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ESSAYS ON LATENT FACTOR MODELS IN FINANCE

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Doctoral Programme of Turku School of Economics

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ISBN 978-951-29-7119-0 (PRINT)
ISBN 978-951-29-7120-6 (PDF)
ISSN 2343-3159 (Painettu/Print)
ISSN 2343-3167 (Verkkojulkaisu/Online)
Painosalama Oy - Turku, Finland 2018

ABSTRACT

This dissertation considers the application of latent factor models to financial risk management and asset pricing. The purpose is to provide solutions for modelling and forecasting the dynamics of asset prices, and to evaluate the performance of the proposed models using empirical data. The applications of the models analysed in the individual research papers vary from risk measurement to arbitrage free term structure modelling, with risk management being common financial theme as the asset pricing models also provide a framework for hedging.

In the first article I consider forecasting Value-at-Risk using models in which the observed stock returns are assumed to be affine functions of independent GARCH-type processes. The models are estimated using various Independent Component Analysis algorithms, and the obtained forecasts are compared with alternative estimates. The empirical results reveal that while most of the tested estimation approaches result in good forecasts under calm market conditions, their behavior during a financial crisis can vary considerably. Estimators based on high order moments or cumulants fared particularly badly, which is likely explained by their sensitivity to outliers.

In the second article I consider a discrete time affine Gaussian model for the joint dynamics of Overnight Indexed Swap rates and Interbank Offered Rates. I present a computationally fast way for estimating the model using least squares regressions. The model is evaluated using European interest rate data, and the results show that it is able to achieve a close and relatively stable fit during challenging market conditions.

The third article considers a quadratic multiple curve models for spot rates. As a special case of the general I derive a multiple curve extensions to the arbitrage free Nelson-Siegel model. The models are estimated using the quadratic Kalman filter under that assumption that the spreads are driven by factors related to liquidity and credit risk.

Keywords: finance, risk management, time series, financial econometrics, asset pricing, term structure models

TIIVISTELMÄ

Tämä väitöskirja tutkii latentteihin faktoreihin perustuvien mallien sovelluksia riskienhallinnassa ja rahoitusinstrumenttien hinnoittelussa. Tarkoituksena on tarjota menetelmiä hintojen ennustamiseen sekä niiden dynamiikan mallintamiseen, ja testata esitettyjä malleja empiirisen aineiston avulla. Artikkeleissa käsiteltävät mallien sovellukset vaihtelevat riskimittojen estimoinnista arbitraasittomiin korkojen aikarakennemalleihin, riskienhallinnan ollessa yhdistävänä rahoitukseen liittyvänä tekijänä.

Ensimmäisessä artikkelissa tarkastelen Value-at-Risk-riskimitan ennustamista käyttäen malleja, joissa havaittujen osaketuottojen oletetaan muodostuvan riippumattomien GARCH-prosessien affiineina muunnoksina. Mallit estimoidaan käyttäen itsenäisten komponenttien analyysiin perustuvia algoritmeja, ja malleja verrataan vaihtoehtoisiin menetelmiin ennusteiden perusteella. Empiiriset tulokset osoittavat, että vaikka suurin osa testatuista menetelmistä kykenee tuottamaan luotettavia ennusteita vakaissa markkinaolosuhteissa, niin kriisien aikana niiden suorituskyvyssä voi ilmetä olennaisia eroja. Erityisesti korkean asteen momentteihin ja kumulantteihin perustuvat menetelmät suoriutuivat huonosti, mikä johtunee kyseisten menetelmien herkkyydestä poikkeaville havainnoille.

Toisessa artikkelissa tarkastelen diskreettiaikaista affiinia Gaussista mallia euriborkorkojen ja yön yli lainojen korkojen yhteisvaihtelulle, sekä esitän laskennallisesti nopean keinon mallin estimoimiseksi pienimmän neliösumman menetelmän avulla. Korkomallia testataan eurooppalaisen aineiston avulla, ja tulokset osoittavat mallin suoriutuvan hyvin vaihtelevissa markkinaolosuhteissa.

Kolmannessa artikkelissa tarkastelen kvadraattista jatkuva-aikaista usean korkokäyrän mallia. Mallin erityistapauksena johdetaan usean korkokäyrän laajennus arbitraasittomaan Nelson-Siegel -malliin. Mallin empiirisessä sovelluksessa EURIBOR ja yön yli lainojen korkojen erotuksen oletetaan johtuvan likviditeetti- ja luottoriskistä. Estimointi suoritetaan kvadraattisen Kalman filtterin avulla.

Asiasanat: rahoitus, aikasarja-analyysi, ekonometria, riskienhallinta, korot

ACKNOWLEDGEMENTS

I would like to thank my supervisors Professor Luis Alvarez and Professor Mika Vaihekoski for the support they provided me. The rigorous lectures of Luis got me interested in finance and financial modeling, when I was still studying for my master's degree. It is almost sure that I wouldn't have considered writing a doctoral dissertation without him. I also wish to express my gratitude to the pre-examiners of this dissertation, Professor Erik Lindström and Professor Pentti Saikkonen, for their insightful comments.

I am grateful to Professor Hannu Oja for introducing me to Independent Component Analysis and many other interesting topics in statistics, as well as allowing me to participate in a reading group alongside his own doctoral students. The study sessions and our discussions were an extremely important motivating factor at the start of my doctoral studies. I am also thankful to Klaus Nordhausen, Markus Matilainen and Joni Virta for many fruitful discussions on mathematical statistics in general and independent component analysis in specific. I would also like to thank Professor Kari Ylinen for making mathematics fun again and providing me with additional material for self-studying modern analysis when I was acquiring the basic mathematical knowledge required for the completion of this dissertation.

I am thankful to Professor Paavo Salminen and all participants in the Finnish Doctoral Programs in Stochastics and Statistics for organizing such a great community and providing sanctuary to a more finance oriented guy like me. I am thankful to my colleagues at the Turku School of Economics, particularly Mika Hannula, Joonas Hämäläinen, Mikko Kepsu, Patrik Nyström, Ville Korpela and Oskari Vähämaa, for multiple interesting discussions and listening to my complaints, when work on some problem wasn't progressing exactly as planned. I would also like to thank Matti Niinikoski, Professor Paavo Okko, Tuula Vilja and all others who have participated in our almost daily discussions over coffee. The last few years wouldn't have been the same without you.

I thank the University of Turku, the Turku School of Economics and the Finnish Doctoral Program in Stochastics and Statistics for funding my doctoral studies. I am thankful to the Turku School of Economics Association for funding the printing of this dissertation

TABLE OF CONTENTS

LIST OF ORIGINAL RESEARCH PAPERS

(1) Market Risk Forecasting with Independent Component Analysis

(2) Affine Multiple Curve Modelling via Least Squares Regressions

(3) Credit Liquidity and the Term Structure of Interest Rates

Part I

INTRODUCTION

1 INTRODUCTORY NOTES

1.1 Motivation

Models have an important role in finance, as they not only help to understand, but also shape the behavior of the financial markets (MacKenzie, 2008). The practical applications of financial models cover a variety of topics from risk management to the pricing of financial assets and entire companies. The field is in a constant state of flux as advances in mathematics and statistics are applied to old problems, or events in the financial markets reveal flaws in existing models. The experiences of the last decade have been particularly consequential, because the financial markets have been under turmoil as a result of the financial crisis of 2007-2008, and the Great Recession and the European debt crisis that followed it. The crises revealed flaws in risk management practices and caused two notable shifts in the fixed income markets as the behavior of the Interbank Offered Rates (IBORs) changed fundamentally and the overall level of interest rates dropped to unprecedented levels.

Before the crisis the IBORs and Overnight Indexed Swap (OIS) rates were practically indistinguishable from each other and they were seen as representing essentially the same risk free rate. Thus it was generally accepted that financial instruments related to the two rates could be modeled separately from each other, with IBORs as the risk free discount rates in standard models for pricing IBOR derivatives (see e.g. Brace, Gatarek and Musiela, 1997). When the theoretical problems in this approach were noted in an article published a mere month before the regime shift, the issue was viewed as a trivial matter (Henrard, 2007, 2010). At the start of the financial crisis the IBORs began to diverge from the OIS rates, as it became obvious that lending to banks was not free of risks. The emergence of spreads, which peaked at over 100 basis points in the euro area, forced the market to adopt a so called multiple curve approached based on OIS discounting for the valuation of IBOR based derivatives, which has motivated research on term structure models consistent with the new regime (see e.g. Kijima, Tanaka and Wong, 2009; Bianchetti and Carlicchi, 2011; Filipović and Trolle, 2017). During the Great Recession that followed the financial crisis, interest rates started falling and the focus of research on term structure models switched to the so called zero lower bound and how interest rates would behave near it (Kim and Singleton, 2012). Eventually the European markets, however, showed that the nominal interest rates can fall below zero.

These significant changes in the financial markets have made it necessary for modern pricing models to not only be able to accommodate a discount rate that is different from the underlying IBORs, but also negative IBORs. Considering how previously fundamental assumptions about the properties of interest rates have turned out to be wrong, it can be considered prudent to approach the new modeling challenge without making superfluous assumptions. The approach chosen in this dissertation is based on latent factors models that make it possible to circumvent the need to identify in advance all factors affecting certain economic phenomena.

Latent factor models in different forms have also found a place in risk management. Independent Component Analysis (ICA) has been of a particular interest in the risk measurement due to promising empirical results (see e.g. Wu, Yu and Li, 2006; Chen, Härdle and Spokoiny, 2007) and the implied tractability. The development of new ICA algorithms, such as gFOBI and gJADE (García-Ferrer, González-Prieto and Peña, 2011; Matilainen, Nordhausen and Oja, 2015), specifically for time series applications has opened new possibilities for financial modeling, but also raises questions as the performance of the algorithms might vary under different market conditions.

1.2 Research objectives, methodology and structure

This dissertation considers the application of latent factor models to financial risk management and asset pricing. The purpose is to provide solutions for modeling and forecasting the dynamics of asset prices, and to evaluate the performance of the proposed models using empirical data. The applications of the models vary in the individual research papers from risk measurement to arbitrage free term structure models, with risk management being common financial theme as the asset pricing models also provide a framework for hedging.

The methods I have chosen to use for extracting the latent variables can be classified into two main categories. In the first two essays I utilize a two-step approach to estimation, where I first extract the latent factors using either ICA or principal component analysis (PCA) and then estimate the remaining model parameters separately using the extracted factors. The third essay utilizes a variant of the Kalman filter to simultaneously estimate the latent factors and all of the model parameters via quasi-maximum likelihood estimation.

The first essay evaluates different approaches for forecasting market risk, i.e. the changes in the value of a portfolio or a financial instrument that are caused by changes in the underlying factors (McNeil, Frey and Embrechts, 2015). The main focus is on comparing the performance of Value-at-Risk (VaR) estimates based on recently developed ICA algorithms to those provided by benchmark methods such as extreme value theory. The empirical results show how well the different forecasting methods function under varying market conditions.

The last two essays consider the dynamics of interest rates in a multiple curve framework. Both essays rely on no-arbitrage theory in order to ensure consistent pricing of financial instruments over the entire term structure. The proposed term structure models can be considered to belong to the affine class of models, as an quadratic term structure model can be expressed as an affine model by augmenting it with pseudo factors (Cheng and Scaillet, 2007). Considering polynomial models of higher order would be unnecessary, as Filipović (2002) showed that affine and quadratic models are the only consistent polynomial term structure models under reasonable conditions.

In the second essay of this dissertation, and the first one on interest rate models, I propose a tractable affine Gaussian models for the joint dynamics of EURIBORs and OIS rates. I also provide a new and computationally efficient method for estimating the multiple curve models in a time series setting by extending the work of Adrian, Crump and Moench (2013) that covered the single curve setting. The related empirical analysis shows how well the model fits the data and provides information about the dynamics of the interest rates and the market price of risk during the Financial Crisis and in the years that followed it.

The third article considers a quadratic multiple curve model for spot rates. As a special case of the general model I derive a multiple curve extensions to the arbitrage free Nelson-Siegel model of Christensen, Diebold and Rudebusch (2011). The models are estimated using the quadratic Kalman filter under the assumption that spreads are driven by factors related to liquidity and credit risk.

The data used in the empirical analysis comes from multiple sources. The stock returns are part of the widely used datasets professor Kenneth French has formed based on data from CRSP and made available on his website. The data on Overnight Indexed Swaps, EURIBOR, Forward Rates and Credit Default Swaps come from Thomson Reuters Datastream. The datasets on German government bond and Pfandbrief yields are published by the Bundesbank on their website. I also utilized survey data published by the European Central Bank. Each essay describes the relevant data in more detail.

The code for the models and their estimation was written mainly in R and Julia. The use of multiple different programming languages was necessary due to the slowness of native R code. In particular, the estimation of continuous time interest rate models using quasi maximum likelihood estimation via Kalman filtering was found to be time consuming, when a good starting value was not available. Some of the ICA algorithms, as is described in more detail in the first article of this dissertation, were implemented using code provided by Markus Matilainen and Klaus Nordhausen.

The first part of the dissertation consists of the introduction, which proceeds as follows. Section 2 considers the general theory of factor models in finance, and provides the theoretical basis for the applications in asset pricing and risk management. Section 3 considers the estimation of latent factor models and provides an overview of the methods used in this dissertation. The extended summaries of the included articles are provided in Section 4. The articles are in the second part of the dissertation.

As the primary focus is on finance and financial econometrics, I will gloss over some of the more technical details, which are covered in standard reference texts on the subject. More specifically, in the context of this dissertation arbitrage free models are also assumed to fulfil the no-free-lunch-with-vanishing risk condition (see e.g. Delbaen and Schachermayer, 1994). Similarly I will not explicitly check the Novikov-condition.

2 ON FACTOR MODELS IN FINANCE

2.1 Asset Pricing and the Dynamics of the Risk Factors

In financial and economic modelling it is common to explain the dynamics of phenomena using a limited number of fundamental factors. In the capital asset pricing model (CAPM) of Treynor (1962), Sharpe (1964), Lintner (1965) and Mossin (1966) the common risk factor was market risk. The arbitrage pricing theory (APT) of Ross (1976) generalized CAPM into a multiple factor framework for pricing assets, but the general theory left the identity of the underlying factors open. Multiple comprehensive textbooks have been written on the subject, and for a thorough treatment of the classical asset pricing theory motivated by economics and utility theory I refer to Cochrane (2005). The approach adopted in this dissertation, however, is based on the no-arbitrage condition without making direct assumptions about the forms of the utility functions of the market participants. The focus is on the dynamics of the asset prices and the risk factors that affect them.

Let the physical probability space, i.e. the probability space under which observations of the data are made, be defined by the triple $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$, where the filtration $\mathcal{F}_t$ denotes the information generated by the factors X_τ on the interval $\tau \in [0, t]$. When the modelled value Y_t, e.g. stock returns, interest rates or the price of a zero coupon bond, is assumed to be given by a factor model, it means that Y_t is a function of the underlying n-vector risk factors X_t. In empirical applications this means that the relation between the X_t and Y_t can be formulated as

$$Y_t = F(X_t, t) + \eta_t,$$

where F is a suitable function and η_t accounts for the pricing errors in term structure models or idiosyncratic risk in excess stock returns. In purely theoretical work the errors are usually excluded.

Affine factor models are probably the most common class of factor models utilized in empirical research on finance, and the first two articles of this dissertation follow this approach for specifying the models. The third article of this doctoral dissertation considers a quadratic model for yields. It should, however, be noted that quadratic models of the form

$$Y_t = A + B'X_t + X_t'CX_t + \eta_t,$$

where $A \in \mathbb{R}$, $B \in \mathbb{R}^n$ and $C \in \mathcal{M}_{n,n}(\mathbb{R})$, can be reformulated as affine models (see e.g. Cheng and Scaillet, 2007) by defining pseudo-factors $Z_t = \text{vech}(X_tX_t')$

and stating the model in terms of the augmented vector of factors $\tilde{X} = (Z_t', X_t')'$. The main benefit of the quadratic specification is its parsimony compared to an equivalent affine model.

In the models I consider, X_t is assumed to be a stationary or covariance stationary ergodic Markov process (see e.g. Ethier and Kurtz, 2009) under $\mathbb{P}$. The first two articles consider discrete time models, where the conditional dynamics of the risk factors can be written in the general form

$$X_t = \mu_t + \Phi_t X_{t-1} + \Sigma_t \epsilon_t,$$

where μ_t is a vector and Φ_t is a square matrix. When μ_t, Φ_t and Σ_t are assumed to be constants and ϵ_t is assumed to be independent and identically normally distributed, the dynamics of X_t are given by a Gaussian vector autoregressive process of the first order. This is the case in the affine Gaussian multiple yield curve model of the second article.

Allowing Σ_t to vary as a function of time and the past values of X_t permits conditional heteroscedasticity, which is useful in the modeling of many financial time series, e.g. stock returns, but it complicates the estimation of the model. A tractable special case is obtained when the individual components of X_t are assumed to be independent of each other. In the ICA models analyzed in the first essay of this dissertation, I consider the case where the individual factors have dynamics of the form

$$x_{i,t} = \mu_i + \phi_i x_{i,t-1} + \sigma_{i,t}\epsilon_{i,t},$$

where $e_{i,t}$ is an i.i.d. process with unit volatility and expectation of zero, and $\sigma_{i,t}$ follows a GARCH$(1,1)$ process

$$\sigma_{i,t}^2 = \alpha_{i,0} + \alpha_{i,1}\left(\sigma_{i,t-1}\epsilon_{i,t-1}\right)^2 + \beta_{i,1}\sigma_{i,t-1}^2$$

or a GJR$(1,1)$ (Glosten, Jagannathan and Runkle, 1993) process

$$\sigma_{i,t}^2 = \alpha_{i,0} + \alpha_{i,1}(\sigma_{i,t-1}\epsilon_{i,t-1})^2 + \beta_{i,1}\sigma_{i,t-1}^2 + \gamma_i \mathbb{1}_{\sigma_{i,t-1}\epsilon_{i,t-1}\leq 0}(\sigma_{i,t-1}\epsilon_{i,t-1})^2,$$

where $\alpha_{i,0} > 0$, $\alpha_{i,1} \geq 0$, $\beta_{i,1} \geq 0$ and $\alpha_{i,1} + \gamma_i \geq 0$ in order to ensure that the variance is positive. The GJR model differs from GARCH by permitting the signs of the innovations to affect the dynamics of conditional volatility. Thus for example negative shocks can have a bigger effect on market volatility than positive shocks.

The continuous time formulation of the state space vector most commonly used in finance, when jumps are not included, is of the form

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t,$$

where W_t is a standard Brownian motion, $\mu(X_t, t)$ is an n-vector and $\sigma(X_t, t)$ is an $n \times k$ matrix such that a solution to the equation exists. This class of state vectors covers for example the affine term structure models of Duffie and Kan (1996), and for a comprehensive analysis of the properties of affine processes I refer to Duffie, Filipović and Schachermayer (2003). For the purposes of this dissertation, it is sufficient to consider the multivariate Ornstein-Uhlenbeck diffusion, which is obtained when σ is a constant and μ is an affine function of X_t. It has the useful property that discrete time observations of the diffusion follow a VAR(1) process.

The dynamics of the risk factors affect the prices of financial assets, and in many practical applications it is important to ensure that the models produce arbitrage free prices. This can be achieved by using the Fundamental Theorem of Asset Pricing that was introduced by Harrison and Kreps (1979) and Harrison and Pliska (1981), and later generalized among others by Delbaen and Schachermayer (1994, 1998). In the chosen approach the prices are derived under the pricing measure $\mathbb{Q}$ that is equivalent to $\mathbb{P}$, i.e. $\mathbb{P}(A) = 0 \Leftrightarrow \mathbb{Q}(A) = 0$. The Radon-Nikodym derivative, $d\mathbb{Q}/d\mathbb{P}$, between the two probability measures is related to market price of risk.

Theorem (Fundamental Theorem of Asset Pricing)**.** *Consider a market model consisting of the price processes* $P_{0,t}, P_{1,t}, \ldots, P_{N,t}$ *on the time interval* $[0,T]$, *where it is assumed that the the numeraire process fulfills the condition* $P_{0,t} > 0$ *for all* $t \geq 0$ *almost surely under the probability measure* $\mathbb{P}$. *The model is said to be arbitrage free if and only if there exists a martingale measure* $\mathbb{Q} \sim \mathbb{P}$ *such that the processes*

$$\frac{P_{0,t}}{P_{0,t}}, \frac{P_{1,t}}{P_{0,t}}, \ldots, \frac{P_{N,t}}{P_{0,t}}$$

are local martingales under $\mathbb{Q}$. *The measure* $\mathbb{Q}$ *is unique if and only if the market model is complete.*

Typically a so called bank account process growing at the risk free rate is chosen as the numeraire, and its value at time zero is fixed as one. Denote the time t value of a bank account process growing at the continuous risk free rate by $B(t) = \exp\left(\int_0^t r_s ds\right)$ and the time t price of a zero coupon bond paying one unit of money at time T by $P(t,T)$. The arbitrage free price of the zero coupon bond at time 0 is then defined as

$$P(0,T) = E_t^{\mathbb{Q}}\left[\frac{1}{B(0,T)}\right] = E_t^{\mathbb{Q}}\left[e^{-\int_0^T r_s ds}\right],$$

where $P(T,T) = B(0,0) = 1$. The prices of many common derivatives such as futures, swaps and European options can be solved similarly by calculating the risk neutral expectation of the discounted amount paid in the future at time T.

It is important to notice that while the Fundamental Theorem of Asset Pricing makes it convenient to formulate theoretical pricing models under the pricing measure $\mathbb{Q}$, the observations of market prices will still be made under the physical measure $\mathbb{P}$. Due to this the dynamics of the asset prices and the state variables driving them have to be known under both probability measures in order to make econometric analysis possible. Given the continuous time dynamics of X_t considered above, the pricing measure $\mathbb{Q}$ can be defined on $\mathcal{F}_T$ as

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{\int_0^T \lambda'(X_\tau)dW_\tau^P - \frac{1}{2}\int_0^T \|\lambda(X_\tau)\|^2 d\tau},$$

where $\lambda(X_\tau)$ is the market price of risk that satisfies the usual condition

$$E^{\mathbb{P}}\left[e^{\int_0^T \lambda'(X_\tau)dW_\tau^P - \frac{1}{2}\int_0^T \|\lambda(X_\tau)\|^2 d\tau}\right] = 1.$$

W_t^Q, the Wiener process under $\mathbb{Q}$, is related to W_t^P by the Girsanov transformation $dW_t^Q = dW_t^P - \lambda(X_t)dt$.

When the market is incomplete, there are several possible martingale measures, and equivalently several possible specifications for the market price of risk. In term structure modelling it is common to circumvent this issue by fixing a specific parametrization for the market price of risk (see e.g. Dai and Singleton, 2000; Duffee, 2002; Cheridito, Filipović and Kimmel, 2007). The determination of the market price of risk is an empirical problem, not a theoretical one, as the actual martingale measure $\mathbb{Q}$ is chosen by the market (Björk, 2004). Assuming a particular structure for the market price of risk is thus equivalent with an assumption about the preferences of the market.

2.2 Financial risk measurement

Risk measurement is an essential part of financial risk management, as it makes it possible for financial institutions to prepare for adverse outcomes by indicating the amount of capital necessary to hold as a buffer against potential losses (McNeil et al., 2015). In the case of market risk, the focus is on measuring losses caused by changes in the asset prices.

Consider a portfolio consisting of N assets with time t prices $P_{1,t}, \ldots, P_{N,t}$ in amounts $n_1, \ldots, n_N$. The value of the portfolio is given by

$$V_{p,t} = n_1 P_{1,t} + \ldots + n_N P_{N,t}.$$

Assuming that the assets don't pay dividends and the composition of the portfolio remains unchanged, the change in portfolio value between times t and $t+1$ is given by

$$V_{p,t+1} - V_{p,t} = n_1 \left(P_{1,t+1} - P_{1,t}\right) + \ldots + n_N \left(P_{N,t+1} - P_{N,t}\right)$$

or in terms of returns as

$$r_{p,t+1} = w_1 r_{1,t+1} + \ldots + w_N r_{N,t+1},$$

where $w_i = n_i P_{1,t}/V_{p,t}$ and $r_{i,t+1} = P_{i,t+1}/P_{i,t} - 1$. The market risk of this portfolio over one period is measured as function of the probability distribution $F_{r_{p,t}}(x) = \mathbb{P}(r_{p,t} \leq x)$. The functions used for measuring the risk are known as risk measures. Among the most commonly used risk measures are volatility, which was used as the basis of the modern portfolio theory of Markowitz (1952), and value-at-risk (VaR), which was originally developed at J.P. Morgan and was later adopted as the basis of capital requirements in the Basel accords.

Given some confidence level $\alpha \in (0, 1)$, the VaR of the portfolio considered above is defined as

$$VaR_\alpha = -\inf\left\{x \in \mathbb{R} : \mathbb{P}(r_{p,t} < x) \geq 1 - \alpha,\right\} = -\inf\left\{x \in \mathbb{R} : F_{r_{p,t}}(x) \geq 1 - \alpha\right\},$$

where minus sign turns the obtained value positive, as losses are in the left tail of the return distribution. The obtained value is a quantile, and it can be interpreted as the maximum loss that is not exceeded at probability α during the forecasting period. However, it provides no information about the magnitude of losses when the VaR level is exceeded, which happens at probability $(1 - \alpha)$. Due to this, VaR is neither a subadditive nor a coherent risk measure in the sense of Artzner, Delbaen, Eber and Heath (1999). Despite this drawback, VaR can be a useful tool, because it is easy to interpret, the related estimates are straightforward to test empirically using any method suitable for the evaluation of quantile forecasts (see e.g. Kupiec, 1995; Christoffersen, 1998), and coherent risk measures such as expected shortfall can be approximated by using VaR estimates calculated at multiple different levels.

Above, I considered the case where the portfolio returns are expressed as a linear function of simple asset returns. If the simple stock returns are assumed to be affine functions of risk factors, as is typical in the asset pricing literature when working with monthly data, aggregation on portfolio level is relatively simple as the portfolio returns themselves will be affine in the same factors. The simple returns, however, are more complicated than logarithmic returns to aggregate over time, whereby compromises, such as the utilization of approximations, have to be made no matter which approach is chosen.

In the first article I use logarithmic returns, and utilize a first order Taylor approximation centered around zero in the estimation of VaR. In the context of financial modeling the first and second order Taylor approximations are usually known as the delta and delta-gamma approximation (see e.g. Britten-Jones and Schaefer, 1999). They can be used to generalize the modeling approach I utilized in the aforementioned article into so called non-linear portfolios that contain derivatives or other financial instruments, whose prices are non-linear functions of the underlying risk factors.

If the risk factors X_{t+1} are normally distributed conditional on the information set $\mathcal{F}_t$ available at time t, the delta and gamma-approximation can be used to calculate estimates of VaR or other risk measures for time $t + 1$ in a straightforward way, as the approximation will have a normal or non-central chi-squared distribution conditional on $\mathcal{F}_t$. In other cases alternative approaches, such as Monte Carlo simulations or saddlepoint approximations (see e.g. Lugannani and Rice, 1980; Broda and Paolella, 2009) are usually utilized in order to estimate the risk measure. If X_t is an affine function of independent random variables whose distributions are known, then the delta approximation can be used to easily calculate the approximate characteristic function of the losses, and a Fourier transform can be used to obtain the density function.

3 ON THE ESTIMATION OF LATENT FACTOR MODELS

Estimation of the proposed models forms a fundamental part of this dissertation, as it is necessary in order to evaluate their performance based on empirical data. While it is possible to estimate some latent factor models without estimating the underlying factors, e.g. by using moment based estimators, I have elected to only use methods that also give estimates of the underlying factors. The estimation approaches considered can be classified into two categories. The first category consists of two-step estimation approaches, where at first the latent state variables are extracted from the data using principal component analysis (PCA) or independent component analysis (ICA) and in the second step the rest of the model parameter estimated. In the second approach all of the model parameters and the latent factor are estimated simultaneously using a quasi-maximum likelihood estimator that utilizes the Kalman filter.

3.1 Principal Component Analysis and Independent Component Analysis

PCA and ICA are blind source separation methods that are closely related to each other. ICA can be considered to be an extension of PCA (Comon, 1994), and PCA is commonly used as the first step in ICA algorithms for dimension reduction, if necessary, and pre-whitening the data. PCA was independently developed by Pearson (1901) and Hotelling (1933). The basic idea of the method is to transform a set of observations into a set of linearly uncorrelated factors known as principal components. It can also be used for dimension reduction, as the principal components are ordered based on the share of the variance in the observations they explain, with the first component having the largest explanatory power.

Let X_t denote observations made of a stationary random vector at time t that have zero empirical mean. Let $\hat{X} = (X_1, \ldots, X_T)'$ be a $T \times n$ data matrix, and let $w_{(i)}$, $i = 1, \ldots, n$, be a loading vector of dimension n. The first principal component $u_{(1)}$ is given by

$$u_{(1)} = \hat{X} w_{(1)},$$

where the loadings satisfy

$$w_{(1)} = \underset{\|w\|=1}{\arg\max} \|\hat{X} w\|^2.$$

A similar procedure can be used to solve solve the kth principal component after subtracting the preceding $k-1$ from the observations:

$$\hat{X}_k = \hat{X} - \sum_{i=1}^{k-1} u_{(i)} w'_{(i)} = \hat{X} - \sum_{i=1}^{k-1} \hat{X} w_{(i)} w'_{(i)}$$
$$w_{(k)} = \text{arg max}_{\|w\|=1} \|\hat{X}_k w\|^2.$$

PCA is inherently connected to the spectral decomposition. This can be seen by considering the sample covariance matrix $\hat{\Sigma}$ of the observed variables $\hat{X}$, which is proportional to $\hat{X}'\hat{X}$. Since the covariance matrix is symmetric, it is orthogonally diagonalizable

$$\hat{\Sigma} = V \Delta V',$$

where the matrix $\Delta = \text{diag}(\lambda_1, \ldots, \lambda_n)'$ has the eigenvalues of $\hat{\Sigma}$ as its diagonal elements in descending order, i.e. $\lambda_1 \geq \lambda_2 \geq \ldots, \geq \lambda_n \geq 0$, and V is an orthogonal matrix, i.e. $V'V = VV' = I$, with the eigenvectors of $\hat{\Sigma}$ as its column vectors. It can be shown that the matrix of principal components can be expressed as

$$U = (u_{(1)}, \ldots, u_{(n)}) = \hat{X}V,$$

and the empirical covariance matrix of the principal components is proportional to Δ. While the spectral decomposition provides an elegant way for presenting the properties of the principal components, it is numerically preferable to calculate the components using the singular value decomposition from the raw data.

ICA is based on the assumption that the observed values X_t are a linear combination of independent components s_t that can't be observed directly. When there is no noise, the system can thus be written as

$$X_t = A s_t, \quad t = 1, \ldots, T$$

where the components of the random vector s_t denote the underlying factors, which are assumed to be stationary, and one can assume without loss of generality that $E(s_t) = 0$ and $Cov(s_t) = I$. ICA algorithms seeks to estimate both the mixing matrix A and s_t using only the observations X_t. The scale of the components is effectively identified for estimation purposes by the assumption about the covariance matrix, but neither the ordering nor the signs of the components can be uniquely identified.

Where PCA utilizes only second order conditions, which are suitable for separating Gaussian random variables, ICA algorithms can utilize more complex conditions, such as negentropy in the case of fastICA (Hyvärinen and Oja, 1997). ICA algorithms require that at most one of the components of s_t is normally distributed, but in the applications I consider this is non-restrictive, because the presence of a time varying volatility means that the components will be at most conditionally normal.

It should be noted that when the extracted factors are utilized in an affine financial model, there is practically no difference between using principal components and independent components as they both span the same vector space and all differences in the results will be explained by the factors that were left out. ICA can however be preferable to PCA, if one is either interested in the extracted factors themselves or uses them in a non-linear model. I explore the latter case in the first article incorporated in this dissertation, where I model the underlying factors as GARCH-processes.

