $f(R)$ theories [124].

5.5 Scalar-tensor theory equivalence

While the main interest in $f(R)$ gravity rose in the early 21st century, it was discovered much earlier that there is a link to scalar tensor theories. For the first time equivalence was shown for R^2 theories [125, 126] and later expanded to general functions $f(R)$ [127, 128, 129] and even more general $f(R, \Box^k R)$ as well.

The equivalence of $f(R)$ theories with scalar tensor or Brans-Dicke theories can be a very useful tool. As there is no unique way to describe the associated fields in classical field theories, the choice can be taken to simplify the treatment as much as possible. In a way this is comparable to the choice of coordinate systems in classical mechanics. As long as the dynamics originating from the field equations remain the same, the description is valid.

As the scalar tensor theories have received much attention over the years, the link provides much understanding of the $f(R)$ models as well. While using the equivalence, one should keep in mind the effects of the conformal transformations involved. Especially, this has an impact on the relevant frame, Jordan or Einstein frame [4, 130]. The characteristic difference between the two frames is whether the scalar field is minimally (Einstein frame) or non-minimally (Jordan frame) coupled to the metric. Both frames have some issues. Namely, in the Jordan frame the energy density of the scalar field is not bounded from below, while the the Einstein frame has problems with equivalence principle (see [4] for discussion).

The standard $f(R)$ action (5.2) can be rewritten with a new field ϕ without altering the dynamical properties

$$S = \frac{1}{2\chi}\int d^4x\sqrt{-g}\Big[f(\phi) + f'(\phi)(R-\phi)\Big] + S_M(g_{\mu\nu},\psi). \tag{5.34}$$

This formulation is in the Jordan frame. The Einstein frame could be reached with a conformal transformation [4, 130]. It should be noted, that this change of frames affects also the matter section of the action.

Upon variation with respect to ϕ, we find $\phi = R$ if $f''(\phi) \neq 0$. This is also the condition for the equivalence. One can check that inserting this to the action above produces the standard $f(R)$ action (5.2). To write the action in a form with a scalar potential, we further redefine $\varphi = f'(\phi)$ and define the potential $V(\varphi) = \phi(\varphi)f'\big(\phi(\varphi)\big) - f(\phi(\varphi))$. Now the action in Jordan frame is recast to

$$S = \frac{1}{2\chi}\int d^4x\sqrt{-g}\Big[\varphi R - V(\varphi)\Big] + S_M(g_{\mu\nu},\psi). \tag{5.35}$$

This is the Brans-Dicke theory with $\omega = 0$ [131][1]. The variation of this action with respect to the metric and the auxiliary field yields the equations

[1] The action of Brans-Dicke theories in the Jordan frame is $S_{BD} = \frac{1}{2\chi}\int d^4x\sqrt{-g}\Big[\varphi R - \frac{\omega}{\varphi}g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - V(\varphi)\Big] + S_M(g_{\mu\nu},\psi)$.

of motion

$$\frac{1}{\varphi}\Big(\nabla_\mu\nabla_\nu\varphi - g_{\mu\nu}\Box\varphi - \frac{V(\varphi)}{2}g_{\mu\nu}\Big) + \frac{\chi}{\varphi}T_{\mu\nu} = G_{\mu\nu}, \tag{5.36}$$

$$R\frac{d\varphi}{d\phi} - \frac{dV}{d\phi} = 0. \tag{5.37}$$

5.6 Weak-field limit

The linear perturbations [66] $h_{\mu\nu}$ of the metric can be written

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}, \tag{5.38}$$

$$g^{\mu\nu} = \tilde{g}^{\mu\nu} - h^{\mu\nu}, \tag{5.39}$$

where the tilde denotes background quantities. The most common background to be used is the Minkowski one, $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$ but for the purposes of the attached papers, this assumption cannot be always made. The Ricci tensor and scalar can be expanded around this background

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} + \delta R_{\mu\nu} + \mathcal{O}(h^2), \tag{5.40}$$

$$R = \tilde{R} + \delta R + \mathcal{O}(h^2). \tag{5.41}$$

The exact form of the curvature perturbation in terms of the metric and its perturbations can be calculated with the knowledge that the connection coefficients are Christoffel symbols[2]. The perturbations of the curvature tensor and the connection can be written

$$\delta R_{\mu\nu} = \nabla_\nu(\delta\Gamma^\lambda_{\mu\lambda}) - \nabla_\lambda(\delta\Gamma^\lambda_{\mu\nu}), \tag{5.42}$$

$$\delta\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\Big[\nabla_\nu(\delta g_{\sigma\mu}) + \nabla_\mu(\delta g_{\sigma\nu}) - \nabla_\sigma(\delta g_{\mu\nu})\Big]. \tag{5.43}$$

Combining these the curvature tensor perturbations can be written in terms of the metric and its perturbations

$$\delta R_{\mu\nu} = \frac{1}{2}\Big[\nabla_\nu\big[g^{\lambda\sigma}\big(\nabla_\lambda(\delta g_{\sigma\mu}) + \nabla_\mu(\delta g_{\sigma\lambda}) - \nabla_\sigma(\delta g_{\mu\lambda})\big)\big] - \tag{5.44}$$

$$-\nabla_\lambda\big[g^{\lambda\sigma}\big(\nabla_\nu(\delta g_{\sigma\mu}) + \nabla_\mu(\delta g_{\sigma\nu}) - \nabla_\sigma(\delta g_{\mu\nu})\big)\big]\Big]. \tag{5.45}$$

[2]It should be stressed, that this treatment is in the metric formalism.

Inserting the perturbed metric (5.38) into these equations yields the first order curvature pertubations

$$\delta R_{\mu\nu} = \frac{1}{2}\Big(\nabla_\mu \nabla_\nu h - \nabla_\mu \nabla^\lambda h_{\lambda\nu} - \nabla_\nu \nabla^\lambda h_{\mu\lambda} + \Box h_{\mu\nu}\Big). \tag{5.46}$$

Using this result and the fact that $\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} = \tilde{g}^{\mu\nu}\delta R_{\mu\nu} - h^{\mu\nu}\tilde{R}_{\mu\nu}$ the scalar curvature perturbations are found

$$\delta R = \Box h - \nabla^\mu \nabla^\nu h_{\mu\nu} - \tilde{R}_{\mu\nu} h^{\mu\nu}. \tag{5.47}$$

In the weak-field limit the functions $f(R)$ can be expanded as well

$$f^n(R) \simeq f^n\big(\tilde{R} + \delta R + \mathcal{O}(h^2)\big) \simeq f^n(\tilde{R}) + f^{n+1}(\tilde{R})R\delta + \mathcal{O}(h^2). \tag{5.48}$$

Following this rule the desired order of the derivatives can be reached. As the equations of motion include the term $\Box f'(R)$, derivatives up to the third order might be needed. Again, up to first order in perturbations the equations of motion can be written as

$$f'(\tilde{R})\Big(\delta R_{\mu\nu} + \frac{\delta R}{2}\Big) - f''(\tilde{R})\nabla^2 \delta R + f'(\tilde{R})\tilde{R}_{\mu\nu} - \frac{f(\tilde{R})}{2} g_{\mu\nu} = \chi T_{\mu\nu}, \tag{5.49}$$

$$3f''(\tilde{R})\Big(\nabla^2 - \partial_0^2\Big)\delta R - f'(\tilde{R})\delta R + f'(\tilde{R})\tilde{R} - 2f(\tilde{R}) = \chi T. \tag{5.50}$$

The last two terms on the left in both the main equations and the trace equation bring zeroeth order contributions. However, the background Einstein solution must exist and therefore the standard Einstein equation $\tilde{R}_{\mu\nu} - \tilde{g}_{\mu\nu}\tilde{R}/2 = \chi T_{\mu\nu}$ is can be used to clean up the non-perturbation terms. Another notice is that in order to have a correct de Sitter solution $f'(R)R - 2f(R) = 0$ must hold, which can be used in many occasions as well.

Due to the gauge invariance a suitable gauge choice, *i.e.* fixing the coordinate system can considerably simplify the equations. The usual choice in the weak-field limit considerations is the *harmonic gauge* [66, 68], in which the coordinate functions x^μ satisfy

$$\Box x^\mu = 0, \tag{5.51}$$

i.e. they must be harmonic functions. Writing down the covariant derivatives explicitly, one finds the condition

$$g^{\alpha\beta}\Gamma^{\mu}_{\alpha\beta} = 0. \tag{5.52}$$

This form of the gauge condition is also known as the de Donder gauge. It is worth noticing that these are equivalent forms even if the connection is not the Levi-Civita one, with Christoffel symbols as coefficients. For the purposes here it is convenient to further plug in the Christoffel symbols to find for the case of static background metric and perturbations (5.38)

$$\partial_\mu h^\mu_\lambda = \frac{1}{2}\partial_\lambda h, \tag{5.53}$$

where $h = h^\mu_\nu$. To make the name conventions even more complex, this form is sometimes called Fock gauge, Einstein gauge, Hilbert gauge or Lorentz gauge [68]. In the first order with static background the condition above can be written with covariant derivatives as well. Another useful form is to examine the trace-reversed perturbations $\bar{h}_{\mu\nu}$, for which

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}h^{\mu\nu}. \tag{5.54}$$

For GR in vacuum, the equations of motion reduce to the simple form of

$$\Box\bar{h}_{\mu\nu} = 0. \tag{5.55}$$

The 00 component of the perturbed metric is related to the familiar Newton gravity potential. In a standard GR case of a star, this one perturbation would be enough to describe the metric. To examine the dynamics of a collapsing dust cloud in $f(R)$ gravity the system cannot be taken as time-independent and another perturbation must be added to the spatial components. Assuming polar coordinates, $x^\mu = (t, r, \theta, \phi) = (t, \mathbf{x})$ as is done in [75], the metric perturbations become

$$h_{00} = -2\phi(t, \mathbf{x}), \tag{5.56}$$

$$h_{11} = 2\Psi(t, \mathbf{x}), \tag{5.57}$$

$$h_{22} = 2\Psi(t, \mathbf{x})r^2, \tag{5.58}$$

$$h_{33} = 2\Phi(t, \mathbf{x})r^2 \sin\theta. \tag{5.59}$$

As in all physical cases, depending on the level of detail the background is always embedded in a larger background. If the background metric is taken to describe the (locally constant) cosmic de Sitter background, the effects of the local neighborhood, Solar System and the Galaxy, must be taken into account in the perturbations. However, if the dynamics of *e.g.* a dust cloud are examined, the galactic background is, again, locally constant and the total perturbation of metric can be written as

$$\phi(t, \mathbf{x}) = \phi_0 + \Phi(t, \mathbf{x}), \tag{5.60}$$

where the first term includes the local neighborhood corrections and the latter term describes the examined object. This reasoning is not valid for the case of large interstellar clouds and galactic dynamic considerations as the term ϕ_0 would not be constant.

Inserting these metric perturbations to the definition of the Ricci scalar and tensor, one can calculate the perturbations of curvature in the collapsing dust cloud scenario to be

$$\delta R = 6\ddot{\Psi} - 2\nabla^2\Phi - 4\nabla^2\Psi, \tag{5.61}$$

$$\delta R_{00} = \nabla^2\Phi - 3\ddot{\Psi}, \tag{5.62}$$

$$\delta R_{ab} = -\nabla^2\Psi + \ddot{\Psi}. \tag{5.63}$$

5.7 Examples of $f(R)$ models

In this section we introduce some $f(R)$ models that have received considerable interest in the past years. While not all of them are considered viable anymore, they possess some characteristics in which they prove valuable in the context of this thesis.

The first model to receive the renewed interest in $f(R)$ theories in the early 2000s is the Carroll-Duvvuri-Trodden-Turner (CDTT) model [132]. In the paper, the authors present that an $f(R)$ modification can cause the accelerated expansion of the Universe. This is achieved by adding a term inversely proportional to the curvature

$$f(R) = R - \frac{\mu^4}{R}, \tag{5.64}$$

where the parameter μ has the unit of mass. The equations of motion for this type models are found by plugging (5.64) into the equations of motion

(5.4) for

$$\Big(1+\frac{\mu^4}{R^2}\Big)R_{\mu\nu}-\frac{1}{2}\Big(1-\frac{\mu^4}{R^2}\Big)Rg_{\mu\nu}+\mu^4\big(g_{\mu\nu}\nabla^\lambda\nabla_\lambda-\nabla_{(\mu}\nabla_{\mu)}\big)R^{-2}=\chi T_{\mu\nu}. \tag{5.65}$$

In vacuum the trace equation states that there is a constant curvature solution, which in this case is easily found to be $R=\pm\sqrt{3}\mu^2$, which correspond to de Sitter and anti de Sitter spaces. This notion and the resulting effective equation of state parameter lead to the rise of $f(R)$ theories as a candidate for solving dark energy problem.

As this model was the first in the new batch of $f(R)$ theories, it is one of the most well-studied. Actually, it was soon discovered that this model is not viable. One can readily check, that this model fails to meet the criteria (5.30a). For the purposes of this thesis, it serves as an excellent example, how an initially interesting model can turn out to be unstable and/or unphysical.

Adding a squared term in the action is one the oldest $f(R)$ type modifications (see *e.g.* [129] and [9]). It received interest even before the new coming of $f(R)$ theories. The basic form is simply

$$f(R)=R+\frac{R^2}{6M^2}, \tag{5.66}$$

where $M\in\mathbb{R}$.

One of the features of R^2 is that it vanishes trivially in the trace equation in vacuum $f'(R)-2f(R)=0$. Much of the interest in these types of actions is due to its successful reproduction of inflation [133].

In most cases adding a term creates more problems than it solves. The most interesting characteristic of the R^2 type of actions is the opposite behaviour, the stabilizing effect. It has been found that adding a squared term into a problematic action can stabilize it. These features are examined in detail in [134]. The main reason for curing instabilities is the ability of the quadratic term to alter the behaviour $\lim_{R\to\infty}f''(R)=0$.

The Hu-Sawicki model [135] is one of the most popular $f(R)$ models due to viability and relatively simple form. It is also one of the models most relevant in the context of this thesis as it is tested in the attached papers [75] and [6]. The model is specifically constructed to survive the Solar System constraints.

The Hu-Sawicki $f(R)$ model is based on an action of the form [135]

$$f(R) = R - \lambda R_c \frac{(R/R_c)^{2n}}{(R/R_c)^{2n} + 1}, \tag{5.67}$$

where the constant parameters $n \in \mathbb{N}$ and $\lambda, R_c \in \mathbb{R}_+$. The parameter R_c, the critical curvature, is of the order of present day curvature. The integer n affects how closely the model mimics the ΛCDM, with higher n having closer resemblance.

Even though the Hu-Sawicki model effectively mimics ΛCDM behaviour, there is no explicit cosmological constant. However, the expansion at high curvature reveals the asymptotic behaviour

$$f(R) \simeq -\lambda R_c + \lambda R_c \left(\frac{R_c}{R}\right)^{2n}. \tag{5.68}$$

Therefore, the limiting the behaviour is that of a cosmological constant. It is also interesting to notice, that this expansion reminds greatly the $1/R$ theories discussed earlier. In the original paper [135] it is shown that with $n = 1$ resembles the behaviour resembles CDTT plus cosmological constant at high curvatures, excluding the problems.

The value of n is bounded from below as has been shown in [136], $n > 0.905$. In the same paper it is shown that low values of n set relatively higher lower bounds for the constant λ.

There are many other models, some of which have already been discarded due to observational or stability issues like the $1/R$ model while others remain viable. These still viable models include the Appleby-Battye model [137], the Starobinsky model [138] and many others such as [139, 140].

Besides all the different models there numerous different approaches, such as the different variational principles and frames mentioned in this chapter. Therefore, the more general constraints for viable models are found, the better.

As an example of the effects of different formalisms, the effects of R^{2n} in the Palatini formalism are studied in [72] and [141]. In this context it can be noted that the instability behaviour is different within the Palatini formalism. For example the Dolgov-Kawasaki instability does not manifest.

5.8 Symmetry considerations

The cosmological principle assumes a homogeneous and isotropic Universe. Theoretically it is based on the much older Copernican principle, according to which states that we are typical observers and therefore other observers in distant galaxies should have similar observations to ours. There are observations backing the homogeneous and isotropic Universe [142], especially the cosmic microwave background (CMB) [143, 144]. The issue is not settled, however [145].

As the homogeneity and isotropy set several symmetries and restrictions on the form of the metric, they are a useful tool in simplifying equations. This is especially true in the case of metric-affine gravity, as there are more degrees of freedom than in the metric case. As is found in the attached paper [67], the degrees of freedom in the connection in metric-affine $f(R)$ can be effectively reduced.

It is important to keep in mind, that while the cosmological principle is useful in large-scale considerations, the symmetry implications do not hold for galactic or Solar System level considerations. There are also other reasons, for which the usefulness of these symmetries must be considered.

While the CMB has been one of the strongest supporters of the cosmological principle, it may also prove to be its downfall. The observed isotropy was first found to be disturbed in temperature dipoles at the level of about per mille, shortly after the initial discovery of CMB [146]. However, this discrepancy was explained by the proper motion of the Solar System [147].

Later satellite missions, COBE [16], WMAP [148] and Planck [15] have deepened the understanding of the CMB.

5.9 Perfect fluid matter

The energy-momentum tensor (1.7) generally describes the flux of the four-momentum across a constant surface ∂V. In cosmology, the matter investigated is most often a fluid, which is described through thermodynamics, *i.e.* with macroscopic quantities like temperature and pressure [68].

In the context of this thesis, the main interest lies in *perfect fluid* matter. In a rest frame a perfect fluid looks isotropic [66]. It is a continuous distribution and its energy-momentum tensor is of the form

$$T_{\mu\nu} = \rho u_\mu u_\nu + p(g_{\mu\nu} + u_\mu u_\nu), \tag{5.69}$$

where u_μ is the four-velocity, $u^\mu = \frac{dx^\mu}{d\tau}$, a unit time-like vector. In rest frame this becomes $u_\mu = (\sqrt{g_{00}}, 0, 0, 0)$. The energy content is described by the energy density $\rho(t)$ and the pressure $p(t)$. A related concept is the energy-momentum four-vector

$$P^\mu = mu^\mu. \tag{5.70}$$

The perfect cosmological fluid can take many forms. It is used to describe interstellar matter and galaxies alike. By taking $p = 0$, the perfect fluid energy-momentum tensor describes dust fields. As there is no pressure, the particles are at rest, with respect to other particles. As galaxies are far away from each other, this seems a fitting description for the "galaxy fluid"and interstellar dust clouds. The cold dark matter of the Universe follows the rules of perfect dust.

The perfect fluid matter has many advantages, of which the diagonality is not least. It is also symmetric and conserved, *i.e.* if no external forces are present, its divergence vanishes

$$\nabla_\mu T^{\mu\nu} = 0. \tag{5.71}$$

5.10 Gravitational waves

One possible interesting set of solutions for the weak-field equations of motion (5.55) are the plane waves for which

$$\bar{h}_{\mu\nu} = C_{\mu\nu} e^{ik_\lambda x^\lambda}. \tag{5.72}$$

Here, k_λ is the wave vector. It can be easily seen, that this leads to $k^\lambda k_\lambda = 0$. This translates to gravitational waves propagating along null geodesics, like photons. Thus they have the same speed c [149], when the wavelength of the gravitational waves is small in comparison to the background curvature radius. This is also related to the fact that for GR the graviton mass is strictly zero $m_g = 0$. As discussed in [2], generalizations must be made to allow for massive gravitons.

Gravitational waves can be caused *e.g.* by inspiraling black holes [150]. The gravitational wave experiments also set an upper limit on the graviton mass, while in $f(R)$ gravity there is necessarily a non-zero mass graviton. The $f(R)$ case of gravitational waves is studied in [6].

Observational bounds on the mass of the graviton are based on the Compton wavelength. As for relativistic energies $E^2 = p^2 + m^2$, for small masses, $m \ll E$ the difference between the velocity of a particle and the speed of light is [151]

$$1 - v \simeq \frac{m^2}{2E^2}. \tag{5.73}$$

The energy can be written as $E = \hbar\omega$ and the Compton wavelength can be used instead of the mass via $\lambda_g = 2\pi\hbar/m_g$. The Compton wavelength is a general property of particles. Roughly speaking, it tells, at which scale the relativistic quantum effects would become non-neglible. A zero-mass particle would be infinite but for all massive particles it is finite. In order to write

$$v_g \simeq 1 - \frac{1}{2}\left(\frac{2\pi}{\lambda_g \omega}\right)^2, \tag{5.74}$$

where v_g is the velocity of the photon. Using the definition of phase speed $v_p = \omega/k$, this can be recast into

$$k^2 = -m_g^2. \tag{5.75}$$

The empty space equations, including a phenomenological massive graviton [152, 2], are $(\Box - m_g^2)h_{\mu\nu} = 0$, which lead to the same equation $k^2 = -m_g^2$ as above. Furthermore, $\Box h = m_g^2 h$, for a plane wave $h \sim e^{ik_\lambda x^\lambda}$. There are 6 degrees of freedom, 5 of which are due to spin-2 nature and one is scalar. As discussed in the attached paper [6], the scalar degree of freedom can be caused by *e.g.* the cosmological constant or the $f(R)$ model contribution.

While these are the equations for trivial $f(R)$ cases, the situation becomes more complex with general $f(R)$ functions. This is discussed in detail in the attached paper [6].

Luku 6

Sources of constraints

While the $1/R$ gravity modifications were the first to reignite the spark of interest in $f(R)$ theories, they were also the first to be rigorously ruled out. They were subjected both to constraints, such as the supernovae [153] and total exclusion due to highly too strong gravitational force to diffuse matter sources in locally de Sitter background [154] and not displaying correct matter era [155] (see also [156] for discussion on the criteria for correct matter era).

Since the days of $1/R$ there have been numerous other proposals for viable theories and also ways to rule out the incorrect ones. In the following I briefly summarize some methods to constrain and rule out candidates for viable $f(R)$ theories.

6.1 Theoretical constraints

The equations of motion of a $f(R)$ theory can be written in a form where the $f(R)$ contribution appears as an effective energy-momentum tensor. If equations (5.4) are rearranged to collect the Einstein tensor, the equations become

$$G_{\mu\nu} = \frac{\chi T_{\mu\nu}}{f'(R)} + g_{\mu\nu}\frac{f(R) - Rf'(R)}{2f'(R)} + \frac{\nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R)}{f'(R)}. \qquad (6.1)$$

This reminds greatly of the standard GR equations, with the first term on the right side only modified by $1/f'(R)$ and would therefore reduce to the standard $\chi T_{\mu\nu}$ for the Einstein-Hilbert Lagrangian. In this manner the rest of the terms, the $f(R)$ contribution, serve as effective matter leading to

$$G_{\mu\nu} = \frac{\chi}{f'(R)}\Big(T_{\mu\nu} + T^e_{\mu\nu}\Big). \tag{6.2}$$

The constructed effective energy-momentum tensor is not a physical quantity and thus does not necessarily follow energy conditions. It can be written in the perfect fluid form, however.

By contracting the Einstein tensor one finds the quantity $G^e = G/f'(R)$, which acts as the effective gravitational coupling in $f(R)$ theories [72]. It also sets one of the most simple viability conditions for $f(R)$ with a physically meaningful R

$$f'(R) > 0. \tag{6.3}$$

This condition of positivity is equivalent to demanding a non-ghost graviton [72]. Due to its simplicity, it is one of the first criteria to consider on constructing a viable $f(R)$ theory. It is worth mentioning that some authors may use different signature leading to reversing this condition and the one discussed next.

6.1.1 Dolgov-Kawasaki instability

Originally in the paper by Dolgov and Kawasaki, it was demonstrated that models with the inverse R term experience instability [108]. It has been later proven, that the same instability can be found in numerous models, which fail to meet the criterion $f''(R) > 0$ [90]. The condition can be found rather simply by examining the weak-field limit of the trace equation (5.5). Before taking the limit, the d'Alembertian can be evaluated to write

$$3\Big(f'''(R)\nabla_\mu R \nabla^\mu R + f''(R)\Box R\Big) + f'(R)R - 2f(R) = \chi T, \tag{6.4}$$

into which the expansion of the Ricci scalar (5.41) can be inserted along with the expansion of $f(R)$ (5.48). In this subsection I use the notation $\delta R = R_1$ for clarity. The background curvature $\tilde{R}$ could in principle be solved from the previous equation by inserting $R = \tilde{R}$ into the equation. Since this background fulfills the equation, the background terms (or the zeroeth order) can be removed. Terms of order higher than the first are disregarded as well. The result in the first order is

$$3\Big[f'''(\tilde{R})\Big(\nabla_\mu\tilde{R}\nabla^\mu R_1+\nabla_\mu R_1\nabla^\mu\tilde{R}\Big)+f^{(4)}(\tilde{R})R_1\nabla_\mu\tilde{R}\nabla^\mu\tilde{R}+f''(\tilde{R})\Box R_1+$$
$$+f'''(\tilde{R})R_1\Box\tilde{R}\Big]+f'(\tilde{R})R_1+f''(\tilde{R})R_1\tilde{R}-2f'(\tilde{R})R_1=0. \tag{6.5}$$

As all the arguments are now $\tilde{R}$, for the rest of this subsection the argument $(\tilde{R})$ has been omitted in numbered equations. It is known that the modification to GR must be relatively small, so it is safe to state that $f(R)=R+\epsilon F(R)$, where $\epsilon>0$ is very small. For $f''(\tilde{R})\neq 0$ we can write

$$3\Big[\frac{F'''}{F''}\Big(\nabla_\mu\tilde{R}\nabla^\mu R_1+\nabla_\mu R_1\nabla^\mu\tilde{R}+R_1\Box\tilde{R}\Big)+\frac{F^{(4)}}{F''}R_1\nabla_\mu\tilde{R}\nabla^\mu\tilde{R}+\Box R_1\Big]+$$
$$+\tilde{R}R_1-\frac{1+\epsilon F'}{\epsilon F''}R_1=0. \tag{6.6}$$

By assuming the metric to be nearly Minkowskian, $g_{\mu\nu}=\eta_{\mu\nu}+\delta g_{\mu\nu}$, the derivatives can be written

$$\Box R=-\ddot{R}+\vec{\nabla}^2R, \tag{6.7}$$
$$\nabla_\mu R\nabla^\mu R=-(\dot{R})^2+(\vec{R})^2. \tag{6.8}$$

Substituting these and rearranging terms in (6.6), one finds

$$\ddot{R}_1-\vec{\nabla}^2R_1+\frac{2F'''}{F''}(\dot{\tilde{R}}\dot{R}_1-\vec{\nabla}\tilde{R}\cdot\vec{\nabla}R_1)+m_e^2R_1=0, \tag{6.9}$$

where the effective mass term is defined as

$$m_e^2\equiv=\frac{1}{3\epsilon F''}-\frac{F^{(4)}}{F''}\nabla_\mu\tilde{R}\nabla^\mu\tilde{R}-\frac{F'''}{F''}\Box\tilde{R}-\frac{\tilde{R}}{3}+\frac{F'}{3F''}. \tag{6.10}$$

As ϵ is very small, we notice the first term in the effective mass becomes the leading term. It should be stressed that the ϵ must be small due to the observational agreement with GR. This effectively reduces the equation to that of a driven harmonic oscillator. As is known from mechanics, the sign must be positive in order to avoid growing instability. Therefore, the necessary criterion is $F''(R)=f''(R)>0$. This result holds also for the empty space $T=0$ which is examined in [90].

In the case of growing perturbations the relaxation time of the solution reveals the time scale of the instability. As this instability was originally found for the $1/R$ theories, the time for the instability to emerge was estimated around 10^{-26}s. Effectively this would cause the curvature perturbations to grow fast out of the weak-field range.

Even though this instability was originally presented as an instability in the matter sector of the equations [108], this is rather a question of convenience. In the form presented here, the instability appears in the gravity sector. In a way this is a telling sign of the interconnection of the gravity and matter contribution to dynamics in modified gravity theories.

Due to the simplicity of the condition $f''(R) > 0$, the Dolgov-Kawasaki instability has become perhaps the best known viability criterion for $f(R)$ gravity along with the ghost condition $f'(R) > 0$. It also presents the possiblu dire consequences of adding higher order terms to the Einstein-Hilbert action.

6.1.2 Ostrogradski instability

The *Ostrogradski instability* [157] does not constrain $f(R)$. Rather, it is a proponent of the viability of $f(R)$ models and therefore it is prudent to mention it here. While to original work took place long before the advent of GR, it has deep implications on modified gravity. It can be used to rule out many possible higher order gravity theories besides $f(R)$ gravity [158, 46].

