UNIVERSITY OF TURKU

ABSTRACT

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Abstract

The value of marketability affects investors as they cannot sell their privately held stocks instantly. This restriction should lead to a discount for lack of marketability, DLOM. The increasing flow of capital to private equity makes the DLOM a vital discount to understand for practitioners.

This thesis studies the existence of DLOM in Europe and what are the factors driving it. This study uses the differences in pre-IPO and IPO prices and matches private stocks to similar public stocks and derives the DLOM using differences in valuations. These DLOMs are explained by factors established by earlier literature, such as, trading restriction period or maturity, risk, size and growth. Moreover, put option methods are used to examine the DLOMs.

The average (median) DLOM for IPO and matched pairs methods are 32% (55%) and 61% (63%), respectively. The main factors driving the DLOM are maturity and risk. In conclusion, the exact relationship between the drivers remains unknown but option models can, on average, explain the discounts. Furthermore, put option methods are useful for an analyst as a starting point for DLOM.

Key words	Marketability, liquidity, private equity

TURUN YLIOPISTO

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Tiivistelmä

Arvopapereiden markkinakelpoisuuden arvo vaikuttaa sijoittajiin, koska he eivät voi myydä yksityisiä osakkeitaan välittömästi. Tämän rajoituksen tulisi johtaa likviditeetin puutoksesta johtuvaan alennukseen (discount for lack of marketability, DLOM). Likviditeettialennuksen ymmärtäminen on tärkeää, koska yksityisiin yrityksiin sijoitetaan yhä enemmän pääomaa.

Tämä tutkielma tutkii likviditeettialennuksen olemassaoloa Euroopassa, ja sitä, mitkä tekijät ajavat sitä. Tässä tutkimuksessa käytetään pörssilistautumisen ja sitä edeltävän arvonmäärityksen, ja vertailukelpoisten yksityisten ja julkisten osakkeiden arvostuksen eroja määritettäessä likviditeettialennusta. Näitä alennuksia selitetään aikaisemman kirjallisuuden esittelemillä tekijöillä, kuten rajoituksen pituudella, riskillä, koolla ja kasvulla. Lisäksi, optiohinnoittelumenetelmiä käytetään alennuksen estimointiin.

Keskimääräinen (mediaani) alennus IPO-tutkimuksessa on 32% (55%) ja vertailukelpoisten yritysten analyysissä 61% (63%). Tärkeimmät likviditeettialennuksen ajurit ovat rajoituksen pituus ja riski. Likviditeettialennuksen ja tärkeimpien tekijöiden tarkka suhde on kuitenkin vielä epäselvä, mutta optiomenetelmät pystyvät selittämään osan alennuksesta. Myyntioptiomenetelmät ovat kuitenkin hyödyllinen lähtökohta analyytikolle, joka estimoi likviditeettialennusta yksityiselle yritykselle.

Avainsanat	Marketability, liquidity, private equity

UNIVERSITY OF TURKU

Turku School of Economics

VALUE OF MARKETABILITY

European evidence

Master's Thesis
in Finance

Author:
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1 INTRODUCTION

1.1 Background and motivation

Marketability of assets is vital for investors willing to trade their assets. Therefore, every investor is affected by the marketability of assets. The value of the marketability of an asset stems from lack of market liquidity; without sufficient supply and demand the asset cannot be bought or sold at its fair price. Market liquidity refers to the ability to trade assets easily with minimal costs (Brunnermeier & Pedersen 2009). The value of market illiquidity can be seen as the price of instant remorse of a transaction; how much does it cost to an investor to reverse the transaction. Assets are not usually completely liquid or illiquid but something in between. Thus, liquidity is a continuum, and every asset can be placed somewhere in this continuum based on their degree of liquidity. (Damodaran 2005a.)

The lack of marketability and market liquidity can cause trade-offs because an illiquid asset will tie capital resulting in the loss of potential gains. In order to avert this trade-off by immediately executing the trade in question, an investor is forced to sell (buy) an asset at lower (higher) price than the fair price of the asset causing immediate losses. Consequently, an illiquid asset with lack of marketability should be traded at a discount. The importance of marketability arises especially in corporate finance and company valuation. The discount for lack of marketability (DLOM) is probably the most common discount applied in company valuation and could account for the largest single impact on company value estimates (Glazer 2005).

Liquidity and marketability matter the most when the market is in a downturn or suffering from an illiquidity crisis. The most illiquid assets cannot be sold to meet investors' immediate cash obligations. (Ang et al. 2014.) Flight to liquidity affects negatively especially illiquid stocks, because investors sell their illiquid assets for their more liquid counterparts (Amihud 2002). It is apparent that a market illiquidity crisis is especially harmful for very illiquid privately owned stocks.

This thesis investigates the value of marketability primarily for privately owned stocks and the factors driving this value by applying option pricing techniques to estimate the value of marketability. In publicly traded stocks, liquidity is usually proxied with the bid–ask spread. The differences between the liquidity publicly traded and privately owned stocks are substantial. Publicly traded stocks can be sold or bought quickly with minimal costs, such as the bid–ask spread and brokerage fees, while it can take several months or years to sell a privately owned stock. Privately owned stocks cannot be sold as quickly as publicly traded stocks mainly because there is no centralized market for them. Owning a private stock leads to a substantial trade-off between time and money. This trade-off is significant because the time value of money is a major component of the DLOM. (Curtiss 2009.)

The DLOM refers primarily to closely owned stocks, such as letter stocks, as in SEC Rule 144, or privately owned stocks (Longstaff 1995; Paglia & Harjoto 2010). The amount of wealth allocated to illiquid assets such as private equity, venture capital and hedge funds is increasing, which underlines the importance of understanding and estimating the discounts, for example the DLOM, for such assets (Chen et al. 2015). The inability to sell privately owned businesses quickly without liquidity costs raises the need for valuation adjustments. The DLOM can have a major effect on private business valuation. (Paglia & Harjoto 2010.)

The size of the estimates of the DLOMs vary between earlier studies. Restricted stock studies commonly report an average DLOM of 30 to 35 percent (Glazer 2005). The DLOM also differs across sectors. Professional services usually have higher DLOMs and healthcare has the lowest estimated DLOMs. Size and profitability of the company play also a role in determining the size of the DLOM. They find nearly 70% discounts for some private companies by matching private transactions to publicly traded counterparts. These findings differ greatly from restricted stock studies. Using restricted stocks, IPO and acquisitions in estimating the DLOM tends to result in undervaluation of the discount. (Paglia & Harjoto 2010.) Business valuation analysts usually apply a 15–25 and 25–35 percent discount on one-year and two-year trading restrictions, respectively (Finnerty 2013).

The DLOM can also be estimated using theoretical option pricing techniques which take the length of the restriction period into account. Longstaff (1995) proposes a look-back put option model which would represent the upper bound of the DLOM and the estimations range from 0.4% up 66% highlighting the model sensitivity to its most important inputs, maturity and volatility. This upper bound could also reflect the maximum bid–ask spread investors accept. The model provides an endogenous measure to the bid–ask spread while earlier research relies on exogenous measures.

Another put option model was introduced by Chaffee (1993) who estimated the DLOM simply with European put options. The shortcoming of a European put is that the price to what an investor would be able to sell their asset, were there no restrictions, is fixed and locked in beforehand. Finnerty (2012, 2013b) uses an Asian put option to model the DLOM. In this approach, the investor is equally likely to sell the asset during the restriction period. Therefore, this model does not require the assumption of superior market timers. Finnerty (2012) reports DLOMs close to market observations on moderate volatilities. Finnerty (2013b) further generalizes the option model to account for longer restriction periods. Ghaidarov (2014) modifies the Asian option model and introduces a forward-starting put which is more coherent with the CAPM framework. The forward-starting put option does not rely on any probability in trading and as such, can be applied to any investment strategy. It also performs better with higher volatilities and longer maturities.

Earlier research has established that the DLOM exists and identified the possible factors driving it. However, most of the academic research on the DLOM is conducted in the US stock market. The US stock market is the most efficient and liquid stock market globally which is why it is important to study DLOM in other markets as well (Bekaert et al. 2007). The DLOMs found in Europe by Klein and Scheibel (2012), for instance, are surprisingly smaller, around 5 percent, which clearly points out a contradiction in the theory; why more illiquid European markets have smaller DLOMs? Furthermore, the DLOM is often used in practice as an automatic discount for illiquid stocks without thorough analysis on the underlying company. DLOM is, however, affected by many company characteristics, such as company size, growth and risk. (Glazer 2005.)

1.2 Research objectives

The fact that earlier studies are primarily based in the world’s most liquid market, the US stock market, raises a fundamental question whether the less liquid markets should have larger discounts for illiquidity. This thesis focuses on the European stock market and the estimation of the DLOM for privately owned stocks. The objective of this study is to investigate the size and characteristics of the DLOM in the European stock market. The research questions are following

- If there is a discount for the lack of marketability, how substantial the discount is?
- Which factors are driving DLOM?

It is especially important to investigate what causes the DLOM in European markets, since the market structure differs from the US markets. Capital market financing is more usual in the US, as in Europe a lot of financing is bank issued. However, small- and medium-sized enterprises (SMEs) prefer debt financing regardless of geography. Many of private companies are SMEs both in Europe and the US, so bank financing is crucial for both regions. (WSBI-ESBG 2015.) In conclusion, with the significant difference in capital market size, Europe may have a less developed PE markets regardless of both being dependent on bank financing. Also, from the risk management point of view, liquidity is of utmost importance during market crashes.

The earlier approaches to estimating the DLOM used a market approach by benchmarking. These studies rely on the comparability of liquidity impaired stocks and stocks with no liquidity restrictions. The difference in comparable stock prices would then represent the DLOM. Two types of comparison are the most common in this approach: restricted letter stock prices to common stock prices, and pre-IPO transaction prices to IPO prices. (Curtiss 2009.)

The use of study averages of these differences in practice without further consideration to the underlying factors affecting the dynamics of the DLOM has faced critique. This approach lacks company-specific information which could affect the size of the DLOM. Therefore, the benchmarking market transaction approach would be inappropriate to use in practice. Also, the time value of money plays quite an important role in the estimation of DLOM. The market transaction approach does not allow for the consideration of the length of the trade-off between time and money. (Curtiss 2009.)