3.2 The Kalman Filter

The Kalman (1960) filter and its extensions, which were famously incorporated in the navigation system of the Apollo Project (see e.g. Grewal and Andrews (2010)), have become popular tools for estimating latent factor models in multiple fields. The use of Kalman filtering in economics was pioneered by Athans (1974), Chong and Cheng (1975) and Burmeister and Wall (1982), and it has been adopted as one of the most commonly used methods for estimating affine term structure models (see e.g. Duffee and Stanton, 2012). The general properties and applications of the Kalman filter have been considered in numerous books and articles, and I refer to Jazwinski (1970), Harvey (1990) and Hamilton (1994) for a more detailed analysis.

The focus of this section is on the linear Kalman filter, but it also gives sufficient insight into the quadratic Kalman filter (Monfort, Renne and Roussellet, 2015), which is based on the transformation of a quadratic system into a linear one via the introduction of additional factors. Other extensions such as the unscentend Kalman filter (Julier and Uhlmann, 1997; Wan and Van Der Merwe, 2000) have also been applied to the estimation of term structure models (see e.g. Christoffersen, Dorion, Jacobs and Karoui, 2014; Filipović and Trolle, 2017), but they are not relevant for the purposes of this dissertation, as I only apply the Kalman filter to the estimation of a quadratic model, whereby the quadratic Kalman filter is the most sensible choice.

I will proceed by describing a linear state space model and explaining how the Kalman filter can be applied to it. Finally, I will describe how the filter can be utilized in maximum likelihood estimation. The maximum likelihood estimator is exact in the case of linear Gaussian models, and can be used as a consistent and asymptotically normal quasi-maximum likelihood estimator provided that sufficient conditions, such as those described in Theorem 2 of Watson (1989), apply.

Let Y_t denote an n-dimensional vector of variables observed at time $t = 1, \ldots, T$, and similarly let X_t denote a state vector of dimension m. In classical Kalman filtering it is assumed that the state space can be written as or approximated by the equations

$$
\begin{aligned}
X_{t+1} &= F_{0,t} + F_{1,t}X_t + v_{t+1};\\
Y_{t+1} &= H_{0,t} + H_{1,t}X_{t+1} + w_{t+1};\\
E[v_t] = 0, \quad E[v_t' v_\tau] &= \begin{cases} Q_t, & t=\tau \\ 0, & t\neq\tau \end{cases};\\
E[w_t] = 0, \quad E[w_t' w_\tau] &= \begin{cases} R_t, & t=\tau \\ 0, & t\neq\tau \end{cases},
\end{aligned}
$$

where the deterministic system matrices $F_{0,t}, F_{1,t}, Q_t, H_{0,t}, H_{1,t}, R_t$ are of dimension $m\times 1, m\times m, m\times m, n\times 1, n\times m$ and $n\times n$, and the innovations v_t and w_t are assumed to be uncorrelated for all lags. The filtering proceeds by recursively using estimated values of the state variables X_t to predict X_{t+1} and Y_{t+1} and their covariances, and then using the observed value of Y_{t+1} to update the predictions. The recursion is initialized using the unconditional expectation and covariance matrix of X_t, i.e. $E[X_t] = \hat{X}_{0|0}$ and $E[(X_t - E[X_t])(X_t - E[X_t])']) = P_{0|0}$.

During each iteration the estimated value $\hat{X}_{t|t}$ is used alongside the matrices F_{0t}, F_{1t} and Q_t to make one period ahead predictions of X_{t+1} and the associated mean squared error matrix conditional on the observed past values of Y_t, $\mathcal{Y}_t = (Y_t', \ldots, Y_1')$. The predictions are given by the equations

$$
\begin{aligned}
\hat{X}_{t+1|t} &= E[X_{t+1}|\mathcal{Y}_t] = F_0 + F_1\hat{X}_{t|t}\\
P_{t+1|t} &= E\left[\left(X_{t+1} - \hat{X}_{t+1|t}\right)\left(X_{t+1} - \hat{X}_{t+1|t}\right)'\right] = F_1 P_{t|t} F_1' + Q_t,
\end{aligned}
$$

The one period ahead predictions of Y_{t+1} and the associated mean squared error matrix are calculated similarly using the predicted values $\hat{X}_{t+1|t}$ as

$$
\begin{aligned}
\hat{Y}_{t+1|t} &= E[Y_{t+1}|\mathcal{Y}_t] = H_0 + H_1\hat{X}_{t+1|t}\\
V_{t+1|t} &= E\left[\left(Y_{t+1} - \hat{Y}_{t+1|t}\right)\left(Y_{t+1} - \hat{Y}_{t+1|t}\right)'\right] = H_1 P_{t+1|t} H_1' + R_t,
\end{aligned}
$$

which are then used to calculate the prediction error $e_{t+1} = Y_{t+1} - \hat{Y}_{t+1|t}$. Finally the state vector and its covariance matrix are updated using the equations

$$
\begin{aligned}
\hat{X}_{t+1|t+1} &= E[X_{t+1}|\mathcal{Y}_{t-1}] = \hat{X}_{t+1|t} + P_{t+1|t}H_1' V_{t+1|t}^{-1} e_{t+1}\\
P_{t+1|t+1} &= E\left[\left(X_{t+1} - \hat{X}_{t+1|t+1}\right)\left(X_{t+1} - \hat{X}_{t+1|t+1}\right)'\right]\\
&= P_{t+1|t} - P_{t+1|t}H_1' V_{t+1|t}^{-1} H_1 P_{t+1|t}
\end{aligned}
$$

Assuming that v_t and w_t are normally distributed, the log-likelihood function is given by

$$
\sum_{t=1}^{T} \log f(e_t, V_{t|t-1})
$$

where

$$f(e_t, V_{t|t-1}) = -\frac{1}{2}\left(n\log(2\pi) + \log\left|V_{t|t-1}\right| + e_t' V_{t|t-1}^{-1} e_t\right).$$

In the setting considered in my third article, the matrices $F_{i,t}$ and $H_{i,t}$ are functions of the unknown model parameters. As is typical to continuous time affine and quadratic interest rate models, at least parts of the matrices will be obtained as numerical solutions to a system of ordinary differential equations. Due to this, I have to rely on numerical derivatives in the optimization of the likelihood function and in the analysis of the asymptotic behaviour of the parameter estimates.

Numerical estimation of high order derivatives can be unreliable. Therefore the estimates of asymptotic covariance matrices in this dissertation will be based on first order numerical derivatives when the Kalman filter is used. Let $f(e_t, V_{t|t-1}; \hat{\theta})$ be the value of the conditional likelihood function conditioned on the parameter values $\hat{\theta}$. The estimator of the asymptotic covariance matrix of the estimated model parameters $\hat{\theta}$ is then given by

$$\widehat{\text{Avar}(\hat{\theta})} = \left[\sum_{t=1}^{T} s(e_t, V_{t|t-1}; \hat{\theta}) s(e_t, V_{t|t-1}; \hat{\theta})'\right]^{-1},$$

where

$$s(e_t, V_{t|t-1}; \theta) = \frac{\partial \log f(e_t, V_{t|t-1}; \theta)}{\partial \theta}.$$

4 SUMMARIES OF THE INCLUDED ESSAYS

(1) Market Risk Forecasting with Independent Component Analysis

In the first essay I consider forecasting VaR for a portfolio of stocks by using ICA to model the dependence structure. I assume that that the vector of asset returns can be written as

$$x_t = As_t,$$

where A is a square mixing matrix and s_t is a vector of independent random variables. Furthermore, I assume that each component $s_{i,t}$ can be written as

$$s_{i,t} = \mu_{i,t} + \sigma_{i,t}\epsilon_{i,t}.$$

where $\mu_{i,t}$ and $\sigma_{i,t}$ are functions of t and past values of $\epsilon_{i,t}$ and $s_{i,t}$, and $\epsilon_{i,t}$ is an i.i.d. random process with volatility of one and expected value of zero. In other words, $s_{i,t}$ is adapted to the filtration $\mathcal{F}_t = \sigma(s_{i,\tau}, \tau < t)$. Thus the x_t and s_t can be expressed as

$$s_t = \mu_t + \Sigma_t\epsilon_t,$$
$$x_t = \mu_t^{(x)} + A_t\epsilon_t$$

where $\Sigma_t = \text{diag}(\sigma_{1,t}, \ldots, \sigma_{n,t}), \mu_t^{(x)} = A\mu_t$ and $A_t = A\Sigma_t$.

I estimate the model in two steps. In the first step I use ICA-algorithms, i.e. PCA, fastICA (Hyvärinen and Oja, 1997), gFOBI or gJADE (García-Ferrer et al., 2011; Matilainen et al., 2015) to estimate $\hat{A}$ and $\hat{s}_t = \hat{A}^{-1}x_t$. In the second step I estimate $\mu_{i,t}$ and $\sigma_{i,t}$ from $\hat{s}_t$ by fitting suitable models. Following previous research on financial time series, I chose to use AR-GARCH and AR-GJR processes for this purpose.

The empirical performance of the different models is tested using data on daily American stock returns covering the years 2000-2014. VaR forecasts for the ICA based models are obtained via Monte Carlo simulations. The ICA estimates are compared with filtered extreme value theory based estimates (McNeil and Frey, 2000). PCA is used as a baseline for the ICA models in order to test whether the use of more complicated algorithms is warranted. The results show that under normal market conditions all the different approaches perform relatively well, with the fastICA seeming to slightly outperform the other ICA algorithms. During the Financial Crisis of 2007-2008 the behavior of the models however changed, and all of the tested model were rejected by the coverage ratio tests. The gFOBI and gJADE based approaches performed particularly badly during this period, which was likely caused by their sensitivity to outliers.

(2) Affine Multiple Curve Modelling via Least Squares Regressions

I propose an affine Gaussian multiple curve model for the term structure of interest rates, and extend the estimation approach of (Adrian et al., 2013) into the multiple curve framework. The OIS rates are used as risk free rates that follow the no-arbitrage condition. EURIBORs are not treated as traded instruments and thus they do not follow the no-arbitrage condition, but the derivative instruments such as forward agreements and swaps that have EURIBOR rate as the underlying instrument are priced using arbitrage free formulas. I primarily use principal components extracted from yields as the risk factors, but also consider the case where they are augmented with a proxy related to liquidity risk.

In the model I assume that under the physical probability measure the risk factors follow a VAR(1) process $\mathbb{P}$ of the form

$$X_{t+1} = \mu + \Phi X_t + v_{t+1}, \quad v_{t+1}|X_t \sim N(0, \Sigma).$$

The arbitrage free price of a risk free bond with maturity n at time t by $P_t^{(n)}$ is given by

$$P_t^{(n)} = E_t^{\mathbb{P}}\left[M_{t+1}P_{t+1}^{(n-1)}\right] = \exp\left(A_n + B_n' X_t\right),$$

where the pricing kernel M_{t+1} is assumed to be of the exponentially affine form

$$M_{t+1} = \exp(-r_t - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\Sigma^{-1/2}v_{t+1}).$$

The continuously compounded risk free rate is denoted by $r_t = -\ln P_t^{(1)}$, i.e. the risk free bank account process $B(t, t+n) = e^{\sum_{i=0}^{n-1} r_{t+i}}$ is used as the numeraire. The market price of risk is assumed to be of the essentially affine form of Duffee (2002)

$$\lambda_t = \Sigma^{-1/2}(\lambda_0 + \lambda_1 X_t).$$

EURIBORs are modeled via tenor specific artificial bonds that are also assumed to be exponentially affine functions of the risk factors:

$$\tilde{P}_t^{(\delta)} = e^{\tilde{A}_\delta + \tilde{B}_\delta' X_t}.$$

I derive explicit pricing formulas for futures and swaps based on arbitrage free pricing theory, and show how to estimate the model using least squares regressions. The performance of the model is demonstrated using euro area interest rate data covering the years 2006-2015. The results show that the model is able to explain most of the market dynamics in the analysed OIS, EURIBOR and EURIBOR FRA rates.

(3) Credit Liquidity and the Term Structure of Interest Rates

I consider a quadratic multiple curve model for the term structures of interest rates, where OIS rates are considered as representing the risk free rate and IBOR related instruments trade at a spread to them. The risk factors are assumed to follow a multivariate Ornstein-Uhlenbeck diffusion under both the risk neutral probability measure $\mathbb{Q}$ and the physical probability measure $\mathbb{P}$:

$$dX_t = (\mu^Q + K^Q X_t)dt + \Sigma dW_t^Q$$
$$dX_t = (\mu^P + K^P X_t)dt + \Sigma dW_t^P,$$

where μ^Q and K^Q can be parametrized independently of μ^P and K^P.

The dynamics OIS rates and IBORs are defined via the prices of risk free zero coupon bonds $P(t,\tau)$ and artificial zero coupon bonds $\tilde{P}(t,\tau)$, which are driven by the spot rate r_t and the spot spread s_t:

$$r_t = \alpha + \beta' X_t + X_t' \Psi X_t$$
$$s_t = \alpha_s + \beta_s' X_t + X_t' \Psi_s X_t,$$

where Ψ and Ψ_s are symmetric positive-semidefinite matrices. Zero coupon bond prices given by

$$P(t,\tau) = E_t^{\mathbb{Q}}\left[e^{-\int_t^{t+\tau} r_\tau d\tau}\right] = e^{A(\tau)+B(\tau)' X_t + X_t' C(\tau) X_t}$$
$$\tilde{P}(t,\tau) = E_t^{\mathbb{Q}}\left[e^{-\int_t^{t+\tau} r_\tau + s_\tau d\tau}\right] = e^{\tilde{A}(\tau)+\tilde{B}(\tau)' X_t + X_t' \tilde{C}(\tau) X_t},$$

where $A(\tau), B(\tau), C(\tau), \tilde{A}(\tau), \tilde{B}(\tau)$ and $\tilde{C}(\tau)$ are the solutions to a system of ordinary differential equations. The price of the artificial bonds are related to the EURIBOR rate of tenor δ through the equation $1 + \delta L(t, t+\delta) = 1/\tilde{P}(t,\delta)$. As a special case of the proposed model I provide a multiple curve extension of the arbitrage free Nelson-Siegel model, which avoids some of the problems typically encountered in the maximum likelihood estimation of more general affine models (see e.g. Hamilton and Wu, 2012).

The models are estimated by quasi maximum likelihood estimation via the quadratic Kalman filter (Monfort et al., 2015) on the basis of European data covering the years 2009-2014. In the empirical part of the study I assume that the spreads are driven by credit and liquidity risk. The results imply that these factors also affect the risk free rates under the physical probability measure.

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Part II

ESSAYS

1

ESSAY 1

Matti Heikkonen

Market Risk Forecasting with Independent Component Analysis

Preprint

Market Risk Forecasting with Independent Component Analysis

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Abstract

Over the recent years there has been increasing interest in the application of independent component analysis (ICA) to the modelling of financial time series. This article evaluates the performance of different ICA methods in forecasting daily Value-at-Risk (VaR) estimates for stock portfolios with time varying volatilities and expected returns. ICA is used to estimate latent factors from logarithmic returns, and the dynamics of said factors are modelled using as AR-GARCH or AR-GJR processes. We find that ICA based methods perform well under normal market conditions and are able to give accurate VaR forecasts at both asset and portfolio level. The results also highlight the importance of the chosen back testing period because structural breaks and outliers in the data, such as those observed during the financial crisis of 2007-2008, can have a more significant effect on some ICA-algorithms than others.

KEYWORDS: market risk, backtesting, risk management, FOBI, JADE, independent component analysis

1 INTRODUCTION

Market risk modelling and the estimation of risk measures such as Value at Risk (VaR) have received increased attention due to the recent financial crisis. Despite extensive research and methodological advances, forecasting VaR has turned out to be challenging, since asset returns do not follow a multivariate normal distribution and they typically exhibit time varying volatilities. The typically high dimensionality of portfolios adds to the problems inherent to the risk forecasting process.

* Department of Accounting and Finance, Turku School of Economics, University of Turku. I thank Hannu Oja, Klaus Nordhausen, Markus Matilainen, Luis Alvarez and everyone, who participated in the Joint Applied Mathematics and Statistics Seminars at the University of Turku in 2015 for comments.

Multiple different approaches to forecasting VaR have been developed and proposed in the existing literature. GARCH-models (Bollerslev, 1986) and their variations have been used to make corrections for conditional heteroskedasticity. Similarly, a plethora of different approaches such as extreme value theory (McNeil and Frey, 2000) have been used for modelling the probability distribution of losses, and there have been extensive comparisons of different estimation methods; see for example Hull and White (1998), Engle and Manganelli (2004), Kuester, Mittnik and Paolella (2006), Mancini and Trojani (2011), Santos, Nogales and Ruiz (2013). The performance of models has typically varied depending on the market conditions of the test period.

This article focuses on the application of independent component analysis (ICA) methods in the estimation of financial risk measures. The methodological choice is motivated by the search for a tractable model, which is capable of accounting for dependencies and non-normality in a data set exhibiting autocorrelations and conditional heteroskedasticity. The ICA approach is based on the assumption that the observed data is formed as a linear combination of latent independent factors. This assumption allows us to estimate the model in two steps: first ICA will be used to extract the underlying factors and the mixing matrix depicting the linear combinations, and then appropriate time series models, e.g. AR(1)-GARCH(1,1), will be fitted to the individual factors. Similar ICA based approaches have been utilized in volatility modelling and VaR forecasting with promising results (see e.g. Wu, Yu and Li, 2006; Chen, Härdle and Spokoiny, 2007, 2010; Broda and Paolella, 2009; García-Ferrer, González-Prieto and Peña, 2012).

We contribute to previous research on the subject by comparing the accuracy of VaR forecasts based on different ICA-algorithms, including the popular fastICA (Hyvärinen and Oja, 1997) and the recent extensions to FOBI and JADE proposed by (García-Ferrer, González-Prieto and Peña, 2011) and (Matilainen, Nordhausen and Oja, 2015) for time series exhibiting stochastic volatility. We also study the effect different conditional volatility models have on the accuracy of the forecasts. The performance of the models is evaluated using daily American stock returns covering the years from 2000 to 2014, which makes it possible to test the forecasting performance of ICA based models under different market conditions. VaR forecasts are made for individual assets and at portfolio level in order to test whether the tested multivariate models will be able to sufficiently depict the time series dynamics on both levels at the same time.

The remainder of this article is organized as follows. The proposed time series model for asset returns and the different ICA methodologies are covered in Section 2. Section 3 presents an empirical analysis and comparison of the different models. Section 4 concludes the paper.

2 ICA METHODOLOGY FOR TIME SERIES

2.1 Asset and portfolio return dynamics

Let $\mathbf{x}_t$ be a vector of asset returns and suppose that they are formed as a linear combination of n underlying factors s_{it} that are independent from each other. The returns can thus be written as

$$\mathbf{x}_t = \mathbf{A}\mathbf{s}_t, \tag{1}$$

where $\mathbf{A}$ is a so-called mixing matrix, and $\mathbf{s}_t = (s_{1,t}, \dots, s_{n,t})$.

Only the asset returns $\mathbf{x}_t$ in equation (1) are assumed to be known as $\mathbf{A}$ and $\mathbf{s}_t$ can't be directly observed. However, if $\mathbf{A}$ were a known invertible square matrix, the underlying factors $\mathbf{s}_t$ could be easily solved from equation (1):

$$\mathbf{s}_t = \mathbf{A}^{-1}\mathbf{x}_t. \tag{2}$$

The process of simultaneously solving $\mathbf{A}$ and $\mathbf{s}_t$ from the observations $\mathbf{x}_t$ is known as blind source separation in signal processing and statistics literature, and multiple different algorithms for solving the problem have been proposed. The simplest approach would be to assume that correlations are sufficient measures of dependence and use principal component analysis (PCA), but using additional criteria will lead to more sophisticated ICA-algorithms, e.g. fastICA (Hyvärinen, 1999) or gFOBI (Matilainen et al., 2015). For an overview of ICA methodology we refer to Hyvärinen, Karhunen and Oja (2001).

The basic model can be extended by allowing for time series dynamics in the factors $\mathbf{s}_t$. Suppose that each component $s_{i,t}$ can be written as:

$$s_{i,t} = \mu_{i,t} + \sigma_{i,t}\epsilon_{i,t}, \tag{3}$$

where $\epsilon_{i,t}$ is an i.i.d. random process with volatility of one and expected value of zero, and $\mu_{i,t}$ and $\sigma_{i,t}$ are functions of t and the past value of $\epsilon_{i,t}$ and $s_{i,t}$. In other words $\mu_{i,t}$ is the expected value and $\sigma_{i,t}$ is the volatility of $s_{i,t}$ conditional on $\mathcal{F}_{t-1}$, the information set available at time $t-1$. The factors $\mathbf{s}_t$ can now be expressed as

$$\mathbf{s}_t = \boldsymbol{\mu}_t + \boldsymbol{\Sigma}_t\boldsymbol{\epsilon}_t, \tag{4}$$

where $\boldsymbol{\Sigma}_t = \mathrm{diag}(\sigma_{1,t}, \dots, \sigma_{n,t})$. We can similarly decompose the representation of returns in equation (1) as

$$\mathbf{x}_t = \boldsymbol{\mu}_t^{(x)} + \mathbf{A}_t\boldsymbol{\epsilon}_t, \tag{5}$$

where $\boldsymbol{\mu}_t^{(x)} = \mathbf{A}\boldsymbol{\mu}_t$ and $\mathbf{A}_t = \mathbf{A}\boldsymbol{\Sigma}_t$.

We aggregate the returns on portfolio level, for a portfolio with asset weights given by vectors $\mathbf{w}_p$, using the approximation

$$r_{p,t} = \mathbf{w}_p'\mathbf{x}_t = \mathbf{w}_p'\boldsymbol{\mu}^{(x)} + \mathbf{w}_p'\mathbf{A}_t\boldsymbol{\epsilon}_t,$$

which is exact for simple returns and coincides with the delta-approximation for logarithmic returns. There are several possible approaches for calculating risk measures of $r_{p,t}$ depending on the additional assumptions one makes. We have strived to minimize the effects of distributional assumptions in order to verify the applicability of ICA-type models to stock markets. Therefore we use the empirical distribution of the estimated independent components and estimate VaR forecasts through Monte Carlo simulations. The chosen method allows one to calculate both asset specific and portfolio level VaR estimates using the same simulations.

Because $r_{p,t}$ is a linear combination of independent random variables, one could alternatively calculate its characteristic function, if the distributions of the components $\boldsymbol{\epsilon}_t$ are known, and combine inverse Fourier transforms with numerical integration in order to calculate VaR or other risk measures (Chen et al., 2010). The saddlepoint approximation of Lugannani and Rice (1980) can also be used, if the cumulant generating functions of the independent components exist (Broda and Paolella, 2009).

2.2 ICA algorithms

In order to describe the ICA estimation process on a general level, we assume that $\mathbf{s}_t$ in equation (1) is stationary, at most one of the components $s_{i,t}$ is normally distributed, and that the mixing matrix $\mathbf{A}$ is a full rank square matrix. The non-normality assumption is necessary for the identification of the independent components, but is non-restrictive in the application we consider since the presence of a time varying volatility means that the components can be at most conditionally normal. Without loss of generality we also assume that $E(\mathbf{s_t}) = 0$ and $\text{Cov}(\mathbf{s}_t) = \mathbf{I}$. This structure implies that $\mathbf{x}_t$ is stationary. The goal is to find an unmixing matrix $\mathbf{W}$ such that the marginal time series in $\mathbf{W}\mathbf{x}_t$ and $\mathbf{s}_t$ are the same up to their signs and order. In the case of ICA algorithms which do not utilize the time series structure of the data, we drop the time index from $\mathbf{x}_t$ and $\mathbf{s}_t$

Let $\mathbf{e}_i$ denote a vector of dimension n, where the ith element is 1 and all other elements are zero, and define $\mathbf{E}^{ij} := \mathbf{e}_i\mathbf{e}_j'$. Using this notation all the possible fourth order moments of an n-dimensional random vector $\mathbf{x}$ are given as elements of the matrices

$$\mathbf{B}^{ij}(\mathbf{x}) := E\left[\mathbf{x}\mathbf{x}'\mathbf{E}^{ij}\mathbf{x}\mathbf{x}'\right] \quad i, j = 1, \ldots, n,$$

and we can write

$$\mathbf{B}(\mathbf{x}) := \sum_i \mathbf{B}^{ii}(\mathbf{x}) = E\left[\mathbf{x}\mathbf{x}'\mathbf{x}\mathbf{x}'\right].$$

The fourth order cumulants of $\mathbf{x}$ are obtained as

$$\mathbf{C}^{ij}(\mathbf{x}) := \mathbf{B}^{ij}(\mathbf{x}) - \mathbf{E}^{ij} - \mathbf{E}^{ji} - tr(\mathbf{E}^{ij})\mathbf{I} \quad i, j = 1, \ldots, n.$$

The classic FOBI and JADE algorithms utilize moment conditions, and start by calculating the standardized vector $\mathbf{x_{st}} = \widehat{\mathrm{Cov}}(\mathbf{x})^{-\frac{1}{2}}\mathbf{x}$, where $\widehat{\mathrm{Cov}}(\mathbf{x})$ is the sample covariance matrix. The FOBI estimate of the unmixing matrix is $\mathbf{W} = \mathbf{U}\widehat{\mathrm{Cov}}(\mathbf{x})^{-\frac{1}{2}}$, where $\mathbf{U}$ is the orthogonal matrix that maximizes $||diag(\mathbf{U}\hat{\mathbf{B}}(\mathbf{x}_{st})\mathbf{U}')||$, i.e. the rows of $\mathbf{U}$ are the eigenvectors of the sample estimate of $\mathbf{B}(\mathbf{x}_{st})$. The JADE estimate is similar, except $\mathbf{U}$ has to maximize $\sum_{i,j} ||diag(\mathbf{U}\hat{\mathbf{C}}^{ij}(\mathbf{x}_{st})\mathbf{U}')||$. The generalized versions of FOBI and JADE, i.e. gFOBI and gJADE, as specified in García-Ferrer et al. (2011) and Matilainen et al. (2015) extend the original algorithms by utilizing lagged fourth order moments and cumulants respectively.

Using similar notation as above, denote lags by $\tau \geq 0$, and define the autocovariance matrices as:

$$\mathbf{\Sigma}_\tau(\mathbf{x}) = E\left[(\mathbf{x}_t - E(\mathbf{x}_t))(\mathbf{x}_{t+\tau} - E(\mathbf{x}_{t+\tau}))'\right].$$

The matrices related to cross moments are

$$\mathbf{B}^{ij}_\tau(\mathbf{x}) = E(\mathbf{x}_{t+\tau}\mathbf{x}_t'\mathbf{E}^{ij}\mathbf{x}_t\mathbf{x}_{t+\tau}') \text{ and } \mathbf{B}_\tau(\mathbf{x}) = \sum_{i=1}^{n} \mathbf{B}^{ii}_\tau(\mathbf{x}).$$

The gFOBI estimator of the unmixing matrix with lags $\tau = 0, \ldots, K$ is $\mathbf{W} = \mathbf{U}\widehat{\mathrm{Cov}}(\mathbf{x})^{-\frac{1}{2}}$, where the orthogonal matrix $\mathbf{U}$ maximizes $\sum_{\tau=0}^{K} ||diag(\mathbf{U}\hat{\mathbf{B}}_\tau(\mathbf{x_{st}})\mathbf{U}^{'})||^2$.

The lagged fourth order cumulants are

$$\mathbf{C}^{ij}_\tau(\mathbf{x}) := \mathbf{B}^{ij}_\tau(\mathbf{x}) - \mathbf{\Sigma}_\tau(\mathbf{x})(\mathbf{E}^{ij} + \mathbf{E}^{ji})\mathbf{\Sigma}_\tau(\mathbf{x})' - \mathrm{tr}(\mathbf{E}^{ij})\mathbf{I}_n, \quad i, j = 1, \ldots, n.$$

The gJADE estimator of the unmixing matrix with lags $\tau = 0, \ldots, K$ is the orthogonal matrix $\mathbf{U}$ that maximizes

$$\sum_{i,j=1}^{n} \sum_{\tau=0}^{K} ||diag(\mathbf{U}\hat{\mathbf{C}}^{\mathbf{ij}}_\tau(\mathbf{x_{st}})\mathbf{U}^{'})||^2. \tag{6}$$

The most popular ICA algorithm in finance has been fastICA, which searches for maximally non-Gaussian orthogonal components. We have opted to use negentropy as a measure of non-Gaussianity over kurtosis, due to it being less sensitive to outliers in the data. The information theoretical entropy of a random variable is related to the information contained in observations of said variable, and it's small for probability distributions which are concentrated on a few values. In the case of a continuous random variable x with density function f, we define its differential entropy H as

$$H(x) := -\int f(x)\log(f(x))\,dx. \tag{7}$$

It is known that Gaussian variables have the largest entropy among all random variables of equal variance. Thus a natural measure of non-Gaussianity can be calculated by subtracting the entropy of the random variable x from the entropy of a Gaussian variable x_g with the same variance. This gives us the definition of negentropy J:

$$J(x) = H(x_g) - H(x). \tag{8}$$

In practice approximations are needed in order to calculate negentropy, since using the definition would require knowledge of the true probability distribution of x. Using a non-quadratic function G, we approximate $J(x)$ in the case of standardized random variables as

$$J(x) \approx \left[E\{G(x)\} - E\left\{G(x_g)\right\}\right]^2. \tag{9}$$

Following previous studies we choose $G(x) = \log\cosh(x)$ and denote its first derivative by $g(x)$ and second derivative by $g'(x)$.

In order estimate the fastICA estimator, we fix the number of components at n and initialize the orthogonal unmixing matrix $\mathbf{U} = (\mathbf{u}_1, \ldots, \mathbf{u}_n)'$ by selecting random values for vectors $\mathbf{u}_i$ and orthogonalizing them. We use the fastICA algorithm of Hyvärinen et al. (2001) that uses symmetric orthogonalization. The unmixing matrix $\mathbf{W} = \mathbf{U}\widehat{\mathrm{Cov}}(\mathbf{x})^{-\frac{1}{2}}$ is solved by repeating the following two steps until the vectors $\mathbf{u}_i$ converge:

1. $\mathbf{u_i} \leftarrow E\left[\mathbf{x}_{st} g(\mathbf{u}_i'\mathbf{x}_{st})\right] - E\left[g'(\mathbf{u}_i'\mathbf{x}_{st})\right]\mathbf{u}_i, \quad i = 1, \ldots, n$
2. $\mathbf{U} \leftarrow (\mathbf{U}\mathbf{U}')^{-1/2}\mathbf{U}$.

The expected values in the algorithm are approximated using sample statistics.

2.3 Volatility and auto-correlation models

In order to fully specify the dynamics of the underlying components in equation (3), we have to choose suitable models for the conditional expectation $\mu_{i,t}$ and the conditional volatility $\sigma_{i,t}$. Following previous research, we have chosen to account for possible autocorrelations by modelling the conditional expectation $\mu_{i,t}$ of the ith component as an AR(1)-process. In the case of σ_{it}, we focus on two different GARCH-type models: GARCH(1,1) and GJR(1,1) (Glosten, Jagannathan and Runkle, 1993). The chosen models allow us to account for different types of volatility dynamics while keeping the number of tested model combinations manageable. Parameters for both volatility models will be estimated using quasi-maximum likelihood estimation, because the exact likelihood functions are not known. In the case of GARCH(1,1) we consider estimation using both normal and student's t-distribution in order to find out whether the distributional assumption affects the results. The chosen estimation approach is standard practice in finance and it has been found to yield acceptable estimates in previous research on stock returns. Sufficient conditions for the consistency and asymptotic normality of the quasi maximum-likelihood estimators have been considered by Bardet and Wintenberger (2009).