The *Ostrogradski theorem* states that there is a linear instability in Hamiltonians that are associated with Lagrangians of higher than one time derivative, which cannot be eliminated by partial integration. This can be demonstrated rather easily by examining a particle in one dimension, with its position given by $q(t)$. The Lagrangian $L(q, \dot{q}, \ldots, q^N)$ depends on the coordinate $q(t)$ and its N first derivatives. The important assumption is that the Lagrangian depends nondegenerately on q^N. The Euler-Lagrange equation can now be written in the usual form

$$\sum_{i=0}^{N} \Big(-\frac{d}{dt}\Big)^i \frac{\partial L}{\partial q^{(i)}} = 0, \tag{6.11}$$

which now contains derivatives up to q^{2N}. In the canonical phase space there are now $2N$ coordinates. The original choice of Ostrogradski is

$$Q_i \equiv q^{(i-1)}, \tag{6.12}$$

$$P_i \equiv \sum_{j=i}^{N} \Big(-\frac{d}{dt} \Big)^{j-i} \frac{\partial L}{\partial q^{(j)}}. \tag{6.13}$$

Due to non-degeneracy the $q^{(N)}$ can be solved in terms P_N and the canonical coordinates Q_i. This implies the existence of a function $A(Q_1, \ldots, Q_N, P_N)$ for which

$$\frac{\partial L}{\partial Q^{(N)}}\bigg|_{\substack{q^{(i-1)}=Q_i \\ q^{(N)}=A}} = P_N. \tag{6.14}$$

The Hamiltonian can now be written as

$$H = P_1 Q_2 + \cdots + P_{N-1} Q_N + P_N A - L(Q_1, \ldots, Q_N, A). \tag{6.15}$$

This clearly produces the correct evolution equations $\dot{Q}_i = \partial H / \partial P_i$ and $\dot{P}_i = -\partial H / \partial Q_i$, which means the Hamiltonian generates time evolution. As there is no explicit time dependence in the Lagrangian, it is the Noether current, meaning it represents the energy of the system. The problem is that the Hamiltonian is linear in the momenta $P_1, \ldots, P_{N-1}$. They are not bounded from below. This creates the instability.

The reason why $f(R)$ escapes the Ostrogradski instability lies in the assumption of non-degeneracy, which does not hold for $f(R)$. The two tensor indices of the second derivative terms of the curvature scalar are contracted, which leads to only a single component of the metric carrying a higher derivative [158]. A reader can consider the degrees of freedom in the metric as the coordinates q_i. This one component acquires a new higher derivative degree of freedom, the energy of which is the opposite to that of the corresponding lower derivative degree of freedom. This follows the line we just examined. The rescue comes from the lower derivative degree of freedom being the Newtonian potential. The potential is of negative energy, but it is fixed by the other metric and matter fields through the constraint on g_{00}. This leaves only the instability of gravitational collapse.

This is the case for all $f(R)$. The same cannot be said of Lagrangians involving derivatives of the curvature scalar. This would lead to higher derivative degrees of freedom, but there would be no additional constraints to come to the rescue. The same would happen for other possible contractions of the Riemann tensor [158]. This leaves $f(R)$ as the stable option among higher order gravity candidates.

6.1.3 Other singularities and constraints

In [159] it was found that with many $f(R)$ models a singularity may manifest even at classical levels. These are caused by non-linear effects in the trace equation. At a finite field value, there is a curvature singularity. This appears with the scalar field at $\phi = 0$, *i.e.* at Einstein theory limit. It requires fine-tuning to suppress the oscillation of the field perturbations by the background field. However, there are ways to cure this type of singularity, such as adding a R^2 term [134].

An investigation into different types of finite-time singularities, which can manifest in $f(R)$ gravity is found in [160]. It is also demonstrated that near the singularity, as curvature becomes exceedingly large, quantum effects become relevant and even dominant. This could contain the situation from becoming singular.

6.1.4 Jeans instability

The *Jeans instability* is the cause of collapse in interstellar dust and gas clouds, which become protostars and later on, stars. In the context of this thesis, I use the term dust cloud, even though the main ingredients are technically gasses. This choice is due to the tendency to call all baryonic matter dust in cosmology.

The instability was originally studied by Sir James Jeans [161], after which the phenomenon is named. The original work relied on non-relativistic Newtonian gravitation, but later studies have extended to GR and even modified gravity [75, 162, 163, 164]. A detailed presentation of the Jeans instability can be found in [23]

Simply put, in order to prevent collapse, a dust cloud must be in hydrostatic equilibrium; the pull of the gravity and the push of the internal pressure must be in balance. In the more involved case of modified gravity, the background space itself affects the system. The most important characteristics of a dust cloud affecting the stability are temperature and density.

The instability manifests in small perturbations of the mass distribution growing to become significant. The perturbations appear in the dust clouds due to outside phenomena, such as supernovae, or as initial anisotropies. Jeans analysis produces a limit for the object in question, the *Jeans mass*. Above this mass limit the cloud collapses and under it, dissipates.

I present here the basics of Jeans analysis as a background material for the attached paper [75]. The object in question is assumed to be described as a collisionless thermodynamic system. As the interstellar dust clouds are cold and thin, this is a reasonable assumption [165, 166]. The Boltzmann equation describes thermodynamic systems, which are not necessarily in equilibrium, in this case non-interacting dust:

$$\frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial t} + (\mathbf{v} \cdot \vec{\nabla}_x) f(\mathbf{x}, \mathbf{v}, t) - (\vec{\nabla}\Phi \cdot \vec{\nabla}_v) f(\mathbf{x}, \mathbf{v}, t) = 0, \tag{6.16}$$

where $\mathbf{v}$ is the three-velocity and $\mathbf{x}$ the three-dimensional coordinates. The Φ is the gravitational potential.

A self-gravitating system of particles in equilibrium is described by a time-independent distribution function $\tilde{f}(\mathbf{x}, \mathbf{v})$ and a potential $\tilde{\Phi}(\mathbf{x})$. The first is the solution of the Boltzmann equation (6.16). The second is the solution of the collisionless Poisson equation, which describes the gravitational field due to mass distribution

$$\vec{\nabla}^2 \Phi(\mathbf{x}, t) = 4\pi G \int f(\mathbf{x}, \mathbf{v}, t) d\mathbf{v}. \tag{6.17}$$

While the background can be seen as an equilibrium system, the interest is in introducing perturbations into the system, which present the beginnings of star formation

$$f(\mathbf{x}, \mathbf{v}, t) = \tilde{f}(\mathbf{x}, \mathbf{v}) + \epsilon f_1(\mathbf{x}, \mathbf{v}, t), \tag{6.18}$$

$$\Phi(\mathbf{x}, t) = \tilde{\Phi}(\mathbf{x}) + \epsilon \Phi_1(\mathbf{x}, t), \tag{6.19}$$

with $\epsilon \ll 1$. As the solutions to the background equations, denoted by tilde, are time-independent, all the time dependence is in the perturbation term. These equations can be simplified in our case. As the equilibrium system is homogenous and static, we can set $\tilde{f}(\mathbf{x}, \mathbf{v}) = \tilde{f}(\mathbf{v})$ and remove the so-called Jeans swindle $\tilde{\Phi}$ [167]. The equations (6.16) and (6.17) become trivial in the zeroeth order and in the first order we find

$$\frac{\partial f_1(\mathbf{x},\mathbf{v},t)}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1(\mathbf{x},\mathbf{v},t)}{\partial \mathbf{x}} - \vec{\nabla}\Phi_1(\mathbf{x},t) \cdot \frac{\tilde{\partial} f(\mathbf{v})}{\partial \mathbf{v}} = 0, \tag{6.20}$$

$$\vec{\nabla}^2\Phi_1(\mathbf{x},t) = 4\pi G \int f_1(\mathbf{x},\mathbf{v},t)d\mathbf{v}. \tag{6.21}$$

In order to find the stable and unstable modes, the equations must be written in Fourier space. Quantities in Fourier space can be written as

$$\Phi(\mathbf{k},\omega) = \int_{\infty}^{\infty}\int_{\infty}^{\infty} e^{2\pi i\mathbf{k}\cdot\mathbf{x}} e^{2\pi i\omega t}\Phi(\mathbf{x},t)d\mathbf{x}\ dt. \tag{6.22}$$

The transformed equations (6.20) and (6.21) read as

$$-i\omega f_1 + \mathbf{v}\cdot(i\mathbf{k}f_1) - (i\mathbf{k}\Phi_1)\cdot\frac{\partial f_0}{\partial \mathbf{v}} = 0, \tag{6.23}$$

$$-k^2\Phi_1 = 4\pi G \int f_1 d\mathbf{v}. \tag{6.24}$$

For clarity I omit writing the arguments of the functions for the rest of the treatment. In the following the variables are those of the Fourier space. The distribution function can now be solved from the first equation for

$$f_1 = \frac{\mathbf{k}\cdot\frac{\partial\tilde{f}}{\partial\mathbf{v}}}{\mathbf{v}\cdot\mathbf{k} - \omega}\Phi_1. \tag{6.25}$$

This can be substituted into (6.24) and since the perturbation of the potential is independent of the velocities and non-zero, the dispersion relation is found to be

$$1 + \frac{4\pi G}{k^2}\int \frac{\mathbf{k}\cdot\frac{\partial\tilde{f}}{\partial\mathbf{v}}}{\mathbf{v}\cdot\mathbf{k} - \omega}d\mathbf{v} = 0. \tag{6.26}$$

As the dust particles are assumed to be moving freely the system can be described by a Maxwell-Bolztmann distribution

$$\tilde{f} = \frac{\tilde{\rho}}{(2\pi\sigma^2)^{3/2}}e^{-(v^2/2\sigma^2)}, \tag{6.27}$$

where the σ^2 is the variance and related to the temperature of the dust and

$\tilde{\rho}$ is the unperturbed density. The coordinates can be freely chosen, and for simplicity I choose $\mathbf{k} = (k, 0, 0)$. Inserting the distribution and the chosen direction, the dispersion equation can be written as

$$1 - \frac{2\sqrt{2\pi}G\tilde{\rho}}{k^2\sigma^3} \int \frac{v_x e^{-(kv_x^2/2\sigma^2)}}{kv_x - \omega} dv_x = 0. \tag{6.28}$$

The unstable modes are those for which $\mathrm{Im}(\omega) > 0$, due to diverging behaviour while $\mathrm{Im}(\omega) < 0$ solutions are oscillating. See [23] and [75] for in-depth discussion on the solutions and details on the integral. Note, that this is the classical limit, without any modified gravity effects. These are introduced in the attached paper [75]. The limiting case for the instability is found for $\omega = 0$, which leads to

$$k^2 = \frac{4\pi G\tilde{\rho}}{\sigma^2} \equiv k_J^2, \tag{6.29}$$

where the *Jeans wavenumber* k_J is introduced. As the wavenumber is not a very descriptive quantity, the *Jeans mass* is defined as the mass contained inside a sphere of diameter λ_J, which is called *Jeans length*

$$\lambda_J^2 \equiv \frac{4\pi^2}{k_J^2} = \frac{\pi\sigma^2}{G\tilde{\rho}}, \tag{6.30}$$

and

$$M_J^2 \equiv \frac{4\pi\tilde{\rho}\lambda_J^3}{24} = \frac{\pi}{6}\sqrt{\frac{1}{\tilde{\rho}}\left(\frac{\pi\sigma^2}{G}\right)^3}. \tag{6.31}$$

In a similar manner, the modified Jeans masses for modified gravity can be found. The most notable differences appear as possibilities of multiple solutions and the more involved behaviour of the phase ω. These are analyzed in detail in [75].

6.2 Observational sources of constraints

Many of the underlying principles discussed in earlier chapters double as observational constraints for $f(R)$ models. As the measurements and observations improve on *e.g.* the CMB, the viable group of $f(R)$ grows more narrow.

Along with the more theory oriented constraints for viable models, there are several classes of astrophysical objects that could be observed for new constraints [168, 169, 170, 171, 172, 173], such as Cepheids. In the attached article [75] a new class of objects is considered for these purposes.

Besides the cosmological and astrophysical observations, particle physics also sets some limits. As these are mostly outside the scope of this thesis, a reader is invited to check [72] for a brief review and see the references there-in.

6.2.1 Bok globules

Bok globules are rather small, in astrophysical sense, clouds of interstellar matter, gas and dust. In the attached paper [75] the stability of these clouds is proposed as a new tool to measure viability of $f(R)$ theories. In this sense the most important property of Bok globules is their mass, which is very close their Jeans mass. As these clouds are dark in terms of radiation, they are hard to spot. Since the spotted Bok globules are nearby, they offer interesting new data to supplement Solar system tests.

Molecular hydrogen and helium are the main ingredients of molecular clouds [174]. Other compounds, especially CO, also appear in the clouds and are important for the observations. For simplicity, constant ratios are normally assumed. A mean molecular weight of $\mu = 2.72$ takes into account both the presence of H_2 and He [175]. While much of the clouds is known, the are still many unknowns, the solving of which could provide a better understanding of the collapse mechanics.

There are multiple observational issues related to molecular clouds in general. Especially the distances are hard to determine due to the kinematic distance ambiguity. In the outer parts of the Milky Way, the observations are unambiguous but it is a different matter in the inner parts. In the inner Milky Way each value of radial velocity along a given line corresponds to two distances on either side of the tangent point, with the exception of the tangent point.

It is also up for debate whether molecular clouds are gravitationally bound or rather short-lived phenomena with lifetimes of millions of years. The calculated values of the Jeans mass (or distance) and the virial parameter in [174] can be compared whether they give the same predictions. However, in almost all the clouds in the Roman-Duval catalogue have virial parameters of less than one, meaning that they are gravitationally bound.

According to observations, many Bok globules exhibit star formation. For example the star-forming Bok globule CB 17 [176] has a temperature of $T_0 = 10.6$K in the center and volume density of $n_H = 2.3 \times 10^5$ cm^{-3}. The core mass is $2.3 \pm 0.3 \mathrm{M}_\odot$. The globule seems to be on the verge of being bound gravitationally. The mean atomic mass per particle in molecular clouds $\mu = 2.32$ [165].

6.2.2 Gravitational waves and gravitons

Massive gravity is an field with extensive literature (see *e.g.* [177] and refences therein.) However, it is entirely possible that gravitons may never really be detected [178]. For the purposes of this thesis, the interest lies in the possibility of massive gravitons in $f(R)$ gravity. In the attached paper [6] it is shown that the mass of the graviton would in many cases be in the Planck scale and thus most likely never within observational limits. However, depending on the examined model, a massive graviton could be predicted, with a mass within observable limits. The emerging massive modes in $f(R)$ gravity are discussed in [179].

As discussed in chapter 5, the standard GR without Λ produces strictly zero-mass gravitons. As this is a special case, most $f(R)$ models are constrained by the upper limit on graviton mass. Several experiments set rather stringent limits on the Compton wavelength as discussed. Much in the same way, the photon mass is widely assumed to be zero, but there is only an upper limit to its mass [180]. In the attached paper [6] the LIGO experiment results on the graviton Compton wavelength are used to find model independent constraints, which apply for all $f(R)$ models. Similar methods could be used to analyze a bit smaller astrophysical objects as well. In [181, 182] binaries are considered for constraining both the graviton mass and $f(R)$ gravity.

There are more stringent limits for the Compton wavelength as well, but these are model dependent. For some models, like the DGP there have been very stringent limits [183]. While the galaxy cluster limits [184] and the weak lensing limits [22] are more constraining than the LIGO results, these are also heavily model dependent. Dark matter is generally added to explain the dynamics of galaxies and clusters. However, it has been shown, that modified gravity can explain this dynamic as well [26]. Therefore, whether the dynamics are due to dark matter, modified gravity or both, the resulting limits for the Compton wavelength vary greatly.

Luku 7

Hamiltonian perturbation analysis

In this chapter I present some results and definitions regarding theory of cosmological perturbations and Hamiltonian mechanics in the context of cosmology. This brief introduction to cosmological perturbations is largely based on the extensive review article [185]. The Hamiltonian approach is mainly based on the books [54] and [55].

7.1 On cosmological perturbations

As discussed in chapter 4 and 5 the assumption of homogeneity and isotropy brings along a multitude of simplifications. In the large-scale considerations this assumption is valid and therefore we look into perturbations to the FRW metric. That is, the cosmological background is the simple idealization and on this background the observed anomalies (such as galaxies) are placed. In [186] it was shown that solutions of the linearized field equations correspond to the solutions of the full nonlinear equations in the case of FRW universes.

In perturbation theory, a discrepancy is added on a background. This perturbation must be increasingly small. If the dynamics of the system are of the sort that the small perturbations cease to be small compared to background, the system is not stable. Then the system cannot be described by this sort of a background and small discrepancies combination. This can be used to rule out possible candidates for $f(R)$ theories.

The approach in perturbation theory is the same as weak in the field limit considerations in section 5.6. Basically, these two methods are the same and some authors use the terms interchangeably. In my work I use the term weak field limit when the background is Minkowskian. In general, perturbations can be added to any form of background.

The difference of cosmological and weak field perturbations is the scale of the background. While weak-field considerations often concentrate on, say, Solar System scale, cosmological perturbations are considered on the large scale. While in the Solar System scale, the effects of the galaxy on the geometry are taken to be static background features, in cosmological scale the galaxies are the perturbations.

Again the metric is split into the background and perturbative part

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}. \tag{7.1}$$

If the $f(R)$ theory is seen as a perturbation of GR (which is reasonable, as it must closely resemble ΛCDM), the background obeys GR and is described by the FRW metric. These result in not only a diagonal metric but also in a diagonal energy-momentum tensor. The added perturbations allow for more generality.

The perturbations of the metric tensor can be split into three categories according to their transformation properties. These are scalar, vector and tensor perturbations [187]. The degree, up to which the perturbations are taken, affects whether all the categories should be taken into account. In the linear approximation the different categories decouple. For second order and higher, the "crossterms"become significant. In the linear case apart from the scalar ones, the perturbations do not have physical effects. The vector perturbations decay in an expanding universe (which is where the interest is) and are not generated in the presence of scalar perturbations. The tensor perturbations result in gravitational waves which are not coupled to energy density and pressure perturbations, which are described by the scalar perturbations.

Especially in the context of perturbation theory, *conformal time* is often used. It is related to the coordinate time by $d\eta = a^{-1}dt$. Using conformal time the FRW line element becomes

$$ds^2 = a^2(\eta)\Big[-d\eta^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\Big], \tag{7.2}$$

now the origins of the name become clear, as this metric is only a conformal transformation away from the the Minkowski metric. The factor a^2 represents $\hat{\Omega}^2$ in (4.34) discussed in chapter 4.

Keeping in mind that the background is FRW and the form of metrics in maximally symmetric space-times, the metric perturbations needs to be of the form

$$h_{\mu\nu} = a^2(\eta) \begin{pmatrix} A & B_a \\ B_a & C_{ab} \end{pmatrix}. \tag{7.3}$$

The temporal component A is already a scalar and its significance therefore clear. As it is related to the Newton gravitational potential it is chosen as 2ϕ, with the multiplier added for convenience.

As for the spatio-temporal components, these can be formed out of a scalar by covariant derivation. As the background is flat, the covariant derivatives become partial ones and the off-diagonals become (enforcing symmetry of the metric) $B_a = -\partial_a \omega$, with an added $-$ sign for convenience.

The spatial part of the metric, describing the maximally symmetric subspace, has two three-indices. Recalling the form of tensors in maximally symmetric spaces, the possible perturbations are found by multiplying a scalar ψ with the induced metric γ_{ab} or by differentiating a scalar, $-\partial_a \partial_b \chi$.

There are now, in total, four scalar perturbations. These correspond to four degrees of freedom. The line element with scalar perturbations can now be written as

$$ds^2 = a^2(\eta)\Big[(1+2\phi)d\eta^2 - \partial_a \omega dx^a d\eta - \big((1-2\psi)\gamma_{ab} + 2\partial_a\partial_b\chi\big)dx^a dx^b\Big]. \tag{7.4}$$

This accounts for all the scalar perturbations. The vector perturbations must be of purely vector nature. This can be enforced by demanding

$$\partial^a F_a = 0. \tag{7.5}$$

While the ∂^a would be a bit troublesome in a general case, in this case the partial and covariant derivatives are interchangeable. Therefore, the partial derivative commutes with the metric. If the rule was not met, the vector F_a could be split into

$$F_a = B_a + \partial_a A, \tag{7.6}$$

where B_a is divergenceless and A is a scalar. There can be no temporal tensor perturbations. The temporal-spatial components of the perturbed metric can be directly written as vectors satisfying the rule (7.5), $-\omega_a$. For the spatial part the three-tensor can be constructed by derivating a vector. The vector perturbations to the metric can now be written as

$$h_{\mu\nu} = -a^2(\eta) \begin{pmatrix} 0 & -\omega_a \\ -\omega_a & \partial_b F_a + \partial_a F_b. \end{pmatrix}. \tag{7.7}$$

This adds up to four vector degrees of freedom in the vector perturbations as there are two three-vectors and two constraints.

Like the vector perturbations, tensor perturbations must fulfill conditions to ensure their purity (in opposition to scalars or vectors). The conditions are

$$\chi_a^a = 0, \tag{7.8}$$

$$\partial^b \chi_{ab} = 0. \tag{7.9}$$

The first one ensures the tensor perturbation cannot be split into having a scalar part and the second one ensures, there is no vector part. Clearly, there cannot be tensor perturbations in the temporal or spatio-temporal parts of the metric. The tensor χ_{ab} is not to be confused with the gravitational coupling χ. The tensor perturbations to the metric are now

$$h_{\mu\nu} = -a^2(\eta) \begin{pmatrix} 0 & 0 \\ 0 & \chi_{ab}. \end{pmatrix} \tag{7.10}$$

There are only two degrees of freedom left in the tensor perturbations as the metric and its perturbations are symmetric and there are four constraints.

In total there are ten degrees of freedom in the metric perturbations. As discussed in the chapter 4, this is exactly the amount of free parameters in the metric. Regardless of the order of expansion, all the perturbations fall into these classes. However, it should be noted that lower order perturbations can affect higher order through terms such as $\omega^a \omega_a$ and in higher order the different categories might couple into terms such as $\psi\chi_{ab}$.

While the higher order perturbations might be initially absent and appear as a consequence of the linear perturbations, this is not the case for the first order. If linear perturbations are present, they must be part of the initial setup. For a system to be stable, all the perturbations must evolve in a stable manner. Then again, to prove a system unstable, it is enough to show that some perturbations are not well-behaved. Therefore, it is possible to choose *e.g.* first order vector perturbations to be absent in the initial setup and find that the tensor perturbations diverge, thus disapproving the system.

In the discussion of perturbations of the metric, it is often omitted that the actual perturbation parameter is the velocity of a test particle v, [66]. This appears as a combination v/c, which allows for the more common parameter, the inverse speed of light c^{-1}. As it is customary to set $c = 1$, as is done in this thesis, it is more clear to think of v as the expansion parameter.

Up to second order, the perturbations are [188]

$$\delta h_{00} = -2a^2\left(\phi^{(1)} + \phi^{(2)}\right), \tag{7.11a}$$

$$\delta h_{0a} = a^2\left(\partial_a\omega^{(1)} + \omega_a^{(1)} + \frac{1}{2}\omega^{(2)}\right), \tag{7.11b}$$

$$\delta h_{ab} = a^2\Big[\nabla_{ab}\chi^{(1)} + 2(\partial_{(a}\chi_{b)}^{(1)}) + \chi_{ab}^{(1)} + \frac{1}{2}\chi_{ab}^{(2)} - \big(2\psi^{(1)} + \psi^{(2)}\big)\delta_{ab}\Big]. \tag{7.11c}$$

Here the quantities $\phi^{(n)}$ are of the order n. The terms of the first order are split into scalar, vector and tensor parts. The derivative operator is $\nabla_{ab} = \partial_a\partial_b - (1/3)\delta_{ab}\nabla^2$, which is trace-free.

There is also gauge freedom [185] which allows for two gauge conditions to further reduce the free parameters. This is due to the fact that general perturbations of the metric are not invariant under change of coordinates. Certain combinations of metric perturbations can be shown to be gauge-invariant [189]

$$\Phi = \phi + (1/a)\partial_\eta\Big[a(\omega - \partial_\eta\chi)\Big], \tag{7.12}$$

$$\Psi = \psi + (\partial_\eta a/a)\partial_\eta(\omega - \partial_\eta\chi). \tag{7.13}$$

These two variables form the basis of the two-dimensional space of gauge-invariant variables. As the Φ and Ψ do not change under change of coordinates, with these variables it is possible to distinguish between physical anomalies and mathematical artifacts. When dealing with physical measurable quantities, a set of coordinates is chosen and gauge modes are canceled. While these quantities are gauge invariant under infinitesimal transformations, they are not necessarily invariant under finite coordinate changes, since (following [185]) the coordinate changes involved are

$$x^\mu \to \hat{x}^\mu = x^\mu + \xi^\mu, \tag{7.14}$$

where ξ^μ is infinitesimal.

The ten degrees of freedom of the metric perturbations can be reduced in several ways. As already mentioned, in the linear order only the scalar degrees of freedom are meaningful. Choosing a gauge also specifies in which coordinate system the perturbations are considered. In general perturbations are not the same in all coordinate systems.

One of the most used gauges is the *longitudinal gauge* (or the conformal-Newton gauge), for which $\chi = 0$ and $\omega = 0$. In the linear order this leaves only two meaningful scalar degrees of freedom to contend with. This gauge fixes the coordinates totally. If the initial coordinate system is (η, x^a), then conditions are imposed into a new coordinate system via the transforms

$$\eta \to \hat{\eta} = \eta - (\omega - \partial_\tau \chi), \tag{7.15}$$

$$x^a \to \hat{x}^a = x^a + \gamma^{ab}\partial_b \chi. \tag{7.16}$$

The line element becomes

$$ds^2 = a(\eta)^2 \Big[(1+\phi)d\eta^2 - (1-2\psi)\gamma_{ab}dx^a dx^b\Big]. \tag{7.17}$$

It is easy to see form (7.12) and (7.13) that in the longitudinal gauge the scalar perturbations equal the gauge invariant variables $\psi = \Psi$ and $\phi = \Phi$. Furthermore, in the case of diagonal energy-momentum tensor (especially, for perfect fluid matter) $\Psi = \Phi$. This one variable acts as the generalization of the Newtonian gravitational potential, hence the name conformal-Newton gauge. Physically the quantities Ψ and Φ are the amplitudes of the perturbations in the metric tensor.

This gauge can only be used if the energy-momentum tensor does not have tensor or vector contribution. The equations of motion would not longer be consistent. Even if such were not initially present, they may manifest due to higher order perturbations. Therefore, the usage of longitudinal gauge is problematic.

In order to study higher order perturbations, which call for dealing with vector and tensor perturbations, a generalization of the longitudinal gauge is in order. This is achieved by the *Poisson gauge* [190, 191], for which the tensor and vector perturbations are included. The gauge conditions of the Poisson gauge are

$$\nabla^a \hat{\chi}_{ab} = 0, \tag{7.18}$$

$$\nabla^a \hat{\omega}_a = 0, \tag{7.19}$$

where all the temporal-spatial perturbations are included in $\hat{\omega}_a = \partial_a \omega + \omega_a$, which is sometimes called the *gravitomagnetic potential* [192]. Besides the scalar term, all the spatial perturbations are collected in $\hat{\chi}_{ab} = \nabla_{ab}\chi + 2(\partial_{(a}\chi_{b)}) + \chi_{ab}$, which is called the *gravitational wave strain*. This can be compared to electromagnetism and the Coulomb gauge with $\nabla^a A_a = 0$. Due to gravitation being of tensor nature rather than vector one, there needs to be more conditions and more potentials. If the case of $\omega_a = \chi_{ab} = 0$ is examined, the Poisson gauge reduces to longitudinal gauge.

The gravitomagnetic potential and the gravitational wave strain have not been detected and are severely constrained to be negligible [191, 193, 194], so discarding the vector and tensor perturbations is justified. As in the scope of this thesis the interest is in finding instabilities, this is a rather moot point. The first order vector and tensor perturbations can be omitted in order to examine scalar perturbations. Therefore, for the rest of the thesis I choose to examine only perturbations with $\omega_a^{(1)} = \chi_{ab}^{(1)} = 0$. Now, due to the conditions (7.5) and (7.8) also $\omega = \chi = \chi_a = 0$.

7.2 On Hamiltonian formulation

In order to look into the Hamiltonian perturbation theory of $f(R)$ gravity we need a (3+1) decomposition. The original ADM formalism was introduced and studied in [195] while this treatment follows [54] and to some extent [55]. Another treatment in in the context of $f(R)$ gravity can be found in [196]. For this approach the underlying geometric structure needs a bit of explaining in addition to chapter to 4.

Let $(M, g_{\mu\nu})$ be a globally hyperbolic space-time, where the hyperbolicity is required for the initial value problem considerations mentioned in chapter 5. The space-time metric $g_{\mu\nu}$ is related to the coordinates x^μ. The integrals in the gravitational action (*e.g.* (5.2)) are over a volume V of the space-time $(M, g_{\mu\nu})$. This volume is further foliated with space-like hypersurfaces Σ_t. These are Cauchy surfaces parametrized by a global (or coordinate) time t and let n^μ be the unit normal vector field to them. The global time is arbitrary and unphysical until one knows the metric. The global time must be a single-valued function of coordinates x^μ though, to ensure nonintersection of the hypersurfaces.