The DLOM can also be estimated using revenue- or profitability-based transaction multiples. Transaction multiples approach rely also on the comparability of publicly traded and privately owned firms and, for example, match them by industry and year of the transaction. (Paglia & Harjoto 2010.) The shortcomings of benchmarking approaches, such as purely comparing stocks with different liquidity by ignoring the trade-off between time and money and relying on study averages without taking the underlying fundamentals into account, raise the need for a more sophisticated method. For example, matching companies properly might be difficult to do as companies are rarely similar enough. Furthermore, the use of pre-IPO and IPO prices could be inaccurate, since many other factors affect IPO pricing, such as future estimations of revenue and profitability growth. IPOs are also commonly underpriced. (Zheng & Stangeland 2007.)

To overcome the inadequacy of benchmarking approach, Chaffe (1993) and Longstaff (1995) estimated the DLOM with a different method. They used option-based modelling to estimate the DLOM. While Chaffe (1993) used a simple European put option with simplified rationale for the underlying problem, mainly because he assumed a fixed strike price that the investor would know beforehand, Longstaff (1995) introduced a model more realistic due to a market-based strike price.

Considering the effect of illiquidity to the investor, the loss of potential gains is the greatest when the asset is overpriced, i.e., the investor cannot sell the asset at the highest fair price due to the cost of illiquidity. Therefore, the value of liquidity can be seen as an option. The option in question is a lookback option, where the payout of the option is the difference between the maximum price during the period and the price at the end of maturity. The maturity of the option represents the period where the investor is unable to sell their stock, for example, in the case of letter stock or privately owned stock. (Longstaff 1995.)

However, Longstaff's (1995) approach requires further assumptions. The investor should have perfect market timing ability to be able to sell the stock at the highest price during the period. Obviously, this is an unrealistic assumption. While the estimates of the DLOM are inaccurate using this method, it provides at least a benchmark for the upper bound of the DLOM. Furthermore, it takes company characteristics into account with the model inputs. Also, an advantage of the option-pricing techniques in estimating the DLOM is the ability to take the length of the non-marketable period of the asset, i.e., the maturity, into account.

Many other option pricing techniques for estimating the DLOM have surfaced in the aftermath of Longstaff's (1995) study. The most notable would be the models proposed by Finnerty (2012) and Ghaidarov (2014), where the assumption of a perfect market timer would be needless. This study focuses on determining whether any of these models can explain the market-based DLOMs. This is done by estimating DLOMs with the option models and comparing them to market-based discounts. The estimation of DLOM with options uses the same observation units than in the market-based discount estimation rather than relying on input averages.

1.3 Structure

This section of the thesis provides background information, motivation and research questions. Additionally, it consists of short review of earlier studies on the discount for lack of marketability. The following section further highlights the theoretical background of this study. First, it highlights the importance of private equity markets and private company valuation. Then it introduces the DLOM and why it is important in corporate finance. The rest of the section focuses on the results of previous studies and the models used to estimate the DLOM. Section 3, the empirical part of the study, begins by introducing the data and motivating the methodology for this study. Subsequently, the results are reported and discussed. Section 4 summarizes and concludes the study.

2 LITERATURE REVIEW

2.1 Background

2.1.1 Flow of capital to private markets

Private equity markets are an important source of funds in corporate finance. They are crucial for different types of firms looking for buyout financing, such as start-ups, middle-market private companies, public firms and firms in financial distress. Since the mid-1980s it has been the fastest growing corporate finance market, compared to public equity and bond markets. (Fenn et al. 1997; Bernstein et al. 2016.) Industries where private equity companies invest grow more significantly than other industries. The rapid growth of private equity has raised concern about its impact on the financial markets since the financial crisis in 2007. (Bernstein et al. 2016.)

On the other hand, the private equity industry has a crucial role in post-crisis recovery. Private equity investors cycle their capital through private companies. The capital invested in private equity yields growth through innovation. Once the initial investor executes its exit from the private placement, the proceeds are reallocated to similar private equity investments that have great growth potential. Furthermore, the superior returns the asset class provides has created an investor demand for private equity. (Caselli & Negri 2018.)

Private equity markets are important for innovation. Compared to industrial R&D, private equity in Europe accounts relatively for more innovation measured in patents. This may imply that the private equity markets are not efficient. Other rationales for the inconsistency of innovation include stricter regulation and less developed exit markets for private equity holders. (Popov & Roosenboom 2009.)

In addition to being an important asset class to investors, private equity is a source for knowledge, funding and skills. The role of private equity in the economy is vast, ranging from economic growth and job creating to developing better businesses and offering investors alternative investment opportunities. Therefore, it is no surprise that private equity has become increasingly important and growing industry. However, private equity has been relatively cyclical. During financial crises, specifically the oil crisis in 1970s, the banking crisis in 1990s, the dot-com bubble in the early 2000s and the financial crisis in 2008, new capital invested to private equity has declined severely. (Caselli & Negri 2018.)

Market liquidity is highly associated with market crashes. Typically, in bull markets securities are bought evenly, whereas during market crises, investors choose to sell their securities in a frenzy. Therefore, the marketability of assets declines severely during market crises. (Chordia et al. 2001.) As this is the case in publicly traded stocks, private equity is even more exposed to this risk as, by definition, it does not have a marketplace to trade in. During market crisis, investors usually cannot sell their most illiquid assets to raise cash to meet their cash obligations (Ang et al. 2014). Being in the more illiquid end of the spectrum, private equity exposes investors to a substantial liquidity risk through the inability to sell.

The phenomenon that investors pivot to more liquid assets during market turmoil is flight to liquidity. The difference between asset liquidity can be seen even within the most liquid asset class, for example, on-the-run government bonds are more liquid than its off-the-run counterpart. (Vayanos 2004.) Furthermore, flight to liquidity negatively affects especially illiquid stocks (Amihud 2002). This further raises the question of the marketability of private equity. Investors value marketability and it should be reflected in private equity valuation. In light of private equity markets being inefficient and exposed to market shocks, and the substantial liquidity risk private equity is exposed to, the marketability-adjusted valuation of private companies is key issue in business valuation.

2.1.2 Lack of marketability in private equity

Private equity valuation is a considerable field in equity valuation. Investors and valuation analysts increasingly need to take issues related to private companies into account. For example, businesses may have start-up-like operations that have to be valued similarly as privately held companies, or companies can grow inorganically through acquisitions of private companies. Company acquisitions often result in notable balances in goodwill which requires impairment testing on a yearly basis according to IFRS and GAAP principles. Fair value estimates of financial statements are often used in impairment testing. (Pinto 2010.)

Private company valuation rationales can be divided into three different categories – transactions, compliance and litigation. Transactions include, for example, capital raising, IPOs, acquisitions and compensation. Early-stage private companies need private financing to boost their growth, and venture capital investors invest in private companies through multiple financing rounds. When a company goes public, it needs to be valuated in order to present an offer price for initial public offering. Naturally, an acquisition relies on the valuation of the target company. Additionally, private company valuation may be needed for tax purposes when employees of a private company are rewarded with company stocks. Beyond transactions, private company valuation is often required in financial reporting and litigation-related disputes. (Pinto 2010.)

Private companies have fewer shareholders and are not registered in a public exchange for trading. Obviously, this lack of share liquidity must affect private company valuations. Furthermore, private company shareholders can be restricted from selling their equity by an agreement between the shareholders. This further restrains trading the stock and results in reduced marketability. (Pinto 2010.)

In order to properly valuate a private company, an appraiser needs to adjust the value of the business they have estimated regardless of their choice of valuation method. Commonly used valuation adjustments reflect lack for control and lack of marketability. The discount for lack of control stems primarily from owning a noncontrolling interest in a privately held company. Lack of control limits the investor’s capabilities in decision-making in the private company, such as selecting management or distributing cash and profits. (Pinto 2010.)

The other commonly used valuation adjustment is discount for lack of marketability. The discount for lack of marketability, or the DLOM, is usually a percentage adjustment to reflect lack of marketability. (Pinto 2010.) DLOM might be the most common discount used for company valuation and it usually carries the largest single impact to valuations. By definition, marketability is the ability to liquidate investments with minimal transaction costs. (Glazer 2005.) Most assets are not completely liquid or illiquid. The marketability of an asset can be seen as a continuum and every asset falls somewhere along it, private equity being deep in the illiquid end of the sphere. (Damodaran 2005a.) The inability to sell assets causes trade-offs between time and money, especially when owning private companies. The search cost which includes the effort and time to find an exit from private company investments is highly associated with illiquidity. The illiquidity of private company stocks is a trade-off between time and money and is a major part of the DLOM. (Curtiss 2009.)

2.2 Market-based approach in estimating DLOM

2.2.1 Restricted stock and IPO studies

The research on DLOM primarily focuses on the difference in marketability and valuation between publicly traded stocks and nonmarketable securities which have similar other attributes than market liquidity. The body of research about DLOM is concentrated to restricted stock and pre-IPO studies. (Paglia & Harjoto 2010; Glazer 2005.) Restricted stock approach studies the difference in price between a public stock and its restricted counterpart. A stock is restricted under the SEC Securities Act Rule 144 from resale. The unregistered stock can be sold when it is registered or after a one-year sale restriction period. The difference between the marketable and nonmarketable security prices is seen as an empirical estimation of the DLOM. (Paglia & Harjoto 2010.)

Restricted stock studies have gained wide traction in practice, for example, the U.S. Tax Courts utilizes restricted stocks as a quantification of the DLOM (Paglia & Harjoto 2010). The findings of restricted stock studies for the magnitude of the average DLOM range from 30 to 35 percent (Glazer 2005). Restricted stock studies generally have a small sample size. Moreover, it is not clear whether the discount implied by these studies can be applied to private companies as the premise of restricted stock studies are within publicly traded stock.

IPO studies focus on the comparison between companies' share price before and following IPO. In this case, the DLOM is the difference between the prices. The weakness of IPO studies is that they usually have a small sample size. Moreover, the characteristics of the company may change drastically between a pre-IPO transaction and IPO. (Paglia & Harjoto 2010.) For example, if a company's growth prospects change during that time, its valuation will probably change as well. Thus, IPO studies may factor in issues unrelated to marketability in the estimated DLOM. The IPO studies also have selection bias involved which questions the applicability of the studies across all private firms (Paglia & Harjoto 2010). Investors are also often required to hold their shares after an IPO. This requirement further extends the nonmarketable period of the asset. Moreover, IPOs are often underpriced which may be partly because of these restrictions and the DLOM they imply. (Longstaff 1995.)