The GARCH(1,1) specification for the conditional volatility is

$$\sigma_{i,t}^2 = \alpha_{i,0} + \alpha_{i,1}(\sigma_{i,t-1}\epsilon_{i,t-1})^2 + \beta_{i,1}\sigma_{i,t-1}^2. \tag{10}$$

The GJR(1,1) specification of a threshold GARCH model by Glosten et al. (1993) gives the volality specification as

$$\sigma_{i,t}^2 = \alpha_{i,0} + \alpha_{i,1}(\sigma_{i,t-1}\epsilon_{i,t-1})^2 + \beta_{i,1}\sigma_{i,t-1}^2 + \gamma_i \mathbb{1}_{\sigma_{i,t-1}\epsilon_{i,t-1}\leq 0}(\sigma_{i,t-1}\epsilon_{i,t-1})^2, \tag{11}$$

where the last term allows the volatility to change differently depending on whether the previous innovation was positive or negative. This is a desired property in financial modelling, since negative shocks have typically a different effect on the volatility of the following days' returns than positive shocks. The positivity of variance is ensured by requiring that $\alpha_{i,0} > 0$, $\alpha_{i,1} \geq 0, \beta_{i,1} \geq 0$ and, in the case of GJR(1,1), $\alpha_{i,1} + \gamma_i \geq 0$.

3 EMPIRICAL ANALYSIS

3.1 Data and methodology

We test the models using two data sets consisting of daily returns for 17 and 30 industry specific American stock portfolios downloaded from the web page of Kenneth French. The returns of each industry portfolio are value weighted averages of the returns in said industry. We convert the returns into logarithmic returns, and form the primary test portfolios by assigning an equal weight on each of the industry specific subportfolios, which essentially act as assets. One day ahead forecasts for VaR are made based on a rolling window of the previous 1000 days' observations, i.e the model is re-estimated before calculating each forecast. The forecast period covers the years 2000-2014.

The ICA algorithms chosen for the tests are FOBI, gJADE and gFOBI. The number of lags used by gJADE and gFOBI was found to have a only a negligible effect on the results, and therefore we report the results using only the lags $K = 1$ and 10. As benchmark models we use the extreme value theory (EVT) based peaks over threshold approach combined with GJR or GARCH filtering as introduced in McNeil and Frey (2000). We also treat PCA as if it were an ICA-algorithm, i.e. we use principal components as the independent components $s_{i,t}$, in order to test whether the use of the more complicated algorithms is worthwhile.

The performance of the difference forecasting methods is evaluated using the unconditional coverage (UC) test of Kupiec (1995), and the independence (IND) and conditional coverage (CC) tests of Christoffersen (1998) that are based on likelihood ratios. The UC test is based on the number of VaR violations, whereas the IND test is used to determine whether the probability of a VaR violation at time t is independent of a previous VaR violation happening at time $t-1$. The CC test is a combination of the UC and IND tests. For these tests we will report the χ^2 statistics with p-values included inside parentheses. We will also report the VaR violation rates, i.e. the number of times the VaR forecasts are exceeded. The tables containing the test results and the number of annual VaR violations are in the appendices.

3.2 Results for the 17 industry specific portfolios dataset

It is well known that the forecasting performance of risk models can be sensitive to the chosen time period. In order to mitigate this problem and isolate model specific failures, we run all tests for both the full forecasting period covering the years 2000-2014 and for shorter subperiods. We start by analyzing the 17 industrial portfolios data set for the complete forecasting period, which covers the recent financial crisis and as such contains a possibly significant structural break in the data. Figure 1 shows the dynamics of the test portfolio and 99 % VaR forecasts based on EVT and fastICA during the forecasting period.

The results in Tables 1-2 of Appendix A reveal that most of the models are rejected by the coverage ratio tests in the case of 95 % and 99 % VaR forecasts. For this test period only the univariate EVT methods and fastICA combined with GJR-volatilities passed the tests at both VaR levels. It is noteworthy that both gJADE and PCA based forecasts fared well only at the lower 95 % VaR level, whereas gFOBI and FOBI based methods were generally rejected even though the passed the independence tests.

The yearly VaR exceedances in Tables 3-4 imply that models are failing the coverage ratio tests mainly due to their poor performance during the 2002 stock market crash and the 2007-2008 financial crisis. These structural breaks in the

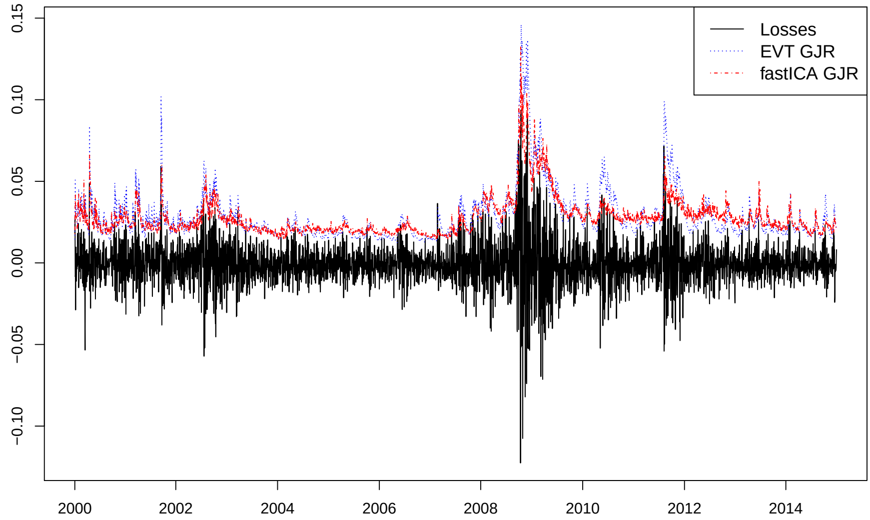

Figure 1: Daily logarithmic losses of the test portfolio and 99 % VaR forecasts given by two estimation methods for the years 2000-2014 based on the 17 industrial portfolios dataset.

data seems to have affected some of the tested models worse than others, but even the best performing models had at least twice as many VaR exceedances as expected during the most recent crisis. We suspect that the extremely poor performance of moment and cumulant based ICA models relative to fastICA is explained by their higher sensitivity to outliers. The tested ICA algorithms also assume that the mixing matrix is constant within the observation window. This assumption seems reasonable under normal market conditions, but it is possible that the sensitivity of asset returns to different factors might have significantly changed during the financial crisis, hence violating one of the core assumptions in the ICA models.

During the pre- and post-crisis years all of the models generally performed well at forecasting 99 % VaR levels as can be seen in Tables 5-6. Under the normal market conditions most tested models had a tendency to overestimate the VaR level as shown by the low number of violations contrary to their performance during the years 2007-2008.

We perform further analysis on the multivariate models by testing their ability to forecast 99 % VaR for the 17 industry specific subportfolios of the data set. In order to make the testing feasible, we use only the GJR-volatility model, as it seemed to outperform normal GARCH(1,1) on the portfolio level and we report only the conditional coverage test statistics.

The results for the years 2000-2014 in Table 7 of Appendix B show that PCA and fastICA had the best performance, with fastICA passing the tests for more industry classes than any other tested model. It is interesting to note that restricting the forecasting period to the pre-crisis years worsens the test results as show in Table 8. During the years 2000-2006 fastICA still seems to have the best performance and generally passes the conditional coverage test, but results for PCA are comparable to those of gFOBI and gJADE. This is in contrast to the post-crisis years 2009-2014, during which all of the models performed well as seen in Table 10. During the crisis years of 2007-2008 all models had subpar forecasting performance as expected, with PCA and fastICA being the best ones.

The results show that gJADE and gFOBI have significant flaws when applied to financial modelling, as they had disastrous performance during the forecasting period 2007-2008 even compared to all other models. Both algorithms also generally underperformed compared to fastICA, which does not rely on high moments or cumulants and is hence less sensitive to the outliers inherent to financial data. This highlights the importance of outlier resistant methods for risk management.

3.3 Results for the 30 industry specific portfolios dataset

In order to get a better idea of how the different methods perform in higher dimensional data sets, we tested the forecasting accuracy of the selected models using the 30 industrial portfolios data set provided by Kenneth French. We were, however, forced to exclude gJADE from this part of the study due to issues related extreme computational time, as the minimization criterion of the algorithm requires evaluating $n^2(K+1)$ matrices, where n is the number dimensions and K is the number of lags. This causes the basic version of gJADE to become unfeasible for high dimensional data sets.

In this data set the results at 95 % VaR level for the whole test period covering the years 2000-2014 were similar to those from the 17 industrial portfolios data set as can be seen in Table 11 of Appendix C. The EVT, PCA and fastICA models performed well, whereas gFOBI and FOBI based models were generally rejected by the tests. At 99 % VaR level the results for the same period show in Table 12 were somewhat surprising, with all models except those based on EVT being rejected by the coverage ratio tests. As previously, this was mainly caused by the extreme number of VaR violations during the financial crisis of 2007-2008, as can be seen in Table 14. In the case of 95 % VaR forecasts, the results in Table 13 show an extreme number of violations for ICA based models also for the year 2002.

Analyzing the pre- and post-crisis years separately gives us additional insight into the performance of the models under different market conditions. During the pre-crisis years of 2000-2006, as shown in Table 15, the coverage ratio tests rejected all PCA models with only the GJR version of these models passing even the independence test, but no other models were rejected by the tests. The results in Table 16 for the post-crisis period of 2009-2014 provided no surprises, and all tested models passed the tests just like in the case of the 17 industrial portfolios data set.

4 CONCLUSIONS

Our results show that ICA methodology has potential as a tool for modelling the dependence between asset returns in the forecasting of risk measures, and it can be used as a theoretically sounder alternative to PCA, but it is not without its own weaknesses. As expected, in VaR forecasting the results are dependent on the chosen test period, but under normal market conditions the ICA models proved to be successful in providing accurate forecasts.

We found that EVT models outperformed ICA methods at portfolio level VaR forecasting, even though fastICA proved to be highly competitive at 99 % VaR level. ICA methodology however has the benefit of providing information about the dependence structure of individual assets, which was shown by testing VaR forecasts for the subportfolios making up the 17 industrial portfolios data set. The factor structure also lends itself easily to both hedging and asset pricing purposes unlike the univariate EVT-models. Overall the results suggest that ICA methods have value in financial modelling, despite their problems in capturing the aberrant market dynamics of the recent financial crisis, and should be considered as an alternative to PCA in modeling non-Gaussian processes.

The main weaknesses of ICA are the assumed stability of the mixing matrix over time, which might not hold under crisis conditions, and the amount of data needed in the estimation process. We, however, note that the issues might be mitigated by using higher or mixed frequency data, since the logarithmic returns can be aggregated to a lower frequency without changing the basic structure of the model. Further research on the issue is needed in order to ascertain the empirical aggregation properties of ICA models.

ICA algorithms can be highly susceptible to outliers and structural breaks in the data as shown by our test results for the stock market crashes of 2001-2002 and the financial crisis of 2007-2008. The ICA methods based on higher moments and cumulants, e.g. gFOBI and gJADE, were found to have the worst performance during the aforementioned years, and as such we suggest using alternative ICA methods, such as fastICA, which are less sensitive to outliers in the modeling and analysis of financial time series. This is in contrast to the promising simulation results of Matilainen et al. (2015). We also found out that increasing the number of lags used by gFOBI or gJADE did not have a noticeable effect on the results.

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APPENDIX A: RESULTS FOR THE 17 INDUSTRIAL PORTFOLIOS DATA SET

Table 1: Violation rates and coverage ratio tests for 95 % VaR forecasts covering the years 2000-2014 of the 17 Industrial portfolios data set. P-values of the coverage ratio tests are reported inside parentheses. ** or * indicates that the model is not rejected by the test at a 0.1 or 0.05 significance level.

	Violations	UC	IND	CC
EVT GARCH	207 (5.5 %)	1.824 (0.177) **	0.175 (0.676) **	1.813 (0.404) **
EVT GJR	204 (5.4 %)	1.282 (0.257) **	1.825 (0.177) **	2.951 (0.229) **
PCA N-GARCH	203 (5.4 %)	1.122 (0.289) **	0.127 (0.721) **	1.103 (0.576) **
PCA t-GARCH	204 (5.4 %)	1.282 (0.257) **	0.103 (0.748) **	1.229 (0.541) **
PCA GJR	198 (5.2 %)	0.48 (0.488) **	2.355 (0.125) **	2.739 (0.254) **
fastICA N-GARCH	221 (5.9 %)	5.548 (0.019)	0.793 (0.373) **	6.024 (0.049)
fastICA t-GARCH	223 (5.9 %)	6.236 (0.013)	1.195 (0.274) **	7.097 (0.029)
fastICA GJR	214 (5.7 %)	3.443 (0.064) *	1.308 (0.253) **	4.499 (0.105) **
gFOBI1 N-GARCH	216 (5.7 %)	3.996 (0.046)	1.818 (0.177) **	5.543 (0.063) *
gFOBI1 t-GARCH	222 (5.9 %)	5.887 (0.015)	1.958 (0.162) **	7.52 (0.023)
gFOBI1 GJR	215 (5.7 %)	3.714 (0.054) *	1.226 (0.268) **	4.679 (0.096) *
gFOBI10 N-GARCH	223 (5.9 %)	6.236 (0.013)	0.291 (0.59) **	6.193 (0.045)
gFOBI10 t-GARCH	220 (5.8 %)	5.218 (0.022)	1.445 (0.229) **	6.356 (0.042)
gFOBI10 GJR	220 (5.8 %)	5.218 (0.022)	0.129 (0.719) **	5.04 (0.08) *
FOBI N-GARCH	221 (5.9 %)	5.548 (0.019)	3.859 (0.049)	9.091 (0.011)
FOBI t-GARCH	225 (6 %)	6.963 (0.008)	4.29 (0.038)	10.901 (0.004)
FOBI GJR	216 (5.7 %)	3.996 (0.046)	3.57 (0.059) *	7.294 (0.026)
gJADE1 N-GARCH	204 (5.4 %)	1.282 (0.257) **	2.281 (0.131) **	3.407 (0.182) **
gJADE1 t-GARCH	210 (5.6 %)	2.457 (0.117) **	2.464 (0.116) **	4.707 (0.095) *
gJADE1 GJR	200 (5.3 %)	0.706 (0.401) **	1.874 (0.171) **	2.462 (0.292) **
gJADE10 N-GARCH	205 (5.4 %)	1.452 (0.228) **	1.406 (0.236) **	2.693 (0.26) **
gJADE10 t-GARCH	210 (5.6 %)	2.457 (0.117) **	2.464 (0.116) **	4.707 (0.095) *
gJADE10 GJR	201 (5.3 %)	0.834 (0.361) **	0.541 (0.462) **	1.248 (0.536) **

Table 2: Violation rates and coverage ratio tests for 99 % VaR forecasts covering the years 2000-2014 of the 17 Industrial portfolios data set. P-values of the coverage ratio tests are reported inside parentheses. ** or * indicates that the model is not rejected by the test at a 0.1 or 0.05 significance level.

	Violations	UC	IND	CC
EVT GARCH	34 (0.9 %)	0.385 (0.535) **	0.619 (0.432) **	1.002 (0.606) **
EVT GJR	38 (1 %)	0.002 (0.965) **	0.741 (0.389) **	0.728 (0.695) **
PCA N-GARCH	53 (1.4 %)	5.545 (0.019)	1.511 (0.219) **	7.064 (0.029)
PCA t-GARCH	52 (1.4 %)	4.877 (0.027)	1.454 (0.228) **	6.338 (0.042)
PCA GJR	58 (1.5 %)	9.449 (0.002)	0.017 (0.896) **	8.606 (0.014)
fastICA N-GARCH	44 (1.2 %)	0.999 (0.318) **	1.039 (0.308) **	2.041 (0.36) **
fastICA t-GARCH	44 (1.2 %)	0.999 (0.318) **	1.039 (0.308) **	2.041 (0.36) **
fastICA GJR	44 (1.2 %)	0.999 (0.318) **	0.395 (0.53) **	1.086 (0.581) **
gFOBI1 N-GARCH	58 (1.5 %)	9.449 (0.002)	1.812 (0.178) **	11.271 (0.004)
gFOBI1 t-GARCH	59 (1.6 %)	10.337 (0.001)	1.875 (0.171) **	12.224 (0.002)
gFOBI1 GJR	59 (1.6 %)	10.337 (0.001)	1.875 (0.171) **	12.224 (0.002)
gFOBI10 N-GARCH	59 (1.6 %)	10.337 (0.001)	1.875 (0.171) **	12.224 (0.002)
gFOBI10 t-GARCH	58 (1.5 %)	9.449 (0.002)	1.812 (0.178) **	11.271 (0.004)
gFOBI10 GJR	58 (1.5 %)	9.449 (0.002)	1.812 (0.178) **	11.271 (0.004)
FOBI N-GARCH	60 (1.6 %)	11.26 (0.001)	0.002 (0.962) **	11.274 (0.004)
FOBI t-GARCH	61 (1.6 %)	12.216 (0)	0 (0.989) **	12.229 (0.002)
FOBI GJR	57 (1.5 %)	8.595 (0.003)	1.749 (0.186) **	10.355 (0.006)
gJADE1 N-GARCH	50 (1.3 %)	3.657 (0.056) *	1.343 (0.246) **	5.007 (0.082) *
gJADE1 t-GARCH	56 (1.5 %)	7.778 (0.005)	0.033 (0.856) **	7.821 (0.02)
gJADE1 GJR	55 (1.5 %)	6.997 (0.008)	1.359 (0.244) **	7.602 (0.022)
gJADE10 N-GARCH	54 (1.4 %)	6.252 (0.012)	0.063 (0.802) **	6.324 (0.042)
gJADE10 t-GARCH	58 (1.5 %)	9.449 (0.002)	3.221 (0.073) *	12.681 (0.002)
gJADE10 GJR	58 (1.5 %)	9.449 (0.002)	0.013 (0.909) **	9.473 (0.009)

Table 3: Annual 95 % VaR forecast violations for the years 2000-2014 of the 17 industrial portfolios data set.

	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014
EVT GARCH	16	15	15	8	10	15	11	19	25	15	13	12	8	10	15
EVT GJR	17	17	15	8	12	14	12	19	24	16	13	10	7	9	11
PCA N-GARCH	15	12	15	8	13	16	12	19	24	12	12	12	8	10	15
PCA t-GARCH	15	14	15	8	13	16	12	19	22	11	13	12	8	11	15
PCA GJR	14	13	16	10	12	14	12	19	22	15	13	11	7	9	11
fastICA N-GARCH	17	17	25	11	11	14	10	21	25	14	14	17	5	6	14
fastICA t-GARCH	16	17	25	11	10	14	10	22	27	16	13	17	5	6	14
fastICA GJR	13	17	26	9	11	14	10	21	25	15	14	17	4	5	13
gFOBI1 N-GARCH	6	13	28	13	12	13	11	26	29	15	13	19	3	5	10
gFOBI1 t-GARCH	6	13	29	12	12	14	11	27	32	16	12	20	3	5	10
gFOBI1 GJR	6	13	27	13	11	13	11	26	31	16	11	20	2	5	10
gFOBI10 N-GARCH	5	15	25	11	12	15	11	25	30	16	15	22	4	5	12
gFOBI10 t-GARCH	6	16	27	10	9	14	11	25	29	16	15	22	4	5	11
gFOBI10 GJR	6	14	26	11	12	14	11	24	31	17	14	20	4	5	11
FOBI N-GARCH	6	14	31	13	12	12	11	28	31	15	13	18	4	4	9
FOBI t-GARCH	6	14	32	14	12	11	11	28	32	15	13	18	5	4	10
FOBI GJR	5	14	27	14	12	12	11	27	30	15	13	19	4	4	9
gJADE1 N-GARCH	11	14	22	12	12	10	10	24	31	12	13	17	3	3	10
gJADE1 t-GARCH	12	14	24	11	11	10	10	25	34	13	12	19	3	3	9
gJADE1 GJR	11	12	24	10	10	10	10	25	32	12	12	18	3	3	8
gJADE10 N-GARCH	11	15	24	11	10	10	10	24	33	12	11	17	3	3	11
gJADE10 t-GARCH	11	15	23	11	12	10	10	25	34	13	12	17	3	3	11
gJADE10 GJR	9	14	27	9	11	9	10	22	33	13	11	17	3	3	10

Table 4: Annual 99 % VaR forecast violations for the years 2000-2014 of the 17 industrial portfolios data set.

	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014
EVT GARCH	1	2	1	0	1	1	3	8	6	0	2	5	0	2	2
EVT GJR	2	1	0	2	2	1	5	6	5	2	3	5	0	2	2
PCA N-GARCH	1	2	3	1	4	4	4	11	9	1	2	5	0	3	3
PCA t-GARCH	2	2	3	1	4	4	4	9	9	1	2	5	0	3	3
PCA GJR	2	1	2	2	7	5	5	10	9	3	3	5	0	2	2
fastICA N-GARCH	3	2	2	1	2	1	2	8	9	0	5	5	0	1	3
fastICA t-GARCH	3	3	2	1	2	1	1	8	9	0	6	4	0	1	3
fastICA GJR	3	2	2	1	2	1	1	11	8	0	5	5	0	1	2
gFOBI1 N-GARCH	2	3	6	2	2	1	1	14	12	1	5	6	0	2	1
gFOBI1 t-GARCH	2	3	6	2	1	1	1	14	12	2	6	6	0	2	1
gFOBI1 GJR	2	2	6	3	1	1	1	14	13	1	6	6	0	2	1
gFOBI10 t-GARCH	2	2	7	1	1	1	1	11	12	2	8	6	0	2	2
gFOBI10 GJR	2	2	6	1	1	1	1	12	13	1	7	6	0	2	3
FOBI N-GARCH	1	3	8	2	2	1	1	13	13	1	6	6	0	2	1
FOBI t-GARCH	1	4	8	2	2	1	1	14	13	1	6	5	0	2	1
FOBI GJR	1	2	6	1	2	1	1	14	14	1	6	5	0	2	1
gJADE1 N-GARCH	3	2	4	1	1	1	1	11	13	0	4	6	0	2	1
gJADE1 t-GARCH	3	2	5	1	1	1	1	12	16	1	4	6	0	2	1
gJADE1 GJR	3	2	4	1	1	1	1	11	17	1	4	6	0	2	1
gJADE10 N-GARCH	3	2	4	1	1	1	1	12	17	0	4	5	0	2	1
gJADE10 t-GARCH	3	2	5	1	1	1	1	14	17	1	4	5	0	2	1
gJADE10 GJR	3	2	4	1	1	1	1	14	17	1	5	5	0	2	1

Table 5: Violation rates and coverage ratio tests for 99 % VaR forecasts covering the years 2000-2006 of the 17 industrial portfolios data set. P-values of the coverage ratio tests are reported inside parentheses. ** or * indicates that the model is not rejected by the test at a 0.1 or 0.05 significance level.

	Violations	UC	IND	CC
EVT GARCH	9 (0.5 %)	5.16 (0.023)	0.093 (0.761) **	5.243 (0.073) *
EVT GJR	13 (0.7 %)	1.33 (0.249) **	3.16 (0.075) *	5.094 (0.078) *
PCA N-GARCH	19 (1.1 %)	0.111 (0.739) **	0.415 (0.519) **	0.528 (0.768) **
PCA t-GARCH	20 (1.1 %)	0.319 (0.572) **	0.46 (0.497) **	0.782 (0.676) **
PCA GJR	24 (1.4 %)	2.118 (0.146) **	0.986 (0.321) **	2.483 (0.289) **
fastICA N-GARCH	13 (0.7 %)	1.33 (0.249) **	0.194 (0.66) **	1.519 (0.468) **
fastICA t-GARCH	13 (0.7 %)	1.33 (0.249) **	0.194 (0.66) **	1.519 (0.468) **
fastICA GJR	12 (0.7 %)	2.02 (0.155) **	3.483 (0.062) *	6.268 (0.044)
gFOBI1 N-GARCH	17 (1 %)	0.02 (0.887) **	0.332 (0.564) **	0.352 (0.839) **
gFOBI1 t-GARCH	16 (0.9 %)	0.15 (0.699) **	0.294 (0.588) **	0.442 (0.802) **
gFOBI1 GJR	16 (0.9 %)	0.15 (0.699) **	0.294 (0.588) **	0.442 (0.802) **
gFOBI10 N-GARCH	15 (0.9 %)	0.405 (0.524) **	0.258 (0.611) **	0.661 (0.719) **
gFOBI10 t-GARCH	15 (0.9 %)	0.405 (0.524) **	0.258 (0.611) **	0.661 (0.719) **
gFOBI10 GJR	14 (0.8 %)	0.796 (0.372) **	0.225 (0.635) **	1.016 (0.602) **
FOBI N-GARCH	18 (1 %)	0.01 (0.922) **	1.827 (0.176) **	1.837 (0.399) **
FOBI t-GARCH	19 (1.1 %)	0.111 (0.739) **	1.645 (0.2) **	1.758 (0.415) **
FOBI GJR	14 (0.8 %)	0.796 (0.372) **	0.225 (0.635) **	1.016 (0.602) **
gJADE1 N-GARCH	13 (0.7 %)	1.33 (0.249) **	0.194 (0.66) **	1.519 (0.468) **
gJADE1 t-GARCH	14 (0.8 %)	0.796 (0.372) **	2.727 (0.099) *	3.519 (0.172) **
gJADE1 GJR	13 (0.7 %)	1.33 (0.249) **	3.16 (0.075) *	5.094 (0.078) *
gJADE10 N-GARCH	13 (0.7 %)	1.33 (0.249) **	0.194 (0.66) **	1.519 (0.468) **
gJADE10 t-GARCH	14 (0.8 %)	0.796 (0.372) **	2.727 (0.099) *	3.519 (0.172) **
gJADE10 GJR	13 (0.7 %)	1.33 (0.249) **	0.194 (0.66) **	1.519 (0.468) **

Table 6: Violation rates and coverage ratio tests for 99 % VaR forceasts covering the years 2009-2014 of the 17 industrial data set. P-values of the coverage ratio tests are reported inside parentheses. ** or * indicates that the model is not rejected by the test at a 0.1 or 0.05 significance level.

	Violations	UC	IND	CC
EVT GARCH	11 (0.7 %)	1.242 (0.265) **	0.162 (0.688) **	1.398 (0.497) **
EVT GJR	14 (0.9 %)	0.083 (0.773) **	0.262 (0.609) **	0.344 (0.842) **
PCA N-GARCH	14 (0.9 %)	0.083 (0.773) **	0.262 (0.609) **	0.344 (0.842) **
PCA t-GARCH	14 (0.9 %)	0.083 (0.773) **	0.262 (0.609) **	0.344 (0.842) **
PCA GJR	15 (1 %)	0.001 (0.979) **	0.301 (0.583) **	0.302 (0.86) **
fastICA N-GARCH	14 (0.9 %)	0.083 (0.773) **	0.262 (0.609) **	0.344 (0.842) **
fastICA t-GARCH	14 (0.9 %)	0.083 (0.773) **	0.262 (0.609) **	0.344 (0.842) **
fastICA GJR	13 (0.9 %)	0.31 (0.578) **	0.226 (0.635) **	0.533 (0.766) **
gFOBI1 N-GARCH	15 (1 %)	0.001 (0.979) **	0.301 (0.583) **	0.302 (0.86) **
gFOBI1 t-GARCH	17 (1.1 %)	0.232 (0.63) **	0.387 (0.534) **	0.622 (0.733) **
gFOBI1 GJR	16 (1.1 %)	0.053 (0.818) **	0.343 (0.558) **	0.397 (0.82) **
gFOBI10 N-GARCH	20 (1.3 %)	1.458 (0.227) **	0.537 (0.464) **	2.001 (0.368) **
gFOBI10 t-GARCH	20 (1.3 %)	1.458 (0.227) **	0.537 (0.464) **	2.001 (0.368) **
gFOBI10 GJR	19 (1.3 %)	0.94 (0.332) **	0.485 (0.486) **	1.43 (0.489) **
FOBI N-GARCH	16 (1.1 %)	0.053 (0.818) **	0.343 (0.558) **	0.397 (0.82) **
FOBI t-GARCH	15 (1 %)	0.001 (0.979) **	0.301 (0.583) **	0.302 (0.86) **
FOBI GJR	15 (1 %)	0.001 (0.979) **	0.301 (0.583) **	0.302 (0.86) **
gJADE1 N-GARCH	13 (0.9 %)	0.31 (0.578) **	0.226 (0.635) **	0.533 (0.766) **
gJADE1 t-GARCH	14 (0.9 %)	0.083 (0.773) **	0.262 (0.609) **	0.344 (0.842) **
gJADE1 GJR	14 (0.9 %)	0.083 (0.773) **	0.262 (0.609) **	0.344 (0.842) **
gJADE10 N-GARCH	12 (0.8 %)	0.692 (0.406) **	0.192 (0.661) **	0.88 (0.644) **
gJADE10 t-GARCH	13 (0.9 %)	0.31 (0.578) **	0.226 (0.635) **	0.533 (0.766) **
gJADE10 GJR	14 (0.9 %)	0.083 (0.773) **	0.262 (0.609) **	0.344 (0.842) **

APPENDIX B: PORTFOLIO SPECIFIC TEST RESULTS

Table 7: Conditional coverage tests for 99 % VaR forecasts covering the years 2000-2014 of the 17 industrial portfolios data. P-values are reported inside parentheses. ** or * indicates that the model is not rejected by the test at a 0.1 or 0.05 significance level.

	PCA	FastICA	gFOBI1	gJADE1
Food	4.311 (0.116) **	0.803 (0.669) **	2.003 (0.367) **	2.003 (0.367) **
Mines	0.803 (0.669) **	4.987 (0.083) *	7.666 (0.022)	19.093 (0)
Oil	8.559 (0.014)	3.238 (0.198) **	10.708 (0.005)	11.982 (0.003)
Clths	3.849 (0.146) **	2.032 (0.362) **	2.108 (0.348) **	0.789 (0.674) **
Durbl	1.18 (0.554) **	0.991 (0.609) **	5.007 (0.082) *	3.33 (0.189) **
Chems	2.388 (0.303) **	0.747 (0.688) **	4.987 (0.083) *	3.296 (0.192) **
Cnsum	7.014 (0.03)	3.843 (0.146) **	10.708 (0.005)	10.133 (0.006)
Cnstr	3.482 (0.175) **	3.197 (0.202) **	5.406 (0.067) *	4.575 (0.102) **
Steel	2.783 (0.249) **	0.741 (0.691) **	1.133 (0.568) **	1.962 (0.375) **
FabPr	2.426 (0.297) **	0.77 (0.68) **	0.991 (0.609) **	2.426 (0.297) **
Machn	0.994 (0.608) **	2.197 (0.333) **	2.662 (0.264) **	0.774 (0.679) **
Cars	3.573 (0.168) **	1.706 (0.426) **	4.404 (0.111) **	3.296 (0.192) **
Trans	0.858 (0.651) **	1.706 (0.426) **	11.274 (0.004)	8.628 (0.013)
Utils	30.774 (0)	13.215 (0.001)	35.254 (0)	19.52 (0)
Rtail	1.471 (0.479) **	0.774 (0.679) **	1.39 (0.499) **	1.002 (0.606) **
Finan	0.991 (0.609) **	1.194 (0.551) **	1.194 (0.551) **	0.894 (0.64) **
Other	2.041 (0.36) **	6.338 (0.042)	5.007 (0.082) *	4.987 (0.083) *

Table 8: Conditional coverage tests for 99 % VaR forecasts covering the years 2000-2006 of the 17 industrial portfolios data. P-values are reported inside parentheses. ** or * indicates that the model is not rejected by the test at a 0.1 or 0.05 significance level.