Besides the global time, let t^μ be a vector field, which could be described as the time flow, for which

$$t^\mu \nabla_\mu t = 1. \tag{7.20}$$

The coordinates on the hypersurfaces maybe unrelated to each other. However, if a congruence of curves σ intersects the hypersurfaces Σ_t and the time flow t^μ is tangent to the congruence, and (7.20) holds, a relation is established, which is usually convenient. If y^a are now coordinates on Σ_t and P is a point on the same surface, one of the curves of the congruence, σ_P, maps the point P to a point P' on another hypersurface $\Sigma_{P'}$ and all the the other hypersurfaces along the time flow. In order to fix the originally arbitrary coordinates, one may set $y^a(P) = y^a(P')$, *i.e.* setting y^a to be constant on all the curves of the congruence.

The construction above defines a new coordinate system in the volume V, which is related to the original coordinates x^μ. Remembering the idea of the time flow,

$$t^\mu = \left(\frac{\partial x^\mu}{\partial t}\right)_{y^a}. \tag{7.21}$$

Using the tetrad formalism [197, 198], the tangent vectors on the hypersurface Σ_t are defined as

$$e^\mu_a = \left(\frac{\partial x^\mu}{\partial y^a}\right)_t. \tag{7.22}$$

These four-vectors and their reciprocal vectors have the following conditions

$$e^a_\mu e^\mu_b = \delta^a_b, \tag{7.23}$$
$$e^a_\mu e^\nu_a = \delta^\mu_\nu. \tag{7.24}$$

In this foliation, the tetrad is Lie transported along σ (to which t^μ is tangent to, which is refered to in the subscript) [56, 58],

$$\mathcal{L}_t e_a^\mu = 0. \tag{7.25}$$

With tetrads it is possible to represent the components of *e.g.* a vector $A^\mu(x)$ in the locally inertial coordinates ξ^a with $A^a = e_\mu^a A^\mu$.

The time flow vector field can be decomposed into parts normal and tangential to the Σ_t. This is the (3+1) decomposition

$$t^\mu = N n^\mu + N^a e_a^\alpha. \tag{7.26}$$

The *lapse* N is a scalar function related to moving in time *i.e.* normal to the hypersurface, in the direction of the normal n^μ. On the surface is defined the vector function *shift*, N^a. Explicitly, the lapse can be written as

$$N = -t^\mu n_\mu. \tag{7.27}$$

This can be interpreted, that N measures the flow of proper time τ with respect to the global time t. The unit normal n^μ to the hypersurface has the following characteristics

$$n_\mu = -N\partial_\mu t, \tag{7.28}$$

$$n_\mu e_a^\mu = 0, \tag{7.29}$$

In order to write the shift function, the metric on the hypersurface is needed. Using the decomposition we can write the infinitesimal change

$$dx^\mu = t^\mu dt + e_a^\alpha dy^a. \tag{7.30}$$

Using equations (7.21), (7.22) and (7.26) the line element can be written as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -(N^2 - N_a N^a) dt^2 + 2N_a dx^a dt + \gamma_{ab} dx^a dx^b, \tag{7.31}$$

where the spatial three-dimensional metric γ_{ab} has been introduced, which describes the geometry for one instant of time

$$\gamma_{ab} = g_{ab} + n_a n_b, \tag{7.32}$$

and is also called the *first fundamental form*. The shift vector, which describes the displacement tangential to Σ_t can now be written as

$$N^a = \gamma^a_b t^b. \tag{7.33}$$

The meaning of these quantities merits a thought. As the main aim in this context is in writing an action in Hamiltonian formalism, the important question is, which are the dynamical variables? As the hypersurfaces Σ_t are identified by following the integral curves of t^μ, the advancing time is related to changes in the spatial metric $h_{ab}(t)$. Therefore, the spacetime $(M, g_{\mu\nu})$ represents the time evolution of a metric on a fixed three-dimensional manifold. Further, as the spatial metric $h_{ab}(t)$ changes, it is the dynamical variable. However, the field variables include N and N_a as these three contain the same information as $g_{\mu\nu}$, which is the standard field variable.

Comparing the line-element (7.31) and the metric perturbations (7.11) allows one to write the perturbations of the lapse, the shift and the induced metric. Due to the transverse condition of the spatio-temporal components, $N^a \partial_a \omega = N^a \omega_a = 0$.

As the examination of dynamics of a system in space-time demands time derivatives, it is useful to take a moment to consider how the covariant derivative of a vector in space-time, say A^μ is related to the covariant derivative of A^μ on a hypersurface. The latter needs to be defined by a connection related to the induced metric γ_{ab}. The intrinsic covariant derivative of a three-vector on the hypersurface can be defined as the projection of the global covariant derivative onto a hypersurface. The projection is achieved with a combination of tetrads

$$\vec{\nabla}_b A_a \equiv \nabla_\nu A_\mu e^\mu_a e^\nu_b. \tag{7.34}$$

It turns out [54] that the associated connection can be written in a similar form to that of four-dimensional Levi-Civita connection

$$\Gamma_{cab} = \frac{1}{2}\big(\partial_b \gamma_{ca} + \partial_a \gamma_{cb} - \partial_c \gamma_{ab}\big). \tag{7.35}$$

For the action integral the volume element of V must be written in terms of the decomposition, therefore the determinant $\sqrt{-g}$ is to be written in terms of h_{ab}, N^a, N. Writing the determinant g gives

$$g = g_{tt}\hat{G}_{tt} + g_{1t}\hat{G}_{1t} + g_{2t}\hat{G}_{2t} + g_{3t}\hat{G}_{3t}, \tag{7.36}$$

where $\hat{G}_{\mu\nu}$ stands for cofactors (see *e.g.* [54]). Only the first term of the sum is non-zero as in all the other terms the first column of the cofactor consists of zeros. The line element (7.31) reveals that $g^{tt} = 1/g_{tt}$ as $g^{\mu\nu}g_{\mu\nu} = \delta^{\mu}_{\nu}$ for all metrics. On the other hand the temporal component of the metric can be written as

$$g^{tt} = g^{\mu\nu}\partial_\mu t \partial_\nu t = -\frac{1}{N^2}, \tag{7.37}$$

using (7.21) and the property of normals $n^a n_a = 1$ on time-like surfaces. The line element can be finally written as

$$\sqrt{-g} = N\sqrt{\gamma}. \tag{7.38}$$

In order to write first the Einstein-Hilbert action and then the $f(R)$ action, the Ricci curvature scalar must be decomposed to the field variables and their derivatives. For this purpose the introduction of the *extrinsic curvature* tensor K_{ab} and scalar K are necessary. The definition is

$$K_{ab} \equiv \nabla_\nu n_\mu e^\mu_a e^\nu_b, \tag{7.39}$$

$$K \equiv h^{ab} K_{ab}. \tag{7.40}$$

The extrinsic curvature is also called the second fundamental form and it describes the effects of embedding a hypersurface Σ in space-time. This quantity is also symmetric due to orthogonality of e^μ_a and n^μ and that the tetrads are Lie transported along each other. The extrinsic curvature is also equal to the Lie derivative of the spatial metric along an unit tangent field ξ^μ

$$K_{ab} = \frac{1}{2}\mathcal{L}_\xi \gamma_{ab}. \tag{7.41}$$

This means the extrinsic curvature describes the rate of change for γ_{ab} or the bending of the hypersurface Σ in space-time. It should be noted that the tangent field ξ^μ in this context is general and does not have to be the vector field t^μ.

As the metric describes how distances change, the extrinsic curvature translates to measure shrinkages and deformations. This is opposed to the intrinsic curvature, contained in the metric. An example would be a cylinder, the surface of which looks clearly curved. However, as is easily experimented, a flat piece of paper can be rolled into a cylinder and rolled open again. The intrinsic curvature of a cylinder is zero, while the extrinsic curvature is non-zero. On the other hand, one cannot roll a piece of paper into a sphere without deformation, in which there is intrinsic curvature as well. The Ricci curvature scalar can now be decomposed as (see [54] for details)

$$R = {}^3R + K^{ab}K_{ab} - K^2 - 2\nabla_\mu(\nabla_\nu n^\mu n^\nu - n^\mu \nabla_\nu n^\nu), \tag{7.42}$$

where the first term ${}^3R$ is the three-dimensional Ricci curvature scalar induced on the hypersurface. As the interest is ultimately not on the extrinsic curvature but the field variables, it must be again decomposed. This can be achieved by considering the time derivative of the induced metric, which is by definition

$$\dot{\gamma}_{ab} = \mathcal{L}_t \gamma_{ab}. \tag{7.43}$$

Since the Lie derivative of a tetrad is zero (7.25)

$$\mathcal{L}_t \gamma_{ab} = \mathcal{L}_t (g_{\mu\nu} e_a^\mu e_b^\nu) = e_a^\mu e_b^\nu \mathcal{L}_t g_{\mu\nu}. \tag{7.44}$$

In metric theories $\nabla_\lambda g_{\mu\nu} = 0$, which leads to

$$\mathcal{L}_t g_{\mu\nu} = \nabla_\nu t_\mu + \nabla_\mu t_\nu. \tag{7.45}$$

Writing the time flow as in (7.26) yields

$$\mathcal{L}_t g_{\mu\nu} = n_\mu \partial_\nu N + n_\nu \partial_\mu N + N(\nabla_\nu n_\mu + \nabla_\mu n_\nu) + 2\nabla_{(\nu}(N_a e^a_{\mu)}). \tag{7.46}$$

The form (7.44) is a projection of the Lie derivative of the metric along $e_a^\mu e_b^\nu$. For this purpose we need the intrinsic covariant derivative (7.34), the normal property (7.29), the definition of the extrinsic curvature and its symmetry. The result is

$$\dot{\gamma}_{ab} = 2NK_{ab} + \vec{\nabla}_a N_b + \vec{\nabla}_b N_a, \tag{7.47}$$

from which the intrinsic curvature can be solved in terms of the desired field variables and their derivatives

$$K_{ab} = \frac{1}{2N}\Big(\dot{\gamma}_{ab} - \vec{\nabla}_a N_b - \vec{\nabla}_b N_a\Big), \tag{7.48}$$

which enables writing the decomposed R (7.42) in terms of the field variables and their derivatives as (it should be noted that some of the terms contribute also to the surface part through [54])

$$R = {}^3R + \Big[(\gamma^{ac}\gamma^{bd} - \gamma^{ab}\gamma^{cd})K_{ab}K_{cd}\Big]. \tag{7.49}$$

As it was stated before, the induced metric γ is the only dynamic variable. Indeed, there are spatial derivatives of the lapse and the shift as well but these sum up to surface terms, which have no impact on the field equations. The volume part of the gravitational Lagrangian can be written using (7.49) and the scalar tensor form of $f(R)$ (5.34)

$$\mathcal{L}_{GV} = f(\phi) + f'(\phi)(K^{ab}K_{ab} + {}^3R - K^2) - f'(\phi)\phi. \tag{7.50}$$

The action integral includes surface terms, which cannot be easily neglected, the foliation of the boundary must be taken into account as well. It is in order to remind of the terminology. The action integral is integrated over hypersurfaces Σ_t. These form the volume bounded by the surface ∂V. One must take care not to confuse the hypersurface and the surface in this context. In this chapter the term hypersurface always refers to the Cauchy surfaces Σ. As well as there is an induced metric on Σ_t there is an induced metric on the boundary ∂V. Let r_a be a unit normal to the boundary ∂V. An associated four-vector is The action integral includes surface terms, which cannot be easily neglected, the foliation of the boundary must be taken into account as well. It is in order to remind of the terminology. The action integral is integrated over hypersurfaces Σ_t. These form the volume bounded by the surface ∂V. One must take care not to confuse the hypersurface and the surface in this context. In this chapter the term hypersurface always refers to the Cauchy surfaces Σ. As well as there is an induced metric on Σ_t there is an induced metric on the boundary ∂V. Let r_a be a unit normal to the boundary ∂V. An associated four-vector is

$$r^\mu = r^a e^\mu_a, \tag{7.51}$$

for which $r^\mu r_\mu = 1$ and $r^\mu n_\mu = 0$. As the tetrads were defined on the Σ_t (7.22), similarly on the boundary, tangent to it, can be defined

$$e^\mu_A \equiv e^\mu_a e^a_A. \tag{7.52}$$

The induced metric on the boundary can now be written as

$$s_{AB} = g_{\mu\nu} e^{\mu}_{A} e^{\nu}_{B}, \tag{7.53}$$

where the capital Latin indices refer to the coordinates on the boundary. Similar to (7.32) the relation to the four-dimensional metric can be written as

$$g^{\mu\nu} = -n^{\mu} n^{\nu} + r^{\mu} r^{\nu} + s^{AB} e^{\mu}_{A} e^{\nu}_{B}. \tag{7.54}$$

To obtain the equations of motion in Hamilton formalism, the action needs to be written in terms of the Hamiltonian and the conjugate momenta. In order to construct the Hamiltonian for gravitation, the action must be written in terms of the fields and their time derivatives. As is apparent from (7.47) and (7.42), the gravitational action depends on $\dot{\gamma}$ through the intrinsic curvature and there are no time derivatives of the lapse or the shift. Therefore, there are no conjugate momenta for N and N^a. They are not dynamical variables and depend only on the choice of the foliation. The action will be of the form

$$S_G = \int_{t_1}^{t_2} dt \Big[\int_{\Sigma_t} p^{ab} \dot{\gamma}_{ab} d^3 y - H_G \Big], \tag{7.55}$$

from which variation yields

$$\delta S_G = \int_{t_1}^{t_2} dt \Big[\int_{\Sigma_t} (p^{ab} \delta\dot{\gamma}_{ab} + \dot{\gamma}_{ab} \delta p^{ab}) d^3 y - \delta H_G \Big]. \tag{7.56}$$

As usual, on the boundary ∂V, the variations vanish

$$\delta N = \delta N^a = \delta\gamma_{ab} = \delta\phi = 0 \tag{7.57}$$

but the conjugate momentum p_{ab} does not necessarily vanish. In comparison, one can be reminded of the discussion in the chapter 5 on the derivatives not generally vanishing on the boundary. Integration by parts of the first term in equation (7.56) enables the following form

$$\delta S_G = \int_{t_1}^{t_2} dt \int_{\Sigma_t} \Big[(\dot{\gamma}_{ab} - \mathcal{H}_{ab})\delta p^{ab} - (\dot{p}^{ab} + \mathcal{P}^{ab})\delta\gamma_{ab} - \Xi\delta\phi + \tag{7.58}$$
$$+ \mathcal{C}\delta N + 2\mathcal{C}_a \delta N^a \Big] d^3y, \tag{7.59}$$

where the variation of the Hamiltonian are written as $\delta H_G/\delta p^{ab} \equiv \mathcal{H}_{ab}$, $\delta H_G/\delta h_{ab} \equiv \mathcal{P}^{ab}$, $\delta H_G/\delta N \equiv \mathcal{C}$, $\delta H_G/\delta N^a \equiv \mathcal{C}_a$ and $\delta H_G/\delta\phi \equiv \Xi$. The equations of motion of the gravitational part in empty space can now be written in the simple form

$$\dot{\gamma} = \mathcal{H}_{ab} \tag{7.60}$$
$$\dot{p}^{ab} = -\mathcal{P}^{ab} \tag{7.61}$$
$$\mathcal{C} = 0 \tag{7.62}$$
$$\mathcal{C}_a = 0 \tag{7.63}$$
$$\Xi = 0. \tag{7.64}$$

Of course, the variations of the Hamiltonian need to be calculated before these equations are of use. So far, the equations are independent of the form of the gravitational action. Another matter to tackle is the boundary terms, which cannot be neglected. This is done in detail in the attached paper [87].

The foliation and the volume V are arbitrary, and one is free to choose V as "cylinder"with the two space-like hypersurfaces Σ_{t_1} and $-\Sigma_{t_2}$ as the top and bottom and a time-like hypersurface $\mathcal{B}$ the surface of the "cylinder"with the normal vectors pointing out of the volume. The entire boundary is now

$$\partial V = \Sigma_{t_1} \cup (-\Sigma_{t_2}) \cup \mathcal{B}. \tag{7.65}$$

As the dependence of $\dot{\gamma}$ in the gravitational action is through the intrinsic curvature, it can be written using the chain rule as

$$p^{ab} = \frac{\partial K_{cd}}{\partial \dot{\gamma}_{ab}} \frac{\partial}{\partial K_{cd}} (\sqrt{-g}\mathcal{L}_G) = \frac{\sqrt{\gamma} f'(\phi)(K^{ab} - Kh^{ab})}{16\pi}. \tag{7.66}$$

This can also be inverted in order to write the intrinsic curvature in terms of the the conjugate momentum

$$K^{ab} = \frac{16\pi}{\sqrt{\gamma}f'(\phi)}\Big(p^{ab} - \frac{1}{2}p\gamma^{ab}\Big), \tag{7.67}$$

where p is the trace of the conjugate momentum p_{ab}. Setting the surface terms aside, the volume part of the Hamiltonian density is now ready to be written,

$$\mathcal{H}_{GV} = p^{ab}\dot{\gamma}_{ab} - \sqrt{-g}\mathcal{L}_{GV}. \tag{7.68}$$

The Hamiltonian is found integrating the Hamiltonian density over the space-like hypersurface Σ_t. Before moving on, it is useful to notice that the first term, related to the conjugate momentum can be rewritten using (7.66) and (7.47)

$$\begin{aligned}
&\frac{\sqrt{\gamma}f'(\phi)(K^{ab} - K\gamma^{ab})}{16\pi}(2NK_{ab} + \vec{\nabla}_a N_b + \vec{\nabla}_b N_a) \\
=&\frac{\sqrt{\gamma}f'(\phi)}{8\pi}\Big[N(K^{ab} - K\gamma^{ab})K_{ab} + (K^{ab} - K\gamma^{ab})\vec{\nabla}_{(a}N_{b)}\Big] \\
=&\frac{\sqrt{\gamma}f'(\phi)}{8\pi}\Big[N\hat{K}^{ab}K_{ab} + \nabla_b\Big(\hat{K}^{ab}N_a\Big) - N_a\nabla_b\hat{K}^{ab}\Big].
\end{aligned} \tag{7.69}$$

Here I have denoted the combination $\hat{K}^{ab} = K^{ab} - K\gamma^{ab}$. Writing in the volume part of the gravitational Lagrangian and integrating over the hypersurface Σ_t the Hamiltonian is found to be

$$\begin{aligned}
H_{GV} =&\frac{1}{16\pi}\int_{\Sigma_t}\Big[N\sqrt{\gamma}\Big(\phi f'(\phi) - f(\phi) - {}^3Rf'(\phi)\Big) - f'(\phi)K\gamma^{ab}K_{ab} + \\
&+\nabla_b\Big(\hat{K}^{ab}N_a\Big) - N_a\nabla_b\hat{K}^{ab}\Big]d^3y.
\end{aligned} \tag{7.70}$$

Applying the Gauss theorem to the second term in (7.69) yields a surface term. The surface part of the Hamiltonian is (see [87] for details)

$$H_S = \frac{1}{8\pi G}\oint_{S_t}\Big[N(k - k_0) - \frac{16\pi N_a p^{ab} r_b}{\sqrt{\gamma}}\Big]f'(\phi)\sqrt{\sigma}d^2\theta. \tag{7.71}$$

Where S_t is the boundary of a hypersurface Σ_t, r^a a unit vector perpendicular to it and σ the trace of the induced metric on S_t. The extrinsic curvature of S_t embedded in the hypersurface has a trace k and k_0 is the extrinsic curvature embedded in flat space.

Inserting the intrinsic curvature in (7.70) allows for writing the action in terms of the field variables and the conjugate momentum

$$\begin{aligned} H_{GV} =& \frac{1}{16\pi}\int_{\Sigma_t}\Big[N\sqrt{\gamma}\Big(\phi f'(\phi) - f(\phi) - {}^3Rf'(\phi)\Big) + \frac{16^2\pi^2 N}{\sqrt{\gamma}f'(\phi)}\Big(p_{ab}p^{ab} - \frac{p^2}{2}\Big) - \\ & -32\pi\sqrt{\gamma}N_a\nabla_b\Big(\frac{p^{ab}}{\sqrt{\gamma}}\Big)\Big]d^3y. \end{aligned} \tag{7.72}$$

Luku 8

Summary of papers

This thesis aims to provide tools and methods for constraining $f(R)$ theories of gravity. While the dark energy problem remains an enormous question, all the possible solutions are to be considered. As briefly reviewed in chapter 6, there are a number of both observational and theoretical ways to rule out or at least constrain $f(R)$ gravity theories. My research strengthens the picture that few $f(R)$ models are viable and even then in only a rather narrow space of parameters.

This thesis present both theoretical tools for simplifying the task of constraint as well as introducing new classes of observational objects as sources of constraints. While the focus is on $f(R)$ theories, many of the methods are also applicable to other modified gravity theories, such as the maximally symmetric considerations in the first attached paper.

The underlying structure and geometry of the spacetime remains an unanswered question. While the rather simple metric approach remains strong, the more elaborate metric-affine approach remains viable too. As the increased generality adds to degrees of freedom and complexity, new tools are required to compare the theoretical findings with the observations.

The widely accepted symmetry considerations are one way to deal with this complexity. In the first attached paper, a method is presented to greatly reduce the complexity involved calculations with a totally arbitrary connection. This can be used *e.g.* when making numerical simulations of stellar bodies, which have are spherically symmetric *i.e.* have a two dimensional maximally symmetric subspace. One of the most useful characteristics of the symmetry and geometry considerations is that they are independent of the chosen model and could be used to not only $f(R)$ models but for any other types of actions as well.

Three of the attached papers include analysis of perturbations. This is not surprising as right after the renewed interest in $f(R)$ theories, it was found out many models experience stability issues, as discussed in the earlier chapter. The instabilities can be found through growing perturbations, which would make the Cauchy problem to be not well-posed and would not have the correct dynamical history.

While the cosmological perturbations are not allowed to grow, this is not always the case in smaller case, as examined in the attached Jeans analysis paper. Local perturbation growth is what leads to structure formation, which is crucial for what we observe - and for the fact that we are able to observe. While this structure formation can happen in virtually all modified gravity models, there are clearly notable differences as shown in the paper. Using different classes of objects of different magnitudes can provide ever better understanding of gravitation. Even if the Bok globules are not very well known, they still offer insight into viable gravity models. This naturally leads to the question, which other astrophysical objects we could use for these purposes.

The last two papers are more observation oriented. These prove once more, that most astrophysical and cosmological observations can be turned into better understanding of gravity and the possibly viable alternatives to ΛCDM. Cosmology can benefit from observations of all magnitudes, whether high-energy particle physics or large-scale astrophysics. While some observations, like the cosmic acceleration, open vast new fields and questions, others reveal the rules inside these fields.

In the following sections I present the key findings in the attached papers.

8.1 Maximal symmetry and metric-affine $f(R)$ gravity

The metric-affine approach to $f(R)$ gravity is in many ways more complex than the metric approach. This is in large part due to the added degrees of freedom, as in metric $f(R)$ there is only one extra degree of freedom (as opposed to GR). If one is to consider the most general case, comparing results with observations and numerical simulations becomes extremely complicated and, in many cases, practically impossible. For these reasons some tidying up is in order.

Using the observed and assumed Copernican principle, isotropy and homogeneity are the basis of many considerations. These are equivalent with maximal symmetry, which we use in in our paper to cut down the degrees of freedom in the affine connection. While we use $f(R)$ theories as a first test case, the method can be used to any types of gravitational actions and modified gravity theories.

The general form of form invariant tensors can be found in many textbooks for ranks 0 to 2, but rank 3 is usually omitted. The treatment requires the vanishing of the Lie derivative of the metric. While the textbooks consider only Riemann space situations, *i.e.* Levi-Civita connection, this is an unnecessary condition.

In our paper we show that the form invariance of tensors in maximally symmetric spaces can be approached in a a more general situation. We find that the necessary condition for this is the vanishing of the non-metricity tensor. Therefore, torsion might still be present, as would be the case for example with fermion matter in many models.

The third rank tensors are found to vanish except for the case of three-dimensional subspaces, which is exactly the case we are interested in. In this case there remains an invariance under cyclic permutations. An interesting consequence is that for higher dimensions, in maximally symmetric (sub)spaces the connection would necessarily be of the Levi-Civita form. This has an impact on gravity theories, which involve more dimensions than 4.

For a three-dimensional subspace our method reduces the 64 degrees of freedom in the connection are reduced to 4. Even without the vanishing of the non-metricity, our method reduces the degrees of freedom by about one half. This result holds for all maximally symmetric subspaces in metric-affine spaces of three dimensions. These quantities are used to calculate the Ricci tensor and scalar.

The equations of motion can now be found for a given theory. In the paper we examine the Einstein-Hilbert Lagrangian as a test case and move on to general $f(R)$ Lagrangians. It is found that for conventional matter the equations of motion do not deviate noticeably from the metric case. This is an alternate way to see that torsion is caused by certain kinds of matter, *i.e.* with non-zero hypermomentum. There remains only a spurious degree of freedom, which could be related to the gauge fixing or the scalar degree of freedom in the literature.

We move on to show that even in the case of ordinary matter the dynamics of our approach are different to metric and Palatini formalism. This is not surprising as in the Palatini case the affine connection plays a different role.

While our test case did not include hypermomentum, such types of matter could be added and dealt with our formalism. The equations of motion would be more complicated, but at the very least, numerically solvable. The same would not necessarily be true for the general case, without using our method.

8.2 Hamiltonian perturbation theory in $f(R)$ gravity

Instabilities found in perturbation analysis are one most common reasons to discard a given $f(R)$ model. For example the well-known Dolgov-Kawasaki instability is due growing perturbations. While the perturbations have been a highly effective tool in the past, they could provide for several more ways to deal with potentially viable theories.

The traditional way to examine perturbations in the metric Lagrangian formalism deals with perturbations in the curvature. Indeed, this is a straightforward method, which is often enough. However, it is not the only possibility and may leave some instabilities undetected.

The reason metric Lagrangian formalism detects only the perturbations in the direction of the curvature is the relation of curvature and the matter content. While for the Einstein-Hilbert action the dependence is algebraic, for $f(R)$ theories this relation might be differential. For this reason examining other perturbations is more fruitful in the case of $f(R)$ theories.

For the purposes of Hamiltonian formulation of the $f(R)$ equations of motion, the surface terms must be revisited. As is discussed earlier in this thesis, the surface terms are generally uninteresting in most Lagrangian considerations. This is not the case in Hamiltonian formulation. For this reason we work in detail through the surface term contribution for $f(R)$ gravity. To my knowledge, there was no previous robust treatment available in the literature.

After the presentation of the equations of motion in Hamiltonian formulation we move to introduce first and second order perturbations to the system. In order to find instabilities it is possible to introduce only certain types of perturbations to the system and examine them. This is on the contrary to proving stability which would require the addition of all perturbations. For this reason we omit the perturbations in the curvature and matter distribution, as these are considered elsewhere. This leads to major simplifications in the treatment.

We choose the Poisson gauge to remove the gauge invariance. In the first order we find three non-trivial equations for large scale perturbations. While two of them are algebraic, only one, that of the conjugate momentum to the induced (three-dimensional) metric. These perturbations are linked to extrinsic curvature, which means they affect the geometry of space. It turns out that for a Universe with growing scale parameter, such as ours, these perturbations are growing. The system does not stay in the range of linear perturbations. We examine some test cases of $f(R)$ to show the breakdown of perturbative approach.

The second order perturbations are added to check if a higher order perturbation approach would fare better. Again, since we are after instabilities we add only certain kind of first order perturbations, scalar, for simplicity. It turns out much similar in the second order as in first order. The first order instability affects the second order which experiences similar behaviour.

While we examine the popular perfect fluid matter, different kinds of matter could lead to further instability issues. Same kind of approach could be for other modified gravity models as well.

8.3 Jeans analysis of Bok globules in $f(R)$ gravity

Modified gravity causes many changes to dynamics of astrophysical objects. The study of stellar objects, Solar System physics and galactic considerations has revealed much about both the desired characteristics and also limitations of alternative gravity theories. It has turned out that seemingly innocent additions to the gravitational action may lead to surprising physics.

In the paper we examine self-gravitating objects and gravitational collapse such as interstellar dust clouds. Perturbation growth in dust clouds is the mechanism through which new stars are formed. The limit for the gravitational collapse is found through Jeans analysis. At the limit the gravitational pull of dust and radiative pressure are equally strong. If the cloud is massive enough, the collapse starts.

We consider the $f(R)$ modified Jeans limit. While this has been examined before in the literature, previous treatments have used assumptions which are generally not met. In our paper we allow the matter density be dynamic, as would be the case in a collapse scenario. We further add a background to the system, which need not be Minkowskian.

The background must be carefully considered in situations where the masses and scales involved are of Solar magnitude. While the surrounding galactic and galaxy cluster certainly have an effect, this effect is constant in the considered scale. Therefore, the field created by the object in question can be considered as a perturbation on this background. In this context it is also possible to use the Birkhoff theorem, which generally is not available for use in $f(R)$ theories. We review the viability considerations in the literature on the Birkhoff theorem in this context.

