UNIVERSITY OF TURKU

REGULARITY OF MINIMIZERS AND SOLUTIONS WITH GENERALIZED ORLICZ GROWTH

Arttu Karppinen

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SARJA – SER. AI OSA – TOM. 633 | ASTRONOMICA – CHEMICA – PHYSICA – MATHEMATICA | TURKU 2020

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ABSTRACT

This thesis studies properties of minimizers of variational integrals and solutions of partial differential equations with generalized Orlicz growth (also known as Musielak–Orlicz growth). This is continuation of regularity theory which is a widely studied field in real analysis. Generalized Orlicz growth generalizes various other growth conditions such as polynomial, Orlicz, variable exponent and double phase growth.

This thesis consists of an introductory section, three published articles and one submitted manuscript.

In the introductory part we give an overview of basic definitions and properties of generalized Orlicz spaces and how they relate to most notable special cases.

In the first article we prove that a gradient of a minimizer has local higher integrability. The proof combines a Caccioppoli inequality, Sobolev–Poincaré inequality and Gehring's lemma.

The second article studies global higher integrability and boundary continuity of a minimizer of an obstacle problem. The first result has similar ingredients as in the first article but the boundary of the set Ω and the obstacle function ψ require additional attention. The second result is based on generalizing Harnack inequalities to the obstacle case and a comparison principle proved in this article.

The third manuscript concerns Hölder continuity results of a minimizer or solution to an obstacle problem. The first result is a Hölder continuity for some $\alpha \in (0, 1)$ provided that the obstacle is Hölder continuous. The second result includes Hölder continuity of a minimizer or a solution for every $\alpha \in (0, 1)$ and Hölder continuity of their gradient for some $\alpha \in (0, 1)$. These maximal regularity results require stricter assumptions compared to the first result, for instance Hölder continuity of the gradient of the obstacle.

The fourth article deals with size of removable sets regarding elliptic partial differential equations with generalized Orlicz growth. The size of this removable set is characterized by intrinsic Hausdorff measure related to the growth condition. The main step is to estimate a Radon measure emerging from the equation by this intrinsic Hausdorff measure.

KEYWORDS: minimizer, solution, regularity, generalized Orlicz spaces, Musielak–Orlicz spaces, obstacle problem, Hölder continuity, higher integrability, removable sets

TURUN YLIOPISTO
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TIIVISTELMÄ

Tässä väitöskirjassa tutkitaan variaatiointegraalien minimoijien ja osittaisdifferentiaaliyhtälöiden ratkaisujen ominaisuuksia yleistetyillä Orlicz-kasvuehdoilla (tunnetaan myös nimellä Musielak–Orlicz-kasvuehdot). Tämä jatkaa säännöllisyysteoriaa, joka on laajasti tutkittu reaalianalyysin osa-alue. Yleistetyt Orlicz-kasvuehdot yleistävät monet muut kasvuehdot kuten polynomikasvun, Orlicz-kasvun, varioivaeksponenttisen kasvun ja double phase -kasvun.

Väitöskirja koostuu johdanto-osuudesta, kolmesta julkaistusta artikkelista ja arvioitavaksi lähetetystä käsikirjoituksesta.

Johdanto-osuudessa annetaan yleiskatsaus perusmääritelmistä sekä ominaisuuksista yleistetyissä Orlicz-avaruuksissa ja niiden vastaavuuksista tärkeimmissä erikoistapauksissa.

Ensimmäisessä artikkelissa todistetaan minimoijan gradientin lokaali korkeampi integroituvuus. Todistus yhdistää Caccioppoli-epäyhtälön, Sobolev–Poincaré-epäyhtälön ja Gehringin lemman.

Toinen artikkeli tutkii esteongelman minimoijan globaalia korkeamapaa integroituvuutta ja reunajatkuvuutta. Ensimmäisessä tuloksessa käytetään samanlaisia aineksia kuin ensimmäisessä artikkelissa, mutta alueen Ω reuna ja estefunktio ψ vaativat tarkempaa huomiota. Toinen tulos pohjautuu esteongelmalle yleistettyihin Harnackin epäyhtälöihin ja vertailuperiaatteeseen, jotka artikkelissa todistetaan.

Kolmas käsikirjoitus tarkastelee esteongelman minimoijan tai ratkaisun Hölder-jatkuvuutta. Ensimmäinen tulos on Hölder-jatkuvuus jollakin $\alpha \in (0, 1)$ kunhan estekin on Hölder-jatkuvuva. Toinen tulos sisältää Hölder-jatkuvuuden kaikilla $\alpha \in (0, 1)$ ja gradientin Hölder-jatkuvuuden jollakin $\alpha \in (0, 1)$. Nämä maksimaaliset säännölisyystulokset vaativat vahvempia oletuksia kuin ensimmäisessä tuloksessa, esimerkiksi estefunktion gradientin tulee olla Hölder-jatkuva.

Neljäs artikkeli käsittelee poistettavien joukkojen kokoa elliptisissä osittaisdifferentiaaliyhtälöissä yleistetyillä Olicz-kasvuehdoilla. Poistettavan joukon koko karakterisoidaan luontaisella Hausdorff-mitalla, joka liittyy kasvuehtoon. Tärkeimmässä vaiheessa yhtälöstä syntyvää Radon-mittaa arvioidaan tällä luontaisella Hausdorff-mitalla.

ASIASANAT: minimoija, ratkaisu, säännöllisyys, yleistetyt Orlicz-avaruudet, Musielak–Orlicz-avaruudet, esteongelma, Hölder-jatkuvuus, korkeampi integroituvuus, poistettavat joukot

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Turku, October 2020 Arttu Karppinen

Contents

List of Original Publications

This dissertation is based on the following original publications, which are referred to in the text by their Roman numerals:

I Petteri Harjulehto, Peter Hästö and Arttu Karppinen. Local higher integrability of the gradient of a quasiminimizer under generalized Orlicz growth conditions. *Nonlinear Anal.*, 2018; 177 special issue in honor of Carlo Sbordone: 543-552.

II Arttu Karppinen. Global continuity and higher integrability of a minimizer of an obstacle problem under generalized Orlicz growth conditions. *Manuscripta Math.*, 2019; to appear.

III Arttu Karppinen and Mikyoung Lee. Hölder continuity of the minimizer of an obstacle problem with generalized Orlicz growth; submitted.