	PCA	FastICA	gFOBI1	gJADE1
Food	6.524 (0.038)	9.609 (0.008)	7.942 (0.019)	7.942 (0.019)
Mines	5.817 (0.055) *	8.747 (0.013)	9.111 (0.011)	19.157 (0)
Oil	12.485 (0.002)	5.569 (0.062) *	12.109 (0.002)	14.15 (0.001)
Clths	19.129 (0)	7.659 (0.022)	8.388 (0.015)	6.689 (0.035)
Durbl	1.461 (0.482) **	0.965 (0.617) **	0.442 (0.802) **	1.461 (0.482) **
Chems	1.837 (0.399) **	6.689 (0.035)	1.016 (0.602) **	3.008 (0.222) **
Cnsum	7.372 (0.025)	0.442 (0.802) **	2.875 (0.238) **	6.761 (0.034)
Cnstr	6.479 (0.039)	5.737 (0.057) *	5.569 (0.062) *	6.479 (0.039)
Steel	8.686 (0.013)	1.119 (0.571) **	1.955 (0.376) **	3.509 (0.173) **
FabPr	0.315 (0.854) **	0.615 (0.735) **	0.615 (0.735) **	0.615 (0.735) **
Machn	0.661 (0.719) **	5.243 (0.073) *	12.706 (0.002)	2.178 (0.337) **
Cars	9.103 (0.011)	0.442 (0.802) **	0.442 (0.802) **	2.387 (0.303) **
Trans	2.178 (0.337) **	1.519 (0.468) **	0.661 (0.719) **	1.519 (0.468) **
Utils	23.478 (0)	10.952 (0.004)	28.868 (0)	15.66 (0)
Rtail	12.706 (0.002)	4.024 (0.134) **	13.833 (0.001)	10.378 (0.006)
Finan	3.942 (0.139) **	5.151 (0.076) *	6.583 (0.037)	7.659 (0.022)
Other	4.024 (0.134) **	0.442 (0.802) **	3.008 (0.222) **	1.016 (0.602) **

Table 9: Conditional coverage tests for 99 % VaR forecasts covering the years 2007-2008 of the 17 industrial portfolios data. P-values are reported inside parentheses. ** or * indicates that the model is not rejected by the test at a 0.1 or 0.05 significance level.

	PCA	FastICA	gFOBI1	gJADE1
Food	18.945 (0)	11.686 (0.003)	27.384 (0)	27.384 (0)
Mines	0.096 (0.953) **	5.8 (0.055) *	13.963 (0.001)	9.626 (0.008)
Oil	2.827 (0.243) **	2.827 (0.243) **	16.393 (0)	13.963 (0.001)
Clths	4.259 (0.119) **	2.892 (0.235) **	13.587 (0.001)	23.066 (0)
Durbl	5.838 (0.054) *	11.686 (0.003)	18.945 (0)	24.45 (0)
Chems	11.686 (0.003)	13.962 (0.001)	44.481 (0)	41.1 (0)
Cnsum	5.838 (0.054) *	5.838 (0.054) *	18.035 (0)	16.384 (0)
Cnstr	13.962 (0.001)	9.566 (0.008)	18.945 (0)	24.45 (0)
Steel	0.892 (0.64) **	4.259 (0.119) **	13.962 (0.001)	11.605 (0.003)
FabPr	5.838 (0.054) *	2.892 (0.235) **	9.566 (0.008)	18.945 (0)
Machn	0.323 (0.851) **	0.892 (0.64) **	5.838 (0.054) *	7.612 (0.022)
Cars	13.962 (0.001)	9.566 (0.008)	24.45 (0)	27.384 (0)
Trans	2.892 (0.235) **	11.686 (0.003)	37.819 (0)	41.1 (0)
Utils	30.43 (0)	11.686 (0.003)	33.585 (0)	30.43 (0)
Rtail	4.259 (0.119) **	5.838 (0.054) *	11.686 (0.003)	18.945 (0)
Finan	54.579 (0)	13.962 (0.001)	30.43 (0)	54.579 (0)
Other	11.686 (0.003)	21.635 (0)	27.384 (0)	28.618 (0)

Table 10: Conditional coverage tests for 99 % VaR forecasts covering the years 2009-2014 of the 17 industrial portfolios data. P-values are reported inside parentheses. ** or * indicates that the model is not rejected by the test at a 0.1 or 0.05 significance level.

	PCA	FastICA	gFOBI1	gJADE1
Food	2.348 (0.309) **	0.969 (0.616) **	0.397 (0.82) **	0.397 (0.82) **
Mines	4.145 (0.126) **	3.01 (0.222) **	3.01 (0.222) **	8.707 (0.013)
Oil	0.88 (0.644) **	0.88 (0.644) **	2.102 (0.35) **	2.102 (0.35) **
Clths	0.969 (0.616) **	0.533 (0.766) **	2.102 (0.35) **	2.102 (0.35) **
Durbl	0.622 (0.733) **	0.88 (0.644) **	0.622 (0.733) **	0.302 (0.86) **
Chems	0.302 (0.86) **	0.344 (0.842) **	0.88 (0.644) **	0.533 (0.766) **
Cnsum	2.111 (0.348) **	2.677 (0.262) **	4.445 (0.108) **	3.178 (0.204) **
Cnstr	0.88 (0.644) **	3.01 (0.222) **	0.533 (0.766) **	0.88 (0.644) **
Steel	2.102 (0.35) **	7.216 (0.027)	5.535 (0.063) *	5.535 (0.063) *
FabPr	0.622 (0.733) **	0.344 (0.842) **	0.88 (0.644) **	0.533 (0.766) **
Machn	1.43 (0.489) **	0.344 (0.842) **	0.533 (0.766) **	0.88 (0.644) **
Cars	0.302 (0.86) **	0.344 (0.842) **	0.344 (0.842) **	0.533 (0.766) **
Trans	0.969 (0.616) **	0.397 (0.82) **	2.677 (0.262) **	1.43 (0.489) **
Utils	2.535 (0.281) **	2.535 (0.281) **	2.205 (0.332) **	3.035 (0.219) **
Rtail	0.622 (0.733) **	0.302 (0.86) **	0.533 (0.766) **	1.398 (0.497) **
Finan	18.216 (0)	3.01 (0.222) **	11.661 (0.003)	11.661 (0.003)
Other	2.001 (0.368) **	0.969 (0.616) **	1.43 (0.489) **	0.622 (0.733) **

APPENDIX C: RESULTS FOR THE 30 INDUSTRIAL PORTFOLIOS DATASET

Table 11: Violation rates and coverage ratio tests for 95 % VaR forecasts covering the years 2000-2014 of the 30 Industrial portfolios data set. P-values of the coverage ratio tests are reported inside parentheses. ** or * indicates that the model is not rejected by the test at a 0.1 or 0.05 significance level.

	Violations	UC	IND	CC
EVT GARCH	206 (5.5 %)	1.633 (0.201) **	0.513 (0.474) **	1.97 (0.373) **
EVT GJR	210 (5.6 %)	2.457 (0.117) **	3.651 (0.056) *	5.894 (0.052) *
PCA N-GARCH	207 (5.5 %)	1.824 (0.177) **	0.009 (0.923) **	1.647 (0.439) **
PCA t-GARCH	210 (5.6 %)	2.457 (0.117) **	0.04 (0.842) **	2.282 (0.319) **
PCA GJR	205 (5.4 %)	1.452 (0.228) **	3.079 (0.079) *	4.365 (0.113) **
fastICA N-GARCH	206 (5.5 %)	1.633 (0.201) **	2.953 (0.086) *	4.41 (0.11) **
fastICA t-GARCH	214 (5.7 %)	3.443 (0.064) *	2.875 (0.09) *	6.066 (0.048)
fastICA GJR	196 (5.2 %)	0.298 (0.585) **	0.836 (0.361) **	1.057 (0.589) **
gFOBI1 N-GARCH	215 (5.7 %)	3.714 (0.054) *	1.186 (0.276) **	4.916 (0.086) *
gFOBI1 t-GARCH	224 (5.9 %)	6.595 (0.01)	3.36 (0.067) *	9.975 (0.007)
gFOBI1 GJR	223 (5.9 %)	6.236 (0.013)	1.807 (0.179) **	8.063 (0.018)
gFOBI10 N-GARCH	218 (5.8 %)	4.587 (0.032)	4.317 (0.038)	8.615 (0.013)
gFOBI10 t-GARCH	225 (6 %)	6.963 (0.008)	7.96 (0.005)	14.57 (0.001)
gFOBI10 GJR	219 (5.8 %)	4.898 (0.027)	4.161 (0.041)	8.761 (0.013)
FOBI N-GARCH	218 (5.8 %)	4.587 (0.032)	3.224 (0.073) *	7.828 (0.02)
FOBI t-GARCH	220 (5.8 %)	5.218 (0.022)	6.234 (0.013)	11.47 (0.003)
FOBI GJR	224 (5.9 %)	6.595 (0.01)	4.368 (0.037)	10.983 (0.004)

Table 12: Violation rates and coverage ratio tests for 99 % VaR forecasts covering the years 2000-2014 of the 30 Industrial portfolios data set. P-values of the coverage ratio tests are reported inside parentheses. ** or * indicates that the model is not rejected by the test at a 0.1 or 0.05 significance level.

	Violations	UC	IND	CC
EVT GARCH	47 (1.2 %)	2.134 (0.144) **	1.186 (0.276) **	3.325 (0.19) **
EVT GJR	49 (1.3 %)	3.108 (0.078) *	0.198 (0.657) **	2.783 (0.249) **
PCA N-GARCH	64 (1.7 %)	15.284 (0)	2.451 (0.117) **	16.678 (0)
PCA t-GARCH	64 (1.7 %)	15.284 (0)	2.451 (0.117) **	16.678 (0)
PCA GJR	70 (1.9 %)	22.265 (0)	0.36 (0.549) **	21.389 (0)
fastICA N-GARCH	56 (1.5 %)	7.778 (0.005)	1.688 (0.194) **	9.476 (0.009)
fastICA t-GARCH	56 (1.5 %)	7.778 (0.005)	1.688 (0.194) **	9.476 (0.009)
fastICA GJR	56 (1.5 %)	7.778 (0.005)	1.688 (0.194) **	9.476 (0.009)
gFOBI1 N-GARCH	65 (1.7 %)	16.371 (0)	2.28 (0.131) **	18.665 (0)
gFOBI1 t-GARCH	68 (1.8 %)	19.817 (0)	2.497 (0.114) **	22.33 (0)
gFOBI1 GJR	67 (1.8 %)	18.638 (0)	2.423 (0.12) **	21.077 (0)
gFOBI10 N-GARCH	62 (1.6 %)	13.206 (0)	0 (0.985) **	13.22 (0.001)
gFOBI10 t-GARCH	61 (1.6 %)	12.216 (0)	0 (0.989) **	12.229 (0.002)
gFOBI10 GJR	61 (1.6 %)	12.216 (0)	0 (0.989) **	12.229 (0.002)
FOBI N-GARCH	69 (1.8 %)	21.026 (0)	0.382 (0.537) **	21.425 (0)
FOBI t-GARCH	66 (1.7 %)	17.489 (0)	0.529 (0.467) **	18.033 (0)
FOBI GJR	66 (1.7 %)	17.489 (0)	0.529 (0.467) **	18.033 (0)

	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014
EVT GARCH	18	12	15	7	12	13	13	21	24	15	13	12	7	10	14
EVT GJR	16	16	16	10	11	13	13	20	25	16	14	10	8	9	13
FHS	17	12	17	7	11	13	13	19	24	14	12	12	7	10	14
PCA N-FHS	15	12	18	7	14	16	14	20	22	14	12	12	7	10	14
PCA t-FHS	14	14	17	7	15	17	14	20	22	14	12	12	7	10	15
PCA GJR-FHS	12	13	15	10	13	15	13	20	25	16	14	10	8	9	12
fastICA N-FHS	12	16	26	10	13	11	10	20	27	12	12	19	2	6	10
fastICA t-FHS	13	17	28	10	13	9	10	22	30	14	12	18	2	6	10
fastICA GJR-FHS	9	12	26	10	12	11	10	19	28	14	10	17	2	5	11
gFOBI1 N-FHS	7	15	26	14	12	13	11	27	33	12	11	19	2	3	10
gFOBI1 t-FHS	7	15	29	14	13	13	11	27	35	14	11	18	3	3	11
gFOBI1 GJR-FHS	7	15	26	14	13	13	11	28	35	14	11	22	2	3	9
gFOBI10 N-FHS	9	15	28	12	12	12	10	26	30	13	14	21	3	3	10
gFOBI10 t-FHS	9	16	33	12	12	13	10	28	31	14	13	18	3	3	10
gFOBI10 GJR-FHS	10	14	29	12	12	12	10	25	32	14	13	20	3	3	10
FOBI N-FHS	7	14	29	13	14	14	10	27	33	13	13	17	3	3	8
FOBI t-FHS	7	15	29	11	14	14	10	27	34	15	13	17	3	2	9
FOBI GJR-FHS	7	15	28	13	14	14	10	29	35	14	12	20	3	2	8

Table 13: Annual 95 % VaR forecast violations for the years 2000-2014 of the 30 industrial portfolios data set.

	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014
EVT GARCH	3	2	3	0	2	2	5	8	9	0	2	6	1	2	2
EVT GJR	4	2	2	2	3	2	6	8	6	2	3	5	0	2	2
FHS	5	2	3	0	4	4	5	11	11	1	3	6	1	2	2
PCA N-FHS	4	2	3	1	6	6	6	11	11	1	3	6	0	2	2
PCA t-FHS	4	2	3	1	6	6	6	11	11	0	2	6	0	3	3
PCA GJR-FHS	5	3	3	2	7	5	6	11	11	3	3	7	0	2	2
fastICA N-FHS	2	3	5	1	2	2	3	8	11	0	9	6	1	1	2
fastICA t-FHS	2	3	5	1	2	2	2	8	12	0	9	6	0	1	3
fastICA GJR-FHS	2	2	5	1	2	2	4	8	12	0	9	5	1	1	2
gFOBI1 N-FHS	3	4	6	2	1	2	3	14	14	0	7	6	0	2	1
gFOBI1 t-FHS	3	4	6	2	2	2	2	14	17	0	7	6	0	2	1
gFOBI1 GJR-FHS	3	3	6	3	2	1	3	14	16	0	7	6	0	2	1
gFOBI10 N-FHS	3	3	7	2	1	2	2	11	14	1	7	6	0	2	1
gFOBI10 t-FHS	3	3	7	2	1	2	2	11	13	1	7	6	0	2	1
gFOBI10 GJR-FHS	3	2	7	2	1	2	2	11	14	1	8	6	0	1	1
FOBI N-FHS	2	3	8	2	2	2	3	15	14	0	9	6	0	2	1
FOBI t-FHS	2	3	8	2	2	2	2	15	15	0	8	5	0	1	1
FOBI GJR-FHS	2	2	7	2	2	1	3	15	15	0	9	5	0	2	1

Table 14: Annual 99 % VaR forecast violations for the years 2000-2014 of the 30 industrial portfolios data set.

Table 15: Violation rates and coverage ratio tests for 99 % VaR forecasts covering the years 2000-2006 of the 30 Industrial portfolios data set. P-values of the coverage ratio tests are reported inside parentheses. ** or * indicates that the model is not rejected by the test at a 0.1 or 0.05 significance level.

	Violations	UC	IND	CC
EVT GARCH	17 (1 %)	0.02 (0.887) **	0.332 (0.564) **	0.352 (0.839) **
EVT GJR	21 (1.2 %)	0.629 (0.428) **	1.399 (0.237) **	1.674 (0.433) **
PCA N-GARCH	28 (1.6 %)	5.275 (0.022)	7.023 (0.008)	11.369 (0.003)
PCA t-GARCH	28 (1.6 %)	5.275 (0.022)	7.023 (0.008)	11.369 (0.003)
PCA GJR	31 (1.8 %)	8.416 (0.004)	2.526 (0.112) **	9.809 (0.007)
fastICA N-GARCH	18 (1 %)	0.01 (0.922) **	0.372 (0.542) **	0.382 (0.826) **
fastICA t-GARCH	17 (1 %)	0.02 (0.887) **	0.332 (0.564) **	0.352 (0.839) **
fastICA GJR	18 (1 %)	0.01 (0.922) **	0.372 (0.542) **	0.382 (0.826) **
gFOBI1 N-GARCH	21 (1.2 %)	0.629 (0.428) **	0.508 (0.476) **	1.14 (0.565) **
gFOBI1 t-GARCH	21 (1.2 %)	0.629 (0.428) **	0.508 (0.476) **	1.14 (0.565) **
gFOBI1 GJR	21 (1.2 %)	0.629 (0.428) **	0.508 (0.476) **	1.14 (0.565) **
gFOBI10 N-GARCH	20 (1.1 %)	0.319 (0.572) **	1.478 (0.224) **	1.8 (0.407) **
gFOBI10 t-GARCH	20 (1.1 %)	0.319 (0.572) **	1.478 (0.224) **	1.8 (0.407) **
gFOBI10 GJR	19 (1.1 %)	0.111 (0.739) **	1.645 (0.2) **	1.758 (0.415) **
FOBI N-GARCH	22 (1.3 %)	1.035 (0.309) **	1.18 (0.277) **	2.219 (0.33) **
FOBI t-GARCH	21 (1.2 %)	0.629 (0.428) **	1.323 (0.25) **	1.955 (0.376) **
FOBI GJR	19 (1.1 %)	0.111 (0.739) **	1.645 (0.2) **	1.758 (0.415) **

Table 16: Violation rates and coverage ratio tests for 99 % VaR forecasts covering the years 2009-2014 of the 30 Industrial portfolios data set. P-values of the coverage ratio tests are reported inside parentheses. ** or * indicates that the model is not rejected by the test at a 0.1 or 0.05 significance level.

	Violations	UC	IND	CC
EVT GARCH	13 (0.9 %)	0.31 (0.578) **	0.226 (0.635) **	0.533 (0.766) **
EVT GJR	14 (0.9 %)	0.083 (0.773) **	0.262 (0.609) **	0.344 (0.842) **
PCA N-GARCH	14 (0.9 %)	0.083 (0.773) **	0.262 (0.609) **	0.344 (0.842) **
PCA t-GARCH	14 (0.9 %)	0.083 (0.773) **	0.262 (0.609) **	0.344 (0.842) **
PCA GJR	17 (1.1 %)	0.232 (0.63) **	0.387 (0.534) **	0.622 (0.733) **
fastICA N-GARCH	19 (1.3 %)	0.94 (0.332) **	0.485 (0.486) **	1.43 (0.489) **
fastICA t-GARCH	19 (1.3 %)	0.94 (0.332) **	0.485 (0.486) **	1.43 (0.489) **
fastICA GJR	18 (1.2 %)	0.53 (0.467) **	0.435 (0.51) **	0.969 (0.616) **
gFOBI1 N-GARCH	16 (1.1 %)	0.053 (0.818) **	0.343 (0.558) **	0.397 (0.82) **
gFOBI1 t-GARCH	16 (1.1 %)	0.053 (0.818) **	0.343 (0.558) **	0.397 (0.82) **
gFOBI1 GJR	16 (1.1 %)	0.053 (0.818) **	0.343 (0.558) **	0.397 (0.82) **
gFOBI10 N-GARCH	17 (1.1 %)	0.232 (0.63) **	0.387 (0.534) **	0.622 (0.733) **
gFOBI10 t-GARCH	17 (1.1 %)	0.232 (0.63) **	0.387 (0.534) **	0.622 (0.733) **
gFOBI10 GJR	17 (1.1 %)	0.232 (0.63) **	0.387 (0.534) **	0.622 (0.733) **
FOBI N-GARCH	18 (1.2 %)	0.53 (0.467) **	0.435 (0.51) **	0.969 (0.616) **
FOBI t-GARCH	15 (1 %)	0.001 (0.979) **	0.301 (0.583) **	0.302 (0.86) **
FOBI GJR	17 (1.1 %)	0.232 (0.63) **	0.387 (0.534) **	0.622 (0.733) **

ESSAY 2

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Affine Multiple Curve Modelling via Least Squares Regressions

Submitted to a journal

Affine Multiple Curve Modelling via Least Squares Regressions

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Abstract

We propose a tractable approach for modelling the joint dynamics of interbank offered rates (IBOR) and Overnight Indexed Swap (OIS) rates and pricing related derivatives such as forward rate agreements. We develop a fast multi-step approach for estimating our model by extending classical affine Gaussian single curve methodology into the multiple curve framework. The performance of the model is demonstrated using euro area interest rate data covering the years 2006-2015, and we also consider the importance of liquidity. The results show that the model is able to achieve a close fit to the analyzed OIS, EURIBOR and EURIBOR forward rates.

JEL classification: C58, E43, G12, G13
KEYWORDS: Term structure of interest rates, Empirical finance, Multiple yield curves, No-arbitrage model, Factor model, OIS, LIBOR

1 INTRODUCTION

Modelling the evolution of interest rates over time is of great importance to most financial institutions. The recent financial crisis caused a significant structural change in the interest rate markets as interbank offered rates (IBOR) diverged from Overnight Indexed Swap (OIS) rates in 2007. The emergence of non-negligible spreads between the two rates has made untenable the assumption that they could both be used as proxies of a single risk free interest rate. The new market situation has made it necessary to create multiple curve models for explaining the term structure of the interest rates and modelling the

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[†] I would like to thank Henri Nyberg (discussant), Luis H. R. Alvarez, Matthijs Lof (discussant), Mika Vaihekoski and the participants at the GSF and FDPE Winter Workshop in Finance 2016 in Helsinki, and at the Turku Finance Research Workshop 2016 for helpful comments and suggestions.

joint dynamics of OIS implied risk free rates and IBOR rates. As a result, a plethora of different modelling approaches (e.g. Henrard 2007; Mercurio 2009; Kijima, Tanaka and Wong 2009; Filipović and Trolle 2013;Moreni and Pallavicini 2014; Grbac, Papapantoleon, Schoenmakers and Skovmand 2015;Grasselli and Miglietta 2016; Cuchiero, Fontana and Gnoatto 2016a) have been proposed for pricing instruments under this new framework.

In this article we aim to provide a tractable model for multiple interest rate curves that can be easily estimated using time series data. We propose an affine Gaussian term structure model, where we have opted to specify the dynamics of both OIS and IBOR rates directly instead of modelling multiplicative spreads. The chosen model structure combined with the assumption that IBOR and OIS rates are driven by the same risk factors enables us to effectively split the estimation of the model into two parts, since the parameters related to the market price of risk can be estimated using only OIS data. This insight allows us to first apply classical affine Gaussian term structure methodology to the risk free interest rates implied by OIS rates, and then estimate a smaller subset of the model parameters using data related to IBOR rates and derivatives. We also consider the effects of market liquidity on the interest rates and risk premiums.

The applicability of our multi curve approach to time series data is important, as previous articles on the evolution of the term structure of interest rates have mainly focused on government bonds (see, e.g. Cochrane and Piazzesi 2005; Diebold and Li 2006; Almeida and Vicente 2008; Christensen, Diebold and Rudebusch 2009; Adrian, Crump and Moench 2013). Most studies on multiple curve models on the other hand have focused on the consistent pricing of different derivatives on a single date. A notable exception is the model proposed by Filipović and Trolle (2013) for the evolution of interbank risk and interest rates over time. The estimation of their model, however, faced issues related to the identification of parameters. There exists, however, a large number of articles that focus on modelling just the IBOR-OIS spreads. The effects of credit and liquidity risk, for example, were considered by Dubecq, Monfort, Renne and Roussellet (2016), who also provide a review of the related literature.

In the empirical implementation of our model we have chosen to work on the basis of the methodology developed by Adrian et al. (2013), because it allows the estimation of an affine Gaussian single curve model via linear regressions without requiring numerical optimization and imposing the bond pricing recursions on parameter estimation unlike in maximum likelihood methods (e.g. Joslin, Singleton and Zhu 2011; Hamilton and Wu 2012; Joslin, Priebsch and Singleton 2014). Fitting the model to the prices of financial derivatives involves non-linear regressions, but fortunately the numerical optimization problem is low dimensional and therefore computationally fast to solve. This is achieved by breaking the parameter estimation into multiple parts and obtaining good initial values for the optimization algorithm. The additional computational burden for fitting the model to multiple tenor specific interest rate curves is insignificant.

We contribute by proposing a tractable multiple curve model for the dynamics of risk free rates, IBORs and IBOR forward rates, deriving explicit pricing formulas for said instruments under our model and developing a multi step approach for estimating the model. The performance of the model and the viability of the estimation approach is demonstrated using three versions of the estimation methodology. The empirical data consists of euro area interest rates with EURIBOR as the IBOR type rate, and with either OIS or German government rates utilized as proxies for the risk free rate. The results provide insight into the dynamics of interest rates inside the euro area.

Our results show that our approach achieves a close fit to the observations and explains the long-term dynamics of forward rates surprisingly well, even when the model is fitted without utilizing forward rate agreement (FRA) data in the estimation process. The model fit, however, becomes worse when the risk factors are extracted from German government bond yields, which suggests that there are important differences between the risk factors driving government bond yields and IBORs. These differences exhibit themselves during the recent European debt crisis which began in 2009. The relative stability of the model is furthermore evaluated and confirmed by an out-of-sample exercise. Our results also support (Adrian et al., 2013) in the conclusion that more than three factors are needed for explaining the dynamics of the risk premium. When a liquidity factor was included as the fourth risk factor, it was found to have a significant effect on the prices of all the other three factors. We also find that including the liquidity factor in the model is important for explaining the dynamics of the spot EURIBOR rates, but its effects on the model's ability to fit FRA rates were minor.

The article proceeds as follows. In Section 2 we explain the dynamics of the different interest rates under our model, derive prices for common IBOR derivatives, and describe our estimation method. Section 3 examines our model through an empirical analysis and discusses the results. Conclusions are in Section 4.

2 THE MODEL

2.1 The term structure of the risk free rates

In line with other modern multiple curve models, we assume that the there is a risk free rate that is separate from the IBORs. In practice this means that the overnight interest rate serves as a proxy for the risk free rate and the term structure is implied by the OIS rates. The assumptions regarding the dynamics and term structure of the risk free rates follow the standard affine Gaussian approach. We assume that the interest rates are driven by an m-vector of state variables X_t that evolve according to the following vector autoregression under the physical probability measure $\mathbb{P}$:

$$X_{t+1} = \mu + \Phi X_t + v_{t+1}, \tag{1}$$

where the shocks v_{t+1} are assumed to be conditionally normally distributed with a covariance matrix Σ.

$$v_{t+1}|X_t \sim N(0, \Sigma).$$

We denote the price of a risk free bond with a maturity of n months at time t by $P_t^{(n)}$, and note that

$$P_t^{(n)} = E_t^{\mathbb{P}}\left[M_{t+1}P_{t+1}^{(n-1)}\right]$$

by the assumption of no arbitrage. The pricing kernel M_{t+1} is assumed to be of the exponentially affine form

$$M_{t+1} = \exp(-r_t - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\Sigma^{-1/2}v_{t+1}),$$

where the continuously compounded risk free rate is denoted by $r_t = -\ln P_t^{(1)}$, i.e. we're using the risk free bank account process $B(t, t+n) = e^{\sum_{i=0}^{n-1} r_{t+i}}$ as the numeraire.

The time varying market price of risk is assumed to be of the essentially affine form (Duffee, 2002)

$$\lambda_t = \Sigma^{-1/2}(\lambda_0 + \lambda_1 X_t),$$

where λ_0 can be seen as controlling the long term mean and λ_1 the short term dynamics of λ_t that depend on X_t. Using this information and a Girsanov transformation, it can be seen that the dynamics of X_{t+1} under the risk neutral pricing measure $\mathbb{Q}$ are given by the vector autoregressive process

$$X_{t+1} = \tilde{\mu} + \tilde{\Phi} X_t + v_{t+1}^{\mathbb{Q}}, \tag{2}$$

where $\tilde{\mu} = \mu - \lambda_0$, $\tilde{\Phi} = \Phi - \lambda_1$, and $v_{t+1}^{\mathbb{Q}}|X_t \sim N(0, \Sigma)$ under $\mathbb{Q}$. This also implies that bond prices are of the exponentially affine form

$$P_t^{(n)} = e^{A_n + B_n' X_t}.$$

Excess return on the risk free bond with a maturity of n months is defined as

$$rx_{t+1}^{(n-1)} = \log P_{t+1}^{(n-1)} - \log P_t^{(n)} - r_t. \tag{3}$$

As noted by Adrian et al. (2013), if we assume that v_{t+1} and $rx_{t+1}^{(n-1)}$ are jointly normally distributed, we can define

$$\beta_t^{(n-1)'} = Cov_t\left[rx_{t+1}^{(n-1)}, v_{t+1}'\right]\Sigma^{-1}.$$

Following (Adrian et al., 2013) we assume that $\beta_t^{(n-1)}$ equals a constant $\beta^{(n-1)}$ by construction, and thus the excess returns can be rewritten as

$$rx_{t+1}^{(n-1)} = \beta^{(n-1)'}(\lambda_0 + \lambda_1 X_t) - \frac{1}{2}\left(\beta^{(n-1)'}\Sigma\beta^{(n-1)} + \sigma^2\right) + \beta^{(n-1)'}v_{t+1} + e_{t+1}^{(n-1)}, \quad (4)$$

where $e_{t+1}^{(n-1)}$ is the return pricing error. By grouping the terms one can see that the excess return is an affine function of the lagged risk factors, the innovation terms from equation 1 and pricing errors.

Under our approach the risk free forward rates at time t and with a tenor of δ months are given by the usual formula

$$L_t(T, T+\delta) = \frac{1}{\delta/12}\left(\frac{P_t^{(T-t)}}{P_t^{(T-t+\delta)}} - 1\right),$$

where $\delta/12$ is the year fraction due to time being denoted in months. In traditional single curve models IBORs and forward IBORs were defined similarly to the risk free rates of our framework, but that is no longer possible under the multiple curve framework. Following recent literature on multiple curve models (Crépey, Grbac and Nguyen 2012, Morino and Runggaldier 2014, Grbac et al. 2015, Cuchiero, Fontana and Gnoatto 2016b), we model the time t IBOR rates using artificial bonds $\tilde{P}(t,T)$. The artificial bonds are assumed to share the same latent factors with the risk free rates. As a result, we apply no-arbitrage conditions to the forward IBOR rates, but the same restriction are not necessary for the non-traded fictitious bonds $\tilde{P}(t,T)$. One could equivalently define the IBOR rates directly as an exponentially affine function of the risk factor, but the bond price construction is useful in understanding the intuition behind the model and connecting it to the classical single curve framework.

2.2 IBOR rates and derivatives

We define tenor specific IBOR curves by assuming that the bond prices $\tilde{P}_t^{(\delta)} = \tilde{P}(t, t+\delta)$ are exponentially affine in the state variables:

$$\tilde{P}_t^{(\delta)} = e^{\tilde{A}_\delta + \tilde{B}_\delta' X_t}. \quad (5)$$

The δ month IBOR rates at time t for the period from t to $t+\delta$ are defined similarly to the risk free rates as

$$L_t^\delta = L_t^\delta(t, t+\delta) = \frac{1}{\delta/12}\left(\frac{1}{\tilde{P}_t^{(\delta)}} - 1\right). \quad (6)$$

One can see that our model is capable of accommodating negative interest rates and the IBOR rates are affine in the spirit of Keller-Ressel, Papapantoleon and Teichmann (2013), i.e. $1 + \frac{\delta}{12} L_t^\delta$ is an exponentially affine function of X_t. The chosen model also allows us to estimate $\tilde{A}_\delta$ and $\tilde{B}_\delta$ via linear regressions as they're assumed to be constant over time.

We have opted to model the IBOR rates directly, but an equivalent specification of the model could be given using the multiplicative spread $S^\delta(t,T)$ between the risk rate free rate $L_t(T, T+\delta)$ and the IBOR rate $L_t^\delta(T, T+\delta)$:

$$S^\delta(t,T) = \frac{1 + \delta L_t^\delta(T, T+\delta)}{1 + \delta L_t(T, T+\delta)}. \tag{7}$$

$S^\delta(t,T)$ is exponentially affine in X_t, and in the case of the the spot rate, it simplifies to

$$S^\delta(t,t) = \frac{P_t^{(n)}}{\tilde{P}_t^{(\delta)}} = e^{\left(A_\delta - \tilde{A}_\delta\right) + \left(B_\delta - \tilde{B}_\delta\right) X_t}. \tag{8}$$

The underlying assumption is that the factors that affect the spread, e.g. credit or liquidity risk, also have an effect on the risk free rate. It can be seen that the affine Gaussian nature of our model allows IBOR rates to drop below the risk free rates. In empirical applications this has not proven to be a problem, and it should be noted that negative spreads between EURIBOR and OIS rates have occurred in 2006.

