Convex operational theories and non-classical features of quantum theory

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LEPPÄJÄRVI, LEEVI: Convex operational theories and non-classical features of quantum theory

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Convex operational theories form a class of physical theories that are built on the operational mixing of states of the system resulting in convex state spaces. Following the operational approach to describe physical experiments, the other operational concepts, such as measurements and state transformations, rise from the properties of the state space. In addition to quantum theory, the convex operational theories include but are not restricted to classical theories and quantum theory of processes.

In the light of recent deep interest in quantum information theory, convex operational theories serve as means to consider information-theoretic principles in a more abstract framework. This allows to compare different types of theories against each other and further study the nature of these principles. Some of these principles can then even be used for different axiomatizations of quantum theory.

This thesis serves to introduce the mathematical concepts related to convex operational theories and then use them to construct this class of theories in the ordered vector space formalism. We use the constructed class of theories to consider the most important aspects of the theories with applications in physical theories such as quantum theory. We study some of the most important non-classical properties of quantum theory in the more abstract framework of convex operational theories including original research on one of these features.

Keywords: convex operational theories, generalized probabilistic theories, quantum theory, quantum theory of processes, convexity, operational approach

TURUN YLIOPISTO
Fysiikan ja tähtitieteen laitos

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Konveksit operationaaliset teoriat muodostavat joukon fysikaalisia teorioita, jotka pohjautuvat tilojen operationaaliselle sekoittamiselle, minkä seurauksena teorian tila-avaruus on konveksi. Fysikaalisten kokeiden kuvaamiseen käytettyjen operationaalisten periaatteiden seurauksena muut teorian määritykseen tarvittavat käsitteet kuten mittaukset ja tilamuunnokset saadaan johdettua tila-avaruuden ominaisuuksien pohjalta. Kvanttiteorian lisäksi konveksit operationaaliset teoriat sisältävät klassiset teoriat ja kvanttiprosessien teorian.

Kvantti-informaatioteorian saaman viimeaikaisen suuren kiinnostuksen valossa konveksit operationaaliset teoriat antavat välineet käsitellä informaatio-teoreettisia periaatteita abstraktimmassa viitekehyksessä. Tämä mahdollistaa eri teorioiden vertaamisen toisiinsa sekä antaa mahdollisuuden tutkia näiden periaatteiden luonnetta. Joitain näistä periaatteista voidaan jopa käyttää kvanttiteorian aksiomatisointiin.

Tämän työn tarkoituksena on esitellä konvekseihin operationaalisiin teorioihin liittyviä matemaattisia rakenteita ja käyttää näitä rakenteita teorioiden formalismin rakentamiseen. Näin muodostettuja teorioita käytetään teorioiden tärkeimpien piirteiden tutkimiseen ja fysikaalisiin teorioihin soveltamiseen. Kvanttiteorian tärkeimpiä ei-klassisia piirteitä tutkitaan abstraktimmassa viitekehyksessä alkuperäistä tutkimusta sisältäen.

Avainsanat: konveksit operationaaliset teoriat, yleistetyt todennäköisyysteoriat, kvanttiteoria, kvanttiprosessien teoria, konveksisuus, operationaalinen lähestymistapa

Contents

Introduction

Quantum theory is by right to be considered as one of the most accurate physical theories: all the predictions of the theory are in an extremely good accordance with the empirical experiments. The reasons behind the success of quantum theory ultimately lie on the deep understanding of its mathematical structure and the ongoing extensive research year after year. Over the decades the motivation behind quantum theory has varied from trying to explain observations that did not have grounds in the classical physics to applying quantum theory in practice.

One of the most resent motivators has been the applicability to information theory leading to the birth of quantum information theory. As we are reaching the quantum domain in the size of the components in classical computers and as the quantum information theory has been seen to hold many significant advantages over the classical information theory, it has triggered an extensive search for applications in quantum technologies. The advantages, for example the speed-ups of classical algorithms and more secure communication protocols, are ultimately a result of the non-classical features of quantum theory such as entanglement, superposition and disturbance caused by measurements [1].

The success of quantum information theory has led the researchers to seek to understand information and information processing in more abstract level as universal information-theoretic principles rather than just as specific features. This has sparked a renewed interest on a class of more abstract theories that were originally used as a tool in the research on the quantum foundations as early as the 1960's. This tool, and the subject of this thesis, is the framework of *convex operational theories*.

The convex operational theories are built upon the idea of operational mixing of the states of the theory resulting in a convex space of states. The operational approach focuses on explaining the mathematical structures in terms that are used to describe physical experiments relying on the statistical analysis of the experiment thus creating a link between states and measurements. Based on the convexity of the state space, we can formulate measurements as affine functions taking a state to a probability that the outcome of the measurement is the one represented by the function and state transformations as affine functions from the state space to another convex space. Furthermore, we can consider joint systems and construct composite state spaces. After fixing the set of states, the theory is (more or less) fixed.

The objectives behind convex operational theories were first considered by Ludwig [2–4], Mielnik [5, 6], Davies & Lewis [7] and Gudder [8]. Their work already led to a class of generalized convex schemes of quantum theory indicating that many principles and features of quantum theory can be formulated in a much more general setting. They showed that these features are not just characteristics of the standard quantum theory but that quantum theory can be considered as a particular instance of these theories leading to different axiomatizations of quantum theory based on these features. The class of theories generated this way not only include quantum theory but classical theories as well.

In the age of quantum information theory the convex operational approach has lifted its head as the framework provides a suitable setting to consider information-theoretic principles in more abstract formalism. The convex operational theories, or nowadays more often referred to as *generalized probabilistic theories (GPTs)*, provide a framework to consider different infromation processing tasks in different theories and then see how different physical principles and phenomena (for example cloning [9], teleportation [10], joint measurability [11], etc.) manifest themselves in different theories.

Studying different physical principles this way not only gives us hints what is so special about quantum theory but also it gives us information on the characteristics of different phenomena themselves as we can study how restricted or generic they are in these theories. One can for instance try to characterize features that are present in any non-classical theories or just in quantum theory. Also work on different axiomatizations of quantum theory has been continued (see for example [12, 13]) but now with the focus on the principles of information processing such as information causality [14], no-signalling and bit commitment [15]. Although the mathematical structure of quantum theory is well understood, the physical principles that would lead us to this structure are still missing.

The focus on information has also led to a more abstract level of quantum information processing as instead of considering quantum states as our objects of interest we can also process quantum processes themselves [16]. Quantum processes are important in quantum information theory as state transformations are an essential part of any manipulation and processing of information. The convex operational theories give an abstract enough setting to consider these scenarios as well.

In this thesis we present one framework for convex operational theories and use it to examine some of the most important non-classical features of quantum theory. In Chapter I we introduce the mathematical structures needed to construct the framework. We start by considering more general notions of convexity based on operational principles and show that they quite naturally fall down under the traditional convexity in vector spaces. In preparation for the construction of the theories, we present the order structures of vector spaces and their tensor products.

In Chapter II we construct the ordered vector space formalism of convex operational theories. We show the connection between convex sets and ordered vector spaces and use this formulation to define a class of operational notions such as effects, measurements and operations. We use natural physical principles to present the composite of individual systems in ordered tensor product spaces. We apply the framework to a class of specific theories by considering the features of quantum theory, quantum theory of processes, polytope theories and classical theories within the constructed framework.

The final chapter is devoted to consider some of the non-classical features of quantum theory in the level of convex operational theories. Chapter III focuses on three task-type information-theoretic features of quantum theory: *cloning*, *broadcasting* and *joint measurability*. We formulate these tasks in convex operational theories and show that they are in fact generic features for classical classical theories such that in any non-classical theory we can prove a no-go theorem for these tasks. We also see how these tasks are not independent of each other but in fact quite naturally linked.

In the last section of Chapter III is included the research part of this thesis. In the research [17], published in Physical Reviw A, we use the notion of noise content to extract intrinsic noise from (physical) observables and apply it to formulate a noise inequality that serves as a sufficient condition for joint measurability of observables. We demonstrate our result by applying it to quantum theory, quantum theory of processes and polytope states spaces. In particular, we consider an example of a newly introduced notion of reverse observables.

This thesis is dedicated to the memory of Väinö Ilmari Leppäjärvi (1939 – 2017).

Chapter I

The mathematical framework

We start our investigation of operational theories by introducing important mathematical concepts that will be needed to describe our class of theories.

1 Convex structures

Convexity is a crucial property for states of any physical system as mixing different state preparations is always possible in any controlled experiment. The notion of convexity is most commonly formulated in vector spaces but in principle there is no reason why it should be so. This leads to a generalized notion of convexity and convex structures. Generalized convexity can be formulated in many ways (see for [8, 18–21]) and here is presented one that supports the ideas behind the operational approach of interpreting physical systems. We will see in this case that under one natural assumption the convex structures can in fact be embedded in vector spaces [18] so that we are left with the traditional notion of convexity.

1.1 Generalized convexity

Let us start by constructing a notion of generalized convexity based on the idea of mixing states in a physical experiment. Suppose we have n different states $\varrho_1, \ldots, \varrho_n$ that are being examined in an experiment. Instead of using the states individually as an input for the experiment, we can decide to take the input to be the mixture of these, i.e. ϱ_1 with probability λ_1, ϱ_2 with probability λ_2, and so on. As we have to use some input, the probabilities have to sum to one. We note that any set of probabilities, called weights, satisfying this must give a valid mixture.

We can consider some basic properties of the mixture. If all the states are the same so that we only mix just one state, then we always take the input for the experiment to be that one state. Thus, with respect to the measurement, this mixture must be equivalent to the one state that it was a mixture of. On the other hand, if the states are different but some state is mixed with a zero-probability, then this state is never used as an input so that the mixture can be expressed without this state.

We also note that as the weights just represent the probabilities of choosing a particular state for the experiment, the input ordering of the states is not fixed for any single run of the experiment. Finally, if we make a mixture of mixtures of states, since we still use as inputs just the specific states as inputs, we can express the mixture of mixtures as a single mixture where the weights are determined accordingly.

These ideas lead us to the following definition. For that, let us denote $\Lambda_n = \{(\lambda_1, \ldots, \lambda_n) \in [0,1]^n \mid \sum_{i=1}^n \lambda_i = 1\}$.

Definition 1.1. A *convex structure* is a set $\mathcal{K}$ equipped with mappings

$$\langle \cdot ; \cdot \rangle_n : \Lambda_n \times \mathcal{K}^n \to \mathcal{K},$$

for which the following conditions hold for all $n \in \mathbb{N},\ n < \infty$:

C-1.
$$\langle \lambda_1, \ldots, \lambda_n; \varrho, \ldots, \varrho \rangle_n = \varrho$$

for all $\varrho \in \mathcal{K}$ and $(\lambda_1, \ldots, \lambda_n) \in \Lambda_n$;

C-2.
$$\begin{aligned}&\langle \lambda_1, \ldots, \lambda_i, \ldots, \lambda_j, \ldots, \lambda_n; \varrho_1, \ldots, \varrho_i, \ldots, \varrho_j, \ldots, \varrho_n \rangle_n \\ &= \langle \lambda_1, \ldots, \lambda_j, \ldots, \lambda_i, \ldots, \lambda_n; \varrho_1, \ldots, \varrho_j, \ldots, \varrho_i, \ldots, \varrho_n \rangle_n\end{aligned}$$

for all $i, j \leq n$, $(\varrho_1, \ldots, \varrho_n) \in \mathcal{K}^n$ and $(\lambda_1, \ldots, \lambda_n) \in \Lambda_n$;

C-3.
$$\begin{aligned}&\langle \lambda_1, \ldots, \lambda_k = 0, \ldots, \lambda_n; \varrho_1, \ldots, \varrho_k, \ldots, \varrho_n \rangle_n \\ &= \langle \lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_n; \varrho_1, \ldots, \varrho_{k-1}, \varrho_{k+1}, \ldots, \varrho_n \rangle_{n-1}\end{aligned}$$

for all $(\varrho_1, \ldots, \varrho_n) \in \mathcal{K}^n$, $(\lambda_1, \ldots, \lambda_n) \in \Lambda_n$ and any $k \leq n$;

C-4.
$$\begin{aligned}&\langle \lambda_1, \ldots, \lambda_i, \ldots, \lambda_n; \varrho_1, \ldots, \varrho_i = \langle \mu_1, \ldots, \mu_m; \varrho'_1, \ldots, \varrho'_m \rangle_m, \ldots, \varrho_n \rangle_n \\ &= \langle \lambda_1, \ldots, \lambda_{i-1}, \lambda_i\mu_1, \ldots, \lambda_i\mu_m, \lambda_{i+1}, \ldots, \lambda_n; \varrho_1, \ldots, \varrho_{i-1}, \varrho'_1, \ldots, \varrho'_m, \varrho_{i+1}, \ldots, \varrho_n \rangle_{n+m-1}\end{aligned}$$

for all $(\varrho_1, \ldots, \varrho_n) \in \mathcal{K}^n$, $(\varrho'_1, \ldots, \varrho'_m) \in \mathcal{K}^m$, $(\lambda_1, \ldots, \lambda_n) \in \Lambda_n$ and $(\mu_1, \ldots, \mu_m) \in \Lambda_m$.

The mappings $\langle \cdot;\cdot \rangle_n$ on $\mathcal{K}$ are called a *convex combination* for each n. The element $\langle \lambda_1,\dots,\lambda_n;\varrho_1,\dots,\varrho_n \rangle_n$ given by a convex combination $\langle \cdot;\cdot \rangle_n$ is called the *mixture* of the elements $\varrho_1,\dots,\varrho_n$ with *weights* $\lambda_1,\dots,\lambda_n$ respectively. Thus, the existence of a convex combination mapping means that a finite set of elements with specific weights form a unique mixture in the convex structure $\mathcal{K}$.

The conditions C–1 — C–4 formulate the ideas presented above: The condition C–1 states that the mixture of an element with itself is just the original element, and condition C–2 means that the mixture is independent of the order of the elements in the mixture. Condition C–3 reads that a mixture which has a weight 0 for some element is independent of that element and does not affect the mixture, and finally condition C–4 illustrates the fact that a mixture of mixture elements can be expressed as a single mixture with weighted weights.

Since the arguments of the convex combination function $\langle \cdot;\cdot \rangle_n$ already indicate the subscript n, it will be left out from here on. The weights of two-element mixtures always satisfy $\lambda_2 = 1-\lambda_1$ so that in order to simplify notations we denote $\langle \lambda;\varrho,\varrho' \rangle \equiv \langle \lambda, 1-\lambda;\varrho,\varrho' \rangle$.

Let us first examine some properties of mixtures.

Proposition 1.2. *Let $\mathcal{K}$ be a convex structure. For a convex combination function $\langle \cdot;\cdot \rangle$ the following properties hold all $n \in \mathbb{N}$:*
a) $\langle 0,\dots,0,1,0,\dots,0;\varrho_1,\dots,\varrho_{k-1},\varrho_k,\varrho_{k+1},\dots,\varrho_n \rangle = \varrho_k$ for all elements $(\varrho_1,\dots,\varrho_n) \in \mathcal{K}^n$ and $1 \le k \le n$.
b) $\langle \lambda_1,\dots,\lambda_k,\lambda_{k+1},\dots,\lambda_n;\varrho,\dots,\varrho,\varrho_{k+1},\dots,\varrho_n \rangle$
$= \left\langle \sum_{i=1}^k \lambda_i,\lambda_{k+1},\dots,\lambda_n;\varrho,\varrho_{k+1},\dots,\varrho_n \right\rangle$ for all $(\lambda_1,\dots,\lambda_n) \in \Lambda_n$, $\varrho,\varrho_{k+1},\dots,\varrho_n \in \mathcal{K}$ and $0 \le k \le n$.

Proof. a) By condition C–2 we can assume that $k=1$. We prove the statement by induction. Let first $n=1$. Then by C–1 we have

$$\langle 1;\varrho_1 \rangle = \varrho_1.$$

Suppose now that a) holds for a mixture of m elements. Now by C–3 we have that

$$\langle 1,0,0,\dots,0;\varrho_1,\varrho_2,\varrho_3,\dots,\varrho_{m+1} \rangle = \langle 1,0,0,\dots,0;\varrho_1,\varrho_2,\varrho_3,\dots,\varrho_m \rangle = \varrho_1.$$

Thus a) holds for all $n \in \mathbb{N}$.

b) For $k=1$ the statement clearly holds since the sum of weights is trivial. If $k=n$ we see by C–1 that

$$\langle \lambda_1,\dots,\lambda_n;\varrho,\dots,\varrho \rangle = \varrho = \langle 1;\varrho \rangle = \left\langle \sum_{i=1}^n \lambda_n;\varrho \right\rangle.$$

Hence, cases $n = 1$ and $n = 2$ are covered.

Let us now fix $k = 2$ and show that b) holds for all $n = 3, 4, \ldots$. This can be achieved by induction with respect to n.

First take $n = 3$, then by C–4 and C–1 we have that

$$\begin{aligned} \langle\, \lambda_1, \lambda_2, \lambda_3; \varrho, \varrho, \varrho_3 \,\rangle &= \left\langle 1 - \lambda_3; \left\langle \frac{\lambda_1}{1-\lambda_3}, \frac{\lambda_2}{1-\lambda_3}; \varrho, \varrho \right\rangle, \varrho_3 \right\rangle \\ &= \langle\, \lambda_1 + \lambda_2, \lambda_3; \varrho, \varrho_3 \,\rangle\,. \end{aligned}$$

Then suppose that b) holds for a mixture of m number of elements when $k = 2$. By C-5 and the induction hypothesis we see that

$$\begin{aligned} &\langle\, \lambda_1\,, \lambda_2, \lambda_3 \ldots, \lambda_{m+1}; \varrho, \varrho, \varrho_3, \ldots, \varrho_{m+1} \,\rangle \\ &= \left\langle 1 - \lambda_{m+1}; \left\langle \frac{\lambda_1}{1-\lambda_{m+1}}, \ldots, \frac{\lambda_m}{1-\lambda_{m+1}}; \varrho, \varrho, \varrho_3, \ldots, \varrho_m \right\rangle, \varrho_{m+1} \right\rangle \\ &= \left\langle 1 - \lambda_{m+1}; \left\langle \frac{\lambda_1 + \lambda_2}{1-\lambda_{m+1}}, \ldots, \frac{\lambda_m}{1-\lambda_{m+1}}; \varrho, \varrho_3, \ldots, \varrho_m \right\rangle, \varrho_{m+1} \right\rangle \\ &= \langle\, \lambda_1 + \lambda_2, \lambda_3, \ldots, \lambda_{m+1}; \varrho, \varrho_3, \ldots, \varrho_{m+1} \,\rangle\,, \end{aligned}$$

so that b) holds for all number of mixtures when $k = 2$.

It remains to show that b) then holds for all $0 \leq k \leq n$. This can be done similarly by induction but now with respect to k. First take $k = 3$:

$$\begin{aligned} &\langle\, \lambda_1, \ldots, \lambda_n\,; \varrho, \varrho, \varrho, \varrho_4, \ldots, \varrho_n \,\rangle \\ &\quad= \left\langle \lambda_1; \varrho, \left\langle \frac{\lambda_2}{1-\lambda_1}, \ldots, \frac{\lambda_n}{1-\lambda_1}; \varrho, \varrho, \varrho_3, \ldots, \varrho_n \right\rangle \right\rangle \\ &\quad= \left\langle \lambda_1; \varrho, \left\langle \frac{\lambda_2 + \lambda_3}{1-\lambda_1}, \ldots, \frac{\lambda_n}{1-\lambda_1}; \varrho, \varrho_3, \ldots, \varrho_n \right\rangle \right\rangle \\ &\quad= \langle\, \lambda_1, \lambda_2 + \lambda_3, \lambda_4, \ldots, \lambda_n; \varrho, \varrho, \varrho_4, \ldots, \varrho_n \,\rangle \\ &\quad= \langle\, \lambda_1 + \lambda_2 + \lambda_3, \lambda_4, \ldots, \lambda_n; \varrho, \varrho_4, \ldots, \varrho_n \,\rangle\,. \end{aligned}$$

Suppose finally that b) holds for all number of elements n in the mixture and for $k = m < n$. Now

$$\begin{aligned} &\langle\, \lambda_1, \ldots, \lambda_{m+1}, \lambda_{m+2}, \ldots \lambda_n; \varrho, \ldots, \varrho, \varrho_{m+2}, \ldots, \varrho_n \,\rangle \\ &\quad= \left\langle \lambda_1; \varrho, \left\langle \frac{\lambda_2}{1-\lambda_1}, \ldots, \frac{\lambda_n}{1-\lambda_1}; \varrho, \ldots, \varrho, \varrho_{m+2}, \ldots, \varrho_n \right\rangle \right\rangle \\ &\quad= \left\langle \lambda_1; \varrho, \left\langle \frac{\sum_{i=2}^{m+1} \lambda_i}{1-\lambda_1}, \lambda_{m+2} \ldots, \frac{\lambda_n}{1-\lambda_1}; \varrho, \varrho_{m+2}, \ldots, \varrho_n \right\rangle \right\rangle \end{aligned}$$

$$= \left\langle \lambda_1, \sum_{i=2}^{m+1} \lambda_i, \lambda_{m+2}, \ldots, \lambda_n; \varrho, \varrho, \varrho_{m+2}, \ldots, \varrho_n \right\rangle$$
$$= \left\langle \sum_{i=1}^{m+1} \lambda_i, \lambda_{m+2}, \ldots, \lambda_n; \varrho, \varrho_{m+2}, \ldots, \varrho_n \right\rangle.$$

Thus b) holds for all number of elements and all number of same elements in the mixture. □

Next proposition shows that in fact we only need to consider two-element mixtures.

Proposition 1.3. *Every mixture can be expressed as a mixture of two elements.*

Proof. Let $(\varrho_1, \ldots, \varrho_n) \in \mathcal{K}^n$ and $(\lambda_1, \ldots, \lambda_n) \in \Lambda_n$ where $n \geq 3$. By condition C–3 we may assume that $\lambda_i \neq 0$ for all $i = 1, \ldots, n$. Since $\sum_{i=1}^n \lambda_i = 1$, especially $\lambda_1 < 1$, and we have that

$$\sum_{i=2}^{n} \frac{\lambda_i}{1-\lambda_1} = 1,$$

so that $(\frac{\lambda_2}{1-\lambda_1}, \ldots, \frac{\lambda_n}{1-\lambda_1}) \in \Lambda_{n-1}$. Hence,

$$\varrho' = \left\langle \frac{\lambda_2}{1-\lambda_1}, \ldots, \frac{\lambda_n}{1-\lambda_1}; \varrho_2, \ldots, \varrho_n \right\rangle$$

is a genuine mixture.

By the condition C–4 we have that

$$\langle \lambda_1; \varrho_1, \varrho' \rangle = \left\langle \lambda_1; \varrho_1, \left\langle \frac{\lambda_2}{1-\lambda_1}, \ldots, \frac{\lambda_n}{1-\lambda_1}; \varrho_2, \ldots, \varrho_n \right\rangle \right\rangle$$
$$= \langle \lambda_1, \ldots, \lambda_n; \varrho_1, \ldots, \varrho_n \rangle.$$

□

In the light of previous proposition, we can restate the most important properties for two-element mixtures.

Corollary 1.4. *Let $\mathcal{K}$ be a convex structure. Then*

i) $\langle 1; \varrho, \varrho' \rangle = \varrho$,

ii) $\langle \lambda; \varrho, \varrho \rangle = \varrho$,

iii) $\langle \lambda; \varrho, \varrho' \rangle = \langle 1-\lambda; \varrho', \varrho \rangle$, *and*

iv) $\langle \lambda; \langle \mu; \varrho, \varrho' \rangle, \varrho'' \rangle = \left\langle \lambda\mu; \varrho, \left\langle \frac{\lambda(1-\mu)}{1-\lambda\mu}; \varrho', \varrho'' \right\rangle \right\rangle$

for all $\varrho, \varrho', \varrho'' \in \mathcal{K}$ *and* $\lambda, \mu \in [0,1]$.

Proof. i), ii) and iii) are just conditions C–3, C–1 and C–2 formulated for mixture of two elements. By C–4 we have for iv) that

$$\langle \lambda; \langle \mu; \varrho, \varrho' \rangle, \varrho'' \rangle = \langle \lambda\mu, \lambda(1-\mu), 1-\lambda; \varrho, \varrho', \varrho'' \rangle = \left\langle \lambda\mu; \varrho, \left\langle \frac{\lambda(1-\mu)}{1-\lambda\mu}; \varrho', \varrho'' \right\rangle \right\rangle .$$

□

Next we see that the traditional notion of convexity is included in our framework of generalized convexity as expected.

1.2 Convex subsets of vector spaces

As a special case of convex structures we can consider convex subsets of vector spaces. For the following basic notions of vector spaces see for example [22, 23].

Definition 1.5. Set $\mathcal{V}$ combined with two operations, vector addition $+$ and scalar multiplication $\cdot$, is a *vector space* over a field $\mathbb{F}$ if the following axioms hold for all vectors $x, y, z \in \mathcal{V}$ and scalars $\alpha, \beta \in \mathbb{F}$:

V-1. $$(x+y)+z = x+(y+z),$$

V-2. $$x+y = y+x,$$

V-3. there exist a zero vector $0 \in \mathcal{V}$ such that
$$0 + x = x$$
for all $x \in \mathcal{V}$,

V-4. for every $x \in \mathcal{V}$ there exists an inverse element $-x \in \mathcal{V}$ such that
$$x + (-x) = 0,$$

V-5. $$\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x,$$

V-6. $$1 \cdot x = x,$$
where 1 is the multiplicative identity in $\mathbb{F}$,

V-7. $$\alpha \cdot (x+y) = \alpha \cdot x + \alpha \cdot y, \text{ and}$$

V-8. $$(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x.$$

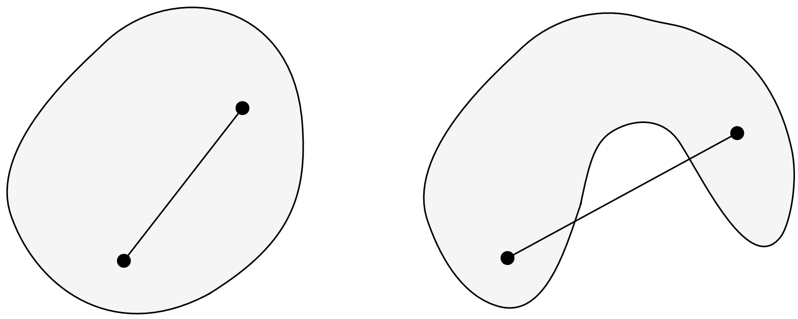

Figure 1: A convex set (left) and a non-convex set (right) in $\mathbb{R}^2$.

Axioms V-1 (associativity), V-3 (neutral element) and V-4 (inverse element) make $\mathcal{V}$ an additive group, and V-2 (commutativity) completes it into an Abelian group. Axioms V-5 to V-8 concern the compatibility of scalar multiplication and vector addition, and the last two are usually referred to as the distributive laws.

A vector space is *finite-dimensional* if there is a finite set of vectors that span the vector space. Basic results of vector spaces is the existence of a *basis*, i.e. a linearly independent set of elements that span the vector space. The *dimension* of a vector space is the number of elements in a basis. If the scalar field $\mathbb{F}$ in Def. 1.5 is taken to be the set of real numbers $\mathbb{R}$, then the vector space is called a *real vector space*.

In our setting we make the following limitation:

All vector spaces are real and finite-dimensional from here onwards.

In vector spaces we can now formulate the traditional notion of convexity.

Definition 1.6. A subset $\mathcal{K}$ of a vector space $\mathcal{V}$ is *convex* if $\lambda x + (1-\lambda)y \in \mathcal{K}$ for all $x, y \in \mathcal{K}$ and $\lambda \in [0,1]$.

Geometrically convexity means that the line segment between any two point of a convex set are included in the set (Fig. 1). We may also consider sums of more than two elements. A linear combination in a vector space $\mathcal{V}$ of the form $\sum_{i=1}^n \lambda_i x_i$, where $x_1, \ldots, x_n \in \mathcal{V}$ and $(\lambda_1, \ldots, \lambda_n) \in \Lambda_n$ is called a convex combination or a convex sum. The next proposition shows that a convex set is closed with respect to finite convex sums [24, Thm. 2.1.4].

Proposition 1.7. *Let $\mathcal{K}$ be a convex subset of a real vector space $\mathcal{V}$. Then for all $n \in \mathbb{N}$,*

$$\sum_{i=1}^n \lambda_i x_i \in \mathcal{K}$$

for all $x_i \in \mathcal{K}$ and $\lambda_i \in [0,1]$ such that $\sum_{i=1}^n \lambda_i = 1$.

Proof. The proof can be adapted from the arguments from proposition 1.3 by induction.

Consider convex sums of n elements. For $n = 1$ the argument is trivial. Suppose then that $\mathcal{K}$ is closed with respect to convex sums of k elements for some $k \in \mathbb{N}$, i.e. $\sum_{i=1}^{k} \lambda_i x_i \in \mathcal{K}$ for all $x_i \in \mathcal{K}$ and $\lambda_i \in [0,1]$ such that $\sum_{i=1}^{k} \lambda_i = 1$ for some $k \in \mathbb{N}$.

Now consider a convex sum of $k+1$ elements, $\sum_{i=1}^{k+1} \lambda_i x_i$. We may suppose that $\lambda_i \neq 0$ for all $i = 1, \ldots, k+1$. Then especially $\lambda_{k+1} < 1$ so that

$$y = \sum_{i=1}^{k} \frac{\lambda_i}{1 - \lambda_{k+1}} x_i$$

is a convex sum of k elements. By the induction hypothesis we have that $y \in \mathcal{K}$, and hence

$$\sum_{i=1}^{k+1} \lambda_i x_i = \sum_{i=1}^{k} \lambda_i x_i + \lambda_{k+1} x_{k+1} = (1 - \lambda_{k+1}) y + \lambda_{k+1} x_{k+1} \in \mathcal{K}$$

since $\mathcal{K}$ is convex. □

The previous proposition confirms that the mapping

$$(\lambda_1, \ldots, \lambda_n, x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} \lambda_i x_i$$

on a convex subset $\mathcal{K}$ of a real vector space $\mathcal{V}$ defines a unique element in $\mathcal{K}$ so that we may see if the conditions of Def. 1.1 hold for it. It is easy to check that the convex sum satisfies the conditions, making $\mathcal{K}$ a convex structure.

Let us then consider some basic concepts and properties of convex sets. These basic concepts can be found for example in [24, 26–28].

Given a subset $V \subset \mathcal{V}$ of a vector space $\mathcal{V}$ we can always construct a convex set containing V (see Fig. 2). Namely, we form a set of all convex combinations of vectors in V. This is called the *convex hull of* V and denoted by $\mathrm{conv}(V)$, i.e.

$$\mathrm{conv}(V) = \left\{ \sum_{i=1}^{n} \lambda_i v_i \,|\, \forall n \in \mathbb{N} : (\lambda_1, \ldots, \lambda_n) \in \Lambda_n,\ v_i \in V\ \forall i = 1, \ldots, n \right\}. \quad (1.1)$$

The convex hull of V is the smallest convex set containing V and the convex hull of a convex set is the set itself. Thus, every convex set is a convex hull of a set. We can ask if for a convex set there is a smallest set that would generate the convex set as its convex hull. We will address this question more closely later on, but essentially this brings us to the notion of extremality.

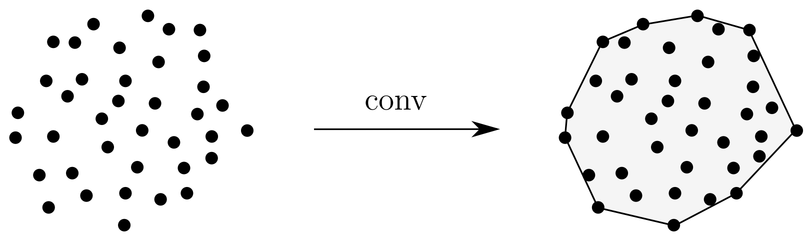

Figure 2: The convex hull of a set of points in $\mathbb{R}^2$.

Let $\mathcal{K}$ be a convex subset of a vector space $\mathcal{V}$. We see that every element $x \in \mathcal{K}$ can be expressed as a trivial mixture $x = \lambda x + (1-\lambda)x$ for every $\lambda \in [0,1]$. We note that in general there might not be any other mixtures to represent an element and we make the following definition.

Definition 1.8. An element $x \in \mathcal{K}$ of a convex set $\mathcal{K}$ is called *extremal* if

$$x = \lambda y + (1-\lambda)z \quad \Rightarrow \quad x = y = z$$

holds for all convex decompositions $x = \lambda y + (1-\lambda)z$ with $y, z \in \mathcal{K}$ and $\lambda \in (0,1)$. The set of extremal points of $\mathcal{K}$ is denoted by $\text{ext}(\mathcal{K})$.

Since extremal points have only a trivial convex decomposition, they can be considered elementary points in the convex set. We note that $\text{conv}(\text{ext}(\mathcal{K})) \subset \mathcal{K}$. Another elementary concept is the face of a convex set.

Definition 1.9. A convex subset $\mathcal{F} \subset \mathcal{K}$ of a convex set $\mathcal{K}$ is called a *face* if for $y, z \in \mathcal{K}$ and $\lambda \in (0,1)$ we have that $\lambda y + (1-\lambda)z \in \mathcal{F}$ implies that $y, z \in \mathcal{F}$.

We see that both the convex set $\mathcal{K}$ itself and the empty set $\emptyset$ are faces of $\mathcal{K}$. They are called *trivial faces*. If $\mathcal{F}$ is a non-trivial face of $\mathcal{K}$, then it is called a *proper face*. We note that a singleton set $\{x\}$ is a face of $\mathcal{K}$ if and only if $x \in \text{ext}(\mathcal{K})$. Also, a face $\mathcal{F}'$ of a face $\mathcal{F}$ of $\mathcal{K}$ is a face of $\mathcal{K}$.

For example, in Fig. 2 we see that the extremal points are exactly the vertices of the set. In addition to the singleton sets of the extremal points, the faces are seen to be the line segments connecting any two adjacent vertices.

The above concepts play important role in convex sets and they will be needed later on. Before introducing a special class of convex sets, we first show the convexity in vector spaces is more than just a particular instance of convex structures.

1.3 Embedding convex structures into vector spaces

Let us consider general convex structures once more. We say that a convex structure $\mathcal{K}$ has a *cancelling property* if the implication

$$\langle \lambda; \varrho', \varrho \rangle = \langle \lambda; \varrho'', \varrho \rangle \quad \Rightarrow \quad \varrho' = \varrho'' \tag{1.2}$$

holds for all $\varrho, \varrho', \varrho'' \in \mathcal{K}$ and $\lambda \in [0,1]$.

We show that the cancelling property guarantees that a convex structure can be identified with a convex subset of a vector space [18, 25].

Proposition 1.10. *Let $\mathcal{K}$ be a convex structure with the cancelling property. Then there exist a vector space $\mathcal{V}$ and an injective map $\varphi : \mathcal{K} \to \mathcal{V}$ such that*

$$\varphi\left(\langle \lambda; \varrho, \varrho' \rangle\right) = \lambda\varphi(\varrho) + (1-\lambda)\varphi(\varrho')$$

for all $\varrho, \varrho' \in \mathcal{K}$ and $\lambda \in [0,1]$.

Proof. We start by constructing the vector space $\mathcal{V}$. For that, consider first the vector space

$$\mathcal{V}_{\mathcal{K}} = \{f : \mathcal{K} \to \mathbb{R} \,|\, f(x) \neq 0 \text{ only for finitely many } x \in \mathcal{K}\}.$$

$\mathcal{V}_{\mathcal{K}}$ is the vector space generated by $\mathcal{K}$ and it has a (canonical) basis $\{\delta_x\}_{x\in\mathcal{K}}$, where

$$\delta_x(y) = \begin{cases} 1, & \text{if } y = x, \\ 0, & \text{otherwise.} \end{cases}$$

We note that the mapping $\psi : x \mapsto \delta_x$ is a bijection between $\mathcal{K}$ and the canonical basis.

We can construct a subspace $\mathcal{U}$ of $\mathcal{V}_{\mathcal{K}}$ by considering the span of the vectors of the form

$$\delta_{\langle \lambda; x, y \rangle} - \lambda\delta_x - (1-\lambda)\delta_y.$$

We then set $\mathcal{V}$ to be the quotient space $\mathcal{V}_{\mathcal{K}}/\mathcal{U}$ and denote by ϕ the linear quotient mapping that takes an element $x \in \mathcal{V}_{\mathcal{K}}$ to some equivalence class in $\mathcal{V}$. We show that the mapping $\varphi = \phi \circ \psi : \mathcal{K} \to \mathcal{V}$ satisfies the required properties.

Clearly φ is well-defined. Let us show the convexity-preserving property of φ. Take $\langle \lambda; x, y \rangle \in \mathcal{K}$ for some $x, y \in \mathcal{K}$ and $\lambda \in [0,1]$. In the trivial cases when $x = y$, $\lambda = 0$ or $\lambda = 1$ the statement is trivial. Thus, we suppose $x \neq y$ and $\lambda \in (0,1)$. Since $\delta_{\langle \lambda; x, y \rangle} - \lambda\delta_x - (1-\lambda)\delta_y \in \mathcal{U}$, we have that $\delta_{\langle \lambda; x, y \rangle}$ is mapped by ϕ to the equivalence class of $\lambda\delta_x + (1-\lambda)\delta_y$ so that since the quotient map is linear, we have that $\varphi(\langle \lambda; x, y \rangle) = \lambda\varphi(x) + (1-\lambda)\varphi(y)$. Thus, φ preserves convexity.

Next we show that φ is injective. For that, take $x, y \in \mathcal{K}$ such that $\varphi(x) = \varphi(y)$. Thus, we have that $\varphi(x) - \varphi(y) = 0$ which means that $\psi(x) - \psi(y) = \delta_x - \delta_y \in \mathcal{U}$. This means that $\delta_x - \delta_y$ is some linear combination

$$\delta_x - \delta_y = \sum_{i=1}^{m} \gamma_i \left[\delta_{z_i} - \lambda_i \delta_{x_i} - (1-\lambda_i)\delta_{y_i}\right] \tag{1.3}$$

for some real numbers $\{\gamma_i\}_{1=1}^{m} \subset \mathbb{R}$, where we have denoted $z_i = \langle \lambda_i; x_i, y_i \rangle$. If we denote $\alpha_i = \max\{\gamma_i, 0\} \geq 0$ and $\beta_i = \max\{-\gamma_i, 0\} \geq 0$, so that $\alpha_i + \beta_i = |\gamma_i|$ and $\alpha_i - \beta_i = \gamma_i$, we can express (1.3) as

$$\begin{aligned} \delta_x + \sum_{i=1}^{m}[\alpha_i\delta_{z_i} + \beta_i\lambda_i\delta_{x_i} + \beta_i(1-\lambda_i)\delta_{y_i}] \\ = \delta_y + \sum_{i=1}^{m}\left[\beta_i\delta_{z_i} + \alpha_i\lambda_i\delta_{x_i} + \alpha_i(1-\lambda_i)\delta_{y_i}\right], \end{aligned} \tag{1.4}$$

where now all the coefficients are positive. We see that both sides of the above equality define the same function F that is non-zero only on finite number of distinct points $w_1, \ldots, w_n$, $n \geq 1$ such that $F(w_i) > 0$ for all $i = 1, \ldots, n$.

Let us first use the expression of the left side of (1.4) for F. First of all, we must have $x, z_i, x_i, y_i \in \{w_1, \ldots, w_m\}$ for all $i = 1, \ldots, m$ since all of these points give a non-zero value for F. Since $w_i \neq w_j$ for all $i \neq j$, we have that

$$\sum_j \delta_x(w_j) = \sum_j \delta_{z_i}(w_j) = \sum_j \delta_{x_i}(w_j) = \sum_j \delta_{y_i}(w_j) = 1 \tag{1.5}$$

for all $i = 1, \ldots, m$. Thus, we have that

$$\sum_j F(w_j) = 1 + \sum_i [\alpha_i + \beta_i\lambda_i + \beta_i(1-\lambda_i)] \tag{1.6}$$

$$= 1 + \sum_i (\alpha_i + \beta_i) = 1 + \sum_i |\gamma_i| = 1 + \gamma, \tag{1.7}$$

where we have denoted $\gamma \equiv \sum_i |\gamma_i|$.