IV Iwona Chlebicka and Arttu Karppinen. Removable sets in elliptic equations with Musielak-Orlicz growth. *J. Math. Anal. Appl.*, 2020; to appear.

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1 Introduction

Regularity theory regarding solutions of nonlinear partial differential has been studied since 1950. The main goal of regularity theory is to prove properties for solutions of partial differential equations, such as existence, uniqueness, continuity, higher integrability etc. Because the equations are rarely solvable explicitly, regularity results often rely on the exact structure of the equation rather than the form of the solution itself. For example, the existence of solutions can be proven by compactness arguments which yield no information other than the mere existence of the solution.

A concept closely related to solutions of (partial) differential equations is a concept of variational minimizers. It can be shown with appropriate assumptions that functions minimizing a given integral can be found by solving Euler–Lagrange equation derived form the integrand and vice versa. Therefore studying minimizers gives an alternative approach regularity theory. There are also situations, when the Euler–Lagrange equations does not exist for a particular variational, due to non differentiability of the integrand for example.

A major breakthrough in regularity theory was done by De Giorgi when he proved $C^{0,\alpha}$-Hölder continuity for some $\alpha > 0$ for solutions of second order elliptic equations in divergence form [15]. His method relied on calculus of variations and a few years later Moser proved the same result with an approach starting from the divergence form equation [32]. Both of these techniques have been generalized extensively to various more general and nonlinear situations.

This thesis concerns regularity theory with generalized Orlicz growth conditions which have many important growth conditions, such as polynomial, variable exponent, double phase and Orlicz growth, as special cases. One of the main motivations is to study regularity theory in suitably general context and therefore unify the theory of previously mentioned special cases, which have been studied independently and with methods specially crafted for those specific functional settings. This philosophy seems to be fruitful as many important results, such as $C^{1,\alpha}$-regularity, have been studied in this general context. However, one should not dismiss the study of various special cases, as they provide valuable insight and techniques which combined together yield successful theory in the general case.

Most of the articles in this thesis concern elliptic obstacle problems. Elliptic problems are often used to model steady states of various phenomena, such heat or potential distribution. Obstacle problems impose an additional constraint that the minimizer or solution u has to lie pointwise above some given obstacle function ψ. Combining these features yields problems which could, for example, be used to model final heat distribution u in a metal plate, which is artificially held above some temperature distribution ψ.

Regularity theory of these elliptic equations has multiple applications in physics and image restoration. Especially the special case having variable exponent growth has been successfully describing electrorheological fluids [36], which change their viscosity under external electric field. Another special case, the double phase growth, has been used to model composite materials [40]. Many of these special cases have been also used for image restoration [10, 18]. Image restoration problems often use BV-spaces and fall outside the scope of this thesis.

1.1 Preliminaries

In this section we give the central definitions and structure conditions of the problems studied in this thesis. First we define generalized Orlicz functions and related function spaces. Then we introduce some structural conditions which are required to obtain regularity results also in the special cases of generalized Orlicz growth. With these basic concepts, we can state our variational integrals and partial differential equations of interest. Lastly, we collect the four most important special cases and describe how structural conditions manifest themselves in the more concrete functionals. Most of the presentation and notation follow the book [19].

1.1.1 Generalized Orlicz and Orlicz–Sobolev spaces

By domain $\Omega \subset \mathbb{R}^n$ we mean open and bounded set and we equip it with standard euclidean distance $|x-y| = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$ and Lebesgue measure μ. By $B(x_0, r)$ we mean an open ball with radius r centred in x_0. If the center is not important, we abbreviate it by B_r and if neither center or radius are important we denote it only by B. A ball with t times the radius is simply denoted by tB. If for open sets A and B we have $\overline{A} \subset B$, we denote it $A \Subset B$. By L-almost increasing function f we mean that f satisfies $f(t) \leqslant Lf(s)$ for any $t < s$ and $L \geqslant 1$. L-almost decreasing is defined similarly.

First we define a generalized Orlicz function which dictates the growth rate of problems in this thesis.

Definition 1.1.1. A function $\varphi : \Omega \times [0, \infty) \to [0, \infty]$ is called generalized Φ-prefunction if it is increasing and satisfies $\varphi(x, 0) = \lim_{t \to \infty} \varphi(x, t) = 0$ and

$\lim_{x\to\infty} \varphi(x,t) = \infty$ for almost every $x \in \Omega$. Additionally, a Φ-prefunction is called a

- weak Φ-function and denoted with $\varphi \in \Phi_w(\Omega)$ if $t \mapsto \frac{\varphi(x,t)}{t}$ is L-almost increasing on $(0,\infty)$ for almost every $x \in \Omega$;
- convex Φ-function and denoted with $\varphi \in \Phi_c(\Omega)$ if $t \mapsto \varphi(x,t)$ is left-continuous and convex for almost every $x \in \Omega$.

If φ is independent of the spatial variable x, it is called an Orlicz function and denoted with $\varphi \in \Phi_w$, for example. As convexity implies the L-almost increasing condition for weak Φ-function, we see that $\Phi_c(\Omega) \subset \Phi_w(\Omega)$.

One of the most important concepts related to minimizers with generalized Orlicz growth is the notion of modular as it allows us to define the relevant function spaces. We denote the Lebesgue measurable functions over set Ω by $L^0(\Omega)$.

Definition 1.1.2. Let $\varphi \in \Phi_w(\Omega)$ and $f \in L^0(\Omega)$. We define the modular of f over set $A \subset \Omega$ by setting

$$\varrho_{\varphi,A}(f) := \int_A \varphi(x, |f(x)|)\, dx.$$

The generalized Orlicz space is defined as

$$L^{\varphi}(\Omega) := \{ f \in L^0(\Omega) \,|\, \varrho_{\varphi,\Omega}(\lambda f) < \infty \text{ for some } \lambda > 0 \}.$$

If no confusion arises, we abbreviate $L^{\varphi}(\Omega)$ as L^{φ}.

The generalized Orlicz space becomes a (quasi)normed space when equipped with the following (quasi)norm

$$\|f\|_{L^{\varphi}(\Omega)} := \inf_{\lambda>0} \left\{ \int_{\Omega} \varphi\left(x, \frac{|f(x)|}{\lambda}\right) dx \leqslant 1 \right\}.$$

If φ is increasing rather than L-almost increasing, then this yields a norm. However, this distinction plays no role in this thesis, so we omit the prefix "quasi".