Following Mercurio (2010), we consider as a traded asset a forward rate agreement at time t, where the floating payment is set at time $t+n$, and the payments are made at time $t+n+\delta$, which is priced using standard risk neutral pricing methods with the risk free bank account process as numeraire. The forward IBOR rate $L_t^\delta(n) = L_t^\delta(t+n, t+n+\delta)$ is defined as the value K that solves the equation

$$E_t^{\mathbb{Q}}\left[\frac{1}{B(t, t+n+\delta)}\left(L_{t+n}^\delta - K\right)\right] = 0.$$

The forward rate that solves the equation is given by

$$L_t^\delta(n) = \frac{1}{P_t^{(n+\delta)}} E_t^{\mathbb{Q}}\left[\frac{1}{B(t, t+n+\delta)} L_{t+n}^\delta\right] \tag{9}$$

Using the longer form of the notation we can see that forward rates are martingales under the forward measure

$$L_t^\delta(t+n, t+n+\delta) = E_t^{\mathbb{Q}^{t+n+\delta}}\left[L_{t+n}^\delta(t+n, t+n+\delta\right].$$

The forward rates can be solved explicitly under the pricing measure $\mathbb{Q}$ by using equations (2) and (6), and the tower property of conditional expectation:

$$L_t^\delta(n) = \frac{1}{\delta/12}\left(\frac{1}{P_t^{(n+\delta)}} E_t^{\mathbb{Q}}\left[\frac{\tilde{P}_{t+n}^{(\delta)}}{B(t,t+n+\delta)}\right] - 1\right) \tag{10}$$

$$= \frac{1}{\delta/12}\left(\frac{1}{P_t^{(n+\delta)}} E_t^{\mathbb{Q}}\left[\frac{1}{B(t,t+n)} e^{(A_\delta - \tilde{A}_\delta) + (B_\delta - \tilde{B}_\delta)' X_{t+n}}\right] - 1\right). \tag{11}$$

In order to evaluate equation (10) we define $C_N = \sum_{k=0}^{N} \tilde{\Phi}^k$ and express the value of X_t in terms of the previous innovations as

$$X_{t+n} = C_{n-1}\tilde{\mu} + \tilde{\Phi}^n X_t + \sum_{i=1}^{n} \tilde{\Phi}^{n-i} v_{t+i}^{\mathbb{Q}}.$$

By grouping terms we can write

$$\frac{1}{B(t,t+n)} = \prod_{i=0}^{n-1} P_{t+i}^{(1)} = e^{nA_1 + B_1' \sum_{i=0}^{n-1} X_{t+i}},$$

where

$$\sum_{i=0}^{n-1} X_{t+i} = \sum_{i=1}^{n-1} C_{i-1}\tilde{\mu} + \sum_{i=1}^{n-1} \tilde{\Phi}^i X_t + \sum_{i=0}^{n-1}\sum_{j=1}^{i} \tilde{\Phi}^{i-j} v_{t+j}^{\mathbb{Q}}$$

$$= \left(\sum_{i=0}^{n-2} C_i\right)\tilde{\mu} + C_{n-1} X_t + \sum_{i=1}^{n-1} C_{n-1-i} v_{t+i}^{\mathbb{Q}}, \quad n \geq 2.$$

In order to calculate the forward rates we define the function

$$\Psi_t(\alpha,\beta,n) := E_t^{\mathbb{Q}}\left[\frac{1}{B(t,t+n)} e^{\alpha + \beta' X_{t+n}}\right]$$

$$= \gamma_t E_t^{\mathbb{Q}}\left[\exp\left(\beta' v_T^{\mathbb{Q}} + \sum_{i=1}^{n-1}\left(B_1' C_{n-1-i} + \beta' \tilde{\Phi}^{n-i}\right) v_{t+i}^{\mathbb{Q}}\right)\right]$$

$$= \gamma_t \exp\left(\frac{1}{2}\left[\beta' \Sigma \beta + \sum_{i=1}^{n-1}\left(B_1' C_{n-1-i} + \beta' \tilde{\Phi}^{n-i}\right) \Sigma \left(B_1' C_{n-1-i} + \beta' \tilde{\Phi}^{n-i}\right)'\right]\right),$$

where $\alpha \in \mathbb{R}, \beta \in \mathbb{R}^m$ and

$$\gamma_t = \exp\left\{nA_1 + \alpha + \left(\beta' C_{n-1} + B_1' \sum_{i=0}^{n-2} C_i\right)\tilde{\mu} + \left(\beta' \tilde{\Phi} + B_1' C_{n-1}\right) X_t\right\}.$$

Using this notation we can rewrite equation (10) as

$$L_t^\delta(n) = \frac{1}{\delta/12}\left(\frac{\Psi_t(A_\delta - \tilde{A}_\delta, B_\delta - \tilde{B}_\delta, n)}{P_t^{(n+\delta)}} - 1\right) =: \tilde{\Psi}_t^\delta(\tilde{A}_\delta, \tilde{B}_\delta, n). \tag{12}$$

It can be seen that $1 + \frac{\delta}{12} L_t^\delta(n)$ is exponentially affine function of X_t.

Consider an interest rate swap (IRS) where two counterparties exchange a stream of fixed rate payments for a stream of floating rate payments indexed to IBOR. Let the tenor structures be

$$t = t_0 < t_1 < \ldots < t_n = T, \quad \delta = t_i - t_{i-1}$$

for the floating rate payments, and

$$t = T_0 < T_1 < \ldots < T_N = T, \quad \Delta = T_i - T_{i-1}$$

for the fixed rate payments, where $\delta \leq \Delta$. The value of the IRS at time t for the payer of the fixed rate K is given by

$$IRS\,(t, T, \delta, \Delta, K) = \sum_{i=1}^{n} \frac{\delta}{12} P_t^{(i \cdot \delta)} L_t^\delta\left((i-1) \cdot \delta\right) - K \frac{\Delta}{12} \sum_{j=1}^{N} P_t^{(i \cdot \Delta)}$$

The fixed rate that makes the value of the IRS equal zero at time t is thus given by

$$S_{t,T,\delta,\Delta} = \frac{\sum_{i=1}^{n} \frac{\delta}{12} P_t^{(i \cdot \delta)} L_t^\delta\left((i-1) \cdot \delta\right)}{\frac{\Delta}{12} \sum_{j=1}^{N} P_t^{(j \cdot \Delta)}}.$$

The swap pricing formula can also be used to infer the rates $L_t^\delta(n)$ from the market quotes of swaps, given that the prices of the risk free bonds are known. This makes it possible to price forward rate agreements without knowing the values of the coefficients $\tilde{A}_\delta$ and $\tilde{B}_\delta$.

2.3 The estimation procedure

We split the estimation of the parameters into essentially two parts. This can be done because under our assumptions the market price of risk and dynamics of the risk factors can be estimated from the OIS implied risk free rates. Both parts of the estimation process utilize the same observations of the state risk factors X_t, which are extracted from observed data as explained in Section 3.1.

In the first part of the process we estimate all the parameters which are required for pricing the risk free zero coupon bonds covered in Section 2.1. Any approach suitable for estimating a single curve affine term structure model, e.g. the maximum likelihood approach of Hamilton and Wu 2012, could be used in this part. We have chosen to use the regression approach of Adrian et al. (2013) due to its tractability and computational speed, and the readily available specification tests which we utilize in the analysis of the results. Their estimation approach is described briefly in the first four steps of the estimation procedure below, and the results are analyzed using the tests presented in their article.

In the second part of the estimation process we estimate the IBOR parameters $\tilde{A}_\delta$ and $\tilde{B}_\delta$ separately for each tenor δ. Initial values for the IBOR parameters are obtained via ordinary least squares (OLS) regressions on the IBOR data, and the estimates can be further improved by applying non-linear least squares (NLS) regression to a data set containing observations of EURIBOR FRAs. The NLS regressions are used to include the information contained in forward rates into the estimation, but this step can be disregarded when even higher computational speed is required and the OLS estimates provide a sufficient fit.

The estimation procedure is as follows:

1. We begin by calculating the observed excess returns on risk free bonds from equation (3) and estimate the VAR(1) model of equation (1) using OLS. The estimated innovations $\hat{v}_{t+1}$ are stacked into the matrix $\hat{V}$ from which we estimate variance-covariance matrix $\hat{\Sigma} = \hat{V}\hat{V}'/T$.

2. Excess returns are stacked over maturities, and regressed using OLS on a constant, lagged pricing factors and pricing factor innovations

$$rx_{t+1} = a + \beta\hat{v}_{t+1} + cX_t + e_{t+1} \tag{13}$$

where a is an $N \times 1$ vector, and β and c are $N \times m$ matrices. The residuals are collected into the matrix $\hat{E} = (e_1, \ldots, e_T)$, which is used to estimate $\hat{\sigma}^2 = \text{tr}(\hat{E}\hat{E}')/NT$. The row vectors $\hat{\beta}^{(i)}$ of the estimated $\hat{\beta}$ are used to construct the matrix $\hat{B}^* = \left[\text{vec}(\hat{\beta}^{(1)}\beta^{(\hat{1})'}), \ldots, \text{vec}(\hat{\beta}^{(N)}\beta^{(\hat{N})'})\right]'$.

3. We use equation (4) to estimate the market price of risk parameters λ_0 and λ_1 from the estimators obtained in the previous steps as

$$\hat{\lambda}_0 = \left(\hat{\beta}'\hat{\beta}\right)^{-1}\hat{\beta}'\left(\hat{a} + \frac{1}{2}\left(\hat{B}^*\text{vec}(\hat{\Sigma}) + \hat{\sigma}^2\iota_N\right)\right)$$
$$\hat{\lambda}_1 = \left(\hat{\beta}'\hat{\beta}\right)^{-1}\hat{\beta}'\hat{c},$$

where ι_N is an $N \times 1$ vector of ones.

4. We generate the OIS yield curve using the modified recursion formula of Adrian et al. (2013). The bond prices are of the exponentially affine form

$$\ln P_t^{(n)} = A_n + B_n' X_t + u_t^{(n)}, \tag{14}$$

and by using the above estimated values, we can obtain the pricing parameters from the system of equations

$$A_n = A_{n-1} + B_{n-1}'(\mu - \lambda_0) + \frac{1}{2}(B_{n-1}'\Sigma B_{n-1} + \sigma^2) + A_1$$
$$B_n' = B_{n-1}'(\Phi - \lambda_1) + B_1'$$
$$A_0 = 0, \quad B_0' = 0, \quad \beta^{(n)} = B_n',$$

where the term $\frac{1}{2}\sigma^2$ arises from allowing maturity specific return fitting errors and incorporating them in the recursion. The derivation of the prices is exact provided that the factor loadings match each other, $B_n = \beta^{(n)}$ for the maturities used in step 2. The recursion is initialized by estimating A_1 and B_1 via OLS from equation (14).

5. The IBOR parameters $\tilde{A}_\delta$ and $\tilde{B}_\delta$ are estimated separately for each tenor δ via OLS from equation (5):

$$\ln \tilde{P}_t^{(\delta)} = \tilde{A}_\delta + \tilde{B}_\delta' X_t + \tilde{u}_t^{(n)}.$$

6. We obtain alternative estimators of $\tilde{A}_\delta$ and $\tilde{B}_\delta$ by simultaneously fitting the model to both IBOR and IBOR forward rates via NLS regression, and using the previously obtained estimators as starting values in the numerical optimization:

$$\min_{\tilde{A}_\delta, \tilde{B}_\delta} \sum_{t=1}^{T} \left[\left(\ln \tilde{P}_t^{(\delta)} - (\tilde{A}_\delta + \tilde{B}_\delta' X_t) \right)^2 + \sum_{i=1}^{k} \left(L_t^\delta (n_i^{FRA^\delta}) - \tilde{\Psi}_t^\delta (\tilde{A}_\delta, \tilde{B}_\delta, n_i^{FRA}) \right)^2 \right],$$

where $n_i^{FRA^\delta}$ is the time in months until the fixing date of the FRA.

One might note that we extract the parameters defining the market price of risk from the OIS rates. This allows us to estimate the tenor specific parameters separately and to keep the NLS estimation computationally fast, while still obtaining a good fit to OIS, IBOR and IBOR forward rates. The non-linear least squares regression used in the last step of our multiple step estimation could be replaced with any other suitable estimation method such as the generalized method of moments (GMM). A simplistic application of GMM, with the assumption of the state vector X_t being orthogonal with respect to the regression errors, would lead to km moment conditions and related $km \times km$ variance-covariance matrix, which in our experience would be numerically almost singular. The NLS approach we employ avoids these problems.

As was noted by Adrian et al. (2013), the equations (3-4) and (14) can be used to show that bond and excess return pricing errors are connected to each other by the equation

$$u_{t+1}^{(n-1)} - u_t^{(n)} + u_t^{(1)} = e_{t+1}^{(n-1)}.$$

In the chosen estimation approach the excess return pricing errors $e_t^{(n)}$ will have essentially negligible autocorrelations, whereas the bond pricing errors $u_t^{(n)}$ will have strong autocorrelations.

3 EMPIRICAL ANALYSIS

3.1 Data and estimation

We consider three different ways of estimating the model and finally perform an out-of-sample exercise on one of the estimation approaches. In the first two approaches we use OIS implied yields to estimate the risk free interest rate curve, and the in the last approach we replace the OIS yields with the yields on German government bonds as a proxy of the risk free rates. The second model specification includes a liquidity risk factor.

The interest rate data consists of OIS, EURIBOR and EURIBOR FRA rates obtained from Thomson Reuters Datastream, and German Pfandbrief and government bond data covering the years 2006-2015. The data is monthly, and the observations are taken on the last trading day of each month. The benchmark for the OIS is the Euro Overnight Index Average (EONIA) rate.

The OIS data we use covers the tenors of $1, 2, \ldots, 12, 18$ and $24, 36, \ldots, 72$ months. OIS rates with higher tenors were discarded, because the market is still developing and thin trading of the instruments might cause some of the data to be unreliable. Furthermore, we interpolate yields for the risk free OIS curve via cubic splines in order to get observations for all maturities $n = 1, \ldots, 72$. Splines were chosen over the Nelson-Siegel-Svensson model, since they offer a more accurate fit to observations and can be fitted without non-convex optimization.

The German Pfandbrief and government bond data sets are published by the Bundesbank, and consist of monthly parameter estimates for the Nelson-Siegel-Svensson model, which are used to calculate the yield observations we use. The government bond yields are expected to differ from OIS rates as the demand for German government bonds has been affected by a flight-to-safety as the result of financial turmoil related to the recent European debt crisis, which began in 2009.

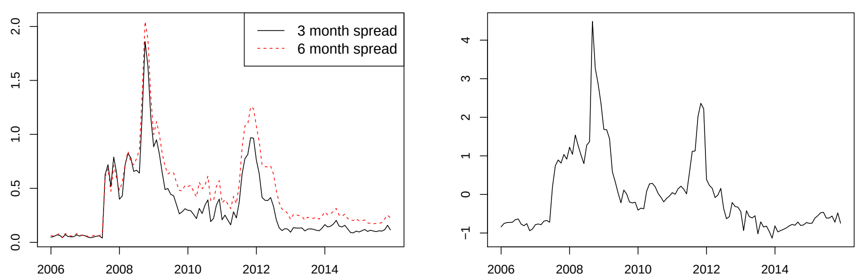
In the first estimation approach we estimate the five-factor specification of our model. We extract principal components from yields of $3, 6, \ldots, 72$ months maturities implied by the OIS rates. Furthermore, we standardize the extracted principal components to have unit variance in order to be used as observations of the state vectors. It should be noted that there would be no significant change in the results, if the principal components were estimated only from yields that were not obtained via interpolation. Due to the limitations of available data, we would be able to use only yield with maturities in full years, in order to keep the maturities of the yields used in the extraction evenly spaced out. Because of this we chose to report only the results obtained via the denser data set that contains some observations obtained through interpolation.

Betas are calculated for risk free bonds with maturities $6, 12, \ldots, 72$ months, where the yield of the bond with maturity $n = 1$ is used as the risk free rate. This gives us 12 beta estimates similar to (Adrian et al., 2013). In step 6 of our estimation procedure we use EURIBOR FRA rates for the 3 and 6 month tenors. In the case of the 3 month tenor the fixing dates are in $n_i^{FRA^\delta} = 1,2,3,4,5,6,7,8,9,12$ months from the observations, and for the 6 month tenor they are in $n_i^{FRA^\delta} = 1,2,3,4,5,6,9,12$ months.

In the second estimation approach we extract only three risk factors from the same OIS implied yields as in the first approach, and we use a liquidity proxy as the fourth priced risk factor. Otherwise the estimation proceeds as in the first approach. The proxy used as the liquidity factor is the first principal component of 3-month German Pfandbrief and government bond yield differential and a factor based on the ECB's quarterly Euro area bank lending survey. The survey factor is calculated as the share of banks reporting that their liquidity position affected the credit standards as applied to the approval of loans or credit lines to enterprises. The quarterly data was converted to monthly frequency by keeping the related factor value constant for the three months covered by the survey answers. We refer to Kempf, Korn and Uhrig-Homburg (2012) for a detailed explanation of the Pfandbrief and their use in the modelling of illiquidity premia.

The liquidity proxy and alongside EURIBOR-OIS spreads is shown in Figure (1). It can be seen that the liquidity factor can likely explain most of the variation in the spreads, but not all of it. The differences between the first two estimation approaches should highlight the effects of the liquidity factor and to what extent the other principal components can account for them.

In the third specification of the model we use the yields of German government bonds as a proxy for the risk free rate in the estimation of a five factor model. The estimation procedure of the risk free curve is identical to the one used by (Adrian et al., 2013) except for the data set, i.e. we extract the principal components from yields of bonds with maturities $3, 6, \ldots, 120$ months, and calculate betas from bonds with maturities $6, 12, \ldots, 60, 84$ and 120 months. The estimation of the parameters unique to EURIBOR and FRA rates is similar to



(a) Spreads between the EURIBOR and OIS rates. (b) Liquidity risk proxy.

Figure 1: EURIBOR-OIS spreads and the liquidity risk proxy.

the first estimation approach. The inclusion of the liquidity proxy in this specification of the model was considered, but it didn't have a significant effect on the conclusions regarding the fit of the model.

The reported out-of-sample exercise is performed based on the first estimation approach, but unreported results imply that a similar exercise based on the second estimation approach would yield similar conclusions. We split the data set in half and use the first five years of data, 2006-2010, to estimate the model. The remaining five years of data covering the years 2011-2015 are used as an out-of-sample period to evaluate the stability of the model fit. The risk factors for the out-of-sample period are extracted using the rotation matrix calculated from the in-sample yields and centered using the same values as the in-sample risk factors, i.e. we use the same affine transformation to extract both in-sample and out-of-sample risk factors by applying it to different data sets. The low number of observations remaining in the in-sample data set is a problem that can't be avoided considering the short history of OIS with tenors over two years. The short in-sample period however allows us to demonstrate that our estimation approach can achieve a good fit even with limited amount of data.

3.2 Results for the first estimation approach

In the first estimation approach we used as risk factors the standardized principal components extracted from only the yields implied by the OIS rates. In the first steps of the estimation, as described in section 2.3, we effectively price the term structure of the risk free interest rates using the methodology of Adrian et al. (2013), which is known to give a good fit on the yield curve. Following their approach we set $\mu = 0$, since the principal components should have zero expected value due to being extracted from demeaned yields.

The estimated model is able to closely fit the risk free yields as can be seen in Figure 2 that also displays the model implied term premiums. The term premium is calculated as the difference between the risk neutral yields $\frac{-1}{n}(A_n^{RF} + B_n^{RF'} X_t)$ and the model implied yields. The parameters A_n^{RF} and B_n^{RF} are obtained by setting λ_0 and λ_1 to zero in the pricing recursions shown in the fourth step of the estimation procedure. It should be noted that the fitted yields remain close to the observations even during the financial crisis of 2007-2008 and the resulting turmoil. The plots also display how the term premiums of the long maturity bonds have disappeared as the interest rates for all maturities haves dropped towards zero, and the term structure has become effectively flat as a result of the European debt crisis.

The summary statistics in Table 1 show that the standard deviations of the yield pricing errors are below 5 basis points for all maturities. The standard deviations related to the shortest maturities are higher than the ones reported by

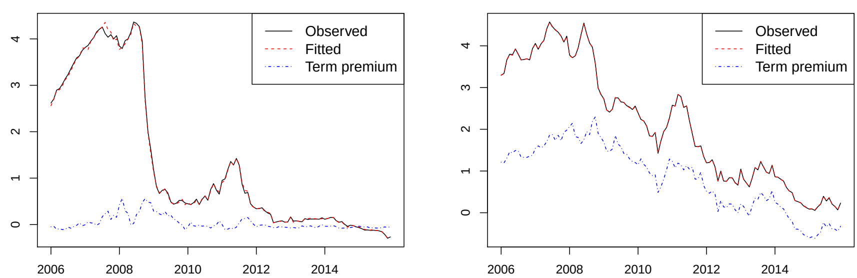
Adrian et al. (2013) for their five factor model. This is likely affected by the fact that we used euro area interest rate data which covers the recent European debt crisis and was not smoothed using the Nelson-Siegel-Svensson model.

Table 1: Summary statistics, rounded to 3 significant digits, for the pricing errors $\frac{1}{n}u_t^{(n)}$ of the monthly OIS yields for the first estimation approach. The sample period is 2006-2016, and the reported statistics include the sample mean, standard deviation, skewness and kurtosis. The autocorrelation coefficients of order one and six are denoted by $\rho(1)$ and $\rho(6)$. The maturity in months is denoted by $n = 3, 6, \ldots, 72$.

Summary statistics	$n = 3$	$n = 6$	$n = 12$	$n = 24$	$n = 60$	$n = 72$
Mean	-0.001	-0.005	-0.008	-0.000	-0.003	-0.002
Standard deviation	0.024	0.039	0.023	0.014	0.004	0.006
Skewness	1.145	2.417	1.230	-2.752	0.245	0.191
Kurtosis	11.072	17.106	10.857	16.447	4.552	3.226
$\rho(1)$	0.375	0.413	0.460	0.411	0.573	0.354
$\rho(6)$	0.056	0.028	0.101	0.001	0.264	0.130

Figure 3 shows that the pricing recursion is able to closely match the betas for the first three factors. The behavior of the fourth and fifth factors is somewhat erratic, but the pricing recursion is able to emulate the general shape of the curve. The good match between the recursion implied values and the β coefficients indicates that the model is able to closely replicate the yield dynamics as expected. An ever closer fit is obtained in the case of yields based on fitted Nelson-Siegel-Svensson curves, as can be seen from the results of the third estimation approach in Section 3.4.

The results in Table 2 reveal that all factors except the fourth one are priced by the market. The last column shows the Wald test statistics and, in parentheses, the p-value of the Wald statistic for the rows of Λ and λ_1 under the null hypothesis that they are zero. The test statistic is asymptotically chi-square distributed



(a) OIS yield fitting and term premium estimates for maturity n=6 (b) OIS yield fitting and term premium estimates for maturity n=72

Figure 2: Five factor model yield fit and term premiums for the first estimation approach.

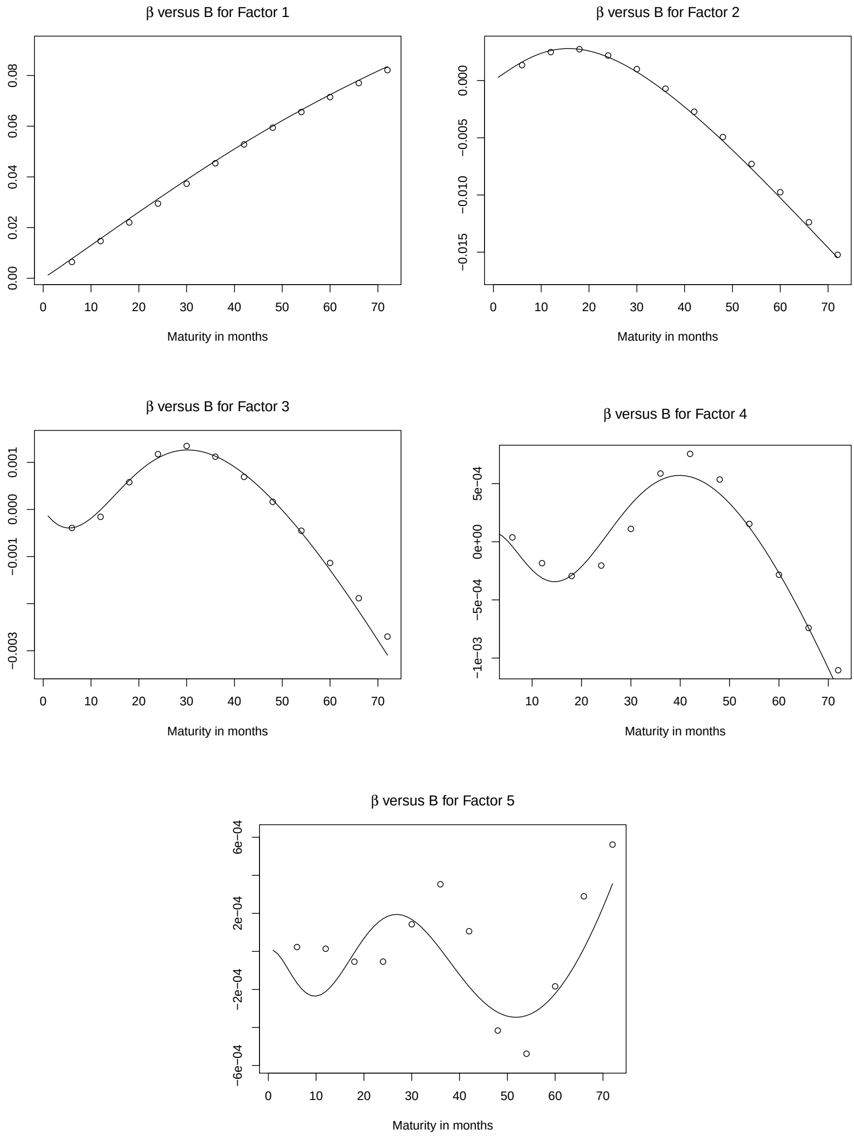


Figure 3: Regression estimates for the coefficients $\beta^{(n)}$ versus the recursion values B_n for the first estimation approach. The $\beta^{(n)}$ coefficients, denoted by circles, are obtained from equation (13) via OLS.

with degrees of freedom equal to the number of columns in the respective matrix. The derivation of the test statistics can be found in the appendices of Adrian et al. (2013).

It is noteworthy that no component of λ_0 was found to be statistically significant, even though the constant component of the level risk (the first element in the vector λ_0) had a relatively high t-statistic. We believe that this result has been affected by the European debt crisis and the flattening of the term structure. It is also important to notice that at least one of the last two factors has a statistically significant effect on the price of risk of every factor that is priced by the market. This supports the conclusion made by Adrian et al. (2013) that affine Gaussian models need more than three risk factors to explain the term structure of interest rates.

The results reveal that level risk, as measured by the first principal component, varies as a function of the slope and curvature factors, as measured by the second and the third principal component, and the fourth factor. This is in contrast to the results of (Adrian et al., 2013), who did not find the third and fourth factors to be significant drivers of level risk in the case of Treasury yields. It is also important to note that level risk has no statistically significant direct effect on the prices of other risk factors. The price of slope risk is again driven only by the fourth factor, whereas the price of curvature risk is affected by the third factor itself and the fifth factor. The price of the fifth factor is affected only by the said factor itself.

Table 2: Market price of risk for the first estimation approach. Statistical significance at a 5 % level is denoted by bold text. W_Λ and W_{λ_1} are Wald-test statistics for the rows of the respective matrices, $\Lambda = (\lambda_0, \lambda_1)$ and λ_1, being zero and the related p-value is in parentheses.

	λ_0	$\lambda_{1,1}$	$\lambda_{1,2}$	$\lambda_{1,3}$	$\lambda_{1,4}$	$\lambda_{1,5}$	W_Λ	W_{λ_1}
PC1	0.015	-0.005	**0.021**	**0.024**	**-0.022**	-0.015	**18.775**	**16.711**
(t-statistic)	(1.431)	(-0.538)	(2.078)	(2.328)	(-2.15)	(-1.46)	(0.005)	(0.005)
PC2	-0.021	-0.046	-0.033	0	**-0.057**	0.01	**12.594**	**11.774**
(t-statistic)	(-0.898)	(-1.951)	(-1.42)	(0.021)	(-2.406)	(0.42)	(0.05)	(0.038)
PC3	0.02	-0.022	0.055	**-0.12**	-0.093	**-0.182**	**20.888**	**20.755**
(t-statistic)	(0.379)	(-0.406)	(1.037)	(-2.257)	(-1.748)	(-3.407)	(0.002)	(0.001)
PC4	0.012	-0.039	0.027	-0.067	-0.054	0.103	7.569	7.514
(t-statistic)	(0.24)	(-0.751)	(0.518)	(-1.298)	(-1.038)	(1.971)	(0.271)	(0.185)
PC5	0.008	-0.062	-0.015	-0.117	-0.159	**-0.454**	**37.469**	**37.462**
(t-statistic)	(0.098)	(-0.759)	(-0.183)	(-1.445)	(-1.967)	(-5.597)	(<0.001)	(<0.001)

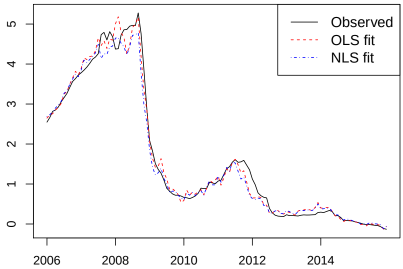
In the last steps of the estimation process, we use the same risk factors as in the previous steps for modelling the dynamics of the EURIBOR rates and related FRAs. The model is able to closely fit the observed EURIBOR rates as shown in Figure 4. Some minor divergence is evident during the European debt crisis, but the results still imply that the factors extracted from only OIS implied yields seem to be able to capture most of the EURIBOR dynamics even under aberrant market conditions. The summary statistics in Table 3 show that the standard deviations of the estimation errors are higher than in the case of the risk free yields. This was an expected result, as the the risk factors were extracted via principal component analysis from the OIS implied yields and would thus explain most of their variance. It should, however, be noted that standard deviations of the errors seems to be higher for the longer tenor EURIBOR interest rates. The model implied multiplicative spreads between the EURIBOR and OIS rates are driven mainly by the third and fourth principal component.

Table 3: Summary statistics of the errors $L_t^\delta - \hat{L}_t^\delta$ based on the EURIBOR OLS estimates of the first estimation approach. The sample period is 2006-2016, and the reported statistics include the sample mean, standard deviation, skewness and kurtosis. The autocorrelation coefficients of order one and six are denoted by $\rho(1)$ and $\rho(6)$. The tenor in months is denoted by $n = 1, 2, \ldots, 12$.

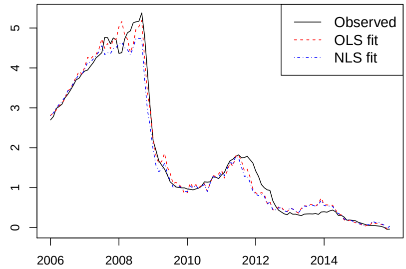
Summary statistics	$n = 1$	$n = 2$	$n = 3$	$n = 6$	$n = 9$	$n = 12$
Mean	0.000	0.000	0.000	0.000	0.000	0.000
Standard deviation	0.170	0.185	0.201	0.218	0.235	0.255
Skewness	0.509	0.333	0.208	0.271	0.376	0.406
Kurtosis	6.766	7.024	6.305	5.137	4.618	4.081
$\rho(1)$	0.530	0.590	0.630	0.715	0.755	0.792
$\rho(6)$	0.105	0.132	0.166	0.167	0.193	0.213

The fit of the model to the FRA rates is shown in Figure 4. The parameters obtained by estimating the model using just EURIBOR data results in an close fit for most of the observation period, with slight temporary divergence during the Financial Crisis of 2007-2008 and the European debt crisis. The NLS estimation that utilizes FRA data, however, is able to eliminate most of the divergence between the observed and model implied FRA rates, without significantly worsening the fit on the EURIBOR rates. It can be seen that the risk factors obtained from just OIS data seem to be able to explain the long term dynamics of the EURIBOR FRAs. The difference between the OLS and NLS estimates for this estimation approach is only small, which indicates that the NLS step of the estimation isn't always necessary. The computational speed of the model can thus be improved by disregarding the NLS step when OLS estimation provides a close enough fit to the observed data.