If $\gamma = 0$, then $\gamma_i = 0$ for all $i = 1, \ldots, m$ so that $\delta_x = \delta_y$ from which it follows that $x = y$ which was the claim. It remains to consider the case when $\gamma > 0$.

We denote by $a' = \frac{a}{1+\gamma}$ and $a^{(j)} = \frac{a}{F(w_j)}$ for any $a \in \mathbb{R}$. Consider the mixture

$$w = \langle F(w_1)', \ldots, F(w_n)'; w_1, \ldots, w_n \rangle .$$

For that we have that a mixture of $x, z_1, \ldots, z_m, x_1, \ldots, x_m, y_1, \ldots, y_m$ with weights $\delta_x(w_j)^{(j)}$, $\alpha_1\delta_{z_1}(w_j)^{(j)}, \ldots, \alpha_m\delta_{z_m}(w_j)^{(j)}$, $\beta_1\lambda_1\delta_{x_1}(w_j)^{(j)}, \ldots, \beta_m\lambda_m\delta_{x_m}(w_j)^{(j)}$, $\beta_1(1-\lambda_1)\delta_{y_1}(w_j)^{(j)}, \ldots, \beta_m(1-\lambda_m)\delta_{y_m}(w_j)^{(j)}$ must equal w_j since the weights are non-zero only if the respective points equal w_j. By using C–3 in Def. 1.1 we can remove the additional 0-weights from the mixture after which all the remaining point must equal w_j so that by C–1 the mixture must result w_j.

If we now plug the previous mixtures of each w_j to the mixture w, we get that w is a mixture of the points $x, z_1, \ldots, z_m, x_1, \ldots, x_m, y_1, \ldots, y_m$ with multiple instances of all of them. By rearranging the terms (C–2), by combining the weights such that there are only single instances of the points $x, z_1, \ldots, z_m, x_1, \ldots, x_m, y_1, \ldots, y_m$ in the mixture (Prop. 1.2 *b*)) and by calculating the respective weights using the identities in (1.5), we finally get that

$$\begin{aligned}
w &= \langle 1', \alpha'_1, \ldots, \alpha'_m, \beta'_1\lambda_1, \ldots, \beta'_m\lambda_m, \beta'_1(1-\lambda_1), \ldots, \beta'_m(1-\lambda_m) \\
&\qquad\qquad ; x, z_1, \ldots, z_m, x_1, \ldots, x_m, y_1, \ldots, y_m \rangle \\
&= \langle 1', \alpha'_1, \ldots, \alpha'_m, \beta'_1, \ldots, \beta'_m; x, z_1, \ldots, z_m, z_1, \ldots, z_m \rangle \\
&= \langle 1', (\alpha_1+\beta_1)', \ldots, (\alpha_m+\beta_m)'; x, z_1, \ldots, z_m \rangle \\
&= \left\langle \frac{1}{1+\gamma}, \frac{\alpha_1+\beta_1}{1+\gamma}, \ldots, \frac{\alpha_m+\beta_m}{1+\gamma}; x, z_1, \ldots, z_m \right\rangle \\
&= \left\langle \frac{1}{1+\gamma}, \frac{\gamma}{1+\gamma}; x, \left\langle \frac{\alpha_1+\beta_1}{\gamma}, \ldots, \frac{\alpha_m+\beta_m}{\gamma}; z_1, \ldots, z_m \right\rangle \right\rangle \\
&= \langle \kappa; x, v \rangle,
\end{aligned}$$

where we used the properties C–1, C–2, C–4 and Prop. 1.2 *b*) and denoted $\kappa = 1' = 1/(1+\gamma)$ and $v = \left\langle \frac{\alpha_1+\beta_1}{\gamma}, \ldots, \frac{\alpha_m+\beta_m}{\gamma}; z_1, \ldots, z_m \right\rangle$.

By repeating the same process by using the expression of the right side of (1.3) for F, we find that $w = \langle \kappa; y, v \rangle$. Thus, $\langle \kappa; x, v \rangle = \langle \kappa; y, v \rangle$ so that it follows from the cancellation property that $x = y$. Hence, φ is injective. We see that the image $\varphi(\mathcal{K})$ is a convex set in $\mathcal{V}$. □

Thus, with the additional cancelling property, the convex structures can be embedded in vector spaces as convex sets. When constructing the convex operational theories we will see that in the case where we take our states in a physical experiment to be elements of a convex structure, the cancelling property is satisfied. Hence, from here on we will focus only on convex subsets of vector spaces.

As an example we consider particular types of convex sets.

1.4 Polytopes

Definition 1.11. A convex subset $\mathcal{P}$ of a vector space $\mathcal{V}$ is called a *polytope* if it is the convex hull of finitely many points.

We see that a polytope always has a finite set of extremal points [28, Section 3.7]. Namely, if $\mathcal{P}$ is a polytope, then $\mathcal{P} = \mathrm{conv}(P)$ for some finite set of points P. Let $x \in \mathrm{ext}(\mathcal{P})$. In particular $x \in \mathrm{conv}(P)$, so that we have by extremality that every convex decomposition of x is trivial so that $x \in P$. Since P is a finite set, $|\mathrm{ext}(\mathcal{P})| < \infty$.

We saw that $\mathrm{ext}(\mathcal{P}) \subset P$. In fact, we can prove the converse if the set P is *convexly independent*, i.e. none of the points in P can be represented as a convex combination of the other points [28].

Proposition 1.12. *If $\mathcal{P}$ is a convex hull of finite set P of convexly independent points, then* $\mathrm{ext}(\mathcal{P}) = P$.

Proof. Above we saw that $\mathrm{ext}(\mathcal{P}) \subset P$. For the converse, we take a point $p_j \in P \equiv \{p_1, \ldots, p_n\}$. Suppose that $p_j = \lambda y + (1-\lambda) z$ for some $y, z \in \mathcal{P}$ and $\lambda \in (0,1)$. Since $\mathcal{P} = \mathrm{conv}(P)$, we have that $y = \sum_{i=1}^{n} \alpha_i p_i$ and $z = \sum_{i=1}^{n} \beta_i p_i$ with some weights $\{\alpha_i\}_i$ and $\{\beta_i\}_i$ so that

$$p_j = [\lambda \alpha_j + (1-\lambda)\beta_j] p_j + \sum_{\substack{i=1 \\ i \neq j}}^{n} [\lambda \alpha_i + (1-\lambda)\beta_i] p_i.$$

We cannot have that $\alpha_j = \beta_j = 0$ since then we would have that p_j is be a non-trivial convex combination of other points in P which would contradict the convex independence of P. The same contradiction is faced for all $\alpha_j \neq 1 \neq \beta_j$. Thus, $\alpha_j = \beta_j = 1$ from which it follows that $y = z = p_j$. Hence, $P \subset \mathrm{ext}(\mathcal{P})$. □

Next consider any polytope $\mathcal{P} = \mathrm{conv}(P)$, where $P = \{p_1, \ldots, p_m\}$. If the set P is convexly independent, we have that $P = \mathrm{ext}(\mathcal{P})$. Otherwise, without loss of generality, we have that p_m can be expressed as a convex combination of other points in P. Thus, $\mathrm{conv}(\{p_1, \ldots, p_m\}) = \mathrm{conv}(\{p_1, \ldots, p_{m-1}\})$. Since P is a finite set, we can similarly continue to remove all convexly dependent points from P so that we are left with a convexly independent set $P' = \{p_1, \ldots, p_n\}$, $n \leq m$, such that $\mathrm{conv}(P') = \mathrm{conv}(P) = \mathcal{P}$. It follows from the Prop. 1.12 that $P' = \mathrm{ext}(\mathcal{P})$. Hence, we have proved the following [28].

Proposition 1.13. *Every polytope is the convex hull of its finite number of extremal points.*

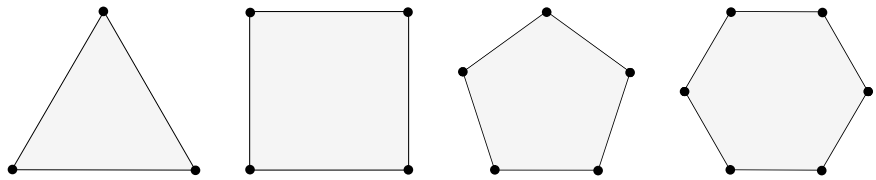

Figure 3: Some of the first regular polygons in $\mathbb{R}^2$.

The above proposition is just a special case of a much more stronger result on convex sets. The result is known as the Krein-Milman theorem and we will only state it here.

Theorem 1.14 (Krein-Milman Theorem)**.** *If $\mathcal{K}$ is a compact convex subset of a finite-dimensional vector space $\mathcal{V}$, then $\mathcal{K}$ is the convex hull of its extremal points.*

The theorem relies on topological properties of vector spaces which can be found in the next section. The theorem can be proved by studying further properties of faces and extremal points (see for example [24, Theorem 2.6.16]).

Examples of polytopes include all regular polygons in $\mathbb{R}^2$ (Fig. 3), i.e., convex sets $P_n = \text{conv}(\{\vec{p}_1, \ldots, \vec{p}_n\})$ such that $\|\vec{p}_1\| = \|\vec{p}_2\| = \ldots = \|\vec{p}_n\|$ and $\vec{p}_i \cdot \vec{p}_{i+1} = \|\vec{p}_i\|^2 \cos\left(\frac{2\pi}{n}\right)$ for all $i = 1, \ldots, n$ where the addition is modulo n. The extremal points of a polygon are its vertices and faces are exactly the sides of the polygon. We will consider polygons and other polytopes later in Chapter II when we consider them as state spaces. In particluar, we will see that a state spaces corresponding to a classical systems form a special class of polytopes, namely simplices [28].

Definition 1.15. A polytope $\mathcal{P}$ with extremal points $\text{ext}(\mathcal{P}) = \{p_0, p_1, \ldots, p_n\}$ is called a *simplex* if $\{p_1 - p_0, \ldots, p_n - p_0\}$ is linearly independent set.

Equivalent way to formulate simplices is to require that the extremal points are *affinely independent* [28, Thm. 3.5.4], i.e. none of them can be expressed as an affine combination[1] of other extremal points. The most recognized feature of simplices is the following characterization [29, Prop. 2.34].

Proposition 1.16. *A polytope $\mathcal{P}$ is a simplex if and only if every element in $\mathcal{P}$ has a unique convex decomposition into extremal elements.*

[1] An affine combination of points is any linear combination such that the coefficients sum to 1. We can define the *affine hull* of $\mathcal{A}$, denoted by $\text{aff}(\mathcal{A})$, as the set of all affine combinations of elements of $\mathcal{A}$.

Proof. Let $\mathcal{S}$ be a simplex and $\text{ext}(\mathcal{S}) = \{s_0, s_1, \dots, s_n\}$ so that the set $\{s_1 - s_0, \dots, s_n - s_0\}$ is linearly independent. Let $x \in \mathcal{P}$ and take two convex decompositions for x so that

$$x = \sum_{i=0}^{n} \alpha_i s_i = \sum_{i=0}^{n} \beta_i s_i.$$

By subtracting $s_0 = \sum_i \alpha_i s_0 = \sum_i \beta_i s_0$ from both convex decompositions and rearranging we have that

$$\sum_{i=1}^{n} (\alpha_i - \beta_i)(s_i - s_0) = 0.$$

Since this is a linear combination of linearly independent vectors resulting the zero vector, we must have that $\alpha_i = \beta_i$ for all $i = 1, \dots, n$. It follows also that $\alpha_0 = 1 - \sum_{i=1}^{n} \alpha_i = 1 - \sum_{i=1}^{n} \beta_i = \beta_0$. Thus, the convex decomposition of x is unique.

Assume then that every element of a polytope $\mathcal{P}$ has a unique convex decomposition into extremal points $\text{ext}(\mathcal{P}) = \{p_0, p_1, \dots, p_m\}$. Suppose the set $P = \{p_i - p_0\}_{i=1}^{m}$ is linearly dependent. Thus, there exists a linear combination of elements of P such that

$$\sum_{i=1}^{m} \gamma_i (p_i - p_0) = 0 \tag{1.8}$$

for some real numbers $\{\gamma_i\}_i$ such that $\gamma_k \neq 0$ at least for some $k \in \{1, \dots, m\}$.

Let us denote

$$\gamma = \sum_{i=1}^{m} \gamma_i, \quad \gamma_{>0} = \sum_{i:\gamma_i > 0} \gamma_i, \quad \gamma_{<0} = \gamma - \gamma_{>0}.$$

We note that $\gamma_{>0} \neq 0$ and/or $\gamma_{<0} \neq 0$, since $\gamma_k \neq 0$.

If now $\gamma = 0$, we have that $\gamma_{>0} = -\gamma_{<0}$ and that $\sum_i \gamma_i p_i = 0$. Thus, we get two equal convex combinations

$$\sum_{i:\gamma_i>0} \frac{\gamma_i}{\gamma_{>0}} p_i = \sum_{j:\gamma_j<0} \frac{\gamma_j}{\gamma_{<0}} p_j.$$

This contradicts the assumption that every element has a unique convex decomposition into extremal elements.

Suppose then that $\gamma \neq 0$. Without loss of generality we then have $\gamma_{>0} > 0$. From (1.8) it follows that $p_0 = \sum_i \frac{\gamma_i}{\gamma} p_i \equiv \sum_i \tilde{\gamma}_i p_i$. Again we get two equal convex combinations

$$\frac{1}{\tilde{\gamma}_{>0}} p_0 + \sum_{j:\tilde{\gamma}_j<0} \frac{-\tilde{\gamma}_j}{\tilde{\gamma}_{>0}} p_j = \sum_{i:\tilde{\gamma}_i>0} \frac{\tilde{\gamma}_i}{\tilde{\gamma}_{>0}} p_i.$$

Again this contradicts the assumption that every element has a unique convex decomposition into extremal elements. Hence, the set P is linearly independent and $\mathcal{P}$ is a simplex. □

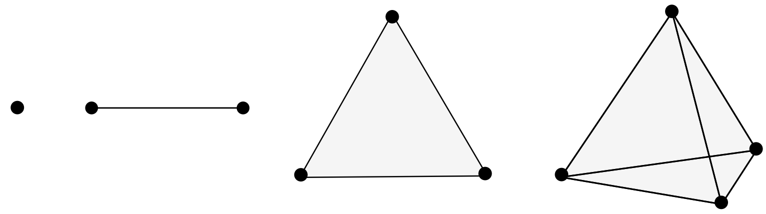

Figure 4: The four simplest simplices: a point, line segment, triangle and tetrahedron.

We note that a simplex with $n+1$ extremal points spans an n-dimensional vector space so that we may call it an *n-simplex.* Thus, for every simplex there exists a canonical vector space. For example we can consider the 0-simplex as a point, 1-simplex as a line segment in $\mathbb{R}$, 2-simplex as a triangle in $\mathbb{R}^2$ and 3-simplex as a tetrahedron in $\mathbb{R}^3$ (Fig. 4).

As was already hinted at the beginning of this section, convexity will play a big role once we start to construct our class of operational theories. However, there is another structure (that is closely related to convexity as we will see) that will have a big part in constructing the theory. This structure is partial orders and ordered vector spaces.

2 Ordered vector spaces

Ordered vector spaces are vector spaces with additional structure induced by a partial order. The basic concepts and properties presented here can be found for example in [30–32]. We begin by a definition of partial order.

Definition 2.1. A *partial order* $\leq$ on a set $\mathcal{P}$ is a binary relation over $\mathcal{P}$ satisfying

PO-1. $x \leq x$,

PO-2. if $x \leq y$ and $y \leq z$, then $x \leq z$, and

PO-3. if $x \leq y$ and $y \leq x$, then $x = y$.

for all $x, y, x \in \mathcal{P}$.

Conditions PO-1, PO-2 and PO-3 are called reflexivity, transitivity and antisymmetry respectively. Set $\mathcal{P}$ equipped with a partial order is called a *partially ordered set* or *poset*. If elements of a poset $\mathcal{P}$ are always comparable, i.e.

PO-4. always either $x \leq y$ or $y \leq x$ for all $x, y \in \mathcal{P}$,

then $\leq$ is said to be a *total order*.

For a partial or total order $\leq$, we also denote $x \geq y$ if $y \leq x$, $x < y$ if $x \leq y$, but $x \neq y$ and similarly $x > y$ if $x \geq y$, but $x \neq y$. We can now consider partial orders in vector spaces.

Definition 2.2. Let $\mathcal{V}$ be a real vector space and $\leq$ a partial order on $\mathcal{V}$. The pair $(\mathcal{V}, \leq)$ is an *ordered vector space* if the following two axioms hold for all $x, y, z \in \mathcal{V}$ and $\lambda \in \mathbb{R}$:

OVP-1. if $x \leq y$, then $x + z \leq y + z$, and

OVP-2. if $x \leq y$ and $\lambda > 0$, then $\lambda x \leq \lambda y$.

Instead of the pair $(\mathcal{V}, \leq)$, we may also call the vector space $\mathcal{V}$ itself an ordered vector space implying there is a partial order $\leq$ such that $(\mathcal{V}, \leq)$ is an ordered vector space. We will see that partial orders in vector spaces are actually characterized by geometrical object called cones.

2.1 Cones

Let $\mathcal{A}$ and $\mathcal{B}$ be subsets of some vector space and $\lambda \in \mathbb{R}$. Denote

$$\mathcal{A} + \mathcal{B} = \{a + b \,|\, a \in \mathcal{A},\ b \in \mathcal{B}\}$$

and

$$\lambda\mathcal{A} = \{\lambda a \,|\, a \in \mathcal{A}\}.$$

Definition 2.3. A subset $\mathcal{C} \subset \mathcal{V}$ of a real vector space $\mathcal{V}$ is a *cone* if $\lambda\mathcal{C} \subset \mathcal{C}$ for all $\lambda \geq 0$. Moreover, a cone $\mathcal{C}$ is *convex* if $\mathcal{C} + \mathcal{C} \subset \mathcal{C}$ and *strict* if $\mathcal{C} \cap -\mathcal{C} = \{0\}$. A strict convex cone is called a *proper cone*.

By the above definition, cones are closed with respect to multiplication by a positive scalar whereas convex cones are additionally closed with respect to vector addition. Cone is therefore proper if and only if it does not contain any non-trivial subspaces of vector space $\mathcal{V}$. We note that convex cones are always convex subsets of the vector space they lie in.

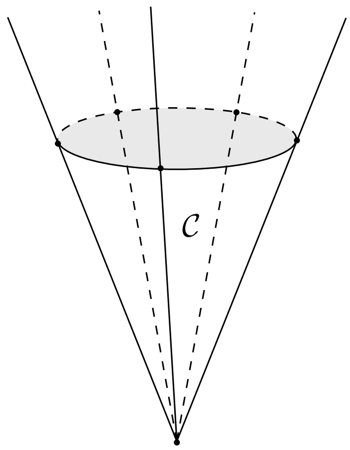

Figure 5: The ice cream cone in $\mathbb{R}^3$. The grey disk serves as a base for the cone.

Examples of cones include all subspaces of vector spaces so that in particular vector spaces are cones. Furthermore, they are convex cones. However, they are not strict cones since any subspace of a vector space multiplied by -1 is the subspace itself. However, we can construct strict cones out of them by considering just positive linear combination of vectors that span them. A prime example of a proper cone is also the ice cream cone in $\mathbb{R}^3$ (Fig. 5) defined for instance as

$$\{(x, y, z) \in \mathbb{R}^3 \,|\, z \geq \sqrt{x^2 + y^2}\}. \tag{2.1}$$

We see that the positive elements of an ordered vector space form a cone [29, Prop. 3.4].

Proposition 2.4. *Let $(\mathcal{V}, \leq)$ be an ordered vector space. Then the set*

$$\mathcal{V}_+ = \{x \in \mathcal{V} \,|\, x \geq 0\}$$

is a proper cone on $\mathcal{V}$.

Proof. By OVP–2, $x \geq 0$ implies $\lambda x \geq 0$ for all $x \in \mathcal{V}_+$, and thus $\lambda \mathcal{V}_+ \subset \mathcal{V}_+$. Similarly by OVP–1 we see that if $y \geq 0$ and $z \geq 0$, then $z + y \geq z \geq 0$, and thus $\mathcal{V}_+ + \mathcal{V}_+ \subset \mathcal{V}_+$.

In order to prove the strictness of $\mathcal{V}_+$ we consider element $x \in \mathcal{V}_+ \cap -\mathcal{V}_+$. This means that both $x \geq 0$ and $-x \geq 0$ hold. By OVP–2 it is clear that $2x \geq 0$. Thus, by OV-P-1 we have that

$$x = 2x + (-x) \geq 0 + (-x) = -x.$$

Similarly

$$-x = 2(-x) + x \geq 0 + x = x,$$

and by PO–3 it follows that $x = -x$ and therefore $x = 0$. $\square$

The cone $\mathcal{V}_+$ on an partially ordered vector space $(\mathcal{V}, \leq)$ is called the *positive cone* of $\mathcal{V}$. Conversely to the previous proposition, we see that any proper cone $\mathcal{C}$ induces a partial order on a vector space $\mathcal{V}$ and that the positive cone $\mathcal{V}_+$ coincides with the cone $\mathcal{C}$ [29, Prop. 3.4].

Proposition 2.5. *Let $\mathcal{C} \subset \mathcal{V}$ be a proper cone on a vector space $\mathcal{V}$. Denote $x \leq y$ if and only if $y - x \in \mathcal{C}$. Then $\leq$ is a partial order on $\mathcal{V}$. Furthermore, $(\mathcal{V}, \leq)$ is an ordered vector space, and $\mathcal{V}_+ = \mathcal{C}$.*

Proof. Clearly $x - x = 0 \in \mathcal{C}$ for all $x \in \mathcal{V}$. Now if $x \leq y$ and $y \leq z$ for all $x, y, z \in \mathcal{V}$, i.e. $y - x \in \mathcal{C}$ and $z - y \in \mathcal{C}$, then $z - x = (z - y) + (y - x) \in \mathcal{C}$ by the convexity of $\mathcal{C}$. Finally, if $x \leq y$ and $y \leq x$ for all $x, y \in \mathcal{V}$, i.e. $x - y \in \mathcal{C}$ and $y - x \in \mathcal{C}$, we have that $y - x = 0$, since $\mathcal{C} \cap -\mathcal{C} = \{0\}$. Thus $\leq$ is a partial order.

Suppose then that $x \leq y$ for all $x, y \in \mathcal{V}$. Firstly, then $y - x = (y + z) - (x + z) \in \mathcal{C}$ and thus $x + z \leq y + z$ for all $z \in \mathcal{V}$. Secondly, also $\lambda(y - x) \in \mathcal{C}$ for all positive $\lambda \in \mathbb{R}$. Hence, $(\mathcal{V}, \leq)$ is an ordered vector space.

Now we see that

$$\mathcal{V}_+ = \{x \in \mathcal{V} \mid x \geq 0\} = \{x \in \mathcal{V} \mid x \in \mathcal{C}\} = \mathcal{C}.$$

□

As the partial order in any ordered vector space is defined by a cone, the order structure can be equivalently studied by studying the properties of cones. Next we formulate some elementary concepts that we will need when dealing with cones [30].

Definition 2.6. A cone $\mathcal{C}$ on a vector space $\mathcal{V}$ is *generating* if $\mathcal{C} - \mathcal{C} = \mathcal{V}$.

Definition 2.7. Let $\mathcal{C}$ be a proper cone in a vector space $\mathcal{V}$. An element $u \in \mathcal{C}$ is said to be an *order unit* (or $\mathcal{C}$-order unit) if for all $x \in \mathcal{V}$ there exists a $\lambda > 0$ such that $x \leq \lambda u$.

Definition 2.8. Let $\mathcal{C}$ be a convex cone in a vector space $\mathcal{V}$. A non-empty convex subset $\mathcal{B} \subset \mathcal{C} \setminus \{0\}$ is a *base* for $\mathcal{C}$ if for every non-zero $x \in \mathcal{C}$ there exists unique $\lambda > 0$ and $b \in \mathcal{B}$ such that $x = \lambda b$.

For example, $1 \in \mathbb{R}_+$ is an order unit in $\mathbb{R}$. Furthermore, the singleton set $\{1\}$ is a base for the cone of positive real numbers $\mathbb{R}_+$. For the ice cream cone (2.1) in $\mathbb{R}^3$ we see that for any fixed $z > 0$, the convex hull of the circle $x^2 + y^2 = z^2$ is a base for the ice cream cone (Fig. 5).

Note that if a convex cone has a base then it is also proper. Indeed, suppose there exists some non-zero element $x \in \mathcal{C} \cap -\mathcal{C}$ where $\mathcal{C}$ is a convex cone with base $\mathcal{B}$. Since $\mathcal{B}$ is a base, there exists $\lambda, \mu > 0$ and $y, z \in \mathcal{B}$ such that $x = \lambda y = -\mu z$. Thus, $\frac{x}{\lambda}, -\frac{x}{\mu} \in \mathcal{B}$. Now

$$\frac{\lambda}{\lambda+\mu}\left(\frac{1}{\lambda}x\right) + \left(1 - \frac{\lambda}{\lambda+\mu}\right)\left(-\frac{1}{\mu}x\right) = \frac{1}{\lambda+\mu}x - \frac{1}{\lambda+\mu}x = 0,$$

but since $\mathcal{B}$ is convex and $0 \notin \mathcal{B}$ this is a contradiction.

For any cone $\mathcal{C}$ we note that even though it might not be generating, the set $\mathcal{C} - \mathcal{C}$ is still the smallest subspace containing $\mathcal{C}$. For order units and generating cones we can prove the following result [30].

Proposition 2.9. *If an ordered vector space $\mathcal{V}$ admits an order unit, then the positive cone $\mathcal{V}_+$ is generating.*

Proof. It is always true that $\mathcal{V}_+ - \mathcal{V}_+ \subset \mathcal{V}$. For the contrary, let u be an order unit in $\mathcal{V}$. For each $x \in \mathcal{V}$ we can find $\lambda' > 0$ such that $x \leq \lambda u$ and $\lambda'' > 0$ such that $-x \leq \lambda' u$. Take $\lambda = \max(\lambda', \lambda'')$ so that $\pm x \leq \lambda u$. Thus we have that $\lambda u \pm x \in \mathcal{V}_+$ from which it follows that

$$x = \frac{1}{2}(\lambda u + x) - \frac{1}{2}(\lambda u - x) \in \mathcal{V}_+ - \mathcal{V}_+.$$

Hence, $\mathcal{V} \subset \mathcal{V}_+ - \mathcal{V}_+$ which proves the claim. □

2.2 Dual space and dual cone

Let $\mathcal{V}$ be a vector space. The set $\mathcal{V}^*$ of all linear functionals on $\mathcal{V}$,

$$\mathcal{V}^* = \{f : \mathcal{V} \to \mathbb{R} \,|\, f \text{ linear}\},$$

is the *(algebraic) dual space* of $\mathcal{V}$ [22].

If we define the usual scalar multiplication and addition of functions on the dual space $\mathcal{V}^*$ of a vector space $\mathcal{V}$,

$$(f+g)(x) = f(x) + g(x), \quad (\lambda f)(x) = \lambda f(x), \quad \forall f, g \in \mathcal{V}^*, \ \lambda \in \mathbb{R},$$

we see that the dual space becomes a vector space itself [22].

For a vector space $\mathcal{V}$ with a basis $\{v_i\}_{i=1}^n$ we can always construct a basis in the dual space $\mathcal{V}^*$. Define linear functionals $v^j \in \mathcal{V}^*$ by

$$v^j(v) = \alpha_j$$

for all $v = \sum_{i=1}^{n} \alpha_i v_i \in \mathcal{V}$ for all $j = 1, \ldots, n$.

Now we see that the set $\{v^j\}_{j=1}^n$ is linearly independent since if $\sum_j \alpha^j v^j = 0$, then

$$0 = \sum_j \alpha^j v^j(v_j) = \alpha^j$$

for all $j = 1, \ldots, n$, and that it spans $\mathcal{V}^*$ since $f = \sum_{i=1}^n f(v_i)v^i$ for all $f \in \mathcal{V}^*$. Thus, the set $\{v^j\}_{j=1}^n$ is a basis for $\mathcal{V}^*$. It is called the *dual basis* of $\{v_i\}_{i=1}^n$. In particular, $\dim(\mathcal{V}) = \dim(\mathcal{V}^*)$ [22].

Apart from dual spaces, we can consider dual spaces of dual spaces. In that case we have the following [22].

Proposition 2.10. *If $\mathcal{V}$ is a finite-dimensional vector space, then $\mathcal{V}^{**} \simeq \mathcal{V}$.*

Proof. Consider the mapping $\Phi : \mathcal{V} \to \mathcal{V}^{**}$ defined by

$$\Phi(x)(f) = f(x)$$

for all $f \in \mathcal{V}^*$. If we take any $x \in \mathcal{V}$ such that $\Phi(x) = 0$, then $f(x) = 0$ for all $f \in \mathcal{V}^*$ implying that $x = 0$. Thus, $\ker(\Phi) = \{0\}$ so that Φ is injective. By the discussion above, we have that $\dim(\mathcal{V}^{**}) = \dim(\mathcal{V}^*) = \dim(\mathcal{V})$ so that by the rank-nullity theorem $\dim(\mathrm{Im}(\Phi)) = \dim(\mathcal{V})$ which then proves the surjectivity of Φ.

Furthermore, Φ preserves the linear structure of $\mathcal{V}$: for all $x, y \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{R}$ we have that $\Phi(\alpha x + \beta y) = \alpha\Phi(x) + \beta\Phi(y)$ which follows from the linearity of functionals in $\mathcal{V}^*$.

□

If $\mathcal{V}$ is now an ordered vector space, it is meaningful to ask how we can define a partial order in the dual space $\mathcal{V}^*$ so that it is connected to the partial order of $\mathcal{V}$. We address this question next.

Let $\mathcal{C}$ be a cone in a vector space $\mathcal{V}$. We say that a functional $f \in \mathcal{V}^*$ is *$\mathcal{C}$-positive* if $f(x) \geq 0$ for all $x \in \mathcal{C}$ and *$\mathcal{C}$-strictly positive* if $f(x) > 0$ for all $x \in \mathcal{C} \setminus \{0\}$. We make the following definition.

Definition 2.11. The set of $\mathcal{C}$-positive functionals,

$$\mathcal{C}^* = \{f \in \mathcal{V}^* \mid f(x) \geq 0 \ \forall x \in \mathcal{C}\},$$

is the *dual cone* of $\mathcal{C}$.

We see that the dual cone $\mathcal{C}^*$ of a cone $\mathcal{C}$ is indeed a cone, since $\lambda f \in \mathcal{C}^*$ for all $f \in \mathcal{C}^*$. In fact, $\mathcal{C}^*$ is always convex even if the original cone $\mathcal{C}$ is not, since $f + g \in \mathcal{C}^*$ for all $f, g \in \mathcal{C}^*$.

If $\mathcal{V}$ is an ordered vector space with a positive cone $\mathcal{V}_+$, we say that a functional is (strictly) positive if it is $\mathcal{V}_+^*$-(strictly) positive. Now the dual cone $\mathcal{V}_+^*$ is the set of positive functionals in $\mathcal{V}^*$, and as it was stated above, $\mathcal{V}_+^*$ is convex. Hence, in order to $\mathcal{V}_+^*$ to induce a partial order in $\mathcal{V}^*$, it must be a proper cone for which we have the following result [31].

Proposition 2.12. *Dual cone $\mathcal{C}^*$ of a cone $\mathcal{C}$ is a proper cone if and only if $\mathcal{C}$ is generating.*

Proof. As the dual cone is always a convex cone, it suffices to only consider the strictness of $\mathcal{C}^*$, i.e. whether $\mathcal{C}^* \cap -\mathcal{C}^* = \{0\}$.

Take $f \in \mathcal{V}^*$. Now we have the following chain of equivalences:

$$\begin{aligned} f(x) = 0 \quad \forall x \in \mathcal{C} - \mathcal{C} \qquad &\Leftrightarrow \qquad f(x) = 0 \quad \forall x \in \mathcal{C} \\ &\Leftrightarrow \qquad f(x) \leq 0 \wedge f(x) \geq 0 \quad \forall x \in \mathcal{C} \\ &\Leftrightarrow \qquad f, -f \in \mathcal{C}^* \\ &\Leftrightarrow \qquad f \in \mathcal{C}^* \cap -\mathcal{C}^*. \end{aligned}$$

Hence, the set of functions in $\mathcal{V}^*$ vanishing on $\mathcal{C} - \mathcal{C}$ coincides with the set $\mathcal{C}^* \cap -\mathcal{C}^*$.

Assume first that $\mathcal{C}$ is generating so that $\mathcal{V} = \mathcal{C} - \mathcal{C}$. Since the zero function is uniquely defined as the only function vanishing on the whole vector space $\mathcal{V}$, we have that $\mathcal{C}^* \cap -\mathcal{C}^* = \{0\}$.

Now assume that $\mathcal{C}^*$ is strict, i.e. $\mathcal{C}^* \cap -\mathcal{C}^* = \{0\}$. Thus, the set of functions vanishing on $\mathcal{C} - \mathcal{C}$ consists only of the zero function. Suppose by contradiction that $\mathcal{V} \neq \mathcal{C} - \mathcal{C}$. Since $\mathcal{C} - \mathcal{C}$ is always a linear subspace of $\mathcal{V}$, there always exists another subspace $\mathcal{S} \subset \mathcal{V}$ such that $\mathcal{V}$ is the direct sum of $\mathcal{C} - \mathcal{C}$ and $\mathcal{S}$, i.e. $\mathcal{V} = (\mathcal{C} - \mathcal{C}) \oplus \mathcal{S}$. If $g \in \mathcal{S}^*$ is any non-zero linear functional on $\mathcal{S}$, we can define a function $f_g : \mathcal{V} \to \mathbb{R}$ by

$$f_g(x) = f_g(y + z) = g(z)$$

for all $x \in \mathcal{V}$, where $y + z$ is the unique direct sum representation of $x \in \mathcal{V}$. Clearly f_g is well-defined and it is straightforward to check that f_g is linear. Hence, $f \in \mathcal{V}^*$ and

$$f_g(y) = f_g(y + 0) = g(0) = 0$$

for all $y \in \mathcal{C} - \mathcal{C}$ which contradicts the fact that the zero function is the only function in $\mathcal{V}$ with that property. Hence, $\mathcal{V} = \mathcal{C} - \mathcal{C}$ so that $\mathcal{C}$ is generating. $\square$

Hence, if the positive cone $\mathcal{V}_+$ of an ordered vector space $\mathcal{V}$ is generating, then the dual cone $\mathcal{V}_+^*$ is a proper cone and thus induces a partial order in $\mathcal{V}^*$ called the *dual order of* $\mathcal{V}$.

For dual cones we can also consider the basic properties of cones. For example in the case a dual cone is a proper cone, we have the following for the order units in the dual cone [30, Thm. 3.5].

Proposition 2.13. *Let $\mathcal{V}$ be an ordered vector space with a generating positive cone $\mathcal{V}_+$. If $u \in \mathcal{V}_+^*$ is an order unit in $\mathcal{V}^*$, then u is strictly positive.*

Proof. Since $\mathcal{V}_+$ is generating, it induces a dual order in $\mathcal{V}^*$. Let u be an order unit in $\mathcal{V}^*$. Since $u \in \mathcal{V}_+^*$, it suffices to show that $u(x) \neq 0$ for all $x > 0$. Take $x > 0$ and by a way of contradiction assume that $u(x) = 0$.

Since u is an order unit, for each $f \in \mathcal{V}^*$ there exists $\lambda > 0$ such that $f(x) \leq \lambda g(x)$. Similarly for $-f \in \mathcal{V}^*$, there exists $\lambda' > 0$ such that $-f(x) \leq \lambda g(x)$. Since $u(x) = 0$, we have that $\pm f(x) \leq 0$ so that $f(x) = 0$ for all $f \in \mathcal{V}^*$. Especially, if now $\{e_i\}_i$ is any basis for $\mathcal{V}$ then for its dual basis $\{e^i\}_i$ we have that $e^i(x) = 0$ for all i so that $x = 0$. This contradicts the fact that $x > 0$. Hence, g is strictly positive. □

For bases we see that the existence of a base in a cone $\mathcal{C}$ is equivalent to having a $\mathcal{C}$-strictly positive functional [33], [30, Thm. 1.47].

Proposition 2.14. *A convex cone $\mathcal{C}$ in a vector space $\mathcal{V}$ has a base if and only if there exists a $\mathcal{C}$-strictly positive functional on $\mathcal{V}$.*

Proof. First let $\mathcal{C}$ be a convex cone in a vector space $\mathcal{V}$ with a base $\mathcal{B}$. Define a function $f : \mathcal{C} \to \mathbb{R}_+$ by $f(0) = 0$ if $0 \in \mathcal{C}$ and for $x \in \mathcal{C} \setminus \{0\}$, $f(x) = \lambda$, where $\lambda > 0$ is the unique real number such that $x = \lambda b$ for some unique $b \in \mathcal{B}$. It is clear that f is $\mathcal{C}$-strictly positive.

In order to show that f is additive take $x, y \in \mathcal{C}$. Since $\mathcal{C}$ in a convex cone, also $x + y \in \mathcal{C}$, so that there exists a unique $b \in \mathcal{B}$ such that $x + y = f(x + y)b$. Also there exists $b_1, b_2 \in \mathcal{B}$ such that $x = f(x)b_1$ and $y = f(y)b_2$. Define

$$b_3 = \frac{f(x)}{f(x) + f(y)} b_1 + \frac{f(y)}{f(x) + f(y)} b_2 \in \mathcal{B},$$

so that

$$x + y = f(x)b_1 + f(y)b_2 = (f(x) + f(y))b_3.$$

Since the above representation is unique, we have that $b = b_3$ and $f(x + y) = f(x) + f(y)$.

Also for each $\mu \geq 0$ there exists a unique $b' \in \mathcal{B}$ such that $\lambda x = f(\lambda x)b'$. Thus

$$f(\lambda x)b' = \lambda x = \lambda(f(x)b_1) = (\lambda f(x))b_1,$$

and the uniqueness of the base representation implies that $b = b'$ and $f(\mu x) = \mu f(x)$.

Let $\mathcal{M}$ be the subspace generated by $\mathcal{C}$, i.e. $\mathcal{M} = \mathcal{C} - \mathcal{C}$. Extend f to $\mathcal{M}$ by $F : \mathcal{M} \to \mathbb{R}$,

$$F(x) = f(x_1) - f(x_2),$$

where $x_1, x_2 \in \mathcal{C}$ are such that $x = x_1 - x_2$. It is now easy to see that F is $\mathcal{C}$-strictly positive and additive.

Take then $\mu \in \mathbb{R}$. If $\mu \geq 0$, it is clear that $\mu x_1, \mu x_2 \in \mathcal{C}$ for all $x = x_1 - x_2 \in \mathcal{M}$, so that by the properties of f we have that

$$F(\mu x) = f(\mu x_1) - f(\mu x_2) = \mu(f(x_1) - f(x_2)) = \mu F(x).$$

On the other hand, if $\mu < 0$, then $-\mu x_1, -\mu x_2 \in \mathcal{C}$ and

$$F(\mu x) = f(-\mu x_2) - f(-\mu x_1) = \mu(f(x_1) - f(x_2)) = \mu F(x).$$

Hence, F is linear on $\mathcal{M}$.

Let then $\mathcal{N}$ be any algebraic complement to $\mathcal{M}$, i.e. any subspace of $\mathcal{V}$ such that $\mathcal{V} = \mathcal{M} \oplus \mathcal{N}$. Define $g : \mathcal{N} \to \mathbb{R}$ by $g(y) = 0$ for all $y \in \mathcal{N}$. Then the mapping $G : \mathcal{V} \to \mathbb{R}$,

$$G(z) = F(x) + g(y),$$

where $z = x + y$ is the unique direct sum representation of $z \in \mathcal{V}$ such that $x \in \mathcal{M}$ and $y \in \mathcal{N}$, is linear and $\mathcal{C}$-strictly positive functional.