If a function f has weak derivatives and they belong to the same generalized Orlicz space, we say that f belongs to a generalized Sobolev space $W^{1,\varphi}(\Omega)$. Norm in this space is defined as

$$\|f\|_{W^{1,\varphi}(\Omega)} = \|f\|_{L^{\varphi}(\Omega)} + \big\| |\nabla f| \big\|_{L^{\varphi}(\Omega)},$$

where $\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right)$ is the weak gradient of f. Later the norm of the gradient is abbreviated as $\|\nabla f\|_{L^{\varphi}(\Omega)}$. It can be shown that both function spaces are complete with respect to given norms. In regularity theory we often need the zero boundary valued Sobolev spaces $W_0^{1,\varphi}(\Omega)$. This space is the closure of compactly supported smooth functions $C_0^\infty(\Omega)$ with respect to the norm of generalized Orlicz–Sobolev space. This definition is sensible at least in the situation where smooth functions are dense in the Orlicz–Sobolev space and this holds in the case where φ satisfies conditions (A0), (A1) and (aDec) introduced below. If two functions u and v satisfy $u - v \in W_0^{1,\varphi}(\Omega)$, we say that these functions have the same boundary values in the Sobolev sense.

As Young and Hölder inequalities are undeniably important tools in regularity theory, we need to introduce a concept of conjugate Φ-function. This is done with Young inequality in mind by defining

$$\varphi^*(x,t) := \sup_{s>0}\{ts - \varphi(x,s)\}.$$

From the definition it is immediate that Young inequality

$$ts \leqslant \varphi(x,t) + \varphi^*(x,s)$$

holds and it can be proven that also the generalized Hölder inequality

$$\iint_{\Omega} fg\,dx \leqslant 2\|f\|_{L^\varphi(\Omega)}\|g\|_{L^{\varphi^*}(\Omega)}$$

holds for $f \in L^\varphi(\Omega)$ and $g \in L^{\varphi^*}(\Omega)$. Note that the constant 2 cannot be dropped.

To achieve most of the regularity theory, we need to impose some additional structure conditions on φ. These conditions imply for example density of smooth functions and boundedness of the maximal function. Their counterparts in special cases have been shown to be sharp in some situations. Note that the first two conditions are automatically satisfied if φ is independent of spatial variable x. We also introduce the notation

$$\varphi_A^+(t) = \operatorname*{ess\,sup}_{x\in A\cap\Omega} \varphi(x,t) \quad \text{and} \quad \varphi_A^-(t) = \operatorname*{ess\,inf}_{x\in A\cap\Omega} \varphi(x,t)$$

and abbreviate $\varphi^\pm(t) := \varphi_\Omega^\pm(t)$. Additionally, we define the inverse of a generalized Orlicz function as

$$\varphi^{-1}(x,\tau) = \inf\{t \geqslant 0 : \varphi(x,t) \geqslant \tau\}.$$

φ^{-1} acts as inverse functions in general, but at points of discontinuity it is chosen to be left-continuous.

1.1.2 Structural conditions

To obtain regularity results, more assumptions are needed for Φ-functions. These conditions have their counterparts in special cases and they are introduced in Section 1.2. Note that all conditions could be formulated in terms of cubes instead of balls.

The first condition restricts the Φ-function to be unweighted.

Definition 1.1.3. We say that $\varphi \in \Phi_w(\Omega)$ satisfies condition (A0) if there exists $\beta \in (0,1]$ such that $\beta \leqslant \varphi^{-1}(x,1) \leqslant 1/\beta$ for almost every $x \in \Omega$.

A simple example of a function on $\Omega = (0,1)$ not satisfying (A0) conditions is $\varphi(x,t) = \frac{t}{x}$ which is easier to see from the equivalent inequality $\varphi(x,\beta) \leqslant 1 \leqslant \varphi(x,1/\beta)$ of the previous definition. Condition (A0) is invariant under conjugation so φ^* satisfies (A0) if and only if φ satisfies it.

The second condition is a jump condition with respect to spatial variable and is also invariant under conjugation.

Definition 1.1.4. We say that $\varphi \in \Phi_w(\Omega)$ satisfies condition (A1) if there exists $\beta \in (0,1)$ such that

$$\beta\varphi^{-1}(x,t) \leqslant \varphi^{-1}(y,t)$$

for every $t \in \left[1, \frac{1}{|B|}\right]$, almost every $x, y \in B \cap \Omega$ and every ball B with $|B| \leqslant 1$.

If φ satisfies (A0), then condition (A1) is equivalent to requiring

$$\varphi(x,\beta t) \leqslant \varphi(y,t)$$

for $\varphi(y,t) \in \left[1, \frac{1}{|B|}\right]$, almost every $x, y \in B \cap \Omega$ and every ball B with $|B| \leqslant 1$. This condition is a requirement for many basic properties of generalized Orlicz-Sobolev spaces such as density of smooth functions.

Sometimes we would like to simplify the latter condition to allow for $t \in \left[1, \frac{1}{|B|}\right]$. This is a different condition and it is called (A1-n) condition. This condition is often more suitable for bounded minimizers and solutions as some results can be stated for either φ satisfying (A1) or the minimizer u being bounded and φ satisfying (A1-n) [21].

As previously introduced (A1) condition is a jump restriction, higher regularity is proven with stronger continuity restriction. This is called "vanishing (A1)" condition. It also has the weak vanishing version, which catches the borderline assumptions in double phase case.

Definition 1.1.5. We say that $\varphi \in \Phi_w(\Omega)$ satisfies condition (wVA1) if for any $\epsilon > 0$, there exists a non-decreasing continuous function $\omega = \omega_\epsilon : [0,\infty) \to [0,1]$ with $\omega(0) = 0$ such that for any small ball $B_r \Subset \Omega$,

$$\varphi^+_{B_r}(t) \leqslant (1+\omega(r))\varphi^-_{B_r}(t) + \omega(r) \text{ for all } t > 0 \text{ with } \varphi^-_{B_r}(t) \in \left[\omega(r), \frac{1}{|B_r|^{1-\epsilon}}\right].$$

Writing the condition without ε results in (VA1) condition. This is the more intuitive condition but generalizes only the strict inequality $\frac{q}{p} < 1+\frac{\alpha}{n}$ of regularity assumption in the double phase case. The weak formulation catches the case where instead of inequality, an equality is also allowed.