Pre-IPO studies usually find average DLOMs close to 45 percent (Glazer 2005). Emory shows in his extensive IPO study series from 1980s to 2000s median DLOMs of

46 percent. Emory's study series consisting of 10 studies all report similar DLOMs. The studies also show the impact of time between a pre-IPO transaction and IPO. The shorter (longer) time between the transactions consistently implied smaller (larger) DLOMs which is why the holding period of private companies is a crucial factor in determining the size of the marketability discount. (Emory 2012.)

2.2.2 Acquisition and benchmark studies

Another approach for quantifying DLOM is the benchmark approach. As private companies do not have a publicly traded market price, the value of such a company can be derived through discounted cash flow methods. However, a DCF relies on the accuracy of future cash flows, and the appropriate discount rates and risk measures are difficult to estimate without problems. Using accrual-based multiples is a typical approach in investment banking and other valuation practitioners. Furthermore, this multiple approach can be used to compare public and private companies to derive an appropriate discount for lack of marketability. (Koeplin et al. 2000.)

Koeplin et al. (2000) use the multiple approach and compare private company acquisitions' valuation multiples to comparable public peers' acquisitions. They use matched pairs analysis where the acquisitions are matched using the same industry and year. Koeplin et al. (2000) argue that the multiples approach yields smaller valuation errors than DCF modeling. They find average DLOMs of 28% and 20% for EBIT and EBITDA multiples, respectively. Using revenue-based multiples the discount is not significant in conventional levels, and book value multiples yield negative average DLOMs.

Block (2007) further extends the analysis of benchmark studies by using similar approach than Koeplin et al. (2000). He also reports a breakdown of DLOMs by industry and finds that manufacturing has the highest discounts as financial sector companies have the smallest DLOMs. Both Koeplin et al. (2000) and Block (2007) use acquisitions in their studies. In Europe, the acquisition studies report much smaller discounts as the discount for privately held businesses relative to their public counterparts are around 5 percent using the acquisition method (Klein & Scheibel 2012).

The value of an investment varies between investors. The reason for perspective-related value is in differing views of future earnings, risk or synergies. For example, acquisitions usually include synergy elements, for example cost synergy through similar operations. Furthermore, the value of a synergy is often misvalued as analyst estimates of the value of synergy usually fail to incorporate practical implementation of the merger and synergy in question. Synergy creation is far more difficult in practice than in an analyst's M&A spreadsheet model. (Damodaran 2005b.) Therefore, acquisition-based DLOMs are criticized for their inaccuracy which is caused by the strategical and synergical components (Paglia & Harjoto 2010).

The shortcomings of acquisition studies needed to be solved, and Paglia and Harjoto (2010) used private transactions in their study and compared them to the valuation of public firms instead of acquisitions. Furthermore, restricted stock, IPO and acquisition studies all suffer from small sample sizes as the comparable firm is harder to find. Considering comparable firms, sample size and comparability is a trade-off pair. The better the comparability, the smaller the sample size. Paglia and Harjoto (2010) eased this trade-off by using all private transactions allowing them to match private and public firms with more detail. They extended the year and industry matching with revenue size also and find DLOMs up to 70 percent, thus violently exceeding the DLOMs from previous studies.

2.3 Modelling and estimation of the DLOM

2.3.1 European put option

The first analytical option model to estimate the DLOM was created by Chaffe (1993) who argued that an investor loses the ability to sell a stock for a fixed time period. In other words, the investor loses a put option on the stock. The value of this put option is considered as the value of marketability. Conversely, the put value divided by the underlying stock price results in DLOM, or at least a major part of it. Chaffe (1993) states that the more marketable a security, the more valuable it is. This relation is clear but the extent and magnitude of it is a more complex problem in valuation. If an investor holds a nonmarketable stock, such as a SEC Rule 144 restricted stock, and buys a plain vanilla put option, they have bought marketability for their nonmarketable stock.

Chaffe (1993) extends his analysis from restricted stocks to private company valuation. The restricted period for SEC Rule 144 has similar nature than private company interests. However, private companies may never experience such a liquidation event as the restricted stock are freely tradable after the restriction period. This liquidation is usually assumed in DLOM studies, such as in pre-IPO studies. Moreover, the strike price in the option model is set to the market price of the stock as of valuation date of the option which results in the option payout being equal to the lost opportunity of a payout of the stock during the restriction period. However, the introduction of option modeling in estimating DLOM allows for taking the holding period into account. The holding period represents a major factor in DLOM, as it is associated with trade-off between time and money.

The Chaffe (1993) model follows a standard Black-Scholes-Merton (1973) option pricing method. Consider a stochastic process

$$dV_t = \mu V_t dt + \sigma V_t dW_t \tag{1}$$

where V_t is the time t price of an asset, μ is the drift, σ is the volatility, and W_t is a standard Wiener process. Under risk neutral valuation techniques, the value of a put option can be expressed as

$$P(V,T) = Xe^{-r(T-t)}N(-d_2) - V_t N(-d_1) \tag{2}$$

where

$$d_1 = \frac{\ln\left(\frac{V_0}{X}\right) + (T-t)(r - \frac{\sigma^2}{2})}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

and r is the risk-free rate, X is the strike price, T is the time to maturity and *N*(.) is the cumulative normal distribution function. Further assuming that the option is at the money, the present value is

$$P(V,T) = V_t\left(2N\left(\frac{\sigma\sqrt{T-t}}{2}\right) - 1\right) \tag{3}$$

With using option valuation techniques, Chaffe (1993) opened a new approach to estimating the DLOM. Especially important factor of option models regarding DLOM is the time to maturity T. The option models also incorporate company risk through the volatility parameter of the stock. However, Chaffe (1993) notes that the vanilla put option does not take the inability to realize gains during the restriction period.

2.3.2 Lookback put option

Longstaff (1995) continued Chaffe's work and idea about options in DLOM modeling. He proposed an option model for the upper bound of the DLOM. In the model, a hypothetical investor has the ability to time markets, i.e., they know when the price of an asset is in its maximum in a given time period. If this investor is restricted to sell their asset for T periods, as the main interpretation for marketability is, their opportunity cost is the difference between the maximum price and the end-of-the-period price of the asset. The present value of this payout is the value of marketability. The option model interpreted from this logic is a lookback option where the strike price is market-based, as opposed to the fixed one in Chaffe model. Furthermore, the model incorporates the inability to realize gains mid-restriction through the assumption of investors having market timing skills.

The model assumptions of markets are similar and rely on the stochastic process presented in equation (1). The difference between the Chaffe (1993) and Longstaff (1993) models are that Longstaff's strike price, the maximum of the stock price during the period, is market-based and follows a different rationale. Longstaff (1995) proposed a model where the investor has superior market timing skills, i.e., can sell the stock at its highest price. As Longstaff (1995) argues, the cost of nonmarketability is highest when the stock price is high and an investor wants to realize gains. Under the market timing assumption, the investor loses the difference between the maximum and the end of the period stock price.

Let M_T denote the payoff at time T to the investor having superior skills, i.e., $M_T=max_{0\leq t\leq T}(V_t e^{r(T-t)})$, and V_T the price at the end of the period. Trading restrictions imply a lost cash flow of $(M_T-V_T)^+$. The present value of such an option is

$$F(V,T) = e^{-rT}E[M_T] - e^{-rT}E[V_T] \tag{4}$$

Under risk neutral pricing, the present value can be expressed as following as the maximum of Brownian motion is known

$$F(V,T) = V_t\left(2+\frac{\sigma^2(T-t)}{2}\right)N\left(\frac{\sigma\sqrt{T-t}}{2}\right)+V_t\frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}e^{\left(-\frac{1}{8}\sigma^2(T-t)\right)}-V_t \tag{5}$$

where V_t is the current price of the stock. The value of the lookback option is proportional to V_t, and hence, the marketability discount is the value of the option divided by V_t. As a result, V_t cancels out and the marketability discount remains the same regardless of the initial stock price. The lookback option is an increasing function to both time to maturity T and volatility σ. This is intuitive since the trading restriction is an important factor in the marketability discount and increased volatility increases the opportunity cost of trading. (Longstaff 1995) Obviously the market timing assumption is very difficult to justify, and Longstaff's (1995) model is usually referred as the upper bound of the DLOM. At the very least, it works as a framework for valuation practitioners.

Brooks (2016) further decomposes the Longstaff model. He constructs the lookback option from two components, vanilla put option and a residual lookback option. Notably, Longstaff's (1995) model includes Chaffe's (1993) put option model through this decomposition. Consequently, the vanilla put option component in Brooks (2016) decomposition is same as equation (2). Assuming that the residual lookback portion is at the money and the risk-free rate equals the dividend yield, its present value can be expressed as

$$L(V,T) = V_t \left(\frac{\sigma^2 (T-t)}{2} N\left(\frac{\sigma\sqrt{T-t}}{2} \right) + \sigma\sqrt{T-t} n\left(\frac{\sigma\sqrt{T-t}}{2} \right) \right) \quad (6)$$

where $n()$ is the probability density function of a standard normal distribution.

The interpretation of the decomposed model further clarifies the aspects of the DLOM. The Chaffe (1993) put option provides a hedge against exposure to downside risk during the marketability restriction period and the residual lookback portion accounts for the loss for a talented or better-informed investor who would have known when to sell at a maximum price. The amount each component contributes to the DLOM is around half with low volatilities and short maturities. The longer (higher) the maturity (volatility), the more does the residual lookback portion contribute, i.e., investor skill is highlighted if the trading restriction is long and the underlying asset risky, volatility being a measure for risk. (Brooks 2016.)

2.3.3 Average-strike put option

The shortcomings of the Longstaff (1995) lookback option, namely the assumption of superior timing skill, inclined Finnerty (2012) to derive another model. Finnerty (2012) uses an average-strike option for DLOM estimation. The Asian option in question relies on a more realistic assumption where an investor is equally likely to sell the stock at any given time during the restriction period, as opposed magically knowing the maximum price.

However, Brooks (2016) criticizes the model for poor justification. Compared to earlier models (Longstaff 1995; Chaffe 1993) the average-strike option has more realistic assumptions and takes the mid-period gain realization inability to account. To be more precise, Longstaff (1995) fails at grasping reality and Chaffe (1993) does not take any investor skill into account.