Then let $f : \mathcal{V} \to \mathbb{R}$ be a $\mathcal{C}$-strictly positive functional. Fix $\alpha > 0$ and denote $\mathcal{B} = \{x \in \mathcal{C} \,|\, f(x) = \alpha\}$. Now each $x \in \mathcal{C}$ has a representation $x = \beta b$, where $\beta = f(x)/\alpha > 0$ and $b = \alpha/f(x)x \in \mathcal{B}$. Suppose then that x has two such representations, i.e. $x = \beta b = \beta' b'$ for some $\beta, \beta' > 0$ and $b, b' \in \mathcal{B}$. Now

$$\beta = \frac{\beta}{\alpha} f(b) = \frac{f(\beta b)}{\alpha} = \frac{f(\beta' b')}{\alpha} = \frac{\beta'}{\alpha} f(b') = \beta'$$

from which it follows also that $b = b'$. Hence, the base representation is unique and $\mathcal{B}$ is a base. □

In particular we see from the proof of the previous proposition that a base $\mathcal{B}$ of a cone $\mathcal{C}$ has to satisfy

$$\mathcal{B} = \{x \in \mathcal{C} \,|\, f(x) = 1\}$$

for some $\mathcal{C}$-strictly positive functional $f \in \mathcal{C}^*$.

2.3 Topological properties

We start by recalling some topological concepts. The general topological concepts presented here can be found in [34]. The parts of topological vector spaces are composed of [33, 35, 36] and the topological properties of cones can be found in [30].

Let $\mathcal{X}$ be a set and τ a collection of subsets of $\mathcal{X}$. We recall that τ is called a *topology* on $\mathcal{X}$ and the pair $(\mathcal{X}, \tau)$ a *topological space* if both the empty set $\emptyset$ and the whole set $\mathcal{X}$ are elements of τ and if any union (finite or infinite) and any finite intersection of elements of τ is an element of τ. Sets $G \in \tau$ are called *open* and their complements $\mathcal{X} \setminus G$ are called *closed*.

A point $x \in E \subset \mathcal{X}$ is called an *interior point* of E, and E the *neighborhood* of x, if there exists an open set $A \subset E$ such that $x \in A$. The set of interior points of $\mathcal{X}$ is denoted by $\text{int}(\mathcal{X})$ and is called the *interior* of $\mathcal{X}$. A subset $A \subset \mathcal{X}$ is open if and only if every point of A is its interior point. A set $B \subset \mathcal{X}$ is called *compact* if every open cover of B has a finite subcover.

As a particular type of topological spaces, we say that a topological space $(\mathcal{X}, \tau)$ is a *Hausdorff space* if for every distinct points $x, y \in \mathcal{X}$, $x \neq y$ there exists open sets E and F such that $x \in E$, $y \in F$ and $E \cap F = \emptyset$. In particular, metric spaces are Hausdorff spaces. For a Hausdorff space every pair of distinct points have disjoint neighborhoods so that we have a clear separation of points. As a consequence this affects the convergence of sequences.

A point $x \in \mathcal{X}$ in a topological space $(\mathcal{X}, \tau)$ is a *limit* of a sequence (x_n) if for every neighborhood E of x there exists $n_0 \in \mathbb{N}$ such that $x_n \in E$ whenever $n \geq n_0$. If a sequence has a limit, then we say that the sequence *converges*. For Hausdorff spaces we have the following [34, Prop. 1.6.11].

Proposition 2.15. *In a Hausdorff topological space any sequence can have at most one limit.*

Proof. Let $\mathcal{X}$ be a Hausdorff topological space, (x_n) a sequence in $\mathcal{X}$. If (x_n) does not converge, then it has no limit and the claim follows. Therefore we may assume that (x_n) converges and has limits $x, y \in \mathcal{X}$. Suppose that x and y are two distinct elements in $\mathcal{X}$, i.e. $x \neq y$. Since $\mathcal{X}$ is Hausdorff, there exists distinct (open) neighbourhoods $E, F \subset \mathcal{X}$ such that $x \in E$ and $y \in F$. If now a sequence $(x_n) \subset \mathcal{X}$ converges to both x and y, there exists $n_1, n_2 \in \mathbb{N}$ such that $x_n \in E$ whenever $n \geq n_1$ and $x_n \in F$ whenever $n \geq n_2$. Now for $n \geq \max n_1, n_2$ we have that $x_n \in E \cap F$ which is a contradiction to the fact that $E \cap F = \{\emptyset\}$. □

As topology can be defined on any set, we can consider topologies on vector spaces. However, we have to make the topology compatible with the vector space structures, namely vector addition and scalar multiplication. This is achieved by continuity of functions.

A function $f : \mathcal{X} \to \mathcal{Y}$ from a topological space $\mathcal{X}$ to a topological space $\mathcal{Y}$ is *continuous at* $x \in \mathcal{X}$ if for every neighborhood V of $f(x)$ there exists a neighborhood U of x such that $f(x) \in V$ whenever $x \in U$. Furthermore, f is *continuous* if it continuous at every point of $\mathcal{X}$.

We now have a natural definition for topological vector spaces [35].

Definition 2.16. Let $\mathcal{V}$ be a real vector space and τ a topology on $\mathcal{V}$. Then the pair $(\mathcal{V}, \tau)$ is called a *real topological vector space* if

TVS-1. the scalar multiplication $(\alpha, x) \mapsto \alpha x$ from $\mathbb{R} \times \mathcal{V}$ to $\mathcal{V}$ is continuous, and

TVS-2. the vector addition $(x, y) \mapsto x + y$ from $\mathcal{V} \times \mathcal{V}$ to $\mathcal{V}$ is continuous.

We can now formulate some results for topological vector spaces that we will need later. Here we settle for just presenting the results and some the implications of the results for our earlier work.

First we see that we can refer to topological properties without specifying the topology if require the topology to be Hausdorff.

Proposition 2.17. *Every Hausdorff topological vector space $\mathcal{V}$ of dimension n is isomorphic with $\mathbb{R}^n$ with the Euclidean topology.*

It is easy to see that the mapping

$$(\alpha_1, \ldots, \alpha_n) \mapsto \sum_{i=1}^{n} \alpha_i v_i$$

from $\mathbb{R}^n$ to $\mathcal{V}$ is an algebraic isomorphism for any basis $\{v_i\}_i$ of $\mathcal{V}$. Furthermore, every isomorhism is of this form. The previous proposition follows by proving that the isomorphism is continuous and has a continuous inverse. The proof of this can be found for example in [30, Thm. 8.32] or [35].

As we have an isomorphism between a vector space and its double dual, we can ask when the same holds for cones. This is characterized by the next proposition [30, Thm. 2.13].

Proposition 2.18. *$\mathcal{C}$ is a closed convex cone in a vector space $\mathcal{V}$ if and only if $(\mathcal{C}^*)^* \simeq \mathcal{C}$.*

From the above proposition we see an immediate implication.

Corollary 2.19. *The dual cone $\mathcal{V}_+^*$ of a closed positive cone $\mathcal{V}_+$ of an ordered vector space $\mathcal{V}$ is generating.*

Proof. From Prop. 2.18 we see that $(\mathcal{V}_+^*)^* \simeq \mathcal{V}_+$. By applying the previous theorem to the cone $\mathcal{V}_+^*$, we see that it is generating since $\mathcal{V}_+$ is a proper cone. □

We see that the statement of the previous proposition is satisfied if the positive cone has a compact base [27, Lemma 8.6].

Proposition 2.20. *Let $\mathcal{C}$ be a cone with base $\mathcal{B}$. If $\mathcal{B}$ is compact, then $\mathcal{C}$ is closed.*

If a cone is closed, we can prove the other direction for Prop. 2.13 [30, Thm. 3.5]

Proposition 2.21. *Let $\mathcal{V}$ be a vector space with a closed generating positive cone $\mathcal{V}_+$. If $u \in \mathcal{V}^*$ is strictly positive, then u is an order unit in $\mathcal{V}^*$.*

An useful notion can be made about the dual cones [30, Thm. 2.13]:

Proposition 2.22. *A dual cone of a cone is closed.*

As we conclude this section, we will next continue with examining further properties of vector spaces by considering how to combine vector spaces and form new ones. This can be accomplished with the tensor product structure.

3 Tensor product of vector spaces

Tensor product of vector spaces is a new vector space that is connected to the given vector spaces in a natural way. We will first consider the algebraic tensor product and then turn to cones and partial orders in them. The material on (algebraic) tensor product of vector spaces presented here can be found in [37–39]. Partial orders in tensor product spaces are covered in [40] and [41].

Let us first consider some properties of bilinear mappings [37]. Let $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ be finite-dimensional real vector spaces of dimensions n, m and l. We say that a mapping $f : \mathcal{U} \times \mathcal{V} \to \mathcal{W}$ is *bilinear* if it is linear with respect to both of its arguments, i.e.

$$\begin{aligned} f(\alpha u_1 + \beta u_2, v) &= \alpha f(u_1, v) + \beta f(u_2, v), \text{ and} \\ f(u, \lambda v_1 + \mu v_2) &= \lambda f(u, v_1) + \mu f(u, v_2) \end{aligned}$$

for all $u, u_1, u_2 \in \mathcal{U}$, $v, v_1, v_2 \in \mathcal{V}$ and $\alpha, \beta, \lambda, \mu \in \mathbb{R}$. We denote the set of bilinear mappings from $\mathcal{U} \times \mathcal{V}$ to $\mathcal{W}$ by $\mathcal{L}(\mathcal{U}, \mathcal{V}; \mathcal{W})$. If $\mathcal{W} = \mathbb{R}$ we say that a bilinear function $f : \mathcal{U} \times \mathcal{V} \to \mathbb{R}$ is a *bilinear form.*

Let $\{u_i\}_{i=1}^n$ be a basis for $\mathcal{U}$ and $\{v_j\}_{j=1}^m$ be a basis for $\mathcal{V}$ and consider a bilinear function $f \in \mathcal{L}(\mathcal{U}, \mathcal{V}; \mathcal{W})$. For each element $u = \sum_i \alpha_i u_i \in \mathcal{U}$ and $v = \sum_j \beta_j v_j \in \mathcal{V}$ we have that

$$f(u, v) = \sum_{i,j} \alpha_i \beta_j f(u_i, v_j).$$

We see that each $f \in \mathcal{L}(\mathcal{U}, \mathcal{V}; \mathcal{W})$ is determined by the mn elements of the set $\{f(u_i, v_j)\}_{i,j} \subset \mathcal{W}$. On the other hand, for arbitrary nm elements $w_{ij} \in \mathcal{W}$, $1 \leq i \leq n$, $1 \leq j \leq m$, we can define a bilinear function $\tilde{f} \in \mathcal{L}(\mathcal{U}, \mathcal{V}; \mathcal{W})$ by

$$\tilde{f}(u, v) = \sum_{i,j} \alpha_i \beta_j u_{ij}$$

for all $u = \sum_i \alpha_i u_i \in \mathcal{U}$ and $v = \sum_j \beta_j v_j \in \mathcal{V}$, so that $\tilde{f}(u_i, v_j) = w_{ij}$ for all i, j.

The set of bilinear functions $\mathcal{L}(\mathcal{U}, \mathcal{V}; \mathcal{W})$ becomes a vector space when we define the linear combination $\alpha f + \beta g$ of elements $f, g \in \mathcal{L}(\mathcal{U}, \mathcal{V}; \mathcal{W})$ with $\alpha, \beta \in \mathbb{R}$ by

$$(\alpha f + \beta g)(u, v) = \alpha f(u, v) + \beta g(u, v)$$

for all $(u, v) \in \mathcal{U} \times \mathcal{V}$.

Furthermore, if $\{w_k\}_{k=1}^l$ is a basis for $\mathcal{W}$ we can construct a basis $\{F_{ijk}\}_{i,j,k}$ for $\mathcal{L}(\mathcal{U}, \mathcal{V}; \mathcal{W})$ by defining the functions $F_{ijk} : \mathcal{U} \times \mathcal{V} \to \mathcal{W}$ by

$$F_{ijk}(u, v) = \alpha_i \beta_j w_k$$

for all $u = \sum_a \alpha_a u_a \in \mathcal{U}$ and $v = \sum_b \beta_b v_b \in \mathcal{V}$. Each F_{ijk} is clearly bilinear, since for example for the first argument we have that

$$F_{ijk}(\lambda u + \lambda' u', v) \;\; = \;\; (\lambda \alpha_i + \lambda' \alpha_i') \beta_j w_k = \lambda F_{ijk}(u, v) + \lambda F_{ijk}(u', v)$$

for all $u = \sum_i \alpha_i u_i, u' = \sum_i \alpha_i' u_i \in \mathcal{U}$, $v = \sum_j \beta_j v_j \in \mathcal{V}$ and $\lambda, \lambda' \in \mathbb{R}$, and the linearity over the second argument can be seen in a similar way. Now for every $f \in \mathcal{L}(\mathcal{U}, \mathcal{V}; \mathcal{W})$ we see that

$$f(u, v) \;\; = \;\; \sum_{i,j} \alpha_i \beta_j f(u_i, v_j) = \sum_{i,j,k} \alpha_i \beta_j \gamma_{ijk} w_k = \left(\sum_{i,j,k} \gamma_{ijk} F_{ijk} \right) (u, v)$$

for all $u = \sum_a \alpha_a u_a \in \mathcal{U}$ and $v = \sum_b \beta_b v_b \in \mathcal{V}$ and where $f(u_i, v_j) = \sum_k \gamma_{ijk} w_k$ is the basis expansion of each $f(u_i, v_j)$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. The linear independence of the set $\{F_{ijk}\}_{i,j,k}$ follows from the fact that $\{w_k\}_k$ forms a basis. Thus, the set $\{F_{ijk}\}_{i,j,k}$ forms a basis of $\mathcal{L}(\mathcal{U},\mathcal{V};\mathcal{W})$ and we note that $\dim(\mathcal{L}(\mathcal{U},\mathcal{V};\mathcal{W})) = \dim(\mathcal{U})\dim(\mathcal{V})\dim(\mathcal{W}) = nml$.

Bilinear mappings are used to construct the tensor product of two vector spaces [39].

Definition 3.1. Let $\mathcal{U}$ and $\mathcal{V}$ be two vector spaces of dimensions $n < \infty$ and $m < \infty$. A pair $(\mathcal{W}, f)$, consisting of a nm-dimensional vector space $\mathcal{W}$ and a mapping $f : \mathcal{U}\times\mathcal{V} \to \mathcal{W}$ is called a *tensor product of* $\mathcal{U}$ *and* $\mathcal{V}$ if

TP-1. $f \in \mathcal{L}(\mathcal{U},\mathcal{V};\mathcal{W})$, and

TP-2. if $\{u_i\}_{i=1}^n$ is a basis for $\mathcal{U}$ and $\{v_j\}_{j=1}^m$ is a basis for $\mathcal{V}$, then $\{f(u_i,v_j)\}_{1\le i\le n,1\le j\le m}$ is a basis for $\mathcal{W}$.

We note that the second condition immediately implies that

$$\dim(\mathcal{W}) = \dim(\mathcal{U})\dim(\mathcal{V}).$$

Before we start studying tensor products we want to make sure that such structures exist [37, 39].

Proposition 3.2 (Existence of tensor products)**.** *For any two finite-dimensional vector spaces there exists a tensor product space.*

Proof. Let $\mathcal{U}$ and $\mathcal{V}$ be two vector spaces of dimensions n and m with bases $\{u_i\}_i$ and $\{v_j\}_j$ respectively, and let $\mathcal{W}$ be any nm-dimensional vector space with a basis $\{w_{ij}\}_{i,j}$. Similarly to what was done above, we can define a bilinear mapping $f \in \mathcal{L}(\mathcal{U},\mathcal{V};\mathcal{W})$ by

$$f(u,v) = \sum_{i,j} \alpha_i\beta_j w_{ij}$$

for all $u = \sum_i \alpha_i u_i \in \mathcal{U}$ and $v = \sum_j \beta_j v_j \in \mathcal{V}$ so that $w_{ij} = f(u_i,v_j)$ for all $i = 1,\ldots,n$ and $j = 1,\ldots,m$.

Let now $\{u_i'\}_i$ and $\{v_j'\}_j$ be any bases for $\mathcal{U}$ and $\mathcal{V}$. The new basis vectors can be represented in the previous bases as

$$u_i' = \sum_k \alpha_{ik}u_k, \quad v_j' = \sum_l \beta_{jl}v_l$$

for some family of real numbers $\{\alpha_{ik}\}_{i,k}, \{\beta_{jl}\}_{j,l} \subset \mathbb{R}$. Since the change of basis is an invertible mapping, we also have representations for the previous bases with respect to the new ones as

$$u_i = \sum_k \tilde{\alpha}_{ik}u_k', \quad v_j = \sum_l \tilde{\beta}_{jl}v_l'$$

for some family of real numbers $\{\tilde{\alpha}_{ik}\}_{i,k}, \{\tilde{\beta}_{jl}\}_{j,l} \subset \mathbb{R}$ such that they satisfy

$$\sum_k \tilde{\alpha}_{ik}\alpha_{kj} = \sum_k \tilde{\beta}_{ik}\beta_{kj} = \delta_{ij}.$$

Now we can express the basis vectors $w_{ij} \in \mathcal{W}$ as

$$w_{ij} = \sum_{k,l} \tilde{\alpha}_{ik}\tilde{\beta}_{jl} f(u'_k, v'_l),$$

since $f(u'_k, v'_l) = \sum_{r,s} \alpha_{kr}\beta_{ls} w_{rs}$. Therefore the vectors $f(u'_i, v'_j)$ span $\mathcal{W}$ and since there are total number of nm of them, they form a basis for $\mathcal{W}$. □

We see that the bilinear mapping f of the tensor product of vector spaces $\mathcal{U}$ and $\mathcal{V}$ is the most universal bilinear mapping on $\mathcal{U} \times \mathcal{V}$ in the sense that any bilinear mapping can be expressed uniquely in terms of f [37, 39].

Proposition 3.3 (Universal property of tensor product)**.** *Let $(\mathcal{W}, f)$ be a tensor product of vector spaces $\mathcal{U}$ and $\mathcal{V}$. If $g \in \mathcal{L}(\mathcal{U}, \mathcal{V}; \mathcal{X})$ for some vector space $\mathcal{X}$, then there exists a unique linear map $h : \mathcal{W} \to \mathcal{X}$ such that $g = h \circ f$.*

Proof. Let $\{u_i\}_i$ and $\{v_j\}_j$ be bases for $\mathcal{U}$ and $\mathcal{V}$. Let $h : \mathcal{W} \to \mathcal{X}$ be a mapping defined by the basis $\{f(u_i, v_j)\}_{i,j}$ of $\mathcal{W}$ such that

$$h\left(\sum_{i,j} \lambda_{ij} f(u_i, v_j)\right) = \sum_{i,j} \lambda_{ij} g(u_i, v_j).$$

Clearly h is well-defined and linear. Now

$$\begin{aligned}(h \circ f)(u, v) &= h(f(u, v)) = h\left(f\left(\sum_i \alpha_i u_i, \sum_j \beta_j v_j\right)\right) = h\left(\sum_{i,j} \alpha_i\beta_j f(u_i, v_j)\right) \\ &= \sum_{i,j} \alpha_i\beta_j g(u_i, v_j) = g\left(\sum_i \alpha_i u_i, \sum_j \beta_j v_j\right) = g(u, v)\end{aligned}$$

for all $u = \sum_i \alpha_i u_i \in \mathcal{U}$ and $v = \sum_j \beta_j v_j \in \mathcal{V}$. The uniqueness of h follows from the fact that $h(f(u_i, v_j)) = g(u_i, v_j)$ which dictates the images of the basis vectors of $\mathcal{W}$. □

From the next proposition we see that the tensor product of two vector spaces is essentially unique so that all the tensor product spaces are isomorphic [37, 39].

Proposition 3.4 (Uniqueness of tensor product)**.** *Let $\mathcal{U}$ and $\mathcal{V}$ be vector spaces, and $(\mathcal{W}, f)$ and $(\mathcal{W}', f')$ their tensor products. Then there exists a unique bijective linear map $e : \mathcal{W} \to \mathcal{W}'$ such that $f' = e \circ f$.*

Proof. Since $f' \in \mathcal{L}(\mathcal{U}, \mathcal{V}; \mathcal{W}')$ and $(\mathcal{W}, f)$ is a tensor product of $\mathcal{U}$ and $\mathcal{V}$, by the previous proposition there exists a unique linear map $e : \mathcal{W} \to \mathcal{W}'$ such that $f' = e \circ f$. All that is left to do is to show the bijectivity of e. This can be achieved by applying the same procedure for $f \in \mathcal{L}(\mathcal{U}, \mathcal{V}; \mathcal{W})$ and $(\mathcal{W}', f')$: there exists a unique linear map $e' : \mathcal{W}' \to \mathcal{W}$ such that $f = e' \circ f'$. Now we see that $f = (e' \circ e) \circ f$, so that $e \circ e' = id$, where id is the unique identity map in $\mathcal{W}$. Similarly we see that also $e' \circ e = id'$ where id' is the unique identity map in $\mathcal{W}'$. This shows that e is bijective and that $e^{-1} = e'$. □

For simplicity, we also call the vector space $\mathcal{W}$ itself as the tensor product of $\mathcal{U}$ and $\mathcal{V}$ and denote $\mathcal{W} = \mathcal{U} \otimes \mathcal{V}$. We also use the notation $f(u, v) = u \otimes v$ for the tensor product mapping f and call it the *canonical mapping* of $\mathcal{U} \otimes \mathcal{V}$. Thus, if $\{u_i\}_i$ and $\{v_j\}_j$ are bases for $\mathcal{U}$ and $\mathcal{V}$ respectively, the basis of $\mathcal{U} \otimes \mathcal{V}$ is then $\{u_i \otimes v_j\}_{i,j}$ and each element $w \in \mathcal{W}$ can be represented in the form $w = \sum_{i,j} \lambda_{ij} u_i \otimes v_j$ for some family $\{\lambda_{ij}\} \subset \mathbb{R}$. We note that $\dim(\mathcal{U} \otimes \mathcal{V}) = \dim(\mathcal{U}) \dim(\mathcal{V})$.

The bilinearity of the canonical map f of $\mathcal{U} \otimes \mathcal{V}$ immediately implies the following properties for the elements of $\mathcal{U} \otimes \mathcal{V}$:

a) $(u_1 + u_2) \otimes v = u_1 \otimes v + u_2 \otimes v$

b) $u \otimes (v_1 + v_2) = u \otimes v_1 + u \otimes v_2$

c) $\lambda(u \otimes v) = (\lambda u) \otimes v = u \otimes (\lambda v)$

for all $u, u_1, u_2 \in \mathcal{U}$, $v, v_1, v_2 \in \mathcal{V}$ and $\lambda \in \mathbb{R}$. Using these identities we also can represent a general element $w = \sum_{i,j} \lambda_{ij} u_i \otimes v_j \in \mathcal{U} \otimes \mathcal{V}$ as $w = \sum_k u'_k \otimes v'_k$ where now $\{u'_k \otimes v'_k\}_k$ does not necessarily form a basis for $\mathcal{U} \otimes \mathcal{V}$ [38].

Let us now explicitly construct a tensor product of two vector spaces $\mathcal{U}$ and $\mathcal{V}$ with bases $\{u_i\}_{i=1}^n$ and $\{v_j\}_{j=1}^m$ respectively [37]. Let $\mathcal{U}^*$ and $\mathcal{V}^*$ be the dual spaces of $\mathcal{U}$ and $\mathcal{V}$ with dual bases $\{u^i\}_{i=1}^n$ and $\{v^j\}_{j=1}^m$ respectively.

Take $\mathcal{W} = \mathcal{L}(\mathcal{U}^*, \mathcal{V}^*; \mathbb{R})$. For each $(u, v) \in \mathcal{U} \times \mathcal{V}$ define a mapping $g_{u,v} : \mathcal{U}^* \times \mathcal{V}^* \to \mathbb{R}$ by

$$g_{u,v}(s, t) = s(u)t(v)$$

for all $(s, t) \in \mathcal{U}^* \times \mathcal{V}^*$. Clearly $g_{u,v}$ is bilinear for each (u, v) since both $\mathcal{U}^*$ and V^* are linear spaces. Thus we can define a mapping $f : \mathcal{U} \times \mathcal{V} \to \mathcal{L}(\mathcal{U}^*, \mathcal{V}^*; \mathbb{R})$ by

$f(u,v) = g_{u,v}$ for all $(u,v) \in \mathcal{U} \times \mathcal{V}$. We see that $f \in \mathcal{L}(\mathcal{U},\mathcal{V};\mathcal{W})$ since

$$\begin{aligned} f\left(\sum_i \alpha_i u_i, \sum_j \beta_j v_j\right)(s,t) &= g_{\sum_i \alpha_i u_i, \sum_j \beta_j v_j}(s,t) = s\left(\sum_i \alpha_i u_i\right) t\left(\sum_j \beta_j v_j\right) \\ &= \sum_{i,j} \alpha_i \beta_j s(u_i) t(v_j) = \sum_{i,j} \alpha_i \beta_j g_{u_i,v_j}(s,t) \\ &= \left(\sum_{i,j} \alpha_i \beta_j f(u_i,v_j)\right)(s,t) \end{aligned}$$

for all $(s,t) \in \mathcal{U}^* \times \mathcal{V}^*$.

As we saw earlier, we can construct a basis $\{F_{ij}\}_{i,j}$ for $\mathcal{L}(\mathcal{U}^*,\mathcal{V}^*;\mathbb{R})$ from functions $F_{ij} : \mathcal{U}^* \times \mathcal{V}^* \to \mathbb{R}$,

$$F_{ij}(y,z) = \alpha^i \beta^j$$

for all $y = \sum_a \alpha^a u^a \in \mathcal{U}^*$ and $z = \sum_b \beta^b v^b \in \mathcal{V}^*$. Now we see that

$$f(u_i,v_j)(y,z) = g_{u_i,v_j}(y,z) = y(u_i)z(v_j) = \alpha^i \beta^j = F_{ij}(y,z)$$

for all $(y,z) \in \mathcal{U}^* \times \mathcal{V}^*$ so that the set $\{f(u_i,v_j)\}_{1\le i\le n, 1\le j\le m}$ forms a basis for $\mathcal{W}$.

Hence, we can identify the tensor product space $\mathcal{U} \otimes \mathcal{V}$ of two finite-dimensional real vector spaces $\mathcal{U}$ and $\mathcal{V}$ with the vector space $\mathcal{L}(\mathcal{U}^*,\mathcal{V}^*;\mathbb{R})$ of bilinear forms on $\mathcal{U}^* \times \mathcal{V}^*$ so that each product element $u \otimes v$, where $u \in \mathcal{U}$ and $v \in \mathcal{V}$, is identified with the bilinear form defined by

$$(u \otimes v)(f,g) = f(u)g(v) \tag{3.1}$$

for all $f \in \mathcal{U}^*$ and $g \in \mathcal{V}^*$.

For the dual space of a tensor product space we have the following [37].

Proposition 3.5. $\mathcal{L}(\mathcal{U},\mathcal{V};\mathbb{R}) \simeq (\mathcal{U} \otimes \mathcal{V})^*$

Proof. Take $g \in (\mathcal{U} \otimes \mathcal{V})^*$ and define a mapping $\Phi_g : \mathcal{U} \times \mathcal{V} \to \mathbb{R}$ by $\Phi_g = g \circ f$, where f is the canonical mapping of $\mathcal{U} \otimes \mathcal{V}$. It is straightforward to check that Φ_g is bilinear. Consider the mapping $g \mapsto \Phi_g$. Since $\Phi_g \in \mathcal{L}(\mathcal{U},\mathcal{V};\mathbb{R})$, by the universal property of tensor products there is only one linear mapping $\tilde{g} \in (\mathcal{U} \otimes \mathcal{V})^*$ such that $\Phi_g = \tilde{g} \circ f$, namely $\tilde{g} = g$. Hence, the mapping $g \mapsto \Phi_g$ is bijective and we see that it is linear, i.e. $\Phi_{\alpha g + \beta g'} = \alpha \Phi_g + \beta \Phi_{g'}$ for all $g, g' \in (\mathcal{U} \otimes \mathcal{V})^*$ and $\alpha, \beta \in \mathbb{R}$. □

As a corollary we have that the dual of the tensor product of vector spaces can be identified with the tensor product of the dual spaces [37, Prop. 1.3].

Corollary 3.6. $(\mathcal{U} \otimes \mathcal{V})^* \simeq \mathcal{U}^* \otimes \mathcal{V}^*$

Proof. By Propositions 3.5 and 2.10 we have that

$$(\mathcal{U} \otimes \mathcal{V})^* \simeq \mathcal{L}(\mathcal{U}, \mathcal{V}; \mathbb{R}) \simeq \mathcal{L}(\mathcal{U}^{**}, \mathcal{V}^{**}; \mathbb{R}) \simeq \mathcal{U}^* \otimes \mathcal{V}^*.$$

□

If we consider more closely the isomorphism (3.1) and the one from Prop. 3.5 we see that we can identify element $g_1 \otimes g_2 \in \mathcal{U}^* \otimes \mathcal{V}^*$ with a linear functional on $\mathcal{U} \otimes \mathcal{V}$ defined by

$$(g_1 \otimes g_2)(u \otimes v) = g_1(u) g_2(v) \tag{3.2}$$

for all $u \in \mathcal{U}$ and $v \in \mathcal{V}$.

We can also consider tensor products of other linear maps between vector spaces, not just the tensor product of linear functionals [37, Thm. 1.2].

Proposition 3.7. *Let $F : \mathcal{U} \to \mathcal{X}$ and $G : \mathcal{V} \to \mathcal{Y}$ be linear maps between vector spaces $\mathcal{U}$ and $\mathcal{X}$, and $\mathcal{V}$ and $\mathcal{Y}$ respectively. There exists a unique linear map $H : \mathcal{U} \otimes \mathcal{V} \to \mathcal{X} \otimes \mathcal{Y}$ such that $H(u \otimes v) = F(u) \otimes G(v)$ for all $u \otimes v \in \mathcal{U} \otimes \mathcal{V}$.*

Proof. Let $f : \mathcal{U} \times \mathcal{V} \to \mathcal{U} \otimes \mathcal{V}$ and $g : \mathcal{X} \times \mathcal{Y} \to \mathcal{X} \otimes \mathcal{Y}$ be the canonical mappings of $\mathcal{U} \otimes \mathcal{V}$ and $\mathcal{X} \otimes \mathcal{Y}$ respectively. We define a mapping $F \times G : \mathcal{U} \times \mathcal{V} \to \mathcal{X} \times \mathcal{Y}$ by

$$(F \times G)(u, v) = (F(u), G(v))$$

for all $(u, v) \in \mathcal{U} \times \mathcal{V}$. We see that then the composite mapping $g \circ (F \times G) : \mathcal{U} \times \mathcal{V} \to mX \otimes \mathcal{Y}$ is bilinear so that $g \circ (F \times G) \in \mathcal{L}(\mathcal{U}, \mathcal{V}; \mathcal{X} \otimes \mathcal{Y})$. By the universal property of the tensor product $\mathcal{U} \otimes \mathcal{V}$ (Prop. 3.3) there exists a unique linear mapping $H : \mathcal{U} \otimes \mathcal{V} \to \mathcal{X} \otimes \mathcal{Y}$ such that $g \circ (F \times G) = H \circ f$. Now

$$H(u \otimes v) = H(f(u, v)) = g((F \times G)(u, v)) = g(F(u), G(v)) = F(u) \otimes G(v)$$

for all $u \otimes v \in \mathcal{U} \otimes \mathcal{V}$. □

The mapping H of the previous proposition is called the *tensor product of linear mappings F and G* and is denoted by $F \otimes G$. For every linear map $F : \mathcal{U} \to \mathcal{X}$ we can also define the *dual map* $F^* : \mathcal{X}^* \to \mathcal{U}^*$ by $F^*(f) = f \circ F$ for all $f \in \mathcal{X}^*$ so that if F and G are as in the previous proposition, we have that

$$\begin{aligned}
((F \otimes G)^*(f \otimes g))(u \otimes v) &= (f \otimes g)((F \otimes g)(u \otimes v)) \\
&= (f \otimes g)(F(u) \otimes G(v)) \\
&= f(F(u))g(G(v)) \\
&= F^*(f)(u)G^*(g)(v) \\
&= (F^*(f) \otimes G^*(g))(u \otimes v) \\
&= ((F^* \otimes G^*)(f \otimes g))(u \otimes v).
\end{aligned}$$

Since the product elements span both $\mathcal{U}\otimes\mathcal{V}$ and $\mathcal{X}^*\otimes\mathcal{Y}^*$, we have that $(F\otimes G)^* = F^*\otimes G^*$.

We note that the above construction for the tensor product of vector spaces can be naturally extended to cover more than two vector spaces. Instead of bilinearity and bilinear mappings one then considers multilinearity and multilinear mappings. One can in particular show that the tensor product is associative such that $(\mathcal{U}\otimes\mathcal{V})\otimes\mathcal{W}\cong\mathcal{U}\otimes(\mathcal{V}\otimes\mathcal{W})$ for any vector spaces $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ so that order in which the tensor products are constructed makes no difference [37]. For simplicity, we only consider tensor product of two vector spaces. With that in mind, we can now start to consider the partial orders in tensor product spaces.

3.1 Tensor product of ordered vector spaces

Let us consider the tensor product structure of two finite-dimensional vector spaces $\mathcal{U}$ and $\mathcal{V}$ that are now partially ordered. Since no partial order was used in the construction of the tensor product of vector spaces, the algebraic set of the tensor product of ordered vector spaces coincides with $\mathcal{U}\otimes\mathcal{V}$. However, the partial order in $\mathcal{U}\otimes\mathcal{V}$ is by no means unique so that the set of positive elements, or the positive cone as we saw earlier, is generally not fixed. Thus, in general there is a freedom in choosing the positive cone. Nevertheless there are two canonical choices, the maximal and the minimal tensor products.

In order to construct a reasonable tensor product space, the partial order there must be somehow linked to the partial orders of the spaces from which it is constructed from. The following definition captures the minimal requirement that the composites of positive elements both in the vector spaces and their duals should be positive [40].

Definition 3.8. A cone $\mathcal{C}_t\subset\mathcal{U}\otimes\mathcal{V}$ in the tensor product of two ordered vector spaces $\mathcal{U}$ and $\mathcal{V}$ is a *tensor cone* if

$$u\otimes v\in\mathcal{C}_t,\quad \forall u\in\mathcal{U}_+,\ v\in\mathcal{V}_+, \tag{3.3}$$

$$e\otimes f\in\mathcal{C}_t^*,\quad \forall e\in\mathcal{U}_+^*\ f\in\mathcal{V}_+^*. \tag{3.4}$$

We note that if $\mathcal{U}_+$ and $\mathcal{V}_+$ are generating cones in $\mathcal{U}$ and $\mathcal{V}$ respectively, then any convex tensor cone $\mathcal{C}_t\subset\mathcal{U}\otimes\mathcal{V}$ is generating. Indeed, since any convex tensor cone contains all positive linear combinations of positive pure tensors, for any element

$\sum_i u_i \otimes v_i \in \mathcal{U} \otimes \mathcal{V}$, we have that

$$\begin{aligned}\sum_i u_i \otimes v_i &= \sum_i (a_i - b_i) \otimes (c_i - d_i) \\ &= \sum_i (a_i \otimes c_i + b_i \otimes d_i) - \sum_i (b_i \otimes c_i + a_i \otimes d_i) \in \mathcal{C}_t - \mathcal{C}_t,\end{aligned}$$

where $u_i = a_i - b_i \in \mathcal{U}$ for some $a_i, b_i \in \mathcal{U}_+$ and $v_i = c_i - d_i \in \mathcal{V}$ for some $c_i, d_i \in \mathcal{V}_+$ for all i.

As was shown in the previous section, in the case of finite-dimensional vector spaces, the tensor product $\mathcal{U} \otimes \mathcal{V}$ may be identified with the set $\mathcal{L}(\mathcal{U}^*, \mathcal{V}^*; \mathbb{R})$ of bilinear functionals on $\mathcal{U}^* \times \mathcal{V}^*$. Hence, the product elements $u \otimes v \in \mathcal{U} \otimes \mathcal{V}$ of the above definition may be identified with bilinear forms defined by

$$(u \otimes v)(f, g) = f(u)g(v) \tag{3.5}$$

for all $f \in \mathcal{U}^*$ and $g \in \mathcal{V}^*$, and the isomorphism of Cor. 3.6 identifies the product elements $f \otimes g \in \mathcal{U}^* \otimes \mathcal{V}^*$ with linear functionals defined by

$$(f \otimes g)(u \otimes v) = f(u)g(v) \tag{3.6}$$

for all $u \in \mathcal{U}$ and $v \in \mathcal{V}$. With these identifications, the conditions of the previous definition are easily checked for various cones in $\mathcal{U} \otimes \mathcal{V}$.

We see, however, that not all tensor cones induce a partial order since they might not be proper or even convex cones. Hence, we arrive to the following definition.

Definition 3.9. A tensor product $\mathcal{U} \otimes \mathcal{V}$ of ordered vector spaces $\mathcal{U}$ and $\mathcal{V}$ with a partial order is an *ordered tensor product* of $\mathcal{U}$ and $\mathcal{V}$ if the partial order is induced by a proper tensor cone.

Let us now consider the maximal and minimal tensor products [40, 41] (also known as the injective and projective tensor products) that were mentioned at the beginning of this subsection.

In the light of the above definitions we can ask what is the minimal proper cone $\mathcal{C}_{\min} \subset \mathcal{U} \otimes \mathcal{V}$ that satisfies the requirements (3.3) and (3.4)? Let us start with the set of products of positive elements, since they must always be included in any tensor cone. We see that they form a cone and even a tensor cone in $\mathcal{U} \otimes \mathcal{V}$. Unfortunately the cone is not a convex one. However we can form a convex cone from them by considering the positive linear combinations, i.e. the conical hull, of such elements. Denote this convex cone by $\mathcal{C}_{\min}$, i.e.

$$\mathcal{C}_{\min} = \left\{ \sum_{i,j} \lambda_{ij} u_i \otimes v_j \,|\, \forall i, j: \; u_i \in \mathcal{U}_+, \, v_j \in \mathcal{V}_+, \, \lambda_{ij} \in \mathbb{R}_+ \right\}.$$

Clearly $\mathcal{C}_{\min}$ is a convex tensor cone since it was constructed that way. It is in fact a proper cone, since if $x \in \mathcal{C}_{\min} \cap -\mathcal{C}_{\min}$ we have that $x = \sum_{i,j} \lambda_{ij} u_i \otimes v_j$ where λ_{ij} must now vanish for all i and j so that $x = 0$. Hence, $\mathcal{C}_{min}$ is a proper tensor cone and we can construct a minimal partial order in $\mathcal{U} \otimes \mathcal{V}$.

Definition 3.10. The *minimal tensor product* of ordered vector spaces $\mathcal{U}$ and $\mathcal{V}$, denoted by $\mathcal{U} \otimes_{\min} \mathcal{V}$, is the ordered tensor product $\mathcal{U} \otimes \mathcal{V}$ ordered by $\mathcal{C}_{\min}$.

In the similar manner as in the case of the minimal tensor product we can ask what could be the maximal proper tensor cone $\mathcal{C}_{\max}$. Let us now consider the second condition of the definition of the tensor cone. Take a product element $f \otimes g \in \mathcal{U}^* \otimes \mathcal{V}^*$, where $f \in \mathcal{U}_+^*$ and $g \in \mathcal{V}_+^*$, acting on a general element $\sum_i u_i \otimes v_i \in \mathcal{U} \otimes \mathcal{V}$. The condition now reads as

$$(f \otimes g) \left(\sum_i u_i \otimes v_i \right) = \sum_i f(u_i) g(v_i) \geq 0. \tag{3.7}$$

Since we are looking for a maximal tensor cone, we can take $\mathcal{C}_{\max}$ as the set consisting of all elements $w \in \mathcal{U} \otimes \mathcal{V}$, where $w = \sum_i u_i \otimes v_i$ for some $\{u_i\}_i \subset \mathcal{U}$ and $\{v_i\}_i \subset \mathcal{V}$, satisfying (3.7) for all $f \in \mathcal{U}_+^*$ and $g \in \mathcal{V}_+^*$, i.e.

$$\mathcal{C}_{\max} = \{w \in \mathcal{U} \otimes \mathcal{V} \,|\, (e \otimes f)(w) \geq 0 \;\; \forall e \in \mathcal{U}_+^*, \;\; f \in \mathcal{V}_+^*\}. \tag{3.8}$$

We can always consider such elements as bilinear forms on $\mathcal{U}^* \times \mathcal{V}^*$ so that we have that

$$\mathcal{C}_{\max} \cong \{w \in \mathcal{L}\left(\mathcal{U}^*, \mathcal{V}^*; \mathbb{R}\right) \,|\, w(f,g) \geq 0 \; \forall f \in \mathcal{U}_+^*, \;\; \forall g \in \mathcal{V}_+^*\}. \tag{3.9}$$

In other words, $\mathcal{C}_{\max}$ is the set of all positive bilinear functionals on $\mathcal{U}^* \times \mathcal{V}^*$. Note that always positive product elements $u \otimes v \in \mathcal{C}_{\max}$, where $u \in \mathcal{U}_+$ and $v \in \mathcal{V}_+$.