We calculate the equations of motion in linear order of perturbations and find the instability limit in the $f(R)$ case and the associated dispersion relation. In the appendix we provide details for evaluating the integrals involved, which are generally not found in the literature. The surprising result is that there are two different instability limits found for $f(R)$ other than the Einstein-Hilbert case. This shows once more the unique nature of GR.

We discuss the physical meaning of the two possible limits and come to the conclusion, that only one of the can be considered physical. The remaining limit coincides with the traditional limit for the Einstein-Hilbert case. We also look into other possible solutions of the dispersion relation and discuss why these solutions can be discarded.

The found Jeans instability limit is dependent on the chosen $f(R)$ model unlike previously found in the literature [162]. Generally, the $f(R)$ contribution from viable models would lower the limit, thus assisting in star formation. The Einstein-Hilbert case provides the highest limit for collapse. At the theoretical maximum, $f(R)$ theories could lower the limit for collapse, to around 65% of the mass limit for GR. For some $f(R)$, however, the prediction does differ from that of GR.

Examining the instability limit and different classes of observed objects shows that Bok globules have physical attributes which places them at the limit. There are Bok globules which have masses just above and below the instability limit. Therefore, observing these globules would allow to measure the predictions of different $f(R)$ models.

Bok globules are considered to be birthplaces of stars and many globules are found to hide a protostar. In this light the assisted star formation in $f(R)$ theories fits with observations. With a more detailed catalogue of Bok globules it would be possible to make constraints for viable models. We demonstrate this with a small sample. For example, for the Appleby-Battye model the method could create a far stricter constraint than the ones before.

8.4 $f(R)$ gravity constraints from gravitational waves

For quantization of gravity much attention has been given to the so far hypothetical mediating particle of gravity, the graviton. In standard GR the graviton has strictly zero mass. There is no experimental proof for this zero mass, and it is likely that there never will be. Much like the case of the photon, there are only more or less stringent upper limits.

The important gravitational wave observation with LIGO provided also a limit for the Compton wavelength of the graviton. This can be interpreted as an upper limit on the mass of the graviton. The most important aspect of this limit is that it is model independent.

There are several previous upper limits on the graviton mass, and while they appear even more stringent than the LIGO one, they are typically highly model dependent. Galaxy dynamics is one of the first sources of a upper limit to graviton mass. It is also a very strict one. However, galaxy dynamics are heavily affected by dark matter, which could be otherwise explained by $f(R)$ gravity or some other modified gravity theory. In this light, the limit cannot be used to constrain modified gravity gravitons without additional investigation.

Gravitational waves reaching Earth and the LIGO instruments affect the local gravitational field. These ripples are exceedingly small, so it is prudent to consider them as perturbations on the local gravitational field, which in this scale is otherwise constant. We calculate the equations of motion in linear order of perturbations and find the graviton dispersion relation.

We find two physically possible modes. The lower one is of Planck scale and would not be detectable by any experiments. It is worth mentioning that while in basic GR the graviton has strictly zero mass, the introduction of cosmological constant, such as in ΛCDM there is a mass. This mass is the lower one of the two we find. Especially, all $f(R)$ models produce a non-zero mass graviton. In what follows I examine the modes to ensure they correspond to massive gravitons.

There are two massive solutions to the dispersion relation

$$3f''(\tilde{R})\Box^2 h - \Big(\frac{f(\tilde{R})f''(\tilde{R})}{f'(\tilde{R})} + f'(\tilde{R})\Big)\Box h + \Big(f(\tilde{R}) - \frac{2f^2(\tilde{R})f''(\tilde{R})}{f'^2(\tilde{R})}\Big)h = 0. \tag{8.1}$$

Therefore, the metric perturbation can be written as a linear combination which tell us the perturbations of the metric can be written as a linear combination

$$h_{\mu\nu} = \mathfrak{h}^{(1)}_{\mu\nu} + \mathfrak{h}^{(2)}_{\mu\nu}, \tag{8.2}$$

where the quantities $\mathfrak{h}^{(i)}$ are the metric perturbations related to the corresponding two mass solutions m_i. As the $\mathfrak{h}^{(i)}$ are related to different masses, they have different time evolution which justifies the linear combination. The components $\mathfrak{h}^{(i)}$ can be further broken down into linear superpositions. As the treatment is identical for both components, I omit the superscript and $\mathfrak{h}_{\mu\nu} \in \{\mathfrak{h}^{(1)}, \mathfrak{h}^{(2)}\}$. Now

$$\mathfrak{h}_{\mu\nu}(x) = a_{\mu\nu}e^{ik\cdot x} + {a_{\mu\nu}}^* e^{-ik\cdot x}, \tag{8.3}$$

where k_μ is the four-momentum related to the corresponding solution m_i. While this is the general form of a solution, the two sets of conditions for it to be the solution for the dispersion relation (8.1) are

$$k_\mu k^\mu = -m_i^2, \tag{8.4}$$

$$k_{\mu\nu} {a^\mu}_\nu = \frac{1}{2} k_\nu {a^\mu}_\mu. \tag{8.5}$$

As the metric is symmetric, the *polarization tensor* is symmetric as well

$$a_{\mu\nu} = a_{\nu\mu}. \tag{8.6}$$

Following mainly the treatment of [66], I show that only two of the components are independent and represent physically meaningful degrees of freedom. With a change of coordinates

$$x^\mu \to x^\mu + \epsilon^\mu(x), \tag{8.7}$$

the new metric perturbations are $\mathfrak{h}'_{\mu\nu}$, with (see [66])

$$\mathfrak{h}'_{\mu\nu} = \mathfrak{h}_{\mu\nu} - \frac{\partial \epsilon_\mu}{\partial x^\nu} - \frac{\partial \epsilon_\nu}{\partial x^\mu}. \tag{8.8}$$

As the transformation is arbitrary, with the choice

$$\epsilon^\mu(x) = i\epsilon^\mu e^{ik\cdot x} - i\epsilon^{*\mu} e^{-ik\cdot x}, \tag{8.9}$$

equation (8.3) becomes

$$\mathfrak{h}'_{\mu\nu}(x) = a'_{\mu\nu} e^{ik\cdot x} + a'^{*}_{\mu\nu} e^{-ik\cdot x}, \tag{8.10}$$

with

$$a'_{\mu\nu} = a_{\mu\nu} + k_\mu \epsilon_\nu + k_\nu \epsilon_\mu. \tag{8.11}$$

The polarization tensors $a_{\mu\nu}$ and $a'_{\mu\nu}$ represent the same physical quantities while the ϵ_μ remains arbitrary. Taking into account the conditions (8.4) and (8.5), there are six degrees of freedom in the polarization tensor and four arbitrary degrees of freedom in ϵ_μ. Therefore, there are only two physically significant degrees of freedom. Taking this into account, we can describe a wave traveling in z-direction with a wave vector

$$k^0 \equiv k, \tag{8.12}$$

$$k^1 = k^2 = 0, \tag{8.13}$$

$$(k_0)^2 - (k_3)^2 = m_i^2. \tag{8.14}$$

The conditions (8.5) yield four equations, which allow expressing a_{0i} and a_{22} in terms of the other six components

$$\frac{a_{13}k_3}{k_0} = a_{01}, \tag{8.15}$$

$$\frac{a_{23}k_3}{k_0} = a_{02}, \tag{8.16}$$

$$\frac{k_3 k_0}{(k_3)^2 + (k_0)^2}(a_{00} + a_{33}) = a_{03}, \tag{8.17}$$

$$a_{00} - a_{11} + a_{33} - \frac{2(k_0)^2}{(k_3)^2 + (k_0)^2}(a_{00} + a_{33}) = a_{22}. \tag{8.18}$$

Under the coordinate transform (8.11) only two of the components remain invariant

$$a'_{11} = a_{11}, \tag{8.19}$$

$$a'_{12} = a_{12}. \tag{8.20}$$

with the transformation of the other components depending on ϵ_μ. Thus, only a_{11} and a_{12} may have a physical significance. The exact same treatment can be used on $\mathfrak{h}^{(2)}_{\mu\nu}$. These two components are related to helicity ± 2, which emphasizes the graviton interpretation [66].

The higher graviton mass mode is only found for non-trivial $f(R)$ models. This mass could be within detectable regime. Until the black hole merger dynamics are better examined in $f(R)$ gravity, it is unknown what is the ratio of these two modes. The LIGO experiment detects the combined effects on the local metric, that is only the combined strength.

We test effect of the upper limit on the Hu-Sawicki model and find that the constraint is not a strict one. However, as the LIGO accuracy improves, this constraint is going to get more stringent. In a similar manner, using a combination of other sources for Compton wavelength limits it would be possible to narrow down the viable regime of $f(R)$ theories. Our treatment can be easily adapted for these considerations.

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Maximal symmetry and metric-affine $f(R)$ gravity

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Abstract
The affine connection in a spacetime with a homogenous and isotropic subspace is derived using the properties of maximally symmetric tensors. The number of degrees of freedom in metric-affine gravity is thereby considerably reduced while the theory allows spatio-temporal torsion and remains non-metric. The Ricci tensor and scalar are calculated in terms of the connection and the field equations derived for the Einstein–Hilbert as well as for $f(R)$ Lagrangians. By considering specific forms of $f(R)$, we demonstrate that the resulting Friedmann equations in the so-called Palatini formalism without torsion and metric-affine formalism with maximal symmetry are in general different in the presence of matter.

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1. Introduction

Based on the cosmological principle derived from the Copernican principle of mediocrity and large-scale observations, standard cosmology assumes a homogeneous and isotropic universe. One finds that there are several studies backing this assumption (e.g. [1–4]). Although the cosmological principle still holds its position as the bedrock of most cosmological models, recently the claim for homogeneity has nonetheless been seriously contested (e.g. [5–7]): one finds this credible as at least on small scales the universe is indeed very inhomogeneous.

The idea of homogeneity and isotropy of the universe has been around for a long time. Its cosmological implications have been studied thoroughly in the context of the metric formalism of the general relativity (GR). Metric-affine formulation of gravity is also an early idea (for its history, see e.g. [8]) based on general concepts of pseudo-Riemannian theory of manifolds where no *a priori* relation between the metric and the connection is assumed. However, there have been few studies into the effects of homogeneity and isotropy on the independent connection in metric-affine gravity, probably because the Einstein–Hilbert action does not make a distinction between the two formalisms.

After the initial interest, metric-affine gravity received only marginal attention until it flared again in the 1970s [9, 10]. There were high hopes that metric-affine gravity might lead us closer to quantum gravity. Failure to do so leads us to put metric-affine gravity aside once again. It functioned merely as curiosity until lately the interest in metric-affine gravity has grown rapidly since Vollick [11] argued that it is possible to explain the accelerating expansion of the universe without the cosmological constant by modifying the Einstein–Hilbert action.

In metric-affine gravity the connection is independent of the metric and has 64 components which are functions of temporal and spatial coordinates. It is clear that by assuming symmetries of the universe, say homogeneity and isotropy, the degrees of freedom should decrease. This is indeed well known to be true also for the affine connection and the consistent use of symmetry principles forms the basis of the present paper. Our aim is to study the structure of metric-affine formalism, in the context of $f(R)$ theories of gravity exploiting the symmetries of homogeneous and isotropic universe, i.e. we seek solutions in the cosmological case. More formal studies of $f(R)$ gravity with torsion have also been conducted recently, see e.g. [12, 13] and references therein.

The difference between metric and metric-affine formalisms is manifested by two important fundamental features. Torsion is allowed in metric-affine gravity unlike in GR (for a review, see [14]). The connection can also deviate from GR in non-metricity. According to Sotiriou [15] both can be induced by matter. However, there is not much experimental evidence to rule out torsion (nor non-metricity) or to prove its existence [16–19]. The debate on the possibility to measure torsion with the data from Gravity Probe B [20, 21] is also interesting. In the latter, it is found that the coupling between the physical objects with the geometrical objects is such that the non-Riemannian geometric quantities couple to the internal degrees of freedom. Therefore, torsion cannot be measured when the experiment does not contain microstructure (spin, dilaton charge and intrinsic shear). One possibility is to use nuclear magnetic resonance gyroscopes instead of mechanical gyroscopes in future experiments. One problem with measurements is the different role it plays in different theories—e.g. in teleparallelism[1] torsion represents the field strength of gravitation while in GR torsion vanishes by definition and curvature geometrizes gravity.

By using symmetry to reduce the degrees of freedom in metric the field equations become much more simple. Comparing the results in standard cosmology and in metric-affine formalism it is possible to better see the role which the independent connection plays. The present study is organized as follows: in section 2 we devise the general tools needed for the following sections. Many parts of this section can be found derived in a slightly different manner in [23]. In section 3, we consider a homogeneous and isotropic space and derive the independent components of the connection and calculate the Ricci tensor and scalar as a function of the found components. The results of section 3 are put into use in section 4. In the case of the Einstein–Hilbert Lagrangian we allow hypermomentum and calculate the Friedmann equation and see how the results relate to standard cosmology. Then we generalize to $f(R)$ actions. In section 5, we discuss our results.

2. Symmetry in spacetime

The symmetry of space can be formalized in terms of isometry and form invariance. A space is form invariant [24] under an isometric coordinate transformation $x \to \bar{x}$ if corresponding metric tensors are related by $\bar{g}_{\alpha\beta}(y) = g_{\alpha\beta}(y)$ for all y. In the case of infinitesimal transformations defined by Killing vectors $\bar{x}^{\mu} = x^{\mu} + X^{\mu}(x)$ this is easily seen to be equivalent

[1] Translation of the original papers of Einstein can be found in [22] with references to the original German versions.

with the requirement of vanishing Lie derivative $\mathcal{L}_X g_{\mu\nu} = 0$ [25, 26]. The Lie derivative can be expressed in terms of Lévi-Civitá connection, i.e. Christoffel symbol $\left\{ {\alpha \atop \mu\nu} \right\}$ as

$$\mathcal{L}_X g_{\mu\nu} = 2\partial_{(\mu} X_{\nu)} - 2X_\alpha \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\}. \tag{1}$$

The affine connection can be most generally written as a sum of a Christoffel symbol, a torsion part and a non-metricity part [9]. However, if the connection is metric, i.e. the non-metricity tensor ($Q_{\alpha\mu\nu} = -\nabla_\alpha g_{\mu\nu}$) vanishes, form invariance can be characterized by the Killing equation

$$\nabla_{(\nu} X_{\mu)} = 0. \tag{2}$$

The Killing equation still allows for a non-zero torsion tensor [27] as connections of the form

$$\Gamma^\alpha_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} + \frac{1}{2} C_{\mu\nu}{}^\alpha, \tag{3}$$

where $C_{\mu\nu}{}^\alpha$ is antisymmetric in the first two indices, fulfil (2) when (1) vanishes. From now on we assume the connection to be of the form (3) in order to use the Killing equation. This is in accordance with the argument of [15] that the connection must be constrained in some way to produce a viable theory.

For a general tensor we require invariance in an infinitesimal isometric transformation as for all y

$$T'^{\mu\nu\ldots}_{\alpha\beta\ldots}(y) = T^{\mu\nu\ldots}_{\alpha\beta\ldots}(y) \tag{4}$$

leading to the conditions

$$0 = \frac{\partial X^\alpha}{\partial x^\mu} T_{\alpha\nu\ldots}(x) + \frac{\partial X^\beta}{\partial x^\nu} T_{\mu\beta\ldots}(x) + \cdots + X^\lambda(x) \frac{\partial}{\partial x^\lambda} T_{\mu\nu\ldots}(x). \tag{5}$$

In a maximally symmetric space, the requirement that the number of independent Killing vectors is maximal, i.e. equations (5) are satisfied, strongly restricts invariant tensors [24].

A form invariant scalar in a maximally symmetric space must always be a constant. For higher rank tensors the invariance equation can be written as

$$\delta^\alpha_\mu T^\beta{}_{\nu\ldots} + \delta^\alpha_\nu T_\mu{}^\beta{}_{\ldots} + \cdots = \delta^\beta_\mu T^\alpha{}_{\nu\ldots} + \delta^\beta_\nu T_\mu{}^\alpha{}_{\ldots} + \cdots. \tag{6}$$

For our purposes the invariance conditions for tensors of ranks 1, 2 and 3 in a four-dimensional spacetime with a maximally symmetric three-dimensional subspace are needed. The first two can be easily found in the literature, e.g. [24]. For rank 3 tensor the result is seldom calculated explicitly. From here on we use Latin indices for the maximally symmetric subspace while the Greek indices refer to four-dimensional spacetime.

The cases of form invariant covariant tensors of ranks 1 and 2 easily yield that

$$A_i = 0, \tag{7a}$$

$$B_{ij} = f g_{ij}, \tag{7b}$$

where the function f does not depend on the coordinates of the maximally symmetric subspace. Applying (6) to a form invariant rank 3 tensor and contracting indices we get three equations

$$(N-1)C_{njk} + C_{jnk} + C_{kjn} = 0, \tag{8a}$$

$$C_{jnk} + (N-1)C_{njk} + C_{nkj} = 0, \tag{8b}$$

$$C_{njk} + C_{knj} + (N-1)C_{jnk} = 0, \tag{8c}$$

where we have adopted a more general notation with N indicating the dimensions of the maximally symmetric subspace. From these we obtain two useful conditions for form invariant tensors: they are invariant under cyclic index permutations,

$$C_{kjn} = C_{nkj}, \tag{9}$$

and they are antisymmeric in the first two indices, except for $N = 3$, since

$$(N-3)C_{[nj]k} = 0. \tag{10}$$

From the set of conditions above ((8*a*), (9), (10)) it follows that all form invariant tensors of rank 3 vanish unless the maximally symmetric subspace is three dimensional, i.e. $N = 3$. As the torsion and non-metricity tensors are of rank 3, they may hence exist only in three-dimensional maximally symmetric (sub)spaces (see also [27]). One might consider that torsion and non-metricity need not be maximally symmetric tensors. However, physically it is not sensible. With $N \neq 3$ the connection is then necessarily the Lévi-Civitá connection.

3. Homogeneous and isotropic space

3.1. Affine connection

A metric with a homogeneous and isotropic subspace can be written in spherical coordinates as [24]

$$\mathrm{d}s^2 = b^2(t)\,\mathrm{d}t^2 - a^2(t)\tilde{g}_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j, \tag{11}$$

where

$$\tilde{g}_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = \frac{1}{1-kr^2}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2 \tag{12}$$

is the metric of the spatial part with $k \in \{-1, 0, 1\}$. Usually a rescaling of the time coordinate is performed [28] to remove the function $b(t)$ but at this point we postpone doing this. This ensures that we can calculate the equations of motion by varying the action with respect to $a(t)$ and $b(t)$ instead of varying with respect to the full metric tensor.

Taking advantage of the symmetries of spacetime, we require that covariant derivative of a maximally symmetric tensor preserves invariance, i.e. maximal symmetry. Thereupon, we can reduce the number of degrees of freedom in the connection by utilizing results of the previous section. First we consider a maximally symmetric covariant vector V_ν. According to (7*a*) only $V_0 \neq 0$ and $V_0 = V_0(t)$. Hence

$$\nabla_0 V_0 = \partial_0 V_0 - \Gamma^0_{00} V_0 = d(t) \Rightarrow \Gamma^0_{00} \equiv c_0(t), \tag{13}$$

where $d(t)$ is some function of time. We see that Γ^0_{00} depends on time only. Moreover

$$0 = \nabla_0 V_i = \partial_0 V_i - \Gamma^\alpha_{0i} V_\alpha \Rightarrow \Gamma^0_{0i} \equiv 0, \tag{14a}$$

$$0 = \nabla_i V_0 = \partial_i V_0 - \Gamma^\alpha_{i0} V_\alpha \Rightarrow \Gamma^0_{i0} \equiv 0, \tag{14b}$$

$$f(t)\tilde{g}_{ij} = \nabla_i V_j = \partial_i V_j - \Gamma^\alpha_{ij} V_\alpha \Rightarrow \Gamma^0_{ij} = -\frac{f(t)}{V_0}\tilde{g}_{ij} \equiv c_n(t)\tilde{g}_{ij}. \tag{14c}$$

Keeping in mind that maximally symmetric contravariant vectors only have one nonvanishing component, one finds that

$$\Gamma^i_{0j} = c_t(t)\delta^i_j. \tag{15}$$

Similar constraints can be derived for rank 2 tensors, for example

$$0 = \nabla_0 B_{0i} = \partial_0 B_{0i} - \Gamma^\beta_{00} B_{\beta i} - \Gamma^\beta_{i0} B_{0\beta} = -\Gamma^i_{00} B_{ii} \tag{16}$$

(no sum in the last form), implying that $\Gamma^i_{00} = 0$. Correspondingly the $0ij$-component gives

$$\Gamma^j_{i0} = c_s(t)\delta^j_i. \tag{17}$$

The discussion above covers 37 components of the connection reducing them to four independent components c_0, c_t, c_n and c_s. The last 27 components are found using the results of section 2 in three dimensions. Assuming that the non-metricity tensor vanishes in the maximally symmetric subspace the connection can be written as

$$\Gamma^k_{ij} = \begin{Bmatrix} k \\ ij \end{Bmatrix} + K(t)\epsilon_{ij}{}^k, \tag{18}$$

where ϵ_{ijk} is the three-dimensional Lévi-Civitá symbol. Note that here the second term, i.e. the contortion tensor, is invariant under cyclic permutations leaving only one degree of freedom.

Thus the connection preserving maximal symmetry in a three-dimensional homogeneous and isotropic subspace can be reduced to four spatio-temporal components $c_i(t)$, one component, $K(t)$, characterizing spatial torsion and the usual metric Christoffel symbols of a maximally symmetric subspace. Their usual metric counterparts are

$$c_0 = 0 \tag{19a}$$

$$c_s = c_t = \frac{\dot a}{a} \tag{19b}$$

$$c_n = a\dot a \tag{19c}$$

$$K = 0 \tag{19d}$$

with $b(t) = 1$.

3.2. The Ricci tensor and scalar

The Ricci tensor and curvature scalar are now straightforwardly calculable. The Ricci tensor is given by [29]

$$R_{\mu\nu} = \partial_\alpha\Gamma^\alpha_{\nu\mu} - \partial_\nu\Gamma^\alpha_{\alpha\mu} + \Gamma^\beta_{\nu\mu}\Gamma^\alpha_{\alpha\beta} - \Gamma^\beta_{\alpha\mu}\Gamma^\alpha_{\nu\beta}. \tag{20}$$

The components $0i$ and $i0$ vanish as they are maximally symmetric vectors of rank 1 in the subspace. The temporal 00 component reads as

$$R_{00} = 3(-\dot c_s + c_0c_s - c_sc_t) \tag{21}$$

and the spatial components can be expressed as

$$R_{ij} = \tilde R_{ij} + (\dot c_n + c_nc_0 + 2c_nc_s - c_tc_n)\tilde g_{ij} + S_{ij}, \tag{22}$$

where $\tilde R_{ij}$ is the standard Ricci tensor of the spatial part. Here the last term carries information on spatial torsion,

$$S_{ij} \equiv \partial_k(K\epsilon_{ji}{}^k) + K\epsilon_{ji}{}^k\begin{Bmatrix} l \\ lk \end{Bmatrix} + 2K\epsilon_{l[j}{}^k\begin{Bmatrix} l \\ i]k \end{Bmatrix} - K^2\epsilon_{li}{}^k\epsilon_{jk}{}^l. \tag{23}$$

As S_{ij} is antisymmetric and g_{ij} symmetric, contraction of the Ricci tensor yields

$$R = -\frac{3}{a^2}\left(2k + \dot c_n + \dot c_s\frac{a^2}{b^2} + 2c_nc_s + C\left(c_n - c_s\frac{a^2}{b^2}\right) - 2K^2\right), \tag{24}$$

where we have used the fact that $\tilde R = -6k/a^2$ and denoted $C \equiv c_0 - c_t$. Note that one can also derive the curvature scalar by using only the torsion tensor instead of the connection, as was done in [27].

4. Field equations

4.1. The Einstein–Hilbert action with hypermomentum

Although our goal is to study the results of the previous section in a general $f(R)$ model, let us first consider the Einstein–Hilbert action in a universe containing matter with non-zero hypermomentum and perfect fluid style energy–momentum. This is by no means the first time these equations of motion are derived (cf [23]) but this provides us a way to fix our notation. We use the energy–momentum and hypermomentum tensors defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\partial S_m}{\partial g^{\mu\nu}}, \qquad \Delta_\alpha{}^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta\Gamma^\alpha_{\mu\nu}}. \tag{25}$$

In the case of the Einstein–Hilbert action we set

$$f(R) = R. \tag{26}$$

We also set $b(t) = 1$ which we know must be a solution [30]. Now the equations of motion are

$$2\kappa\frac{T_i^i}{3} = -\frac{1}{a^2}(2k + c_n(C+2c_s) - 2K^2 + 3\dot{c}_n - 2\dot{c}_s) + 3(Cc_s - \dot{c}_s) \tag{27a}$$

$$2\kappa T^0{}_0 = -\frac{3}{a^2}(2k + c_n(C+2c_s) - 2K^2 + \dot{c}_n) + 3(Cc_s - \dot{c}_s) \tag{27b}$$

$$\frac{\Delta_0{}^{00}}{6} = c_s - \frac{c_n}{a^2} \tag{27c}$$

$$\frac{a^2}{6}\Delta_0{}^{ij}\tilde{g}_{ij} = \frac{\dot{a}}{a} - C - 2c_s \tag{27d}$$

$$\frac{\Delta_i{}^{i0}}{6} = C + \frac{3a\dot{a} - 2c_n}{a^2} \tag{27e}$$

$$\frac{a^2}{12}\Delta_k{}^{ij}\epsilon_{ij}{}^k = K. \tag{27f}$$

From these equations the functions c_i can be solved

$$c_n = -\frac{a}{24}\left(2a\Delta_0{}^{00} + a\Delta_i{}^{i0} + a^3\Delta_0{}^{ij}\tilde{g}_{ij} - 24\dot{a}\right) \tag{28a}$$

$$C = \frac{1}{12}\left(\Delta_i{}^{i0} - a^2\Delta_0{}^{ij}\tilde{g}_{ij} - 12\frac{\dot{a}}{a}\right) \tag{28b}$$

$$c_s = \frac{1}{24}\left(2\Delta_0{}^{00} - \Delta_i{}^{i0} - a^2\Delta_0{}^{ij}\tilde{g}_{ij}\right) + \frac{\dot{a}}{a} \tag{28c}$$

Inserting these into the equations of motion for a and b and choosing perfect fluid energy–momentum tensor we find the Friedmann equation

$$H^2 = \frac{\kappa\rho_0}{3a^3} - \frac{k}{a^2} + HA_1 + A_2 + a^2A_3 - a^4A_4. \tag{29}$$

with

$$A_1 = \frac{\Delta_0{}^{00}}{4} + \frac{a^2}{6}\Delta_0{}^{ij}\tilde{g}_{ij} \tag{30a}$$

$$A_2 = \frac{1}{144}\left(\left(\Delta_0{}^{00}\right)^2 - \Delta_0{}^{00}\Delta_i{}^{i0} + \frac{\left(\Delta_i{}^{i0}\right)^2}{4}\right) \tag{30b}$$

$$A_3 = \frac{1}{144}\left(\left(\Delta_k{}^{ij}\epsilon_{ij}{}^{k}\right)^2 + \frac{\Delta_0{}^{ij}\Delta_k{}^{k0}\tilde{g}_{ij}}{2} - \Delta_0{}^{00}\Delta_0{}^{ij}\tilde{g}_{ij}\right) \tag{30c}$$

$$A_4 = \frac{1}{192}\left(\Delta_0{}^{ij}\tilde{g}_{ij}\right)^2. \tag{30d}$$

From this form we clearly see that setting hypermomentum to zero yields general relativity as was to be expected.

4.2. A General f(R) Lagrangian without hypermomentum

Modified gravity theories in which the Lagrangian is a function of the curvature scalar have received much attention (e.g. [31–34]). Adding terms of the type R^n is a natural and simple modification to the general relativity. This type of terms can produce early time inflation [35] and late time accelerating expansion [31].

Further motivation for the $f(R)$ gravity can be found in the fact that is equivalent to a certain class of scalar tensor theories [36]. One might ask why restrict to functions of only the curvature scalar. Simplicity is one reason but there are also underlying problems with Lagrangians depending upon more than one time derivative [37, 38]. Functions of curvature scalar only avoid the linear instability troubling other possibilities.

However, using the metric formalism there are problems with the $f(R)$ gravity [39]. Most of the work on $f(R)$ gravity is done in metric formalism. Here we look into the possibilities of using metric-affine formalism and maximal symmetry together with $f(R)$ gravity.