The last two conditions restrict the growth rate of φ. Essentially they exclude the often problematic cases of L^1- and L^∞-spaces in a global sense.

Definition 1.1.6. We say that $\varphi \in \Phi_w(\Omega)$ satisfies $(\mathrm{aInc})_p$ condition if $t \mapsto \frac{\varphi(x,t)}{t^p}$ is L-almost increasing and $(\mathrm{aDec})_q$ condition if $t \mapsto \frac{\varphi(x,t)}{t^q}$ is L-almost decreasing. If $(\mathrm{aInc})_p$ is satisfied for some $p > 1$, we abbreviate by saying that φ satifies (aInc). The same holds also for (aDec).

These conditions restrict the Φ-function to grow faster than t^p but slower than t^q globally. The almost-part in the definition allows for local deviations in the growth rate and yields crude estimates $c_1 \min\{t^p, t^q\} \leqslant \varphi(x,t) \leqslant c_2 \max\{t^p, t^q\}$.

1.1.3 Variational integrals and elliptic partial differential equations

The starting point of regularity theory is to define the minimizing problem or partial differential equation. We start by fixing a boundary function $f : \partial\Omega \to \mathbb{R}$, which has an extension to $W^{1,\varphi}(\Omega)$, and define a minimizer $u \in W^{1,\varphi}(\Omega)$ of a φ-energy, with same boundary values as f in the Sobolev sense, by

$$\int_\Omega \varphi(x, |\nabla u|)\, dx \leqslant \int_\Omega \varphi(x, |\nabla v|)\, dx$$

for all v such that $v - f \in W_0^{1,\varphi}(\Omega)$. If φ is differentiable with respect to the t variable and satisfies (aDec), we can obtain the Euler–Lagrange equation related to the minimizing problem. In a weak form, the equation is

$$\int_\Omega \frac{\varphi'(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla v\, dx = 0,$$

for all $v \in W_0^{1,\varphi}(\Omega)$ [20]. Since the Euler–Lagrange equation requires differentiability of φ, studying minimizers seems the more natural context. However some results, such as maximal regularity of the minimizer in III, require differentiability of the energy functional already in the special cases. Therefore it is useful to utilize also the equation formulation when it is available. Also some questions, such as size of removable sets in IV, are proven with machinery that relies on the equation formulation and linear operators emerging from them. In addition, all of the special cases introduced in Section 1.2 are in fact differentiable. A non-differentiable functional would be for example $\varphi(t) = \max\{t, t^2\}$.

Sometimes we want to study only the interior of the set Ω and we change our focus to local minimizers without a mention of a boundary function f. In generalized Orlicz setting we define a local quasiminimizer u to satisfy

$$\int_{\Omega' \cap \{u \neq v\}} \varphi(x, |\nabla u|)\, dx \leqslant Q \int_{\Omega' \cap \{u \neq v\}} \varphi(x, |\nabla v|)\, dx$$

for every $\Omega' \Subset \Omega$ and every v satisfying $u - v \in W_0^{1,\varphi}(\Omega')$. Quasiminimizers have been studied in [21, 22]. Having $Q = 1$ yields a local minimizer. By requiring the test function v to be greater than u pointwise almost everywhere in previous definition, we call u a global or local (quasi)superminimizer. Quasiminimizers correspond to solutions, which are defined by means of differential operator $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ satisfying conditions listed in Section 1.3.4.

1.1.4 Obstacle problem

II-IV tackle with a modification of the minimizing problem called an obstacle problem. In addition to possible boundary function f we introduce an obstacle function $\psi : \Omega \to [-\infty, \infty]$, which is not necessarily a generalized Orlicz function, and define the class of admissible functions as

$$\mathcal{K}_\psi^f(\Omega) := \{v \in W^{1,\varphi}(\Omega) : v \geqslant \psi \text{ almost everywhere in } \Omega, v - f \in W_0^{1,\varphi}(\Omega)\}.$$

Then we say that a function $u \in \mathcal{K}_\psi^f(\Omega)$ is a minimizer of the $\mathcal{K}_\psi^f(\Omega)$-obstacle problem if

$$\int_\Omega \varphi(x, |\nabla u|)\, dx \leqslant \int_\Omega \varphi(x, |\nabla v|)\, dx$$

for all $v \in \mathcal{K}_\psi^f(\Omega)$. If ψ is identically $-\infty$, then u is called just a minimizer. Obstacle problem has been studied for example in the following articles [6, 7, 33]. There is also research done on double-sided obstacle problems, where the minimizer must lie pointwise between two obstacles [16, 24].

In most results, regularity of the obstacle restricts the regularity of the minimizer of an obstacle problem. This is not unexpected as it is quite possible that the minimizer and the obstacle are equal in some open subset of Ω. Therefore we always assume that the obstacle is in fact a generalized Orlicz function with some Hölder continuity or its gradient has some sort of higher integrability.

It is quite easy to deduce that minimizer of an obstacle problem is always a superminimizer of the corresponding variational problem. For this reason DeGiorgi–Moser theory yields partial results automatically to the obstacle problem also, namely the very weak Harnack inequality which bounds the infimum of the minimizer below with a L^p-norm of the minimizer.

1.2 Notable special cases

The main special cases of interest in generalized Orlicz growth conditions can be divided to polynomial, Orlicz, variable exponent and double phase growth conditions. Naturally, there exists also other special cases such as multi phase case or variable exponent double phase case. In this section we inspect each main special case in detail and discuss how the structural conditions apply in these special cases.

In all of the cases below, local Hölder continuity of the gradient has been obtained, but with widely different techniques. The overall method has been similar, but the proofs have heavily relied on the specific structure of the growth rate. However, as generalized Orlicz spaces contain all these special cases, every case is now covered by one theory.