We note that if $w \in \mathcal{C}_{\max}$, then

$$\left(\sum_{i,j} \mu_{ij} e_i \otimes f_j \right)(w) \geq 0 \tag{3.10}$$

for all $e_i \in \mathcal{U}_+^*$, $f_j \in \mathcal{V}_+^*$ and $\mu_{ij} \in \mathbb{R}_+$ for all i and j. This also works another way around since $e \otimes f$ is just a special case of an element of the form (3.10). Hence, $\mathcal{C}_{\max}$ is a convex cone which also makes it a tensor cone.

However, it may happen that $\mathcal{C}_{\max}$ is not a proper cone as it may not be a strict one. In a similar manner as in the proof of Prop. 2.12 we see that the set $\mathcal{C}_{\max} \cap -\mathcal{C}_{\max}$ coincides with the set of bilinear forms $w \in \mathcal{U} \otimes \mathcal{V}$ which vanish on

$\left(\mathcal{U}_+^* - \mathcal{U}_+^*\right) \times \left(\mathcal{V}_+^* - \mathcal{V}_+^*\right)$. Thus, w may not vanish on all of $\mathcal{U}^* \times \mathcal{V}^*$. If now $\mathcal{U}_+^*$ and $\mathcal{V}_+^*$ are generating cones, then $\mathcal{C}_{\max} \cap -\mathcal{C}_{\max} = \{0\}$ and $\mathcal{C}_{\max}$ is a proper tensor cone. For example, the generating property of $\mathcal{U}_+^*$ and $\mathcal{V}_+^*$ is guaranteed by Cor. 2.19 and Prop. 2.9 if $\mathcal{U}_+$ and $\mathcal{V}_+$ are closed or if both $\mathcal{U}_+^*$ and $\mathcal{V}_+^*$ admit order units.

From now on we will assume that $\mathcal{C}_{\max}$ is a proper tensor cone in which case we make the following definition.

Definition 3.11. The *maximal tensor product* of ordered vector spaces $\mathcal{U}$ and $\mathcal{V}$, denoted by $\mathcal{U} \otimes_{\max} \mathcal{V}$, is the ordered tensor product $\mathcal{U} \otimes \mathcal{V}$ ordered by $\mathcal{C}_{\max}$.

We can show that there is a duality between maximal and minimal tensor cones. Naturally we have that $\mathcal{C}_{\max} = (\mathcal{U} \otimes_{\max} \mathcal{V})_+$ and $\mathcal{C}_{\min} = (\mathcal{U} \otimes_{\min} \mathcal{V})_+$. Suppose that $\mathcal{U}_+^*$ and $\mathcal{V}_+^*$ are proper cones (so that $\mathcal{U}_+$ and $\mathcal{V}_+$ are generating cones). Then we can also consider minimal and maximal tensor cones in $\mathcal{U}^* \otimes \mathcal{V}^*$. Denote

$$\mathcal{D}_{\min} = (\mathcal{U}^* \otimes_{\min} \mathcal{V}^*)_+, \quad \mathcal{D}_{\max} = (\mathcal{U}^* \otimes_{\max} \mathcal{V}^*)_+.$$

Note that in order for $\mathcal{D}_{\max}$ to be a proper cone, we must have that $(\mathcal{U}_+^*)^* = \mathcal{U}_+$ and $(\mathcal{V}_+^*)^* = \mathcal{V}_+$ which is always guaranteed as $\mathcal{U}_+^*$ and $\mathcal{V}_+^*$ are closed by Prop. 2.22.

Consider first $\mathcal{D}_{\min}$. We have that

$$\mathcal{D}_{\min} = \left\{ \sum_{i,j} \mu_{ij} e_i \otimes f_j \,|\, \forall i,j: \ e_i \in \mathcal{U}_+^*, \ f_j \in \mathcal{V}_+^*, \ \mu_{ij} \in \mathbb{R}_+ \right\}.$$

Now in our finite-dimensional setting we have that

$$\begin{aligned} \mathcal{D}_{\min}^* &= \{ w \in (\mathcal{U}^* \otimes \mathcal{V}^*)^* \,|\, w(g) \geq 0 \ \ \forall g \in \mathcal{D}_{\min} \} \\ &\cong \left\{ w \in \mathcal{U} \otimes \mathcal{V} \ | \ \left(\sum_{i,j} \mu_{ij} e_i \otimes f_j \right)(w) \geq 0 \ \ \forall \sum_{i,j} \mu_{ij} e_i \otimes f_j \in \mathcal{D}_{\min} \right\} \\ &= \left\{ w \in \mathcal{U} \otimes \mathcal{V} \ | \ (e \otimes f)(w) \geq 0 \ \ \forall e \in \mathcal{U}_+^*, \ f \in \mathcal{V}_+^* \right\} \\ &= \mathcal{C}_{\max}. \end{aligned} \tag{3.11}$$

Similarly for $\mathcal{D}_{\max}$, if $\mathcal{U}_+$ and $\mathcal{V}_+$ are closed, we have that

$$\begin{aligned} \mathcal{D}_{\max} &= \{ g \in \mathcal{U}^* \otimes \mathcal{V}^* \ | \ (u \otimes v)(g) \geq 0 \ \ \forall u \in \mathcal{U}_+^{**}, \ v \in \mathcal{V}_+^{**} \} \\ &\cong \{ g \in (\mathcal{U} \otimes \mathcal{V})^* \ | \ g(u \otimes v) \geq 0 \ \ \forall u \in \mathcal{U}_+, \ v \in \mathcal{V}_+ \} \\ &= \{ g \in (\mathcal{U} \otimes \mathcal{V})^* \ | \ g(w) \geq 0 \ \ \forall w \in \mathcal{C}_{\min} \} \end{aligned}$$

Now

$$\begin{aligned} \mathcal{D}_{\max}^* &= \{ w \in (\mathcal{U}^* \otimes \mathcal{V}^*)^* \,|\, w(g) \geq 0 \ \ \forall g \in \mathcal{D}_{\max} \} \\ &\cong \{ w \in \mathcal{U} \otimes \mathcal{V} \,|\, g(w) \geq 0 \ \ \forall g \in \mathcal{D}_{\max} \} \\ &= \mathcal{C}_{\min} \end{aligned} \tag{3.12}$$

As we noted, all positive linear combinations of positive product elements are positive bilinear forms so that $\mathcal{C}_{\min} \subset \mathcal{C}_{\max}$. Furthermore, the construction of $\mathcal{C}_{\min}$ and $\mathcal{C}_{\max}$ ensure that if $\mathcal{C}_t$ is any proper tensor cone in a tensor product space, then $\mathcal{C}_{\min} \subset \mathcal{C}_t \subset \mathcal{C}_{\max}$.

We can also consider tensor products of linear maps in ordered tensor product spaces. As was shown in Prop. 3.7, every pair of linear functions $F : \mathcal{U} \to \mathcal{X}$ and $G : \mathcal{V} \to \mathcal{Y}$ define a unique linear map $F \otimes G : \mathcal{U} \otimes \mathcal{V} \to \mathcal{X} \otimes \mathcal{Y}$ for vector spaces $\mathcal{U}$, $\mathcal{V}$, $\mathcal{X}$ and $\mathcal{Y}$. If now the vector spaces are ordered, we can consider the positivity of $F \otimes G$ if F and G are positive, i.e. $F(\mathcal{U}_+) \subset \mathcal{X}_+$ and $G(\mathcal{V}_+) \subset \mathcal{Y}_+$.

With the minimal tensor product we find that if $w = \sum_{i,j} \lambda_{ij} u_i \otimes v_j \in (\mathcal{U} \otimes_{\min} \mathcal{V})_+$, then

$$(F \otimes G)(w) = \sum_{i,j} \lambda_{ij} F(u_i) \otimes G(v_j) \in (\mathcal{X} \otimes_{\min} \mathcal{Y})_+,$$

so that the tensor product of positive maps is always positive on the minimal tensor product. Similarly $(F^* \otimes G^*)(\mathcal{X}^* \otimes_{\min} \mathcal{Y}^*)_+) \subset (\mathcal{U}^* \otimes_{\min} \mathcal{V}^*)_+$ as it follows from the positivity of F and G that $F^*(\mathcal{X}^*_+) \subset \mathcal{U}^*_+$ and $F^*(\mathcal{Y}^*_+) \subset \mathcal{V}^*_+$.

With the maximal tensor product we can prove the same, namely if $w = \sum_i u_i \otimes v_i \in (\mathcal{U} \otimes_{\max} \mathcal{V})_+$, then for all $f \in \mathcal{X}^*_+$ and $g \in \mathcal{Y}^*_+$ we have that

$$\begin{aligned}
(f \otimes g)((F \otimes G)(w)) &= \sum_i (f \otimes g)(F(u_i) \otimes G(v_i)) && (3.13)\\
&= \sum_i f(F(u_i)) g(G(v_i)) && (3.14)\\
&= \sum_i F^*(f)(u_i) G^*(g)(v_i) && (3.15)\\
&= \sum_i (F^*(f) \otimes G^*(g))(u_i \otimes v_i && (3.16)\\
&= (F^*(f) \otimes G^*(g))(w) \geq 0 && (3.17)
\end{aligned}$$

so that indeed $(F \otimes G)(\mathcal{U} \otimes_{\max} \mathcal{V})_+) \subset (\mathcal{X} \otimes_{\max} \mathcal{Y})_+$.

This concludes our introduction to the mathematical framework needed in formulating convex operational theories. In next chapter we use the tools of this chapter to construct the operational class of convex theories.

Chapter II

Convex operational theories

The key features behind convex operational theories are present in its naming: the operational approach focuses on specifying the mathematical structures that are needed to describe a physical experiment theoretically and lies on the statistical analysis of the experiment, and in such experiments the mixing of states leads us to convex state spaces.

In any physical experiment some properties of some physical system are being observed. Thus, prior to being observed, the physical system has gone through some preparation procedures that have prepared the system in a *state* that contains information about the systems properties. The system is then taken and a *measurement* is applied leading to the registration of some measurement outcome that is then affiliated to some property of the system. Hence, the state of the system and the measurement outcomes have to be linked, and furthermore, they have to be linked in a probabilistic way so that a probability of registering a specific outcome when the system is in a particular state can be determined.

Furthermore, if the outputs of two preparation devices give the same outcome probabilities for all measurements, we have no way of telling them apart. In this way the preparation devices form equivalence classes so that the state of the system is actually an equivalence class of the preparation devices. This reflects the fact that the state may be prepared in different ways. Similarly we may consider two measurement devices that give the same statistics for all states; once again we may identify the measurement with an equivalence class of such measurement devices. Thus, also equivalent measurements may be performed with different set-ups.

In this chapter we consider the operational concepts more closely. The operational approach described here can be found in [42, 43]. The framework presented here is fairly standardized and can be found for example in [29, 44–46].

4 States

We will start with the states of a physical system. As was already noted, the states can be considered as an equivalence class of preparation devices. We can then consider mixing of these devices so that we come to a clear operational meaning for convexity.

4.1 Convexity as an operational principle

Consider two preparation procedures that prepare the system in two different states s_1 and s_2 and suppose we fix the measurement so that the same measurement can be performed on either of the states. Say that we then alternate the preparation devices so that the state s_1 is used in the measurement with probability λ and the state s_2 is used with the probability $1-\lambda$ for some $\lambda \in [0,1]$.

After a set of measurements the outcomes form a probability distribution and by that point we cannot say which preparation device was actually used in every single run of the measurement. Thus, because the alternation of states is physically possible and the measurement outcomes are then known to form a legitimate probability distribution, the alternation, or *mixing*, of states is to be considered as a new preparation device [43]. This combination of preparations is called the *mixture* of s_1 and s_2 with weights λ and $1-\lambda$ respectively. This leads to the notion of (generalized) convexity and convex structures that were defined and studied in Chapter I.

As was also observed in Chapter I, the cancellation property (1.2) plays a crucial role in embedding convex structures into vector spaces. In the operational set-up, we see that this property is satisfied. Indeed, suppose we have two preparation devices: one that mixes two states ϱ and ϱ' with weights λ and $1-\lambda$, and one that then mixes states ϱ and ϱ'' with the same weights λ and $1-\lambda$ respectively.

Suppose then that when we make some measurements, we notice that the outcome probability distributions for both preparation devices are equal, thus the both devices represent the same state. However, due to the statistical correspondence between states and measurements, since we know that the same state ϱ was present in the mixtures with the same weight λ, the outcome statistics for ϱ' and ϱ'' can be restored. Because the mixed states were equal and we do the same manipulations for the mixed states statistics, it follows that also the measurement statistics for ϱ' and ϱ'' must coincide. Since this holds for all measurements, we are forced to conclude that $\varrho' = \varrho''$ and the cancellation property is satisfied.

Hence, in the operational framework Prop. 1.10 shows that the convex structures can be identified with convex sets in real vector spaces. In order to simplify the treatment of the theory, the underlying vector spaces are taken to be finite-dimensional.

In addition we make two technical assumptions. First, we consider our finite-dimensional vector space to be equipped with the unique Hausdorff topology of Prop. 2.17. This assumption is natural in the sense that if we consider a sequence of states so that the states are getting closer together, i.e. the sequence is converging, the limit state or the state that the sequence is closing in to must be unique, which is the case if we have a Hausdorff topology as we saw in Prop. 2.15.

Second, we take the convex set of states to be compact with respect to our Hausdorff topology. In finite-dimensional setting compactness is equivalent to the set being closed and bounded [34, Thm. 3.2.8]. The closeness is a natural assumption in a sense that if we have a sequence of states that converge we require the limit to be a state as well. If not, we can always extend our set to its closure. The boundedness of the state space is purely technical assumption.

The compactness then guarantees by the Krein-Milmann Theorem 1.14 that the set of extreme points of the state space is enough to characterize the whole state space since every state can be expressed as a convex combination of extremal states (also called *pure states*). We note that the convex deceomposition of a state into pure states is not unique unless the state space is a simplex. The compactness assumption is a common one in most of the related works (see for example [9, 47]).

Now we are ready to define the state space of our theory.

Definition 4.1. The *state space* $\mathcal{S} \subset \mathcal{V}$ of a convex operational theory is a compact convex subset of a finite-dimensional real Hausdorff vector space $\mathcal{V}$.

We will see that state spaces can be expressed as bases for proper cones and thus linking convex sets and partial orders in vector spaces.

4.2 State space as a compact base for a generating positive cone

The state space can be neatly connected to the ordered vector spaces that were studied in the previous chapter. In general there is no canonical way to do this so we present one which follows the one presented in [29, Appendix B] and [48]. Also [49] assumes the similar construction. For other types of formulations, see for example [41, 50].

Suppose we have a state space $\mathcal{S} \subset \mathcal{V}$ in a real vector space $\mathcal{V}$ of dimension d. Consider the linear span of $\mathcal{S}$, $\mathrm{span}(\mathcal{S})$, and denote $\dim(\mathrm{span}(\mathcal{S})) = d_{\mathcal{S}}$. By Prop. 2.17, $\mathrm{span}(\mathcal{S})$ is now isomorphic to $\mathbb{R}^{d_{\mathcal{S}}}$ with the natural Hausdorff topology. Since this isomorphism is continuous (as is required from an isomorphism between topological spaces) and $\mathcal{S}$ is compact, the resulting image $\mathcal{S}'$ of $\mathcal{S}$ is also compact. Moreover, the isomorphism preserves convexity so that $\mathcal{S}'$ is also convex. In $\mathbb{R}^{d_{\mathcal{S}}}$ we now have that either $\mathrm{aff}(\mathcal{S}') = \mathbb{R}^{d_{\mathcal{S}}}$ or $\mathrm{aff}(\mathcal{S}')$ is an affine hyperplane[2] in $\mathbb{R}^{d_{\mathcal{S}}}$ [29]. We see that the former is the case exactly when $0 \in \mathrm{aff}(\mathcal{S}')$.

Namely, if $\mathrm{aff}(\mathcal{S}')$ is an affine hyperplane and we suppose that $0 \in \mathrm{aff}(\mathcal{S}')$, then for the linear function $f \in \mathbb{R}^{d_{\mathcal{S}}}$ that defines the hyperplane we have that $f(x) = 0$ for all $x \in \mathrm{aff}(\mathcal{S}')$. Now if we take any vector $y \in \mathrm{span}(\mathrm{aff}(\mathcal{S}'))$, then as y is some linear combination of vectors in $\mathrm{aff}(\mathcal{S}')$, the linearity of f implies that $f(y) = 0$. Thus, $y \in \mathrm{aff}(\mathcal{S}')$ so that $\mathrm{span}(\mathrm{aff}(\mathcal{S}')) = \mathrm{aff}(\mathcal{S}') \neq \mathbb{R}^{d_{\mathcal{S}}}$. This contradicts the fact that $\mathrm{span}(\mathcal{S}') = \mathbb{R}^{d_{\mathcal{S}}}$ so that the zero vector cannot be contained in $\mathrm{aff}(\mathcal{S}')$.

Thus, in the case $\mathrm{aff}(\mathcal{S}') = \mathbb{R}^{d_{\mathcal{S}}}$ we can consider $\mathcal{S}'$ to be embedded into $\mathbb{R}^{d_{\mathcal{S}}+1}$ so that it is convex isomorphic with the set $\{(s,1) \in \mathbb{R}^{d_{\mathcal{S}}+1} \,|\, s \in \mathcal{S}'\}$ whose affine hull is an affine hyperplane in $\mathbb{R}^{d_{\mathcal{S}}+1}$ and which spans $\mathbb{R}^{d_{\mathcal{S}}+1}$. Hence, for both of these cases we have a vector space $\mathcal{A}$ and a compact convex set $\mathcal{S}^{\mathcal{A}}$ that spans $\mathcal{A}$ and is convex isomorphic to $\mathcal{S}$. Furthermore, $\mathrm{aff}(\mathcal{S}^{\mathcal{A}})$ is an affine hyperplane in $\mathcal{A}$.

Consider now a subset $\mathcal{A}_+$ of the vector space $\mathcal{A}$, where

$$\mathcal{A}_+ = \{\alpha s \in \mathcal{A} \,|\, \alpha \geq 0,\ s \in \mathcal{S}^{\mathcal{A}}\}.$$

We see that $\mathcal{A}_+$ is a cone since $\lambda \alpha s \in \mathcal{A}_+$ for all $\lambda, \alpha \geq 0$ and $s \in \mathcal{S}^{\mathcal{A}}$. It is also a convex cone, since for all $\alpha, \beta \geq 0$ such that $\alpha + \beta \neq 0$ we have that

$$\alpha s + \beta t \quad = \quad \alpha s + \beta t = (\alpha+\beta)\left(\frac{\alpha}{\alpha+\beta}s + \frac{\beta}{\alpha+\beta}t\right) \in \mathcal{A}_+ \qquad (4.1)$$

for all $s, t \in \mathcal{S}^{\mathcal{A}}$ since $\mathcal{S}^{\mathcal{A}}$ is convex. If $\alpha = \beta = 0$, then $\alpha s + \beta t = 0 \in \mathcal{A}_+$ for all $s, t \in \mathcal{S}^{\mathcal{A}}$.

We show that $\mathcal{A}_+$ is also a proper cone of $\mathcal{A}$. Since $\mathrm{aff}(\mathcal{S}^{\mathcal{A}})$ is an affine hyperplane in $\mathcal{A}$, it does not contain the zero vector as was just observed. Thus, there exists a linear functional $f \in \mathcal{A}^*$ such that $f(s) = \alpha > 0$ for all $s \in \mathcal{S}^{\mathcal{A}}$. If we now take $x \in \mathcal{A}_+ \cap -\mathcal{A}_+$, $x \neq 0$, we have that $x = \beta s$ for some $\beta > 0$ and $s \in \mathcal{S}^{\mathcal{A}} \cap -\mathcal{S}^{\mathcal{A}}$. Thus, $-s \in \mathcal{S}^{\mathcal{A}}$, so that $f(-s) = -\alpha < 0$ which is a contradiction. Hence, $x = 0$ so

[2] An affine hyperplane in a d-dimenional vector space $\mathcal{V}$ is a $d-1$-dimensional affine subspace of $\mathcal{V}$. A subset $\mathcal{H} \subset \mathcal{V}$ is a hyperplane if and only if $\mathcal{H} = \{x \in \mathcal{V} \,|\, f(x) = \alpha\}$ for some linear functional $f \in \mathcal{V}^*$ and $\alpha \in \mathbb{R}$.

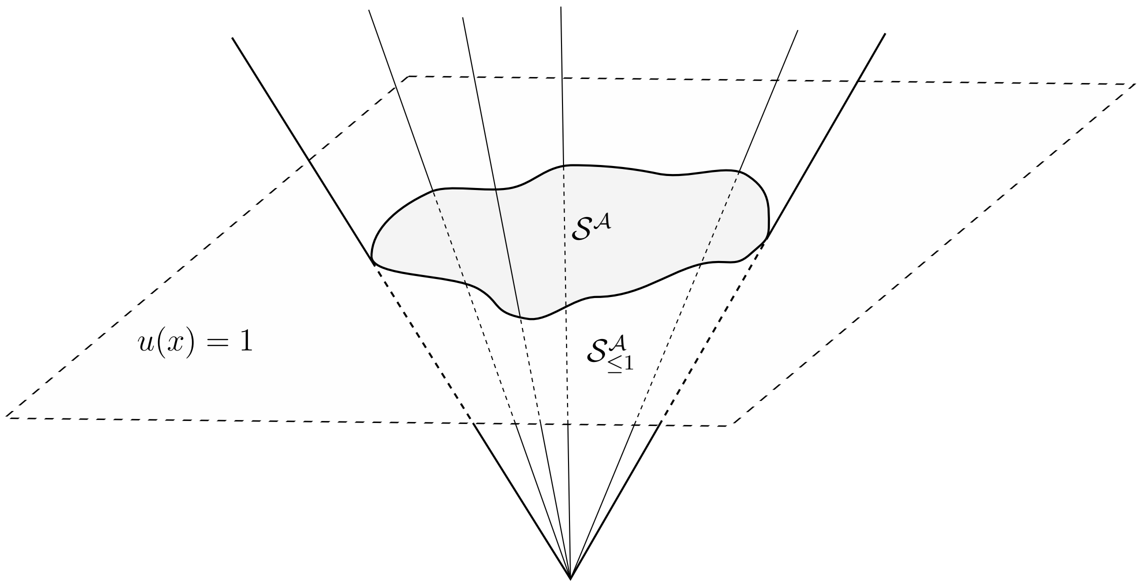

Figure 6: State space $\mathcal{S}^{\mathcal{A}}$ as a base for the positive cone $\mathcal{A}_+$.

that $\mathcal{A}_+ \cap -\mathcal{A}_+ = \{0\}$ and $\mathcal{A}_+$ is a proper cone. Furthermore, $\mathcal{A} = \mathcal{A}_+ - \mathcal{A}_+$ since $\mathcal{S}^{\mathcal{A}}$ spans $\mathcal{A}$ so that $\mathcal{A}_+$ is generating cone.

It is clear that the set $\mathcal{S}^{\mathcal{A}}$ is actually a base for $\mathcal{A}_+$. Hence, $\mathcal{A}_+$ is a proper generating cone in $\mathcal{A}$ with a compact base $\mathcal{S}^{\mathcal{A}}$. Furthermore, $\mathcal{A}_+$ is closed by Prop. 2.20, and by Prop. 2.14 we have that there exists a strictly positive functional $u \in \mathcal{A}^*$ such that

$$\mathcal{S}^{\mathcal{A}} = \{x \in \mathcal{A}_+ \,|\, u(x) = 1\}.$$

By Prop. 2.21, u is an order unit in $\mathcal{A}^*_+$.

In conclusion, if $\mathcal{S}$ is a state space of a convex operational theory, there exists a finite-dimensional vector space $\mathcal{A}$, a closed proper generating cone $\mathcal{A}_+$ in $\mathcal{A}$ and an order unit $u \in \mathcal{A}^*$ such that $\mathcal{S}$ is convex isomorphic with a compact base $\mathcal{S}^{\mathcal{A}} = \{x \in \mathcal{A}_+ \,|\, u(x) = 1\}$ of $\mathcal{A}_+$ (Fig. 6). We note that in this case the dual cone $\mathcal{A}^*_+$ is also closed proper generating cone by Prop. 2.22 and Prop. 2.12 and Cor. 2.19. For a state space $\mathcal{S}$ we may thus denote $\mathcal{S}^{\mathcal{A}}$ if we want to emphasize that we consider the state space to be a subset of an ordered vector space $\mathcal{A}$ by the above construction.

Measuring and manipulating states are not always perfect in the sense that the measurement device sometimes fails to recognice the states or some part of the initial ensemble of states may be destroyed in the process. For this reason we can consider adding an empty outcome so that the predicted probabilities of a measurement add up to a number less than one. Thus, we can extend our notion of states by considering the set $\mathcal{S}^{\mathcal{A}}_{\leq 1} = \{x \in \mathcal{A}_+ \,|\, u(x) \leq 1\}$ whose elements are called the *subnormalized states* (Fig. 6) [21]. For example, if we consider a beam of light as our state, then a subnormalized state could be the same beam whose intensity has been decreased. In particular we see that $0 \in \mathcal{S}^{\mathcal{A}}_{\leq 1}$ so that it can be interpreted as the zero state, i.e., a state whose intensity has been decreased so much that no measurement or processing device can even read it.

As we have now defined the set of states for our class of theories, we are ready to move on to measurements. Let us start with the most elementary measurements.

5 Effects

As was noted at the beginning of the previous section, the states and measurements are linked in a statistical way so that the theory does not predict with certainty which outcome the measurement results but rather it gives the probability of an outcome occuring. This basic statistical framework can be found in [42, 43].

5.1 Basic statistical framework

Consider the setting of the beginning of the previous section where we had two preparation procedures that prepare the system in two different states s_1 and s_2 respectively before making some measurement. By construction of the mixture state, we can make the same measurement on both of these states individually. By making measurements to these states we get probability distributions describing the outcome probabilities for both states.

Say we then make a new measurement for a mixed state of s_1 and s_2 with weights λ and $1 - \lambda$ for some $\lambda \in [0, 1]$ so that we get one more outcome probability distribution. Since the mixing of states is statistical in nature, we cannot know which state was measured in each single measurement. Hence, the outcome probability distribution of the measurement of the mixed state must be equal to the mixing of the probability distributions of the separate measurements of the states.

We call an elementary event corresponding to the outcome of the measurement an *effect*. Effect describes the statement that the outcome of the measured physical observable takes some specific outcome or belongs to a certain subset of outcomes for some state $s \in \mathcal{S}$. Then it is natural for each effect to associate a function describing the probability of the statement of the effect being true for the state that was measured. Thus, effects can be presented as functions from the set of states $\mathcal{S}$ to the interval $[0, 1]$.

Now the convexity preserving property can be further specified. A functional $f : \mathcal{S} \to \mathbb{R}$ on a state space $\mathcal{S}$ is called *affine* if it preserves the convexity of the states, i.e.

$$f(\lambda s_1 + (1-\lambda)s_2) = \lambda f(s_1) + (1-\lambda)f(s_2)$$

for all $s_1, s_2 \in \mathcal{S}$ and $\lambda \in [0,1]$. We denote the set of affine functionals on $\mathcal{S}$ by $\mathcal{F}(\mathcal{S})$. We are now ready to formulate the definition of effect space in convex operational theories.

Definition 5.1. The *effect space* $\mathcal{E}(\mathcal{S})$ of a convex operational theory with a state space $\mathcal{S}$ consists of affine functionals on $\mathcal{S}$ taking values in the unit interval $[0,1]$, i.e.

$$\mathcal{E}(\mathcal{S}) = \{e \in \mathcal{F}(\mathcal{S}) \,|\, 0 \leq e(s) \leq 1 \ \ \forall s \in \mathcal{S}\}.$$

If the system is in a state $s \in \mathcal{S}$, then $e(s) \in [0,1]$ is interpreted as the probability that the effect $e \in \mathcal{E}(\mathcal{S})$ is observed. The set of affine functionals $\mathcal{F}(\mathcal{S})$ form a real vector space and the effect space form a convex subset of $\mathcal{F}(\mathcal{S})$.

There are two effects of particular interest, the zero effect o and the unit effect u defined as

$$o(s) = 0, \quad u(s) = 1$$

for all states $s \in \mathcal{S}$. Hence, the zero effect corresponds to an event that never happens whereas the unit effect depicts an event that always happens. In fact, with these two effects we can give an equivalent definition for the effects on $\mathcal{S}$ by

$$\mathcal{E}(\mathcal{S}) = \{e \in \mathcal{F}(\mathcal{S}) \,|\, o \leq e \leq u\},$$

where the partial order is defined so that for two effects $e, f \in \mathcal{E}(\mathcal{S})$ we denote that $e \leq f$ if and only if $e(s) \leq f(s)$ for all $s \in \mathcal{S}$. For each effect e we can also define a *complement effect* $u - e$ which corresponds to the event that the event of e does not happen.

5.2 Effect space as a an intersection of two cones

Let us now consider $\mathcal{S}$ in the ordered vector space formalism presented in the previous section. We saw that there exists a finite-dimensional vector space $\mathcal{A}$, a closed and generating positive cone $\mathcal{A}_+$ with a compact base $\mathcal{S}^{\mathcal{A}}$ and an order unit u in $\mathcal{A}_+^*$ such that $\mathcal{S}$ is convex isomorphic to $\mathcal{S}^{\mathcal{A}} = \{x \in \mathcal{A} \,|\, u(x) = 1\}$. We see that the order unit in $\mathcal{A}_+^*$ is exactly the unit effect on $\mathcal{S}$.

If we instead consider effects on the set of subnormalized states $\mathcal{S}^{\mathcal{A}}_{\leq 1}$, it is natural to assume that if we make an empty measurement, i.e., measurement on the zero state $0 \in \mathcal{S}_{\leq 1}$, the probability that an outcome is obtained is zero. For affine functionals this means that the mapping fixes the origin so that the translation part of an affine mapping is zero so that effects are actually linear functionals. We can emphasize this point even further.

We will show that every effect e has an extension to a linear functional $\tilde{e}$ on $\mathcal{A}$ [43, Prop. 2.30]. We will do this in parts. First we set $\tilde{e}(0) = 0$ as was just discussed. For each $e \in \mathcal{E}(\mathcal{S}^{\mathcal{A}})$ we can define $\tilde{e} : \mathcal{A}_+ \to \mathbb{R}$

$$\tilde{e}(x) = u(x)e(x/u(x))$$

for all $x \in \mathcal{A}_+ \setminus \{0\}$. Since $\mathcal{S}^{\mathcal{A}}$ is a base for $\mathcal{A}_+$ and since the order unit is a linear functional, we have that for each $x \in \mathcal{A}_+$,

$$\tilde{e}(x) = \lambda e(s),$$

where $x = \lambda s$ for some unique $\lambda \in \mathbb{R}$ and $s \in \mathcal{S}^{\mathcal{A}}$. Using Eq. (4.1), we see that

$$\tilde{e}(\alpha x + \beta y) = \alpha \tilde{e}(x) + \beta \tilde{e}(y)$$

for all $x, y \in \mathcal{A}_+$ and $\alpha, \beta \geq 0$.

Since $\mathcal{A}_+$ is a generating cone, we can further define (and rename) $\tilde{e} : \mathcal{A} \to \mathbb{R}$ by

$$\tilde{e}(x) = \tilde{e}(t) - \tilde{e}(u)$$

for all $x = t - u \in \mathcal{A}$, where $t, u \in \mathcal{A}_+$. It is simple enough to check that now, indeed,

$$\tilde{e}(\alpha x + \beta y) = \alpha \tilde{e}(x) + \beta \tilde{e}(y)$$

for all $x, y \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{R}$ so that $\tilde{e} \in \mathcal{A}^*$. Since $\mathrm{span}(\mathcal{S}^{\mathcal{A}}) = \mathcal{A}$, the extension $\tilde{e} \in \mathcal{A}^*$ for each $e \in \mathcal{E}(\mathcal{S}^{\mathcal{A}})$ is unique.

In $\mathcal{A}^*$ we have the dual ordering induced by the positive cone $\mathcal{A}_+$. We see now that for each $e \in \mathcal{E}(\mathcal{S}^{\mathcal{A}})$ we have that $\tilde{e} \geq 0$ since $\tilde{e}(x) = \lambda e(s) \geq 0$ for all $x \in \mathcal{A}_+$, where $\lambda > 0$ is the unique positive real number and $s \in \mathcal{S}^{\mathcal{A}}$ is the unique state such that $x = \lambda s$. On the other hand we see that $\tilde{e} \leq u$ since now $\tilde{e}(x) = \lambda e(s) \leq \lambda = u(x)$ for all $x \in \mathcal{A}_+$. Hence, the set of the extended effects constitutes of the linear extensions $\tilde{e} \in \mathcal{A}^*$ of each effect $e \in \mathcal{E}(\mathcal{S}^{\mathcal{A}})$.

We saw that effects can be extended to linear functionals on $\mathcal{A}$. What remains to see if there are other elements of $\mathcal{A}^*$ that are effects when restricted to $\mathcal{S}^{\mathcal{A}}$. In

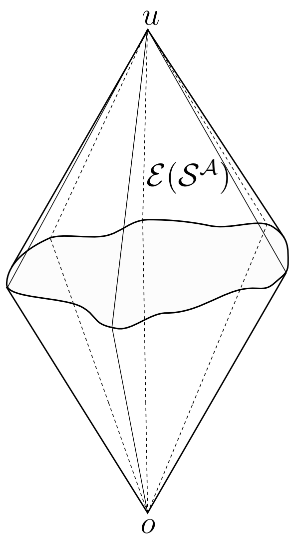

Figure 7: Effect space $\mathcal{E}(\mathcal{S}^{\mathcal{A}})$ as an intersection of the cones $\mathcal{A}_+^*$ and $u - \mathcal{A}_+^*$.

addition to being linear, the extended effects $\tilde{e}$ lie between the zero functional o and order unit u. Let us take a linear functional $g \in \mathcal{A}^*$ such that $o \leq g \leq u$. When restricted to $\mathcal{S}^{\mathcal{A}}$, we see that $g_{|\mathcal{S}^{\mathcal{A}}} \in \mathcal{E}(\mathcal{S}^{\mathcal{A}})$ so that we can extend it to $\tilde{g_{|\mathcal{S}^{\mathcal{A}}}}$. By the uniqueness of the linear extension we have that $g = \tilde{g_{|\mathcal{S}^{\mathcal{A}}}}$. Thus, all linear functionals on $\mathcal{A}$ that are bounded by o and u are given by the extensions of effects. Hence, we can consider the extended effects in stead of the effects themselves since the two set coincide on the state space.

Thus, the above discussion has led as to conclude that the effect space form a convex subset of the dual space of the vector space $\mathcal{A}$ of the state space $\mathcal{S}^{\mathcal{A}}$, or more formally

$$\mathcal{E}(\mathcal{S}^{\mathcal{A}}) = \{e \in \mathcal{A}^* \,|\, o \leq e \leq u\}.$$

Thus, in fact $\mathcal{E}(\mathcal{S}^{\mathcal{A}})$ can be represented as an intersection of two cones (Fig. 7) , namely

$$\mathcal{E}(\mathcal{S}^{\mathcal{A}}) = \mathcal{A}_+^* \cap \left(u - \mathcal{A}_+^*\right).$$

Next we use the effects to form a measurement of an observable.

6 Observables

6.1 Measurements on states

In the previous section we associated effects with elementary events or questions that we can ask about the values of the measurement outcomes. Let us now take a state and consider that we have a collection of such effects so that the probabilities of observing these effects sum up to one. This means that some effect from this collection is always observed.

Thus, the collection of effects form a complete description of the possible values of the measurement of some physical quantity; an *observable*. We can therefore associate observables with a collection of effects such that for any state the probability that some effect is observed is one. Note that if our probabilities do not sum up to one, we can always complete the collection with a complement of the sum of the other effects: this guarantees that the normalization then holds.

We note that when the outcome set is finite, the observable is determined by as many effects as there are possible outcomes since an effect can be assigned for each of these outcomes. In order to simplify the treatment, we only consider observables with a finite outcome set. The treatment of observables with continuous sets of outcomes is neglected. More formal description of observables is given by the following definition.

Definition 6.1. Let $\mathcal{S}$ be a state space of a convex operational theory. An *observable with a finite number of outcomes* is a mapping $\mathsf{A} : x \mapsto \mathsf{A}_x$ from a finite outcome set Ω to $\mathcal{E}(\mathcal{S})$ such that $\sum_{x\in\Omega} \mathsf{A}_x(s) = 1$ for all $s \in \mathcal{S}$. The set of observables on $\mathcal{S}$ with outcome set Ω is denoted by $\mathcal{O}(\mathcal{S}, \Omega)$ and all observables on $\mathcal{S}$ by $\mathcal{O}(\mathcal{S})$.

If $e \in \mathcal{E}(\mathcal{S})$ is any effect, then we can form a *binary* observable E with outcome set $\Omega = \{+, -\}$ by defining $\mathsf{E}_+ = e$ and $\mathsf{E}_- = u - e$. An observable T on an outcome set Ω is said to be *trivial* if the outcome probabilities are state-independent for each outcome, i.e., $\mathsf{T}_x(s) = \mathsf{T}_x(s')$ for all $s, s' \in \mathcal{S}$ for each $x \in \Omega$. It follows that T is then of the form $\mathsf{T}_x = p_x u$ for all $x \in \Omega$, where $p : x \mapsto p_x$ is a probability distribution on Ω. The set of trivial observables on $\mathcal{S}$ are then denoted by $\mathcal{T}(\mathcal{S}, \Omega)$ and $\mathcal{T}(\mathcal{S})$.

6.2 Mixtures and post-processings of observables

An useful feature of observable is that similarly as the states, the observables can be mixed too [43]. If we have two observables A and B with outcome sets Ω and Λ respectively, for each $\lambda \in [0, 1]$ we can define an observable C with an outcome set $\Gamma \equiv \Omega \cup \Lambda$ by

$$\mathsf{C}_z = \lambda \mathsf{A}_z + (1 - \lambda)\mathsf{B}_z \tag{6.1}$$

for all $z \in \Gamma$, where $\mathsf{A}_z = o$ if $z \notin \Omega$ and $\mathsf{B}_z = o$ if $z \notin \Lambda$. We say that C is then a mixture of A and B. By this extension of outcome sets we can always assume that the outcome sets of the mixture is the same as those of the observables that are mixed.

As trivial observables are state-independent, they give no information about the state that is measured. Thus, if we mix a trivial observable T with a non-trivial observable $\mathcal{A}$, the measurement statistics that is acquired by measuring the mixed observable is somewhat "fuzzier" than the one we would have obtained by measuring just A. Hence, we can interpret trivial observable as noise in the measurement.

Another way to form new observables from known ones is to process their measurement outcomes. This is called the coarse-graining of observables [43]. A transition map $\nu : \Omega \to \Lambda$ between outcome spaces Ω and Λ can be represented as a right stochastic matrix. The elements $(\nu_{xy})_{x\in\Omega,\ y\in\Lambda}$ of ν satisfy $0 \leq \nu_{xy} \leq 1$ and $\sum_y \nu_{xy} = 1$. Each matrix element ν_{xy} represents a probability for the transition $x \mapsto y$ occurring.