The analysis in a general $f(R)$ theory with matter follows along similar lines as above. We assume that the matter Lagrangian $\mathcal{L}_m$ does not depend explicitly on the connection, i.e. the hypermomentum is zero. This does not necessarily hold for the cosmic fluid but it still possesses some interesting characteristics. In this case the gravitational Lagrangian is given by $\mathcal{L} = ba^3 f(R(a, b, C, c_n, \dot{c}_n, c_s, \dot{c}_s, K))$ and the field equations are now

$$2\kappa\frac{T_i^i}{3} = f(R) + \frac{2}{a^2}(2k + 2c_n c_s + \dot{c}_n + Cc_n - 2K^2)f'(R) \tag{31a}$$

$$2\kappa T^0{}_0 = f(R) + \frac{6}{b^2}(\dot{c}_s - Cc_s)f'(R) \tag{31b}$$

$$0 = f'(R)\left(\frac{c_s}{b^2} - \frac{c_n}{a^2}\right) \tag{31c}$$

$$f''(R)\dot{R} = f'(R)\left(C + 2c_s - \frac{\dot{b}}{b} - \frac{\dot{a}}{a}\right) \tag{31d}$$

$$f''(R)\dot{R} = -f'(R)\left(C - 2c_n\frac{b^2}{a^2} - \frac{\dot{b}}{b} + 3\frac{\dot{a}}{a}\right) \tag{31e}$$

$$0 = f'(R)K. \tag{31f}$$

If $f'(R) \neq 0$, the third and the last equations are readily solvable,

$$c_s = \frac{b^2}{a^2}c_n, \qquad K = 0. \tag{32}$$

Summing equations (31*d*) and (31*e*) and using (32) we find

$$C = \frac{\dot{b}}{b} - \frac{\dot{a}}{a}. \tag{33}$$

Combining (31*a*), (31*b*), (32) and (33) gives

$$\frac{b^2}{a^2}c_n^2 + k + c_n\left(\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right) - \dot{c}_n = \frac{\kappa a^2\left(T_i^i - 3T^0{}_0\right)}{6f'(R)}. \tag{34}$$

Because the curvature scalar R can be expressed in terms of c_n, a and b, equation (34) is a nonlinear first-order equation for c_n. It can be solved, at least in principle, for a given $f(R)$.

In the absence of matter summing the first two equations gives the trace equation,

$$f'(R)R - 2f(R) = 0. \tag{35}$$

This differential equation is readily solved for

$$f(R) = cR^2 \tag{36}$$

with some constant c. Because by plugging in a given $f(R)$ we can solve the equation for a constant R, empty space is necessarily a space with constant scalar curvature. Equations (31*d*) and (31*e*) then yield

$$c_s = \frac{\dot{a}}{a}. \tag{37}$$

Thus we end up with same components for the connection as for the case of the Einstein–Hilbert action without matter. We can hence conclude that in a homogeneous and isotropic space without matter, the metric-affine formalism results in the same equations as metric formalism. As an easy check shows, adding the cosmological constant leaves the situation unaltered. Therefore, the possible new effects of metric-affine formalism are due to matter.

With matter that is not coupled to the independent connection, we still get equations (32)–(34). The trace equation, however, changes. If the matter energy–momentum tensor is of perfect fluid form with $T_0^0 = -\rho$ and $T_i^i = 3p$ we have

$$f'(R)R - 2f(R) = \kappa(3p - \rho). \tag{38}$$

Here we note that in the special case of radiation filled universe the right-hand side vanishes and once again we reproduce the results of metric formalism. Moreover, if the hypermomentum were present all the aforementioned equations would change. Even the simple (31*f*) would become non-trivial and giving $K \propto (a^3 f'(R))^{-1}$. As the nature of the gravitation–matter coupling is not completely clear even this approach has some potential interest.

Although a radiation-dominated universe reproduces the metric cosmology, this is not a general property. For example, if we choose $f(R) = R + \lambda R^2$, with λ some small constant, and examine a non-relativistic matter filled universe, the trace equation (38) yields

$$R = \kappa\rho = \frac{\kappa\rho_0}{a^3}, \tag{39}$$

where ρ_0 is a constant and we have rescaled time so that $b = 1$. From equations (32), (33) and (31*d*) we then get

$$c_n = \frac{a\dot{a}(a^3 - \kappa\rho_0\lambda)}{2\kappa\rho_0\lambda + a^3}. \tag{40}$$

Clearly we need $\rho_0 = 0$ in order to reproduce $c_n = \dot{a}a$ (i.e. the metric solution), leaving empty space as the only possibility. If, however, we allow for non-Lévi-Civitá connections there are other possibilities. Inserting equation (40) into (31*b*) and (39) we can eliminate $\ddot{a}$ to obtain an effective Friedmann equation

$$H^2 = -\frac{(2\kappa\lambda + a^3)\left(\frac{1}{2}\kappa^2\lambda\rho_0^2 + 6\kappa\lambda ka - \kappa\rho_0 a^3 + 3ka^4\right)}{3a^3(a^3 - \kappa\lambda\rho_0)^2}. \tag{41}$$

If we expand this equation in λ, the result is

$$H^2 = \frac{\kappa\rho_0}{3a^3} - \frac{k}{a^2} + \left(\frac{7\kappa^2\rho_0^2}{6a^6} - \frac{6\kappa\rho_0 k}{a^5}\right)\lambda + \mathcal{O}(\lambda^2). \tag{42}$$

The limit $\lambda \to 0$ coincides with standard cosmology as expected. Note, that the correction $\propto a^{-6}$ can be created also by adding non-metric matter coupling, i.e. hypermomentum, as in [27], but here it is created solely by the form of the gravitational Lagrangian. Comparing equation (41) to the results in the so-called Palatini formalism[2] [40–42] we find that they agree.

This raises the question, whether our maximally symmetric approach generally coincides with the results in the more commonly considered Palatini formalism. Another reason to suspect similar results is that in [12] metric-affine formalism with fully vanishing non-metricity is found to produce the same dynamics as the Palatini formalism.

In order to answer this question, we consider a toy model where the Lagrangian is of the form $f(R) = R^n$. Here we can check that equation (34) accepts this kind of Lagrangian and easily see that it is acceptable. Following the procedure above results in an effective Friedmann equation

$$H^2 = -\frac{4n^2k}{(n-3)^2a^2} - \frac{2n(n+1)}{3(n-3)^2}A^{\frac{1}{n}}, \tag{43}$$

where $A = \frac{\kappa\rho_0}{(n-2)a^3}$. The corresponding equation in the Palatini formalism reads as

$$H^2 = \frac{2n\left((1-n)A^{\frac{1}{n}}a^3 + 2\kappa\rho_0 A^{\frac{1-n}{n}} - 6nka\right)}{3a^3((7n+6)n-9)}. \tag{44}$$

Hence, we see that the coincidence in the λR^2 model was an exception: the maximally symmetric formalism and the Palatini formalism in general lead to different dynamical equations. The difference is pronounced in the case of $n = 3$, where the Palatini formalism is well behaved but here we find that our approach is singular in the sense that no Friedmann equation can be derived. Note, that there is also singularity at $n = 2$ in both cases as the trace equation for $f(R) = R^2$ holds only in empty space. Our result should be compared with the result of [12] where it was found that metric-affine formalism with torsion only does coincide with Palatini formalism.

A much studied special case of the $f(R)$ models is the $f(R) = R - \frac{\mu^4}{R}$ (cf [11, 31]). Inserting this $f(R)$ and a dust-dominated universe yields the Friedmann equation. The corrections to general relativity in this formalism are most easily seen when we expand in μ

$$H^2 = \frac{\kappa\rho_0}{3a^3} - \frac{k}{a^2} + \frac{2a^3(9ka - 4\kappa\rho_0)}{3\kappa^2\rho_0^2}\mu + \mathcal{O}(\mu^2). \tag{45}$$

Clearly this is different from that in metric formalism. This is important as in [39] it is shown that in the metric formalism this action leads to instability effectively ruling it out as a viable

[2] The term Palatini formalism is widely used but misleading as it was in fact Einstein who first varied independently metric and connection [8]. We have adopted the convention of Sotiriou, i.e. we will call $f(R)$ theories in which the matter action is chosen to be independent of the connection, $f(R)$ theories of gravity in the Palatini formalism, to make the distinction from metric-affine $f(R)$ gravity [15].

model. As the different Friedmann equations mean different dynamics it is possible to avert this instability.

5. Conclusions

In this paper, we have studied a homogeneous and isotropic spacetime with a maximally symmetric formalism in $f(R)$ theories of gravity. Even though this is not the most general case homogenous and isotropic spacetime is an important special case in cosmology. The effects of homogeneity and isotropy in the standard Einstein–Hilbert case has been discussed before [27] but here we have shown that even in more general $f(R)$ theories, only one spurious extra degree of freedom appears in empty space.

Interesting possibilities begin to emerge, when one includes matter in the system. In the case of the Einstein–Hilbert action, the addition of matter without hypermomentum does not change the solutions of the field equations from those of metric formalism. New types of solutions appear only if the matter Lagrangian has an explicit dependence on the connection [27], in which case the connection is even less determined for general $f(R)$ Lagrangians. These results are in accordance with those of [15] where it was argued that torsion is caused by the antisymmetric part of the hypermomentum.

However, even for ordinary matter (i.e. no hypermomentum), the construction of the Friedmann equations reveals that the maximally symmetric formalism is dynamically different from the corresponding Palatini formalism although they may coincide in some special cases. This appears to be a consequence of inclusion of spatio-temporal non-metricity. Indeed the difference between the two formalisms is due to the fact that in the Palatini formalism torsion is assumed to vanish *a priori* whereas here only spatial non-metricity is assumed to vanish. Therefore the degrees of freedom in these two approaches are dissimilar resulting in a differently constrained system. Physically it is unclear which approach one should adopt. As there is almost no evidence for torsion, the usual pick would be Palatini formalism. Metric-affine formalism, however, is more general and is based on the explicit use of the cosmological principle.

In [15] it is argued that all constraints on non-metricity also place a constraint on the form of the Lagrangian and should therefore be avoided. We agree with the first argument. Our equation (34) is an example of these constraints. However, in this case the constraint allows non-trivial forms of $f(R)$. Thus, we do not see the necessity for the latter argument. We find that our formalism reduces to Palatini formalism only in special cases. This is not in contradiction with [15] since our assumption of vanishing spatial components of non-metricity differs from the assumption of [15] (i.e. the Weyl vector vanishes).

In all cases a spurious degree of freedom which has little or no physical meaning remains. It emerges because two components of the connection appear only as a certain combination in the Lagrangian. As they affect the physics of the universe only via this combination, their geometrical interpretation can be found if there are non-metric matter couplings present.

The cosmological consequences of the maximally symmetric formalism are an interesting possible direction of studies as well as generalization to spherically symmetric systems. Both are likely to give at least some constraints for a given $f(R)$ theory. Cosmological data (e.g. CMB and supernova data) could be fitted to metric-affine gravity models with maximal symmetry in order to find the constraints in this formalism.

Furthermore, although isotropy is commonly accepted there have been numerous articles investigating the possibility of an inhomogeneous universe [5–7, 43, 44], motivating further study of the connection in an inhomogeneous and isotropic space. These results could be used to ease the usage of metric-affine formalism in spherically symmetric universes.

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Hamiltonian perturbation theory in $f(R)$ gravity

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Hamiltonian perturbation theory is used to analyze the stability of $f(R)$ models. The Hamiltonian equations for the metric and its momentum conjugate are written for the $f(R)$ Lagrangian in the presence of perfect fluid matter. The perturbations examined are perpendicular to R. As perturbations are added to the metric and momentum conjugate to the induced metric instabilities are found, depending on the form of $f(R)$. Thus the examination of these instabilities is a way to rule out certain $f(R)$ models.

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I. INTRODUCTION

The question of dark energy has been at the heart of cosmology since the discovery of accelerating expansion of the universe [1]. The traditional picture of general relativity with ordinary relativistic or nonrelativistic matter in a homogeneous and isotropic universe meets severe problems when accommodating it to current cosmological observations. The conflicting observational evidence comes mainly from supernova light curves [1,2], CMB anisotropies [3,4], and large scale structures [5,6]. This has lead to several suggested remedies. Perhaps the most popular way is to add some nonconventional matter to the universe. Among these the simplest possibility is no doubt to use the cosmological constant. A review of the subject can be found in [7]. In any case, the key aspect is the negative pressure of the new matter which boosts the expansion of the universe. Other considerations include more general distribution of matter, i.e. nonhomogeneous or nonisotropic universe (see e.g. [8]).

Besides these two, a lot of effort has been put into studies on generalizations and modifications of general relativity. For example, metric-affine theories (see e.g. [9]), scalar-tensor theory (see e.g. [10,11]), brane-world gravity (see e.g. [12]), and more general Lagrangians have been considered. In the present paper we are especially interested in $f(R)$ gravity models in which the Einstein-Hilbert action is replaced by a function of the curvature scalar R [13–18]. None of these modifications is free of problems and this is indeed the case of $f(R)$ gravity as well. As for any model, the cosmological observations issue some constraints (see e.g. [19,20]) as do the observations in the solar system (see e.g. [21–26]). The opinions are still divided on the viability of $f(R)$ theories of gravity. There are numerous approving studies (see e.g. [27,28]) as well as skeptical ones (see e.g. [29,30]).

As the actual universe is not homogeneous and isotropic but contains local perturbations, additional challenges for $f(R)$ theories emerge from stability analysis [31–33]. An acceptable cosmological model has to be stable against perturbations in the metric and the mass distribution. However, stability analysis is customarily done only in the direction of R, i.e. only curvature perturbations are considered. This is motivated, in particular, in the case of general relativity, where the relation between space-time curvature and the matter density is a simple one: the trace of Einstein equations implies $R \propto \rho$. This in turn implicates a simple and direct relation between the perturbations in matter and curvature. This is not the only possibility. In a $f(R)$ model, the relation is more complicated due to appearance of function $f(R)$ and higher derivative terms in the field equations. The phase space is considerably larger and metrics corresponding a given matter distribution ambiguous. The physical acceptability, however, of a model requires general stability; also stability against perturbations which keep curvature constant, perpendicular to R.

The Hamiltonian formulation of general relativity has been around since the work of Arnowitt, Deser, and Misner [34].[1] Hamiltonian formulation has also surfaced in the works of Ashtekar [35]. The first papers on the subject often neglected the boundary terms, however, later works have clarified these details (e.g. [36–38]). Hamiltonian formulation has not received too much interest in contemporary papers. In particular and to our knowledge, the use of Hamiltonian formulation on perturbations of $f(R)$ theories has not been studied so far. The main interest has been in specific choices for the function $f(R)$.

In the present paper we look into perturbations using Hamiltonian formalism of $f(R)$ theories. While the technique has not yet been applied to general $f(R)$ theories with perturbations, it is a useful tool in studying the stability of $f(R)$ models: with it is simple to study perturbations perpendicular to R. As in classical mechanics, the Hamiltonian is written as a functional of the fields and their canonical momenta. However, in a geometric theory like general relativity and $f(R)$ theories, some complications appear due to constraints between field components. The two main aspects of the canonical Hamiltonian formalism are that the field equations are of the first order in the time derivatives and that time is distinguished from

[1]The ideas were first seen in the long out of print *Gravitation: An Introduction to Current Research*. The authors have later on released the article on ArXiv as cited.

other coordinates. For writing the Hamiltonian equations, we must thus foliate the region of space-time with spacelike hypersurfaces. Finally, the resulting field equations for the perturbations are then analyzed for instabilities. The conventions and details of the formalism can be found in [39].

As shall be seen, the formulation presents a nondynamic field. We therefore use a combination of Hamiltonian and Lagrangian formalisms, i.e., we do not perform the Legendre transformation on all the fields. Using partial Legendre transformation (see e.g. [40,41] for discussions on the transformation) is possible due to writing the $f(R)$ action in the form of a scalar-tensor theory. This way we avoid many of the usual complications of higher order theories. The other possible ways to proceed with full Legendre formulation would include constraints via the Dirac-Bergmann algorithm [42–45] and transformation to the Einstein metric [46]. In this case, however, the action can be written in a form where the singularity of the Lagrangian can be avoided. As shall be seen, our treatment does not suffer from singularity. Therefore, we find the method of partial Legendre transformation the simplest tool.

The paper is organized as follows. In Sec. II we write the Hamiltonian field equations. The 3 + 1 decomposition and foliation of the space-time are also presented. The first order perturbations are added to the system in Sec. III. We also take a look at second order perturbations in Sec. IV. In Sec. V we summarize our results.

II. HAMILTONIAN FORMULATION

In this section we mainly follow the treatment of [39]. Another overview of Hamiltonian (and Lagrangian) formulation of general relativity can be found in [47]. As $f(R)$ theories of gravity can be written as scalar-tensor theories [21], we start by writing the $f(R)$ action as

$$\begin{aligned} S = S_G + S_M &= \frac{1}{16\pi}\int_V f(R)\sqrt{-g}d^4x + S_s + S_M \\ &= \frac{1}{16\pi}\int_V \sqrt{-g}(f(\varphi) + f'(\varphi)R - f'(\varphi)\varphi)) \\ &\quad \times d^4x + S_s + S_M, \end{aligned} \tag{1}$$

where V is a volume of space-time, S_s is a surface term which we will cover later on, S_M is the matter term, and φ is the auxiliary scalar field. Variation with respect to φ would lead to the equation $f''(\varphi)(R - \varphi) = 0$. Therefore $R = \varphi$ unless $f''(\varphi) = 0$. If $f''(\varphi) = 0$ we would not get a field equation for φ from (1). From here on we make the assumption $f''(\varphi) \neq 0$ unless stated otherwise.

For the purposes of writing the action in terms of the Hamiltonian, a 3 + 1 decomposition is needed. We foliate the space-time volume V with spacelike hypersurfaces Σ_t of constant time. We shall use the Greek alphabet for space-time and the Latin alphabet for space. By decomposing the line element to (three) scalar, (three) vector, and (three) tensor parts as

$$ds^2 = -N^2dt^2 + h_{ab}(dy^a + N^a dt)(dy^b + N^b dt), \tag{2}$$

one defines the lapse N, the shift N_a, and the induced metric $h_{ab} = g_{\mu\nu}e^\mu_a e^\nu_b$ on the hypersurface Σ_t. By using these introduced quantities the invariant volume element reads $\sqrt{-g}d^4x = N\sqrt{h}dtd^3x$. The Ricci scalar can be written in terms of the extrinsic curvature K_{ab}, n_α the unit normal to the boundary ∂V, and $\tilde{R}$ the induced Ricci scalar of the (three dimensional) metric h_{ab} as

$$R = \tilde{R} + K^{ab}K_{ab} - K^2 - 2\nabla_\alpha(\nabla_\beta n^\alpha n^\beta - n^\alpha\nabla_\beta n^\beta). \tag{3}$$

Extrinsic curvature is the measure of shrinkage and deformation of an object upon being moved a unit interval of proper time into the enveloping space-time. It can be written as a function of the induced metric, the lapse, and the shift, which appear to be the fields we are finally interested in:

$$K_{ab} = \frac{1}{2N}(\dot{h}_{ab} - N_{a|b} - N_{b|a}). \tag{4}$$

The surface terms of the action (1) are not of special interest in this paper. However, it is not trivial that these parts do not affect the results. Generally, the surface term must be added to the action in order to avoid the need for further boundary conditions (e.g. [48]).

By choosing the space-time volume V such that, its boundary can be written as a union of two spacelike hypersurfaces Σ_{t_2}, $-\Sigma_{t_1}$ with normals pointing outwards and a timelike hypersurface $\mathcal{B}$, i.e. $\partial V = \Sigma_{t_2} \cup (-\Sigma_{t_1}) \cup \mathcal{B}$. The surface term reads

$$\begin{aligned} S_s &= \frac{1}{8\pi}\oint_{\partial V} \epsilon f'(\varphi)K|h|^{1/2}d^3y - S_0 \\ &= \frac{1}{8\pi}\bigg(\int_{\Sigma_{t_1}} f'(\varphi)K\sqrt{h}d^3y - \int_{\Sigma_{t_2}} f'(\varphi)K\sqrt{h}d^3y \\ &\quad + \int_{\mathcal{B}} f'(\varphi)\mathcal{K}\sqrt{-\gamma}d^3y\bigg) - S_0, \end{aligned} \tag{5}$$

where $\epsilon = n^\alpha n_\alpha$. Here $S_0 = \frac{1}{8\pi}\oint_{\partial V}\epsilon K_0\sqrt{|h|}d^3y$ is a nondynamical subtraction term, the purpose of which is to prevent the integral from diverging in the limit when the spatial boundary S_t is pushed to the infinity. The constant K_0 is the extrinsic curvature of the boundary ∂V embedded in flat space-time. In the last term γ is the induced metric on $\mathcal{B}$ and $\mathcal{K}$ is the extrinsic curvature scalar of $\mathcal{B}$. However, this is not the only term contributing to the surface part. The term $f'(\varphi)R$ from (1) produces surface and volume terms, namely,

$$\int_V f'(\varphi)R\sqrt{-g}d^4x = \int_{t_1}^{t_2} dt \int_{\Sigma_t} f'(\varphi)(\tilde{R} + K^{ab}K_{ab} - K^2)N\sqrt{h}d^3y - 2\oint_{\partial V} f'(\varphi) \times (\nabla_\beta n^\alpha n^\beta - n^\alpha \nabla_\beta n^\beta)d\Sigma_\alpha. \quad (6)$$

When combining these surface contributions the first two terms in (5) are eliminated. The only surface term left from (6) is

$$\begin{aligned} &-2\int_{\mathcal{B}} f'(\varphi)(\nabla_\beta n^\alpha n^\beta - n^\alpha \nabla_\beta n^\beta)d\Sigma_\alpha \\ &= -2\int_{\mathcal{B}} f'(\varphi)(\nabla_\beta n^\alpha)n^\beta r_\alpha \sqrt{-\gamma}d^3y \\ &= 2\int_{\mathcal{B}} f'(\varphi)(\nabla_\beta r_\alpha)n^\beta n^\alpha \sqrt{-\gamma}d^3y, \end{aligned} \quad (7)$$

where r_α is the perpendicular unit vector of the boundary S_t of Σ_t, i.e. $r^\alpha r_\alpha = 1$ and $r^\alpha n_\alpha = 0$. Summing the remaining surface terms we obtain

$$S_S = 2\oint_{S_t} (k - k_0)f'(\varphi)N\sqrt{\sigma}d^2\theta. \quad (8)$$

We have also introduced the induced metric on the boundary S_t, $\sigma_{AB} = h_{ab}e^a_A e^b_B$, and σ is its trace. The extrinsic curvature of S_t embedded in Σ_t is $k_{AB}e^a_A e^b_B \nabla_b r_a$, k is its trace, and similarly k_0 with the embedding in flat space. The constant k_0 comes from the subtraction term.

We now have the surface part of the action ready for construction of the Hamiltonian. We shall see later on that the surface term (8) is indeed canceled in the process of calculating the field equations. Many of the technical details were omitted and we refer the reader to [39]. The generalization to $f(R)$ is easy.

In the Hamiltonian formulation field equations are found for fields and their momentum conjugates. Here the fields are h_{ab}, N, N_a, and φ. It turns out that in the case of $f'(R)$ gravity we need only the momentum conjugate to the induced metric h_{ab}. This is due to the fact that it is the sole dynamical variable. The lapse and the shift are related to the arbitrary foliation so this should not be unexpected. Thus we perform a partial Legendre transformation [40]. The momentum conjugate can be written using the extrinsic curvature

$$p^{ab} = \frac{\partial K_{cd}}{\partial \dot{h}_{ab}} \frac{\partial}{\partial K_{cd}}(\sqrt{g}\mathcal{L}_G) = \frac{\sqrt{h}f'(\varphi)(K^{ab} - Kh^{ab})}{16\pi}. \quad (9)$$

For evaluating $\partial K_{cd}/\partial \dot{h}_{ab}$, the extrinsic curvature was written as a function of the induced metric given in the formula (4).

For writing the Hamiltonian density $\mathcal{H} = p^{ab}\dot{h}_{ab} - \sqrt{-g}\mathcal{L}$, we still need the volume part of the action. We write the gravitation part of the action without the surface part (which we include later on) as

$$S_{\rm GV} = \frac{1}{16\pi}\int_T dt\left\{\int_{\Sigma_t}[f(\varphi) + f'(\varphi)(K^{ab}K_{ab} + \tilde{R} - K^2) - f'(\varphi)\varphi]N\sqrt{h}d^3x\right\}. \quad (10)$$

After some manipulations the volume part of the Hamiltonian density can be cast to the form

$$\begin{aligned} \mathcal{H}_G &= p^{ab}\dot{h}_{ab} - \sqrt{-g}\mathcal{L}_G \\ &= \frac{N\sqrt{h}}{16\pi}\{f'(\varphi)[K^{ab}K_{ab} - K^2 - \tilde{R} + \varphi] - f(\varphi)\} \\ &\quad + \frac{\sqrt{h}f'(\varphi)}{8\pi}[(K^{ab} - Kh^{ab})N_a]_{|b} \\ &\quad - \frac{\sqrt{h}f'(\varphi)}{8\pi}(K^{ab} - Kh^{ab})_{|b}N_a. \end{aligned} \quad (11)$$

To express the Hamiltonian density in terms of adequate variables, i.e. induced metric h_{ab} and its momentum conjugate p_{ab}, we need to rewrite the extrinsic curvature. By inverting (9) we get

$$\sqrt{h}K^{ab}f'(\varphi) = 16\pi(p^{ab} - \tfrac{1}{2}ph^{ab}) \equiv \hat{p}^{ab} - \tfrac{1}{2}\hat{p}h^{ab}. \quad (12)$$

It should be noted here that the sole momentum conjugate is invertible and the Lagrangian is hence not singular. This would be the case had the original $f(R)$ Lagrangian not been written using the scalar field. Using this inverted equation the Hamiltonian can be written as a function of the momentum conjugate. Now the volume part of the gravitational Hamiltonian is obtained by integrating $\mathcal{H}_G$ over the hypersurface Σ_t:

$$H_G = \frac{1}{16\pi}\int_{\Sigma_t}\Big\{N\sqrt{h}[\varphi f'(\varphi) - f(\varphi) - \tilde{R}f'(\varphi)] + \frac{N}{\sqrt{h}f'(\varphi)}\Big(\hat{p}_{ab}\hat{p}^{ab} - \frac{\hat{p}^2}{2}\Big) \quad (13)$$

$$- 2\sqrt{h}N_a\Big(\frac{\hat{p}^{ab}}{\sqrt{h}}\Big)_{|b}\Big\}d^3x. \quad (14)$$

Similarly we get the surface part of the gravitational Hamiltonian by taking the appropriate terms and integrating over the hypersurface Σ_t:

$$H_S = \frac{1}{8\pi}\oint_{S_t}\left[N(k - k_0) - \frac{N_a\hat{p}^{ab}r_b}{\sqrt{h}}\right]f'(\varphi)\sqrt{\sigma}d^2\theta. \quad (15)$$

The latter term is produced by applying the Gauss theorem to the middle term of (11) when integrating over the density.

We obtain the field equations by varying the action with respect to N, N^a, h_{ab}, p_{ab}, and φ. These are all treated as independent variables. The equations will be greatly simplified by the fact that we have only one dynamic field, the induced metric. We have the normal boundary conditions for the variations vanishing on the boundary,

$$\delta N = \delta N^a = \delta h_{ab} = \delta \varphi = 0. \tag{16}$$

The full Hamiltonian H includes both surface and volume parts as well as a matter part S_M. Since we can write variation of the action as

$$\delta S = \int_{t_1}^{t_2} dt \bigg[\int_{\Sigma_t} (p^{ab} \delta \dot{h}_{ab} + \dot{h}_{ab} \delta p^{ab}) d^3 y - \delta H \bigg], \tag{17}$$

the Hamiltonian equations are of the form

$$\begin{aligned} &\dot{h}_{ab} = \frac{\partial \mathcal{H}_G}{\partial p}, \qquad \dot{p}^{ab} = -\frac{\partial \mathcal{H}_G}{\partial h} + \frac{\partial S_M}{\partial h}, \\ &\frac{\partial \mathcal{H}_G}{\partial N_a} = 0, \qquad \frac{\partial \mathcal{H}_G}{\partial N} = \frac{\partial S_M}{\partial N}, \qquad \frac{\partial \mathcal{H}_G}{\partial \varphi} = 0. \end{aligned} \tag{18}$$

To simplify the field equations, we can choose the foliation to be such that $N_a = 0$ and hence $h_{ab} = g_{ab}$, when the effects of the surface terms vanish. This choice removes one field equation, that of N_a, and the other ones are much simplified. After tedious calculations we end up with equations

$$\begin{aligned} -\frac{\dot{p}^{ab}}{N\sqrt{h}} = G^{ab} f'(\varphi) + \frac{h^{ab}}{2}\bigg(\varphi f'(\varphi) - f(\varphi) - 16\pi P \\ - \frac{p^{cd} p_{cd} - \frac{p^2}{2}}{h f'(\varphi)}\bigg) + \frac{2p^a_c p^{cb} - p p^{ab}}{h f'(\varphi)}, \end{aligned} \tag{19a}$$

$$\begin{aligned} 16\pi\sqrt{h}\rho = (\tilde{R} + K^2 - K^{ab}K_{ab} - \varphi) f'(\varphi)\sqrt{h} \\ + f(\varphi)\sqrt{h}, \end{aligned} \tag{19b}$$

$$\dot{h}_{ab} = \frac{2N}{\sqrt{h} f'(\varphi)}\bigg(p_{ab} - \frac{1}{2} p h_{ab}\bigg), \tag{19c}$$

$$\varphi - \tilde{R} = \frac{\frac{\hat{p}^2}{2} - \hat{p}_{ab}\hat{p}^{ab}}{h(f'(\varphi))^2}, \tag{19d}$$

where $\tilde{G}^{ab} = \tilde{R}^{ab} - \frac{1}{2}\tilde{R}h^{ab}$. For technical details we refer the reader to [39]. The treatment is for the Einstein-Hilbert action which can be straightforwardly generalized to the $f(R)$ case. Note, however, that in the derivation of Eq. (19d) further use is made of the assumption $f''(R) \neq 0$. Otherwise, we would get a trivial equality. As can be seen in Eqs. (19a) and (19b) we have also added matter:

$$\frac{\delta S_m}{\delta N} = -\frac{\sqrt{g}}{2} T_{00} \frac{\delta g^{00}}{\delta N} = -\sqrt{h}\rho, \tag{20}$$

$$\frac{\delta S_m}{\delta h_{ab}} = -\frac{\sqrt{g}}{2} T^{ab} \frac{\delta g_{ab}}{\delta h_{ab}} = -\frac{N\sqrt{h}}{2} P h^{ab}, \tag{21}$$

which is of the perfect fluid form.