1.2.1 Polynomial growth

The prototype of a nonlinear elliptic partial differential equation is the p-Laplace equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$$

for $p > 1$, which is the Euler–Lagrange equation of the p-energy integral

$$\inf \int_\Omega |\nabla w|^p \, dx,$$

that is, the choice of $\varphi(x,t) = t^p$. These problems have been widely studied since 1960's and their solutions or minimizers are known to have Hölder continuous gradients [38]. This result is also in some sense optimal and cannot be improved and therefore restricts the results in all of the cases described later. As φ is spatially independent Φ-function, it satisfies (A0) and (A1) automatically and clearly satisfies (aInc)$_p$ and (aDec)$_p$.

As polynomial growth is a special case of every special case introduced below, it naturally has the most complete theory. However, some effects, such as Lavrentiev phenomenon [41], do not appear in this case. For a small sample of articles regarding polynomial growth and its extensions the reader can refer to [25, 26, 30, 38].

1.2.2 Orlicz growth

Orlicz growth can be seen as the spatially independent case of generalized Orlicz growth, that is $\varphi(x,t) = \varphi_0(t)$. An example of Orlicz type function would be $\varphi_0(t) = t^p \log(e + t)$. Higher regularity for Orlicz variational functionals was proven by Lieberman in 1990's [29] and similar regularity of double phase and generalized Orlicz growth rely greatly on this result. Lieberman's result was improved by relaxing the assumptions in [4]. In general, Orlicz functions satisfy (A0), (A1) and (wVA1) automatically and conditions (aInc) and (aDec) have practically identical definitions. Other results obtained in Orlicz setting are studied for example in [9, 11, 39].

Orlicz growth also shows when dealing with $\varphi^{\pm}$-functions of a generalized Orlicz function φ, as these are often well defined Orlicz functions. Another way to use

Orlicz functions is to make a regularized approximation of the generalized Orlicz function as was done for example in [23] and III.

1.2.3 Variable exponent growth

The variable exponent case is obtained by choosing $\varphi(x,t) = t^{p(x)}$ for a function $p : \Omega \to [1,\infty]$. The underlying partial differential equation is the $p(x)$-Laplacian

$$-\operatorname{div}\left(\frac{1}{p(x)}|\nabla u|^{p(x)-2}\nabla u\right) = 0.$$

Sometimes the $\frac{1}{p(x)}$ is left out, as it is only a scalar. In this special case the condition (A1) is called log-Hölder continuity of $\frac{1}{p}$, that is there exists a constant C such that

$$\frac{1}{p(x)} - \frac{1}{p(y)} \leqslant \frac{C}{\log\left(e + \frac{1}{|x-y|}\right)}.$$

An equivalent and sometimes more descriptive formulation of this condition is that $|B|^{\frac{1}{p(x)}-\frac{1}{p(y)}}$ is bounded for every x and y satisfying $x, y \in B$. Here, both (A1) and (A1-n) are equivalent conditions, where as in the double phase case they are distinct.

Already in the variable exponent case it is known that some additional assumptions are needed to guarantee the density of smooth functions. Most often the choice is the above mentioned log-Hölder continuity and only a few regularity results have been proven without assuming it.

$C^{1,\alpha}$-regularity of variable exponent growth was proven by Acerbi and Mingione [1]. Many other results from classical polynomial growth have been extended to this non-classical case such as regularity of superharmonic functions [3] and stability properties of the problem [27]. See also [2, 17, 28, 31, 34].

1.2.4 Double phase growth

In the double phase case we choose $\varphi(x,t) = t^p + a(x)t^q$, where $p < q$ and $a : \Omega \to [0,\infty]$ is a scaling function which describes the mixture rates in composite materials. As the function a is allowed to vanish, at some places φ grows as t^p and at some places as perturbed t^q. In this situation (A0) restricts a to be bounded and (A1) in essence forces a to be Hölder continuous with exponent $\frac{n}{p}(q-p)$ and (A1-n) with exponent $q-p$. Lastly, (aInc) and (aDec) are satisfied with p and q. In this case it has been shown, that the Hölder continuity condition is sharp to acquire Hölder regularity of minimizers.

Double phase case was last of special cases mentioned in this section to which maximal regularity has been proven. It was done by Mingione, Baroni and Colombo and the proofs rely on carefully quantifying and inspecting in each small ball whether the functional is in p-phase or (p,q)-phase [5, 13]. In recent years more and more results from the classical case have been extended to double phase case, for example Calderón–Zygmund theory [8] and estimates on removable sets [12]. See also [14, 18, 35, 37].

1.3 Structure of the thesis

This thesis consists of four research articles or manuscripts and they consider different aspects of regularity of minimizers of variational integrals or solutions of partial differential equations with generalized Orlicz growth. The first paper considers a local quasiminimizer where as papers II and III study obstacle problem directly and IV utilizes results of these papers.

1.3.1 Local higher integrability of a quasiminimizer

I concerns local higher integrability of a local quasiminimizer. By definition, a minimizer has a gradient which is φ-integrable but the main result of this article shows that actually the gradient has better integrability properties. The assumption (aDec)$^\infty$ means that for some $q > 1$, the mapping $t \mapsto \frac{\varphi(x,t)+1}{t^q}$ is L-almost decreasing and is called doubling at infinity.

Theorem 1.3.1. *Let $\varphi \in \Phi_w(\Omega)$ satisfy assumptions (A0), (A1), (aInc) and (aDec)$^\infty$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and suppose $u \in W^{1,\varphi}_{loc}(\Omega)$ be a local quasiminimizer of the φ-energy. Then there exists $\varepsilon > 0$ such that*

$$\varphi(\cdot, |\nabla u|) \in L^{1+\varepsilon}_{\mathrm{loc}}(\Omega).$$

More specifically, it is proven that under the assumptions of the theorem, for a small positive ε and any $\Omega' \Subset \Omega$, the following inequality holds

$$\left(\fint_{\Omega'} \varphi(x, |\nabla u|)^{1+\varepsilon}\, dx\right)^{\frac{1}{1+\varepsilon}} \leqslant C\left(\fint_{\Omega'} \varphi(x, |\nabla u|)\, dx + 1\right) < \infty.$$

This result is a preversion of so called Caldéron–Zygmund theory, where integrability of a solution to an inhomogeneous differential equation is controlled by the integrability of the inhomogeneous part. This result is useful because it is essentially a reverse Hölder inequality.