Examples of such maps include the relabeling of the measurement outcomes [17]. For any function $f : \Omega \to \Lambda$ between outcome spaces Ω and Λ we can define the matrix elements of the transition map ν^f induced by f as

$$\nu^f_{xy} = \begin{cases} 1, & \text{if } f(x) = y, \\ 0, & \text{otherwise.} \end{cases}$$

A particular example of such relabeling is the copying of measurement outcomes. Namely, if we define a relabeling function $c : \Omega \to \Omega \times \Omega$ for some outcome set Ω by $c(x) = (x, x)$ for all $x \in \Omega$, then the transition ν^c takes any element x to (x, x).

With the stochastic matrix ν and an observable A with an outcome set Ω we can form a new observable $\nu \circ \mathsf{A}$ with outcome set Λ by defining

$$(\nu \circ \mathsf{A})_y = \sum_{x\in\Omega} \nu_{xy} \mathsf{A}_x \tag{6.2}$$

for all $y \in \Lambda$. It is straightforward to check that this indeed defines an observable. We note that this way of forming new observables defines a preorder in the set of observable $\mathcal{O}(\mathcal{S})$ [43] so that we can make the following definition.

Definition 6.2. We say that an observable $\mathsf{B} \in \mathcal{O}(\mathcal{S}, \Lambda)$ is a *post-processing* of an observable $\mathsf{A} \in \mathcal{O}(\mathcal{S}, \Omega)$ if there exists a right stochastic matrix $\nu : \Omega \to \Lambda$ such that $\mathsf{B} = \nu \circ \mathsf{A}$.

Just as in the case of mixing observables with trivial observables, we can sometimes interpret post-processings of an observable as noisy versions of the observable.

7 Operations

Instead of measuring states, we can consider processing or transforming them. The general ideas for state transformations that are used here can be found in [21, 43]. As state preparators only produces states as outputs and measurements take states as inputs producing classical outputs, a general state transformation takes a state as an input producing state as an output. Thus, state transformations can be considered as mappings from some state space to another. A physical example of such a transformation is an optical fiber. These transformations are called operations.

We note that not every mapping $\tau : \mathcal{S} \to \mathcal{S}'$ from a state space $\mathcal{S}$ to a state space $\mathcal{S}'$ can be a transformation. Namely, by the probabilistic nature of mixing, the transformed state of a mixture of states must equal to the mixture of transformed states. That is, the operation map must preserve convexity, i.e. it must be affine.

Let us consider operations in the ordered vector space framework. Thus, for a state space $\mathcal{S}^{\mathcal{A}}$ we have the set of subnormalized states $\mathcal{S}^{\mathcal{A}}_{\leq 1}$. It may happen that a state transformation is not perfect so that it destroys some part of the system one way or another. Then we may consider the output state to be subnormalized. In the optical fiber example, a light beam passing through the fiber usually has a decrease in its intensity so that it can be considered as a subnormalized state. As we may continue to do some other transformations on the output state, our most general operations take subnormalized states also as an input.

Hence, for an operation $\tau : \mathcal{S}^{\mathcal{A}}_{\leq 1} \to \mathcal{S}^{\mathcal{B}}_{\leq 1}$ we have that $u_B(\tau(s)) \leq u_A(s)$ for all $s \in \mathcal{S}^{\mathcal{A}}_{\leq 1}$. It follows that $\tau(0_A) = 0_B \in \mathcal{S}^{\mathcal{B}}_{\leq 1}$ for the zero state $0_A \in \mathcal{S}^{\mathcal{A}}_{\leq 1}$. As in the case of effects, we can then uniquely extend the operation into a linear function from $\mathcal{A}$ to $\mathcal{B}$. As a state space can be expressed as a base for the positive cone, the operation must map positive elements to positive elements, i.e. it must be a positive function. Thus, we have arrived for the following definition.

Definition 7.1. Let $\mathcal{S}^{\mathcal{A}}$ and $\mathcal{S}^{\mathcal{B}}$ be two state spaces. A linear mapping $\tau : \mathcal{A} \to \mathcal{B}$ is called an *operation* if $\tau(\mathcal{A}_+) \subset \mathcal{B}_+$ and $u_B(\tau(x)) \leq u_A(x)$ for all $x \in \mathcal{A}_+$. Furthermore, τ is a *channel* if the previous inequality is an equality which is equivalent with $\tau(\mathcal{S}^{\mathcal{A}}) \subset \mathcal{S}^{\mathcal{B}}$.

For operation τ we interpret $u_B(\tau(s))$ as a probability of the transmission of the state s during an operation τ. Channels thus correspond to perfect transformations so that if we take a (normalized) state as an input we get a (normalized) state as an output. In many cases we assume our transformations to be perfect so that we mainly consider channels instead of general operations.

As was the case with states and measurement, also channels can be mixed. Namely, if $\tau : \mathcal{A} \to \mathcal{B}$ and $\chi : \mathcal{A} \to \mathcal{B}$ are channels on a state space $\mathcal{S}^{\mathcal{A}}$, then $\lambda\tau(s)+(1-\lambda)\chi(s) \in \mathcal{S}^{\mathcal{B}}$ so that the mixture $\lambda\tau+(1-\lambda)\chi$ is a channel on $\mathcal{S}^{\mathcal{A}}$. Another way to form new channels is the *concatenation of channels*, that is, for the channels $\tau : \mathcal{A} \to \mathcal{B}$ and $\upsilon : \mathcal{B} \to \mathcal{V}$ we may define the composite mapping $\upsilon \circ \tau : \mathcal{A} \to \mathcal{C}$ and we see that it is indeed a channel because $(\upsilon \circ \tau)(\mathcal{S}^{\mathcal{A}}) = \upsilon(\tau(\mathcal{S}^{\mathcal{A}})) \subset \upsilon(\mathcal{S}^{\mathcal{B}}) \subset \mathcal{S}^{\mathcal{C}}$.

We will consider some properties of operations that are presented in [21]. As an operation is a linear mapping, we can define its dual so that for an operation $\tau : \mathcal{A} \to \mathcal{B}$ we can define $\tau^* : \mathcal{B}^* \to \mathcal{A}^*$ as

$$\tau^*(g)(x) = g(\tau(x)) \tag{7.1}$$

for all $g \in \mathcal{B}^*$ and $x \in \mathcal{A}$. We see that τ^* is also linear.

By using the dual map, we see that each operation determines an effect $e_\tau \in \mathcal{A}^*$ by

$$e_\tau(x) = \tau^*(u_B)(x) = (u_B \circ \tau)(x) \tag{7.2}$$

for all $x \in \mathcal{A}$. Clearly $e_\tau \in \mathcal{A}^*$ is linear. For all $x = \alpha s \in \mathcal{A}_+$ we have that

$$e_\tau(x) = u_B(\tau(x)) = \alpha u_B(\tau(s)) \leq \alpha u_A(s) = u_A(x),$$

and since both τ and u_B are positive,

$$e_\tau(x) = u_B(\tau(x)) = \alpha u_B(\tau(s)) \geq 0 = o(x)$$

so that indeed $o \leq e_\tau \leq u_A$. Thus, e_τ is an effect on $\mathcal{S}^{\mathcal{A}}$.

Thus, each operation determines a unique effect. On the other hand we see that an effect can be determined by multiple operations. Indeed, if we take $e \in \mathcal{E}(\mathcal{S}^{\mathcal{A}})$ and a fixed $s \in \mathcal{S}^{\mathcal{B}}$, we can define a mapping $\tau_s : \mathcal{A} \to \mathcal{B}$ by

$$\tau_s(x) = e(x)s$$

for all $x \in \mathcal{A}$. Now τ_s is linear since e is linear, and $u_B(\tau_s(s')) = e(s') \leq u_A(s')$ for all $s' \in \mathcal{A}_+$. We see that now τ_s determines e since

$$\tau_s^*(u_B)(x) = u_B(\tau_s(x)) = e(x)u_B(s) = e(x)$$

for all $x \in \mathcal{A}$. Thus, τ_s defines the same effect for all $s \in \mathcal{S}^{\mathcal{B}}$.

8 Composite systems

Until now we have only considered operational concepts such as transformations and measurements on single systems. In order to truly characterize a theory, a way to form compound systems is needed. This is because sometimes a physical system may be composed into parts so that the total system is actually sum of its parts. This is the case when we consider systems interacting with each other.

One of the most important examples is that of an open system in contact with its environment. In order to see the effects of an environment on a system, we must consider the system and the environment as a compound system with some interaction between them. By ignoring the environmental degrees of freedom we can see the effects on the system.

The material presented here is mostly based on [51, 52].

8.1 Physical composites

Let us consider two state spaces $\mathcal{S}^{\mathcal{A}} \subset \mathcal{A}$ and $\mathcal{S}^{\mathcal{B}} \subset \mathcal{B}$ of systems A and B in ordered vector spaces $\mathcal{A}$ and $\mathcal{B}$ respectively. We are interested in the joint system $A + B$ with a state space $\mathcal{S}^{\mathcal{AB}}$ in some vector space $\mathcal{AB}$. Since we want our compound system to have a legitimate state space, we require that there exists a closed, generating positive cone $\mathcal{AB}_+$ and an order unit $u_{AB} \in \mathcal{AB}^*$ such that $\mathcal{S}^{\mathcal{AB}} = \{s \in \mathcal{AB}_+ \,|\, u_{AB}(s) = 1\}$ is a compact base for $\mathcal{AB}_+$. Now the question is how the local state spaces $\mathcal{S}_A$ and $\mathcal{S}_B$ and their respective vector spaces $\mathcal{A}$ and $\mathcal{B}$ are related to the composite state space $\mathcal{S}^{\mathcal{AB}}$ and its vector space $\mathcal{AB}$.

By the dualities $\mathcal{A} \cong \mathcal{A}^{**}$ and $\mathcal{B} \cong \mathcal{B}^{**}$ we can consider states acting on effects rather than effects acting on states. Thus, we may identify every state $s \in \mathcal{A}$ with an element $\tilde{s} \in \mathcal{A}^{**}$ such that $\tilde{s}(a) = a(s)$ for all $a \in \mathcal{A}^*$ and $\tilde{s}(u_A) = 1$, and similarly for states in $\mathcal{B}$ and $\mathcal{AB}$. From now on we use this identification wherever it seems convenient without explicit remarks.

We start our investigation of the composite system by assuming that for each pair of effects in $\mathcal{E}(\mathcal{S}^A) \times \mathcal{E}(\mathcal{S}^B)$ there exists a product effect in $\mathcal{E}(\mathcal{S}^{\mathcal{AB}})$, i.e., there exists a function $\pi : \mathcal{E}(\mathcal{S}^A) \times \mathcal{E}(\mathcal{S}^B) \to \mathcal{E}(\mathcal{S}^{\mathcal{AB}})$ so that for each $(e, f) \in \mathcal{E}(\mathcal{S}^{\mathcal{A}}) \times \mathcal{E}(\mathcal{S}^{\mathcal{B}})$ we have that $\pi(e, f) \in \mathcal{E}(\mathcal{S}^{\mathcal{AB}})$. This is a requirement that measurements can be performed locally. Similarly we want to be able to prepare our states independently so that for each pair of $(s, t) \in \mathcal{S}^{\mathcal{A}} \times \mathcal{S}^{\mathcal{B}}$ there exists a state $s_{AB} \in \mathcal{S}^{\mathcal{AB}}$.

Some natural assumptions about the state space $\mathcal{S}^{\mathcal{AB}}$ can now be made:

i) for each state $s \in \mathcal{S}^{\mathcal{AB}}$ the composition $s' = s \circ \pi : \mathcal{E}(\mathcal{S}^A) \times \mathcal{E}(\mathcal{S}^B) \to \mathbb{R}$ defines a joint probability $s(\pi(e,f))$ for each pair of effects $(e,f) \in \mathcal{E}(\mathcal{S}_A) \times \mathcal{E}(\mathcal{S}_B)$; if $s_{AB} \in \mathcal{S}^{\mathcal{AB}}$ is a state that is prepared locally from $s \in \mathcal{S}^{\mathcal{A}}$ and $t \in \mathcal{S}^{\mathcal{B}}$, then the compatibility of local preparations and local measurements is assumed to hold so that $s_{AB}(\pi(e,f)) = s(e)t(f)$ for all $(e,f) \in \mathcal{E}(\mathcal{S}^{\mathcal{A}}) \times \mathcal{E}(\mathcal{S}^{\mathcal{B}})$,

ii) the *non-signalling principle* holds for the joint probabilities, i.e. the marginal probability distribution for the outcomes of a measurement on one of the systems is not affected by the measurement performed on the other system; this leads to well-defined marginal and conditional states and we assume that the conditional states defined for each subsystem belong to the respective state spaces, and

iii) the joint probabilities respect the *local tomography principle* [13], i.e. the joint state $s \in \mathcal{S}^{\mathcal{AB}}$ is determined by the joint probabilities on all pairs of effects $(e,f) \in \mathcal{E}(\mathcal{S}_A) \times \mathcal{E}(\mathcal{S}_B)$.

8.2 Tensor product of local state spaces

We show that under the assumptions presented above, the vector space $\mathcal{AB}$ is isomorphic to $\mathcal{A} \otimes \mathcal{B}$. This is achieved equivalently by showing that $\mathcal{AB}^* \cong \mathcal{A}^* \otimes \mathcal{B}^*$. We first give the mathematical formulations for the above assumptions.

The local tomography principle in iii) states that if

$$s'_1(e,f) = s'_2(e,f),$$

for all $(e,f) \in \mathcal{E}(\mathcal{S}_A) \times \mathcal{E}(\mathcal{S}_B)$ for some $s_1, s_2 \in \mathcal{S}^{\mathcal{AB}}$, then we have that $s_1 = s_2$. Similarly the non-signalling condition in ii) can be expressed as a requirement that for each state $s \in \mathcal{S}^{\mathcal{AB}}$

$$\sum_{y \in \Lambda} s'(e, \mathsf{B}_y) = \sum_{y' \in \Lambda'} s'(e, \mathsf{B}'_{y'})$$

and

$$\sum_{x \in \Omega} s'(\mathsf{A}_x, f) = \sum_{x' \in \Omega'} (\mathsf{A}'_{x'}, f)$$

holds for all effects $e \in \mathcal{E}(\mathcal{S}^{\mathcal{A}})$ and $f \in \mathcal{E}(\mathcal{S}^{\mathcal{B}})$ and all observables $\mathsf{A} \in \mathcal{O}(\mathcal{S}_A, \Omega)$, $\mathsf{A}' \in \mathcal{O}(\mathcal{S}_A, \Omega')$, $\mathsf{B} \in \mathcal{O}(\mathcal{S}_B, \Lambda)$ and $\mathsf{B}' \in \mathcal{O}(\mathcal{S}_B, \Lambda')$ with some outcome sets Ω, Ω', Λ and Λ' respectively.

The non-signalling condition implies that we can define marginal states $s_A : \mathcal{E}(\mathcal{S}^{\mathcal{A}}) \to \mathbb{R}$ and $s_B : \mathcal{E}(\mathcal{S}^{\mathcal{B}}) \to \mathbb{R}$ by

$$s_A(e) = \sum_y s'(e, \mathsf{B}_y), \quad s_B(f) = \sum_y s'(\mathsf{A}_x, f)$$

for all $e \in \mathcal{E}(\mathcal{S}^{\mathcal{A}})$ and $f \in \mathcal{E}(\mathcal{S}^{\mathcal{B}})$ so that they are indeed well-defined since they are independent of the measurements performed on the other system. Furthermore, for each $e \in \mathcal{E}(\mathcal{S}^{\mathcal{A}})$ and $f \in \mathcal{E}(\mathcal{S}^{\mathcal{B}})$ we can define conditional states $s_{A|f} : \mathcal{E}(\mathcal{S}^{\mathcal{A}}) \to \mathbb{R}$ and $s_{B|e} : \mathcal{E}(\mathcal{S}^{\mathcal{B}}) \to \mathbb{R}$ by

$$s_{A|f}(e') = \frac{s'(e', f)}{s_B(f)}, \quad s_{B|e}(f') = \frac{s'(e, f')}{s_A(e)}$$

for all $e' \in \mathcal{E}(\mathcal{S}^{\mathcal{A}})$ and $f' \in \mathcal{E}(\mathcal{S}^{\mathcal{B}})$. The conditional states thus gives conditional probabilities for measurements when some measurement is performed on the other system.

We assume that these conditional states are indeed states so that for each $s \in \mathcal{S}^{\mathcal{AB}}$ we have that $s_{A|f} \in \mathcal{S}^{\mathcal{A}}$ and $s_{B|e} \in \mathcal{S}^{\mathcal{B}}$ for all $e \in \mathcal{E}(\mathcal{S}^{\mathcal{A}})$ and $f \in \mathcal{E}(\mathcal{S}^{\mathcal{B}})$. The state conditions $s_{A|f}(u_A) = 1$ and $s_{B|e}(u_B) = 1$ then imply that $s_A(e) = s'(e, u_B)$ and $s_B(f) = s'(u_A, f)$ for all effects e and f.

For each $s \in \mathcal{S}^{\mathcal{AB}}$ and $e \in \mathcal{E}(\mathcal{S}^{\mathcal{A}})$ we define $\hat{s} : \mathcal{E}(\mathcal{S}^{\mathcal{A}}) \to \mathbb{R}^{\mathcal{E}(\mathcal{S}^{\mathcal{B}})}$ such that

$$\hat{s}(e)(f) = s'(e, f)$$

for all $e \in \mathcal{E}(\mathcal{S}^{\mathcal{A}})$ and $f \in \mathcal{E}(\mathcal{S}^{\mathcal{B}})$. We then notice that $s_{B|e} = \hat{s}(e)/s_A(e)$ so that from the assumption $s_{B|e} \in \mathcal{S}^{\mathcal{B}}$ it follows that $\hat{s}(e) \in \mathcal{B}^{**}$ for each $e \in \mathcal{E}(\mathcal{S}^{\mathcal{A}})$. We can define the dual map $\hat{s}^*$ of $\hat{s}$ so that $\hat{s}^*(f)(e) = \hat{s}(e)(f)$. We note that then for each $f \in \mathcal{E}(\mathcal{S}^{\mathcal{B}})$ we have that $\hat{s}^*(f) = s_B(f) s_{A|f}$. From the conditional state assumption $s_{A|f} \in \mathcal{S}^{\mathcal{A}}$ it similarly follows that $\hat{s}^*(f) \in \mathcal{A}^{**}$ for each $f \in \mathcal{E}(\mathcal{S}^{\mathcal{B}})$.

Hence, $\hat{s}(e)$ is linear on $\mathcal{E}(\mathcal{S}^{\mathcal{B}})$ for each $e \in \mathcal{E}(\mathcal{S}^{\mathcal{A}})$ and $\hat{s}^*(f)$ is linear on $\mathcal{E}(\mathcal{S}^{\mathcal{A}})$ for each $f \in \mathcal{E}(\mathcal{S}^{\mathcal{B}})$ whenever the linear combination of effects is defined, i.e. it remains an effect on the respective effect space. This means that

$$s'\left(\sum_i \alpha_i e_i, f\right) = \sum_i \alpha_i s'(e_i, f), \quad s'\left(e, \sum_j \beta_j f_j\right) = \sum_j \beta_j s'(e, f_j),$$

for all $e, \sum_i \alpha_i e_i \in \mathcal{E}(\mathcal{S}^{\mathcal{A}})$ and $f, \sum_j \beta_j f_j \in \mathcal{E}(\mathcal{S}^{\mathcal{B}})$. Thus, s' is a bilinear form on $\mathcal{E}(\mathcal{S}^{\mathcal{A}}) \times \mathcal{E}(\mathcal{S}^{\mathcal{B}})$. However, since $\mathcal{E}(\mathcal{S}^{\mathcal{A}})$ and $\mathcal{E}(\mathcal{S}^{\mathcal{B}})$ span $\mathcal{A}^*$ and $\mathcal{B}^*$ respectively, the bilinear form s' can be uniquely extended to a bilinear form on whole $\mathcal{A}^* \times \mathcal{B}^*$. We denote this extension again by s'. For each state $s \in \mathcal{S}^{\mathcal{AB}}$ the composition s' thus uniquely extends to a bilinear form on $\mathcal{A}^* \times \mathcal{B}^*$. Since this holds for all states s, which are elements of $\mathcal{AB}^{**}$, it follows that π (or its extension) is bilinear on $\mathcal{A}^* \times \mathcal{B}^*$.

Let us then consider the space $\mathcal{A}^* \otimes \mathcal{B}^*$. Since $\pi \in \mathcal{L}(\mathcal{A}^*, \mathcal{B}^*; \mathcal{AB}^*)$, by Prop. 3.3 there exists a unique linear function $h : \mathcal{A}^* \otimes \mathcal{B}^* \to \mathcal{AB}^*$ such that $\pi = h \circ \tau$, where τ is the canonical mapping of $\mathcal{A}^* \otimes \mathcal{B}^*$. Thus, instead of $\pi(a, b)$ we may write

$h(a \otimes b)$ for all $a \in \mathcal{A}^*$ and $b \in \mathcal{B}^*$. The dual map h^* of h is defined by $h^*(z) = z \circ h$ for all $z \in \mathcal{AB}^{**}$. It remains to show that h is the required isomorphism by showing the injectivity of h and h^*.

In order to prove the injectivity of h we note that since elements of the form $e \otimes f \in \mathcal{A}^* \otimes \mathcal{B}^*$, where $e \in \mathcal{E}(\mathcal{S}^\mathcal{A})$ and $f \in \mathcal{E}(\mathcal{S}^\mathcal{B})$, span $\mathcal{A}^* \otimes \mathcal{B}^*$ it suffices to consider only these product effects. Suppose $h(e \otimes f) = 0$ for some $e \otimes f \in \mathcal{A}^* \otimes \mathcal{B}^*$, where e and f are effects on the respective spaces. Then for all pairs of states $(s,t) \in \mathcal{S}^\mathcal{A} \times \mathcal{S}^\mathcal{B}$ there exists a state $s_{AB} \in \mathcal{S}^{\mathcal{AB}}$ such that $0 = s_{AB}(h(e \otimes f)) = s(e)t(f)$. Since this holds for all s, t it follows that $e \otimes f = 0$ and so h is injective.

So far we have not used the local tomography principle. We see that now the local tomography principle is equivalent with the injectivity of h^*, or equivalently, the surjectivity of h. Suppose $s_1 \circ h = h^*(s_1) = h^*(s_2) = s_2 \circ h$ for some states $s_1, s_2 \in \mathcal{S}^{\mathcal{AB}}$. Then we have the following chain of equivalences and implications:

$$\begin{aligned} h^*(s_1) &= h^*(s_2) \\ \Leftrightarrow \quad s_1(h(a \otimes b)) &= s_2(h(a \otimes b)) \quad \forall a \otimes b \in \mathcal{A}^* \otimes \mathcal{B}^* \\ \Leftrightarrow \quad s_1(\pi(a,b)) &= s_2(\pi(a,b)) \quad \forall (a,b) \in \mathcal{A}^* \times \mathcal{B}^* \\ \Leftrightarrow \quad s_1(\pi(e,f)) &= s_2(\pi(e,f)) \quad \forall (e,f) \in \mathcal{E}(\mathcal{S}^\mathcal{A}) \times \mathcal{E}(\mathcal{S}^\mathcal{B}) \\ \Rightarrow \quad s_1 = s_2, & \end{aligned}$$

where the last implication follows from the local tomography principle. Since $\mathcal{AB}_+$ is generating and the states form a base for the positive cone, h^* is injective on all of $\mathcal{AB}$. Hence, h is a linear bijection so that $\mathcal{A}^* \otimes \mathcal{B}^* \cong \mathcal{AB}^*$ from which it follows that we can consider our composite state space $\mathcal{S}^{\mathcal{AB}}$ as a subset of $\mathcal{A} \otimes \mathcal{B}$.

We can now identify $\pi(e,f) \in \mathcal{E}(\mathcal{S}^{\mathcal{AB}})$ for each pair $(e,f) \in \mathcal{E}(\mathcal{S}^\mathcal{A}) \times \mathcal{E}(\mathcal{S}^\mathcal{B})$ with $e \otimes f \in \mathcal{A}^* \otimes \mathcal{B}^*$. Then also $u_A \otimes u_B$ corresponds to the order unit $u_{AB} \in \mathcal{AB}^*$. Similarly a state $S \in \mathcal{S}^{\mathcal{AB}}$ corresponding to a pair $(s,t) \in \mathcal{S}^\mathcal{A} \times \mathcal{S}^\mathcal{B}$ of independently prepared states can be identified with a product state $s \otimes t \in \mathcal{A} \otimes \mathcal{B}$ so that $(e \otimes f)(s \otimes t) = e(s)f(t)$.

What remains in order to characterize the state space is to find the positive cone $\mathcal{AB}_+$ for $\mathcal{AB}$. The requirements now are that each product state $s \otimes t$ belongs to $\mathcal{AB}_+$ and each product effect $e \otimes f$ belongs to the dual cone $\mathcal{AB}_+^*$. We note that this is then just requiring that $\mathcal{AB}_+$ is a proper tensor cone so that $\mathcal{A} \otimes \mathcal{B}$ is an ordered tensor product of $\mathcal{A}$ and $\mathcal{B}$. However, as we saw in Chapter I, this kind of positive cone is highly non-unique. Thus, the state space $\mathcal{S}^{\mathcal{AB}}$ is not completely determined by the local state spaces $\mathcal{S}^\mathcal{A}$ and $\mathcal{S}^\mathcal{B}$.

What we also saw in Chapter I was that there are two limiting cases for the positive cone $\mathcal{AB}_+$, namely the minimal and the maximal tensor cones. Thus, we can define the *maximal state space* $\mathcal{S}^{\mathcal{A}} \otimes_{\max} \mathcal{S}^{\mathcal{B}}$ by

$$\mathcal{S}^{\mathcal{A}} \otimes_{\max} \mathcal{S}^{\mathcal{B}} = \{s \in (\mathcal{A} \otimes_{\max} \mathcal{B})_+ \,|\, (u_A \otimes u_B)(s) = 1\}$$

and the *minimal state space* $\mathcal{S}^{\mathcal{A}} \otimes_{\min} \mathcal{S}^{\mathcal{B}}$ by

$$\mathcal{S}^{\mathcal{A}} \otimes_{\min} \mathcal{S}^{\mathcal{B}} = \{s \in (\mathcal{A} \otimes_{\min} \mathcal{B})_+ \,|\, (u_A \otimes u_B)(s) = 1\}.$$

Since every positive tensor cone lies between the maximal and minimal cones, every joint state space $\mathcal{S}^{\mathcal{AB}}$ of $\mathcal{S}^{\mathcal{A}}$ and $\mathcal{S}^{\mathcal{B}}$ lies between the maximal and minimal state spaces.

We note that every element $s \in (\mathcal{A} \otimes_{\min} \mathcal{B})_+$ can be expressed as

$$s = \sum_i \mu_i a_i \otimes b_i = \sum_i \alpha_i \beta_i \mu_i s_i^{\mathcal{A}} \otimes s_i^{\mathcal{B}}$$

where $\mu_i \geq 0$, $a_i = \alpha_i s_i^{\mathcal{A}} \in \mathcal{A}_+$ and $b_i = \beta_i s_i^{\mathcal{B}} \in \mathcal{B}_+$ for some $\alpha_i, \beta_i > 0$, $s_i^{\mathcal{A}} \in \mathcal{S}^{\mathcal{A}}$ and $s_i^{\mathcal{B}} \in \mathcal{S}^{\mathcal{B}}$ for all i. If we denote $\lambda_i = \mu_i \alpha_i \beta_i$ for all i then the normalization condition $(u_A \otimes u_B)(s) = 1$ becomes $\sum_i \lambda_i = 1$ so that

$$\mathcal{S}^{\mathcal{A}} \otimes_{\min} \mathcal{S}^{\mathcal{B}} = \left\{ \sum_i \lambda_i s_i^A \otimes s_i^B \,|\, \sum_i \lambda_i = 1,\ s_i^A \in \mathcal{S}^{\mathcal{A}},\ s_i^B \in \mathcal{S}^{\mathcal{B}} \ \ \forall i \right\}.$$

This then agrees the minimal requirement that the joint state space should include all convex combinations of product states.

We can also express $\mathcal{S}^{\mathcal{A}} \otimes_{\max} \mathcal{S}^{\mathcal{B}}$ explicitly as

$$\begin{aligned} \mathcal{S}^{\mathcal{A}} \otimes_{\max} \mathcal{S}^{\mathcal{B}} \;\; = \;\; & \{s \in \mathcal{A} \otimes \mathcal{B} \,|\, (u_A \otimes u_B)(s) = 1, \\ & \qquad (a \otimes b)(s) \geq 0 \ \ \forall a \in \mathcal{A}_+^*,\ b \in \mathcal{B}_+^*\}. \end{aligned}$$

The maximal state space is the largest set of states that respect the non-signalling condition. Finally we can define our state space for a composite system.

Definition 8.1. Let $\mathcal{S}^{\mathcal{A}}$ and $\mathcal{S}^{\mathcal{B}}$ be state spaces of systems A and B with ordered vector spaces $\mathcal{A}$ and $\mathcal{B}$ respectively. If $\mathcal{C}_t$ is any proper tensor cone in $\mathcal{A} \otimes \mathcal{B}$ satisfying $\mathcal{C}_{\min} \subset \mathcal{C}_t \subset \mathcal{C}_{\max}$ then

$$\mathcal{S}^{\mathcal{A}} \otimes \mathcal{S}^{\mathcal{B}} := \{s \in \mathcal{C}_t \,|\, (u_A \otimes u_B)(s) = 1\}$$

is a *composite state space* of system $A + B$.

Thus, state space of a composite system $A + B$ is any ordered tensor product lying between the minimal and maximal tensor products and normalized by the order unit $u_A \otimes u_B$. We also note that from the dualities (3.11) and (3.12) it follows that for the maximal state space then effect space is minimal and similarly for the minimal state space the effect space is maximal.

Since the convex combinations of product states are always contained in any composite, we make the following definition.

Definition 8.2. For a composite state space $\mathcal{S}^{\mathcal{A}} \otimes \mathcal{S}^{\mathcal{B}}$ of two state spaces $\mathcal{S}^{\mathcal{A}}$ and $\mathcal{S}^{\mathcal{B}}$ we say that a state $s \in \mathcal{S}^{\mathcal{A}} \otimes \mathcal{S}^{\mathcal{B}}$ is *separable* if it is in $\mathcal{S}^{\mathcal{A}} \otimes_{\min} \mathcal{S}^{\mathcal{B}}$ and *entangled* otherwise.

We further emphasize that for minimal state space there are entangled effects whereas for maximal state space all effects are separable. Any composite state space strictly between them has both entangled effects and states.

9 Applications

We can now consider some specific state spaces within convex operational theories. We can apply our framework to identify the operational concepts defined in the other sections of this chapter on these state spaces and see some special properties of these theories.

9.1 Quantum theory

The most important application of convex operational theories is the *quantum theory*. In this section we will see how quantum theory fits in the framework of convex theories. For the basic results in quantum theory that are presented here, see [42, 43, 53]. For quantum theory in the convex operational theories, see [29, 44]

Let $\mathcal{H}$ be a finite-dimensional Hilbert space and denote by $\mathcal{L}(\mathcal{H})$ the set of linear operators $A : \mathcal{H} \to \mathcal{H}$ and by $\mathcal{L}_s(\mathcal{H})$ the set of selfadjoint operators in $\mathcal{L}(\mathcal{H})$. We say that an operator $A \in \mathcal{L}(\mathcal{H})$ is positive iff $\langle \psi \,|\, A\psi \rangle \geq 0$ for all $\psi \in \mathcal{H}$, and the set of positive operators is denoted by $\mathcal{L}_+(\mathcal{H})$. It follows that every positive operator is necessarily selfadjoint so that $\mathcal{L}_+(\mathcal{H}) \subset \mathcal{L}_s(\mathcal{H})$. We define the trace norm $\|\cdot\|_1$ on $\mathcal{L}(\mathcal{H})$ by

$$\|A\|_1 = \mathrm{tr}\,[|A|]$$

for all $A \in \mathcal{L}(\mathcal{H})$, where $|A| = \sqrt{A^*A} \in \mathcal{L}_+(\mathcal{H})$ denotes the absolute value of A.

The set of states $\mathcal{S}(\mathcal{H})$ in finite-dimensional quantum theory is given by the set of positive trace-one linear operators on $\mathcal{H}$. Since all positive operators are also selfadjoint, we can further consider density matrices as a subset of $\mathcal{L}_s(\mathcal{H})$, so that

$$\mathcal{S}(\mathcal{H}) = \{\varrho \in \mathcal{L}_s(\mathcal{H}) \,|\, \varrho \geq 0,\ \mathrm{tr}\,[\varrho] = 1\}.$$

Selfadjoint operators on $\mathcal{H}$ form a real finite-dimensional vector space as every real linear combination of selfadjoint operators is selfadjoint. As the trace norm induces a Hausdorff topology, $\mathcal{L}_s(\mathcal{H})$ is a real Hausdorff topological vector space.

Clearly now $\mathcal{S}(\mathcal{H})$ is convex: all positive linear combinations of positive operators are positive and since trace is a linear functional, the trace of a mixture of states is one. $\mathcal{S}(\mathcal{H})$ is also bounded since $\|\varrho - \sigma\|_1 \leq 2$ for all $\varrho, \sigma \in \mathcal{S}(\mathcal{H})$. The set $\mathcal{L}_+(\mathcal{H})$ is closed in $\mathcal{L}_s(\mathcal{H})$, and since trace is a continuous function, the set of unit trace operators $\mathrm{tr}^{-1}(\{1\})$ is also closed so that $\mathcal{S}(\mathcal{H}) = \mathcal{L}_+(\mathcal{H}) \cap \mathrm{tr}^{-1}(\{1\})$ is closed. Hence, $\mathcal{S}(\mathcal{H})$ is a compact convex subset of $\mathcal{L}_s(\mathcal{H})$ and therefore is a suitable state space in the framework of convex operational theories.

We see that $\mathcal{L}_+(\mathcal{H})$ forms a convex cone. Furthermore, it is quite clearly a proper cone. Therefore $\mathcal{L}_+(\mathcal{H})$ induces a partial order in $\mathcal{L}_s(\mathcal{H})$ making it an ordered vector space. Since the trace is a strictly positive functional on $\mathcal{L}_+(\mathcal{H}) \setminus \{0\}$ we have by Prop. 2.14 that $\mathcal{S}(\mathcal{H})$ is a base for $\mathcal{L}_+(\mathcal{H})$. Every selfadjoint operator $A \in \mathcal{L}_s(\mathcal{H})$ can be written as a difference of two positive operators $A_+ = \frac{1}{2}(|A| + A)$ and $A_- = \frac{1}{2}(|A| + A)$ as $A = A_+ - A_-$ so that $\mathcal{L}_+(\mathcal{H})$ actually generates $\mathcal{L}_s(\mathcal{H})$. The trace is now an order unit in $\mathcal{L}_s(\mathcal{H})^*$.

Hence, for finite-dimensional quantum theory we have finite-dimensional real Hausdorff ordered vector space $\mathcal{A} = \mathcal{L}_s(\mathcal{H})$, a closed generating positive cone $\mathcal{A}_+ = \mathcal{L}_+(\mathcal{H})$ and an order unit $u = \mathrm{tr} \in \mathcal{L}_s(\mathcal{H})^*$ such that the state space $\mathcal{S}(\mathcal{H})$ is a compact base for $\mathcal{L}_+(\mathcal{H})$.

The effect space $\mathcal{E}(\mathcal{H})$ now reads as

$$\mathcal{E}(\mathcal{H}) = \{e \in \mathcal{L}_s(\mathcal{H})^* \,|\, o \leq e \leq \mathrm{tr}\}.$$

The duality between $\mathcal{L}(\mathcal{H})$ and its (complex) dual space is given by an isomorphism $A \mapsto f_A$, where $f_A(B) = \mathrm{tr}\,[AB]$ for all $B \in \mathcal{L}(\mathcal{H})$. It follows that for each $e \in \mathcal{L}(\mathcal{H})^*$ there exists a unique $E \in \mathcal{L}(\mathcal{H})$ such that $e(B) = \mathrm{tr}\,[EB]$ for all $B \in \mathcal{L}_s(\mathcal{H})$. Since $e(B) \in \mathbb{R}$ for all $B \in \mathcal{L}_s(\mathcal{H})$ we have that $E \in \mathcal{L}_s(\mathcal{H})$.

We see that $e \geq o$ is equivalent with the condition that $\mathrm{tr}\,[EB] \geq 0$ for all $B \in \mathcal{L}_+(\mathcal{H})$. Using the spectral decomposition for A we see that $e \geq o$ if and only if $E \geq O$. Similarly $e \leq \mathrm{tr}$ if and only if $E \leq I$, where I is the identity operator on $\mathcal{H}$.

Conversely, if $E \in \mathcal{L}_s(\mathcal{H})$ satisfies the previous operator inequalities, we see that $\mathrm{tr}\,[E\,\cdot\,]_{|\mathcal{L}_s(\mathcal{H})} \in \mathcal{E}(\mathcal{H})$. Hence, we can identify our effects with positive unit bounded selfadjoint operators, i.e.,

$$\mathcal{E}(\mathcal{H}) = \{E \in \mathcal{L}_s(\mathcal{H}) \,|\, O \leq E \leq I\}.$$

The effect operators form a convex set in $\mathcal{L}_s(\mathcal{H})$ with the set of projection operators $\mathcal{P}(\mathcal{H}) = \{P \in \mathcal{L}_s(\mathcal{H}) \,|\, P^2 = P\}$ as its set of extremal points. The complement of an effect $E \in \mathcal{E}(\mathcal{H})$ is now $I - E \in \mathcal{E}(\mathcal{H})$.

It is evident from the previous that now observable A with a finite outcome set $\Omega = \{x_1, \ldots, x_n\}$ is just a collection of effect operators $\mathsf{A}(x_i) \in \mathcal{E}(\mathcal{H})$, $i = 1, \ldots, n$, such that $\sum_{i=1}^{n} \mathsf{A}(x_i) = I$. Any trivial observable T on Ω now consists of operators $p(x_i)I$, where $p : \Omega \to [0, 1]$ is any probability distribution on Ω.

Observables with finite outcomes are just a special instance of observables and in quantum theory general observables can be described by so called *positive operator-valued measures* (POVMs). A POVM is a mapping $\mathsf{A} : \Sigma \to \mathcal{E}(\mathcal{H})$ from a σ-algebra of an outcome set Ω to the set of effects such that $\mathsf{A}(\emptyset) = 0$, $\mathsf{A}(\Omega) = 1$ and $\mathsf{A}(\cup_i X_i) = \sum_i \mathsf{A}(X_i)$ for any sequence of disjoints sets $\{X_i\}_i \subset \Sigma$. We note that in the case of a finite Ω the definition of a POVM A reduces to our previous definition of observables with finite number of outcomes.

Let $\mathcal{S}(\mathcal{H})$ and $\mathcal{S}(\mathcal{H}')$ be two state spaces with Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$. State transformations in quantum theory are described by quantum channels, i.e., *completely positive* trace-preserving (CPTP) linear maps $\Phi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}')$. Similarly we can define quantum operations as completely positive trace-non-increasing linear maps. Quantum channels (operations) are clearly channels (operations) also by our definition since completely positive maps are always positive.

We see that in quantum theory we have require the channels to be completely positive whereas in convex operational theories we did not consider this. The reason why channels are required to be completely positive lies on the structure of composite systems: if we pair a state of a system S with another state of an ancillary system A and consider the channel on the compound system $S + A$ such that the channel acts on the system S with identity channel acting on the ancillary system A, we still require the mapping to remain positive since all we did was introduce an ancillary system that did not even interact with the original system. It turns out that not all positive maps remain positive in this scenario so that a more stronger notion of positivity is needed, namely complete positivity.

However, in the framework of convex operational theories, in general one tends not to use the notion of complete positivity. Namely, in quantum theory the complete positivity is defined so that the positivity of the channel on the compound system is required to hold for all dimensions of the ancillary Hilbert space. In convex operational theory this kind of dependency on the size of the ancillary system is however not readily formulated.