Continuing with the stochastic process introduced in equation (1) the Finnerty (2012) Asian option present value is

$$A(V,T) = V_t e^{-q(T-t)} \left(N\left(\frac{v\sqrt{T-t}}{2}\right) - N\left(-\frac{v\sqrt{T-t}}{2}\right)\right) \tag{7}$$

where

$$v\sqrt{T-t} = \sqrt{\sigma^2(T-t) + \ln\left(2\left(e^{(T-t)\sigma^2} - \sigma^2(T-t) - 1\right)\right) - 2\ln\left(e^{(T-t)\sigma^2} - 1\right)} \tag{8}$$

and ln() is the natural logarithm function.

Brooks (2016) decomposes the volatility time term. With positive volatility and time to maturity, σ>v>0 holds. For conventional volatilities and time to maturities, the proportion of v from σ is stable, i.e., $\pi = \frac{v}{\sigma}$ is tightly around 57%. Considering a plain vanilla forward-starting put, the average-strike put option is the same except for the inclusion of the proportion term.

Equation 8 implies that a maximum volatility time term exists and cannot exceed ln(2). Using the maximum of the volatility time term yields a cap of 32.28% to the DLOM. Most empirically observed DLOMs exceed this cap. (Brooks 2016.) As a critique, Ghaidarov (2009) proposed a very similar Asian option model where the volatility time term is different and does not have such a limitation.

2.3.4 Forward-starting put option

Ghaidarov (2014) continued his work with DLOMs and introduced yet another option model. This model is a forward-starting put option, where the strike price is set to the date of investor's choosing. Ghaidarov (2014) argues that Finnerty's Asian option has limited rationale. The limitation in Finnerty's argumentation is in the evenly distributed probability of selling the asset. The probability for selling on a given day is exogenous through the restriction period length. The longer the period, the greater the probability of selling due to shorter period where the probability is evenly distributed.

Consider an investor who owns two shares of the same company with different restriction periods. The stock with the shorter restriction period is more likely to be realized even though the stocks are similar in every other way. Ghaidarov (2014) further argues that the decision to sell a stock is not only driven by the length of the restriction period, but also capital need, realizing gains or change in view on the company- or macroeconomic-related factors. One could argue that the stock with longer restriction period would be more likely to be sold in the absence of restrictions to limit the exposure of time increasing losses of nonmarketability.

The present value of the forward-starting put option by Ghaidarov (2014) is under conditions from equation (1) as follows:

$$F(V,T) = V_t e^{-q(T-t)} \left(N\left(\frac{\sigma\sqrt{T-t}}{2}\right) - N\left(-\frac{\sigma\sqrt{T-t}}{2}\right) \right) \tag{9}$$

Except for the different volatility term, the model is similar to Asian options. The forward-starting put can be used with any kind of trading strategy, as it does not require market timing skills or evenly distributed probability of selling. Ghaidarov (2014) shows that forward-starting put serves as a more conservative upper bound for DLOM than Asian options. Furthermore, the forward-starting put can be seen as a cap for the bounds of Asian options. As the average-strike put options rely on the assumption of equally likely trading days, an investor might want to sell their stock on a day that the stock price is higher than the average. Therefore, it is intuitive that the forward-starting put, where the investor can choose the day to sell, is a maximum cap for the upper bound of the Asian option. Trading restrictions would impose an investor holding an average-strike put option above zero opportunity loss. In comparison, Ghaidarov (2014) shows that holding a forward-starting put option would fully offset the costs of opportunity loss.

The forward-starting put model manages to reflect empirical discounts from restricted stock studies quite well. However, restricted stock studies exogenously state a restriction period. Often, the lack of marketability does not have such certainty, especially in the case of private equity.

2.4 Further analysis of the option models

2.4.1 Theta convexity

The size of the marketability discount increases in time. Naturally, the longer the restriction, the larger the discount. Option models are consistent with this fact of the DLOM. However, the DLOM seems to grow slower in respect to maturity with greater maturities. The option models as a function of maturity are shown in Figure 1. The shape of the curve is concave in every option model. The Finnerty (2012) model is often referred as the lower bound of DLOM, whereas Longstaff's (1995) lookback serves as an upper bound.

The Ghaidarov (2014) model falls in between as a tighter upper bound, as it does not restrict the volatility-maturity term as opposed to Finnerty (2012) model.

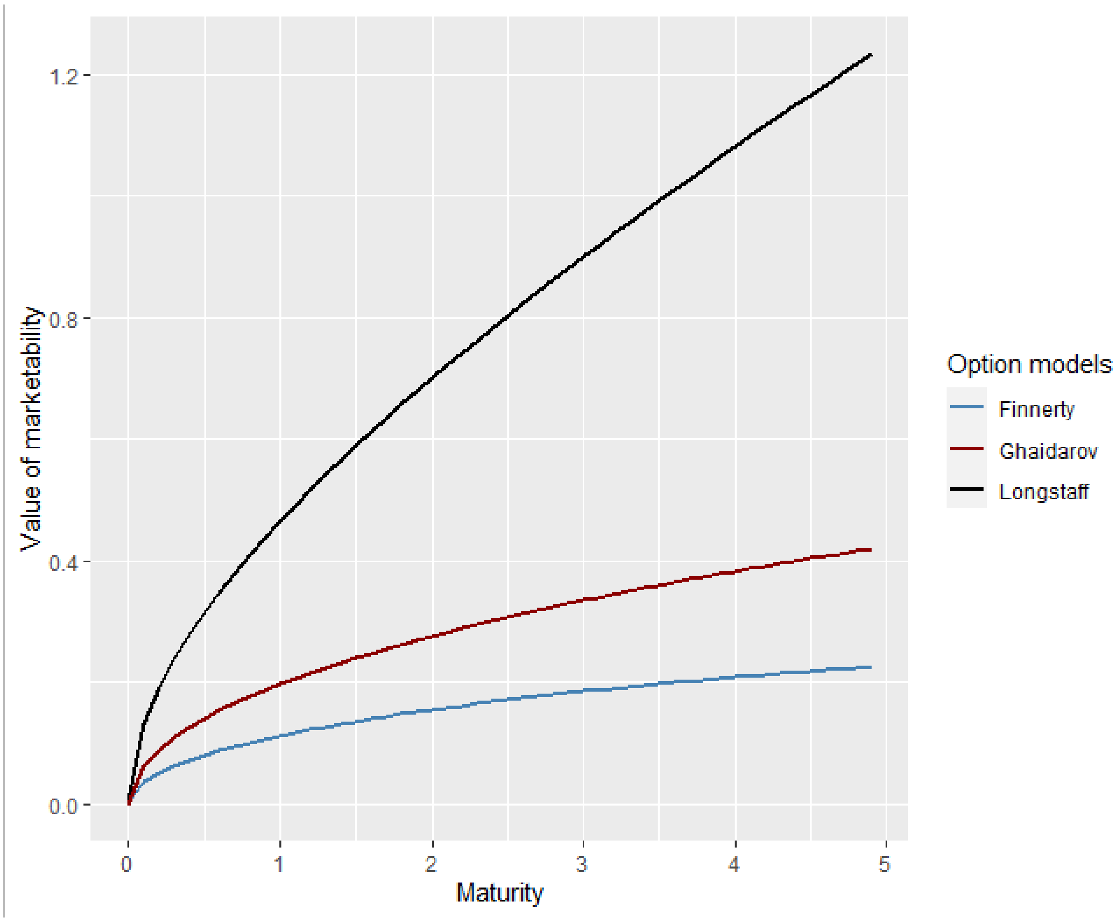

Figure 1 DLOM as function of maturity, σ=50%

Longstaff appears to be more concave than other option models, at least at low maturities. It is also clearly more sensitive to the maturity parameter than the other two options. Furthermore, the lookback put option tends to produce DLOMs over 100% which obviously, is against common sense. To account for the shape of the curve more accurately, a second derivative in respect to maturity is calculated. The second derivative acts as a measure for curvature. Figure 2 shows the Theta convexity of the option models using 50% annual volatility.

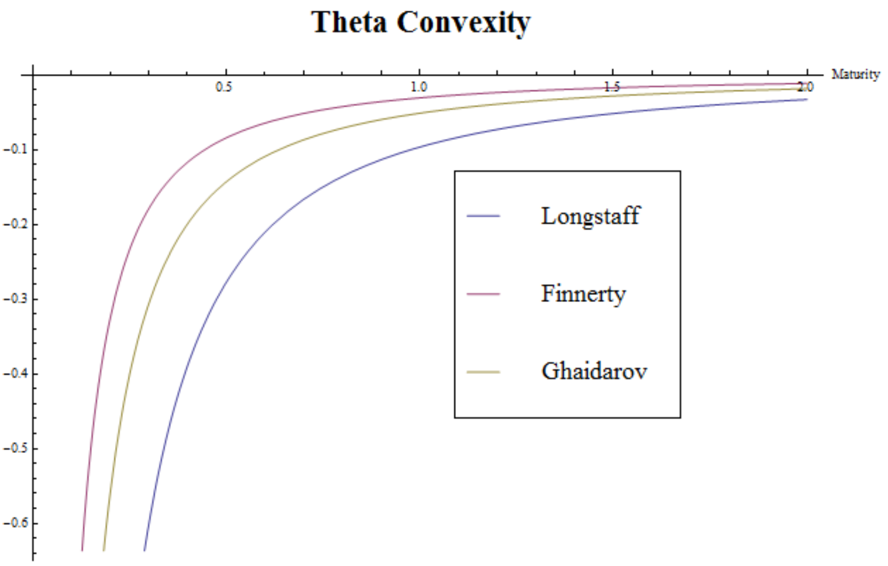

Figure 2 Theta convexities of the option models, σ=50%

The Longstaff (1995) lookback option is much more sensitive to maturity; it is both steeper and more concave as a function of maturity. Too concave a curve could overestimate the discount especially at lower maturities. The refined models of Finnerty (2012) and Ghaidarov (2014) show a slightly smaller negative convexity and may be better suited for estimating the DLOM.

2.4.2 Vega convexity

Option prices are also generally increasing relative to volatility. The interpretation that riskier assets should have higher marketability discounts is consistent with the behavior of option prices relative to rising volatility. The option prices as a function of volatility are shown in Figure 3.