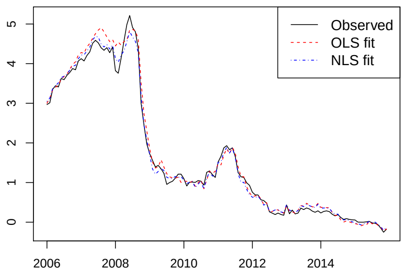
The fit of the model is good considering the observation period of the data and the model's time homogeneous and purely Gaussian nature. The model captures the time dynamics of the volatile FRA rates surprisingly well without incorporating jump or square root processes. It should also be noted that our model was estimated and evaluated using data covering a period of ten years, as opposed to calibrating the model to observed prices on only a single date, as is commonly done in articles on derivative pricing (see e.g. Cuchiero et al. 2016b, Grasselli and Miglietta 2016). This is important especially for the purposes of risk management, where a model should be able to explain observed asset price dynamics over time with minimal recalibration.



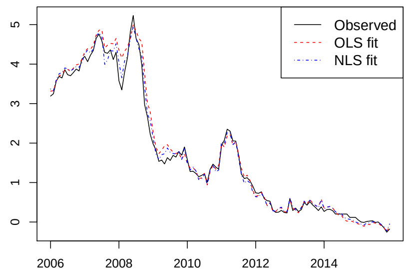
(a) Fit of the estimates for 3 month EURIBOR rate using OLS and NLS estimation.



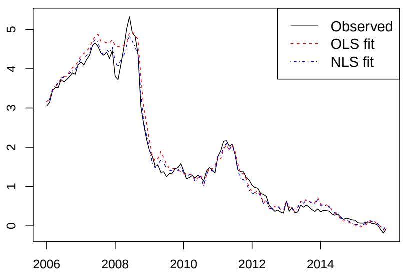
(b) Fit of the estimates for 6 month EURIBOR rate using OLS and NLS estimation.



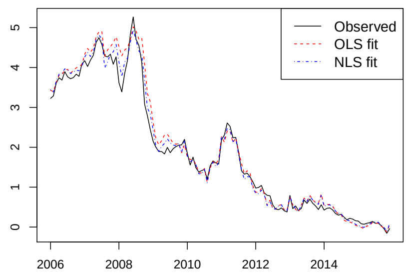
(c) Fit of the estimates for 3 month EURIBOR FRA with fixing date in n=6 months.



(d) Fit of the estimates for 3 month EURIBOR FRA with fixing date in n=12 months.



(e) Fit of the estimates for 6 month EURIBOR FRA with fixing date in n=6 months.



(f) Fit of the estimates for 6 month EURIBOR FRA with fixing date in n=12 months.

Figure 4: EURIBOR and FRA fit of the OLS and NLS estimates for the first estimation approach

3.3 Results for the second estimation approach

In the second estimation approach three risk factors were extracted from OIS implied yields and the fourth risk factor corresponds to liquidity risk. The estimated model is again able to closely fit the risk free yields despite the slight change in the risk factors. The summary statistics of the yield pricing errors in Table 4 show that the means and standard deviations of the yield pricing errors remain low and are comparable to the results of the first estimation approach for the shorter maturities.

Table 4: Summary statistics, rounded to 3 significant digits, for the pricing errors $\frac{1}{n}u_t^{(n)}$ of the monthly OIS yields for the second estimation approach. The sample period is 2006-2016, and the reported statistics include the sample mean, standard deviation, skewness and kurtosis. The autocorrelation coefficients of order one and six are denoted by $\rho(1)$ and $\rho(6)$. The maturity in months is denoted by $n = 3, 6, \ldots., 72$.

Summary statistics	$n = 3$	$n = 6$	$n = 12$	$n = 24$	$n = 60$	$n = 72$
Mean	-0.005	-0.013	-0.021	-0.015	0.017	0.008
Standard deviations	0.034	0.024	0.032	0.025	0.021	0.027
Skewness	-1.005	-0.368	0.270	0.019	0.784	0.322
Kurtosis	4.960	2.192	2.192	2.935	3.010	4.366
$\rho(1)$	0.769	0.868	0.832	0.866	0.934	0.675
$\rho(6)$	0.414	0.725	0.471	0.589	0.624	0.308

Table 5 shows the parameters estimates for the market price of risk and the related test statistics. It can be seen that the results differ from those obtained using the first estimation approach, as only the first three factors are priced by the market. The liquidity risk factor itself is not priced, but it drives the prices of all the other risk factors. This is despite the fact that the slope risk, as measured by the second principal component, and curvature risk, as measured by the third principal component, affect only their own market prices.

The alternative way of extracting the risk factors allows the OLS estimates to obtain a closer fit to the observed EURIBOR rates, as we can see from Figure 5 and Table 6. The fit remains close even during the recent European debt crisis even though some deviations are still observable during the years 2011 and 2012. The standard deviations of the errors are lower than in the first estimation approach and they're similar for all tenors. This shows that all of the effects of the liquidity factor are not captured by the first few principal components of the yields.

The parameter estimates obtained by linear regressions on just the EURIBOR data are also able to explain most of the EURIBOR FRA dynamics as shown in Figure 5. The fit is close for most of the sample period, but there are temporary spikes of divergence between the observed rates and the model implied values. However, NLS estimation seems to eliminate most of the in-sample divergence

Table 5: Market price of risk for the second estimation approach, where the fourth risk factor corresponds to liquidity risk. Statistical significance at a 5 % level is indicated by bold text. W_Λ and W_{λ_1} are Wald-test statistics for the rows of the respective matrices, $\Lambda = (\lambda_0, \lambda_1)$ and λ_1, being zero and the related p-value is in parentheses.

	λ_0	$\lambda_{1,1}$	$\lambda_{1,2}$	$\lambda_{1,3}$	$\lambda_{1,4}$	W_Λ	W_{λ_1}
PC1	0.014	0.016	0.008	-0.018	**0.066**	**34.234**	**31.984**
(t-statistic)	(1.484)	(1.461)	(0.823)	(-1.378)	(4.606)	(<0.001)	(<0.001)
PC2	-0.024	-0.017	**-0.052**	-0.052	**0.079**	**10.617**	**9.664**
(t-statistic)	(-0.98)	(-0.618)	(-2.025)	(-1.517)	(2.145)	(0.06)	(0.046)
PC3	0.041	0.037	0.031	**-0.276**	**0.269**	**14.601**	**14.155**
(t-statistic)	(0.697)	(0.56)	(0.498)	(-3.291)	(2.947)	(0.012)	(0.007)
Liquidy factor	-0.014	-0.022	0.031	0.082	0.073	0.538	0.53
(t-statistic)	(-0.082)	(-0.115)	(0.17)	(0.313)	(0.245)	(0.991)	(0.97)

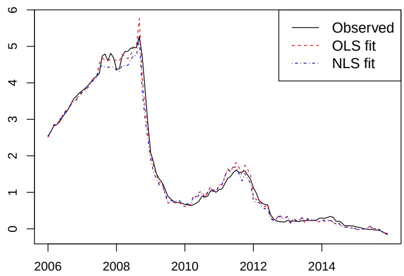
Table 6: Summary statistics of the errors $L_t^\delta - \hat{L}_t^\delta$ based on the EURIBOR OLS estimates of the second estimation approach. The sample period is 2006-2016, and the reported statistics include the sample mean, standard deviation, skewness and kurtosis. The autocorrelation coefficients of order one and six are denoted by $\rho(1)$ and $\rho(6)$. The tenor in months is denoted by $n = 1, 2, \ldots, 12$.

Summary statistics	$n=1$	$n=2$	$n=3$	$n=6$	$n=9$	$n=12$
Mean	0.000	0.000	0.000	0.000	0.000	0.000
Standard deviations	0.143	0.145	0.135	0.130	0.141	0.158
Skewness	1.623	1.230	0.497	0.569	0.667	0.588
Kurtosis	9.642	9.537	7.245	5.956	5.450	4.557
$\rho(1)$	0.436	0.466	0.383	0.375	0.463	0.572
$\rho(6)$	0.179	0.303	0.278	0.127	0.156	0.214

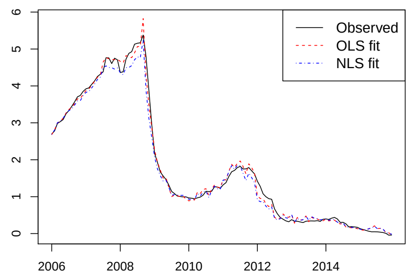
and allows us to obtain an ever better fit to the FRAs, but it comes at the cost of a slightly worse fit to the EURIBOR rates. The overall quality of the fit to the FRA rates is similar to the first estimation approach, and the inclusion of the liquidity factor didn't cause a noticeable improvement unlike in the case of the EURIBOR rates.

3.4 Results for the third estimation approach

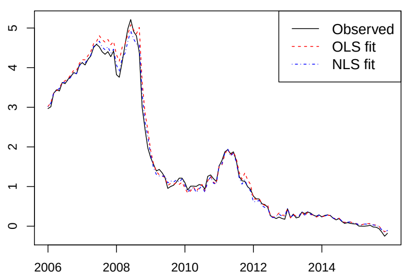
In the third estimation approach we used the yields on German government bonds as a proxy for the risk free interest rates. This data set is comparable in nature to the one used by (Adrian et al., 2013), and the fit of the model on the yields is similarly as shown in Figure 6. The plots reveal similarities between evolution of the German government bond yields and the OIS rates used in the estimation approaches of the previous sections. The general trend for yields of all maturities has been downward, with the shortest maturity yields falling fastest in the wake of the financial crisis. The longer maturity yields remained



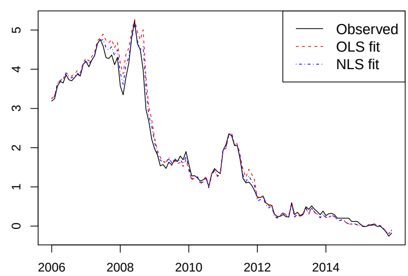
(a) Fit of the estimates for 3 month EURIBOR rate using OLS and NLS estimation.



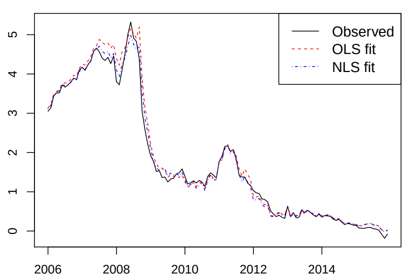
(b) Fit of the estimates for 6 month EURIBOR rate using OLS and NLS estimation.



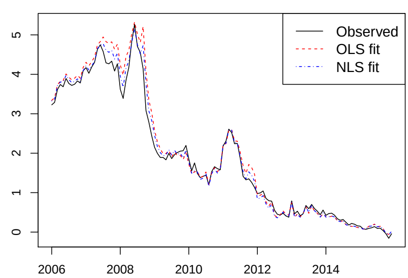
(c) Fit of the estimates for 3 month EURIBOR FRA with fixing date in n=6 months



(d) Fit of the estimates for 3 month EURIBOR FRA with fixing date in n=12 months



(e) Fit of the estimates for 6 month EURIBOR FRA with fixing date in n=6 months



(f) Fit of the estimates for 6 month EURIBOR FRA with fixing date in n=12 months

Figure 5: EURIBOR and FRA fit for the NSL and OLS estimates of the second estimation approach.

higher for longer, but the flattening of the yield curve is evident and as is the fall in the term premiums during the European debt crisis.

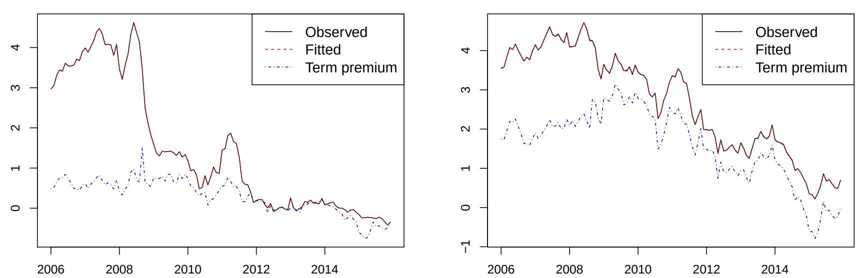
Table 7 reports the summary statistics of German government yield pricing errors. The means of the yield fitting errors are low for all maturities, and the standard deviations of the errors remain below one basis point for all maturities above 8 months. We suspect that the high volatility of the errors for low maturity yields were caused by the fast drops in rates in 2009.

Table 7: Summary statistics for the pricing errors $\frac{1}{n}u_t^{(n)}$ of the German government bond yields. The sample period is 2006-2016, and the reported statistics include the sample mean, standard deviation, skewness and kurtosis. The autocorrelation coefficients of order one and six are denoted by $\rho(1)$ and $\rho(6)$. The maturity in months is denoted by $n = 3, 6, \ldots, 120$.

Summary statistics	$n = 3$	$n = 6$	$n = 12$	$n = 24$	$n = 60$	$n = 120$
Mean	-0.008	-0.013	-0.005	-0.001	-0.005	-0.010
Standard deviations	0.027	0.020	0.004	0.003	0.004	0.006
Skewness	-2.065	-2.527	-0.913	-1.207	0.606	0.217
Kurtosis	16.584	19.643	9.092	5.208	3.848	3.005
$\rho(1)$	0.398	0.418	0.532	0.644	0.914	0.723
$\rho(6)$	0.115	0.269	0.214	0.156	0.599	0.291

The overall fit of the model to the risk free yields seems to be better than in the case of the OIS data, and the pricing recursion is able to closely match the betas for all factors as shown in Figure 7. The match is close even for the fourth and fifth factors, which proved to be problematic with the OIS implied yields. We believe that this might be related to the German government bond data being smoothed by the Nelson-Siegel-Svensson model. It should also be noted that the market for OIS with tenors over two years is relatively new, which might affect the quality of the OIS data used in the first two estimation approaches.

Table 8 shows the estimation results and test statistics for the market price of



(a) German government bond yield fitting and term premium estimates for maturity $n = 24$.
(b) German government bond yield fitting and term premium estimates for maturity $n = 120$.

Figure 6: Five factor model yield fit and term premiums for the third estimation approach.

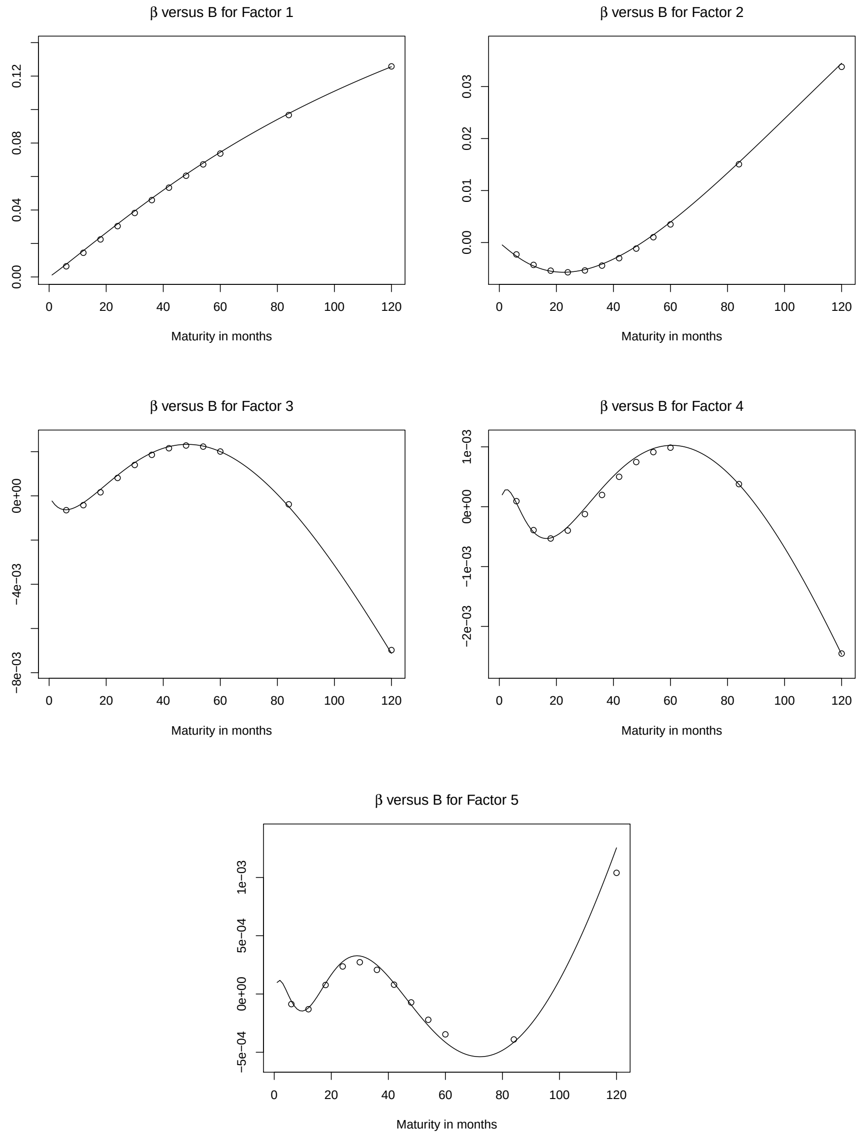


Figure 7: Regression estimates for the coefficients $\beta^{(n)}$ versus the recursion values B_n for the third estimation approach. The $\beta^{(n)}$ coefficients, denoted by circles, are obtained from the equation (13) via OLS

risk. As can be seen, at the 5 % significance level all factors except the slope risk, as measured by the second principal component, are priced. This is in line with results of Adrian et al. (2013) for Treasury yield data. The level risk as measured by the first principal component is driven by the slope risk and the fourth factor. Neither the third nor the fifth factor affect it, which shows a difference between the pricing of the OIS implied yield and the yields of German government bonds. The constant component of the level risk was not found to be statistically different from zero at a 5% significance level.

The only factor that has a statistically significant effect on the market price of curvature risk is the curvature risk itself, as measured by the third principal component. It is not affected by the level risk or the fourth factor, unlike in the case of Treasury yields as shown by Adrian et al. (2013). Curvature risk together with the fourth factor, however, play a role in driving the market price of the fourth factor. The fifth factor affects solely its own market price.

Table 8: Market price of risk estimated from German government bonds in the third specification of our model. Statistical significance at 5% level is indicated by bold text. W_Λ and W_{λ_1} are Wald-test statistics for the rows of the respective matrices, $\Lambda = (\lambda_0, \lambda_1)$ and λ_1, being zero and the related p-value is in parentheses.

	λ_0	$\lambda_{1,1}$	$\lambda_{1,2}$	$\lambda_{1,3}$	$\lambda_{1,4}$	$\lambda_{1,5}$	W_Λ	W_{λ_1}
PC1	0.019	-0.006	**-0.027**	-0.011	**0.034**	-0.005	**18.76**	**15.98**
(t-statistic)	(1.643)	(-0.564)	(-2.4)	(-0.983)	(2.988)	(-0.439)	(0.005)	(0.00)
PC2	0.005	0.05	-0.031	0.041	-0.043	-0.008	10.151	10.119
(t-statistic)	(0.186)	(1.897)	(-1.171)	(1.542)	(-1.641)	(-0.304)	(0.118)	(0.072)
PC3	-0.028	0.055	-0.104	**-0.211**	0.094	-0.007	**16.82**	**16.605**
(t-statistic)	(-0.444)	(0.875)	(-1.662)	(-3.354)	(1.474)	(-0.116)	(0.010)	(0.005)
PC4	0.067	-0.123	0.151	**0.265**	**-0.415**	-0.07	**36.638**	**35.879**
(t-statistic)	(0.809)	(-1.486)	(1.815)	(3.149)	(-4.736)	(-0.84)	(<0.001)	(<0.001)
PC5	-0.07	-0.034	0.040	-0.046	-0.074	**-0.389**	**22.185**	**21.596**
(t-statistic)	(-0.836)	(-0.398)	(0.469)	(-0.537)	(-0.828)	(-4.535)	(0.001)	(0.001)

We also used the factors extracted from the German government bond yields to model the dynamics of the EURIBOR rates. Figure 8 shows that the OLS fit on EURIBOR rates is generally close only until the year 2010, after which a clear divergence can be observed for all maturities. It is noteworthy that after they diverge, the model implied EURIBOR rates seem to exhibit a higher volatility than the observations. We suspect that the European debt crisis and the resulting flight-to-quality has affected German government bond yields in ways which are unrelated to the movements of EURIBOR rates. These effects would exhibit themselves in the principal components extracted from the bond yields thus worsening the fit of the model on EURIBOR rates.

The summary statistics of the EURIBOR fitting errors shown in Table 9 reveal that the fitting problems are similar for all tenors, and the standard deviations of the errors are higher than in the other estimation approaches. The results indicate that the first two estimation approaches should be preferred and that the risk factors for explaining EURIBOR dynamics should not be extracted from German government bonds.

Table 9: Summary statistics of the errors $L_t^\delta - \hat{L}_t^\delta$ based on the EURIBOR OLS estimates of the third estimation approach. The sample period is 2006-2016, and the reported statistics include the sample mean, standard deviation, skewness and kurtosis. The autocorrelation coefficients of order one and six are denoted by $\rho(1)$ and $\rho(6)$. The tenor in months is denoted by $n = 1, 2, \ldots, 12$.

	$n = 1$	$n = 2$	$n = 3$	$n = 6$	$n = 9$	$n = 12$
Mean	-0.000	-0.000	-0.000	-0.000	-0.000	-0.001
Standard deviations	0.332	0.357	0.373	0.360	0.355	0.358
Skewness	-1.876	-1.722	-1.447	-1.414	-1.460	-1.424
Kurtosis	7.486	6.893	5.475	5.095	5.148	4.929
$\rho(1)$	0.780	0.827	0.838	0.840	0.838	0.840
$\rho(6)$	0.198	0.259	0.310	0.315	0.317	0.336

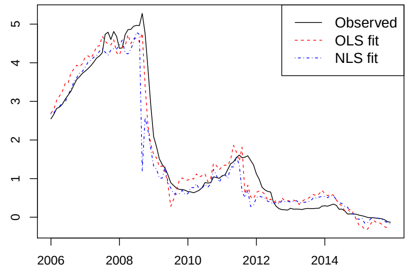
The fit on the FRA rates shows similar behavior as in the case of the EURIBOR rates, as seen in Figures 8, and the fitted FRA rates are more volatile than the observed ones. The biggest divergence between the model implied and observed forward rates is in October 2009 and it's caused by an extreme value of the fourth principal component. It is somewhat peculiar that this is also the month when Greece announced that it had been understating its deficit figures for years. Further research into the common risk factors of euro area government bonds might thus be warranted, but it is beyond the scope of this article.

NLS estimation is able to improve the fit on FRA rates by smoothing the most extreme divergences and achieves a relatively good fit. This is however achieved at the cost of an even worse fit on the observed EURIBOR rates. Especially the 3 months EURIBOR interest rates exhibit major divergence between the model fit obtained via NLS estimation and the observations during the years 2008 and 2009.

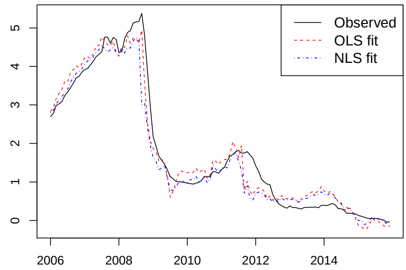
3.5 Out of Sample Exercise

We evaluate the out of sample performance of the model and our estimation approach by splitting the data set in half. The first five years of data, 2006-2010, are used to estimate the model, and the fit is compared against the out-of-sample period covering the years 2011-2015. In practice one would re-estimate the parameter value more often, but we consider this extreme exercise in order to illustrate the relative stability of the model fit.

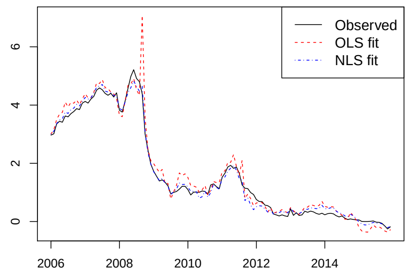
The model fit to OIS yields remains extremely close during the out-of-sample period, even though some slight divergence is visible during the years 2014 and 2015 as can be seen in Figure 9. The summary statistics of the risk free yield pricing errors for the in-sample period of 2006-2010 and out-of-sample period



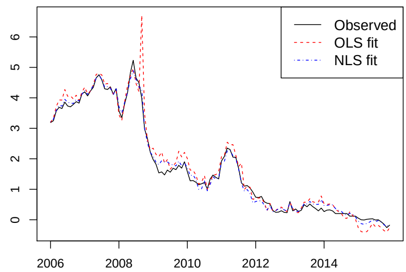
(a) Fit of the estimates for 3 month EURIBOR rate using OLS and NLS estimation.



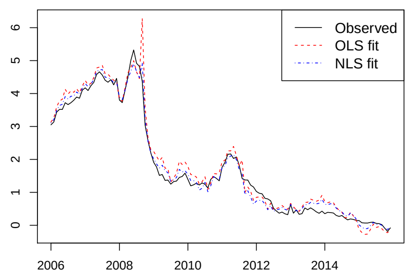
(b) Fit of the estimates for 6 month EURIBOR rate using OLS and NLS estimation.



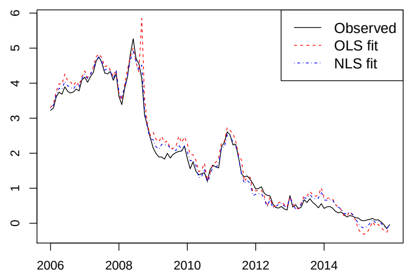
(c) Fit of the estimates for 3 month EURIBOR FRA with fixing date in n=6 months



(d) Fit of the estimates for 3 month EURIBOR FRA with fixing date in n=12 months



(e) Fit of the estimates for 6 month EURIBOR FRA with fixing date in n=6 months



(f) Fit of the estimates for 6 month EURIBOR FRA with fixing date in n=12 months

Figure 8: EURIBOR and FRA fit for the NSL and OLS estimates of the third estimation approach.

of 2011-2015 are shown in Tables 10-11. It can be clearly seen that the fit of the model remains good during the out-of-sample period. It should be noted that the out-of-sample period has relatively stable market conditions compared to the in-sample period, which covers the Financial Crisis and the resulting drop in interest rates.

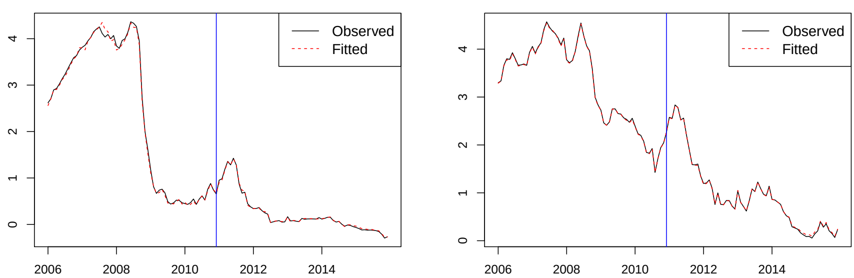
Table 10: Summary statistics for the pricing errors $\frac{1}{n}u_t^{(n)}$ of the OIS yields for the in-sample period of 2006-2010.

Summary statistics	$n = 3$	$n = 6$	$n = 12$	$n = 24$	$n = 60$	$n = 72$
Mean	-0.009	-0.018	-0.024	-0.008	-0.015	-0.012
Standard deviations	0.032	0.054	0.029	0.018	0.006	0.010
Skewness	2.130	2.114	1.541	-1.226	-0.119	0.074
Kurtosis	11.584	10.750	8.342	6.811	2.438	3.328
$\rho(1)$	0.446	0.436	0.433	0.350	0.714	0.311
$\rho(6)$	0.002	-0.005	-0.038	0.025	0.264	0.100

Table 11: Summary statistics for the pricing errors $\frac{1}{n}u_t^{(n)}$ of the OIS yields for the out-of-sample period of 2011-2015.

Summary statistics	$n = 3$	$n = 6$	$n = 12$	$n = 24$	$n = 60$	$n = 72$
Mean	0.008	0.004	-0.001	0.004	0.003	0.004
Standard deviations	0.024	0.018	0.011	0.015	0.009	0.016
Skewness	0.213	-0.108	-1.039	0.334	-0.336	0.690
Kurtosis	2.924	4.797	4.142	2.220	2.195	2.763
$\rho(1)$	0.811	0.434	0.469	0.886	0.861	0.851
$\rho(6)$	0.458	-0.155	0.125	0.614	0.525	0.531

The limited number of observations in the in sample period affects t-statistics related to the market price of risk making it harder to statistically distinguish the parameter values from zero. Table 12 shows that only the first, third and fifth



(a) OIS yield fitting and term premium estimates for maturity n=6

(b) OIS yield fitting and term premium estimates for maturity n=72

Figure 9: Five factor model yield fit and term premiums for the in sample period of 2006-2010 and out of sample period of 2011-2015.

factors seem to be priced based on the Wald test statistics even though each row of the matrix λ_1 has values that are individually significant. The prices of the first two factors are driving by the seconds and third factor, whereas the fourth and fifth factor drives the market prices of the last three factors.

Table 12: Market price of risk obtained from the in sample period of 2006-2010. Statistical significance at a 5 % level is denoted by bold text. W_Λ and W_{λ_1} are Wald-test statistics for the rows of the respective matrices, $\Lambda = (\lambda_0, \lambda_1)$ and λ_1, being zero and the related p-value is in parentheses.

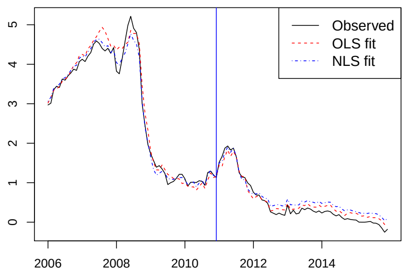
	λ_0	$\lambda_{1,1}$	$\lambda_{1,2}$	$\lambda_{1,3}$	$\lambda_{1,4}$	$\lambda_{1,5}$	W_Λ	W_{λ_1}
PC1	0.022	0.008	**0.044**	**0.055**	0.03	0.027	**19.17**	**17.865**
(t-statistic)	(1.149)	(0.415)	(2.302)	(2.872)	(1.503)	(1.394)	(0.004)	(0.003)
PC2	-0.031	-0.073	**-0.146**	-0.058	0.053	-0.105	9.629	9.473
(t-statistic)	(-0.469)	(-1.085)	(-2.194)	(-0.871)	(0.772)	(-1.573)	(0.141)	(0.092)
PC3	0.007	0.004	0.049	-0.102	**0.218**	0.127	**12.806**	**12.768**
(t-statistic)	(0.1)	(0.053)	(0.652)	(-1.358)	(2.819)	(1.684)	(0.046)	(0.026)
PC4	-0.027	0.008	0.069	0.189	**-0.273**	0.128	10.563	10.428
(t-statistic)	(-0.251)	(0.078)	(0.644)	(1.747)	(-2.453)	(1.179)	(0.103)	(0.064)
PC5	-0.058	0.017	-0.073	-0.078	**-0.241**	**-0.506**	**24.06**	**23.784**
(t-statistic)	(-0.511)	(0.15)	(-0.637)	(-0.673)	(-2.031)	(-4.359)	(0.001)	(<0.001)

Figure 10 shows the fit of the model to the FRA and EURIBOR rates. It can be seen that the model seems to capture the dynamics of the observed FRA rates even during the out of sample period of 2011-2015. A gradual difference in level between the model implied and observed rates is visible towards the end of the year 2012, and the fit of the model is worse to the EURIBOR rates than to the related FRA rates. It is noteworthy that NLS-estimation seem to only improve the fit of the model during the in sample period. In practical applications re-estimation of the model could be used to eliminate the divergence.