The proof starts by proving for any $u \in W^{1,1}(B)$ with $\|\nabla u\|_{L\varphi(B)} < 1$ the following Sobolev–Poincaré inequality

$$\fint_B \varphi\left(x, \beta_0 \frac{|u-u_B|}{\mathrm{diam}(B)}\right) dx \leqslant \left(\fint_B \varphi(x, |\nabla u|)^{\frac{1}{s}}\, dx\right)^s + 1,$$

where $s \in [1, p]$ with $s < \frac{n}{n-1}$ and $\beta_0 = \beta_0(n, s, \beta, p, q)$. This is combined with Caccioppoli inequality

$$\fint_B \varphi(x, |\nabla u|)\, dx \leqslant C \left(\fint_{2B} \varphi\left(x, \frac{|u - u_{2B}|}{\operatorname{diam}(B)}\right) dx + 1 \right),$$

where u is a local quasiminimizer in Ω and $2B \Subset \Omega$. The chain of inequalities is a set up for Gehring's lemma, which yields the higher integrability. The Sobolev–Poincaré inequality is actually a property of Orlicz–Sobolev space and therefore independent of the underlying minimizing problem. The minimizing property of u is only used to achieve the Caccioppoli inequality and its proof is quite routine application of a suitable test function in the quasiminimizing condition and "hole filling" technique. The proof of the Sobolev–Poincaré inequality is obtained in three steps: a Jensen type estimate, a singular Jensen type inequality and the final Sobolev–Poincaré inequality.

1.3.2 Global regularity of a minimizer to an obstacle problem

In II the previous result is improved to a global result and also boundary continuity of the minimizer is studied. Both results are done for the obstacle problem and the boundary is assumed to have some regularity: capacity condition for boundary continuity and measure density condition for higher integrability. It is also proven that the former implies the latter if the exponent q from (aDec) is strictly less than the dimension of the underlying euclidean space.

The beginning of the article consists of proving the standard properties of obstacle problems such as minimizers u of an obstacle problem being superminimizes, comparison principle, continuity of u and local minimality of u in the set $\{u > \psi\}$. These results lay the ground work for later results related to obstacle problems.

The first main result is to extend boundary continuity of an ordinary minimizer also to the case of obstacle problem.

Theorem 1.3.2. *Let $\varphi \in \Phi_c(\mathbb{R}^n)$ be strictly convex and satisfy (A0), (A1), (A1-n), (aInc)$_p$ and (aDec)$_q$. Let $\psi \in C(\Omega)$ and $f \in C(\overline{\Omega}) \cap W^{1,\varphi}(\Omega)$ be such that $\mathcal{K}^f_\psi(\Omega) \neq \emptyset$ and let u be the continuous minimizer of the $\mathcal{K}^f_\psi(\Omega)$-obstacle problem. If $x_0 \in \partial\Omega$ satisfies the capacity fatness condition then*

$$\lim_{x \to x_0} u(x) = f(x_0).$$

Capacity fatness condition means that

$$C_\varphi(B(x_0, r) \setminus \Omega, B(x_0, 2r)) \geqslant c_* C_\varphi(B(x_0, r), B(x_0, 2r)),$$

where c_* is a positive constant less than 1, $x_0 \in \partial\Omega$ and $r \leqslant R$ for some R. The capacity C_φ over a set E in generalized Orlicz spaces is defined as

$$C_\varphi(E, \Omega) = \inf_{u \in S_\varphi(E,\Omega)} \int_\Omega \varphi(x, |\nabla u|)\, dx,$$

where the infimum is taken over the set $S_\varphi(E,\Omega)$ of all functions $u \in W_0^{1,\varphi}(\Omega)$ with $u \geqslant 1$ almost everywhere in an open set containing E. This proof relies on the similar result without the obstacle and properties mentioned in the previous paragraph.

The second main result is the global higher integrability result.

Theorem 1.3.3. *Suppose that $\varphi \in \Phi(\mathbb{R}^n)$ satisfies conditions (A0), (A1), (aInc)$_p$ and (aDec)$_q$. Additionally suppose that the measure density condition is fulfilled at every point $x_0 \in \partial\Omega$ with a constant c_*, and let u be the minimizer of the $\mathcal{K}_\psi^f(\Omega)$-obstacle problem, where $\psi, f \in W^{1,\varphi}(\Omega)$ and $\varphi(\cdot,|\nabla\psi|), \varphi(\cdot,|\nabla f|) \in L^{1+\delta}(\Omega)$ for a $\delta > 0$ and $\mathcal{K}_\psi^f(\Omega) \neq \emptyset$. Then there exist $\varepsilon > 0$ and a constant $C = C(n,p,q,\beta,c_*)$ such that $\varphi(x,|\nabla u|) \in L^{1+\varepsilon}(\Omega)$ and*

$$\fint_\Omega \varphi(x,|\nabla u|)^{1+\varepsilon}\,dx \leqslant C\bigg[\bigg(\int_\Omega \varphi(x,|\nabla u|)\,dx\bigg)^{1+\varepsilon} + \int_\Omega \varphi(x,|\nabla\psi)^{1+\varepsilon}\,dx + \fint_\Omega \varphi(x,|\nabla f|)^{1+\varepsilon}\,dx + 1\bigg].$$

The measure density condition is defined similarly as capacity fatness condition, but capacities are replaced with Lebesgue measures.

The global higher integrability has similar strategy to the proof of local higher integrability. Now the Caccioppoli inequality is done in two parts: for balls near the boundary and for balls far away from the boundary. Also the setup for Sobolev–Poincare for balls crossing the boundary of Ω needs additional work compared to the local version. The final reverse Hölder inequality depends also on the modulars of the boundary function and its gradient and the gradient of the obstacle.

1.3.3 Maximal regularity for the obstacle problem

III consists of three results divided into two theorems. The first one improves the local continuity of the minimizer of an obstacle problem to local Hölder continuity and the latter two yield maximal regularity: Hölder continuity for every constant $\alpha \in (0,1)$ and Hölder continuity of the gradient $|\nabla u|$ for some constant $\alpha > 0$. The second theorem has a stronger assumption (wVA1) instead of (A1) used in the first theorem. Here $\partial_t\varphi$ means the derivative with respect to the t-variable of φ. In the article a slightly different notation for the class of admissible functions, that is $\mathcal{K}_\psi^\varphi(\Omega)$ or $\mathcal{K}_\psi(\Omega)$, have been used. The class of functions is similar as in the rest of the thesis, but results are local and therefore without the boundary function f.