Even though we assumed from the start that $f''(R) \neq 0$, it is worthwhile to take a look at the case of the Einstein-Hilbert Lagrangian. If in Eq. (1) we choose $f(R) = R$, the equality is trivial, and only a variation of a constant resulting in a trivial field equation. As the assumption of $f''(R) \neq 0$ is needed in the field equations only in (19d), the equations would stay the same except for this one equation which is irrelevant. From Eq. (19b) we get the familiar Friedmann equation for the background,

$$\mathcal{H}^2 = \frac{8\pi\rho_0}{3a}, \tag{22}$$

in a matter dominated universe ($\rho = a^{-3}\rho_0$). Here $\mathcal{H} = a'(\eta)/a(\eta)$ is the conformal Hubble parameter. We will need this background result later on when we insert the asymptotic background solution into the equations. Namely, we can solve $a'(\eta)$ from this equation.

III. FIRST ORDER PERTURBATIONS

In this section we add first order perturbations to the metric and the momentum conjugate. In general relativity the trace equation connects curvature and matter density [for fixed equation of state $p = p(\rho)$] by a simple relation $R = \kappa(\rho - 3P)$, where $\kappa = 8\pi G$. The perturbations would be connected correspondingly: $\delta R = \kappa(\delta\rho - 3\delta P)$. As the trace equation in $f(R)$ gravity is $f'(R)R - 2f(R) + 3\Box f'(R) = \kappa(\rho - 3P)$ there are more freedom in metrics that produce a given mass configuration. Indeed, the relation between curvature and matter distribution is no more an algebraic one, but defined by a differential equation. Thus the phase space of metrics is larger and there are perturbations keeping R and thus ρ fixed. This is manifested by the statement that Birkhoff's theorem[2] is no more valid in the traditional form in $f(R)$ theories [52,53]. Since there are number of studies of the perturbations along R (e.g. [31]), we are now interested in the opposite and do not introduce perturbations to matter but perturbations perpendicular to R only, i.e. $\delta R = \delta\rho = 0$.

We may add the most general first order perturbations to the metric. These include scalar, vector, and tensor perturbations. In light of the recent observations and for simplicity, we examine the case of the spatially flat Friedmann-Robertson-Walker metric. The perturbations in first order can now be written as [54]

$$g_{00} = \tilde{g}_{00} - 2a^2\Phi, \tag{23a}$$

$$g_{0a} = \tilde{g}_{0a} + a^2(\partial_a\omega + \omega_a), \tag{23b}$$

$$g_{ab} = \tilde{g}_{ab} + a^2(-2\Psi\delta_{ab} + \nabla_{ab}\chi + \partial_a\chi_b + \partial_b\chi_a + \chi_{ab}), \tag{23c}$$

where tilde denotes the background part and the vectors ω^a and χ^a are transverse (i.e. $\partial^a\omega_a = 0$, $\partial^a\chi_a = 0$) and χ_{ab} is trace free and symmetric tensor (i.e. $\partial^a\chi_{ab} = 0$, $\chi^a_a = 0$). Comparing the elements in (23) and the line element (2) to find the perturbations in the first order for lapse, shift, and

[2]Birkhoff proved the so-called Birkhoff theorem in 1923 [49]. However, two years earlier a less known Norwegian physicist Jebsen presented the idea in [50]. The history of the theorem is examined in [51].

the induced metric, we obtain

$$N = \tilde{N} + a\Phi, \tag{24a}$$

$$N_a = \tilde{N}_a + a^2(\partial_a \omega + \omega_a) \equiv \tilde{N}_a + a^2 \hat{\omega}_a, \tag{24b}$$

$$h_{ab} = \tilde{h}_{ab} + a^2(-2\Psi\delta_{ab} + \nabla_a\nabla_b\chi + \partial_a\chi_b + \partial_b\chi_a + \chi_{ab})$$
$$\equiv \tilde{h}_{ab} + a^2(-2\Psi\delta_{ab} + \hat{\chi}_{ab}). \tag{24c}$$

The standard practice of splitting the perturbations into scalar, vector, and tensor parts [55] is motivated by the reason that in a linear theory these modes decouple. Moreover, each of them has a clear physical interpretation [56]. The first order vector perturbations are not generated in the presence of scalar perturbations and dissipate over time. Tensor perturbations cause gravitational waves which do not couple to first order scalar perturbations. Therefore, we may omit vector and tensor perturbations in the first order case and assume $\omega_a = 0$, $\chi_{ab} = 0$. We can further simplify the metric for our purposes by choosing an appropriate gauge. We choose to use the Poisson gauge [54] which is a generalization of the much used longitudinal gauge. The gauge conditions are

$$\nabla \cdot \hat{\chi} = 0, \tag{25a}$$

$$\nabla \cdot \hat{\omega} = 0. \tag{25b}$$

Since ω^a and χ_a are transverse vectors and χ_{ab} is a symmetric, transverse, and trace-free tensor, we have $\omega = \chi = \chi_a = 0$. Along with the physical meaning of the perturbations discussed above, the perturbed metric simplifies to

$$N = \tilde{N} + a\Phi, \tag{26a}$$

$$N_a = \tilde{N}_a, \tag{26b}$$

$$h_{ab} = \tilde{h}_{ab} - 2a^2\Psi\delta_{ab}. \tag{26c}$$

As the dynamical components of the metric are coupled to their momentum conjugates, we are to add perturbations also to the conjugates. Only the induced metric h_{ab} has a conjugate p_{ab}, and hence for the perturbed one we write

$$p_{ab} = \tilde{p}_{ab} + \Theta\delta_{ab} \tag{27}$$

having the same structure as (26c).

In the following we work mostly, unless otherwise stated, in conformal time instead of standard coordinate time. So we have $ds^2 = -a(\eta)^2 d\eta^2 + a(\eta)^2\delta_{ab}dx^a dx^b$, where the conformal time η is related to standard coordinate time by $d\eta = a^{-1}dt$. A prime denotes derivatives with respect to the conformal time and a dot denotes derivatives with respect to the coordinate time. This choice of background metric corresponds to $\tilde{R} = \tilde{G}_{ab} = 0$ and $\sqrt{\tilde{h}} = a^3$. Also, we now have $\tilde{p}_{ab} = -2f'(\varphi)a^3a'$. Since we wrote the $f(R)$ theory using a scalar in (1) we have $\varphi \sim R$. Perturbing φ would produce perturbations parallel to R which we are not interested in.

Equations (19) for the chosen background metric and scalar field are now given in a fairly simple form. This reads

$$16\pi Pa^4 = 2(a')^2 f'(\varphi) - a^4(f(\varphi) - \varphi f'(\varphi)) - 4a(a'' f'(\varphi)$$
$$+ a'\varphi' f''(\varphi)), \tag{28a}$$

$$16\pi\rho a^3 = \frac{f'(\varphi)(6(a')^2 - a^4\varphi)}{a} + a^3 f(\varphi), \tag{28b}$$

$$\varphi = \frac{6(a')^2}{a^4}. \tag{28c}$$

We get only three nontrivial equations as (19c) produces only a trivial identity. These equations, satisfied for any acceptable matter, are used to simplify the perturbation equations derived later. In the following we assume a matter filled universe with $P = 0$ and $\rho = \rho_0/a^3$.

By adding the perturbations introduced in (24) and (27) to the equations of motion (19), we get three equations for the large scale perturbations (i.e. space independent perturbations):

$$\Psi = \frac{\Theta}{10a^3 a' f'(\varphi)}, \tag{29a}$$

$$\Theta' = \left(\frac{3a'}{a} + \frac{a''}{a} + \frac{12a' f''(\varphi)(aa'' - 2(a')^2)}{a^5 f'(\varphi)}\right)\Theta, \tag{29b}$$

$$\Phi = 0, \tag{29c}$$

where we have used the background Eqs. (28a) and (28c) to simplify Eq. (29b). We immediately notice that there remains only one dynamic equation while the other two are algebraic. The background equation for the induced metric can be used to eliminate the second time derivative of the scale parameter. Equation (29b) is thus written as

$$\Theta' = \left(\frac{5a'}{a} - \frac{4\pi\rho_0}{a' f'(\varphi)}\right)\Theta. \tag{30}$$

The behavior of perturbation is clearly dependent on the form of the function $f(\varphi)$ explicitly via its derivatives. Moreover, it is found that the time derivative of Ψ is zero and therefore by Eq. (29a) we can write

$$\Theta = Ca^3 a' f'(\varphi), \tag{31}$$

where C is a constant. So, in a universe with growing $a(\eta)$ the perturbations in momentum conjugate increase. The perturbations of the metric tensor, however, behave differently: the temporal part vanishes and the spatial perturbations are constant. So, the system leaves the linear perturbative regime and ultimately suffers linear instability.

Although asymptotic analysis is ultimately irrelevant for a linearized unstable system, we take a look to some examples to get a better feeling of the evolution. As known, the simple function $f(R) = R - \mu^4/R$ results asymptotic

Einstein–de-Sitter behavior. Now $a(t) = e^{\Lambda t}$, and coordinate and conformal times are related by $\eta = -\frac{e^{-\Lambda t}}{\Lambda} + c$ so that $a(\eta) = \Lambda^{-1}(c - \eta)^{-1}$. In the high curvature limit we get

$$\Theta(\eta) = \hat{C}\frac{36\Lambda^4 + \mu^4}{36(c - \eta)^5\Lambda^8},$$

where $\hat{C}$ is a constant. The result can also be written more intuitively in coordinate time as

$$\Theta(t) = Ce^{4\Lambda t}\left(1 + \frac{e^{4\Lambda t}\mu^4}{36\Lambda^4}\right),$$

and thus the perturbations increase as time goes to infinity. Here C is another constant. Ultimately the first order perturbation theory breaks down; it is not applicable in this case. Similar behavior can be seen explicitly for another often used $f(R) = R - \mu^2 R^2$.

Even though we have not included perturbations in matter, it is worthwhile to check what would happen if we did include these perturbations. For a moment we consider $\rho = \tilde{\rho} + \sigma$, where σ is a perturbation. It turns out that no density perturbations are present, i.e. the perturbation equation is $\sigma = 0$. This is not surprising as the matter perturbations are coupled to the temporal perturbation of the metric which is also zero. These vanish unless φ (which is essentially R) is perturbed, too.

As we have found, the only dynamical equation is (29b) for the momentum conjugate, while the two other equations determine how metric perturbations follow it; they are constraint equations. If these constraints were to be discarded, we would end up with nondiagonal perturbations in the metric. Moreover, nonexistence of temporal perturbations is connected with the orthogonality of perturbations to curvature. As it appears that the spatial perturbations in the metric do not grow or vanish in time, there is a flat direction of phase space, where any spatial first order perturbation is possible and stable.

IV. SECOND ORDER PERTURBATIONS

We have now seen that the first order perturbation predicts that $f(R)$ theories suffer instability which invalidates first order expansion; Eq. (30) reveals that we cannot use first order perturbation theory. The next check would be to consider second order perturbations, which might give us further understanding of the perturbations involved. We first write the most general form of the metric and the momentum conjugate as

$$N = \tilde{N} + a(\Phi^{(1)} + \Phi^{(2)}), \tag{32a}$$

$$N^a = \tilde{N}^a + a\sum_{r=1}^{2}(\partial_a\omega^{(r)} + \omega_a^{(r)}), \tag{32b}$$

$$h_{ab} = \tilde{h}_{ab} + a^2\Big[-(2\Psi^{(1)} + \Psi^{(2)})\delta_{ab} + \sum_{r=1}^{2}(\nabla_a\nabla_b\chi^{(r)} + \partial_a\chi_b^{(r)} + \partial_b\chi_a^{(r)} + \chi_{ab}^{(r)})\Big], \tag{32c}$$

$$p_{ab} = \tilde{p}_{ab} + (\Theta^{(1)} + \Theta^{(2)})\delta_{ab}, \tag{32d}$$

where the upper index i denotes the order of the perturbation. As we have chosen to work in the Poisson gauge [55], we have $\omega^{(r)} = \chi^{(r)} = \chi_a^{(r)} = 0$. The vector perturbations $\omega_a^{(r)}$ and $\chi_{ab}^{(r)}$ still remain, however, and some extra attention has to be paid to them. In general the scalar, vector, and tensor perturbations do not decouple any more in the second order perturbation theory. First order vector perturbations contribute to the second order scalar perturbations by terms like $\omega^a\omega_a$ and vice versa. However, first order perturbations do not manifest themselves if not present initially. Since we are now interested in showing the instability of the system, it is sufficient that some initial condition reveals unstable behavior. In particular, we are free to choose initial condition $\omega_a(0) = 0$ for the vector perturbations. First order tensor perturbations can omit them as well. Note that, if we were trying to show the stability of the system, the burden of proof would be much heavier: we should show that any choice of initial conditions leads to a stable system.

As mentioned, vector and tensor perturbations in second order cannot be discarded by similar arguments. They are strongly affected by first order scalar perturbations. However, the second order scalar perturbations are again independent of the tensor and scalar perturbations of the second order. Therefore, for our purposes, it is sufficient to study only second order scalar perturbations, which can be performed rather simply. We write the relevant perturbation equations for second order in the same manner as in the previous section. We obtain

$$\Psi^{(2)\prime} = 0, \tag{33a}$$

$$\Theta^{(2)} = -\frac{3}{20a^3a'f'(\varphi)}(\Theta^{(1)})^2 + 5a^3a'f'(\varphi)\Psi^{(2)}, \tag{33b}$$

$$\Phi^{(2)} = 0. \tag{33c}$$

Thus metric perturbation $\Phi^{(2)}$ still vanishes and $\Psi^{(2)}$ is again constant related to the perturbation of the momentum conjugate $\Theta^{(2)}$ by (33b). The perturbation in the momentum conjugate is still depending on the form of the $f(R)$. For $f(R) = R - \mu^4/R$, the result is the same as in the first order; the perturbation of the temporal part disappears, the spatial part remains constant, and the momentum conjugate is the only dynamical variable. It is clear that the instabilities in the first order propagate to the second order as the metric perturbations behave in exactly the same way in both first and second order. Thus $f(R)$ models may be inherently unstable up to second order when examining perturbations perpendicular to R. Because of the similar form of the first and the second order scalar perturbations, one might conjecture that it is a more general feature of the theory.

V. DISCUSSION

Traditionally the stability analysis is performed in the Lagrangian formalism and the analysis parallel to R has been carried out before in several papers (e.g. [31]). Many of the interesting $f(R)$ models have been found to be inherently unstable in the past [33,57]. However, stability analysis has not yet been used to the full extent as long as the studies concentrate on curvature perturbations only. By using a combination of Hamiltonian and Lagrangian formulation (via the partial Legendre transform), we examined the large scale cosmological perturbations perpendicular to R with nonrelativistic matter. These perturbations are fairly easy to examine within this formulation. The analysis could have been done using the full Hamiltonian formulation. In that case, however, we would have been faced by complications. The normal treatment of these complications would be the Dirac-Bergmann algorithm.

The found instabilities are noticeably different to those of previous works (e.g. [33]). Because of the constraint $\dot{R} = 0$ diagonal perturbations of the metric and the momentum conjugate are related to each other. The temporal part of the metric showed up to be constrained by the equation of the momentum conjugate. Moreover, the spatial part of the metric is forced to vanish. If these constraints were not satisfied, we would have nontrivial perturbations of nondiagonal elements of the metrics like g_{0i}.

The perturbations of the momentum conjugate turn out to be the most interesting ones. The equation depends explicitly on the form of the function $f(R)$. Some choices of $f(R)$ lead clearly to an unstable cosmological model, but seemingly not all. We have studied some well-known $f(R)$ functions and find them unstable. Albeit the physical interpretation of the perturbation momentum conjugate is unfortunately not as clear as that of the metric perturbations, Eq. (9) demonstrates the relation between the momentum conjugate and the extrinsic curvature. In the $3 + 1$ decomposition the intrinsic curvature $\tilde{R}$ defines how the hypersurface is curved, whereas the extrinsic curvature defines how each slice is curved relative to the enveloping space-time.

As the perturbations were not well behaved in this context, further studies would be relevant in order to find the limits of these constraints. Fruitful directions would likely be investigating the effects of other types of matter. Also, it would be prudent to examine the case where the metric can include shift (i.e. $N_a \neq 0$). It is clear from the form of (11) that such a generalization would affect the following equations of motion deeply as the last term would be nonzero. This is understandable as the metric would now include spatiotemporal elements. It is also possible to study more general theories with the Lagrangian depending also on, for example, $R_{\mu\nu}R^{\mu\nu}$ or the Gauss-Bonnet term.

It appears that with Hamiltonian formulation of perturbations can be used to constrain the spectrum of cosmologically acceptable $f(R)$ theories. While there are several physical arguments to judge the $f(R)$ theories like cosmological observations and solar system behavior, stability analysis is one important tool to rule out ill-behaved models out of numerous possible modified theories of gravity. With continued studies it is possible to find the ones best describing the observed behavior of the universe.

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RESEARCH ARTICLE

Jeans analysis of Bok globules in $f(R)$ gravity

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Abstract We examine the effects of $f(R)$ gravity on Jeans analysis of collapsing dust clouds. We provide a method for testing modified gravity models by their effects on star formation as the presence of $f(R)$ gravity is found to modify the limit for collapse. In this analysis we add perturbations to a de Sitter background. As the standard Einstein-Hilbert Lagrangian is modified, new types of dynamics emerge. Depending on the characteristics of a chosen $f(R)$ model, the appearance of new limits is possible. The physicality of these limits is further examined. We find the asymptotic Jeans masses for $f(R)$ theories compared to standard Jeans mass. Through this ratio, the effects of the $f(R)$ modified Jeans mass for viable theories are examined in molecular clouds. Bok globules have a mass range comparable to Jeans masses in question and are therefore used for comparing different $f(R)$ models. Viable theories are found to assist in star formation.

Keywords Modified gravity · F(R)theories · Bok globules · Jeans analysis

1 Introduction

The standard cosmological model, also called the concordance model [1], is based on general relativity (GR) combined with cold dark matter and the cosmological constant. It explains nicely almost all observational data, in particular the accelerated expansion of the Universe [2,3]. However, the nature and smallness of the cosmological constant is highly problematic as there is no natural way to generate such an extreme

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parameter [4]. Therefore, there are various competing models including new forms of matter [5,6], inhomogeneous cosmologies [7] and modified gravitation theories [8]. The alternative explanations are all constrained by local experiments showing that GR works well in stellar system and galactic scales. Therefore, the cause of the accelerated expansion must be restricted to large scales.

One widely studied class of modified gravity theories involves replacing the scalar curvature of the Einstein-Hilbert action by a more general function $f(R)$. This leads to the equations of motion with fourth order derivatives in contrast to the second order differential equations of GR. A large number of different $f(R)$ theories have been under scrutiny (see *e.g.* [9–13]).

The most important characteristic of these models is the generated accelerated expansion. The first proposed models were quickly discarded as problems arose with stability and solar system constraints. As studies have found theoretical and observational constraints on possible models (*e.g.* [14]), the viable models have become more refined.

The challenge of $f(R)$ theories is surviving the strict solar-system bounds and simultaneously creating the accelerated expansion at late times. These theories can be interpreted as introducing a scalar degree of freedom [15] which may cause considerable deviation from GR around the Sun. A viable model should therefore include a mechanism to hide the new effects on high curvature regimes [16]. This is achieved by $f(R)$ models where the squared mass of the scalar is large in the large curvature region [17]. The same condition is set by the high-redshift observations of the cosmic microwave background (CMB) [18].

Besides explaining the accelerated expansion, $f(R)$ theories have been shown to have other benefits. It may be related to the early inflationary expansion of the Universe [19]. Moreover, it has been shown that with modified gravity the rotation curves of spiral galaxies and the halos of the core clusters could be explained without dark matter [20–22], and $f(R)$ theories have also been shown to give possible solutions to problems related to other objects such as neutron stars [23].

In the present article, we study some of the most successful $f(R)$ theories by considering the structure formation. This is done using the Jeans instability analysis of self-gravitating systems, where *e.g.* star formation can be examined. Instabilities in self-gravitating systems were first studied by Jeans [24]. As this was before the advent of the GR, the analysis was restricted to non-relativistic, Newtonian gravity. Later on, Jeans analysis has been upgraded to use GR and some works have further extended it to modified gravity [25,26].

For $f(R)$ models, it is possible to find further constraints for viable models [25]. We generalize the method and apply it to molecular clouds. These could offer a new class of objects to measure the viability of $f(R)$ models as there is ample observational data on large molecular clouds [27]. However, the masses of large clouds are several magnitudes higher than the Jeans masses, but in the smaller Bok globules the cloud masses are close to the well-known Jeans limit. Therefore, the $f(R)$ modified Jeans mass and the standard Jeans mass may yield different predictions on whether a globule is about to collapse.

2 Equations of motion

In f(R) gravity the Einstein-Hilbert Lagrangian is not set *a priori* to be the linear $f(R) = R$. The function f(R) is an analytic function of the curvature scalar R. If we set the requirement of no higher derivatives than second degree, the function reduces to R and the Einstein-Hilbert Lagrangian is obtained. With a generalized function it is possible to find a better match to observations than the simplest choice, $f(R) = R$.

We consider general f(R) modifications to the Einstein-Hilbert action,

$$\mathcal{A} = \frac{1}{2\chi}\int d^4x\sqrt{-g}\Big(f(R) + 2\chi\mathcal{L}_m\Big), \tag{1}$$

where $\chi = \frac{8\pi G}{c^4}$ is the coupling of gravitational equations. The latter term $\mathcal{L}_m$ is the minimally coupled matter Lagrangian. With $f(R) = R$ the action would reduce to the standard Einstein-Hilbert action. There are several restrictions to the possible form of the function f(R) which are further discussed in Sect. 7. The signature of the metric is $-, +, +, +$, the Riemann curvature tensor is $R^{\alpha}_{\beta\mu\nu} = \partial_\mu\Gamma^{\alpha}_{\beta\nu} - \partial_\nu\Gamma^{\alpha}_{\beta\mu} + \Gamma^{\alpha}_{\kappa\mu}\Gamma^{\kappa}_{\beta\nu} - \Gamma^{\alpha}_{\kappa\nu}\Gamma^{\kappa}_{\beta\mu}$ and the Ricci tensor is $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$. Using standard metric variational techniques we find the field equations and the trace equation

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu f'(R) + g_{\mu\nu}\Box f'(R) = \chi T_{\mu\nu} \tag{2}$$

$$3\Box f'(R) + f'(R)R - 2f(R) = \chi T, \tag{3}$$

where $T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\sqrt{-g}\mathcal{L}_m}{\delta g^{\mu\nu}}$ is the energy-momentum tensor and $T = T^{\alpha}_{\alpha}$. A prime is used to denote the derivatives with respect to R. As we are about to examine collapsing molecular clouds (*i.e.* relatively thin matter), we are going to use a weak field approximation for the metric. The background is assumed to be of de Sitter form with perturbations added to the diagonal elements [28]. In polar coordinates $x^\mu = (t, r, \theta, \phi) = (t, \mathbf{x})$ we have a diagonal metric up to $\mathcal{O}(3)$ with

$$g_{00} = -\big(h(r) + 2\phi(t, \mathbf{x})\big), \tag{4}$$

$$g_{11} = 1/h(r) + 2\Psi(t, \mathbf{x}), \tag{5}$$

$$g_{22} = (1 + 2\Psi(t, \mathbf{x}))r^2, \tag{6}$$

$$g_{33} = (1 + 2\Psi(t, \mathbf{x}))r^2 \sin\theta \tag{7}$$

The expansion parameter is v, the velocity of a test particle[1] [29]. Note, that this form of metric tensor corresponds to the first order post-Newtonian approximation in quasi-Minkowskian coordinates.

The perturbation in the temporal component of the metric, $\phi(t, \mathbf{x})$ corresponds to the Newton gravitational potential. It can be further broken up as $\phi(t, \mathbf{x}) = \phi_0 + \Phi(t, \mathbf{x})$,

[1] The expansion parameter can be equivalently c^{-1} as the velocity appears as a combination v/c. As it is customary to set $c = 1$, we prefer v as the expansion parameter.

where the constant ϕ_0 refers to the local environment around the object in question. For example, for the galactic potential we would have $\phi_0 \approx 2 \times 10^{-6}$ [30] (with c set to unity). For large scale considerations this constant term must be discarded as there is no constant background. Due to the derivatives involved in calculating the curvature tensor and scalar, this constant term does not affect curvature.

The non-diagonal elements must have odd powers as the time reversal transformation (as well as other coordinate reflections) should change the sign. Therefore, they are at least of order $\mathcal{O}(3)$. For the case of a weak field limit, terms of $\mathcal{O}(3)$ and higher are discarded. The Ricci scalar can be expanded around the background as

$$R \simeq R_0 + R^{(2)}(t, \mathbf{x}) + \mathcal{O}(4), \tag{8}$$

where $R^{(n)}$ denotes that the quantity is $\mathcal{O}(n)$. For de Sitter background $R_0 = 4\Lambda$. As the derivatives of *f*(*R*) appear in the equations of motion, we need an expansion for this function as well

$$f^n(R) \simeq f^n\left(R_0 + R^{(2)} + \mathcal{O}(4)\right) \simeq f^n(R_0) + f^{n+1}(R_0)R^{(2)} + \mathcal{O}(4), \tag{9}$$

which can be iterated for the desired order. In our case the first order is sufficient *i.e.* $f'(R) \simeq f'(R_0) + f''(R_0)R^{(2)} + \mathcal{O}(4)$. Not all the characteristics of *f*(*R*) models manifest at this order. However, in the scope of this paper, we concentrate on the lowest order effects on stability. If differences between GR and *f*(*R*) appear in a lower order, they are not likely to be canceled in higher orders. Inserting these into (2) we have in second order

$$f'(R_0)\left(R_{tt}^{(2)} + \frac{R^{(2)}}{2}\right) - f''(R_0)\nabla^2 R^{(2)} + f'(R_0)R_{tt}^{(0)} - \frac{f(R_0)}{2}g_{tt} = \chi T_{tt}^{(0)}, \tag{10}$$

$$3f''(R_0)\left(\nabla^2 - \partial_0^2\right)R^{(2)} - f'(R_0)R^{(2)} + f'(R_0)R_0 - 2f(R_0) = \chi T^{(0)}, \tag{11}$$

where ∇^2 is the spatial flat Laplacian. The flatness of the Laplacian is due to the quasi-Minkowskian nature of the metric [29]. In contrast to [25], we have included the time derivatives as collapsing clouds are time-dependent and dynamic. For a viable *f*(*R*) theory to have a de Sitter solution, $f'(R)R = 2f(R)$ must hold. This is in order to achieve the cosmic acceleration. Observations (*e.g.* Planck results [31]) show that the current evolution of the Universe is close to de Sitter behaviour. Therefore, the Λ background and the solution associated to it must exist as well as the solutions for the matter on the foreground. For the background the solution is $f'(R_0)R = 2f(R_0)$. The deviation of the current curvature from the de Sitter space is due to the matter content, which is caused locally, as the curvature scalar is a local quantity. In the first order expansion [34] the Birkhoff theorem is valid and allows us to separate the background. This leads to cancellation of the last two terms in (11).

With the same substitutions and $R_{tt}^{(0)} = -\Lambda$ we look at the last two terms in (10) to find them equal to $2f'(R_0)\Lambda(\phi_0 + \Phi(t, \mathbf{x}))$. The dynamic term would be of higher order. The constant term is clearly small as well but deserves a closer look. As part of the background it effectively works as a source of curvature. In that sense it is best compared to the other sources, *i.e.* the energy momentum tensor on the right side of the equation.