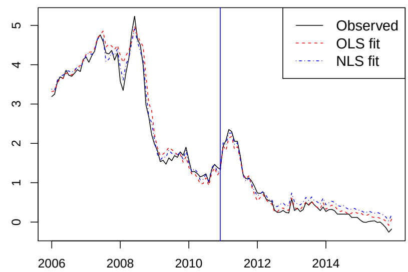
The summary statistics for the pricing errors of the EURIBOR rates are shown in Tables 13 and 14. During the out of sample period the fit of the model is worse for the longer tenor EURIBOR rates as can be seen from the means and standard deviations of the errors. As the Figure 10 showed, a persisting difference in the levels of the model implied and observed rates appears during the out of sample period, which explains the large autocorrelations of the errors.

Table 13: Summary statistics of the errors $L_t^\delta - \hat{L}_t^\delta$ based on the EURIBOR OLS estimation for the in sample period of 2006-2010.

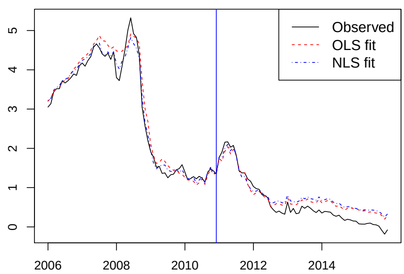
	$n = 1$	$n = 2$	$n = 3$	$n = 6$	$n = 9$	$n = 12$
Mean	0.000	0.000	0.000	0.000	0.000	0.000
Standard deviations	0.204	0.218	0.228	0.225	0.228	0.240
Skewness	0.500	0.202	-0.030	-0.023	0.089	0.194
Kurtosis	4.867	4.756	4.665	4.686	4.787	4.596
$\rho(1)$	0.415	0.501	0.522	0.580	0.603	0.643
$\rho(6)$	0.142	0.181	0.181	0.062	0.026	-0.023



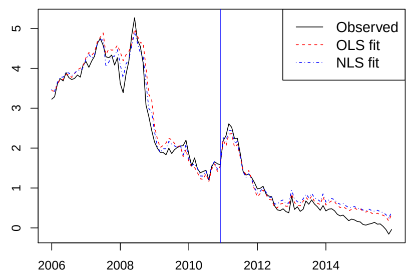
(a) Fit of the estimates for 3 month EURIBOR FRA with fixing date in n=6 months.



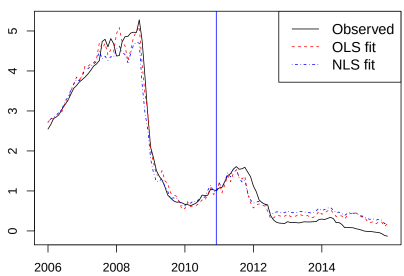
(b) Fit of the estimates for 3 month EURIBOR FRA with fixing date in n=12 months.



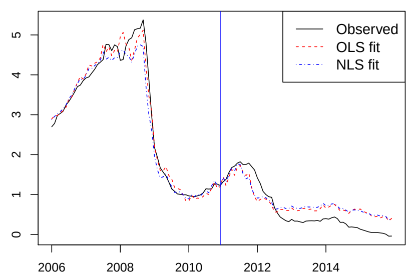
(c) Fit of the estimates for 6 month EURIBOR FRA with fixing date in n=6 months.



(d) Fit of the estimates for 6 month EURIBOR FRA with fixing date in n=12 months.



(e) Fit of the estimates for 3 month EURIBOR rate using OLS and NLS estimation.



(f) Fit of the estimates for 6 month EURIBOR rate using OLS and NLS estimation.

Figure 10: EURIBOR and FRA fit of the OLS and NLS estimates. The vertical line indicates the start of the out of sample period covering the years 2011-2015.

Table 14: Summary statistics of the errors $L_t^\delta - \hat{L}_t^\delta$ based on thef EURIBOR OLS estimation for the out of sample period of 2011-2015.

	$n = 1$	$n = 2$	$n = 3$	$n = 6$	$n = 9$	$n = 12$
Mean	-0.022	-0.027	-0.084	-0.158	-0.183	-0.191
Standard deviations	0.171	0.188	0.229	0.294	0.348	0.390
Skewness	1.361	1.423	1.302	1.048	0.887	0.805
Kurtosis	5.024	5.274	4.369	3.125	2.624	2.320
$\rho(1)$	0.878	0.881	0.912	0.945	0.956	0.962
$\rho(6)$	0.434	0.462	0.557	0.652	0.680	0.699

4 CONCLUSIONS

In this article we proposed an affine Gaussian model for modelling the joint dynamics IBOR and OIS rates and pricing related interest rate derivatives. We developed a simple and computationally fast approach for estimating the model and implemented a version of it that was based on the methodology of Adrian et al. (2013). Our multiple curve approach maintains the computational benefits of their methodology and readily permits all the extensions described in the Section 4 of their article, e.g. unspanned factors and the use of daily data.

Our empirical analysis shows that the model achieves a close in sample fit to the observed EURIBOR and OIS rates, and that the long term dynamics of the two types of rates can be mostly explained by shared risk factors. The model fit is good even when all the risk factors are extracted from just OIS yields, even though the model implied EURIBORss show minor divergence from observed rates mainly during the European debt crisis. The same risk factors can also explain the dynamics of the EURIBOR FRA rates. This suggests that the EURIBOR-OIS spreads should be modelled jointly with the OIS implied yield curve in order to get a coherent view of the market dynamics. The inclusion of a liquidity risk factor leads to an improvement on model fit to the EURIBORs as it can explain most of the differences between the EURIBOR and OIS rates, but the difference on the fit to the EURIBOR FRA rates is negligible. It should be noted that our liquidity proxy is correlated with the other risk factors, whereas Dubecq et al. (2016) assumed in their model that the liquidity and credit risk factors were independent from the factors driving the risk free interest rate.

Risk factors extracted from the German government bonds on the other hand were less useful in explaining the dynamics of EURIBORs, which indicates that there are differences between the risk factors driving government bond rates and EURIBORs. Our out-of-sample exercise shows that the models ability to explain the dynamics of EURIBOR and OIS rates isn't limited to the in-sample period. In the case of FRA rates the model interest rates started slowly diverging from the observed rates during the second year of the out-of-sample period.

The results on the five and four factor specifications of our model indicate that more than three factors can be necessary to price the term structure of interest rates, which is in line with results of Adrian et al. (2013). While their effect on the yields might be small, the fourth and fifth factor play an important role in determining the dynamics of the risk premiums for factors as expected. The results on the second specification of our model imply that liquidity risk has a significant effect on driving the market price of risk. It is possible that some of its effects are captured by the higher principal components of yields.

There were, however, noteworthy surprises as the slope risk was priced and the level risk, as a measured by first principal component, didn't have a significant effect on the price of any factor in any of the used estimation approaches. This is in contrast to Adrian et al. (2013) whose results implied that slope risk, as measured by the second principal component, was not priced in the four or five factor specifications of theor model. The results suggests a difference between the euro area and American interest rate dynamics, which is likely explained by the recent European debt crisis that is covered by our data set.

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ESSAY 3

Heikkonen, Matti

Credit, Liquidity and the Term Structure of Interest Rates

Credit, Liquidity and the Term Structure of Interest Rates

Matti Heikkonen*

Abstract

We propose a parsimonious quadratic short rate model for explaining the term structure of Overnight Indexed Swap rates and Interbank Offered Rates. A multiple curve extension to the arbitrage free affine Nelson-Siegel model is derived as a special case of the general quadratic model. The models are estimated using the quadratic Kalman filter, and they obtain a close fit to euro area data covering the years 2009-2014. The results provide insight into the dynamics of the interest rates and spreads, and the effects liquidity and credit risk have on them under the physical and risk free probability measures.

JEL classification: E43, G12, G21
KEYWORDS: Credit Risk, Liquidity Risk, Interbank Market, EURIBOR, Multi-Curve model, Quadratic term-structure model, Short rate model

1 INTRODUCTION

The financial crisis of 2007-2008 had a significant impact in the interbank markets globally. Before the financial crisis Interbank Offered Rates (IBORs) and Overnighted Indexed Swap (OIS) rates were practically indistinguishable from each other, and it was generally accepted that financial instruments related to the two rates could be modeled separately from each other (see e.g. Henrard, 2007, 2010). This assumption, however, became untenable during the financial crisis when increased liquidity and credit risk brought the interbank trading to a virtual halt and caused significant spreads between IBORs and OIS rates (Michaud and Upper (2008); González-Páramo (2011)) as shown in Figure 1. The topic is not only important for market participants, but also has policy implications for

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central banks seeking to control market stress.

A growing body of literature has focused on the spreads, and there has been great interest in finding the underlying factors and disentangling their effects. We add to the existing literature on spreads and link it to the framework of multiple curve term structure models by considering a quadratic term structure model (QTSM) for the joint dynamics of the risk free OIS rates and their spreads with IBORs. The proposed model is parsimonious and easily accommodates a negative lower bound on yields. We evaluate the empirical performance of the proposed model using data on Euro Interbank Offered Rates (EURIBORs) and OIS rates under the assumption that the spreads are driven by factors related to credit and liquidity risk.

The proposed model belongs to the quadratic class of term structure models (see e.g. Ahn, Dittmar and Gallant 2002, Leippold and Wu 2002, Leippold and Wu 2003, Chen, Filipović and Poor 2004, Jiang and Yan 2009). However, it should be noted that the introduction of additional pseudo-factors makes it possible to convert quadratic models into affine models, as was shown by Cheng and Scaillet (2007), but it comes at the cost of the parsimonious presentation offered by the quadratic class. Therefore our approach is essentially grounded in the tradition of affine yield curve models that have been popular in finance since the seminal works of Vasicek (1977), Cox, Ingersoll Jr and Ross (1985) and Duffie and Kan (1996). Due to their tractability, this class of models is also well suited for econometric estimation and the pricing of derivatives. The models can accommodate different price structures, and their properties and ability to model the yield structures of government bonds have been well studied (see Dai and Singleton, 2000, 2002; Duffee, 2002; Cochrane and Piazzesi, 2005; Collin-Dufresne, Goldstein and Jones, 2008; Duffee and Stanton, 2012) and adaptations have been made for modeling LIBOR rates (Keller-Ressel, Papapantoleon and Teichmann, 2013).

Our approach for incorporating spreads is related to the extensive existing literature on multiple curve models (e.g. Henrard, 2007; Kijima, Tanaka and Wong, 2009; Mercurio, 2009; Bianchetti, 2010; Grbac, Papapantoleon, Schoenmakers and Skovmand, 2015; Nguyen and Seifried, 2015; Cuchiero, Fontana and Gnoatto, 2016b,a; Crépey, Macrina, Nguyen and Skovmand, 2016; Filipović, Larsson and Ttrolle, 2017) and it nests the short spread model of Grbac, Meneghello and Runggaldier (2016). As special case of the model, we derive a multiple curve extension to the arbitrage free Nelson-Siegel (AFNS) model (Christensen, Diebold and Rudebusch, 2009, 2011) and explain how similar extensions can be easily derived for the AFNS models with stochastic volatility considered by Christensen, Lopez and Rudebusch (2014a, 2015).

In the empirical part of the article, we consider specifications of the model that contain five factors. The first three factors affect the risk free spot rate and the last two factors drive the spot spread between EURIBORs and OIS rates.

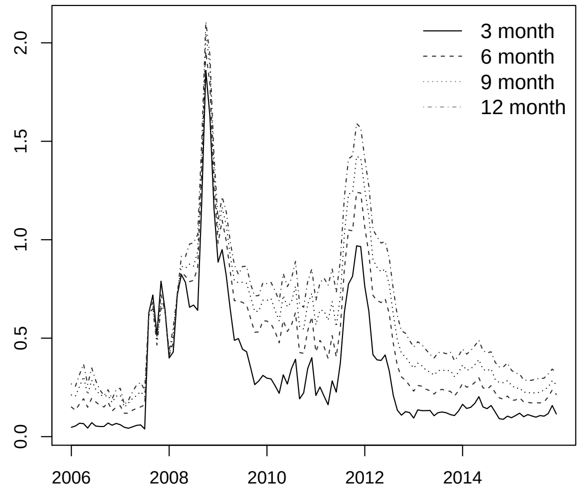


Figure 1: The EURIBOR-OIS spreads during the years 2006-2015 calculated using EURIBOR and OIS rates of the same tenor.

The last two factors correspond to liquidity and credit risk, and they are allowed to depend on the factors driving the risk free rate unlike in the model considered by Dubecq, Monfort, Renne and Roussellet (2016). Estimation is done using the Quadratic Kalman Filter (QKF) of Monfort, Renne and Roussellet (2015).

Our identification scheme is based on the use of proxies for credit risk and liquidity, and is similar to approaches used by Kempf, Korn and Uhrig-Homburg (2012) and Dubecq et al. (2016). Our credit proxy is based on Credit Default Swap (CDS) data, and it can be motivated by the results of Longstaff, Pan, Pedersen and Singleton (2011), who found that the first principal component could account for most of the variation in the spreads in the case of sovereing CDS data. The liquidity proxy is based on European Central Bank's (ECB's) survey. and the spreads between Pfandbrief and German government bond yields which have been analyzed in detail by Kempf et al. (2012).

The article proceeds as follows. In Section 2 we provide a review of previous research on spreads. In Section 3 we present the proposed short rate model and the variant of it that extends the arbitrage free Nelson-Siegel model. Section 4 provides a discussion of the data, the estimation approach and the results. Concluding remarks are made in Section 5.

2 PREVIOUS RESEARCH ON SPREADS

This article focuses on modelling the dynamics of the risk free rate and EURIBORs. As such, it is closely related to the existing literature on spreads. The primary focus of said studies has been on liquidity and credit risk, whose effects on the interbank market and the fixed income markets in general have been considered in numerous articles after the financial crisis with partly contradictory results. The effects of policy actions, such as the Federal Reserve's Term Auction Facility, have been considered by Taylor and Williams (2009), Wu (2011) and McAndrews, Sarkar and Wang (2016). Gyntelberg and Wooldridge (2008) studied the causes of the spreads, and found that both credit and liquidity risk were behind the divergence of the interbank rates during the financial turmoil of 2007. Angelini, Nobili and Picillo (2011) studied the spreads between unsecured and secured interbank rates, and according to their results the spreads were mainly driven by aggregate factors such as aggregate risk aversion rather than bank-specific factors, but for example central bank interventions were not found to be an important determinant. Smith (2012) emphasized the importance of time-varying risk premia, while noting that credit and liquidity factors also played an important role. According to Christensen, Lopez and Rudebusch (2014b) central bank liquidity operations helped lower the premium and the spreads between LIBORs and Treasuries would have been even higher otherwise.

Kempf et al. (2012) studied the term structure of illiquidity premia, and found that volatilities in the asset markets drove short-term liquidation risk whereas the long-term economic outlook had a dominant effect on the long term liquidation risk. The results of Filipović and Trolle (2013) imply that liquidity was the main driver of spreads at the start of the financial crisis, but the dynamics changed after the bankruptcy of Lehman Brothers and the default component became dominant. It was the almost sole driver of spreads in their observation period after May 2009, including the European debt crisis. Schwarz (2016) on the other hand found liquidity to be more important in explaining EURIBOR-OIS spreads than credit risk, whereas according to Dubecq et al. (2016) liquidity was on average more important during the years 2007-2013, but credit risk accounted for most of the spreads at the end of that period. The existing literature thus suggests that both credit and liquidity factors play an important role in explaining the interbank spreads, but their relative importance varies over time.

The spreads in government have been studied in parallel to the interbank spreads. The results of Codogno, Favero and Missale (2003) imply that the spreads between government bond yields in the euro area are mainly driven by differences in credit risk. While credit quality is the main driver, according to Beber, Brandt and Kavajecz (2008) liquidity still has a non-trivial effect on the spreads which is heightened during periods of large flows into or out of the bond market. Monfort and Renne (2013) suggest that there is a causal relationship between periods of credit and liquidity stress. In the U.S. bond market the effects of liquidity and the related flight-to-quality premium has been studied for example by Longstaff (2004).

3 THE MODEL

3.1 The General Framework

We assume that under the risk neutral pricing measure $\mathbb{Q}$ the risk free spot rate r_t is a quadratic function of the n dimensional vector of risk factors X_t, which follow a multivariate Ornstein-Uhlenbeck process. Their dynamics can be represented as the system of equations

$$r_t = \alpha + \beta' X_t + X_t' \Psi X_t \tag{1}$$

$$dX_t = \left(\mu^Q + K^Q X_t\right) dt + \Sigma dW_t^Q, \tag{2}$$

where W_t is a standard Brownian motion, and the matrix Ψ is symmetric and positive semi-definite. If we further restrict Ψ to be positive definite, it can be easily seen that the lower bound of r_t is given by $\alpha - \frac{1}{4}\beta'\Psi^{-1}\beta$. K^Q is assumed to be negative semi-definite, which means that some of the risk factors are allowed to follow unit root processes under the pricing measure. The model can be easily modified to accommodate a deterministic shift extension (see Brigo and Mercurio, 2001) by replacing α with a deterministic function ϕ_t.

The market price of risk is assumed to be of the essentially affine form (Duffee, 2002)

$$\lambda_t = \lambda_0 + \lambda_1 X_t.$$

Given the market price of risk, it can be seen that the dynamics of the state vector X_t under the physical probability measure $\mathbb{P}$ are given by

$$dX_t = (\mu^P + K^P X_t) dt + \Sigma dW_t^P,$$

where $\mu^P = \mu^Q + \lambda_0$ and $K^P = K^Q + \lambda_1$. The eigenvalues of K^P are assumed to negative in order to ensure that the process is stationary under the physical probability measure. In the empirical part of this article we parametrize the dynamics of the risk factors independently under the two probability measures in order to improve the numerical stability of the estimation process.

The time t price of a risk free zero coupon bond with maturity for τ is given by

$$P(t,\tau) = E_t^Q\left[\exp\left(-\int_0^\tau r_{t+u}du\right)\right] = \exp\left(A(\tau) + B(\tau)'X_t + X_t'C(\tau)X_t\right),$$

where $A(\tau)$, $B(\tau)$ and $C(\tau)$ are the solution to the following system of ordinary differential equations

$$\begin{aligned}
\frac{dA(\tau)}{d\tau} &= \mathrm{tr}\left[\Sigma\Sigma' C(\tau)\right] + \frac{1}{2}B(\tau)'\Sigma\Sigma' B(\tau) + B(\tau)'\mu^Q - \alpha, & A(0) &= 0\\
\frac{dB(\tau)}{d\tau} &= 2C(\tau)\Sigma\Sigma' B(\tau) + K^{Q'}B(\tau) + 2C(\tau)\mu^Q - \beta, & B(0) &= 0\\
\frac{dC(\tau)}{d\tau} &= 2C(\tau)\Sigma\Sigma' C(\tau) + C(\tau)K^Q + K^{Q'}C(\tau) - \Psi, & C(0) &= 0.
\end{aligned}$$

We define IBOR dynamics using artificial bonds $\tilde{P}(t,\tau)$, which are similar to the risk free bonds $P(t,\tau)$, but have a spread relative to the risk free rate r_t. The spot spread s_t is assumed to be of the quadratic form

$$s_t = \alpha_s + \beta_s' X_t + X_t'\Psi_s X_t,$$

where Ψ_s is a symmetric positive semi-definite matrix. The time t IBOR of tenor δ is then given by the equation

$$1 + \delta L(t, t+\delta) = \frac{1}{\tilde{P}(t,\delta)},$$

where

$$\tilde{P}(t,\tau) = E_t^Q\left[\exp\left(-\int_0^\tau (r_{t+u} + s_{t+u})\,du\right)\right] = \exp\left(\tilde{A}(\tau) + \tilde{B}(\tau)'X_t + X_t'\tilde{C}(\tau)X_t\right).$$

We have elected to use the same parameters for all tenors, but tenor specific spot spreads and deterministic shift extensions can be used when more flexibility is necessary. The spreads can be bounded to be strictly non-negative by requiring Ψ_s to be positive definite and imposing the restriction $\alpha_s - \frac{1}{4}\beta_s'\Psi_s^{-1}\beta_s \geq 0$. It should, however, be noted that negative spot spreads do not imply negative spreads between IBORs and OIS rates, and the dynamics of X_t can make negative values of s_t unlikely even when they're theoretically possible. Permitting negative spot spreads can thus be useful for facilitating the estimation of the model by providing additional flexibility in the parametrization.

3.2 An Extension to the Arbitrage Free Nelson-Siegel Model

The quadratic multiple curve model proposed in the previous section can be modified so that the risk free rates are given by the AFNS-model of Christensen et al. (2009, 2011). This is achieved by applying simple restrictions to the model parameters in equations (1) and (2). The risk free yield curve is of the AFNS form, if the spot rate is a sum of the first two risk factors

$$r_t = X_{1,t} + X_{2,t},$$

and Σ and K^Q are block diagonal matrices of the form:

$$\begin{aligned} K^Q &= \mathrm{diag}(\overline{K}^Q, \underline{K}^Q) \\ \Sigma &= \mathrm{diag}(\overline{\Sigma}, \underline{\Sigma}), \end{aligned}$$

where $\overline{\Sigma}$ is a 3×3 matrix and

$$\overline{K}^Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\lambda & \lambda \\ 0 & 0 & -\lambda \end{pmatrix}, \quad \lambda > 0.$$

The risk free short rate is thus driven by the first three risk factors that are independent from the other risk factors under the risk free measure $\mathbb{Q}$. Under the physical measure $\mathbb{P}$ the model can admit dependence between all of the risk factors, since K^P is allowed to vary freely. A similar approach can be used to form linear quadratic multiple curve extensions to the AFNS models with stochastic volatility considered by Christensen et al. (2015). The extensions would require additional restrictions on K^P in order to ensure that the factors driving the stochastic volatility remain non-negative, i.e. all the non-diagonal elements on the relevant rows of the mean reversion matrix would be zeros.

It can be shown that under this specification the price of the risk free zero coupon bond with maturity of τ is

$$P(t,\tau) = \exp\left(A(\tau) + B_1(\tau)X_{1,t} + B_2(\tau)X_{2,t} + B_3(\tau)X_{3,t}\right),$$

where the values of the functions $B_i(\tau)$ are

$$\begin{aligned} B_1(\tau) &= -\tau \\ B_2(\tau) &= -\frac{1 - e^{-\lambda\tau}}{\lambda} \\ B_3(\tau) &= \tau e^{-\lambda\tau} + B_2(\tau). \end{aligned}$$

A solution for the yield adjustment term $A(\tau)$ can also be derived in analytical form (see Christensen et al., 2011). In addition to analytical formulas, the AFNS-model also gives well defined interpretations for the three factors driving the risk free rate, as $X_{1,t}$ defines the level of the yield curve, $X_{2,t}$ the slope and $X_{3,t}$ the curvature. The interpretations thus align with those of the first three principal components extracted from yields that are commonly used as risk factors in discrete time term structure models.

The tractability of the model for EURIBORs can be further improved by assuming that spot spreads are not affected by the factors that drive the risk free rate. We partition the state variables X_t into two vectors $\overline{X}_t = (X_{1,t}, X_{2,t}, X_{3,t})'$ and $\underline{X}_t = (X_{4,t}, \ldots, X_{n,t})'$, where only the factors $\underline{X}_t$ affect the spreads. The spot spreads and the prices of the artificial bonds related to EURIBORs are then simplified to

$$s_t = \alpha_s + \beta_s' \underline{X}_t + \underline{X}_t' \Psi_s \underline{X}_t$$
$$\tilde{P}(t,\tau) = P(t,\tau) E_t^Q \left[\exp\left(- \int_0^\tau s_{t+u} du \right) \right].$$

4 EMPIRICAL ANALYSIS

4.1 Data and the Identification of the Credit and Liquidity Proxies

Our interest rate data consists of weekly observations of EURIBOR and OIS rates obtained from Thomson Reuters Datastream, and German Pfandbrief and government bond data covering the years 2009-2014. The benchmark for the OIS is the Euro Overnight Index Average (EONIA) rate. The German Pfandbrief and government bond data sets are published by Bundesbank, and consist of parameter estimates for the Nelson-Siegel-Svensson model, which are used to calculate the yields. The OIS data we use covers the tenors of 1, 3, 6, 9, 12, 24, 36, 48 and 60 months. The EURIBOR data covers the tenors of 1, 3, 6, 9 and 12 months. We extract the prices of the zero coupon bonds $P(t,\tau)$ and $\tilde{P}(t,\tau)$ from the EURIBOR and OIS data, and calculate the yields $y_t(\tau) = -\log(P(t,\tau))/\tau$ and $\tilde{y}_t(\tau) = -\log\left(\tilde{P}(t,\tau)\right)/\tau$ from the bond prices.

We identify the liquidity risk factor by using as a proxy the first principal component of 3-month and 5 year German Pfandbrief and government bond yield differentials and a factor based on the ECB's quarterly Euro area bank lending survey. The Pfandbrief is regarded as having credit risk comparable to the German government bonds, but it is a less liquid instrument (Kempf et al., 2012). The survey factor is calculated as the share of the banks reporting that their liquidity position affected the credit standards as applied to the approval of loans or credit lines to enterprises. The quarterly data was converted to weekly frequency by keeping the related factor value constant for the three months covered by the survey answers. Credit risk is identified using as a proxy the first principal component of Credit Default Swap rates of 37 euro area banks similar to Dubecq et al. (2016). The sample variance of both proxies is standardized to one and the resulting time series are shown in Figure 2.

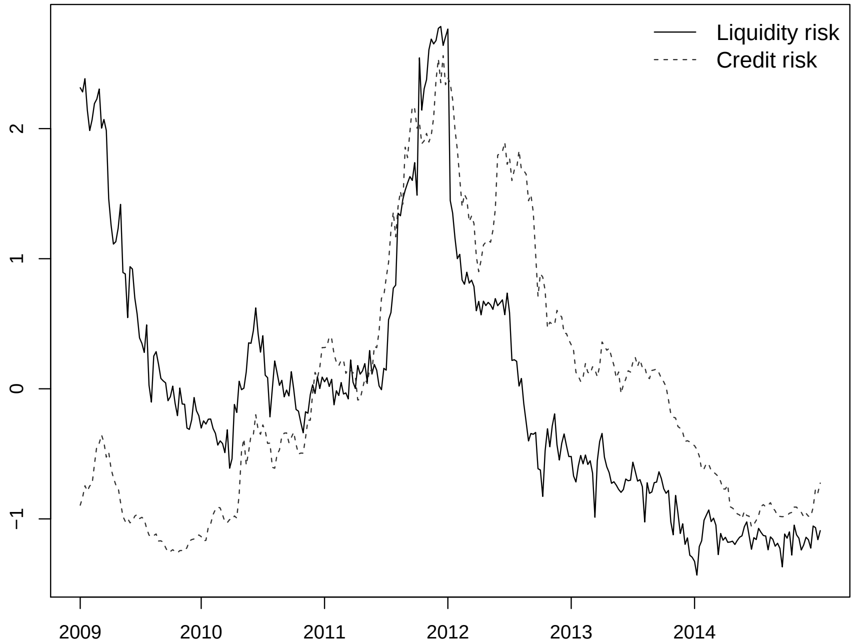


Figure 2: The credit and liquidity proxies.

4.2 Model specifications and estimation method

We estimate a five factor version of the full quadratic model where the first three factors affect only the risk free rates and the last two factors, which we associate with credit risk and liquidity, affect the short spread s_t. This allows us two decompose the state vector into two parts $X_t = \left(\overline{X}_t', \underline{X}_t'\right)'$, where $\overline{X}_t = (X_{1,t}, X_{2,t}, X_{3,t})'$ and $\underline{X}_t = (X_{4,t}, X_{5,t})'$. Using this notation the dynamics of the model under the physical probability measure $\mathbb{P}$ can be written as

$$\begin{aligned} dX_t &= \left(\mu^P + K^P X_t\right) dt + \Sigma dW_t \\ r_t &= \alpha + \beta' \overline{X}_t + \overline{X}_t' \Psi \overline{X}_t \\ s_t &= \alpha_s + \beta_s' \underline{X}_t + \underline{X}_t' \Psi_s \underline{X}_t, \end{aligned}$$

Observations of X_t made at discrete times follow a vector autoregressive process, whose dynamics are described in Appendix A.

We follow the identification restrictions of Ahn et al. (2002), and restrict β and β_s to be vectors of zeros, Σ to be a diagonal matrix and K^P to be an upper triangular matrix. Ψ and Ψ_s are restricted to be positive semi-definite matrices, and their diagonal elements are set equal to one in order to improve the numerical stability of the estimation. The liquidity risk and the credit risk factor are identified by associating them with the related proxies denoted by $p_{l,t}$ and $p_{c,t}$.

We consider two different specifications of the general QTSM model. In the first specification no additional restrictions are placed on the parameter, whereas in the second specification K^Q is restricted to be a block diagonal upper triangular matrix so that $\overline{X}_t$ and $\underline{X}_t$ are independent of each other under the pricing measure. Without the restriction on K^Q, the yields are given by a quadratic function of all the risk factors, even though $\underline{X}_t$ does not affect the risk free spot rates.

The general model is compared with our multiple curve extension to the AFNS-model, where we restrict K^P to be an upper triangular matrix and Σ to be a diagonal matrix similar to the general model. The AFNS model would be well identified with less restrictions, but the chosen approach was adopted because under it all three estimated models have the same basic form with each having increasingly stricter restrictions than the previous one. This approach also allows us to show how even this restricted version of the AFNS model is able to achieve an extremely close fit to the data.

We use the quadratic Kalman filter of Monfort et al. (2015) as described in Appendix B to estimate the models via quasi-maximum likelihood methods and to obtain approximations of the underlying latent factors. The estimation is based on the observed OIS implied yields $y_t(\tau)$, and EURIBOR implied yields $\tilde{y}_t(\tau)$ and the proxies $p_{i,t}$ described in Section 4.1. We associate the fourth risk factor with liquidity risk and the fifth risk factor with credit risk. The full set of observation equations for the previously specified maturities τ, EURIBOR tenors $\tilde{\tau}$ and proxies are defined as

$$
\begin{aligned}
y_t(\tau) &= \frac{-1}{\tau} \ln P(t,\tau) = A(\tau) + B(\tau)' X_t + X_t' C(\tau) X_t + \epsilon_{\tau,t}, \\
\tilde{y}_t(\tilde{\tau}) &= \frac{-1}{\tilde{\tau}} \ln \tilde{P}(t,\tilde{\tau}) = \tilde{A}(\tilde{\tau}) + \tilde{B}(\tilde{\tau})' X_t + X_t' \tilde{C}(\tilde{\tau}) X_t + \epsilon_{\tilde{\tilde{\tau}},t}, \\
p_{l,t} &= A_l + C_l X_{4,t}^2 + \epsilon_{l,t} \\
p_{c,t} &= A_c + C_c X_{5,t}^2 + \epsilon_{c,t}.
\end{aligned}
$$

Because the filter is initialized using unconditional moments of the factors, the first forecasts are typically more inaccurate that the latter forecasts, which utilize information contained in the observed time series. Due to this we chose to ignore the first forecasts in the calculation of the log likelihood function. This choice doesn't affect the asymptotic properties of the estimates. The covariance matrix of the observation errors is fixed to be a diagonal matrix with all error terms having standard deviations of 5 basis points.

4.3 Parameter Estimates

The parameter estimates for the dynamics of X_t in the full quadratic model are shown in Table 1. The mean reversion matrices K^P and K^Q show a meaningful difference between behavior of the risks factors under the physical and risk neutral probability measures. Under the physical measure the fourth and fifth factor, which are related to liquidity and credit risk, have a significant effect on the three factors driving the risk free spot rate. However, under the risk neutral measure the effect of these factors is significantly smaller as shown by the parameter values in the 3×2 upper right block of K^{Q}. While said parameters are mostly statistically significant also under the pricing measure, their economic significance can be considered to be minor due to their low values. It is noteworthy that the factor related to credit risk follows a unit root process under the risk neutral pricing measure $\mathbb{Q}$, as the mean reversion rate is zero. This is in line with the research in credit risk, where the intensities are often found to be non-stationary under $\mathbb{Q}$ (see e.g. Dubecq et al., 2016). Under the physical probability measure the credit risk factor is stationary with a slow rate of mean reversion.