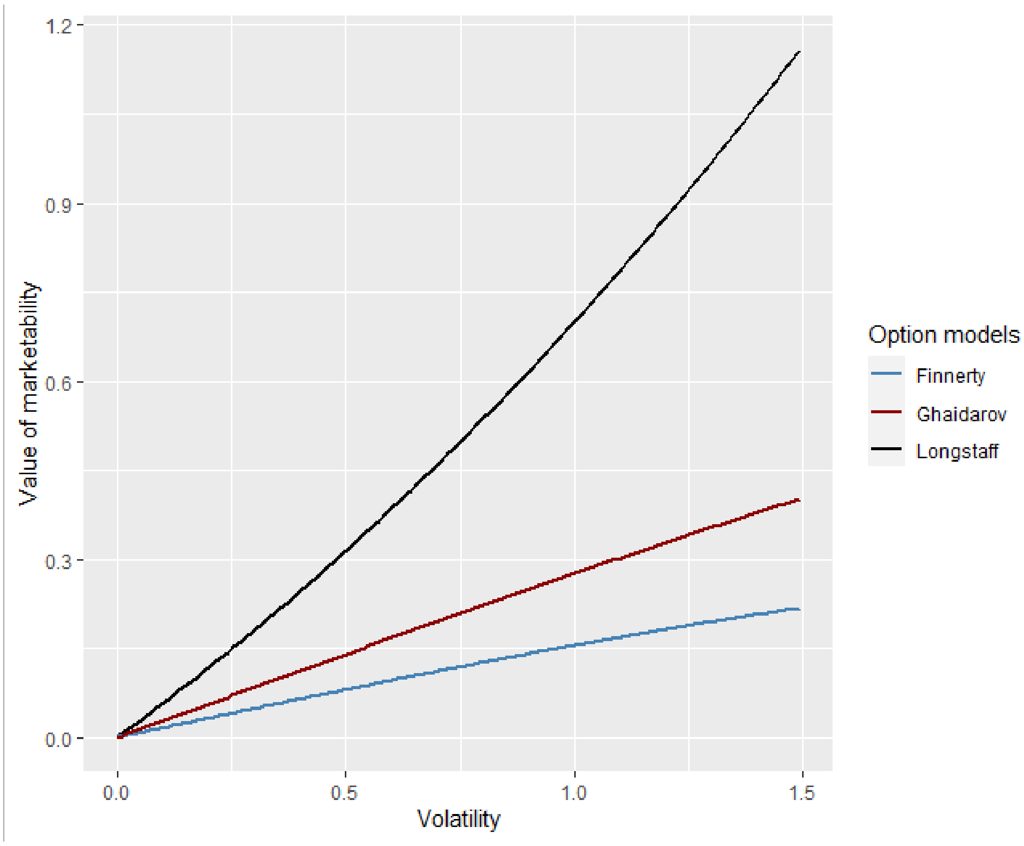

Figure 3 DLOM as a function of volatility, T=1

Figure 3 exhibits that the models are not similar in shape. The lookback put option is convex, and the other two seem to be almost linear in relationship of volatility and option fair value. To analyze the Vega convexity of the models, the second derivative in respect to volatility is shown in Figure 4.

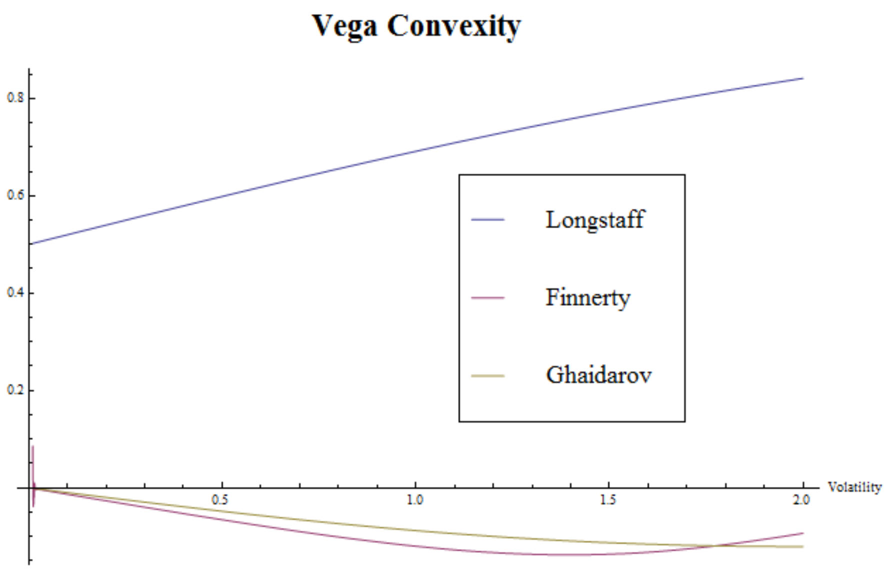

Figure 4 Vega convexities of the option models, T=1

Figure 4 clearly states that the Longstaff (1995) lookback put option is convex, while other models are slightly concave. The negative Vega convexity is consistent with CAPM framework (Ghaidarov 2014). Meulbroek (2001) estimates a required return for a restricted investor using CAPM and Sharpe ratios. The implied lower bound of the DLOM in Meulbroek's (2001) model is a concave relative to volatility. The curvatures of the models in respect to both maturity and volatility suggest that the Finnerty (2012) and the Ghaidarov (2014) models might be theoretically more consistent and better suited for DLOM estimations. In conclusion, the option models exhibit a non-linear relationship between DLOM and both maturity and volatility. Next section, the empirical study, examines the existence of DLOM in European private equity markets and the relationship between DLOM and company-specific factors.

3 DATA AND METHODOLOGY

3.1 Data

The dataset of the study consists of European private placements, IPOs and publicly traded stock data. The time span of the study is roughly the last 20 years, from 1.1.2001 to 10.5.2021. The longer period is justified as the data on private company transactions is quite rare and partly incomplete. Data from IPOs and private deals are from SDC Platinum and public stock data from Refinitiv Eikon. The data in this study is divided into two different datasets. First, IPO data is collected to analyze the DLOM. Secondly, a matched pairs strategy is used, where a private company stock is matched with its liquid counterpart, a publicly traded stock. The data is managed with R and Excel, and the analysis is conducted with R.

Initially, all IPOs and private placements in Europe during the past 20 years are in the dataset, totalling 9251 and 7862, respectively. IPOs that could not be matched with pre-IPO deals by company-specific CUSIP code were excluded from the study. Also, to make sure that the deals were pre-IPO deals, the private placements after the IPO of each company were eliminated to prevent deals executed after the IPO because these would not represent nonmarketable stock properly. Bad data was also eliminated, such as missing values. In the cases where there were multiple pre-IPO deals for the company, only the latest transaction prior to the IPO was included. After these eliminations, the IPO dataset consists of 58 observations. The DLOMs were then estimated as a simple percentage difference between the nonmarketable and marketable stock market values. Even after cleaning the data, two outliers were eliminated as the estimated values showed more than 500% premium for lack of marketability.

The matched pairs strategy utilises the same private equity deals as in the initial IPO data. The private companies are matched using the NAICS 3-digit Subsector code, year of the deal and revenue. (see Paglia & Harjoto 2010.) The company in the same subsector in the same year with the absolute closest revenue was matched with the private company. Further cleaning of the data was required for similar reasons as in the IPO study. However, there are minor differences in the matched pairs data.

Implementing the option method into the matched pairs section requires the use of market-based maturity. The holding periods, or maturity when estimating DLOMs with options, is proxied with the length between deals of the same company. In matched pairs dataset, only the companies that have multiple deals in the data were included. Furthermore, the very last deal of each company in the remaining dataset was removed because the holding period could not be estimated.

As the matched pairs method uses valuation multiples, negative values for revenue and EBITDA for both public and private companies were eliminated when applicable at the later stages of the study. For example, using revenue multiples I did not remove negative EBITDA observations. Furthermore, missing stock return data from the public company caused the exclusion in the matched pairs data. Negative DLOMs were also removed from the data (see Paglia & Harjoto 2010). After cleaning, the data consisted of 95 matched pair observations. The dataset would be further narrowed down for different parts of the analysis in case of missing data.

3.2 Methodology

This study examines the discount for lack of marketability for private companies. The methodology is based on comparing option pricing techniques to market observations of the DLOM. Section 3.3 will be structured followingly: First, in both subsections, the market based DLOMs (IPO or matched pairs) will be estimated. Secondly, the option models are estimated and compared to market-based discounts. Third, the comparison and relationship between the factors and DLOM are further analyzed.

In option modeling, first, the upper bound for DLOM is derived with Longstaff's (1995) lookback put option for upper bound of the discount. This method, however, relies on the assumption that investors possess a superior skill for market timing, i.e., they know when the stock price is the highest during the period. Furthermore, this option model may yield DLOMs in excess the value of the stock. Hence, the DLOM is capped at 100% for obvious reasons.

Secondly, Finnerty (2012) and Ghaidarov (2014) methods are used. Finnerty (2012) extends the simple lookback option by using an Asian option to derive the value of marketability. This approach assumes that an investor would be equally likely to sell the stock at any time during the restricted holding period which is more realistic than Longstaff (1995) model. However, the Finnerty average strike model tends to understate the DLOM at high volatilities of approximately over 150% and yield a lower bound for the DLOM (Elmore 2017).

Ghaidarov (2014) proposes an option model where the investor can choose when they will sell the stock during the period. This forward-starting put option can be priced in the beginning while the strike price would be fixed by setting it to the market price of the day of the investor's choosing. The investor does not need a superior market timing skill or to know the stock prices of the period beforehand. Ghaidarov (2014) shows that the DLOMs are consistent with the observed restricted stock prices also with higher volatilities.

Understating the DLOM for high volatilities is problematic especially for the valuation of small private companies. Consider an example of a venture capitalist investing in a small private business. The goal of the venture capitalist is to boost the growth and value of the business before their exit. This implies high volatility for the values of small companies. High level of volatility in privately owned stocks is also highlighted by Chaffe (1993) who concludes that private stocks are likely to have a volatility over 50% based on volatilities of small companies in OTC market.

The models' market-based inputs are volatility and maturity. The estimation of the inputs can be quite prone to error. The volatility of a private company is unobservable since they are not listed, and maturity depends on the holding period of the investor. While the restricted stock volatility can be matched with its publicly traded counterpart, the volatility proxy for a private company is more complex. The volatility of a private company is usually proxied by matching the company with a comparable publicly traded company (Elmore 2017). This can be done, for example, with similar size and industry (see Paglia & Harjoto 2010).

In the IPO section, the volatility will be proxied with the first-year volatility of the continuous stock returns after the IPO. In the matched pairs section, the volatility of the continuous stock returns of the matched public company will be used. The matched company volatility will be estimated from the year prior to the private company deal date. Volatility will be estimated from daily returns and transformed to yearly volatility assuming the convention of 250 trading days per year in both sections.