Consider quantum systems A and B with a state spaces $\mathcal{S}(\mathcal{H}_A)$ and $\mathcal{S}(\mathcal{H}_B)$ respectively. The state space of the composite system $A+B$ is then build around the vector space $\mathcal{H}_A \otimes \mathcal{H}_B$ with an inner product defined on the product elements by

$$\langle\, \psi_A \otimes \psi_B \,|\, \varphi_A \otimes \varphi_B \,\rangle = \langle\, \psi_A \,|\, \varphi_A \,\rangle \,\langle\, \psi_B \,|\, \varphi_B \,\rangle\,,$$

which can then be extended to the whole space by linearity. Thus, in the finite-dimensional case, the inner product space $\mathcal{H}_A \otimes \mathcal{H}_B$ is also a Hilbert space so that we take the composite state space to be $\mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$.

We see that the composite state space of two quantum systems is neither maximal nor minimal. Indeed, it is easy to check that all separable states in $\mathcal{L}_s(\mathcal{H}_A)\otimes\mathcal{L}_s(\mathcal{H}_B)$ are included in $\mathcal{S}(\mathcal{H}_A\otimes\mathcal{H}_B)$ but on the other hand we have entangled states as well. An example of a positive trace-one operator on $\mathcal{H}_A\otimes\mathcal{H}_B$, $d = \dim(\mathcal{H}_A) \leq \dim(\mathcal{H}_B)$, is the maximally entangled state $|\psi_+\rangle\langle\psi_+|$, where

$$\psi_+ = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} \varphi_i^A \otimes \varphi_i^B$$

for an orthonormal basis $\{\varphi_i^A\}_i$ for $\mathcal{H}_A$ and an orthonormal subset $\{\varphi_i^B\}_i$ of $\mathcal{H}_B$. Similarly we have that all entangled states in $\mathcal{S}(\mathcal{H}_A\otimes\mathcal{H}_B)$ are included in $(\mathcal{L}_s(\mathcal{H}_A)\otimes_{\max} \mathcal{L}_s(\mathcal{H}_B))_+$ but we also have operators in the latter set that are positive only on all separable effects (such "states" are called entanglement witnesses) so that they are not positive in $\mathcal{L}_s(\mathcal{H}_A \otimes \mathcal{H}_B)$. Thus, in quantum theory we have that

$$\mathcal{S}^{\mathcal{L}_s(\mathcal{H}_A)} \otimes_{\min} \mathcal{S}^{\mathcal{L}_s(\mathcal{H}_B)} \subset \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B) \subset \mathcal{S}^{\mathcal{L}_s(\mathcal{H}_A)} \otimes_{\max} \mathcal{S}^{\mathcal{L}_s(\mathcal{H}_B)},$$

where the inclusions are strict.

9.2 Quantum theory of processes

The power of the general framework of convex operational theories can be seen when considering quantum channels. We will see that instead of quantum states, we can take the system of interest to be the set of quantum channels and we can consider the properties of the channels in this framework similarly to any other theory. For the state space consisting of quantum channels, we call the operational theory as *quantum theory of processess*. Here we will only focus on states, effects and observables of quantum theory of processes. The material presented here is found in [16, 54, 55].

For a fixed basis of $\mathcal{H}_A$ there exists an isomorphism between linear maps $\mathcal{E}$: $\mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$ and operators on $\mathcal{L}(\mathcal{H}_B \otimes \mathcal{H}_A)$ given by

$$\mathcal{E} \mapsto \Phi_{\mathcal{E}} = (\mathcal{E} \otimes \mathcal{I}_A)(\Psi_+),$$

where $\mathcal{I}_A$ is the identity channel on $\mathcal{L}(\mathcal{H}_A)$ and Ψ_+ is the unnormalized maximally entangled state in $\mathcal{S}(\mathcal{H}_A)$, $\Psi_+ = d|\psi_+\rangle\langle\psi_+|$. This well known result is called the *Choi-Jamiolkowksi isomorphism* (see [56] for proof) and the operator $\Phi_{\mathcal{E}}$ as the *Choi operator* of $\mathcal{E}$.

It is known that the linear map $\mathcal{E}$ is completely positive if and only if the Choi operator of $\mathcal{E}$ is positive [56]. Furthermore, $\mathcal{E}$ is trace-preserving if and only if $\mathrm{tr}_B\,[\Phi_{\mathcal{E}}] = I_A$ [16]. Thus, if $\mathcal{E}$ is a quantum channel, this means that for the corresponding Choi operator $\Phi_{\mathcal{E}}$ we have that $\frac{1}{d}\Phi_{\mathcal{E}} \in \mathcal{S}(\mathcal{H}_B \otimes \mathcal{H}_A)$. As any mixture of quantum channels is a channel, the set of channels form a convex subset of $\mathcal{S}(\mathcal{H}_B \otimes \mathcal{H}_A)$. This motivates us to consider channels as our systems of interest so that we can consider channels as states in the framework of convex operational theories.

Let us denote the set of quantum channels from $\mathcal{L}(\mathcal{H}_A)$ to $\mathcal{L}(\mathcal{H}_B)$ by $\mathcal{C}(\mathcal{H}_A, \mathcal{H}_B)$. Thus, we consider the state space $\mathcal{S} = \mathcal{C}(\mathcal{H}_A, \mathcal{H}_B)$. How about effects and observables? We will first construct a reasonable measurement set-up for channels inside quantum theory and then consider how it is related to the observables in convex operational theories.

Consider an unknown quantum channel $\mathcal{E} \in \mathcal{C}(\mathcal{H}_A, \mathcal{H}_B)$. The measurement $\mathcal{M}$ of the channel can be composed of three steps: preparing a test state in a test system that is composed of the input system of the channel and an ancilla, applying the channel on the input system and an identity channel on the ancillary system, and measuring the transformed test state by an observable. Thus, the measurement is specified by the Hilbert space $\mathcal{H}_{\mathrm{anc}}$, the test state $\Psi \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_{\mathrm{anc}})$ and the observable $\mathsf{M} \in \mathcal{O}(\mathcal{S}(\mathcal{H}_B \otimes \mathcal{H}_{\mathrm{anc}}), \Omega)$ with some outcome set Ω.

With this set-up $\mathcal{M}$ for the channel $\mathcal{E}$, the probability $p_x(\mathcal{E}, \{\Psi, \mathsf{M}\})$ that when measuring M we get outcome $x \in \Omega$ when the test system is in the state Ψ can be expressed as

$$p_x(\mathcal{E}, \{\Psi, \mathsf{M}\}) = \mathrm{tr}\,[(\mathcal{E} \otimes \mathcal{I}_{\mathrm{anc}})(\Psi)\mathsf{M}_x]\,.$$

We can define a CP linear map $\mathcal{I}_A \otimes \mathcal{R}_\Psi$, where $\mathcal{R}_\Psi : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_{\mathrm{anc}})$, such that

$\mathcal{I}_A \otimes \mathcal{R}_\Psi$ takes Ψ_+ to Ψ so that by considering the dual map we have that

$$p_x(\mathcal{M}, \mathcal{E}) = \mathrm{tr}\left[(\mathcal{E} \otimes \mathcal{I}_A)(\Psi_+)(\mathcal{I}_B \otimes \mathcal{R}_\Psi^*)(\mathsf{M}_x)\right] = \mathrm{tr}\left[\Phi_\mathcal{E} \mathcal{M}_x\right],$$

where we have defined the operators $\mathcal{M}_x = (\mathcal{I}_B \otimes \mathcal{R}_\Psi^*)(\mathsf{M}_x)$ on $\mathcal{H}_B \otimes \mathcal{H}_A$. These operators are called *channel effects* and they form a *process positive operator measure* (PPOVM). The PPOVM elements are seen to be positive and they satisfy a normalization $\sum_x \mathcal{M}_x = I_B \otimes \varrho_A$, where $\varrho_A \in \mathcal{S}(\mathcal{H}_A)$. It has been shown [54] that every collection of operators satisfying those properties forms a valid PPOVM with some ancillary system, test state and POVM.

Similarly to POVMs, we note that the measurement $\mathcal{M}$ can be expressed as a mapping $\mathcal{M} : x \mapsto \mathcal{M}_x$ the set of outcomes to the PPOVM elements. Furthermore, each $\mathcal{M}_x$ can be expressed as an affine mapping $\mathcal{E} \mapsto p_x(\mathcal{E}, \{\Psi, \mathsf{M}\}) \in [0, 1]$. Thus, the PPOVM elements are effects in the framework of convex operational theories.

However, unlike in the standard quantum theory, the affine maps do not have a one-to-one correspondence with the PPOVM elements as two PPOVMs $\mathcal{M}$ and $\mathcal{R}$ are equivalent if and only if for all outcomes x we have that $\mathcal{M}_x - \mathcal{R}_x = I_B \otimes A_x$ for some $A_x \in \mathcal{L}(\mathcal{H}_A)$ such that $\mathrm{tr}\left[A_x\right] = 0$ [57]. Nevertheless, all admissible mappings $\mathcal{E} \mapsto p_x(\mathcal{E})$ from channels to probabilities can be implemented by some PPOVM so that all effects have some corresponding PPOVM element for some PPOVM.

9.3 Polytope theories

A class of state spaces are formed when there are only finite number of pure states. This is the case if and only if the state space is a polytope.

Definition 9.1. A state space $\mathcal{S}$ is called a *polytope state space* if $\mathcal{S}$ is a polytope.

As a particular instance of polytope state spaces we consider the polygon state spaces.

Definition 9.2. A state space $\mathcal{S}$ is called a *polygon state space* if $\mathcal{S}$ is isomorphic to a regular polygon in $\mathbb{R}^2$.

We consider polygon state spaces parameterized (almost) as in [58] so that they are embedded in $\mathbb{R}^3$ and lying on the $z = 1$ plane. A polygon state space $\mathcal{S}_n$ with n number of vertices is then given by the convex hull of pure states

$$s_k = \begin{pmatrix} r_n \cos\left(\frac{2k\pi}{n}\right) \\ r_n \sin\left(\frac{2k\pi}{n}\right) \\ 1 \end{pmatrix}, \quad k \in I_n \tag{9.1}$$

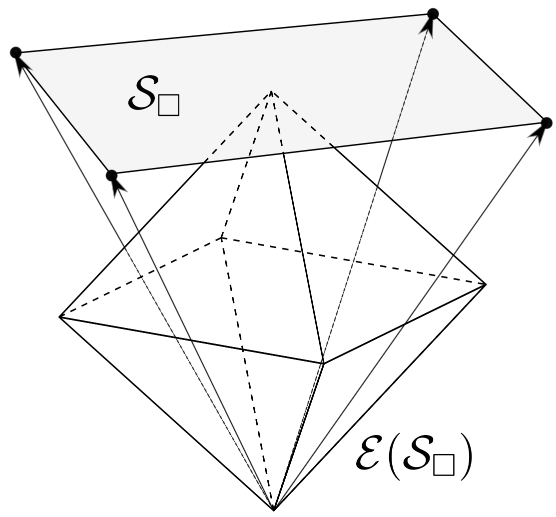

Figure 8: Square state space and its effect space.

where we have denoted $r_n = \sec\left(\frac{\pi}{n}\right)$ and $I_n = \{1, \ldots, n\} \subset \mathbb{N}$.

The polygons are two-dimensional so that the effects can be represented by linear functionals on $\mathbb{R}^3$. If we define $e(s) = e \cdot s$ as the Euclidean dot product we see that the effects are elements in $\mathbb{R}^3$. Hence, we can express each $e \in \mathcal{E}(\mathcal{S}_n)$ as $e = \left(\vec{a}, \frac{1}{2} - \alpha\right)^T \in \mathbb{R}^3$, where $\vec{a} = (a_x, a_y)$.

In the case when n is even, the effect space $\mathcal{E}(\mathcal{S}_n)$ is given by a convex hull of the non-trivial extremal effects

$$e_k = \frac{1}{2}\begin{pmatrix} \cos\left(\dfrac{(2k-1)\pi}{n}\right) \\ \sin\left(\dfrac{(2k-1)\pi}{n}\right) \\ 1 \end{pmatrix}, \quad k \in I_n \tag{9.2}$$

together with the unit effect $u = (0,0,1)^T$ and the zero effect $o = (0,0,0)^T$. We note that the convex hull of the non-trivial effects also forms a regular n-polygon in the $z = \frac{1}{2}$ plane. Fig. 8 represents the the case when $n = 4$ so that the state space $\mathcal{S}_\square \equiv \mathcal{S}_4$ is the square state space.

On the other hand, when n is odd, we define

$$f_k = \frac{1}{1+r_n}\begin{pmatrix} \cos\left(\dfrac{(2k-1)\pi}{n}\right) \\ \sin\left(\dfrac{(2k-1)\pi}{n}\right) \\ 1 \end{pmatrix}, \quad k \in I_n \tag{9.3}$$

and see that the set of extremal effects is then $\{o, u, f_1, \ldots, f_n, u - f_1, \ldots, u - f_n\}$. In this case we note that the the non-trivial extremal effects are not on a single plane any more.

In both cases the observables are then naturally composed of effects that sum to the unit effect u. Similarly operations, channels and composite systems can be constructed in polygon theories accordingly. Here we will only focus on states and effects. We only note that with some composites (for example the maximal state space of two square spaces) we can form so called PR-boxes [59] where states possess correlations stronger than any entangled quantum states.

Let us consider the even polygons more closely. For that, let us define $E_n : \mathbb{R}^2 \to \mathbb{R}$ by

$$E_n(\vec{a}) = \max_{k \in I_n} \left[r_n \left(\cos\left(\frac{2k\pi}{n}\right) a_x + \sin\left(\frac{2k\pi}{n}\right) a_y \right) \right]. \tag{9.4}$$

and $S_n : \mathbb{R}^2 \to \mathbb{R}$ (as a rescaling and rotated version of E_n) by

$$S_n(\vec{r}) = \max_{k \in I_n} \left[\cos\left(\frac{(2k-1)\pi}{n}\right) r_x + \sin\left(\frac{(2k-1)\pi}{n}\right) r_y \right]. \tag{9.5}$$

When n is even so that the state space is symmetric, it is straightforward to check that E_n and S_n define norms on $\mathbb{R}^2$. We call them the *effect space norm* and *state space norm* respectively.

If we consider the effect condition $0 \leq e(s_k) \leq 1$ on the extremal states $\{s_k\}_{k \in I_n}$, we see that $e = \left(\vec{a}, \frac{1}{2} - \alpha\right)^T$ is an effect if and only if

$$E_n(\vec{a}) + |\alpha| \leq \frac{1}{2}. \tag{9.6}$$

Now non-trivial extremal effects have $\alpha = 0$ so that $E_n(\vec{a}) \leq \frac{1}{2}$ gives a regular polygon in the $z = \frac{1}{2}$ plane.

By the duality of states and effects we can instead consider states acting on effects. Thus, we see that the condition for a vector $s = (\vec{r}, 1) \in \mathbb{R}^3$, where $\vec{r} = (r_x, r_y) \in \mathbb{R}^2$, to be a state is $0 \leq s \cdot e_k \leq 1$ for all $k \in I_n$. With the state space norm we see that $s = (\vec{r}, 1)$ is a state if and only if

$$S_n(\vec{r}) \leq 1. \tag{9.7}$$

We see that the state norm induces the polygon state space as its unit ball. In the odd polygon case we note that S_n does similarly determine the state space as its unit ball, but in this case S_n is not a norm as is it is not homogeneous.

9.4 Classical theories

Let us consider classical theories where the state of the system is identified with a point in phase space, that is, a multidimensional space where each degree of freedom of the system is represented by an axis in the space [60]. Then the state of the system is specified by the coordinates of the state space. For example in classical mechanics, the state of an object moving through 3-dimensional space can be specified by knowing its position and momentum which then correspond to a point in the phase space $\mathbb{R}^3 \times \mathbb{R}^3$.

Considering the phase space as the set of states works fine for simple systems where the phase space coordinates are easily determined. However, if we have a more complex system, for example we have a large number of systems, it is in general impossible to determine the exact coordinates of the phase space for the system. In this case one only considers the statistical ensemble of states and focuses on the statistical properties of the system. Then all we can do is to give a probability that the state is some part of the phase space. Hence, the notion of states must be extended to include all probability distributions on the phase space [60].

For simplicity, we consider the case when the phase space is finite. Let us consider the set of probability distributions on some finite phase space Ω. Without loss of generality we take $\Omega = \{1, \dots, n\}$ so that we can express each probability distribution p on Ω as a vector $\vec{p} = (p_1, \dots, p_n)^T \in \mathbb{R}^n$, where $p_i := p(i)$ for all $i = 1, \dots, n$ [43]. Now $p_i \in [0, 1]$ for all $i = 1, \dots, n$ and $\sum_{i=1}^n p_i = 1$. With this identification, it is straightforward to check that the set of probability distributions is a compact convex subset of $\mathbb{R}^n$ so that we may consider it as a state space, denoted by $\mathcal{S}_C$, for a convex operational theory.

Let $\{\vec{e}_i\}_{i=1}^n$ denote the standard basis for $\mathbb{R}^n$ so that $\vec{e}_i$ is the ith column of the identity matrix on $\mathbb{R}^n$. We then have that

$$\vec{p} = \sum_{i=1}^{n} p_i \vec{e}_i.$$

Clearly now every basis element $\vec{e}_i$ corresponds to a probability distribution

$$\delta_i(j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases}$$

that is called the *Dirac measure* on $i \in \Omega$.

The Dirac measures form the set of extremal elements of $\mathcal{S}_C$. As the corresponding vectors $\{\vec{e}_i\}_{i=1}^n$ form a basis for $\mathbb{R}^n$, the decomposition of a probability measure p into extremal elements is unique so that by Prop. 1.16 the state space $\mathcal{S}_C$ is necessarily a simplex. Hence we make the following definition.

Definition 9.3. A state space $\mathcal{S}_C$ is called *classical* if $\mathcal{S}_C$ is a simplex.

The effects on a classical state space $\mathcal{S}_C$ can be identified with vectors $\vec{f} = (f_1, \ldots, f_n)^T \in \mathbb{R}^n$ where $f_i \in [0,1]$ for all $i = 1, \ldots, n$ so that the probability of measuring $\vec{f}$ when the system is in the state $\vec{p}$ is $f(p) = \vec{f} \cdot \vec{p} = \sum_{i=1}^n f_i p_i$. The effect space has 2^n extremal effects of the form $\left(\frac{1}{2} \pm \frac{1}{2}, \ldots, \frac{1}{2} \pm \frac{1}{2}\right)^T \in \mathbb{R}^n$ [43]. An observable on $\mathcal{S}_C$ is composed of effect vectors $\{\vec{f_j}\}_j$ such that $\sum_j \vec{f_j} = (1, \ldots, 1)^T \in \mathbb{R}^n$.

Channels on $\mathcal{S}_C$ are exactly the right stochastic matrices $\nu : \Omega \to \Lambda$ from a phase space $\Omega = \{1, \ldots, n\}$ to another phase space $\Lambda = \{1, \ldots, m\}$ that were considered in post-processings of observables. For a state $\vec{p}$ we have that $\nu : \vec{p} \mapsto \vec{p'}$, where

$$p'_j = \sum_{i=1}^n \nu_{ij} p_i$$

for all $j = 1, \ldots, m$. Indeed, $p'_j \in [0,1]$ for all $j = 1, \ldots, m$ and

$$\sum_{j=1}^m p'_j = \sum_{i=1}^n \sum_{j=1}^m \nu_{ij} p_i = \sum_{i=1}^n p_i = 1.$$

Thus, ν is a channel. As linear maps between $\mathbb{R}^n$ and $\mathbb{R}^m$ are just matrices, the only maps preserving the normalization of states are the stochastic matrices.

It has been shown that a state space $\mathcal{S}$ is simplex if and only if the maximal and the minimal state spaces coincide when $\mathcal{S}$ is composed with any other state space [41]. Thus, when we consider a composite system of classical state spaces, the joint state space is always fixed. Furthermore we see that if the maximal and minimal state spaces of the composite system coincide, then one of the subsystems must be classical.

Chapter III

Non-classical features of quantum theory

Quantum theory is known to hold features that we do not experience in classical theories. We saw for example that the composite systems of classical state spaces do not contain any entangled states since the tensor product of the composite state space was fixed to be the minimal one. Classical theories are said to be *local* whereas quantum theory is a *nonlocal theory*. This is because with the minimal composite state space, the outcome probability distributions of local measurements on the composite system are not correlated whereas in quantum theory the correlations can be even experimentally verified [61].

However, in Chapter II we saw that we can construct theories that are neither classical nor quantum but that nevertheless contain entangled states as well: for any state space we can fix the tensor product to be the maximal one, then the composite system always contains entangled states. Thus, entanglement is not strictly quantum feature but in fact in fact generic amongst the non-classical theories. In fact, we can construct theories that are even more nonlocal than quantum theory [62]; an example of which is the PR-box that was briefly mentioned in the previous chapter.

In the present chapter we will focus on some of the non-classical features of quantum theory. In particular we will consider a few task-type features where we ask if some operational task can be completed in convex operational theories and if it can, then we wish to see in which type of theories. Such tasks include *cloning* [9, 63], *distinguishability of states* [64, 65], *broadcasting* [9, 63] and *joint measurability* (or *compatibility*) of observables [17, 50, 66] and they have been extensively studied in quantum theory [11, 67–71].

We will fill formulate the tasks in the framework of convex operational theories and study see which theories these tasks can be completed in. We find that the tasks presented here are actually all classical in the sense that in non-classical theories we can formulate a no-go theorem for each of these tasks. Hence, no-cloning, no-broadcasting and incompatibility are generic features of all non-classical theories. Furthermore we will see how closely these task are connected.

As a part of the task of joint measurability of observables, original research on conditions of compatibility in convex operational theories is presented [17]. We will introduce the notion of noise content of an observable to see how much intrinsic noise is present in the observable and use it to formulate an inequality that serves as a sufficient criteria for compatibility. We see that the criteria takes different shapes in different theories.

Other tasks that are not considered here contain for example *teleportation* [10] and *steering* [47, 72]. Although these tasks can be accomplished in quantum theory, they can be shown to be non-generic in the convex operational theories such that there exists even non-classical theories where these tasks cannot be accomplished. In the present work we will limit ourselves outside of these tasks.

We start our investigation on the tasks of cloning and distinguishability of states.

10 Cloning

Consider a process where one takes a state of a state space $\mathcal{S}$ and makes two identical copies of it. This process is called *cloning* of the state and the physical device realising this process is the *cloning device*. If we consider the state to be unknown, then the cloning device must be applicable for any state so that it clones every state of $\mathcal{S}$. In this case we say that the device is a *universal cloning device* and that $\mathcal{S}$ admits *universal cloning*. Instead of universal cloning we may also consider cloning devices that clone some particular subset S of $\mathcal{S}$. In this case we say that the cloning device *clones* S. As a particular instance we may consider the cloning of pure states of $\mathcal{S}$ resulting in *universal cloning of the pure states*. We follow the works done in [9, 65]

In the framework of convex operational theories processing of states is handled with operations. After the cloning we end up with two clones of the state so that the output state space of the operation is to be considered the composite state space $\mathcal{S} \otimes \mathcal{S}$ with some tensor product. As the operation then transforms states into states, the operation is actually a channel. Hence we arrive at the following definition.

Definition 10.1. A finite set of states $S = \{s_j\}_{j=1}^n \subset \mathcal{S}$ in a state space $\mathcal{S}^{\mathcal{A}}$ is called *clonable* if there exists a channel $\tau : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ such that $\tau(s_j) = s_j \otimes s_j$ for all $j = 1, \ldots, n$. If S is clonable by τ, then we say that τ *clones* S.

Thus if a channel τ clones $\mathcal{S}^{\mathcal{A}}$, the state space admits universal cloning. We will see that cloning is closely connected to another operational task, namely distinguishability of states.

10.1 Distinguishability of states

Consider we have a set of states such that the states that the set is composed of are known to us. As a task we can ask if we have any means of telling them apart. Since states are observed by making measurements, the task is to find a measurement such that the outcome of the measurement tells with certainty which state was measured. This task is known as *distinguishability of states*.

Definition 10.2. A finite set of states $S = \{s_j\}_{j=1}^n \subset \mathcal{S}^{\mathcal{A}}$ in a state space $\mathcal{S}^{\mathcal{A}}$ is called *(jointly) distinguishable* if there exists an observable $\mathsf{A} \in \mathcal{O}(\mathcal{S}^{\mathcal{A}}, \Omega)$ with an outcome set $\Omega \cong \{1, \ldots, n\}$ such that $\mathsf{A}_i(s_j) = \delta_{ij}$ for all $s_j \in S$. Then A is called *distinguishing* for S.

Let us start the examination of distinguishability with the classical case. For classical state spaces we see that all the pure states of the state space can be distinguished [29, Prop. 3.31].

Proposition 10.3. *Let $\mathcal{S}^{\mathcal{A}}$ be a state space with $\dim(\mathcal{A}) = n$. There are n jointly distinguishable states $\{s_1, \ldots, s_n\} \subset \mathcal{S}^{\mathcal{A}}$ if and only if $\mathcal{S}^{\mathcal{A}}$ is a simplex with $\mathrm{ext}(\mathcal{S}^{\mathcal{A}}) = \{s_1, \ldots, s_n\}$.*

Proof. We recall from Chapter II that by our construction of the state space, $\mathcal{S}^{\mathcal{A}}$ lies in a n-dimensional ordered vector space $\mathcal{A}$ such that $\mathrm{aff}(\mathcal{S}^{\mathcal{A}})$ is an affine hyperplane in $\mathcal{A}$ with $0 \notin \mathrm{aff}(\mathcal{S}^{\mathcal{A}})$.

Let first $\mathcal{S}^{\mathcal{A}}$ be a $(n-1)$-simplex with extremal states $\mathrm{ext}(\mathcal{S}) = \{s_1, \ldots, s_n\}$. It follows that $\mathrm{ext}(\mathcal{S})$ must be linearly independent set since otherwise we could use them to form an affine decomposition for the zero vector so that $0 \in \mathrm{aff}(\mathcal{S})$ which would be a contradiction. Thus, $\mathrm{ext}(\mathcal{S})$ forms a basis for $\mathcal{A}$. By construction, the dual basis $\{e_i\}_{i=1}^n \subset \mathcal{A}^*$ of $\mathrm{ext}(\mathcal{S})$ satisfies

$$e_i(s_j) = \delta_{ij}.$$

For all $i = 1, \ldots, n$ we have that $e_i \geq o$. This follows from $e_i(x) = \alpha\lambda_i \geq 0$, where $x = \alpha s \in \mathcal{A}_+$ is the unique base decomposition of x so that $\alpha > 0$ and $s \in \mathcal{S}$ such that $s = \sum_j \lambda_j s_j$ is a unique convex decomposition of s into pure states. Similarly $e_i \leq u$ since now $e_i(x) = \alpha\lambda_i \leq \alpha = u(x)$. Furthermore we have that

$$\sum_{i=1}^{n} e_i(s) = \sum_{i,j=1}^{n} \lambda_j e_i(s_j) = \sum_{i=1}^{n} \lambda_i = 1$$

so that the dual basis actually forms a distinguishing observable for $\mathrm{ext}(\mathcal{S})$.

Suppose then that there exists an observable A that distinguishes the set S of n states $S = \{s'_1, \ldots, s'_n\}$. We note that the set S is linearly independent. Otherwise there would exist some $j \in \{1, \ldots, n\}$ such that s'_j could be expressed as a linear combination of other states in S as

$$s'_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i s'_i$$

so that by the distinguishibility of A

$$1 = \mathsf{A}_j(s'_j) = \sum_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i \mathsf{A}_j(s'_i) = 0$$

which would be a contradiction.

Thus, S forms a basis for $\mathcal{A}$ so that each state $s \in \mathcal{S}^{\mathcal{A}}$ can be expressed uniquely as

$$s = \sum_{i=1} \gamma_i s'_i.$$

Now from the normalization $\sum_j \mathsf{A}_j(s) = 1$ it follows that $\sum_i \gamma_i = 1$ and the positivity and distinguishability of A implies that $0 \leq \mathsf{A}_j(s) = \gamma_j$ for all $j = 1, \ldots, n$. Hence, every element of $\mathcal{S}^{\mathcal{A}}$ has a unique convex decomposition into elements of S. It remains to show that actually $S = \mathrm{ext}(\mathcal{S}^{\mathcal{A}})$.

From the previous observation we have that $\mathcal{S}^{\mathcal{A}} = \mathrm{conv}(S)$ so that $\mathrm{ext}(\mathcal{S}^{\mathcal{A}}) \subset S$. Since $\mathcal{A} = \mathrm{span}(\mathcal{S}^{\mathcal{A}}) = \mathrm{span}(\mathrm{ext}(\mathcal{S}^{\mathcal{A}}))$, the set of extremal states $\mathrm{ext}(\mathcal{S}^{\mathcal{A}})$ must contain at least n-elements. But since S only contains n elements we have that $S = \mathrm{ext}(\mathcal{S}^{\mathcal{A}})$. □

Since measurement outcomes form probability distributions for each state, the distinguishability of states can be examined from these probabilities. In particular, the closeness of probability distributions can be determined by the means of (classical) probability theory. We consider one of these measures more closely.

Definition 10.4. The *fidelity* of two states s_1 and s_2 in a state space $\mathcal{S}$ is defined as

$$F(s_1, s_2) = \inf_{\mathsf{A} \in \mathcal{O}(\mathcal{S})} \sum_x \sqrt{\mathsf{A}_x(s_1)\mathsf{A}_x(s_2)}. \tag{10.1}$$

The expression of which the infimum is taken from is called the *Bhattacharyya coefficient* of two probability distributions and it measures the overlap, or relative closeness, of the distributions. The fidelity is then defined by taking the optimal measurement that most distinguishes s_1 and s_2.

We see that the fidelity satisfies some important general properties [65].

Proposition 10.5. *Let $\mathcal{S}$ and $\mathcal{S}'$ be two state spaces and $s_1, s_2 \in \mathcal{S}$. The following properties hold for the fidelity:*

i) $0 \leq F(s_1, s_2) \leq 1$,

ii) $F(s_1, s_2) = 1$ *if and only if* $s_1 = s_2$,

iii) $F(s_1, s_2) = 0$ *if and only the states* s_1 *and* s_2 *are distinguishable,*

iv) $F(\tau(s_1), \tau(s_2)) \geq F(s_1, s_2)$ *for all channels* $\tau : \mathcal{S} \to \mathcal{S}'$,

v) $F\left(\sum_i \lambda_i s_i, \sum_i \lambda'_i s'_i\right) \geq \sum_i \sqrt{\lambda_i \lambda'_i} F(s_i, s'_i)$ *for all convex combinations of states* $\{s_i\}_i, \{s'_i\}_i \subset \mathcal{S}$, *and*

vi) $F(s_1, s_2)F(s'_1, s'_2) \geq F(s_1 \otimes s'_1, s_2 \otimes s'_2)$ *for all states* $s_1, s_2 \in \mathcal{S}$ *and* $s'_1, s'_2 \in \mathcal{S}'$ *and all composites* $\mathcal{S} \otimes \mathcal{S}'$.

Proof. The parts *i)* and *ii)* are clear: there always exists a measurement that gives different probabilities for different states so that when we consider the Bhattacharyya coefficient as an inner product of two vectors in $\mathbb{R}^n$ (n is the number of outcomes of the optimal observable) with entries as the square roots of the respective probabilities, the properties follow. See [65] for *iii)*.

iv) Let $\tau : \mathcal{S} \to \mathcal{S}'$ be a channel. Let $\varepsilon > 0$. Then for any two states $s_1, s_2 \in \mathcal{S}$ there exists the optimal observable $\mathsf{A}' \in \mathcal{O}(\mathcal{S}', \Omega)$ with an outcome set Ω such that

$$F(\tau(s_1), \tau(s_2)) + \varepsilon \geq \sum_{x \in \Omega} \sqrt{\mathsf{A}'_x(\tau(s_1))\mathsf{A}'_x(\tau(s_2))}$$

However, we see that the mapping $\mathsf{A} : x \mapsto \mathsf{A}_x$, where $\mathsf{A}_x := \mathsf{A}'_x \circ \tau$, defines an observable on $\mathcal{S}$. The observable A might not be optimal for the fidelity of s_1 and s_2 so that we have

$$F(\tau(s_1), \tau(s_2)) + \varepsilon \geq \sum_{x \in \Omega} \sqrt{\mathsf{A}_x(s_1)\mathsf{A}_x(s_2)} \geq F(s_1, s_2).$$

Since ε was arbitrary, this proves the claim.

v) For any $n \in \mathbb{N}$, take any two sets of states $\{s_i\}_{i=1}^n, \{s'_i\}_{i=1}^n \subset \mathcal{S}$. Let $s = \sum_i \lambda_i s_i$ and $s' = \sum_i \lambda'_i s'_i$ be any two convex combinations of these states respectively. For $\varepsilon > 0$ and s and s' we take the optimal measurement $\mathsf{A} \in \mathcal{O}(\mathcal{S}, \Omega)$ so that

$$\begin{aligned} F(s,s') + \varepsilon &\geq \sum_x \sqrt{\mathsf{A}_x(s)\mathsf{A}_x(s')} \\ &= \sum_x \sqrt{\sum_i \lambda_i \mathsf{A}_x(s_i)} \sqrt{\sum_i \lambda'_i \mathsf{A}_x(s'_i)} \\ &\geq \sum_x \sum_i \sqrt{\lambda_i \lambda'_i} \sqrt{\mathsf{A}_x(s_i)\mathsf{A}_x(s'_i)} \\ &\geq \sum_i \sqrt{\lambda_i \lambda'_i} F(s_i, s'_i), \end{aligned}$$

where on the third line we have used the Cauchy-Schwarz inequality for vectors $(\lambda_1 \mathsf{A}_x(s_1), \ldots, \lambda_n \mathsf{A}_x(s_n))$ and $(\lambda'_1 \mathsf{A}_x(s'_1), \ldots, \lambda'_n \mathsf{A}_x(s'_n))$ in $\mathbb{R}^n$. Since ε was arbitrary, this proves the claim.

vi) For any $\varepsilon > 0$, let $\mathsf{A} \in \mathcal{O}(\mathcal{S})$ and $\mathsf{A}' \in \mathcal{O}(\mathcal{S}')$ be the optimal measurement for s_1, s_2 and s'_1, s'_2 respectively so that

$$F(s_1, s_2) + \varepsilon \geq \sum_x \sqrt{\mathsf{A}_x(s_1)\mathsf{A}_x(s_2)}$$

and

$$F(s'_1, s'_2) + \varepsilon \geq \sum_{x'} \sqrt{\mathsf{A}'_{x'}(s'_1)\mathsf{A}'_{x'}(s'_2)}.$$

Now the mapping $\mathsf{B} : xx' \mapsto \mathsf{B}_{xx'}$, where $\mathsf{B}_{xx'} := \mathsf{A}_x \otimes \mathsf{A}'_{x'}$, is an observable in $\mathcal{S} \otimes \mathcal{S}'$ for any tensor product. Then

$$\begin{aligned} (F(s_1,s_2)+\varepsilon)(F(s'_1,s'_2)+\varepsilon) &\geq \left[\sum_x \sqrt{\mathsf{A}_x(s_1)\mathsf{A}_x(s_2)}\right]\left[\sum_{x'} \sqrt{\mathsf{A}'_{x'}(s'_1)\mathsf{A}'_{x'}(s'_2)}\right] \\ &= \sum_{xx'} \sqrt{\mathsf{B}_{xx'}(s_1 \otimes s'_1)\mathsf{B}_{xx'}(s_2 \otimes s'_2)} \\ &\geq F(s_1 \otimes s'_1, s_2 \otimes s'_2). \end{aligned}$$

Since ε was arbitrary, this proves the claim.

□

10.2 No-cloning theorems

The fidelity can now be used to show the connection of distinguishability and clonability of any two states [65].

Proposition 10.6. *Two distinct states $s_1 \neq s_2$ in a state space $\mathcal{S}^{\mathcal{A}}$ are clonable if and only if they are distinguishable.*

Proof. Suppose that s_1 and s_2 are jointly distinguishable so that there exists an observable A such that $\mathsf{A}_i(s_j) = \delta_{ij}$ for $i, j = 1, 2$. Then a channel $\tau : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ defined by

$$\tau(x) = \mathsf{A}_1(x)s_1 \otimes s_1 + \mathsf{A}_2(x)s_2 \otimes s_2$$

for all $x \in \mathcal{A}$ clones s_1 and s_2.

For the other direction we suppose that s_1 and s_2 are clonable by some channel $\tau' : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$. From the properties *iv)* and *vi)* of Prop. 10.5 it follows that

$$F(s_1, s_2) \leq F(\tau'(s_1), \tau'(s_2)) = F(s_1 \otimes s_1, s_2 \otimes s_2) \leq F(s_1, s_2)^2$$

which is only possible when either $F(s_1, s_2) = 0$ or $F(s_1, s_2) = 1$. But these are exactly the cases when s_1 and s_2 are either distinguishable or equal. Since s_1 and s_2 were distinct, they cannot be equal so that they must be distinguishable. □

It turns out that in any theory with at least two distinct states there exists a pair of states that are not distinguishable. Namely, if $\mathcal{S}$ is a state space with two distinguishable states $s_1, s_2 \in \mathcal{S}$, then for a fixed $0 < \lambda < 1$ there is a state $s = \lambda s_1 + (1 - \lambda)s_2$ such that $s \neq s_{1,2}$. By property *v)* of Prop. 10.5 we have that

$$F(s_1, s) \geq \lambda F(s_1, s_1) + (1 - \lambda)F(s_1, s_2) = \lambda > 0$$

so that s_1 and s are not distinguishable. This observation leads to the impossibility of universal cloning in any non-trivial theory.

Theorem 10.7 (No universal cloning)**.** *In any convex operational theory there is no universal cloning machine that would clone any unknown state.*

Proof. As was noted above, any non-trivial state space has an indistinguishable pair. By Prop. 10.6 this pair is not clonable by the same cloning device. □

The previous no-cloning theorem contains classical theories as well. However, the difference between classical and non-classical theories can be seen when we try to clone just the pure states, i.e. we consider the universal cloning of pure states.

As we observed, classical state spaces are characterized by the fact that all pure states can be distinguished. Thus, if $S = \{s_1, \ldots, s_n\} \subset \mathrm{ext}(\mathcal{S})$ is any set of pure states in a classical state space $\mathcal{S}_C$, then there exists an observable A with an outcome set $\Omega = \{1, \ldots, n\}$ such that $\mathsf{A}_i(s_j) = \delta_{ij}$ for all $i, j = 1, \ldots, n$. Then, just in the proof of Prop. 10.6, we can define a channel $\tau : \mathcal{S}_C \to \mathcal{S}_C \otimes \mathcal{S}_C$ by

$$\tau(s) = \sum_{i=1}^{n} \mathsf{A}_i(s) s_i \otimes s_i \tag{10.2}$$

which then clones all states in S. Hence, classical theories admit universal cloning of pure states.

The key point now is that classical theories are actually the only ones with universal cloning of pure states. This is based on the generalization on Prop. 10.6 which states the following [63].

Proposition 10.8. *Every finite set of states is clonable if and only if it is distinguishable.*

We note that the sufficiency of distinguishability for cloning follows from the cloning machine (10.2).

Thus, if there is a universal cloning machine for pure states on a state space $\mathcal{S}$, then any finite set $S = \{s_1, \ldots, s_n\}$ of pure state can be distinguished by some observable B. Then for any two convex compositions of the states of S that equal the same state, i.e. for any convex coefficients $\{\lambda_i\}_i$ and $\{\mu_i\}_i$ such that $s := \sum_i \lambda_i s_i = \sum_i \mu_i s_i \in \mathcal{S}$, it follows that

$$\lambda_i = \mathsf{B}_i(s) = \mu_i$$

for all $i = 1, \ldots, n$, so that the convex decomposition of s is unique. Since by Prop. 1.16 this is the case only for state spaces that are simplices and hence classical. Hence, the universal cloning of pure states cannot be achieved in any non-classical theory so that we conclude with the following theorem.

Theorem 10.9 (No universal cloning of pure states)**.** *Universal cloning of pure states is only possible in classical theories.*

As a remark we note that Prop. 10.6 is not enough to prove the previous theorem namely because it only deals with pairwise distinguishability. In fact, if we have a state space $\mathcal{S}$ which admits universal cloning of pure states, then all that Prop. 10.6 gives us is that all *pairs* of pure states are distinguishable. Pairwise distinguishability does not however guarantee that all of them are jointly distinguishable.