Table 1: Parameter estimates for the dynamics of X_t in the QTSM models. The standard deviations of the estimates are inside parentheses, and the parameter values that have been fixed as zeroes due to the model specification or on the basis of preliminary analysis have been marked with a hyphen.

(a) Parameter estimates for the full QTSM model.

K_1^P	-0.1035 (0.0286)	-14.3026 (3.8516)	-1.4385 (0.2820)	-5.0647 (1.1192)	-6.5292 (1.2117)
K_2^P	-	-169.4034 (55.4537)	-16.1775 (3.8265)	-61.8712 (9.1583)	-79.8013 (18.3164)
K_3^P	-	-	-3.2389 (0.6341)	-0.8699 (0.1107)	-0.9609 (0.1831)
K_4^P	-	-	-	-0.0515 (0.0111)	-0.0962 (0.0361)
K_5^P	-	-	-	-	-8.56e-04 (3.07e-04)
$\mu^{P\prime}$	0.0965 (0.0231)	1.0603 (0.3447)	0.6158 (0.1808)	0.0060 (0.0041)	8.85e-04 (3.62e-04)
K_1^Q	-0.4842 (0.0671)	10.5765 (3.056)	2.4993 (0.5152)	-0.0035 (7.46e-04)	0.0127 (0.0042)
K_2^Q	-	-65.6923 (14.6999)	-35.6399 (6.0876)	0.0019 (6.08e-04)	-0.0685 (0.0208)
K_3^Q	-	-	-0.2163 (0.0484)	-1.28e-04 (4.08e-05)	0.0061 (0.0022)
K_4^Q	-	-	-	-20.2238 (1.9977)	-5.9885 (1.5669)
K_5^Q	-	-	-	-	-
$\mu^{Q\prime}$	2.1546 (0.9186)	-9.7906 (5.1191)	0.0574 (0.0103)	5.539 (1.1329)	-0.1388 (0.0238)
diag(Σ)	0.0035 (6.57e-04)	1.53e-04 (6.47e-05)	0.0033 (3.23e-04)	0.0208 (0.0042)	0.0151 (0.0018)

(b) Parameter estimates for the QTSM model with block diagonal K^{Q}.

K_1^P	-0.1044 (0.0154)	-14.303 (1.5337)	-1.3584 (0.1984)	-4.8843 (0.4778)	-6.5279 (0.44)
K_2^P	-	-169.6195 (24.25)	-15.3322 (1.9259)	-59.7226 (7.3822)	-79.804 (9.5321)
K_3^P	-	-	-3.3244 (0.3549)	-0.8424 (0.0698)	-0.9843 (0.1538)
K_4^P	-	-	-	-0.0531 (0.0065)	-0.1002 (0.0175)
K_5^P	-	-	-	-	-8e-04 (1.78e-04)
$\mu^{P\prime}$	0.0349 (0.0079)	0.3348 (0.0382)	0.6474 (0.126)	0.0068 (8.79e-04)	9.62e-04 (1.14e-04)
K_1^Q	-0.4793 (0.0643)	10.5389 (1.218)	2.5592 (0.3114)	-	-
K_2^Q	-	-65.0424 (8.2707)	-34.3479 (4.7747)	-	-
K_3^Q	-	-	-0.2187 (0.0283)	-	-
K_4^Q	-	-	-	-21.2125 (0.9374)	-7.0866 (0.554)
K_5^Q	-	-	-	-	-
$\mu^{Q\prime}$	2.1564 (0.2841)	-9.9561 (0.5322)	0.0554 (0.0094)	5.6859 (0.6774)	-0.1444 (0.0152)
diag(Σ)	0.0034 (3.68e-04)	1.54e-04 (1.37e-05)	0.0035 (3.93e-04)	0.0229 (0.0033)	0.0159 (0.0012)

The parameter estimates for factor dynamics of the quadratic model with a block diagonal mean reversion rate matrix K^Q under the risk free pricing measure are extremely close to those of the more general model considered above. The only exception to this rule are the first two values of μ^P differ between the two model, which are differ by approximately three standard deviations.

The multiple curve AFNS model, which is a restricted version of the quadratic models considered above, allows for a clearer interpretation of the factor dynamics. The first three rows of K^P in Table 2 show that the three factors, i.e. level, slope and curvature, driving the risk free rate are affected by liquidity and credit risk. The positive coefficient related to liquidity risk on the first row of K^P indicate that high levels in that risk factor also related to higher interest rate levels and vice versa. This might be partially explained by the actions of the European Central Bank in response to worsening liquidity (see González-Páramo, 2011).

The AFNS and the QTSM specifications for the short rate lead to different parameter estimates for the dynamics of the two factors driving the spreads. The differences in μ_4^Q, μ_5^Q, $K_{4,4}^Q$ and $K_{4,5}^Q$ will affect $A(\tau)$, $B(\tau)$ and $C(\tau)$, which will also be reflected in the filtered estimates of the risk factors.

Table 2: Parameter estimates for the dynamics of X_t in the multiple curve AFNS model. The standard deviations of the estimates are inside parentheses, and the parameter values that have been fixed as zeroes due to the model specification or on the basis of preliminary analysis have been marked with a hyphen. The parameter driving the rate of mean reversion under the pricing measure was estimated as $\lambda = 0.4864$ with a standard deviation of of 0.0130)

K_1^P	-0.0162 (0.0073)	3.3572 (0.9358)	0.6374 (0.3377)	2.8663 (0.6638)	-0.2527 (0.1304)
K_2^P	-	-7.7966 (1.7475)	-0.3996 (0.7104)	-6.0687 (1.2420)	-0.3182 (0.1844)
K_3^P	-	-	-1.9561 (0.6906)	-0.7285 (0.6372)	1.1042 (0.4990)
K_4^P	-	-	-	-0.1494 (0.054)	-0.0050 (0.0047)
K_5^P	-	-	-	-	-0.0177 (0.0095)
$\mu^{P\prime}$	-0.1046 (0.0435)	0.1575 (0.0825)	0.0801 (0.0823)	0.0083 (0.0057)	-
$\mu^{Q\prime}$	-	-	-	1.2178 (0.1591)	-0.1053 (0.0153)
$K_{4,4}^Q$	-6.8935 (0.8442)	$K_{4,5}^Q$	1.0499 (0.9886)	$K_{5,5}^Q$	-
diag(Σ)	0.0020 (0.0017)	9e-06 (1.66e-05)	0.0152 (8.23e-04)	0.0164 (0.0022)	0.0156 (0.0019)

The parameters defining the relation between the risk factors and the spot rate, the spreads and the proxies are shown in Table 3, and they are similar for both the QTSM models as expected. It is noteworthy that the estimated parameters permit negative spreads as shown by the values of α_s. However, the actual spot spreads, as implied by the models and the factors extracted using the quadratic Kalman filter, stay positive during the entire in sample period.

Table 3: Parameter estimates for all of the models, with standard deviations shown inside parentheses.

(a)

	QTSM	QTSM-b
α	-0.136 (0.0385)	-0.1427 (0.0284)
$\Psi_{2,1}$	-0.0374 (0.0119)	-0.0292 (0.0033)
$\Psi_{3,1}$	0.0925 (0.0374)	0.1109 (0.023)
$\Psi_{3,2}$	0.688 (0.095)	0.657 (0.0657)
α_s	-0.0251 (0.0077)	-0.0257 (0.0053)
$\Psi_{s,2,1}$	0.9211 (0.0264)	0.9222 (0.0176)
A_l	-10.4145 (3.0515)	-9.5215 (1.8566)
A_c	-5.8632 (0.9875)	-5.599 (0.5906)
C_l	69.0115 (15.736)	62.7211 (9.4801)
C_c	66.096 (13.9173)	61.5657 (5.3523)

(b)

	AFNS
α_s	-0.0048 (0.0015)
$\Psi_{s,2,1}$	0.8066 (0.0693)
A_l	-3.8138 (0.6858)
A_c	-2.197 (0.3216)
C_l	331.9056 (77.1482)
C_c	275.1046 (64.1559)

4.4 In sample forecasts and the filtered time series

We start the analysis of the model fit to the data on the basis of the means of errors and root mean squared errors, which are summarized in Table 4. All of the models were able to achieve close fits with the means of errors staying below one basis point. As expected, the estimates based on the general QTSM specifications of the model having the lowest RMSEs for most of the observed time series, with the statistic being mostly below 3 basis points. Based on these statistics, the most general specification of the model achieved the best fit to the data as expected. The AFNS specification of the model achieved the most uneven fit to the data, but it was generally competitive with the more general models. It should also be noted that the quasi maximum likelihood estimation utilized the forecasting errors, and therefore in the estimation we essentially optimized the accuracy of the forecasts instead of the fit of the model on a given date considered in this paragraph.

The differences in accuracy between the in-sample forecasts of the different models are small, with RMSE statistics in Table 5 favouring the parsimonious multiple curve AFNS model for most of the time series. The difference between the highest and the lowest RMSE is less than one basis point for each of the observed time series. The results suggests that the AFNS specification is sufficiently flexible when compared to the more general quadratic specifications, and it can be preferable in practical applications as it is more parsimonious, easier to estimate and offers factor structure with clear interpretation. Further research with a larger dataset that allows out-of-sample testing is, however, needed to verify this conclusion. The forecasts alongside the observations are shown in

Table 4: Mean and root mean squared errors of the model fit to the observed yields and proxies based on the updated estimates of the factors obtained from the quadratic Kalman filter as defined in Appendic B. In each row the lowest RMSE value and the mean closest to zero are bolded.

	QTSM		QTSM-block		AFNS	
	mean	RMSE	mean	RMSE	mean	RMSE
1m OIS	0.0023	**0.0310**	0.0010	0.0313	**0.0002**	0.0459
3m OIS	0.0022	0.0194	0.0042	0.0199	**0.0006**	**0.0193**
6m OIS	-0.0021	**0.0221**	**-0.0016**	0.0226	-0.0019	0.0266
9m OIS	**-0.0044**	**0.0208**	-0.0049	0.0218	-0.0054	0.0325
12m OIS	**-0.0014**	**0.0206**	-0.0020	0.0214	-0.0062	0.0336
24m OIS	**0.0061**	**0.0322**	0.0085	0.0334	-0.0017	0.0336
36m OIS	**0.0038**	**0.0273**	0.0040	0.0284	0.0070	0.0393
48m OIS	**-0.0014**	**0.0220**	-0.0059	0.0238	0.0042	0.0262
60m OIS	**0.0049**	**0.0306**	0.0085	0.0323	-0.0029	0.0418
1m EURIBOR	**0.0001**	**0.0192**	0.0004	0.0198	0.0030	0.0256
3m EURIBOR	0.0051	0.0266	0.0063	0.0270	**-0.0035**	**0.0222**
6m EURIBOR	0.0061	**0.0210**	**0.0061**	0.0212	0.0104	0.0235
9m EURIBOR	**-0.0070**	**0.0160**	-0.0075	0.0168	0.0012	0.0182
12m EURIBOR	0.0080	**0.0360**	0.0081	0.0363	**-0.0028**	0.0361
p_l	0.0019	**0.0153**	0.0018	0.0153	**0.0008**	0.0153
p_c	**-0.0005**	0.0242	-0.0006	0.0251	0.0011	**0.0177**

the figures of Appendix C.

It should be noted that even though the forecasting errors are generally small, they exhibit autocorrelations which are significant in the case of the EURIBORs and the shortest maturity OIS rates. This observation is likely to be explained by the nature of the data or the assumptions underlying the models. The autocorrelations might be explained by a missing explanatory variable, which has only a minor effect overall level of the interest rates. It is also possible that the model parameters are not constant over the entire observation period, which covers the Great Recession and the European debt crisis. It is however questionable, whether implementing a regime switching model would be worth the additional complexity considering the good explanatory power of our relative simple models. One should also note the differences between the short term dynamics of the relatively volatile OIS rates, which are based on actual transactions, and the stable EURIBOR rates.

The quadratic Kalman filter provides estimates of the latent factors, which allows us to calculate the model implied spot rates r_t and spreads s_t shown in Figure 3. The estimated spot rates of all three models have similar dynamics and reflect the events observable in the movements of the OIS rates such as the

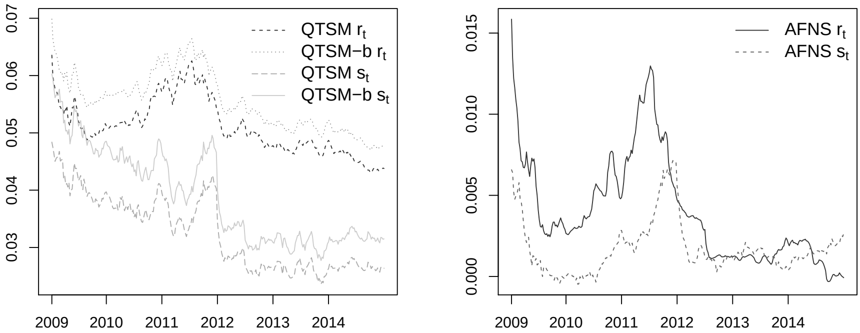
Table 5: Mean and root mean squared errors of the model implied forecasts obtained from the quadratic Kalman filter for the observed yields and proxies. In each row the lowest RMSE value and the mean closest to zero are bolded. The Kalman filter forecasting equations are shown in Appendix B.

	QTSM		QTSM-block		AFNS	
	mean	RMSE	mean	RMSE	mean	RMSE
1m OIS	0.0032	0.0567	0.0028	**0.0564**	**-0.0023**	0.0632
3m OIS	0.0030	0.0508	0.0059	0.0510	**-0.0016**	**0.0457**
6m OIS	-0.0017	0.0555	**-0.0002**	0.0560	-0.0037	**0.0528**
9m OIS	-0.0043	**0.0585**	**-0.0038**	0.0593	-0.0068	0.0603
12m OIS	-0.0018	**0.0630**	**-0.0013**	0.0639	-0.0073	0.0650
24m OIS	0.0043	0.0856	0.0077	0.0860	**-0.0017**	**0.0811**
36m OIS	**0.0009**	**0.0889**	0.0022	0.0890	0.0077	0.0892
48m OIS	**-0.0047**	0.0866	-0.0081	0.0872	0.0054	**0.0843**
60m OIS	0.0016	**0.0905**	0.0061	0.0918	**-0.0014**	0.0927
1m EURIBOR	0.0032	0.0442	0.0039	0.0448	**0.0024**	**0.0442**
3m EURIBOR	0.0094	0.0407	0.0113	0.0417	**-0.0023**	**0.0323**
6m EURIBOR	**0.0104**	0.0412	0.0112	0.0420	0.0124	**0.0376**
9m EURIBOR	-0.0032	0.0357	**-0.0028**	0.0370	0.0033	**0.0341**
12m EURIBOR	0.0111	0.0476	0.0122	0.0492	**-0.0008**	**0.0443**
p_l	0.0217	0.1731	0.0206	0.1729	**-0.0010**	**0.1723**
p_c	-0.0017	0.1058	-0.0018	0.1061	**0.0006**	**0.1027**

drop in 2009, when central central bank policy caused interest rates in general to fall. The spot spreads implied by the different models also show similarities, but there is an interesting difference in 2014, when the spot spreads implied by the AFNS specification are rising coinciding with a similar change in the credit risk proxy, whereas those implied by the full quadratic models vary near the same level they have exhibited since the middle of 2012.

The differences in the levels of the spot rates and spreads explained by the differences in the model specification and parameter estimates as mentioned earlier in Section 4.3. In particular, the constant terms in the yield spreads $\tilde{y}_t(\tau) - y_t(\tau)$ implied by the AFNS and the QTSM models are different, which translates to a difference in the level of the filtered factors.

Estimates of the first three factors in the AFNS model are show in Figure 4. The level factor captures the overall fall in interest rate levels that characterizes the observation period, with temporary upward spikes in 2011 and 2013. The slope factor behaves essentially as a mirror image of the level factor. The fact that the absolute values of both factors are similar reflects the flattening of the interest rate curve as the interest rates fall close to zero.



(a) The spot rates implied by the QTSM models. (b) The spot rates implied by the AFNS model.

Figure 3: In-sample estimates of the model implied spot rates r_t and spreads s_t calculated on the basis of updated factor estimates obtained from the quadratic Kalman filter, as described in Appendix B.

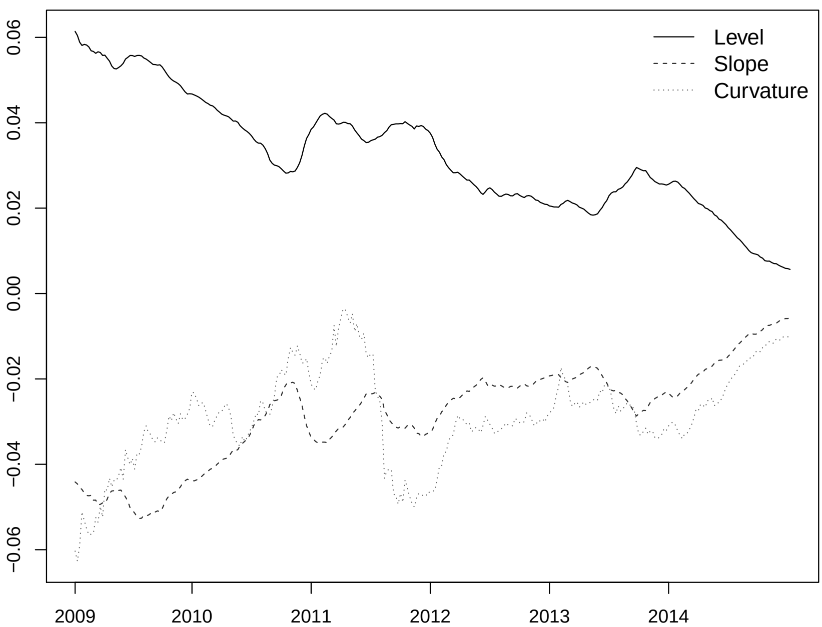


Figure 4: The level, slope and curvature factors of extended the AFNS model obtained from the quadratic Kalman filter.

5 CONCLUSION

We introduce a parsimonious quadratic multiple curve term structure model where the spreads between the spot rates driving IBORs and OIS rates are quadratic functions of risk factors. As a special case of the class of models we derive a multiple curve extension to the arbitrage free Nelson-Siegel model of Christensen et al. (2011). The proposed models are tractable and can be parametrized in such a way that yields or spreads have an arbitrary lower bound.

We estimate the pameters of the models using quasi maximum likelihood estimation via the quadratic Kalman filter of Monfort et al. (2015). The results show that the models are able to obtain close fits to European interest data during the tumultuous years that followed the financial crisis of 2007-2008, but more data is required in order to evaluate the out of sample performance of the proposed models. We identify the effects of liquidity and credit risk via proxies and find that they affect the factors driving the risk free rate. However, the parameters in the matrix K^{Q} defining the dependence between the factors affecting the spreads and the spot rates under the risk neutral pricing measure $\mathbb{Q}$ were extremely small, which suggests that the dependence is important mainly under the physical probability measure. Restricting the factors driving the spreads to be independent of the factors driving the risk free rates can be used to derive tractable and parsimonious pricing formulas such as our extension to the arbitrage free Nelson-Siegel model.

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APPENDIX A: CONDITIONAL DISCRETE TIME DYNAMICS OF THE RISK FACTORS

The dynamics of the risk factors under the probability measure $\mathbb{P}$ follow the n-dimensional Ornstein-Uhlenbeck process

$$dX_t = (\mu^P + K^P X_t)dt + \Sigma dW_t^P.$$

When K^P is diagonalizable, its eigen decomposition is

$$K^P = U\Lambda U^{-1},$$

where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Using this representation, the well known conditional expectation and variance of X_t (see e.g. Ahn et al., 2002) are given by the equations

$$E^P\left[X_{t+\tau}|X_t\right] = U\Lambda^{-1}\left(e^{\tau\Lambda} - I\right)U^{-1}\mu^P + Ue^{\tau\Lambda}U^{-1}X_t$$
$$\text{Var}\left(X_{t+\tau}|X_t\right) = U\left[\frac{v_{ij}\left(e^{\tau(\lambda_i+\lambda_j)} - 1\right)}{\lambda_i + \lambda_j}\right]_{ij} U',$$

where v_{ij} are elements of the matrix

$$V = U^{-1}\Sigma\Sigma' U'^{-1}.$$

The conditional distribution of the risk factors is defined as

$$X_{t+\tau}|X_t \sim N\left(E^P\left[X_{t+\tau}|X_t\right], \text{Var}\left(X_{t+\tau}|X_t\right)\right).$$

APPENDIX B: QUADRATIC KALMAN FILTER

The QKF is based on converting the quadratic model into an affine one. Following the notation of Monfort et al. (2015), we write the physical dynamics of the state space model as

$$X_t = \mu + \Phi X_{t-1} + \Omega\epsilon_t, \quad \epsilon_t \sim IIN(0, I)$$
$$Y_t = A + BX_t + \sum_{k=1}^{m} e_k X_t' C^{(i)} X_t + D\eta_t, \quad \eta_t \sim IIN(0, I),$$

where Y_t is an m-vector, $\Omega\Omega' = \Sigma$, ϵ_t and ν_t are assumed to be independent of each other, and e_k is a standard basis vector, i.e. a column vector whose ith element is equal to one and all other elements are equal to zeroes. The latent factors X_t are stacked into the augmented state vector $Z_t = (X_t', \text{Vec}(X_t X_t'))'$. Under the physical probability measure the dynamics of the observations Y_t and latent factors Z_t can be written as

$$
\begin{aligned}
Z_t &= \tilde{\mu} + \tilde{\Phi} Z_{t-1} + \tilde{\Sigma}_{t-1}^{1/2} \epsilon_t, \quad \epsilon_t \sim IIN(0, I) \\
Y_t &= A + \tilde{B} Z_t + D\eta_t, \quad \eta_t \sim IIN(0, I)
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\mu} &= \begin{pmatrix} \mu \\ Vec(\mu\mu' + \Sigma \end{pmatrix}) \\
\tilde{\Phi} &= \left(\begin{array}{c|c} \Phi & 0 \\ \hline \mu \otimes \Phi + \Phi \otimes \mu & \Phi \otimes \Phi \end{array}\right), \\
\tilde{\Sigma}_{t-1} \equiv \tilde{\Sigma}(Z_{t-1}) &= \left(\begin{array}{c|c} \Sigma & \Sigma \Gamma_{t-1} \\ \hline \Gamma_{t-1}' \Sigma & \Gamma_{t-1} \Sigma \Gamma_{t-1}' + (I + \Lambda_n)(\Sigma \otimes \Sigma), \end{array}\right) \\
\tilde{B} &= \left(\begin{array}{c|c} & Vec\left[C^{(1)}\right]' \\ B & \vdots \\ & Vec\left[C^{(m)}\right]' \end{array}\right)
\end{aligned}
$$

and $\Gamma_{t-1} = I \otimes (\mu + \Phi X_{t-1}) + (\mu + \Phi X_{t-1}) \otimes I$, and Λ_n is an $n^2 \times n^2$ matrix partitioned into $n \times n$ blocks such that the block (i, j) is $e_j e_i'$.

The filter is initialized using the values

$$Z_{0|0} = \tilde{\mu}^u \text{ and } P_{0|0} = \tilde{\Sigma}^u,$$

where $\tilde{\mu}^u$ and $\tilde{\Sigma}^u$ are the unconditional mean and variance of Z_t. The prediction and measurement equations are

$$
\begin{aligned}
Z_{t|t-1} &= \tilde{\mu} + \tilde{\Phi} Z_{t-1|t-1} \\
P_{t|t-1} &= \tilde{\Phi} P_{t-1|t-1} \tilde{\Phi}' + \tilde{\Sigma}_{t-1} \\
Y_{t|t-1} &= \tilde{A} + \tilde{B} Z_{t|t-1} \\
M_{t|t-1} &= \tilde{B} P_{t|t-1} \tilde{B}' + DD'
\end{aligned}
$$

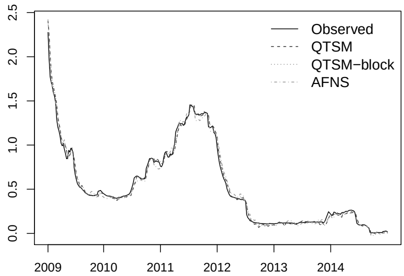
and the Kalman gain is given by

$$K_t = P_{t|t-1} \tilde{B}' M_{t|t-1}^{-1}.$$

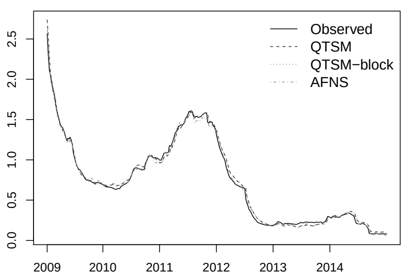
The updated states are given by the equations

$$
\begin{aligned}
Z_{t|t} &= Z_{t|t-1} + K_t (Y_t - Y_{t|t-1}) \\
P_{t|t} &= P_{t|t-1} - K_t M_{t|t-1} K_t'.
\end{aligned}
$$

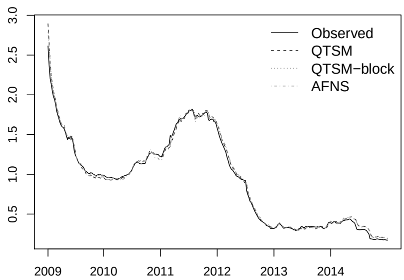
APPENDIX C: SUPPLEMENTARY FIGURES AND TABLES



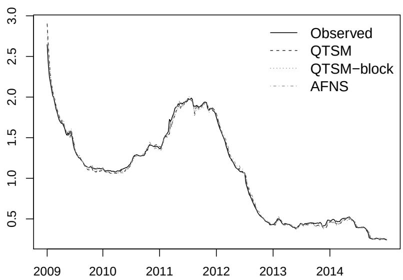
(a) 1 month EURIBOR



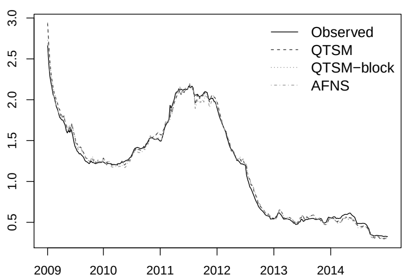
(b) 3 month EURIBOR



(c) 6 month EURIBOR.

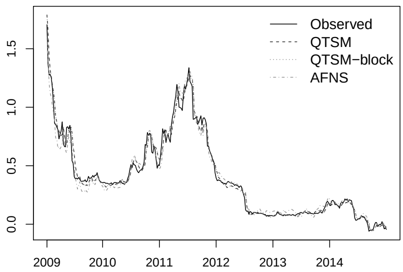


(d) 9 month EURIBOR

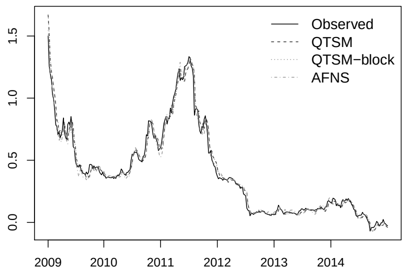


(e) 12 month EURIBOR

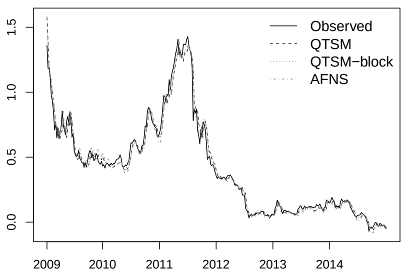
Figure 5: In-sample forecasts of the EURIBOR implied bond yields.



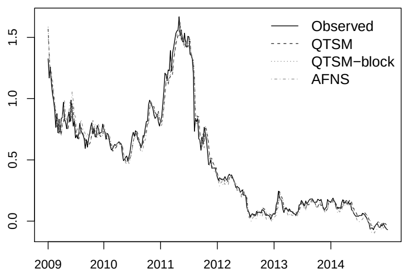
(a) 1 month OIS



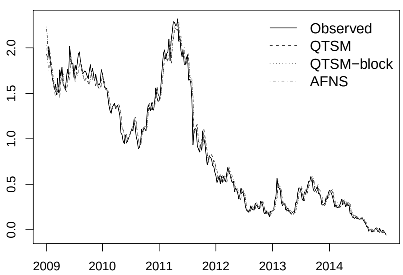
(b) 3 month OIS



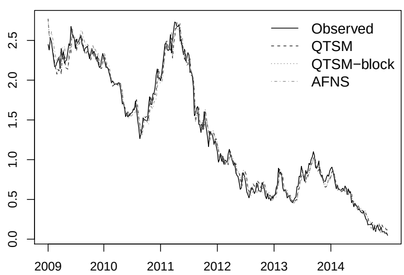
(c) 6 month OIS



(d) 1 year OIS

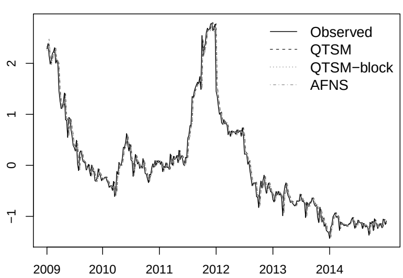


(e) 3 year OIS

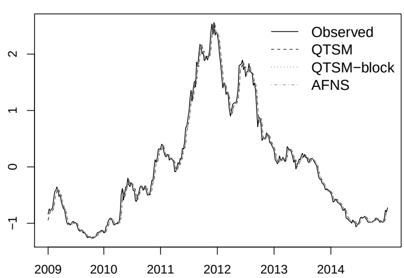


(f) 5 year OIS

Figure 6: In-sample forecasts of the OIS implied bond yields.



(a) Liquidity risk proxy



(b) Credit risk proxy

Figure 7: In-sample forecasts of the liquidity and credit risk proxies

Table 6: Autocorrelations of the forecast errors.

	QTSM		QTSM-block		AFNS	
	$\rho(1)$	$\rho(6)$	$\rho(1)$	$\rho(6)$	$\rho(1)$	$\rho(6)$
1m OIS	0.5401	-0.1573	0.5365	-0.1585	0.5950	0.0713
3m OIS	0.5223	0.0953	0.5231	0.1005	0.3486	-0.0146
6m OIS	0.4047	0.1574	0.4179	0.1737	0.2794	0.1277
9m OIS	0.2642	0.1055	0.2875	0.1266	0.2455	0.1686
12m OIS	0.2112	0.0987	0.2342	0.1139	0.1961	0.1546
24m OIS	0.2169	0.1040	0.2240	0.1030	0.1142	0.0231
36m OIS	0.1643	0.0950	0.1704	0.0959	0.1812	0.1134
48m OIS	0.1158	0.0388	0.1272	0.0494	0.1290	0.0220
60m OIS	0.1727	0.0817	0.1973	0.1090	0.2805	0.1218
1m EURIBOR	0.7507	0.2866	0.7539	0.2977	0.7572	0.2528
3m EURIBOR	0.8200	0.5126	0.8245	0.5337	0.7310	0.3504
6m EURIBOR	0.7630	0.4337	0.7657	0.4406	0.7248	0.4245
9m EURIBOR	0.6410	0.3000	0.6600	0.3306	0.6283	0.4152
12m EURIBOR	0.7238	0.4444	0.7354	0.4631	0.7114	0.5576
p_l	-0.0890	0.0696	-0.0891	0.0693	-0.0999	0.0723
p_c	0.3687	0.0929	0.3740	0.0944	0.2728	0.0760

Annales Universitatis Turkuensis

Turun yliopisto
University of Turku

Painopaikka, Turku , Finland 2018

ISBN 978-951-29-7119-0 (PRINT)
ISBN 978-951-29-7120-6 (PDF)
ISSN 2343-3159 (Painettu/Print) | ISSN 2343-3167 (Verkkojulkaisu/Online)