The holding period of a private stock is unknown beforehand. For restricted stocks, maturity is exogenous for legislative reasons, but private stocks do not always have a strict restriction. The illiquidity of a private stock is more related to demand and search costs because it requires more effort to find a buyer for a private stock, and this effort along with the time it takes to find a buyer, is the illiquidity cost for privately owned stocks (Curtiss 2009). Usually, estimating the expected holding period requires professional judgement, and it often is at least a few months (Elmore 2017). In this study, the maturity in the IPO section of this study is the difference between IPO date and last private deal date. This naturally limits the maturity and is quite intuitive. The private equity investor is forced to hold the stock for that time.

For matched pairs, the maturity is the time between private equity deals. Assuming the length of a holding period can be estimated from data results in an estimation of maturity of the option. For example, a private company may go public in the future, but the time of the IPO is uncertain, or VC's may plan to sell (see Finnerty 2013b). In this study, the maturity in matched pairs section is the difference in time between deals of the same stock.

The IPO method uses ex-post data, and its primary goal is to ascertain the existence of the DLOM and test the option models and the underlying factors driving DLOM. In the matched pairs section, a more practical approach is used. The data is ex-ante, if I assume that the investor who is selling the stock has an exit planned, knows that they can sell at that exit and sells accordingly. The estimated holding periods, i.e., the time between deals, act as trading restrictions. The deals are liquidation events, i.e., opportunities to sell and as such, the exits for the investor as planned. I assume that the investor planned the exit beforehand and is actually able sell at the exit. In my dataset, these exits are the next deal date on the same company. This approach allows for determining DLOM before a deal and allows for studying if analysts could estimate a proper DLOM practically before the deal.

For the sake of my analysis, the dividends of private companies in the dataset are assumed to be zero; the dividend data is scarce and disregarding them is not a crucial error in the analysis. All in all, dividends are generally expected to cause the fall of the underlying stock ex-dividend, but the analysis concentrates on companies not trading at a public exchange. Hence, there is no underlying stock price movement to be corrected with dividends. The option models for DLOM also assume dividends to be zero.

The complex nature of the DLOM and scarcity of private company data forces such assumptions to be made. For example, proxying the risk associated with private stocks by using public company volatility may be somewhat inaccurate, and the uncertainty of the maturity can have a considerable impact on the DLOM. However, as the DLOM is a complicated discount especially for private companies without trading restrictions, this approach offers a way to account for it and provides framework for determining the DLOMs for private companies. Furthermore, estimating the inputs for option models is beneficial for possible practical applications.

Finnerty (2013b) argues that the DLOMs produced by a model should be consistent with the market-based observations of the DLOM. However, the comparison to restricted stock studies is problematic due to them not representing DLOM only; other factors unrelated to liquidity could affect the differences between the prices between common stock and restricted stocks, such as capital scarcity effects, discount for lack of control and information asymmetry (Robak 2007). Similarly, other estimating methods for DLOM could suffer from these effects. As these factors are very difficult to control, the internal validity of the results in this study may be declined.

The next subsection discusses the results of this thesis with existing literature. First, in the IPO analysis, the existence of DLOM in European markets is confirmed using t statistics. Further in the IPO analysis, the market-based estimated DLOMs are compared to option model outputs. Then, the differences between the market-based and option implied DLOMs are analyzed with t-tests to confirm whether the mean of the difference is statistically different from zero. Moreover, other company-specific factors are analyzed if they affect DLOM.

Second, in matched pairs section, the market-based estimations DLOMs are again estimated. Then the put option methods are used to derive DLOMs using ex-ante data. Then they are compared to the DLOMs observed in the European market. The differences between the option implied and market-based DLOMs are again analyzed using matched pairs t-tests. Furthermore, other aspects, such as growth, size and profitability, related to DLOM are analyzed to factor in components that option models may fail to account for in both sections.

3.3 Results and discussion

3.3.1 IPO

First, I will test whether there is a positive DLOM to be observed in the European market. This is done using the differences between the market values implied by pre-IPO and IPO transactions. After the data elimination described in Section 3.1, the dataset consists of 56 observations. Descriptive statistics of the data are shown in Table 1.

Table 1 Descriptive statistics for the IPOs

	DLOM	*T*	*Value at IPO*	*Value at private deal*	*σ*
Mean	0.32	3.33	839.32	448.43	0.51
Std.Dev	0.69	3.07	2003.91	1716.33	0.29
Min	-2.12	0	1.5	0.6	0
Q1	0.09	0.99	36.5	15.15	0.33
Median	0.55	2.32	105.45	37.85	0.51
Q3	0.86	5.07	784.85	181.55	0.66
Max	0.99	13.06	13371.3	12539	1.68
Skewness	-1.46	1.11	4.63	6.29	1.23
SE.Skewness	0.32	0.32	0.32	0.32	0.35
Kurtosis	1.81	0.59	24.87	41.06	3.62
N. Valid	56	56	56	56	46
Pct. Valid	100	100	100	100	82.14

Table 1 shows the descriptive statistics. The estimated DLOM appears to be quite positive judging from the distribution of it. The median DLOM of 55 percent appears to be larger than in the Koeplin et al. (2000) study but is in line with the median DLOM of 46 percent reported by Emory's study series (2012). The first used acquisition-based estimation while the latter estimated the differences between pre-IPO and IPO valuations. It seems that in Europe, the DLOM appears to be generally larger than in the US. To be more concise in my analysis, I run t tests to confirm that the mean is different from zero. Table 2 shows these results.

Table 2 T-tests for the IPO implied DLOM

	n	*Estimate*	*t statistic*	*p*	*Conf. low*	*Conf. high*
H_1: *DLOM≠0*	56	0.32	3.46	0.00104	0.135	0.507
H_1: *DLOM>0*	56	0.32	3.46	<0.001	0.166	NA

Both one- and two-sided t tests are statistically significant at conventional levels of alpha. This result suggests that there is discount for lack of marketability in Europe.

Interestingly, previous studies from Europe find smaller DLOMs, even smaller than in the US (Klein & Scheibel 2012).

Leaning on to the assumption that DLOM exist, I study whether option models are of use when determining DLOMs and if they can explain the discounts. I use the lookback put option, average-strike put option and the forward-starting put option to model DLOMs using equations (5), (7) and (9), respectively. The data and methods introduced in 3.1 and 3.2 are used in determining the fair values of the options.

As option fair values are always positive when maturity is positive, which is the case with DLOM, I also eliminate negative values of the market-based DLOM (see Paglia & Harjoto 2010) and observations where there were not enough return data after the IPO to estimate the volatility input for the option models. After the elimination, there were a total of 36 observations left. I compare the fair values of the options, i.e. the estimated DLOM, to market-based DLOMs from the differences between pre-IPO and IPO stock prices.

The estimated error term in the for the option fair values is the difference between market-based DLOM and corresponding option value. I use matched pairs t-tests for the error terms of each option to validate if they differ from zero. The t-test has a null hypothesis that the mean of the difference is zero. Conversely, same results would be obtained if we tested whether the market-based DLOMs and option fair values have statistically the same mean. The results of the paired t-tests are reported in Table 3. The reference group is the market-based estimated DLOM, and the estimates are positive when the option model overstates the market-based DLOM, and vice versa. The confidence intervals are estimated with an alpha of 5%.

Table 3 Matched pairs t-tests for option model errors

Option models	*n*	*Error mean*	*t statistic*	*p*	*Conf. low*	*Conf. high*
Lookback put	36	0.129	2.340	0.0251	0.0171	0.0242
Average-strike put	36	-0.440	-9.376	<0.001	-0.536	-0.345
Forward-starting put	36	-0.272	-5.377	<0.001	-0.375	-0.169

We see that the average-strike and forward-starting put option models underestimate the DLOM in Europe as they are statistically significantly different from zero. Longstaff's model seems to be the closest one but overstates the DLOM in general having a p-value of 2.42% and thus rejecting the null hypothesis at stricter confidence levels. Finnerty's average-strike lookback severely understates the DLOM. On average, the market-based DLOM lands between the Finnerty's (2014) average-strike put and Longstaff's (1995) lookback put options. This is in line with previous research done in the US as Elmore (2017) states that the average-strike put and the lookback put option tend to yield lower and upper bounds of the DLOM, respectively. However, Ghaidarov (2014) shows that his forward-starting put fits surprisingly well to US data on the DLOMs instead of only serving as a tighter upper bound. European evidence contradicts this, as the forward-starting put tends to understate the European DLOM.

On average, the average-strike put option yield lower bounds of the DLOM as it understates the DLOM, and vice versa for the lookback put option. As opposed to Ghaidarov (2014), the forward-starting put option does not produce a tighter upper bound as it understates the DLOM on average. Instead, it might work as a tighter lower bound than the average-strike put option. However, the option models do not perform that well in separate observations. For example, the number of market-observed DLOMs that land in between the forward-starting put option and the lookback put option is 19, meaning roughly half of the observations. Overstating half of the observations, the forward-starting put option does not perform well as a lower bound either. In comparison, the average-strike put does not seem to perform much better as a lower bound as it produces a looser bound and the number of market-based DLOMs between the bounds only increases by one, totaling 20 out of 36. Figure 5 plots the error terms of the options relative to maturity.