Consider a particular example of square state space $\mathcal{S}_\Box$. We have four extremal points $s_1, s_2, s_3, s_4 \in \mathcal{S}_\Box$ such that $s_1 + s_3 = s_2 + s_4$. We can now define observables

E and F with outcome set $\{+,-\}$ by

$$\mathsf{E}_+(s_1) = \mathsf{E}_+(s_2) = \mathsf{E}_-(s_3) = \mathsf{E}_-(s_4) = 0, \quad \mathsf{E}_-(s_1) = \mathsf{E}_-(s_2) = \mathsf{E}_+(s_3) = \mathsf{E}_+(s_4) = 1$$
$$\mathsf{F}_-(s_1) = \mathsf{F}_+(s_2) = \mathsf{F}_+(s_3) = \mathsf{F}_-(s_4) = 0, \quad \mathsf{F}_+(s_1) = \mathsf{F}_-(s_2) = \mathsf{F}_-(s_3) = \mathsf{F}_+(s_4) = 1.$$

We see that E distinguishes all pairs $\{s_1, s_3\}, \{s_1, s_4\}, \{s_2, s_3\}$ and $\{s_2, s_4\}$ and similarly F all pairs $\{s_1, s_2\}, \{s_1, s_3\}, \{s_4, s_2\}$ and $\{s_4, s_3\}$. Thus, all pairs of pure states are distinguishable but since the state space is non-classical, not all pure state are jointly distinguishable. Hence, Prop. 10.8 is truly needed.

11 Broadcasting

As a generalization of cloning, we may weaken the cloning condition such that instead of requiring a channel to transform a state into a product state with itself, we demand the both reduced states of the transformed state to match the original state. This task as known as *broadcasting* of states and the channel a *broadcasting device*. Similar way to cloning, we can consider *universal broadcasting* or *universal broadcasting of pure states* as tasks to broadcast all states or just the pure states with the same device respectively. We see that the results of the previous section give us a no-go theorem for broadcasting as well. The material presented here follows [9, 63].

11.1 No universal broadcasting

In order to make a rigorous definition of broadcasting let us consider reduced states more closely.

Let $\mathcal{S}^{\mathcal{A}}$ and $\mathcal{S}^{\mathcal{B}}$ be state spaces in ordered vector spaces $\mathcal{A}$ and $\mathcal{B}$ respectively. On the vector space $\mathcal{A} \otimes \mathcal{B}$ we can define *partial units* $M_B : \mathcal{A} \otimes \mathcal{B} \to \mathcal{A}$ and $M_A : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B}$ by

$$a(M_B(x)) = (a \otimes u_B)(x), \quad b(M_A(x)) = (u_A \otimes b)(x)$$

for all $x \in \mathcal{A} \otimes \mathcal{B}$, $a \in \mathcal{A}^*$ and $b \in \mathcal{B}^*$, where u_A and u_B are order units in $\mathcal{A}^*$ and $\mathcal{B}^*$ respectively. We note that the partial units $M_{A,B}$ are channels. We recall that the reduced states $s_{A,B}$ of a state $s \in \mathcal{S}^{\mathcal{A}} \otimes \mathcal{S}^{\mathcal{B}}$ are defined as

$$e(s_A) = (e \otimes u_B)(s), \quad f(s_B) = (u_A \otimes f)(s)$$

for all $e \in \mathcal{E}(\mathcal{S}^{\mathcal{A}})$ and $f \in \mathcal{E}(\mathcal{S}^{\mathcal{B}})$. Hence, we see that $s_A = M_B(s)$ and $s_B = M_A(s)$ for all states $s \in \mathcal{S}^{\mathcal{A}} \otimes \mathcal{S}^{\mathcal{B}}$.

With state spaces $\mathcal{S}^{\mathcal{A}}$, $\mathcal{S}^{\mathcal{B}}$ and $\mathcal{S}^{\mathcal{C}}$ in vector spaces $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ respectively, we define for any linear map $\tau : \mathcal{C} \to \mathcal{A} \otimes \mathcal{B}$ the marginal mappings as $\tau_A = M_B \circ \tau$ and $\tau_B = M_A \circ \tau$. In the case that τ is a channel we see that $\tau_A : \mathcal{C} \to \mathcal{A}$ and $\tau_B : \mathcal{C} \to \mathcal{B}$ are channels as well.

Definition 11.1. A finite set of states $S = \{s_j\}_{j=1}^n \subset \mathcal{S}^{\mathcal{A}}$ in a state space $\mathcal{S}$ is called *broadcastable* if there exists a channel $\tau : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ such that $\tau_A(s_j) = \tau_B(s_j) = s_j$ for all $j = 1, \ldots, n$. We then say that τ *broadcasts* S.

We see that cloning is a stronger form of broadcasting, that is, if a set of states is clonable then it is also broadcastable. Indeed, suppose that there is a channel τ that clones a set of states $S = \{s_j\}_{j=1}^n$. Then $\tau_A(s_j) = \tau_B(s_j) = s_j$ for all $j = 1, \ldots, n$ so that τ broadcasts S.

We will see that the impossibility of universal broadcasting in non-classical theories actually follows from the no-cloning theorems [63]. The key role in this observation plays the following proposition [41].

Proposition 11.2. *Let $\mathcal{S}^{\mathcal{A}} \otimes \mathcal{S}^{\mathcal{B}}$ be a composite state space of $\mathcal{S}^{\mathcal{A}}$ and $\mathcal{S}^{\mathcal{B}}$ and $s \in \mathcal{S}^{\mathcal{A}} \otimes \mathcal{S}^{\mathcal{B}}$. If either marginal $s_A \in \mathcal{S}^{\mathcal{A}}$ or $s_B \in \mathcal{S}^{\mathcal{B}}$ of s is a pure state in the respective state space, then $s = s_A \otimes s_B$.*

Proof. Without loss of generality, we may suppose that s_A is a pure state in $\mathcal{S}^{\mathcal{A}}$. From the definition of the conditional states it follows that

$$s(e \otimes f) = s_A(e) s_{B|e}(f) = s_B(f) s_{A|f}(e)$$

for all $e \otimes f \in \mathcal{E}(\mathcal{S}^{\mathcal{A}}) \otimes \mathcal{E}(\mathcal{S}^{\mathcal{B}})$. If we consider any observable $\mathsf{B} \in \mathcal{O}(\mathcal{S}^{\mathcal{B}}, \Omega_B)$ with some outcome set Ω_B, we have that

$$s_A = \sum_{y \in \Omega_B} s_B(\mathsf{B}_y) s_{A|\mathsf{B}_y}.$$

But this is just s_A represented as a convex combination of the states $s_{A|\mathsf{B}_y}$ with weights $s_B(\mathsf{B}_y)$. Since s_A is pure, either $s_B(\mathsf{B}_y) = 0$ or $s_{A|\mathsf{B}_y} = s_A$ for each $y \in \Omega_B$. Since the observable B was chosen arbitrarily, we have that for either case then $s(e \otimes f) = s_A(e) s_B(f)$ for all $e \otimes f \in \mathcal{E}(\mathcal{S}^{\mathcal{A}}) \otimes \mathcal{E}(\mathcal{S}^{\mathcal{B}})$. It follows from the duality $(\mathcal{A}^* \otimes \mathcal{B}^*)^* \cong \mathcal{A} \otimes \mathcal{B}$ that s must be of the form $s = s_A \otimes s_B$. □

Now we can prove the no-broadcasting theorem for non-classical state spaces. Suppose we have a universal broadcasting device τ for pure states on a state space $\mathcal{S}$. Thus, $\tau_A(s) = \tau_B(s) = s$ for all pure states $s \in \mathcal{S}$. Since s is pure and $\tau_A(s)$

and $\tau_B(s)$ are just the reduced states of the state $\tau(s)$, it follows from the previous proposition that $\tau(s) = s \otimes s$. Thus, τ is actually a universal cloning of pure states on $\mathcal{S}$ from which it follows that $\mathcal{S}$ is a classical state space. Hence we have the following.

Theorem 11.3 (No universal broadcasting of pure states)**.** *Universal broadcasting of pure states is only possible in classical theories.*

As the pure states cannot be broadcast in any non-classical theory, the theory cannot admit universal broadcasting either. However, for some sets of states there might be some device that broadcasts the set. We will look further into the properties of such sets of states next.

11.2 The set of broadcastable states

We see that already Prop. 10.8 tells us something about the sets of states that can be broadcast. Namely, let us consider a set $S = \{s_i\}_{i=1}^n$ of jointly distinguishable states in a state space $\mathcal{S}$. Then by the proposition, the set S can also be cloned so that if we take the convex hull $\mathrm{conv}(S)$ of S in $\mathcal{S}$, we can broadcast every element of $\mathrm{conv}(S)$ simply by cloning its pure states that are now included in the set S. We can even use the cloning device in (10.2) and as was noted, any cloning device is also a broadcasting device.

Furthermore, if we consider a channel τ and some convex subset of states S', we see that if τ broadcasts the extremal points of S', then τ broadcasts whole S'. Indeed, let $\mathrm{ext}(S') = \{s'_1, \dots, s'_m\}$. If now τ broadcasts $\mathrm{ext}(S')$ then for every $s = \sum_i \lambda'_i s'_i \in S'$ we have that

$$\tau_A(s) = \tau_A\left(\sum_{i=1}^m \lambda'_i s'_i\right) = \sum_{i=1}^m \lambda'_i \tau_A(s'_i) = \sum_{i=1}^m \lambda'_i s'_i = s,$$

and similarly $\tau_B(s) = s$.

To conclude, for a convex hull of any set of distinguishable states there always exists a broadcasting device. Also for a fixed broadcasting device, the set of states it broadcasts is a convex subset of the states. In order to say more about the sets of broadcastable states, we have to look more closely into properties of convex subsets of convex sets. The work presented here is adapted from [9, 63].

Let $\mathcal{K} \subset \mathcal{S}$ be a convex subset of a convex set $\mathcal{S}$. We say that a linear mapping $P : \mathcal{S} \to \mathcal{S}$ is a *compression* of $\mathcal{S}$ onto $\mathcal{K}$ if $P(\mathcal{S}) = \mathcal{K}$ and $P(P(s)) = P(s)$ for all $s \in \mathcal{S}$. Then we can also consider P as a surjective mapping $P : \mathcal{S} \to \mathcal{K}$.

This means that if $k \in \mathcal{K}$, then there exists $s \in \mathcal{S}$ such that $P(s) = k$ so that $P(k) = P(P(s)) = P(s) = k$.

We can now prove the following lemma.

Lemma 11.4. *For any linear map $R : \mathcal{S} \to \mathcal{S}$ there exists a compression of $\mathcal{S}$ onto the fixed points of R.*

Proof. Let us define a sequence of functions $(P_n)_n$ from $\mathcal{S}$ to itself where

$$P_n(s) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} R^i(s)$$

for all $s \in \mathcal{S}$. Since $\mathcal{S}$ is compact, the sequence $(P_n)_n$ converges to some limit function $P : \mathcal{S} \to \mathcal{S}$. If $s \in \mathcal{S}$ is a fixed point of R, i.e. $R(s) = s$, then clearly $P(s) = s$. Thus, the fixed points of R are included in the range of P. For the converse we have that if $P(s') = s$ for some $s, s' \in \mathcal{S}$, then

$$\begin{aligned} R(s) &= R(P(s')) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} R^{i+1}(s') && (11.1)\\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} R^i(s') - \lim_{n\to\infty} \frac{1}{n} R(s') + \lim_{n\to\infty} \frac{1}{n} R^{n+1}(s') && (11.2)\\ &= P(s') = s && (11.3) \end{aligned}$$

so that s is a fixed point of R. Thus, the range of P is exactly the fixed points of R. Moreover, $P(s')$ is a fixed point of R so that $P(P(s')) = P(s')$ for any $s' \in \mathcal{S}$. □

Let now both $\mathcal{S}$ and $\mathcal{K}$ be compact. If we consider $\mathcal{S} = \mathcal{S}^{\mathcal{A}}$ as a state space in the ordered vector space formalism in a vector space $\mathcal{A}$, we may form a state space $\mathcal{K}^{\mathcal{B}}$ that lies in some ordered vector space $\mathcal{B}$ such that $\mathcal{K}$ is isomorphic with $\mathcal{K}^{\mathcal{B}}$ by some isomorphism $\iota : \mathcal{K}^{\mathcal{B}} \to \mathcal{K}$. Since $\mathcal{K}^{\mathcal{B}}$ spans $\mathcal{B}$ we can extend ι into an injection from $\mathcal{B} \to \mathcal{A}$.

For any compression $P : \mathcal{S} \to \mathcal{S}$ we can define a linear function $P' = \iota^{-1} \circ P : \mathcal{S}^{\mathcal{A}} \to \mathcal{K}^{\mathcal{B}}$ such that $P'(\mathcal{S}^{\mathcal{A}}) = \mathcal{K}^{\mathcal{B}}$ and $P'(\iota(P'(s))) = P'(s)$ for all $s \in \mathcal{S}^{\mathcal{A}}$. Since $\mathcal{S}^{\mathcal{A}}$ spans $\mathcal{A}$, this can be uniquely extended to a linear function from $\mathcal{A}$ to $\mathcal{B}$. We call this extension again a compression and denote it again by P.

For this extension we still have that $P(\mathcal{S}^{\mathcal{A}}) = \mathcal{K}^{\mathcal{B}}$, and now furthermore $P(\mathcal{A}_+) = \mathcal{B}_+$ and $P(\iota(P(x))) = P(x)$ for all $x \in \mathcal{A}$. Moreover, if $k \in \mathcal{K} \subset \mathcal{S}^{\mathcal{A}}$, then $\iota(P(k)) = k$, and if $k' \in \mathcal{K}^{\mathcal{B}}$, then $P(\iota(k')) = k'$. We note that P is a surjection since for any $y = \alpha k_1 - \beta k_2 \in \mathcal{B}$, where $k_1, k_2 \in \mathcal{K}^{\mathcal{B}}$, we have that $P(s_1) = k_1$ and $P(s_2) = k_2$ for some $s_1, s_2 \in \mathcal{S}^{\mathcal{A}}$ so that if we denote $x = \alpha s_1 - \beta s_2 \in \mathcal{A}$, then $P(x) = y$.

For P, we can also consider the dual map $P^* : \mathcal{B}^* \to \mathcal{A}^*$ defined by $P^*(g) = g \circ P$. Since P is surjective, the dual map P^* is injective. If $x \in \mathcal{A}$ with a base decomposition $x = \alpha s_1 - \beta s_2$, we have that

$$P^*(u_B)(x) = u_B(P(x)) = \alpha u_B(P(s_1)) - \beta u_B(P(s_2)) = \alpha - beta = u_A(x).$$

Thus, $P^*(u_B) = u_A$. Since $P(\mathcal{A}_+) = \mathcal{B}_+$, for the linear map $P \otimes P : \mathcal{A} \otimes \mathcal{A} \to \mathcal{B} \otimes \mathcal{B}$ we have that $(P \otimes P)((\mathcal{A} \otimes_{\max} \mathcal{A})_+) \subset (\mathcal{B} \otimes_{\max} \mathcal{B})_+$. Furthermore, if $s \in \mathcal{S}^{\mathcal{A}} \otimes_{\max} \mathcal{S}^{\mathcal{A}}$, then

$$\begin{aligned} (u_B \otimes u_B)(P \otimes P)(s) &= (P^* \otimes P^*)(u_B \otimes u_B)(s) = (P^*(u_B) \otimes P^*(u_B))(s) \\ &= (u_A \otimes u_A)(s) = 1 \end{aligned}$$

so that $(P \otimes P)(\mathcal{S}^{\mathcal{A}} \otimes_{\max} \mathcal{S}^{\mathcal{A}}) \subset \mathcal{K}^{\mathcal{B}} \otimes_{\max} \mathcal{K}^{\mathcal{B}}$.

Now we are ready to prove the main result of this section [63].

Theorem 11.5. *A set of states is broadcastable if and only if it lies in a simplex generated by jointly distinguishable states.*

Proof. As was already shown, a convex hull of jointly distinguishable states is always broadcastable. For the other direction we consider any channel $\tau : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and denote by $\mathcal{C} \subset \mathcal{S}^{\mathcal{A}}$ the set of states broadcast by τ on a state space $\mathcal{S}^{\mathcal{A}}$.

Let us define an isomorphism $\Theta : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ by $\Theta(x \otimes y) = y \otimes x$ for all $x \otimes y \in \mathcal{A} \otimes \mathcal{A}$. We also see that $\Theta^*(f \otimes g) = g \otimes f$ for all $f \otimes g \in \mathcal{A}^* \otimes \mathcal{A}^*$. For τ, we can then define $\tau' := \frac{1}{2}(\tau + \Theta \circ \tau) : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$. If now $s \in \mathcal{C}$, then

$$\begin{aligned} f((\Theta \circ \tau)_A(s)) &= (u_A \otimes f)(\Theta(\tau(s))) = \Theta^*(u_A \otimes f)(\tau(s)) \\ &= (f \otimes u_A)(\tau(s)) = f(\tau_B(s)) = f(s) \end{aligned}$$

for all $f \in \mathcal{A}^*$, which implies that $\tau'_A(s) = \frac{1}{2}(\tau_A(s) + (\Theta \circ \tau)_A(s)) = s$. Similarly we have that $\tau'_B(s) = s$ for all $s \in \mathcal{C}$ so that τ' broadcasts every state in $\mathcal{C}$. Thus, if we denote by $\mathcal{K}$ the set of states broadcast by τ', then $\mathcal{C} \subset \mathcal{K}$. As was noted earlier, both $\mathcal{C}$ and $\mathcal{K}$ are convex subsets of $\mathcal{S}^{\mathcal{A}}$. We will show that $\mathcal{K}$ is generated by some set of jointly distinguishable states in $\mathcal{S}^{\mathcal{A}}$.

Let us consider the marginal map $\tau'_A : \mathcal{A} \to \mathcal{A}$. Clearly, if $s \in \mathcal{K}$, then s is a fixed point of τ'_A. Conversely, if s' is a fixed point of τ'_A, then

$$\tau'_B(s') = \frac{1}{2}(\tau'_B(s') + (\Theta \circ \tau)_B(s')) = \frac{1}{2}(\tau'_B(s') + \tau_A(s')) = \frac{1}{2}(\tau'_B(s') + s')$$

from which it follows that $\tau'_B(s') = s'$. Thus, $s' \in \mathcal{K}$ and $\mathcal{K}$ is exactly the set of fixed points of τ_A (or similarly τ_B). We note that the same does not hold for $\mathcal{C}$ and the fixed points of τ_A, since if s is a fixed point of τ_A, we might not have $\tau_B(s) = s$.

By Lemma 11.4 there exists a compression $P : \mathcal{S}^{\mathcal{A}} \to \mathcal{K}$ so that we can consider it as a compression $P : \mathcal{A} \to \mathcal{B}$ onto an ordered vector space $\mathcal{B}$ such that $\mathcal{K} = \mathcal{K}^{\mathcal{B}}$. We define a linear map $\chi : \mathcal{B} \to \mathcal{B} \otimes_{\max} \mathcal{B}$ by

$$\chi(y) = (P \otimes P)(\tau'(\iota(y)))$$

for all $y \in \mathcal{B}$. Since τ' is a channel, by the properties of $P \otimes P$, we have that χ is a channel. We see that χ broadcasts $\mathcal{K}^{\mathcal{B}}$: if $k \in \mathcal{K}^{\mathcal{B}}$, then

$$\begin{aligned} f(\chi_A(k)) &= (f \otimes u_B)(\chi(\iota(k))) && (11.4)\\ &= (f \otimes u_B)(P \otimes P)(\tau'(\iota(k))) && (11.5)\\ &= (P^* \otimes P^*)(f \otimes u_B)(\tau'(\iota(k))) && (11.6)\\ &= (P^*(f) \otimes u_A)(\tau'(\iota(k))) && (11.7)\\ &= P^*(f)(\tau'_A(\iota(k))) = P^*(f)(\iota(k)) && (11.8)\\ &= f(P(\iota(k))) = f(k) && (11.9) \end{aligned}$$

for all $f \in \mathcal{B}^*$. Thus, $\chi_A(k) = k$ for all $k \in \mathcal{K}^{\mathcal{B}}$ and similarly we can show that then also $\chi_B(k) = k$.

Hence, χ is a universal broadcasting machine on $\mathcal{K}^{\mathcal{B}}$. In particular, χ broadcasts all the pure states of $\mathcal{K}^{\mathcal{B}}$ so that from Prop. 11.2 we then have that χ clones all the pure states. By Prop. 10.8 it then follows that the pure states are jointly distinguishable so that $\mathcal{K}^{\mathcal{B}}$ is a simplex. It follows that $\iota(\mathcal{K}^{\mathcal{B}}) = \mathcal{K} \subset \mathcal{S}^{\mathcal{A}}$ is a simplex.

For the extremal states $\mathrm{ext}(\mathcal{K}^{\mathcal{B}}) = \{k'_1, \ldots, k'_n\}$ we have an observable A' on $\mathcal{K}^{\mathcal{B}}$ such that $\mathsf{A}'_i(k'_j) = \delta_{ij}$. Define a mapping $\mathsf{A} : \{1, \ldots, n\} \to \mathcal{A}^*$ by $i \mapsto \mathsf{A}_i$ where $\mathsf{A}_i := P^*(\mathsf{A}'_i)$. Since A' is an observable on $\mathcal{K}^{\mathcal{B}}$ and $P(\mathcal{S}^{\mathcal{A}}) = \mathcal{K}^{\mathcal{B}}$, we have that $\mathsf{A}_i \geq o$ for all $i = 1, \ldots, n$ and

$$\sum_{i=1}^{n} \mathsf{A}_i(s) = \sum_{i=1}^{n} P^*(\mathsf{A}'_i(s)) = \sum_{i=1}^{n} \mathsf{A}'_i(P(s)) = 1$$

for all $s \in \mathcal{S}^{\mathcal{A}}$ so that A is an observable on $\mathcal{S}^{\mathcal{A}}$. If we denote $k_j = \iota(k'_j)$, then

$$\mathsf{A}_i(k_j) = \mathsf{A}'_i(P(\iota(k'_j))) = \mathsf{A}'_i(k'_j) = \delta_{ij}.$$

Hence, $\mathcal{K}$ is the simplex generated by a jointly distinguishable states $\{k_i\}_i$ and the set of states $\mathcal{C}$ broadcast by τ is a subset of $\mathcal{K}$. □

In quantum theory the previous theorem gets the following form.

Theorem 11.6. *If a set of quantum states is broadcastable then the states commute with each other.*

Proof. Consider a state space $\mathcal{S}(\mathcal{H}) = \{\varrho \in \mathcal{L}_s(\mathcal{H}) \,|\, \varrho \geq 0,\ \mathrm{tr}\,[\varrho] = 1\}$ for some Hilbert space $\mathcal{H}$. Let $\mathcal{C}$ be a set of states broadcast by a quantum channel $\Phi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$. By Thm. 11.5, $\mathcal{C}$ is contained in a simplex $\mathcal{K}$ of distinguishable states.

Take now two states $\varrho, \varrho' \in \mathcal{K}$. Since they are distinguishable, there exists an effect operator $E \in \mathcal{E}(\mathcal{H})$ such that $\mathrm{tr}\,[E\varrho] = 1$ and $\mathrm{tr}\,[E\varrho'] = 0$. Consider now the spectral decompositions of ϱ and ϱ', i.e.

$$\varrho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i| \quad \varrho' = \sum_j \mu_j |\varphi_j\rangle\langle\varphi_j|$$

for some rank-one projectors $|\psi_i\rangle\langle\psi_i|$ and $|\varphi_j\rangle\langle\varphi_j|$, weights $\lambda_i, \mu_j \in (0, 1)$ for all i, j with $\sum_i \lambda_i = \sum_j \mu_j = 1$.

Now the condition $\mathrm{tr}\,[E\varrho] = 1$ becomes

$$\sum_i \lambda_i \langle\, \psi_i \,|\, E\psi_i \,\rangle = 1$$

which is only satisfied if $\langle\, \psi_i \,|\, E\psi_i \,\rangle = 1$ for all i. Thus, E has an eigenvalue 1 for all the eigenvectors $\psi_i \in \mathcal{H}$. Similarly the condition $\mathrm{tr}\,[E\varrho'] = 0$ is equivalent with $|\varphi_j\rangle\langle E\varphi_j| = 0$ for all j so that $\varphi_j \in \mathcal{H}$ belongs to the eigenspace of E corresponding to the eigenvalue 0. Since the eigenvectors corresponding to different eigenvalues are orthogonal, we have that $\varrho\varrho' = \varrho'\varrho = 0$. Thus, in particular the states in $\mathcal{C}$ commute with each other. □

12 Joint measurability

Consider an operational task, where for two observables A and B we try make a measurement such that the full measurement outcome statistic of both of these observables can be extracted from this measurement. If this kind of measurement exists, we call this measurement a *joint measurement* for A and B and say that they are *jointly measurable* or *compatible*. It is convenient to require that the method of extraction is that we get the measurement statistics of A and B as marginal distributions of the outcome probability distribution of the joint measurement. Also, the task of joint measurability can be extended to cover joint measurements of multiple observables. Thus, we arrive at the following definition.

Definition 12.1. Observables $\mathsf{A}^{(1)}, \ldots, \mathsf{A}^{(m-1)}$ and $\mathsf{A}^{(m)}$ with outcome sets $\Omega_1, \ldots, \Omega_{m-1}$ and Ω_m respectively are *compatible* if there exists an observable G with outcome set $\Omega_1 \times \cdots \times \Omega_m$ such that for all $i = 1, \ldots, m$ we have that

$$\sum_{\substack{j=1 \\ j \neq i}}^{m} \sum_{x^{(j)} \in \Omega_j} \mathsf{G}_{x^{(1)} \cdots x^{(m)}} = \mathsf{A}^{(i)}_{x^{(i)}} \tag{12.1}$$

for each outcome $x^(i) \in \Omega_i$. The observable G is then called the *joint observable* of observables $\mathsf{A}^{(1)}, \ldots, \mathsf{A}^{(m-1)}$ and $\mathsf{A}^{(m)}$. If a joint observable does not exist, then the observables are called *incompatible.*

Let us start with a classical state space $\mathcal{S}_C$ with extremal states $\mathrm{ext}(\mathcal{S}) = \{s_1, \ldots, s_n\}$. Does there exist incompatible observables on $\mathcal{S}_C$? The answer is expectedly no [50]. Consider any m observables $\mathsf{A}^{(1)}, \ldots, \mathsf{A}^{(m-1)}$ and $\mathsf{A}^{(m)}$ on $\mathcal{S}_C$ with outcome sets $\Omega_1, \ldots, \Omega_{m-1}$ and Ω_m respectively. As $\mathcal{S}_C$ is classical, by Prop. 10.3 there exists an observable $\mathcal{C}$ that distinguishes all the pure states of $\mathcal{S}_C$. By Prop. 1.16 any $s \in \mathcal{S}_C$ has a unique convex decomposition $s = \sum_i \lambda_i s_i$ into pure states so that

$$\mathsf{C}_i(s) = \lambda_i$$

for all $i = 1, \ldots, n$. As the convex decomposition is unique, in order to determine any observable it is enough to determine the values of the effects on the extremal points. Now we can form a joint observable G for $\{\mathsf{A}^{(i)}\}_i$ by

$$\mathsf{G}_{xy}(s) = \sum_{i=1}^{n} \mathsf{C}_i(s) \mathsf{A}^{(1)}_{x^{(1)}}(s_i) \cdots \mathsf{A}^{(m)}_{x^{(m)}}(s_i) \tag{12.2}$$

for all all states $s \in \mathcal{S}_C$ and outcomes $x^{(i)} \in \Omega_i$ for all $i = 1, \ldots, m$. It is easy to see that we get observables $\mathsf{A}^{(1)}, \ldots, \mathsf{A}^{(m-1)}$ and $\mathsf{A}^{(m)}$ as marginals of G. Thus we have proved the following.

Proposition 12.2. *All observables on a classical state space are compatible.*

Joint measurability can also be formulated in another way [17]. Consider the measurement of a joint observable. The measurement outcomes are then classical symbols and thus can be processed by the means of classical transformations or channels. As the measurement statistics contains all the information about the measurement of the original observables, it should be expected that by applying different classical transformations on the obtained outcomes would lead back to the original observables. In another words, we would expect that the observables would be post-processings of the joint observable.

We can now actually show that for a collection of observables the existence of a joint observables is equivalent to the fact that they are post-processings of a single observable [17].

Proposition 12.3. *Observables* $\mathsf{A}^{(1)},\dots,\mathsf{A}^{(m-1)}$ *and* $\mathsf{A}^{(m)}$ *are compatible if and only if there exists an observable* C *and classical channels* $\nu^{(1)},\dots,\nu^{(m-1)}$ *and* $\nu^{(m)}$ *such that* $\mathsf{A}^{(i)}=\nu^{(i)}\circ\mathsf{C}$ *for all* $i=1,\dots,m$.

Proof. Let us first assume that the observables $\mathsf{A}^{(1)},\dots,\mathsf{A}^{(m-1)}$ and $\mathsf{A}^{(m)}$ with outcome sets $\Omega_1,\dots,\Omega_{m-1}$ and Ω_m are compatible. Thus, there exists a joint observable G with an outcome set $\Omega\equiv\Omega_1\times\cdots\times\Omega_m$. Let us consider projection functions $\pi_k:\Omega\to\Omega_k$ for all $k=1,\dots,m$ such that

$$\pi_k(\vec{x})=x^{(k)}$$

for all $\vec{x}=(x^{(1)},\dots,x^{(m)})\in\Omega$. We can then form a class of classical channels $\nu^{(k)}:\Omega\to\Omega_k$ by

$$\nu^{(k)}_{\vec{x}y}=\begin{cases}1, & \text{if } \pi_k(\vec{x})=y\\ 0, & \text{otherwise.}\end{cases}$$

By post-processing the joint observable G with the channels $\nu^{(k)}$ we see that

$$(\nu^{(k)}\circ\mathsf{G})_{x^{(k)}}=\sum_{\vec{x}\in\Omega}\nu^{(k)}_{\vec{x}x^{(k)}}\mathsf{G}_{\vec{x}}=\sum_{\substack{j=1\\ j\neq k}}^{m}\sum_{x^{(j)}\in\Omega_j}\mathsf{G}_{x^{(1)}\cdots x^{(m)}}=\mathsf{A}^{(k)}_{x^{(k)}}. \tag{12.3}$$

Hence, $\mathsf{A}^{(k)}=\nu^{(k)}\circ\mathsf{G}$ for all $k=1,\dots,m$ so that each $\mathsf{A}^{(k)}$ is a post-processing of G.

Suppose then that each $\mathsf{A}^{(k)}$ is a post-processing of some observable C. Thus, there exists classical channels $\mu^{(1)},\dots,\mu^{(m-1)}$ and $\mu^{(m)}$ such that

$$\mathsf{A}^{(i)}=\mu^{(i)}\circ\mathsf{C} \tag{12.4}$$

for all $i=1,\dots,m$. We denote

$$\mathsf{G}'_{x^{(1)}\cdots x^{(m)}}=\sum_{y}\prod_{j=1}^{m}\nu^{(j)}_{yx^{(j)}}\mathsf{C}_y \tag{12.5}$$

for all $x^{(j)}\in\Omega_j$, $j=1,\dots,m$. It is straightforward to check that G' is an observable, and that

$$\mathsf{A}^{(i)}_{x^{(i)}}=\sum_{\substack{j=1\\ j\neq i}}^{m}\sum_{x^{(j)}\in\Omega_j}\mathsf{G}'_{x^{(1)}\cdots x^{(m)}}. \tag{12.6}$$

Thus, G' is a joint observable for $\mathsf{A}^{(1)},\dots,\mathsf{A}^{(m-1)}$ and $\mathsf{A}^{(m)}$. □

As a specific type of observables we have the observable with only two outcomes, namely the binary observables. For binary observables we see that the compatibility requirement reduces to a set of functional inequalities that must be satisfied.

Compatibility of binary observables

Let us consider two binary observables A and B on a state space $\mathcal{S}^{\mathcal{A}}$. Suppose that they are compatible so that there exists a joint observable G such that

$$\mathsf{G}_{++} + \mathsf{G}_{+-} = \mathsf{A}_{+} \tag{12.7}$$
$$\mathsf{G}_{-+} + \mathsf{G}_{--} = u - \mathsf{A}_{+} \tag{12.8}$$
$$\mathsf{G}_{++} + \mathsf{G}_{-+} = \mathsf{B}_{+} \tag{12.9}$$
$$\mathsf{G}_{+-} + \mathsf{G}_{--} = u - \mathsf{B}_{+}. \tag{12.10}$$

If we denote $g \equiv \mathsf{G}_{++}$, $a \equiv \mathsf{A}_{+}$ and $b \equiv \mathsf{B}_{+}$, then express the effects $\mathsf{G}_{\pm\pm}$ as

$$\mathsf{G}_{++} = g \tag{12.11}$$
$$\mathsf{G}_{+-} = a - g \tag{12.12}$$
$$\mathsf{G}_{-+} = b - g \tag{12.13}$$
$$\mathsf{G}_{--} = u - a - b + g. \tag{12.14}$$

From the positivity of the effects $\mathsf{G}_{\pm\pm}$ it follows that the inequalities

$$g \geq o \tag{12.15}$$
$$a \geq g \tag{12.16}$$
$$b \geq g \tag{12.17}$$
$$u + g \geq a + b \tag{12.18}$$

are satisfied.

On the other hand suppose that the inequalities (12.15)–(12.18) are satisfied for some $g \in \mathcal{A}^*$. Then we can define a joint observable G according to (12.11)– (12.14) so that the inequalities guarantee that the elements $\mathsf{G}_{\pm\pm}$ are valid effects and form an observable.

Hence we have proved the following [47].

Proposition 12.4. *Two binary observables* A *and* B *on* $\mathcal{S}^{\mathcal{A}}$ *are compatible if and only if there exists a functional* $g \in \mathcal{A}^*$ *satisfying the inequalities* (12.15)–(12.18).

We will show next that in any non-classical theory there exists an incompatible pair of binary observables.

12.1 Existence of incompatibility in non-classical theories

Indeed, it was shown in [50] that incompatibility is a generic non-classical feature. The results, that we follow also here, relies heavily on the properties of faces of convex sets. In particular, we consider two specific type of faces, the exposed faces and the maximal faces.

Definition 12.5. A subset $\mathcal{C} \subset \mathcal{K}$ of a convex set $\mathcal{K}$ is called an *exposed face* of $\mathcal{K}$ if there exists an affine function $f \in \mathcal{F}(\mathcal{K})$ such that $\mathcal{C} = \{x \in \mathcal{K} \,|\, f(x) = \max_{y \in \mathcal{K}} f(y)\}$. If $\mathcal{C}$ is a singleton set, then $\mathcal{C}$ is called an *exposed point*.

Definition 12.6. A proper face $\mathcal{F} \subset \mathcal{K}$ of a compact convex set $\mathcal{K}$ is called a *maximal face* of $\mathcal{K}$ if for every $x \in \mathcal{K} \setminus \mathcal{F}$ we have that $\mathrm{conv}(\mathcal{F} \cup \{x\}) \cap \mathrm{int}(\mathcal{K}) \neq \emptyset$.

Since the convex set in Def. 12.5 is compact, every affine functional truly attains its maximum value so that it is valid to define the exposed face as we did. We also see that every exposed face actually is a face. Namely, if $x \in \mathcal{C} \subset \mathcal{K}$ for some exposed face $\mathcal{C}$ defined by some affine functional f, then if $x = \lambda y + (1-\lambda) z$ for some $y, z \in \mathcal{K}$, $0 < \lambda < 1$, then

$$\max_{x' \in \mathcal{K}} f(x') = f(x) = \lambda f(y) + (1-\lambda) f(z) \leq \lambda f(y) + (1-\lambda) \max_{x' \in \mathcal{K}} f(x')$$

so that $\max_{x'} f(x') \leq f(y) \leq \max_{x'} f(x')$ from which it follows that $y \in \mathcal{C}$ and similarly for z. Thus, $\mathcal{C}$ is a face.

Since every face with only one point is an extremal point, every exposed point is an extremal point. Furhtermore, the set of exposed points is dense in $\mathrm{ext}(\mathcal{K})$ [26, Thm. 18.6]. It is also worth noting that every face of a closed set is closed [26, Cor. 18.1.1]. On the maximal faces we note that we require the maximal faces to be non-trivial so that $\mathcal{K}$ itself is not a maximal face. One can show that every maximal face is an exposed face [50].

For example in the case of the regular polygons we see that the vertices of the polygon are both extremal and exposed points whereas the maximal faces are the edges of the polygon. For compact convex sets we can prove the following result on maximal faces.

Proposition 12.7. *Let $x \in \mathcal{S}^{\mathcal{A}}$ be a pure state of a state space $\mathcal{S}^{\mathcal{A}}$ and $\mathcal{F}$ a maximal face such that $x \notin \mathcal{F}$. If there exists another pure state $y \in \mathcal{S}^{\mathcal{A}}$ such that $y \notin \mathcal{F}$ and $y \neq x$, then there exists a pair of incompatible observables.*

Proof. Let $\mathcal{S}$ be a state space and x and $\mathcal{F}$ as stated in the proposition. Suppose that there exists an extreme point $y \in \mathcal{S}$ such that $y \notin \mathcal{F}$ and $y \neq x$. Since all extremal points are faces and since $\mathcal{S}$ is closed, we have that the sets $\mathcal{F}$, $\{x\}$ and $\{y\}$ are closed. Because $y \notin \mathcal{F}$ and $y \notin \{x\}$, there exists an open neighbourhood N of y such that $x \notin N$ and $N \cap \mathcal{F} = \emptyset$.

As the set of exposed points is dense in the set of extremal points of $\mathcal{S}$, y can be expressed as a limit point of a sequence of exposed points so that there exists a point y' of the sequence such that $y' \in N$. Similarly for x we can find an exposed point x' that is in some open neighbourhood of x such that $x' \neq y'$ and $x \notin \mathcal{F}$.

Denote $\mathcal{Q}_1 = \mathcal{F}$, $\mathcal{Q}_2 = \{x'\}$ and $\mathcal{Q}_3 = \{y'\}$. Since now $\mathcal{Q}_i$ is an exposed face for all $i = 1, 2, 3$, by definition there exists affine functionals $f'_i \in \mathcal{F}(\mathcal{S})$ such that for all $i = 1, 2, 3$

$$\mathcal{Q}_i = \{s \in \mathcal{S} \mid f'_i(s) = f'_{\text{min},i}\}, \tag{12.19}$$

where we have denoted $\min_{s\in\mathcal{S}} f'_i(s) = f'_{\text{min},i}$. If we further denote $\max_{s\in\mathcal{S}} f'_i(s) = f'_{\text{max},i}$ for all i, we can define new affine functionals $f_i \in \mathcal{F}(\mathcal{S})$ by

$$f_i(s) = \frac{f'_i(s) - f'_{\text{min},i}}{f'_{max,i} - f'_{min,i}} \tag{12.20}$$

so that $\min_{s\in\mathcal{S}} f_i(s) = 0$ and $\max_{s\in\mathcal{S}} f_i(s) = 1$. Hence, we have affine functionals $f_i \in \mathcal{F}(\mathcal{S})$ such that

$$\mathcal{Q}_i = \{s \in \mathcal{S} \mid f_i(s) = 0\}, \tag{12.21}$$

and

$$\max_{s\in\mathcal{S}} f_i(s) = 1$$

for all $i = 1, 2, 3$.

We consider now $\mathcal{S}^{\mathcal{A}} \cong \mathcal{S}$ in an ordered vector space $\mathcal{A}$. By construction, the functions f_i are effects on $\mathcal{S}$ so that we can extend them to linear functionals in $\mathcal{A}^*$ (denoted again by f_i) such that they are effects on $\mathcal{S}^{\mathcal{A}}$. We can construct three binary observables $\mathsf{A}^{(1)}$, $\mathsf{A}^{(2)}$ and $\mathsf{A}^{(3)}$ on $\mathcal{S}^{\mathcal{A}}$ such that

$$\mathsf{A}^{(i)}_+ = f_i, \quad \mathsf{A}^{(i)}_- = 1 - f_i \tag{12.22}$$

for all $i = 1, 2, 3$.