Theorem 1.3.4. *Let $\Omega \subset \mathbb{R}^n$ be a domain and $\varphi \in \Phi_w(\Omega)$ satisfy* (aInc), (aDec), (A0), *and* (A1). *Let u be a solution to the $\mathcal{K}_\psi(\Omega)$-obstacle problem. Suppose that the obstacle $\psi \in C^{0,\beta}_{\mathrm{loc}}(\Omega)$ for some $\beta \in (0,1)$. Then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0,1)$.*

Theorem 1.3.5. *Let $\varphi \in \Phi_w(\Omega)$ and $\varphi(x, \cdot) \in C^1([0, \infty))$ for any $x \in \Omega$ with $\partial_t \varphi$ satisfying* (A0), $(\text{Inc})_{p-1}$, $(\text{Dec})_{q-1}$ *for some $1 < p \leqslant q$. Let $u \in \mathcal{K}_\psi(\Omega)$ be a solution to the $\mathcal{K}_\psi(\Omega)$-obstacle problem.*

(i) If φ satisfies (wVA1), then $u \in C^{0,\alpha}_{\text{loc}}(\Omega)$ for any $\alpha \in (0, 1)$.

(ii) If φ satisfies (wVA1) with

$$\omega(r) \lesssim r^\delta \text{ for all } r \in (0, 1] \text{ and for some } \delta > 0,$$

then $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$ for some $\alpha \in (0, 1)$.

Naturally the obtained constants depend also on the Hölder continuity constant of the obstacle, as the minimizer and the obstacle might be equal almost everywhere in some open subset of Ω. In this article some parts have also been proven with the use of weak formulation of the partial differential equation which easily implies the first result also to the minimizing problem. In the higher regularity results we already assume C^1-continuity of the Φ-function so minimizers and solutions are identical.

The first continuity result follows from carefully reproducing the Hölder continuity proof of a regular minimizer and handling the obstacle appropriately. Hölder continuity of the obstacle allows to change between supremum and infimum of the obstacle and trapping the minimizer between them. If the minimizer is larger than the supremum of the obstacle, the situation is essentially a regular minimization problem and the Hölder continuity follows also in that case.

Higher regularity requires a more refined control of the values of the Φ-function with respect to the spatial variable x. The article follows the structure introduced in [23] and compares the minimizer of an obstacle problem to another minimizer of an obstacle problem with Orlicz growth and a regular minimizer with the same Orlicz growth. By estimating the L^1-distance of these different minimizers we can transfer the already known higher regularity of minimizer of the Orlicz-Laplacian equation to minimizer u.

1.3.4 Removable sets

IV explores removable sets for elliptic partial differential equation with generalized Orlicz growth. Here we assume that the differential operator $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following conditions

1. $x \mapsto A(x, \cdot)$ is measurable and $z \mapsto A(\cdot, z)$ is continuous.
2. $|A(x, z)| \leqslant c_1 \varphi(x, |z|) / |z|$.
3. $c_2\, \varphi(x, |z|) \leqslant A(x, z) \cdot z$.

A function u is $\mathcal{A}_\varphi$-harmonic if it solves

$$\langle \mathcal{A}_{\varphi(\cdot)} u, w\rangle := \int_\Omega A(x, \nabla u)\cdot \nabla w \, dx = 0 \quad \text{for all} \quad w \in C_0^\infty(\Omega).$$

The simplest formulation of the main result is as follows.

Theorem 1.3.6. *Suppose $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, is a bounded open set and A satisfies the conditions above with a $\Phi_c(\Omega)$–function $\varphi : \Omega \times [0,\infty) \to [0,\infty)$ satisfying (A0), (A1), (aInc)$_p$ and (aDec)$_q$ with some $1 < p \leqslant q \leqslant n$. Suppose that E is a closed set in Ω and $u \in C^{0,\theta}(\Omega)$ with $0 < \theta \leqslant 1$ is $\mathcal{A}_\varphi$–harmonic in $\Omega \setminus E$. If $\mathcal{H}_{\mathcal{J}_{\theta,\varphi}}(E) = 0$, then u is $\mathcal{A}_\varphi$–harmonic in Ω.*

The basic question can be formulated as follows: If we have a solution u in some set Ω apart from a small closed set $E \subset \Omega$, how large (and in what sense) can E be so that we can extend u as a solution to all of Ω. In the end the size of the set E is measured with an intrinsic Hausdorff-type measure of the form

$$\mathcal{J}_{\theta,\varphi}(B_R(x_0)) \leqslant R^{-\theta} \int_{B_R(x_0)} \varphi(x, R^{\theta-1})\, dx,$$

which combines information of φ and the Hölder continuity constant θ of the solution u. Main novelty in this article is to abandon Hölder inequality in the standard proof and replace it with carefully set up Young inequality. In the generalized Orlicz case, Hölder inequality seems insufficient as the Luxemburg norm is harder to estimate than the standard modular expression and usually resorts to crude norm-modular estimates with exponent p and q from (aInc)$_p$ and (aDec)$_q$. The analysis also relies on the previous results about solutions to obstacle problems and the main proof estimates a Radon measure given by Riesz representation theorem applied to the obstacle problem.

List of References

[1] E. Acerbi and G. Mingione: Regularity results for a class of functionals with nonstandard growth, Archive for rational mechanics & analysis 156, (2001), 121-140 .

[2] E. Acerbi and G. Mingione: Gradient estimates for the $p(x)$-Laplacean system, Journal für die reine und angewandte Mathematik 584, (2005), 117-148.

[3] T. Adamowicz, A. Björn and J. Björn: Regularity of p($\cdot$)-superharmonic functions, the Kellogg property and semiregular boundary points, Ann. Inst. H. Poincare Anal. Non Lineaire 31, no. 6, (2014), 1131-1153.

[4] W. Arriagada and J. Huentutripay: Improved bounds for solutions of φ-Laplacians, Opuscula Math. 38, no. 6, (2018), 765-777.

[5] P. Baroni, M. Colombo and G. Mingione: Regularity for general functionals with double phase. Calc. Var. Partial Differential Equations 57, no. 2, Art. 62, (2018), 48 pp.