The perturbation terms of the Ricci scalar and the tt component of the Ricci tensor can be calculated from the perturbed metric (4)

$$R^{(2)} = 6\ddot{\Psi} - 2\nabla^2\Phi - 4\nabla^2\Psi, \tag{12}$$

$$R^{(2)}_{tt} = \nabla^2\Phi - 3\ddot{\Psi}. \tag{13}$$

For the equations of motion the energy-momentum tensor must be defined. We use the perfect fluid form

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu}. \tag{14}$$

with p being the pressure and ρ the mass density. As the molecular clouds consist of dust, we further set ($p = 0$) and obtain

$$-2f'(R_0)\nabla^2\Psi + f''(R_0)\big(2\nabla^4\Phi + 4\nabla^4\Psi - 6\nabla^2\ddot{\Psi}\big) = \chi\rho + 2f'(R_0)\Lambda\phi_0, \tag{15}$$

$$f''(R_0)\big(5\nabla^2\ddot{\Psi} + \nabla^2\ddot{\Phi} - \nabla^4\Phi - 2\nabla^4\Psi\big) - f'(R_0)\left(\ddot{\Psi} + \frac{\nabla^2\Phi}{3} + \frac{2\nabla^2\Psi}{3}\right) = -\frac{\chi\rho}{6}. \tag{16}$$

On the right side of the equations, the elements of the energy-momentum tensor are of the zeroth order due to the coupling constant χ being second order. We have omitted the term $-18f''(R_0)\ddddot{\Psi}$ as it is of higher order due to the multiple time derivatives and, therefore, being smaller.

The relation of the time derivatives and spatial derivatives and the order merits a mention. The derivatives have an effect on the expansion order (see [29], chapter 9). The time derivatives raise the the order since

$$\frac{\partial}{\partial_i} \sim \frac{1}{r}, \tag{17}$$

$$\frac{\partial}{\partial_t} \sim \frac{v}{r}. \tag{18}$$

This calls for a question whether also second order perturbations should be included. In the linear Jeans analysis the perturbations examined are arbitrarily small and we may assume $\Phi \gg \Phi^2$. It is also worth mentioning that while first order perturbations can cause second order perturbations to appear, the second order perturbations cannot cause first order perturbations to appear. For this reason we have not included quadratic terms in the Ricci tensor (12) and scalar (13).

We can rescale (15) by dividing it with $f'(R_0)$ on both sides. This way, the effect of a chosen $f(R)$ model is incorporated into the ratio $f''(R_0)/f'(R_0)$ and leads to a scaled gravitational constant $\chi/f'(R_0)$. For simplicity we aim to set $f'(R_0) = 1$.

For GR, we have $f'(R) = 1$ and measurements of the gravitational constant indicate its relative error is $\delta G/G < 1.2 \times 10^{-4}$ [32]. Using the expansion $f'(R) \simeq f'(R_0) + f''(R_0)R^{(2)}$ the solar system results and the $f'(R_0)$ can be related. Since the perturbation $R^{(2)} \ll 1$ by definition, the second term would be small as well, unless $f''(R_0) \gg 1$, which would cause physical anomalies. Therefore, the effective deviation is

$$\frac{\delta G}{G} = \left|\frac{f'(R) - 1}{f'(R)}\right| < 1.2 \times 10^{-4}, \tag{19}$$

and we set $f'(R_0) = 1$.

The value of the cosmological constant should be extremely small [4]. Therefore, the second term on the right side of (15) would be small as well. For dust clouds, such as the Bok globules, the ratio of the terms is $\left|\frac{\Lambda\phi_0}{\chi\rho}\right| \sim 10^{-5}$. As the the galactic potential originates from the matter content of the Milky Way, the second term corresponds to a constant part of ρ. In the scale of a single dust cloud the background ρ is constant. It can also be argued that to this order the Birkhoff theorem can be applied in this weak field approach. In low orders of expansion and a static Ricci scalar (here, the galactic background), the Birkhoff theorem is valid [33,34]. Therefore, the net effect on the globule would be negligible. On these grounds we remove the second term in the following treatment.

For $f''(R_0) = 0$ with a static potential $\ddot{\Phi} = 0$ the standard Poisson equation of $\nabla^2\Phi = 4\pi G\rho$ is reached. In the Newtonian case the only perturbation considered is static Φ, omitting the spatial perturbation Ψ.

3 Collapse in a self-gravitating collisionless system

A self-gravitating system of particles in equilibrium is described by a time-independent distribution function $f_0(\mathbf{x}, \mathbf{v})$ and a potential $\Phi_0(\mathbf{x})$. They are the solutions of the collisionless Boltzmann equation and the Poisson equation

$$\nabla^2\Phi(\mathbf{x}, t) = 4\pi G \int f(\mathbf{x}, \mathbf{v}, t)d\mathbf{v}, \tag{20}$$

$$\frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial t} + (\mathbf{v} \cdot \nabla_x) f(\mathbf{x}, \mathbf{v}, t) - (\nabla\Phi \cdot \nabla_v) f(\mathbf{x}, \mathbf{v}, t) = 0. \tag{21}$$

Here $\mathbf{v}$ and $\mathbf{x}$ are spatial velocity and position vectors and the ∇ operates in the three spatial dimensions. In the Newtonian limit Φ_0 is just the gravitational potential of the metric (4).

Following standard methods (*e.g.* [35]) we linearize these two equations and write them in Fourier space to obtain (for clarity we omit writing the variables)

$$-i\omega f_1 + \mathbf{v} \cdot (i\mathbf{k} f_1) - (i\mathbf{k}\Phi_1) \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0, \tag{22}$$

$$-k^2\Phi_1 = 4\pi G \int f_1 d\mathbf{v}. \tag{23}$$

We can now solve for

$$f_1 = \frac{\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}}{\mathbf{v} \cdot \mathbf{k} - \omega}\Phi_1. \tag{24}$$

For the purposes of Jeans analysis, we need to consider small perturbations to the equilibrium and linearize the equations of motion. We write the mass distribution function $\rho = \int f(\mathbf{x}, \mathbf{v}, t) d\mathbf{v}$ in (15) and write the equations in Fourier space to get the following equations

$$k^2 \Psi_1 + k^2 \alpha (k^2 \Phi_1 + 2k^2 \Psi_1 - 3\omega^2 \Psi_1) = 4\pi G \int f_1 \, d\mathbf{v}, \tag{25}$$

$$3k^4 \alpha \left(\frac{\omega^2}{k^2} (5\Psi_1 + \Phi_1) - \Phi_1 - 2\Psi_1 \right) - k^2 \Phi_1 - 2k^2 \Psi_1 + 3\omega^2 \Psi_1 = -4\pi G \int f_1 d\mathbf{v}. \tag{26}$$

We have denoted $f''(R_0) = \alpha$, which conveys the effects of *f*(*R*) theories. From these two equations we can solve for

$$\Psi_1 = -\frac{k^2 (1 + 2\alpha k^2 - 3\alpha\omega^2)}{(1 + 4\alpha k^2)(k^2 - 3\omega^2)} \Phi_1, \tag{27}$$

which can be inserted back into (25) to obtain

$$\frac{k^4 \Big(1 + 3\alpha (k^2 - \omega^2 + 3\alpha\omega^4) \Big)}{(1 + 4\alpha k^2)(k^2 - 3\omega^2)} \Phi_1 = -4\pi G \int f_1 d\mathbf{v}. \tag{28}$$

With the solved linearized matter distribution (24) we reach the dispersion relation

$$4\pi G \int \Big(\frac{\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}}{\mathbf{v} \cdot \mathbf{k} - \omega} \Big) d\mathbf{v} + \frac{k^4 \Big(1 + 3\alpha (k^2 - \omega^2 + 3\alpha\omega^4) \Big)}{(1 + 4\alpha k^2)(k^2 - 3\omega^2)} = 0. \tag{29}$$

In the standard case, $\alpha = 0$, the limit for instability is found at

$$k^2_{\omega=0} = \frac{4\pi G \rho_0}{\sigma^2} \equiv k_J^2. \tag{30}$$

Which is called the Jeans wavenumber. With this we can define the Jeans mass as the mass which was initially inside a sphere of diameter λ_J:

$$\lambda_J^2 \equiv \frac{4\pi^2}{k_J^2} = \frac{\pi \sigma^2}{G \rho_0}, \quad M_J \equiv \frac{4\pi \rho_0}{3} \Big(\frac{\lambda_J}{2} \Big)^3 = \frac{\pi}{6} \sqrt{\frac{1}{\rho_0} \Big(\frac{\pi \sigma^2}{G} \Big)^3}. \tag{31}$$

The Jeans length λ_J is the limit beyond which the perturbations are unstable, experiencing exponential growth. On the other hand $\lambda_J < 2\pi / k_J$ perturbations in stellar systems the response is strongly damped even though the system contains no friction [35]. The Jeans mass, however, is more useful for our purposes of probing the stability of interstellar clouds. If the mass of the cloud exceeds Jeans mass M_J, it will collapse.

4 Jeans instability limit in the $f(R)$ case

To discuss the case $f''(R_0) \neq 0$ we return to the dispersion relation (29). It can be recast into (see the appendix 9.1 for details)

$$1 - \sqrt{\pi} x e^{x^2} \mathrm{erfc}(x) = \frac{k^4 \left(1 + 3\alpha(k^2 + \omega_I^2 + 3\alpha\omega_I^4)\right)}{k_J^2 (1 + 4\alpha k^2)(k^2 + 3\omega_I^2)}, \tag{32}$$

with $x = \frac{|\omega_I|}{\sqrt{2}k\sigma}$ and $\omega_I = -i\omega$. The left side is a bounded monotonously decreasing function in respect with x. The limit for instability is found at $\omega = \omega_I = x = 0$ where the left side of (32) reduces to unity.

$$k_J^2 - \frac{k^2(1 + 3\alpha k^2)}{1 + 4\alpha k^2} = 0. \tag{33}$$

which can be simplified into $\alpha k^4 + \left(1 - 4\alpha k_J^2\right)k^2 - k_J^2 = 0$. If $\alpha = 0$, the equation would be of lower order and produce only the standard solution. If $\alpha \neq 0$, several solutions are possible. Besides the apparently excluded case $-1/3k^2 < \alpha < -1/4k^2$ where k^2 would be negative, (33) can be solved for

$$k^2 = k_\pm^2 = \frac{-1 + 4\alpha k_J^2 \pm \sqrt{1 + 4\alpha k_J^2 + 16\alpha^2 k_J^4}}{6\alpha}. \tag{34}$$

With (31) we can write the *f*(*R*) modified Jeans mass as

$$\tilde{M}_{J\pm} = \left(\frac{6\alpha k_J^2}{-1 + 4\alpha k_J^2 \pm \sqrt{1 + 4\alpha k_J^2 + 16(\alpha k_J^2)^2}}\right)^{3/2} M_J \equiv \beta_\pm M_J. \tag{35}$$

To reach a real mass the expression inside the brackets must be positive. It is apparent that for the β_- solution, we must have $\beta < 0$ to avoid complex masses. This equation describes the relation of standard GR Jeans mass for a self-gravitating stellar system and one described with *f*(*R*) gravity.

4.1 Solutions for non-zero ω

The dispersion Eq. (32) can be solved with $\omega = 0$ to get the instability limit (34) but there might be other solutions as well. The physical meaning of these $\omega \neq 0$ solutions merits a brief inspection.

Solving the dispersion equation for $k(\omega)$ is difficult and unnecessary. Examination of the derivatives on both sides of the Eq. (32) is sufficient to reveal the existence of at least one non-zero solution.

These non-zero ω solutions are also present in the standard case of $\alpha = 0$. If we examine a case in which $k = k_J - \delta k$, we notice that this corresponds to a mass slightly over the Jeans mass, $M = M_J + \delta M$. In the mean time, this k would require a non-zero ω for the dispersion relation to hold. The physical interpretation is, that when the object (*e.g.* a dust cloud) has a mass exceeding the Jeans mass, it will collapse even if it has a small initial radial velocity.

These non-zero solutions would appear at high values of ω. However, the original expansion around the background would break upon leaving the neighborhood of $\omega = 0$. For this reason it is possible to examine only the case of $\omega = 0$.

4.2 Characterization of k_- and k_+

In the standard case of Jeans stability analysis, there is only one limit for unstable modes. With a more general *f*(*R*) case the situation changes and there are possibly two limits for instability (35). The appearance of other limits is in a way expected as *f*(*R*) theories allow for an additional degree of freedom (see it e.g. [31]) which is perhaps best illustrated through the scalar-tensor theory equivalence. The physical significance of these two limits must be addressed.

The standard Jeans mass should be recovered with $f(R) = R$ which corresponds to $\alpha = 0$. Upon examining (34), this is leads to $k_+^2 = k_J^2$ and k_-^2 has the asymptotic behaviour $-1/3\alpha$, which diverges. Therefore, GR would have the standard Jeans mass and there would be no other meaningful limit, as is to be expected. With this observation the solution k_+ can be labeled as the generalization of the standard Jeans wave number.

The addition of a more complicated *f*(*R*) leads to different results depending on the sign of α. For $\alpha > 0$, the k_-^2 solution would translate to a negative Jeans mass. The result is one modified Jeans mass. This modified mass is lower than the standard one. For a dust cloud this means assisted star formation.

For negative α the situation is more interesting as there are two positive solutions for k^2. The new solution k_- would produce a considerably lower limit, converging to zero at $\alpha = 0$. The physical meaning of this limit must be addressed. If it translates into a lower limit for collapse, the effects for *e.g.* star formation would be observable. The k_+ solution refers to higher Jeans mass than standard case. In this case the expansion of the cosmic background counteracts the collapse.

The limit for instability was found earlier by setting $\omega = 0$, *i.e.* the mass distribution is time-independent, stable. All the contracting modes must fulfill the dispersion equation (32). This is the case for $k < k_+$ as in the standard case. However, for $k > k_-$ there are no solutions (32). The left side of the equation is monotonous and has an upper limit 1. The right side has values over 1 when $\alpha < 0$ and $k > k_-$.

As the dispersion relation does not hold with $k > k_-$, the limit k_- is not a limit for collapsing modes. However, the with $\omega = 0$ this corresponds to a stable mode. There are three forces in action in a collapse scenario, mass causing the collapse, thermal movement keeping it apart and, with an *f*(*R*) model, the background effect. With $k > k_+$ the mass is not sufficient to counteract the thermal movement, so the the mass distribution, *e.g.* a dust cloud, starts to oscillate, with non-zero ω.

As can be observed from (32), temperature does not appear explicitly in the dispersion relation with $\omega = 0$. Thus, for k_- the balancing forces are mass and the expansion of the background. With non-zero ω the equation will not hold. It would require extreme fine-tuning to reach this state and would be lost due any external perturbation. Therefore, it will not be physically meaningful. One more reason to discard the k_- solution is the Dolgov-Kawasaki instability, which is covered in Sect. 7. With the Dolgov-Kawasaki instability and non-negativity of the Jeans mass both $\alpha > 0$ and $\alpha < 0$ are denied for k_-. For these reasons we restrict to β_+ and $\tilde{M}_{J+}$ for the following treatment and omit the subscript signs.

Even though in our case there remains only the modified Jeans mass and one instability, in other situations these new instabilities might endure. In [36] instabilities and collapse were studied in oscillating backgrounds. These situations demand the inclusion of higher order derivative terms in the equations of motion. This difference allows for different instabilities to manifest in situations like black hole formation.

5 Comparison of Jeans masses in GR and $f(R)$ models

We derive the range within which *f*(*R*) models fall compared to the GR. Using the definition of Jeans mass (31) and the derived *f*(*R*) Jeans mass (35) we can write

$$\tilde{M}_J = \beta \frac{\pi}{6} \sqrt{\frac{1}{\rho_0} \left(\frac{\pi \sigma^2}{G} \right)^3}. \tag{36}$$

If we are to examine star formation, ρ_0 is the interstellar medium density (ISM) and σ is the velocity dispersion of particles due to temperature,

$$\rho_0 = m_H n_H \mu, \ \sigma^2 = \frac{k_B T}{m_H} \tag{37}$$

with n_H being the number of particles, μ the mean molecular weight (check [37] for values in molecular clouds), k_B the Boltzmann constant and m_H the proton mass. With these we compute the behaviour of $\tilde{M}_J$ for a given *f*(*R*) model described by β

$$\tilde{M}_J = \beta \frac{\pi T^{3/2}}{6 m_H^2} \sqrt{\frac{1}{n_H \mu} \left(\frac{\pi k_B}{G} \right)^3}. \tag{38}$$

The asymptotic value for $\beta \to \infty$ and $\beta \to 0$ are easily found to be

$$\lim_{\beta \to \infty} \tilde{M}_J = (3/4)^{3/2} M_J \tag{39}$$

$$\lim_{\beta \to 0} \tilde{M}_J = M_J \tag{40}$$

$$\lim_{\beta \to -\infty} \tilde{M}_J = \infty \tag{41}$$

Therefore for β, inserting the values of the constants,

$$\frac{\tilde{M}_J}{M_J} \in (0.649519, 1]. \tag{42}$$

We see now, that *f*(*R*) gravity can cause a considerably lower limit for gravitational collapse. For theories with positive β_+, this would assist in star formation. For negative values, the effect is inverse and would lead to reduced star formation. In the following section we will use compare these limits to observations of Bok globules.

6 Jeans mass limit in Bok globules

Bok globules are clouds of interstellar gas and dust. These dark clouds are relatively hard to spot and therefore all the observed globules are located nearby, on the galactic scale. The cloud cores are cold at temperatures of around 10 K. Most of the observed globules are isolated and of simple shape. Masses of these globules tend to be less than $100M_\odot$ with many around $10M_\odot$. This is considerably less than the large molecular clouds in the Milky Way, which are several orders of magnitude greater (e.g. [27]).

Bok globules have masses and the corresponding Jeans masses of the same order. Therefore, we can observe Bok globules for which the classic Jeans mass and the *f*(*R*) corrected Jeans mass give a different prediction for stability. There are observations of hundreds of globules [38–40], with estimates of the total number of globules in the Milky Way at tens of thousands [41]. It has been found [38,42,43] that most of the Bok globules experience star formation with one or more star forming cores.

The formation of the globules themselves is a process not well understood. It is possible that they form as condensations of diffuse gas in relative isolation. Another explanation is that the globules form as dense cores of larger interstellar clouds [44]. This agrees with the greater density. The presence of large external masses of stellar winds may also play a role in starting the collapse and star formation.

The observation of Bok globules is somewhat problematic. Extensive tables on their properties are not yet readily available. The kinetic temperatures are calculated from ammonia observations [39]. With excitation and kinetic temperatures known, the molecular hydrogen number density can be found. The reported masses are calculated for the globule cores assuming homogeneous distribution and spherical symmetry. However, the majority of globules as a whole are not spherical but elliptical [38].

For these reason the physics of Bok globules are not yet completely understood. For our purposes of looking into the stability of the clouds, the observations are sufficient as a demonstration for the viability of the method. Better accuracy in measurements of density and temperature would provide for a more accurate study. A more accurate modeling of the collapse would also take into account other forces, such as magnetic or turbulent. Nevertheless, the following will serve as a feasibility study on using Bok globules for constraints (Table 1).

The chosen globules are the ones in [39] which have calculated kinetic temperatures, hydrogen number densities and masses. The dark cloud names are those given in [45].

Table 1 For selected Bok globules we present the name, kinetic temperature, particle number, mass, conventional Jeans Mass, lowest possible Jeans mass due to *f*(*R*) gravity, stability prediction from Jeans mass and stability reported in [46]

	T (K)	n_{H_2} (cm^{-3})	M	M_J	$\tilde{M}_J$	Prediction	Stability
CB 87	11.4	$(1.7 \pm 0.2) \times 10^4$	2.73 ± 0.24	9.6	6.2	Stable	Stable
CB 110	21.8	$(1.5 \pm 0.6) \times 10^5$	7.21 ± 1.64	8.5	5.5	MD	Unstable
CB 131	25.1	$(2.5 \pm 1.3) \times 10^5$	7.83 ± 2.35	8.1	5.3	MD	Unstable
CB 134	13.2	$(7.5 \pm 3.3) \times 10^5$	1.91 ± 0.52	1.8	1.2	Unstable	Unstable
CB 161	12.5	$(7.0 \pm 1.6) \times 10^4$	2.79 ± 0.72	5.4	3.5	Stable	Unstable
CB 184	15.5	$(3.0 \pm 0.4) \times 10^4$	4.70 ± 1.76	11.4	7.4	Stable	Unstable
CB 188	19.0	$(1.2 \pm 0.2) \times 10^5$	7.19 ± 2.28	7.7	5.0	MD	Unstable
FeSt 1-457	10.9	$(6.5 \pm 1.7) \times 10^5$	1.12 ± 0.23	1.4	0.94	MD	Unstable
Lynds 495	12.6	$(4.8 \pm 1.4) \times 10^4$	2.95 ± 0.77	6.6	4.3	Stable	Unstable
Lynds 498	11.0	$(4.3 \pm 0.5) \times 10^4$	1.42 ± 0.16	5.7	3.7	Stable	Stable
Coalsack	15	$(5.4 \pm 1.4) \times 10^4$	4.50	8.1	5.3	Stable	Stable

Masses are in the units of solar masses $M_\odot$. MD stands for the case where the prediction depends on the chosen *f*(*R*) parameter

For our purposes of comparing the Jeans masses, we use the observational data from several Bok globules in [46].

The listed errors in parameters are 1σ deviations. The reported temperatures are effective temperatures which include the kinetic and the turbulent nonthermal component [46].

We notice that the modified *f*(*R*) Jeans masses offer better agreement than the conventional Jeans masses. In four of the globules, (CB 110, CB 131, CB 188 and FeSt 1-457) the mass exceeds the modified Jeans mass for some of the theories but not the conventional Jeans mass. In fact only one of the observed globules, CB 134, has a mass exceeding the conventional Jeans mass. Clearly, having a lower limit for the collapse due to *f*(*R*) gravity agrees with observations.

There is a disagreement on three globules, (CB 161, CB 184 and Lynds 495). Even though the mass of the globule is lower than the critical mass, a collapse can occur. In these cases however, it is due to some external force, *e.g.* a shock wave from a supernova.

According to [46] the globules with disagreement (CB 161, CB 184 and Lynds 495) in the prediction are "marginally unstable" which is a state with a considerably longer lifetime. These perturbations take considerably more time to dissipate. During that time some external force to begin the collapse is more likely to take place. Therefore, it is not directly contradictory to our findings (Fig. 1).

The globule CB 188 is found to have a protostar. As the prediction for this globule depends on the chosen *f*(*R*) theory, it supports the *f*(*R*) modified Jeans mass which can also been seen in 6. The conventional zone for collapse is within the error of the measured mass of CB 188, but the collapse better fits the modified Jeans mass. For the globule mass to be sufficient the Jeans mass modification coefficient should be

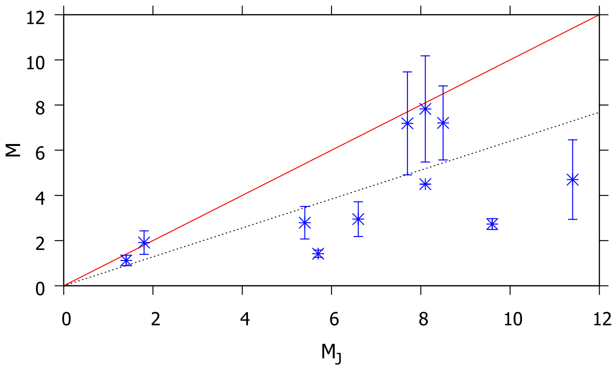

Fig. 1 The masses of the examined Bok globules are presented with the graphs of the conventional Jeans mass, *solid line*, and the lowest *f*(*R*) modified *dotted line*. The units on both axes the solar mass units. The part above the *solid line* is the stable zone, whereas the one below the *dotted line* is the collapsing region. The part between the *lines* depends on the chosen *f*(*R*) model

$\beta < 0.93$. With the definition of the Jeans mass (30) we have for this globule

$$f''(R_0) > 0.052 k_J^2 = 0.052 \frac{4\pi G m_H^2 n_H \mu}{k_B T} \approx 1.3 \times 10^{-31} \mathrm{m}^{-2}. \tag{43}$$

The most interesting globule in the sample is FeSt 1-457, for which the non-altered Jeans mass is well above the observed mass even with the error taken into account. For FeSt 1-457 the coefficient would be the even lower $\beta < 0.78$ at the best estimate and the same as for CB 188, $\beta < 0.93$, for the upper limit. With the upper limit we would have a constraint for *f*(*R*) models

$$f''(R_0) > 1.3 \times 10^{-31} \mathrm{m}^{-2}. \tag{44}$$

In this sense the *f*(*R*) modified Jeans mass appears to better predict the collapse of globules. As the mass necessary to make clouds collapse is less, this has an effect on the forces holding the collapse at bay. This would imply that the counteracting forces, *e.g.* turbulence, do not need to be as strong in molecular clouds as with conventional Jeans mass.

We stress that this is not a stringent constraint for *f*(*R*) theories. As it is produced by a single data point, it rather serves as a feasibility study. With a larger dataset, preferably with smaller error bars, it is possible to find a constraint for *f*(*R*) models and other types of modified gravity as well.

7 Constraints for $f(R)$ models

In this section we take a look at specific *f*(*R*) models which are considered viable. The treatment of the previous chapter is subjected to these models. In the literature, there are numerous general constraints on viable models. First we take a look at these known constraints.

There are several necessary conditions for *f*(*R*) models to satisfy [47]. Some of the conditions are based on mathematical properties and some are related to observations of the Universe. Some of the most simple and fundamental ones are the ghost and the Dolgov-Kawasaki criteria. To avoid ghosts and anti-gravity the condition $f'(R) > 0$ is necessary [11]. The Dolgov-Kawasaki singularity is avoided with $f''(R) > 0$ [14].

The Dolgov-Kawasaki criterion effectively rules out the somewhat unambiguous solutions of k_-. It should be noted that this criterion must be satisfied at $R \geq R_a$, where R_a is the present day curvature. On examining perturbations on a Minkowskian background as in [25], these constraints do not need to hold and the k_- solution is not ruled out.

Most cosmological constraints come from far-away objects such as supernovae [3] and large scale structures [2,48] but nearby objects can be considered as well. Several types of astrophysical objects have been considered for constraints in the literature [30,49]. These include cepheids, red giant stars, water masers and relatively closer dwarf galaxies [50–52].

Next we will examine some specific *f*(*R*) models and the constraints set by the globule observations.

7.1 Hu-Sawicki model

The Hu-Sawicki models produce the accelerated expansion and satisfy both cosmological and solar-system constraints. There are three parameters, for which there are some constraints. The *f*(*R*) function reads as

$$f(R) = R - \lambda R_c \frac{(R/R_c)^{2n}}{(R/R_c)^{2n} + 1}, \; n, \lambda, R_c > 0. \tag{45}$$

The critical curvature R_c is of the order of the present day curvature. The larger the n the longer the model mimics ΛCDM. It has been found that there is also a lower limit for $n > 0.905$ [47] and in [53] it is found that for $n = 1$ the λ must be large $\lambda \gg 20$.

The only non-zero values for $f''(R_0)$ are with $n = 1/2$ and $n = 1/4$. These are both ruled out by the condition $\lambda > 0.905$. For all viable Hu-Sawicki models, the Jeans limit is the same as for GR.

7.2 Starobinsky model

The Starobinsky model [11] is of the form

$$f(R) = R - \lambda R_0 \left(\left(1 + \frac{R^2}{R_0^2} \right)^{-n} - 1 \right). \tag{46}$$

This yields the condition $2n\lambda/R_0 > 1.3 \times 10^{-31}\,\mathrm{m}^{-2}$ for the globule FeSt 1-457. In [11] it is found that $n \geq 0$ and $\lambda > 8/3\sqrt{3}$. As the R_0 is of the order of the cosmological constant, *i.e.* very small, the condition is necessarily satisfied. The Starobinsky model exhibits assisted star formation due to the added *f*(*R*) effects.

Both the Starobinsky model and the Hu-Sawicki model have similar expansions in the high curvature regime. These lead to a condition $n > 0.9$ [54]. The shared condition is due to the similar expansion.

7.3 Appleby-Battye model

In the Appleby-Battye model the $f(R)$ is tailored to agree with the cosmology constraints as well as the stability issues. The form is

$$f(R) = \frac{1}{2}R + \frac{1}{2a}\log[\cosh(aR) - \tanh(b)\sinh(aR)], \tag{47}$$

with a and b being the model parameters. This leads to $f''(R) = \frac{a}{2}\mathrm{sech}^2(aR - b)$. In [13], it is found that $a \approx \frac{2b}{R_0} \approx \frac{b}{6H_0^2}$. Therefore

$$\frac{b}{12H_0^2}\mathrm{sech}^2(b) > 1.3 \times 10^{-31}\mathrm{m}^{-2}, \tag{48}$$

which in turn implies that roughly $b < 78$. Another constraint is that $8e^{-2b} \ll 2R_0$, which is satisfied around $b = 46$, which leaves us a range of roughly $50 < b < 75$. The Appleby-Battye model requires fine-tuning due to the existing constraints. With the globule observations it is even more so.