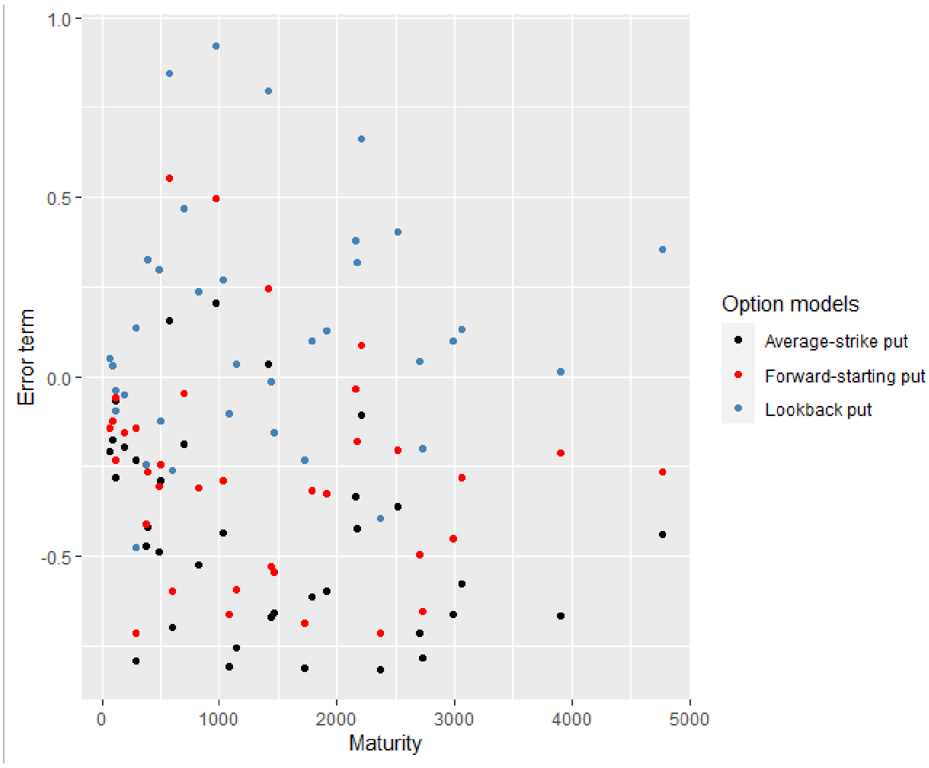

Figure 5 Error terms relative to maturity using DLOMs implied by market values

The observations are quite widely spread leaving room for model improvement. Interestingly, the forward-starting put option tends to yield quite inaccurate, but on average, lower values than the market-based DLOM. This also contradicts with Ghaidarov's (2014) note that the forward-starting put would be a tight upper bound.

As estimating the market-based DLOM with absolute market values can be ambiguous and may fail to take factors affecting valuation into account, using market value multiples links the valuation to the fundamentals of the company. I use market value to sales (MV/Sales) multiple for this, as revenue is the most frequently observable and does not lead to major data loss due to missing values. The more appropriate numerator when using revenue as the denominator in the multiple would be enterprise value since otherwise, there is a mismatch between the numerator and the denominator. Enterprise value is the value of the company to all its stakeholders, and revenue belongs to both stockholders and creditors. However, I am forced to use market value in the multiple as private companies usually lack in information regarding debt and other liabilities. Furthermore, MV/Sales, or Price/Sales, is commonly used in the industry as a valuation multiple.

The results for the performance of the option models using MV/Sales multiples are reported in Table 4. The data has been again filtered so that negative DLOMs and missing values have been omitted.

Table 4 Matched pairs t-tests for option model errors using IPO revenue multiple implied DLOM

Option models	*n*	*Error mean*	*t statistic*	*p*	*Conf. low*	*Conf. high*
Lookback put	18	-0.038	-0.614	0.547	-0.168	0.092
Average-strike put	18	-0.526	-9.612	<0.001	-0.641	-0.411
Forward-starting put	18	-0.392	-7.335	<0.001	-0.505	-0.280

From Table 4 we see clearly that the lookback put generally performs better than the other two option models. Using the revenue-based multiple yields a p-value of over 50% implying that the lookback put mean is statistically equal to market-based observations of the DLOM. Rather than serving only as an upper bound, Longstaff's (1995) lookback put option does not actually overstate the DLOMs but, in general, is performing relatively well. Figure 6 shows the individual observations of the error terms as of maturity of the option.

Figure 6 Error terms relative to maturity using DLOMs implied by revenue multiples

The data points of Longstaff's (1995) lookback put option are more gathered around zero while the other two options clearly tend to understate the DLOM. As the observations are drastically spread-out, the models might fail to account for some factors of the marketability discount. The forward-starting put option seems to estimate the lower bound rather than a tighter upper bound, as opposed to Ghaidarov (2014).

For a more thorough analysis, I will run linear regressions for the DLOM to establish the determinants of the DLOM. The dependent variable is the market-based DLOM, and independent variables include maturity, risk, size and growth as earlier literature has suggested. As I am interested whether these factors influence DLOM, I only use observations where DLOM is positive, i.e., in the presence of a marketability discount. Therefore, negative DLOMs have been eliminated from the regressions.

Table 5 shows the outputs of the regression models and model diagnostics can be found in Appendix 5. As I eliminated negative DLOMs, the number of observations is slightly smaller and varies between regression models. The number of observations for each model are reported in Table 5. Some models show violations of regression model assumptions, such as heteroscedasticity, and skewness and kurtosis of the residuals. A linear regression relies also on that the independent variables are not highly correlated. The correlation tables can be found at Appendix 4. To summarize multicollinearity issues, no major correlations were found between the independent variables. The violations of the model assumptions should be considered when interpreting the results and drawing conclusions.

In the models in Table 5, the residuals are very close in the Q-Q plot found in Appendix 5. Appendix 5 displays also skewness, kurtosis and Shapiro-Wilk test p-values of the model residuals. Every model suffers from a small degree of heteroscedasticity as the residuals are not spread equally along the fitted values. Also, the residuals tend to be bimodal, light-tailed and negatively skewed, although the deviation from normal distribution remains small (Appendix 5).

Table 5 IPO regressions using all factors

	Model 1		**Model 2**		**Model 3**		**Model 4**	
Predictors	Estimates	p	Estimates	p	Estimates	p	Estimates	p
(Intercept)	0.5603 ***	<0.001	0.4619 ***	<0.001	0.6527 ***	<0.001	0.5217 **	0.007
Maturity	0.0437 **	0.004	0.0457 **	0.001	0.0426 **	0.003	0.0426 *	0.014
Volatility	-0.1819	0.232			-0.2627	0.081	-0.0753	0.762
Market value					-0.0002 *	0.033	-0.0001	0.193
Revenue growth							0.0002	0.741
Observations	37		44		37		26	
R2 / R2 adjusted	0.283 / 0.241		0.220 / 0.201		0.376 / 0.320		0.338 / 0.212	

* *p<0.05* ** *p<0.01* *** *p<0.001*

Table 5 shows that growth and volatility factors are not statistically significant at a 5 percent confidence level in any model. However, it appears that the most significant is time, which is proxied with the option maturity, i.e. time between pre-IPO deal and IPO. Maturity is, in fact, a main driver for option models as well. Also, market value is statistically significant at the third model. As it proxies size, larger companies tend to earn slightly smaller DLOMs. Volatility is barely non-significant with the lowest p-value of 8.1 percent. Growth, proxied by one-year revenue growth before the deal, fails to explain DLOM as it is statistically significant at any conventional level.

Revenue size could have been used as a proxy for size but it was insignificant in every model and not reported. Market value performed better as a predictor for size as it is barely insignificant at a 10 percent confidence level with a p-value of 0.115 in the model where the predictors are maturity, volatility and market value. Moreover, this model has the second highest R^2 and the highest adjusted R^2 out of all models. In conclusion, the larger the maturity and volatility, the larger the DLOM. Additionally, a larger company by market value may earn a slightly smaller DLOM although not statistically significantly. The relationships implied by the results are in line with previous research regarding the DLOM (see Paglia & Harjoto 2010; Longstaff 1995; Ghaidarov 2014).

However, the model specification needs some attention. As the DLOM is defined as inability to sell during a given time, it should be zero if the maturity is zero. In other words, if there is no time between the opportunities to trade, the DLOM should be zero. For example, an investor can trade securities every day at an exchange where there should not be any marketability discount due to this ability. However, private equity does not have the benefit of being able to trade whenever wanted, thus leading to the existence of DLOM. For this reason, I assume the intercept to be zero as the DLOM should be nonexistent in the absence of trading restrictions.

Coercing zero intercept might inflate the R-squared value but for this model, it is justified for the rationale behind DLOM. The inflation of R-squared stems from the fact that coercing the intercept to zero increases total sum of squares, i.e. the total variance of the dependent variable, relatively more than the residual sum of squares, i.e. the unexplained variance. The results for the regressions where the intercept is coerced to zero are reported in Table 6. Again, the models are generally slightly skewed left but are quite close to a normal distribution. The tails in models 5 and 7 are slightly light implying a platykurtic distributions for the residuals, and model 6 residuals have a leptokurtic distribution (Appendix 5).

Table 6 IPO regressions and the intercept assumed zero

	Model 5		**Model 6**		**Model 7**		**Model 8**	
Predictors	*Estimates*	*p*	*Estimates*	*p*	*Estimates*	*p*	*Estimates*	*p*
Maturity	0.0863 ***	<0.001	0.1210 ***	<0.001	0.0869 ***	<0.001	0.0697 ***	<0.001
Volatility	0.4046 **	0.001			0.4061 **	0.001	0.5632 ***	0.001
Market value					-0.0000	0.874	0.0000	1.000
Revenue growth							-0.0001	0.903
Observations	37		44		37		26	
R^2 / R^2 adj	0.803 / 0.791		0.707 / 0.700		0.804 / 0.785		0.852 / 0.824	

* *p<0.05* ** *p<0.01* *** *p<0.001*

With the modified model specification, we see that the main drivers for DLOM are time and risk. For example, a one-year trading restriction yields 8.6 percent higher DLOMs on average. The other factors, size and growth, are not statistically significant although size could cause multicollinearity issues with volatility as it is sometimes used as a proxy for risk. However, the correlation between the two is roughly -0.21, and thus, should not be concerning (Appendix 4).

The simple linear regression assumes linear relationship between the dependent and independent variables. The literature on option models for DLOM, however, assume a non-linear relationship. For example, recall that the DLOM as a function of maturity is concave with every option model as the Theta convexity is negative. For volatility, only Longstaff's (1995) lookback put option is convex while the other models are concave. Nevertheless, none of the option models is linear in relation to maturity or volatility.

Due to this non-linear relationship implied by option studies, I extend my regression analysis to predict using the option models. As the underlying factors for the option models are maturity and volatility, earlier regression results advocate the use of option models to estimate DLOM. However, assuming non-linear relationships allows me to use simple linear regressions using option models as predictors. Again, the intercept is assumed zero because of the nature of the DLOM. Table 7 shows results for this regression analysis. The model residuals are slightly left-skewed and have positive excess kurtosis, meaning the distribution is fat-tailed to some degree (Appendix 5).