Suppose now that all observables are compatible. Specifically then $\mathsf{A}^{(1)}$ is pairwise compatible with both $\mathsf{A}^{(2)}$ and $\mathsf{A}^{(3)}$ so that there exists functionals $g_{2,3} \in \mathcal{A}^*$ such that

$$g_j \geq o \tag{12.23}$$

$$f_1 \geq g_j \tag{12.24}$$

$$f_j \geq g_j \tag{12.25}$$

$$u + g_j \geq f_1 + f_j \tag{12.26}$$

for $j = 2, 3$.

It follows from the first three inequalities that $g_j(z) = 0$ for all $z \in \operatorname{conv}(\mathcal{Q}_1 \cup \mathcal{Q}_j)$ for $j = 2, 3$. Namely, if $q_1 \in \mathcal{Q}_1$, then $0 = f_1(q_1) \geq g_j(q_1) \geq 0$ and if $q_j \in \mathcal{Q}_j$, then $0 = f_j(q_j) \geq g_j(q_j) \geq 0$ so that $g_j(z) = 0$ for all $z \in \mathcal{Q}_1 \cup \mathcal{Q}_j$. Since g_2 and g_3 are in particular affine, they vanish also on the convex hulls of $\mathcal{Q}_1 \cup \mathcal{Q}_2$ and $\mathcal{Q}_1 \cup \mathcal{Q}_3$ respectively.

Since $\mathcal{Q}_1 = \mathcal{F}$ is a maximal face and $x', y' \notin \mathcal{F}$, then

$$\operatorname{conv}(\mathcal{Q}_1 \cup \mathcal{Q}_j) \cap \operatorname{int}(\mathcal{S}) \neq \emptyset, \quad j = 2, 3. \tag{12.27}$$

Thus, there exists an interior point $z \in \operatorname{int}(\mathcal{S})$ such that $g_2(z) = g_3(z) = 0$. Since they are also positive, they must be zero functions $g_2 = g_3 = o$. From the last inequality it then follows that

$$f_1 + f_j \leq u, \quad j = 2, 3.$$

This implies that if $f_1(s) = 1$ for some $s \in \mathcal{S}$, then $f_j(s) = 0$ for $j = 2, 3$. We note that there exists at least one such state, since f_1 reaches its maximal value 1 on $\mathcal{S}^{\mathcal{A}}$. Thus,

$$\emptyset \neq \{s \in \mathcal{S}^{\mathcal{A}} \mid f_1(s) = 1\} \subset \mathcal{Q}_j, \quad j = 2, 3.$$

We remember that $\mathcal{Q}_2 = \{x'\}$ and $\mathcal{Q}_3 = \{y'\}$ are singleton sets so that $x' = y'$ which is a contradiction. □

We still need one technical result on maximal faces that will not be proved here.

Lemma 12.8. *For every boundary point x of a compact convex set $\mathcal{K}$ there exists maximal faces $\mathcal{F}_1$ and $\mathcal{F}_2$ such that $x \in \mathcal{F}_1$, $x \notin \mathcal{F}_2$.*

For a complete proof, see [50]. The proof begins by showing that every boundary point of $\mathcal{K}$ is contained in some maximal face of $\mathcal{K}$. Then it is shown that any boundary point x cannot be included in all maximal faces so that there must exist some other maximal face that does not contain x.

With the help of the lemma, we can show that the statement of the previous proposition is always satisfied in any non-classical theory.

Theorem 12.9. *There exists an incompatible pair of observables in any non-classical state space.*

Proof. Let $\mathcal{S}^{\mathcal{A}}$ be a state space that is not a simplex. Thus, there exists an extremal element $x \in \operatorname{ext}(\mathcal{S}^{\mathcal{A}})$ such that it can be given as an affine combination $x = \sum_{i=1} \alpha_i y_i$

of some other pure states $y_1, \ldots, y_n \in \mathrm{ext}(\mathcal{S}^{\mathcal{A}})$, $y_i \neq x$ for all i, with coefficients $\alpha_i \in \mathbb{R}$, $\sum_i \lambda_i = 1$ (so that at least one of them is negative as otherwise the affine combination would be a convex combination). By Lemma 12.8 there exists a maximal face $\mathcal{F}$ such that $x \notin \mathcal{F}$. Thus, if we show that $y_i \notin \mathcal{F}$ for some $i = 1, \ldots, n$ the claim follows from Prop. 12.7.

Suppose that $y_i \in \mathcal{F}$ for all $i = 1, \ldots, n$. Since $\mathcal{F}$ is a maximal face, there exists (possibly by rescaling and shifting the values as in the proof of Prop. 12.7) a positive functional $f \in \mathcal{A}^*_+$ such that

$$\mathcal{F} = \{s \in \mathcal{S}^{\mathcal{A}} \,|\, f(s) = 0\}.$$

Thus, $f(y_i) = 0$ for all $i = 1, \ldots, n$, and since f is affine, it follows that

$$f(x) = \sum_{i=1}^{n} \alpha_i f(y_i) = 0$$

so that $x \in \mathcal{F}$ which is a contradiction. □

12.2 Incompatibility of channels and no-broadcasting

The existence of incompatibility in non-classical theories actually prohibits universal broadcasting. To see this, we first consider compatibility of channels. We follow the approach presented in [73] (see also [74]) and generalize it in the framework of convex operational theories. For simplicity, we consider the compatibility of two channels. The compatibility of channels can now be defined with the help of marginal channels that were introduced at the beginning of the broadcasting section.

Definition 12.10. Let $\mathcal{S}^{\mathcal{A}}$, $\mathcal{S}^{\mathcal{B}}$ and $\mathcal{S}^{\mathcal{C}}$ be states spaces in vector spaces $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. Two channels $\tau : \mathcal{C} \to \mathcal{A}$ and $\chi : \mathcal{C} \to \mathcal{B}$ are *compatible* if there exists a channel $\gamma : \mathcal{C} \to \mathcal{A} \otimes \mathcal{B}$ such that

$$\gamma_A(c) = (M_B \circ \gamma)(c) = \tau(c), \quad \gamma_B(c) = (M_A \circ \gamma)(c) = \chi(c)$$

for all $c \in \mathcal{C}$. Otherwise they are *incompatible*.

We see that the compatibility of channels captures the idea behind broadcasting. Namely, if the identity channel $\mathcal{I}$ on a state space $\mathcal{S}^{\mathcal{A}}$ is compatible with itself, then the joint channels $\gamma : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ satisfies

$$\gamma_A(s) = \gamma_B(s) = s \tag{12.28}$$

for all $s \in \mathcal{S}$. This is nothing but the broadcasting condition so that then γ is actually a universal broadcasting device on $\mathcal{S}^{\mathcal{A}}$. Hence, the universal no-broadcasting is equivalent to the incompatibility of identity channels on the state space.

However, we can show that if there exists a pair of incompatible channels on that state space, then the identity channels are also incompatible [73].

Proposition 12.11. *Let $\mathcal{I} : \mathcal{C} \to \mathcal{C}$ be the identity channel on a state space $\mathcal{S}^{\mathcal{C}}$. If $\mathcal{I}$ is compatible with itself, then all pairs of channels on $\mathcal{S}^{\mathcal{C}}$ are compatible.*

Proof. Let $\tau : \mathcal{C} \to \mathcal{A}$ and $\chi : \mathcal{C} \to \mathcal{B}$ be any two channels on $\mathcal{S}^{\mathcal{C}}$. Since the identity channel is compatible, there exists a channel $\gamma : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ such that

$$\gamma_A(x) = \gamma_B(x) = x$$

for all $x \in \mathcal{C}$. Let us consider the concatenation channel υ of $\tau \otimes \chi$ and γ, i.e. $\upsilon := (\tau \otimes \chi) \circ \gamma : \mathcal{C} \to \mathcal{A} \otimes \mathcal{B}$. Since both γ and $\tau \otimes \chi$ are channels, υ is a channel. Now we see that for all $f \in \mathcal{A}^*$ we have

$$\begin{aligned} f(\upsilon_A(x)) &= (f \otimes u_B)((\tau \otimes \chi)(\gamma(x))) = (\tau^* \otimes \chi^*)(f \otimes u_B)(\gamma(x)) \\ &= \gamma^* \left((\tau^* \otimes \chi^*)(f \otimes u_B)\right)(x) = \gamma^* \left(\tau^*(f) \otimes \chi^*(u_B)\right)(x) \\ &= \gamma^* \left(\tau^*(f) \otimes u_C\right)(x) = \left(\tau^*(f) \otimes u_C\right)(\gamma(x)) \\ &= \tau^*(f)(\gamma_A(x)) = \tau^*(f)(x) = f(\tau(x)) \end{aligned}$$

for all $x \in \mathcal{C}$. Thus, $\upsilon_A = \tau$ and similarly $\upsilon_B = \chi$. Hence, υ is a joint channel for τ and χ so that they are compatible. □

The previous proposition shows that universal no-broadcasting is equivalent to the existence of an incompatible pair of channels. Thus, in order to have another proof for the no-broadcasting theorem in all non-classical theories, we need to show that there always exists an incompatible pair of channels. We show this by relaying on the results of the previous section which tells us that incompatible observables exist in any non-classical theories.

But how does the compatibility of channels relate to joint measurability of observables? Consider an observable A on a state space $\mathcal{S}$ with outcomes sets Ω. For Ω we can construct a Hilbert space $\ell^2(\Omega)$ as the set functions from Ω to the complex numbers $\mathbb{C}$. Such functions form a Hilbert space with the inner product

$$\langle\, f \,|\, g \,\rangle = \sum_{x \in \Omega} \overline{f(x)} g(x).$$

Now $\ell^2(\Omega)$ has a basis $\{\delta_x\}_{x\in\Omega}$, where δ_x is the Dirac measure on x. As Ω is finite, $\ell^2(\Omega)$ is finite-dimensional. Thus, we use it to construct a quantum state space as

$$\mathcal{S}(\ell^2(\Omega)) = \{\varrho \in \mathcal{L}_s(\ell^2(\Omega)) \,|\, \varrho \geq 0,\ \mathrm{tr}\,[\varrho] = 1\}.$$

With the observable A and the state space $\mathcal{S}(\ell^2(\Omega))$ we can construct a channel $\tau^{\mathsf{A}} : \mathcal{A} \to \mathcal{L}_s(\ell^2(\Omega))$ by

$$\tau^{\mathsf{A}}(a) = \sum_{x\in\Omega} \mathsf{A}_x(a)|\delta_x\rangle\langle\delta_x| \tag{12.29}$$

for all $a \in \mathcal{A}$. For $s \in \mathcal{S}^{\mathcal{A}}$ we have that since the projectors $|\delta_x\rangle\langle\delta_x|$ are clearly states in $\mathcal{S}(\ell^2(\Omega))$ and $\{\mathsf{A}_x(s)\}_x$ is a set of convex coefficients, $\tau^{\mathsf{A}}(s) \in \mathcal{S}(\ell^2(\Omega))$ so that τ^{A} is a channel. Now we can prove the equivalence between the compatibility of observables and the channels that the observables define [73].

Proposition 12.12. *Let A and B be two observables on a state space $\mathcal{S}^{\mathcal{A}}$ with outcome sets Λ and Ω. The observables A and B are compatible if and only if the channels τ^{A} and τ^{B} are compatible.*

Proof. Let A and B be compatible and denote by G their joint observable such that $\sum_{x\in\Omega} \mathsf{G}_{xy} = \mathsf{A}_y$ and $\sum_{y\in\Lambda} \mathsf{G}_{xy} = \mathsf{B}_x$. We define a channel $\upsilon : \mathcal{A} \to \mathcal{L}_s(\ell^2(\Omega)\otimes\ell^2(\Lambda))$ by

$$\upsilon(a) = \sum_{x,y} \mathsf{G}_{xy}(a)|\delta_x\otimes\delta_y\rangle\langle\delta_x\otimes\delta_y| \tag{12.30}$$

for all $a \in \mathcal{A}$. Similarly to (12.29), we see that υ is a channel. Now

$$\upsilon_A(a) = \mathrm{tr}_A\,[\upsilon(a)] = \sum_{x,y} \mathsf{G}_{xy}(a)|\delta_y\rangle\langle\delta_y| = \sum_{y} \mathsf{A}_y(a)|\delta_y\rangle\langle\delta_y| = \tau^{\mathsf{A}}(a)$$

for all $a \in \mathcal{A}$ and similarly $\upsilon_B = \tau^{\mathsf{B}}$. Thus, the channels τ^{A} and τ^{B} are compatible.

Let then τ^{A} and τ^{B} be compatible channels and denote by υ their joint channel such that $\upsilon_A = \tau^{\mathsf{A}}$ and $\upsilon_B = \tau^{\mathsf{B}}$. We define a joint observable $\mathsf{G} : \Omega\times\Lambda \to \mathcal{E}(\mathcal{S}^{\mathcal{A}})$ by

$$\mathsf{G}_{xy}(a) = \mathrm{tr}\,[\upsilon(a)|\delta_x\rangle\langle\delta_x|\otimes|\delta_y\rangle\langle\delta_y|] \tag{12.31}$$

for all $a \in \mathcal{A}$. Now clearly $\mathsf{G}_{xy} \geq o$ and $\sum_{x,y} \mathsf{G}_{xy}(s) = 1$ for all $s \in \mathcal{S}^{\mathcal{A}}$. Furthermore,

$$\begin{aligned}
\sum_{x\in\Omega} \mathsf{G}_{xy}(a) &= \mathrm{tr}\left[\upsilon(a)(I \otimes |\delta_y\rangle\langle\delta_y|)\right] \\
&= \mathrm{tr}\left[|\delta_y\rangle\langle\delta_y|\mathrm{tr}_a\left[\upsilon(a)\right]\right] \\
&= \mathrm{tr}\left[|\delta_y\rangle\langle\delta_y|\upsilon^{\mathsf{A}}(a)\right] \\
&= \mathrm{tr}\left[|\delta_y\rangle\langle\delta_y| \sum_{y'\in\Lambda} \mathsf{A}_{y'}(a)|\delta_{y'}\rangle\langle\delta_{y'}|\right] \\
&= \mathrm{tr}\left[\mathsf{A}_y(a)|\delta_y\rangle\langle\delta_y|\right] \\
&= \mathsf{A}_y(a)
\end{aligned}$$

for all $a \in \mathcal{A}$ and similarly $\sum_{y\in\Lambda} \mathsf{G}_{xy} = \mathsf{B}_x$. □

Now we get the no-broadcasting theorem as a simple corollary.

Corollary 12.13 (No-broadcasting). *There does not exist a universal broadcasting machine on any non-classical state space.*

Proof. Suppose that there exists a universal broadcasting machine on a non-classical state space $\mathcal{S}$. As was noted, the existence of a universal broadcasting machine on a state space $\mathcal{S}$ is equivalent to compatibility of the identity channels on $\mathcal{S}$. By Prop. 12.11, all pairs of channels on $\mathcal{S}$ are compatible. In particular, for any two observables A and B the channels τ^{A} and τ^{B} are compatible. Nevertheless, by Thm. 12.9 there exists an incompatible pair of observables on $\mathcal{S}$ so that by Prop. 12.12 the channels defined by these observables are incompatible. This contradicts the fact that all channels on $\mathcal{S}$ are compatible. □

12.3 Necessary condition for incompatibility of observables

So far we have seen that incompatibility of observables is a generic feature for all non-classical theories. From the operational perspective that is still not enough since although we know that some measurements cannot be performed jointly we would also like to characterize which kind of measurements cannot be performed jointly. Here we present an inequality that recognizes if a collection of observables is can be measured jointly. Thus, it serves as a necessary condition for incompatibility of observables. The material presented here is a result from original research [17] conducted with the thesis supervisor Teiko Heinosaari and associate professor Sergey Filippov from Moscow Institute of Physics and Technology.

Noise content of an observable

We start our investigation by looking at the intrinsic noise, or the noise content of an observable. Instead of adding noise to the observable we see for the noise that is already present. We see that then the incompatibility of observables then limits the amount of noise that a collection of observables can have. We note that this agrees with the fact that any set of observables can be made compatible if sufficient amount of noise is added to them [66].

As we saw in Chapter II, one way to form new observables from known ones is to mix them. We can also consider this backwards: we want to express a known observable as a mixture of some two other observables. This can always be done (non-trivially) if the observable is not extremal. In particular we want to limit one of the observables to belong in some subset of observables that we take to represent noise in our system (most typically the trivial observables). As the mixture is then not arbitrary, it causes limitations in the weight of the mixture. Hence, we make the following definition.

Definition 12.14. Let $\mathcal{N} \subset \mathcal{O}(\mathcal{S})$ be a subset of observables describing the noisy observables on a state space $\mathcal{S}$. For an observable $\mathsf{A} \in \mathcal{O}(\mathcal{S})$ we define the *noise content of* A *with respect to* $\mathcal{N}$ as

$$w(\mathsf{A};\mathcal{N}) = \sup\{0 \leq \lambda \leq 1 \,|\, \mathsf{A} = \lambda \mathsf{N} + (1-\lambda)\mathsf{B} \text{ for some } \mathsf{N} \in \mathcal{N} \text{ and } \mathsf{B} \in \mathcal{O}\}. \tag{12.32}$$

If $\mathcal{N} = \mathcal{T}$, we say that $w(\mathsf{A};\mathcal{T})$ is then just the noise content of A.

We see that we can express the noise content $w(\mathsf{A};\mathcal{N})$ of an observable $\mathsf{A} \in \mathcal{O}(\mathcal{S},\Omega)$ as

$$w(\mathsf{A};\mathcal{N}) = \sup\{0 \leq \lambda \leq 1 \,|\, \mathsf{A}_x \geq \lambda \mathsf{N}_x \text{ for all } x \in \Omega \text{ for some } \mathsf{N} \in \mathcal{N}\}. \tag{12.33}$$

Namely, if $\mathsf{A} = \lambda\mathsf{N} + (1-\lambda)\mathsf{B}$ for some $0 \leq \lambda \leq 1$ as in (12.32), then $\mathsf{A}_x \geq \lambda\mathsf{N}_x$ for all outcomes x. Conversely, if $\mathsf{A}_x \geq \lambda\mathsf{N}_x$ for some λ for all $x \in \Omega$, then

$$\mathsf{A} = \lambda\mathsf{N} + (1-\lambda)\tilde{\mathsf{A}},$$

where

$$\tilde{\mathsf{A}} = \frac{1}{1-\lambda}\left(\mathsf{A} - \lambda\mathsf{N}\right)$$

is an observable.

As trivial observables do not provide any information about the measured state, they can be thought as noise in the measurement. In fact, in the prototypical case $\mathcal{N} = \mathcal{T}$ we have the following explicit form for the noise content $w(\,\cdot\,;\mathcal{T})$.

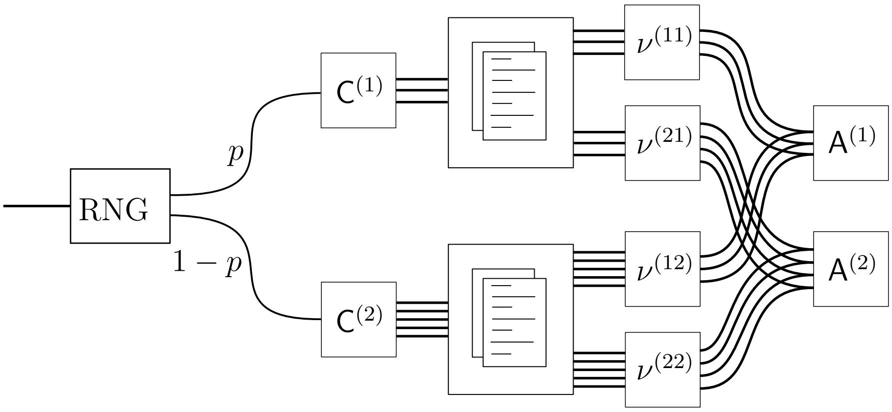

Figure 9: Joint measurement scheme for two observables $\mathsf{A}^{(1)}$ and $\mathsf{A}^{(2)}$. The joint measurement consists of random choice between two observables $\mathsf{C}^{(1)}$ and $\mathsf{C}^{(2)}$ after which the post-processings are applied separately for the copied outcomes of $\mathsf{C}^{(1)}$ and $\mathsf{C}^{(2)}$ in order to obtain $\mathsf{A}^{(1)}$ and $\mathsf{A}^{(2)}$.

Proposition 12.15. *Let $\mathsf{A} \in \mathcal{O}(\mathcal{S}, \Omega)$ be an observable on state space $\mathcal{S}$. Then $w(\mathsf{A}; \mathcal{T}) = \sum_{x \in \Omega} \inf_{s \in \mathcal{S}} \mathsf{A}_x(s)$.*

Proof. Denote $a_x = \inf_{s \in \mathcal{S}} \mathsf{A}_x(s)$ and $a = \sum_{x \in \Omega} a_x$. Let $\mathsf{T} \in \mathcal{O}(\mathcal{S}, \Omega)$ be any trivial observable and take some λ, $0 \leq \lambda \leq 1$, such that $\mathsf{A}_x \geq \lambda \mathsf{T}_x$ for all $x \in \Omega$. The partial order is determined in the set of states so that by taking the infimum over $\mathcal{S}$ and then summing over all the outcomes in Ω we have that

$$a = \sum_{x \in \Omega} \inf_{s \in \mathcal{S}} \mathsf{A}_x(s) \geq \sum_{x \in \Omega} \lambda \inf_{s \in \mathcal{S}} \mathsf{T}_x(s) = \lambda \sum_{x \in \Omega} \mathsf{T}_x(s') = \lambda,$$

where $s' \in \mathcal{S}$ is any state since T_x is state-independent for all $x \in \Omega$. Thus, we have an upper bound for λ so that $w(\mathsf{A}; \mathcal{T}) \leq a$. Now, if $a = 0$, then $w(\mathsf{A}; \mathcal{T}) = 0 = a$, and if $a \neq 0$, then by defining T by $\mathsf{T}(s) = a_x/a$ we see that the upper bound is attained and $w(\mathsf{A}; \mathcal{T}) = a$. □

Joint measurement scheme and the incompatibility inequality

Let us consider a joint measurement scheme from which our incompatibility inequality can be extracted. For that, let us construct another equivalent formulation for joint measurability.

Let $\mathsf{A}^{(1)}, \ldots, \mathsf{A}^{(m-1)}$ and $\mathsf{A}^{(m)}$ be observables on a state space $\mathcal{S}$ with outcome sets $\Omega_1, \ldots, \Omega_{m-1}$ and Ω_m. Instead of considering one observable which the m observables would be post-processings of, let us consider a collection $\{\mathsf{C}^{(i)}\}_{i=1}^m$ of m observables with outcome set Λ, and m^2 classical channels $\nu^{(jk)} : \Lambda \to \Omega_j$ such that

$$\mathsf{A}^{(i)} = \sum_{j=1}^{m} p_j \nu^{(ij)} \circ \mathsf{C}^{(j)} \tag{12.34}$$

for all $i = 1, \dots, m$ for some probability distribution $(p_j)_j$. Fig. 9 depicts the case when the joint measurement scheme is applied to two observables $\mathsf{A}^{(1)}$ and $\mathsf{A}^{(2)}$.

Clearly, if $p_k = 1$ for some $k = 1, \dots, m$, then every $\mathsf{A}^{(i)}$ is a post-processing of $\mathsf{C}^{(k)}$. Thus, every collection of compatible observables can be written in the form of (12.34).

On the other hand we see that if the observables can be expressed as in (12.34), then there exists an observable C which the observables $\{\mathsf{A}^{(i)}\}_{i=1}^m$ are then post-processings of. In fact, we can take C to be just the mixture $\sum_{j=1} p_j \mathsf{C}^{(j)}$ but now with an extra outcome indicating which observable $\mathsf{C}^{(j)}$ was measured so that we may apply the right classical channel $\nu^{(jk)}$ according to the mixture. Hence, we define observable C as

$$\mathsf{C}_{yk} = p_k \mathsf{C}^{(k)}_y \tag{12.35}$$

for all outcomes $y \in \Lambda$, $k \in \{1, \dots, m\}$, and new classical channels $\nu^{(i)} : \Lambda \times \{1, \dots, m\} \to \Omega_i$ as

$$\nu^{(i)}_{ykx} = \nu^{(ik)}_{yx} \tag{12.36}$$

so that

$$(\nu^{(i)} \circ \mathsf{C})_x = \sum_{y,j} \nu^{(ij)}_{yx} p_j \mathsf{C}^{(j)}_y = \sum_{j=1}^{m} p_j (\nu^{(ij)} \circ \mathsf{C}^{(j)})_x = \mathsf{A}^{(i)}_x \tag{12.37}$$

for all $x \in \Omega_i$. Thus, if a collection of observables can be written in the form of (12.34), then they are compatible.

Now we are ready to formulate and prove the incompatibility inequality from the special case of the joint measurability scheme that was described above.

Theorem 12.16. *If* $\mathsf{A}^{(1)}, \dots, \mathsf{A}^{(m-1)}$ *and* $\mathsf{A}^{(m)}$ *are* m *observables such that*

$$\sum_{i=1}^{m} w(\mathsf{A}^{(i)}; \mathcal{T}) \geq m - 1, \tag{12.38}$$

then they are compatible.

Proof. Consider the compatibility condition (12.34). Suppose that the classical channels cannot be chosen arbitrarily, but instead we limit them such that $\nu^{(ij)} \circ \mathsf{C}^{(j)} = \mathsf{T}^{(i)}$ for some trivial observable $\mathsf{T}^{(i)}$ for all $j \neq i$. Then the compatibility condition reduces to

$$\mathsf{A}^{(i)} = p_i \nu^{(ii)} \circ \mathsf{C}^{(i)} + (1 - p_i) \mathsf{T}^{(i)} \tag{12.39}$$

for all $i = 1, \ldots, m$. Because of this extra limitation, we cannot guarantee that any set of compatible observables can be written in this form.

However, since we now have that

$$\sum_{i=1}^{m} w(\mathsf{A}^{(i)}; \mathcal{T}) \geq m - 1, \tag{12.40}$$

we can define a probability distribution $(p_j)_j$ by

$$p_j = 1 - w(\mathsf{A}^{(j)}; \mathcal{T}), \quad j = 1, \ldots, m-1 \tag{12.41}$$

$$p_m = 1 - \sum_{j=1}^{m-1} p_j \tag{12.42}$$

so that

$$w(\mathsf{A}^{(i)}; \mathcal{T}) \geq 1 - p_i \tag{12.43}$$

for all $i = 1, \ldots, m$. It follows then from the definition of the noise content that each $\mathsf{A}^{(i)}$ can be expressed in the form of (12.39). Thus, since this expression is just a special case of (12.34), this means that the observables must be compatible. □

The previous theorem can be written in an equivalent form.

Corollary 12.17. *If $\mathsf{A}^{(1)}, \ldots, \mathsf{A}^{(m-1)}$ and $\mathsf{A}^{(m)}$ are m incompatible observables, then*

$$\sum_{i=1}^{m} w(\mathsf{A}^{(i)}; \mathcal{T}) < m - 1. \tag{12.44}$$

We will see that for different state spaces our result takes different kind of forms. As an application, we consider quantum theory, quantum theory of processes and polytope theories.

Quantum theory

In finite-dimensional quantum theory we have that $\mathcal{S} = \mathcal{S}(\mathcal{H})$ for some finite-dimensional Hilbert space $\mathcal{H}$ and that every observable $\mathsf{A} \in \mathcal{O}(\mathcal{S}, \Omega)$ is described as a collection of effect operators $\mathsf{A}_x \in \mathcal{E}(\mathcal{H})$ such that $\sum_{x\in\Omega} \mathsf{A}_x = I$. In quantum theory Cor. 12.17 takes the following form.

Corollary 12.18. *If $\mathsf{A}^{(1)}, \ldots, \mathsf{A}^{(m-1)}$ and $\mathsf{A}^{(m)}$ are m incompatible observables in quantum theory, then the sum of the minimal eigenvalues of all their effects is smaller than $m - 1$.*

Proof. For the observables $\mathsf{A}^{(i)}$ we have that

$$\inf_{s\in\mathcal{S}} \mathsf{A}^{(i)}_{x^{(i)}}(s) = \min_{\psi\neq 0} \frac{\langle\, \psi \,|\, \mathsf{A}^{(i)}_{x^{(i)}}\psi \,\rangle}{\langle\, \psi \,|\, \psi \,\rangle}. \tag{12.45}$$

for all $i = 1, \ldots, m$. These are nothing but the smallest eigenvalues of the effect operators $\mathsf{A}^{(i)}_{x^{(i)}}$ so that it follows from Prop. 12.15 that the noise content $w(\mathsf{A}^{(i)}; \mathcal{T})$ of each observable $\mathsf{A}^{(i)}$ is the sum of the minimal eigenvalues of its effect operators. The claim follows from Cor. 12.17. □

Let us consider a particular example. In any convex operational theory, for an observable A on a state space $\mathcal{S}$ with an outcome set Ω of n elements, we can define the *reverse observable* A^{r} of A by

$$\mathsf{A}^{\mathrm{r}}_y = \frac{1}{n-1} \sum_{\substack{x\in\Omega \\ x\neq y}} \mathsf{A}_x = \frac{1}{n-1}\,(u - \mathsf{A}_y)$$

for all $y \in \Omega$. We see that the reverse observable A^{r} is actually a post-processing of A. Indeed, if we define a classical channel $\nu^{\mathrm{r}} : \Omega \to \Omega$ by $\nu^{\mathrm{r}}_{xy} = 0$ if $x = y$ and $\nu^{\mathrm{r}}_{xy} = \frac{1}{n-1}$ for all $x \neq y$, it is clear that $\nu^{\mathrm{r}} \circ \mathsf{A} = \mathsf{A}^{\mathrm{r}}$.

The operational meaning of the post-processing is that when we measure A and obtain some outcome x, we roll a fair dice with $n-1$ sides each of which correspond to an outcome $y \neq x$ and we choose the outcome corresponding to the result of the rolling of the dice and take that to be the outcome of the new observable A^{r}.

We can now consider the compatibility of the reverse versions of m *regular rank-1 POVMs* $\mathsf{A}^{(1)}, \ldots, \mathsf{A}^{(m-1)}$ and $\mathsf{A}^{(m)}$, i.e. POVMs with $\mathsf{A}^{(i)}_x = \frac{d}{n} P^{(i)}_x$ for all $i = 1, \ldots, m$, where n is the (same) number of outcomes for each observable, d is the dimension of the underlying Hilbert space and $P^{(i)}_x$ are one-dimensional projections on the Hilbert space. For the reverse versions of $\mathsf{A}^{(i),\mathrm{r}}$ we have that

$$\begin{aligned} \min_{\psi\neq 0} \frac{\langle\, \psi \,|\, \mathsf{A}^{(i),\mathrm{r}}_x \psi \,\rangle}{\langle\, \psi \,|\, \psi \,\rangle} &= \min_{\psi\neq 0} \frac{1}{n-1} \frac{\langle\, \psi \,|\, I - \frac{d}{n} P^{(i)}_x \psi \,\rangle}{\langle\, \psi \,|\, \psi \,\rangle} \\ &= \frac{1}{n-1} + \frac{d}{n(n-1)} \min_{\psi\neq 0} \left[- \frac{\langle\, \psi \,|\, P^{(i)}_x \psi \,\rangle}{\langle\, \psi \,|\, \psi \,\rangle} \right] \\ &= \frac{1}{n-1} - \frac{d}{n(n-1)} = \frac{n-d}{n(n-1)} \end{aligned}$$

for all $i = 1, \ldots, m$.

Now Prop. 12.15 gives us that $w(\mathsf{A}^{(i),\mathrm{r}}, \mathcal{T}) = \frac{n-d}{n-1}$ for all $i = 1, \ldots, m$ by Cor. 12.18 the reverse observable are compatible if

$$m \frac{n-d}{n-1} \geq m - 1 \quad \Leftrightarrow \quad n \geq m(d-1) + 1.$$

We see for example that m reverse regular rank-1 qubit observables (i.e. $d = 2$) are compatible for all $n \geq m + 1$. Similarly three reverse regular rank-1 qutrit observables (i.e. $m = d = 3$) are compatible for all $n \geq 7$. However, one can easily come up with an example of such reverse qutrit observables with $n = 3$ outcomes that are incompatible showing that our inequality condition is not trivial. By quantum examples we can also demonstrate that the inequality condition for incompatibility is indeed not a sufficient one [17].

Quantum theory of processes

In finite-dimensional quantum theory of transformations we have that $\mathcal{S} = \mathcal{C}(\mathcal{H}_A, \mathcal{H}_B)$ for some Hilbert spaces $\mathcal{H}_{A,B}$. Every observable $\mathsf{A} \in \mathcal{O}(\mathcal{S}, \Omega)$ is described as a collection of PPOVM-elements $\mathsf{A}_x \in \mathcal{L}_+(\mathcal{H}_A \otimes \mathcal{H}_B)$ such that $\sum_{x\in\Omega} \mathsf{A}_x = \varrho \otimes I$ for some density operator $\varrho \in \mathcal{S}(\mathcal{H}_A)$. In quantum theory of processes Cor. 12.17 takes the following form.

Corollary 12.19. *If $\mathsf{A}^{(1)}, \ldots, \mathsf{A}^{(m-1)}$ and $\mathsf{A}^{(m)}$ are m incompatible PPOVMs, then the sum of the minimal eigenvalues of all their effects is smaller than $m - 1$.*

Proof. Let us consider any observable $\mathsf{A} \in \mathcal{O}(\mathcal{S}, \Omega)$ described by PPOVM elements A_x, $x \in \Omega$, satisfying the normalization $\sum_x \mathsf{A}_x = \varrho \otimes I$ for some density operator ϱ. We denote by a_x the minimal eigenvalue of the PPOVM element A_x for each $x \in \Omega$ and $a = \sum_x a_x$.

Let us define a trivial observable T by

$$\mathsf{T}_x = \frac{a_x}{a} \varrho \otimes I.$$

It now follows that since

$$\mathsf{A}_x \geq a_x I \otimes I \geq a_x \varrho \otimes I\,, \tag{12.46}$$

we can define a valid observable A' by

$$\mathsf{A}'_x = \frac{1}{1-a}(\mathsf{A}_x - a_x \varrho \otimes I). \tag{12.47}$$

Now clearly

$$\mathsf{A} = a\mathsf{T} + (1-a)\mathsf{A}'\,, \tag{12.48}$$

from which it follows that

$$w(\mathsf{A}; \mathcal{T}) \geq a. \tag{12.49}$$

Hence, if $\mathsf{A}^{(1)}, \ldots, \mathsf{A}^{(m-1)}$ and $\mathsf{A}^{(m)}$ are m incompatible PPOVMs, then by Cor. 12.17 the sum of the noise contents of the observables is smaller than $m - 1$ so that the claim follows from (12.49). □

We note that in the case of PPOVMs the expression for the noise content as a sum of minimal eigenvalues of the PPOVM-elements no longer holds, but nevertheless it does give lower bound so that the compatibility condition takes the same form as in the case of POVMs.

Polytope theories

In polytope state spaces the state space $\mathcal{S}$ is the convex hull of a finite number of extreme points $\{s_1, \ldots, s_n\} = \mathrm{ext}(\mathcal{S})$. In polytope state spaces Cor. 12.17 takes the following form.

Corollary 12.20. *If $\mathsf{A}^{(1)}, \ldots, \mathsf{A}^{(m-1)}$ and $\mathsf{A}^{(m)}$ are m incompatible observables on a polytopic state space $\mathcal{S}$, then the sum of minimal values of all of their effects on $\mathrm{ext}(\mathcal{S})$ is smaller than $m-1$.*

Proof. We can write every state $s \in \mathcal{S}$ as a convex combination $s = \sum_i \lambda_i s_i$ of the extremal elements in $\mathrm{ext}(\mathcal{S})$ with some weights $\{\lambda_i\}_{j=1}^n$ so that

$$\mathsf{A}^{(i)}_{x^{(i)}}(s) = \sum_j \lambda_j \mathsf{A}^{(i)}_{x^{(i)}}(s_j) \geq \sum_j \lambda_j \min_k \mathsf{A}^{(i)}_{x^{(i)}}(s_k) = \min_k \mathsf{A}^{(i)}_{x^{(i)}}(s_k) \tag{12.50}$$

for all $i = 1, \ldots, m$ and outcomes $x^{(i)}$. Since this holds for every state, we have that $\inf_{s \in \mathcal{S}} \mathsf{A}^{(i)}_{x^{(i)}}(s) = \min_k \mathsf{A}^{(i)}_{x^{(i)}}(s_k)$. The claim follows from Prop. 12.15 and Cor. 12.17. □

In the case of square state space $\mathcal{S}_\square = \mathrm{conv}(\{s_1, s_2, s_3, s_4\})$, where s_1, s_2, s_3, s_4 are the extremal states of the state space such that $s_1 + s_3 = s_2 + s_4$, we can find observables where our inequality actually serves as both necessary and sufficient condition for incompatibility. Namely, we can define two observables E^α and F^β on $\mathcal{S}_\square$ by

$$\mathsf{E}^\alpha_+(s_1) = \mathsf{A}^\alpha_+(s_2) = \alpha, \quad \mathsf{E}^\alpha_+(s_3) = \mathsf{A}^\alpha_+(s_4) = 1,$$
$$\mathsf{F}^\beta_+(s_1) = \mathsf{B}^\beta_+(s_4) = \beta, \quad \mathsf{F}^\beta_+(s_2) = \mathsf{B}^\beta_+(s_3) = 1.$$

We find that $w(\mathsf{E}^\alpha; \mathcal{T}) = \alpha$ and $w(\mathsf{F}^\beta; \mathcal{T}) = \beta$ so that from Cor. 12.20 we see that then E^α and F^β are compatible if $\alpha + \beta \geq 1$. It is easy to give the observables as mixtures with the maximal noise contents as

$$\begin{aligned} \mathsf{E}^\alpha &= \alpha \mathsf{T} + (1-\alpha)\mathsf{E} \\ \mathsf{F}^\beta &= \beta \mathsf{T} + (1-\beta)\mathsf{F}, \end{aligned}$$

where T is the trivial binary observable with $\mathsf{T}_+ = u$ and $\mathsf{T}_- = o$, and $\mathsf{E} \equiv \mathsf{E}^0$ and $\mathsf{F} \equiv \mathsf{F}^0$ are the observables from the example after the no-cloning Thm. 10.9.

The sufficiency of the inequality follows from [66], where it was shown that the observables E and F are in fact maximally incompatible. This means that the noise that is required to mix with them in order to make their noisy versions compatible is enough to make any other pair of observables compatible too. More precisely, it was shown that the observables $\lambda\mathsf{E} + (1-\lambda)\mathsf{T}_1$ and $\mu\mathsf{F} + (1-\mu)\mathsf{T}_2$ are incompatible for all choices of trivial observables T_1 and T_2 if and only if $\lambda + \mu > 1$.

Conclusions

We have used the operationally natural assumption of convexity of states to formulate a general class of convex operational theories that include both quantum and classical theory. We have introduced the basic mathematical concepts used in the construction of these theories and used them to examine the most important properties of the theories. We have illustrated these properties by applying the framework to class of important theories. Finally the framework was used to formulate and examine some of the most important non-classical features of quantum theory in a general setting and see how they manifest themselves in different theories. Original research on one of these features, joint measurability, was conducted and we were able to obtain a non-trivial sufficient condition for joint measurability of observables as a result of this research.

We note however that the approach presented here is not the only one in the framework of generalized probabilistic theories that serve as a generalized setting for quantum theory. There have for example been approaches considering the logical structures [75] and categories [76] as a starting point to construct similar theories. One future endeavor would be to try to unite some of these different approaches, and steps into this direction have already been made (see [77]). Nevertheless, the approach presented here serves as a relatively simple introduction to recognizing general features of a physical theory and taking them to a more abstract settings.

We have seen that the convex operational theories serve as a powerful tool to consider different features of different theories and give us means to compare them. The power of these theories can be summarised in the fact that with the same framework we can even consider the general aspects of three types of physical theories, namely quantum theory, quantum theory of processes and classical theories. The applicability of the framework to quantum theory of process allows us to consider the general features of the higher-order quantum computation where the standard quantum information theory is taken to a more abstract level. Future aspects include surveying this approach more closely.

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