7.4 Tsujikawa model

The Tsujikawa model is described by

$$f(R) = R - \lambda R_c \tanh\left(\frac{R}{R_c}\right), \tag{49}$$

with R_c and λ being positive model parameters. The Tsujikawa and Appleby-Battye models have similarities, but for the purposes of our treatment, the behaviour is different. This model has $f''(R_0) = 0$. Therefore, the Tsujikawa yields the same predictions for collapse in the globules as the conventional GR gravity.

8 Conclusions

We have examined the effects of $f(R)$ gravity on collisionless collapse, especially the limit of instability. The mass distribution is allowed to be time-dependent to better describe a collapse event. The examination is based on de Sitter background with perturbations.

We have found that with the addition of $f(R)$, the limit for collapse can be different. It is also found that with certain models, for which $f''(0) < 0$ a new limit is present.

This second limit is found to have no physical consequences and is ruled out due to the Dolgov-Kawasaki instability with present day curvatures.

In reference [25] an analysis similar to ours is done. However, the *f*(*R*) parameter is fixed as $\alpha = -\frac{1}{k_j^2} = -\frac{\sigma^2}{4\pi G\rho_0}$ and the background is also taken to be Minkowskian. These are unnecessary constraints on the models, restricting to a fixed Jeans limit. Therefore, our results are more general.

It is found that *f*(*R*) models can affect star formation by lowering the limit for collapse. For viable models the result is assisted star formation. This is in agreement with the observed collapse behaviour in Bok globules. The globules experience star formation at rates higher than standard Jeans analysis would suggest.

The effects of *f*(*R*) on Jeans mass are constrained for models passing the Dolgov-Kawasaki criterion. We find the lower limit for Jeans mass that a model could reach. The upper limit coincides with GR. In the extreme a modification can lower the required mass for collapse by around one third.

The modified Jeans limit, as well as the standard limit, are found to be of the order of Bok globules. These gas clouds and their collapse behaviour can be examined for agreement with *f*(*R*) modified predictions.

We have used a small test sample of Bok globules to demonstrate that it is possible to obtain a constraint for some *f*(*R*) models. This constraint is based on lowering the collapse to a level agreeing with the amount of protostars in Bok globules. With a larger data set and better understanding of the physics in these clouds, a strong limit might be obtained.

In our linearized approach, not all the characteristics of *f*(*R*) models are present. It is also possible, that in a higher order, more theories would experience changes to stability. However, it is unlikely that these further changes would cancel the phenomena caused by the lower order terms.

Some of the examined viable *f*(*R*) models revert to the standard GR value in regard of the modified Jeans limit. This is due to the modifications to the Jeans limit appearing only as $f''(R) \neq 0$. The Hu-Sawicki model and the Tsujikawa model do not experience any modifications as their acceptable parameter space does not allow for $f''(R_0) \neq 0$.

The Starobinsky model allows for the modified Jeans limit, which fit the observations well. For the Appleby-Battye model we find to obtain a considerably lower Jeans mass, which would better fit observations, the constraints on the model become even more stringent.

A more detailed collapse model, including *e.g.* turbulence, could provide a more accurate limit. Understanding the effects of modified gravity in star formation could lead to better understanding of the demands for a viable gravity theory.

The methodology we have developed in this article can be applied to more extensive datasets on Bok globules as they become available. Similar treatment can also be subjected to protogalaxies (in reference [55] galactic disks are examined). Perhaps the most interesting possibility is to extend a similar treatment to other modified gravity theories such as scalar-tensor gravity. The effects on Jeans mass are likely to appear due to most modifications. Theories that raise Jeans mass inhibit star formation and therefore, are not favored by observations.

Appendix

Dispersion relation integral

We examine the integral part of the dispersion relation

$$4\pi G \int \left(\frac{\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}}{\mathbf{v} \cdot \mathbf{k} - \omega} \right) d\mathbf{v} + \frac{k^4 \Big(1 + 3\alpha (k^2 - \omega^2 + 3\alpha\omega^4) \Big)}{(1 + 4\alpha k^2)(k^2 - 3\omega^2)}$$
$$\equiv I + \frac{k^4 \Big(1 + 3\alpha (k^2 - \omega^2 + 3\alpha\omega^4) \Big)}{(1 + 4\alpha k^2)(k^2 - 3\omega^2)} = 0 \tag{50}$$

The distribution of particle speeds in a stellar system follows the Maxwell-Boltzmann distribution. Therefore, we have for the background matter distribution $f_0(\mathbf{v})$

$$f_0 = \frac{\rho_0}{\sqrt{2\pi\sigma^2}} e^{-(v^2/2\sigma^2)}. \tag{51}$$

The coordinate system is arbitrary and we are free to choose $\mathbf{k} = (k, 0, 0)$:

$$I = -\frac{2\sqrt{2\pi} G\rho_0}{\sigma^3} \int \frac{k v_x e^{-v_x^2/(2\sigma^2)}}{k v_x - \omega} dv_x. \tag{52}$$

We make a substitution of $v_x = \sqrt{2}\sigma x$ to reach

$$-\frac{4\sqrt{\pi} G\rho_0}{\sigma^2} \int \frac{x e^{-x^2}}{x - \omega/(\sqrt{2}\sigma k)} dx. \tag{53}$$

The problematic part is the singularity at $x = \omega/\sqrt{2}\sigma k$. Depending on ω, whether it is imaginary or not, the integration path must be chosen accordingly. We are interested in the unstable modes for which $\mathrm{Im}(\omega) > 0$, which is also the most simple case. We notice that the integral has a close resemblance to a plasma dispersion function (*e.g.* [35] p. 787)

$$Z(w) = i\sqrt{\pi} e^{-w^2} \big(1 + \mathrm{erf}(iw) \big), \quad \big(\mathrm{Im}(w) > 0 \big) \tag{54}$$
$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-s^2}}{s - w} ds \tag{55}$$

where $\mathrm{erf}(z)$ is an error function. The w derivative is found to be

$$\frac{dZ(w)}{dw} = -\frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{s e^{-s^2}}{s - w} ds \tag{56}$$

In general we have

$$Z^{(n)}(w) = \frac{d^n Z(w)}{dw^n} = \frac{n!}{\sqrt{n}} \int_{-\infty}^{\infty} ds \frac{e^{-s^2}}{(s-w)^{n+1}} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{ds}{s-w} \frac{d^n(e^{-s^2})}{ds^n}. \tag{57}$$

For Hermite polynomials $H_n(s)$ holds the equality

$$\frac{d^n}{ds^n}(e^{-s^2}) = (-1)^n e^{-s^2} H_n(s) \tag{58}$$

known as the Rodrigues formula. With this equality the derivatives of $Z(w)$ can be written as

$$\frac{d^n Z(w)}{dw^n} = \frac{(-1)^n}{\sqrt{n}} \int_{-\infty}^{\infty} ds \frac{H_n(s)e^{-s^2}}{s-w}. \tag{59}$$

We can further use the Hermite polynomials in writing the powers of the variable s with the relation

$$s^n = \frac{1}{2^n} \sum_{m=0}^{M} d_m(n) H_{n-2m}(s) \tag{60}$$

with the coefficients $d_m(n)$ found in most tables and $M \equiv [n/2]$ and therefore

$$Z_n(w) = \frac{1}{2^n} \sum_{m=0}^{[n/2]} (-1)^{n-2m} d_m(n) \frac{d^{n-2m} Z(w)}{dw^{n-2m}}. \tag{61}$$

This can be used to solve the integral in I

$$\int \frac{xe^{-x^2}}{x-w} dx = 1 + wZ(w). \tag{62}$$

The imaginary part of $wZ(w)$ must vanish for the dispersion relation to be satisfied. For that to happen, we must have $\mathrm{Re}(w) = 0$. We further mark $\omega = i\omega_I$ and write the plasma dispersion function in a different form (54)

$$1+wZ(w) = 1+\mathrm{i}w\sqrt{\pi}e^{-w^2}\big[1+\mathrm{erf}(\mathrm{i}w)\big] = 1 - \frac{\sqrt{\pi}\omega_I}{\sqrt{2}k\sigma}\exp\Big(\frac{\omega_I^2}{2k^2\sigma^2}\Big)\mathrm{erfc}\Big(\frac{\omega_I}{\sqrt{2}k\sigma}\Big) \tag{63}$$

with $\mathrm{erfc}(z) \equiv 1 - \mathrm{erf}(z)$ being the complementary error function and $w = \omega/\sqrt{2}k\sigma$, $\mathrm{erf}(-z) = -\mathrm{erf}(z)$. Finally we have

$$I = -\frac{4\pi G\rho_0}{\sigma^2}\left[1 - \frac{\sqrt{\pi}\omega_I}{\sqrt{2}k\sigma}\exp\left(\frac{\omega_I^2}{2k^2\sigma^2}\right)\mathrm{erfc}\left(\frac{\omega_I}{\sqrt{2}k\sigma}\right)\right]. \tag{64}$$

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$f(R)$ gravity constraints from gravitational waves

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Abstract

The recent LIGO observation sparked interest in the field of gravitational wave signals. Besides the gravitational wave observation the LIGO collaboration used the inspiraling black hole pair to constrain the graviton mass. Unlike general relativity, $f(R)$ theories have a characteristic non-zero mass graviton. We apply the constraint on the graviton mass to viable $f(R)$ models to find the effects on model parameters. We find it possible to constrain the parameter space with the gravity wave based observations. We make a case study for the popular Hu-Sawicki model and find a parameter bracket. The result generalizes to other $f(R)$ theories and can be used to contain the parameter space.

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I. INTRODUCTION

The recent observation of gravitational waves [1] confirmed the hundred years ago predicted gravitational waves. In the early years of general relativity different models for gravitation were considered as well. For a long time these alternatives to general relativity were little more than a curiosity as the observations of that time did not call for anything else. Many of these modified theories of gravity were ruled out for theoretical reasons but others remained viable.

Cosmic microwave background [2] and supernova observations [3, 4] lead to the discovery of accelerating expansion of the Universe. It can be argued that this discovery and a growing body of observations lead to a renaissance in cosmology. The accelerated expansion could be explained with the cosmological constant, but there are some fundamental problems with the cosmological constant [5] and the ΛCDM or concordance model [6]. Therefore, the modified gravity theories which received little interest for decades have become relevant once again.

The $f(R)$ theories (see *e.g.* [7, 8] for reviews), or fourth order theories, which generalize the Einstein Hilbert Langrangian to be a function of the curvature scalar, have received considerable attention in the 21st century. In [9] it was shown that the accelerating expansion could be explained with a $f(R)$ modification. Since then, more viable models have been proposed (*e.g.* [10–15]).

In standard general relativity the graviton, which mediates the gravitational force, has a zero mass. In order to give graviton a mass some generalization is needed, namely taking a set background metric [16]. General relativity is a unique theory given a certain set of postulates[1] and the path of least change is fixing the background metric.

It is possible to add a term to the Einstein Hilbert action causing a massive graviton [16, 17]. There are a number of different terms that produce a massive graviton but most of these fail to reach the correct Newtonian limit [18, 19]. However, while in general relativity the graviton naturally has a zero mass, this is not the case for $f(R)$ gravity [7].

In $f(R)$ gravity the graviton has *a priori* a non-zero mass. As the $f(R)$ theories are explicitly higher order theories, this in not in contradiction with the demands of constructing a massive graviton for general relativity. The higher order contribution in the field equations adds up to effective graviton mass term. This link between graviton mass and model dependence can be converted into boundaries for viable $f(R)$ models.

Solar system observations have set several bounds on the mass of the graviton. As the dynamics of the solar system are found to follow general relativity extremely closely, these bounds are rather stringent. If the Newtonian potential is modified with the graviton mass,

[1] A metric theory with field equations of linear second order in derivatives, satisfies the Newtonian weak field limit and does not depend on any prior geometry.

the Kepler laws produce a limit for the Compton wavelength of the graviton [20, 21]. As the Compton wavelength is related to the mass [22] via $\lambda_g = h/m_g c$ this translates to a bound on graviton mass.

Inspiraling binaries are a known source of gravitational waves and a possibility to commit graviton mass measurements [20, 21, 23]. Before the LIGO experiments the graviton mass has been bounded by binary pulsars [17] instead of a pair of black holes. Similar studies have been done in the context of $f(R)$ gravity [24]. Assuming a non-zero mass m_g graviton would cause the gravitational potential to be of the Yukawa form $r^{-1}e^{-m_g rc/h}$. The exponential dependence would cause a cut-off of the gravitational interaction at large distances, namely larger than the Compton wavelength. Such a cut-off has not been observed in the solar system [20] or galaxy clusters [25]. Therefore, these observations set an upper limit for the mass of the graviton m_g.

The galaxy cluster limits for the graviton mass are rather stringent ones with $m_g c^2 < 2 \times 10^{-29}$eV [25], but are model dependent regarding *e.g.* dark matter assumptions. These are not directly applicable to $f(R)$ theories as they modify the effects and need of dark matter [26–29]. For the time being, the best model independent bounds for the graviton mass are those from the recent LIGO observations $m_g c^2 < 1.2 \times 10^{-22}$eV [22]. If a super massive black hole binary is detected in the future, it could introduce a several orders of magnitude more stringent limit [23]. The gravitational wave based bounds arise from the dynamics of gravitation and as such are model-independent.

In the following we will examine the naturally occurring graviton mass in the $f(R)$ [30]. There have been several studies into constraining $f(R)$ theories with both theoretical and observational means (*e.g.* [8, 31–36]). There have also been previous studies into $f(R)$ gravity in the context of binaries and related graviton mass [37, 38]. With the recent LIGO upper limit on the graviton mass we can further constrain the model parameters of viable $f(R)$ theories such as the Hu-Sawicki model [10].

II. EQUATIONS OF MOTION

In the following we derive the equations of motion describing gravitational waves and graviton mass arising from the $f(R)$ contribution. We examine a $f(R)$ modified gravitational action[2]

$$\mathcal{A} = \frac{1}{2\chi} \int d^4x \sqrt{-g} \Big(f(R) + 2\chi \mathcal{L}_m \Big), \tag{1}$$

[2] The signature of the metric is $-,+,+,+$, the Riemann curvature tensor is $R^{\alpha}_{\beta\mu\nu} = \partial_\mu \Gamma^{\alpha}_{\beta\nu} - \partial_\nu \Gamma^{\alpha}_{\beta\mu} + \Gamma^{\alpha}_{\kappa\mu}\Gamma^{\kappa}_{\beta\nu} - \Gamma^{\alpha}_{\kappa\nu}\Gamma^{\kappa}_{\beta\mu}$ and the Ricci tensor is $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$.

where $\chi = \frac{8\pi G}{c^4}$ is the coupling of gravitational equations. The latter term $\mathcal{L}_m$ is the minimally coupled matter Lagrangian. Following standard metric variational techniques we find the field equations and the trace equation

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu f'(R) + g_{\mu\nu}\Box f'(R) = \chi T_{\mu\nu} \quad (2)$$

$$3\Box f'(R) + f'(R)R - 2f(R) = \chi T, \quad (3)$$

where $T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\sqrt{-g}\mathcal{L}_m}{\delta g^{\mu\nu}}$ is the energy-momentum tensor and $T = T^\alpha_\alpha$. The prime is used to denote the derivatives with respect to R. We study the linear perturbations $h_{\mu\nu}$ and write

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}, \quad (4)$$

where $\tilde{g}_{\mu\nu}$ is the background metric. In general we use tilde to denote the quantities calculated with the background metric. The Ricci tensor and scalar can be expanded around the background as

$$R_{\mu\nu} \simeq \tilde{R}_{\mu\nu} + \delta R_{\mu\nu} + \mathcal{O}(h^2), \quad (5)$$

$$R \simeq \tilde{R} + \delta R + \mathcal{O}(h^2). \quad (6)$$

As the first derivative of $f(R)$ appears in the equations of motion, we need an expansion for this function as well, *i.e.* $f'(R) \simeq f'(\tilde{R}) + f''(\tilde{R})\delta R + \mathcal{O}(4)$. This expansion can be plugged into (3) for

$$f''(\tilde{R})(3\Box\delta R + \tilde{R}\delta R) - f'(\tilde{R})\delta R = 0. \quad (7)$$

As we are primarily interested in the propagation of gravitational waves in empty space, we set $T_{\mu\nu} = 0$. The variations of the Ricci tensor and scalar can be written in terms of the metric perturbation $h_{\mu\nu}$ (*e.g.* [39])

$$\delta R_{\mu\nu} = \frac{1}{2}\Big(\nabla_\mu\nabla_\nu h - \nabla_\mu\nabla^\lambda h_{\lambda\nu} - \nabla_\nu\nabla^\lambda h_{\mu\lambda} + \Box h_{\mu\nu}\Big), \quad (8)$$

$$\delta R = \delta(g^{\mu\nu}R_{\mu\nu}) = \Box h - \nabla^\mu\nabla^\nu h_{\mu\nu} - \tilde{R}_{\mu\nu}h^{\mu\nu}. \quad (9)$$

As the case is gauge invariant we fix the gauge to be the harmonic gauge with

$$\nabla_\mu h^\mu_\lambda = \frac{1}{2}\nabla_\lambda h, \quad (10)$$

which further implies $\nabla^\mu\nabla^\nu h_{\mu\nu} = \frac{1}{2}\Box h$.

For a viable $f(R)$ theory to have a de Sitter solution, the background equations of (2) and (3) for empty space, $f'(\tilde{R})\tilde{R} = 2f(\tilde{R})$ and $\tilde{R}_{\mu\nu} = g_{\mu\nu}\frac{f(\tilde{R})}{2f'(\tilde{R})}$, must hold. Using these equalities and the harmonic gauge we find

$$3f''(\tilde{R})\Box^2 h - \Big(\frac{f(\tilde{R})f''(\tilde{R})}{f'(\tilde{R})} + f'(\tilde{R})\Big)\Box h + \Big(f(\tilde{R}) - \frac{2f^2(\tilde{R})f''(\tilde{R})}{f'^2(\tilde{R})}\Big)h = 0. \tag{11}$$

The graviton dispersion relation $k^2 = -m_g^2$ reveals that the plane wave solution $h \sim e^{ik\cdot x}$ fulfills $\Box h = m_g^2 h$. Therefore, we can write

$$3f''(\tilde{R})m_g^4 - \Big(\frac{f(\tilde{R})f''(\tilde{R})}{f'(\tilde{R})} + f'(\tilde{R})\Big)m_g^2 + \Big(f(\tilde{R}) - \frac{2f^2(\tilde{R})f''(\tilde{R})}{f'^2(\tilde{R})}\Big) = 0, \tag{12}$$

for non-zero perturbations. Thus we obtain two solutions for m_g^2

$$m_1^2 = \frac{f'^2(\tilde{R}) - 2f(\tilde{R})f''(\tilde{R})}{3f'(\tilde{R})f''(\tilde{R})}, \tag{13}$$

$$m_2^2 = \frac{1}{2}\tilde{R}, \tag{14}$$

which tell us the perturbations of the metric can be written as a linear combination

$$h_{\mu\nu} = h_{\mu\nu}^{(1)}e^{ik^{(1)}\cdot x} + h_{\mu\nu}^{(2)}e^{ik^{(2)}\cdot x}, \tag{15}$$

where the quantities $h_{\mu\nu}^{(i)}$ and $k_\mu^{(i)}$ are the metric perturbation and four-momentum related to the corresponding solution m_i.

We have found two physically viable solutions for a non-zero graviton mass. The first solution (13) resembles the stability criterion of [40, 41]. Basically this criterion tells us, that the square of the graviton mass must not be negative. The mass is often derived with the well-known $f(R)$ theory scalar-tensor theory equivalence [42–44]. This solution is not available when $f''(R) = 0$, such as in the case of GR.

The second solution (14) does not depend on $f''(R)$ and would hold even for GR. This solution is related to having $\delta R = 0$ in (7). In the case of empty space GR we would have $\tilde{R} = 0$ and $m_2 = 0$ as is to be expected. Clearly, there is a well-behaved GR limit, $f''(R) \to 0$, for the second solution. Since for this solution $\delta R = 0$, in the situation $\tilde{R} = 0$ the perturbation of the metric would be simply

$$\delta R \sim h_{\mu\nu}^{(1)}e^{ik^{(1)}\cdot x} \tag{16}$$

and only the scalar modes would manifest. Therefore, m_2 solutions do not effect scalar perturbations while the tensor perturbations are affected by both the solutions.

For the first solution, the GR limit is problematic as it diverges as $f''(R) \to 0$. This reveals an interesting fact that even though $f(R)$ models have to closely resemble GR, it cannot be infinitely close. This can be compared to the result of the forbidden Higuchi mass range of the graviton [45–47]. The emergence of these massive modes in $f(R)$ gravity is discussed in detail in [30].

We notice that the second solution is extremely small with $m_2 \sim \sqrt{\Lambda}$, which easily passes all constraints on graviton mass. Therefore, our attention concentrates on the first solution, which can be constrained. It is unknown which mass state of gravitons inspiraling black holes would emit. Mergers in $f(R)$ gravity would need to be studied further to be able to distinguish between these two. To our knowledge, such studies have not yet been conducted.

Another, often overlooked, fact is that for GR with Λ, $f(R) = R + \Lambda$, we would have a non-zero graviton mass, $m_2^2 = 2\Lambda$. This is due to relaxing the assumptions of GR [16]. Even though this is mathematically clear, the physical consequences are debatable, see *e.g.* [48] and references therein for discussion.

For the case of $f(R)$ gravity, there is the extra scalar degree of freedom like with the cosmological constant. A massive graviton always implies extra degrees of freedom. Due to the added degrees of freedom, the gravitational waves with Λ or $f(R)$ are different to those caused by plain GR. However, this does not affect the relation to observations.

The LIGO observations provide a lower limit for the Compton wavelength of the graviton [1]. A finite Compton wavelength in general, would translate to a massive theory and therefore, extra degrees of freedom. The measurements detect perturbations of the metric, $h_{\mu\nu}$, which can be written as a linear combination of the modes associated with masses (13) and (14). It is not known, what is the ratio of these two modes caused by the black holes but the total contribution is constrained.

In the following, we shall take a closer look at specific models and use the Hu-Sawicki model as a case study to demonstrate the procedure.

III. VIABLE $f(R)$ MODELS AND GRAVITON MASS CONSTRAINTS

There have been numerous studies to constrain viable $f(R)$ models [10, 32, 49]. The most stringent bound with $\tilde{R}$ is

$$|f'(\tilde{R}) - 1| < 4 \times 10^{-7}, \tag{17}$$

constraining the parameters of the $f(R)$ function. Here, and for the rest of the paper we assume natural units. With the graviton mass we can find another bound for these parameters.

The popular Hu-Sawicki model [10] is constructed to evade the solar system tests and produce the observed late-time cosmology. A truly viable model needs to fulfil the high curvature regime constraints as well as provide the accelerated expansion of the Universe, which appears at low curvature regimes. The Hu-Sawicki model is of the form

$$f(R) = R - \mu R_c \frac{\left(\frac{R}{R_c}\right)^{2n}}{b\left(\frac{R}{R_c}\right)^{2n} + 1}, \tag{18}$$

with μ, R_c, b positive constants and $n \in \mathbb{N}$. Inserting this into the de Sitter criterion, $\tilde{R}f'(\tilde{R}) - 2f(\tilde{R}) = 0$, we can solve for b

$$b_{\pm} = -1 + \mu \pm \sqrt{\mu(\mu - 2n)}. \tag{19}$$

As the action must be real, b must have a real value as well. This leads to a constraint $\mu > 2n$. The constant R_c is a free scaling parameter and for simplicity we have chosen $R_c = \tilde{R}$. The bound (17) translates to

$$|f'(\tilde{R}) - 1| = \frac{2n\mu}{(1 + b_{\pm})^2} < 4 \times 10^{-7} \tag{20}$$

For b_- we have

$$|f'(\tilde{R}) - 1| = \frac{2n\mu}{(\mu - \sqrt{\mu(\mu - 2n)})^2} = \frac{2n}{\mu\left(1 - \sqrt{1 - \frac{2n}{\mu}}\right)^2} < 4 \times 10^{-7}. \tag{21}$$

With the condition $\mu > 2n$ the square root can be expanded as a series. This results in $|f'(\tilde{R}) - 1| \sim \mu < 10^{-7}$ which is in clear contradiction with $\mu > 2n$. Therefore we must choose $b = b_+$, for which we find

$$\frac{2n\mu}{(\mu + \sqrt{\mu(\mu - 2n)})^2} \sim \frac{2n\mu}{4\mu^2} = \frac{n}{2\mu} < 4 \times 10^{-7} \tag{22}$$

when $\mu >> 1$. This further translates to $\mu > 10^6$. Here we have assumed $n \sim 1$. For viable models this is a reasonable assumption [8]. In any case the maximum effect of n is one magnitude for viable models. As $\mu >> 1$ we can write the square of the graviton mass as a series of $x = 1/\mu$

$$m_g^2 = \tilde{R}\left(\frac{2}{3n(1 + 2n)x} - \frac{2n(5 + 2n)}{3(1 + 2n)^2}x\right) + \mathcal{O}(x^2). \tag{23}$$

Therefore, we have $nm_g^2/\tilde{R} \sim \mu$. As the gravitational wave observations set an upper limit for the graviton mass we find a upper limit for μ as well. We can write the relation of the background curvature to the cosmological constant as $\tilde{R} = 4\Lambda$. Using the density parameter Ω_Λ we can also write

$$\Lambda = 3H_0^2\Omega_\Lambda, \tag{24}$$

where H_0 is the Hubble parameter. Using the Planck collaboration results [50] and the LIGO results, we can now constrain the parameter μ in Hu-Sawicki models (again assuming $n \sim 1$)

$$10^{20} > \mu > 10^6. \tag{25}$$

We can see now, that the model is contained to a certain bracket, which is yet too constraining. However, with further more accurate measurements it is possible to further narrow down the bracket. As the gravitational wave constraints are independent of *e.g.* solar constraints, these offer valuable proof to the limits of $f(R)$ and scalar tensor gravity as well.

It is also interesting to notice, that the galaxy cluster limit for the graviton mass is 7 orders of magnitude tighter than the LIGO limit. If we could apply this limit, the upper limit would be of the same order as the lower limit, causing severe fine-tuning issues. However, we stress that the model dependent galaxy cluster result cannot be used directly with $f(R)$ theories.

Similar procedures can be subjected to other $f(R)$ models as, such as the Starobinsky model [11], which is described by

$$f(R) = R + \lambda R_0\Big(\big(1 + \frac{R^2}{R_0^2}\big)^{-n} - 1\Big). \tag{26}$$

with λ and R_0 positive constants and $n \in \mathbb{N}$. For the Starobinsky model, we can follow similar procedures to find $10^{-20} < \lambda < 10^{-8}$ with the similar assumption $n \sim 1$. In a similar manner constraints could be found on any other viable model as well.

IV. DISCUSSION

We have studied $f(R)$ theories and the naturally emerging massive graviton. With bounds on the graviton mass produced by the gravitational wave observations it is possible to constrain $f(R)$ theories. As a case study, we concentrate on the Hu-Sawicki model. For this model we find an upper limit for the free parameter in addition to the lower limit previously presented in the literature. While the free parameter bracket is still wide, it tells a story of fine-tuning. As the massive graviton is characteristic of $f(R)$ theories and massive Brans-Dicke theories, the viability of these models is more and more under question.

The same procedure can be subjected to other $f(R)$ theories as well. As there is a known connection between $f(R)$ gravity and scalar-tensor gravity (*e.g.* [51]), these theories are also a possible target for application.[3].

The LIGO measurement accuracy is expected to rise in the future [1, 52] with the construction of additional measuring stations. As these are likely to bring down the upper limit for the graviton mass, the bracket found for the free parameter for the Hu-Sawicki model is bound to narrow down even further.

Space-based detection of gravitational waves in the future with eLISA or similar programs are expected to give constraints on the graviton mass [53–55]. Single observations with the space-based devices are expected to reach a two magnitudes more precise measurement than the LIGO. However, as the there are multiple events during the mission, the total accuracy is expected to be 3 orders magnitude better. This will lead to a considerably tighter bracket for viable $f(R)$ models.

Related to these limits, besides other things, detection of a non-zero graviton mass would have far-reaching consequences for $f(R)$ theories and naturally GR itself. As the $f(R)$ models predict a massive graviton, the detected mass would further fine-tune the possible parameter space. On the other hand it would spell disaster for standard GR and emphasize the need for modified gravity.

Another possibility would be to use the so-far model dependent graviton mass constraints from galaxy clusters. In order to achieve this, the effects of modified gravity on dynamics and dark matter assumptions have to be carefully considered. As these model dependent limit a far tighter than the LIGO limits, they could provide far more stringent constraints and even rule out theories considered viable.

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