[6] S-S. Byun, Y. Cho and J. Oh: Gradient estimates for double phase problems with irregular obstacles, Nonlinear Anal. 177, part A, (2018), 169-185.

[7] S-S. Byun, K-A. Lee, J. Oh and J. Park: Nondivergence elliptic and parabolic problems with irregular obstacles, Math. Z. 290, (2018), 973-990.

[8] S-S. Byun, S. Ryu and P. Shin: Calderón–Zygmund estimates for ω-minimizers of double phase variational problems, Appl. Math. Lett. 86 (2018), 256-263.

[9] S. Challal and A. Lyaghfouri: Removable Sets for A-Harmonic Functions, Journal for Analysis and its Applications Volume 30 (2011), 421-433.

[10] Y. Chen, S. Levine, and M. Rao. Variable Exponent, Linear Growth Functionals in Image Restoration. SIAM J. Appl. Math. 66, (2006), 1383-1406.

[11] I. Chlebicka: Regularizing effect of the lower-order terms in elliptic problems with Orlicz growth, Isr. J. Math. 236, (2020), 967-1000.

[12] I. Chlebicka and C. De Filippis: Removable sets in non-uniformly elliptic problems. Annali di Matematica 199, (2020), 619-649 .

[13] M. Colombo and G. Mingione: Regularity for double phase variational problems, Arch. Ration. Mech. Anal. 215 no. 2, (2015), 443-496.

[14] C. De Filippis: Regularity results for a class of non-autonomous obstacle problems with (p, q)-growth, J. Math. Anal. Appl., to appear. Doi: 10.1016/j.jmaa.2019.123450

[15] E. De Giorgi: Sulla diferenziabilitia e l'analiticitta delle estremali degli integrali multipli regolari, Mem. Accad. Sci. Torino 3 (3), (1957), 25-43.

[16] M. Eleuteri, P. Harjulehto, T. Lukkari: Global regularity and stability of solutions to obstacle problems with nonstandard growth, Rev. Mat. Complut. 26, (2013), 147-181.

[17] L. Fang, L. Zhibin and P. Ling: Variable exponent functionals in image restoration, Appl. Math. Comput. 216, no. 3, (2010), 870-882.

[18] P. Harjulehto and P. Hästö: Double phase image restoration, J. Math. Anal. Appl., to appear. Doi: 10.1016/j.jmaa.2019.123832

[19] P. Harjulehto and P. Hästö: *Orlicz spaces and Generalized Orlicz spaces*, Lecture Notes in Mathematics, vol. 2236, Springer, Cham, 2019.

[20] P. Harjulehto, P. Hästö and R. Klén: Basic properties of generalized Orlicz spaces, Nonlinear Anal. 143 (2016), 155-173.

[21] P. Harjulehto, P. Hästö and M. Lee: Hölder continuity of quasiminimizers and ω-minimizers of functionals with generalized Orlicz growth, Ann. Sc. Norm. Super. Pisa Cl. Sci., to appear.

[22] P. Harjulehto, P. Hästö and O. Toivanen: Hölder regularity of quasiminimizers under generalized growth conditions, Calc. Var. Partial Differential Equations 56 no. 2, (2017), article:22.

[23] P. Hästö and J. Ok: Maximal regularity for local minimizers of non-autonomous functionals, Preprint.

[24] T. Kilpeläinen and W. P. Ziemer: Pointwise regularity of solutions to nonlinear double obstacle problems, Ark. Mat. Volume 29, Number 1-2, (1991), 83-106.

[25] T. Kilpeläinen and X. Zhong: Removable sets for continuous solutions of quasilinear elliptic equations, Proc. Amer. Math. Soc. 130, (2002), 1681-1688.

[26] J. Kinnunen and N. Shanmugalingam: Regularity of quasi-minimizers on metric spaces, Manuscripta Math. 105, (2001), 401-423.

[27] J. Lang and O. Méndez: Γ-convergence of the energy functionals for the variable exponent p(·)-Laplacian and stability of the minimizers with respect to integrability, J. Anal. Math. 134 no. 2, (2018), 575-596.

[28] V. Latvala, T. Lukkari and O. Toivanen: The fundamental convergence theorem for $p(\cdot)$-superharmonic functions, Potential Anal 35, (2011), 329-351.

[29] G. M. Lieberman: On the natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva, Comm. Partial Differential Equations 16 no. 2-3, (1991), 311-361.

[30] P. Lindqvist: On the equation $\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2}u = 0$, Proc. Amer. Math. Soc. 116, (1992), 583-584.

[31] A. Lyaghfouri: Removable sets for Hölder continuous $p(x)$-harmonic functions, Analysis and Applications Vol. 10, No. 02, (2012), 199-206.

[32] J. Moser: On Harnack' s theorem for elliptic differential equations, Comm. Pure Appl. Math. 14 (1961), 577-591.

[33] J. Ok: Regularity results for a class of obstacle problems with nonstandard growth, J. Math. Anal. Appl. 444, (2016), 957-979.

[34] J. Ok: Gradient estimates for elliptic equations with $L^{p}(\cdot)\log L$ growth, Calc. Var. 55, Art. 26, (2016).

[35] J. Ok: Partial regularity for general systems of double phase type with continuous coefficients, Nonlinear Anal. 177 Part B, (2018), 673-698 (Special issue in honor of Carlo Sbordone for his 70th birthday).

[36] K. R. Rajagopal and M. Růžička: On the modeling of electrorheological materials, Mech. Research Comm. 23, (1996), 401-407.

[37] F. Maeda, Y. Mizuta, T. Ohno, and T. Shimomura: Sobolev's inequality for double phase functionals with variable exponents, Forum Mathematicum, 31(2),(2019), 517-527.

[38] N. Uraltseva: Degenerate quasilinear elliptic systems, Zap. Nauchn. Sem Leningrad. Otdel. Mat. Inst. Steklov (LOMI), 7 (1968).

[39] L. Wang, F. Yao, S. Zhou and H. Jia: Optimal regularity for the Poisson equation, Proc. Amer. Math. Soc. 137, (2009), 2037-2047

[40] V. V. Zhikov: On some variational problems, Russian J. Math. Phys. 5 no. 1, (1997), 105-116.

[41] V. V. Zhikov: Lavrentiev phenomenon and homogenization for some variational problems, Composite Media and Homogenization Theory, (1995), 273-288.

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