Table 7 IPO regressions, option models as predictors

	Model 9		**Model 10**		**Model 11**	
Predictors	*Estimates*	*p*	*Estimates*	*p*	*Estimates*	*p*
Longstaff	0.7661 ***	<0.001				
Ghaidarov			1.4957 ***	<0.001		
Finnerty					3.0296 ***	<0.001
Observations	36		36		36	
R^2 / R^2 adj	0.774 / 0.768		0.699 / 0.691		0.778 / 0.771	

* *p<0.05*   ** *p<0.01*   *** *p<0.001*

The interpretation of the model coefficients in Table 7 is quite different. A statistically significant beta coefficient over (under) 1 would imply that the option model understates (overstates) DLOM on average, as it requires a larger (smaller) beta to explain the DLOM. As per earlier literature and the results of this thesis, the lookback put option overstates the DLOM while the forward-starting put and the average-strike put understate the DLOM. The forward-starting put seems to produce tighter lower bounds than the average-strike put option. Ghaidarov (2014) suggests that the forward-starting put could be used as an upper bound, or even as a proxy for the DLOM. Even though market value was statistically significant in earlier regression models, it lost significance when used as a predictor along with the option models (Appendix 1).

Figure 7 plots the market-based DLOMs against each option estimation and the slope of the regressions where option models are predictors. The average-strike put is eliminated from the figure for clarity. Furthermore, it does not seem to produce very good results as per earlier results of this thesis.

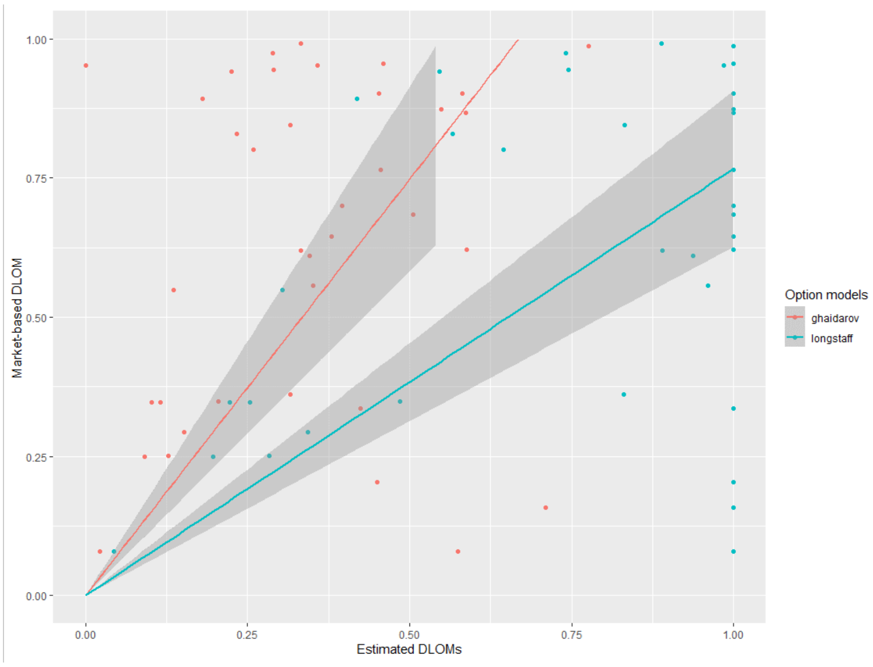

Figure 7 Regressions of IPO DLOMs vs. option models

The grey area represents the confidence interval of the model predictions with a confidence level of 5 percent. The different colors of the plot represent the market-based relative to each option model estimation. In total, two different option models are used and every colored regression lines are estimated from the similar color observations. The plot can be interpreted so that a slope going on the diagonal would be generally a good fit. In that case, the market-based DLOMs would equal the estimated DLOMs on average. Steeper slope understates the DLOM and vice versa. The fit of the slope against the data points highlights how well the models can explain the observed DLOMs from the market. The lookback put option slope is right below the diagonal while the forward-starting put option is above it. This finding is somewhat different from earlier literature. Longstaff's (1995) lookback put option is generally considered to be an upper bound. However, Figure 7 shows that many of the lookback put option estimates are capped at 100%. This suggests that the lookback put option does not perform well with greater maturities and volatilities.

The results of the IPO analysis suggest that the lookback put is an upper bound, although there are DLOMs that exceed the upper bound. Moreover, instead of being an upper bound, the lookback put option could be useful as a proxy for the DLOM itself. In comparison, Ghaidarov (2014) argues, that the forward-starting put would not only be an upper bound but a tighter one while the results show that it consistently understates the DLOM on average. As for validity of the results, the internal validity of the IPO method can be questioned as there might be other factors difficult to control affecting the DLOM. Furthermore, the regression model diagnostics in Appendix 5 show that apart from models 2 and 11, every model has normally distributed residuals at a 5 percent significance level.

3.3.2 Matching private and public companies

As the existence of European DLOM is established but the magnitude and reasons remain somewhat unclear, I incorporate another method in estimating the DLOM. Also, the possible internal validity issues call for another method to be used for robustness of the empirical part of this thesis. This section matches privately held companies to a public company. This section is also with ex-ante data regarding data used for analytical option model DLOM estimation. The criteria for matching are the same 3-digit NAICS Subsector code, and similar revenue size for the year of the private equity transaction. The financial data for the public companies have been lagged by one year because the financial data concerning specific private equity deals are usually data from previous year (see Paglia & Harjoto 2010). Furthermore, the data regarding option models is ex-ante so that an analyst could use this method in practise when determining a DLOM for a private equity deal. This section follows a very similar pattern to the IPO section.

The market-based DLOMs are estimated similarly as in the IPO section with the exception that market value multiples are used instead of absolute market values. The initial dataset consists of 839 observations. After eliminating missing values of the revenue multiple implied DLOMs (total 588), non-positive DLOMs (total 148) and missing or zero public match volatility caused by missing stock return data (total 8), a total of 95 observations remains in the data. Table 8 shows the descriptive statistics of the dataset.

Table 8 Descriptive statistics for private and public matches

	DLOM MV/S	*T*	*Private company revenue*	*Public match revenue*	*σ*
Mean	0.61	0.82	88.61	88.17	0.59
Std.Dev	0.27	1.48	224.21	224.31	0.47
Min	0.03	0	0.1	0.07	0.11
Q1	0.43	0.1	1.5	1.31	0.35
Median	0.63	0.33	9.2	8.73	0.45
Q3	0.85	0.83	27.5	26.73	0.73
Max	1	9.42	973.7	979.29	2.98
Skewness	-0.32	3.58	2.97	2.97	2.79
SE.Skewness	0.25	0.25	0.25	0.25	0.25
Kurtosis	-1.02	14.57	7.64	7.72	9.59
N.Valid	95	95	95	95	95
Pct.Valid	100	100	100	100	100

The descriptive statistics show that the matching has succeeded well on average as the mean and median of the company size between private and public companies are relatively close together. Next, the option models are used to estimate an analytical DLOM and the error terms are again tested for significance using matched pairs t-tests. Table 9 shows the matched pairs t-tests. The confidence intervals are estimated using a 5% confidence level.

Table 9 Matched pairs t-tests for option model errors using revenue multiple implied DLOM

Option models	*n*	*Error mean*	*t statistic*	*p*	*Conf. low*	*Conf. high*
Lookback put	95	-0.275	-6.140	<0.001	-0.364	-0.186
Average-strike put	95	-0.531	-17.611	<0.001	-0.591	-0.471
Forward-starting put	95	-0.462	-13.646	<0.001	-0.530	-0.395

The results of the t-tests imply that every option model understates the market-based DLOM. The DLOMs seem quite high using the MV/Sales multiple as the mean DLOM is 61%. Using MV/EBITDA multiple would take the profitability of the company into account. I further extract data by eliminating negative EBITDAs and negative implied

DLOMS leaving only 9 observations. The results of this analysis must be interpreted with extreme caution because there is so few observations. The descriptive statistics of this data is in Appendix 2. The median (mean) DLOM is 48 (49) percent. The problem is that private firms suffer from being unprofitable, causing small sample size and loss of statistical power. The results for the matched pairs t-tests are in Table 10.

Table 10 Matched pairs t-tests for option model errors using EBITDA multiple implied DLOM

Option models	*n*	*Error mean*	*t statistic*	*p*	*Conf. low*	*Conf. high*
Lookback put	9	-0.142	-0.922	0.378	-0.484	0.201
Average-strike put	9	-0.411	-4.006	0.0025	-0.640	-0.182
Forward-starting put	9	-0.328	-2.688	0.0228	-0.601	-0.056

The t-tests show that the lookback put performs the best. The other option models' error terms are significantly different from zero. However, the mean of the lookback put option's error term is still 14.2% which has considerable economic significance. The small number of observations may be the reason why the t-test implies true zero for the lookback put while the mean is actually quite heavily negative. On average, the lookback put option understates the DLOM by 14.2%. The small sample size may lead to loss in statistical power of the t-test. To conclude, the t-tests suggest that option models understate the market-based DLOMs that are estimated using market multiples rather than absolute market values.

The matched pairs method follows the method Paglia and Harjoto (2010) used. The results are also in line with it. The market-based DLOMs found are larger than in earlier literature as the mean DLOMs are 61 and 50 percent using MV/Sales and MV/EBITDA multiples, respectively. Paglia and Harjoto (2010) report corresponding mean DLOMs of 75% and 50% for revenue- and EBITDA-based multiples, respectively. Corresponding means of this thesis are 61% and 49%.

To conduct a more sound analysis, I run regressions where I predict DLOMs using factors established in earlier literature similarly as in the IPO subsection. First, I run regressions with the factors being the independent variables and assuming linear relationship. The regression results are in Table 11. Model diagnostics are reported in Appendix 6. The residuals of the models in Table 11 appear to be slightly skewed left and thin-tailed.

Table 11 Company match regressions using all factors

	Model 12		Model 13		Model 14		Model 15	
Predictors	*Estimates*	*p*	*Estimates*	*p*	*Estimates*	*p*	*Estimates*	*p*
(Intercept)	0.6962 ***	**<0.001**	0.6484 ***	**<0.001**	0.7258 ***	**<0.001**	0.7092 ***	**<0.001**
Maturity	-0.0423 *	**0.024**	-0.0413 *	**0.027**	-0.0428 *	**0.020**	-0.0457 *	**0.015**
Volatility	-0.0798	0.177			-0.0998	0.094	-0.0533	0.455
Market value					-0.0003	0.075	-0.0003	0.170
Revenue growth							0.0107	0.728
Observations	95		95		95		89	
R^2 / R^2 adjusted	0.070 / 0.050		0.051 / 0.041		0.102 / 0.072		0.097 / 0.054	

* *p<0.05* ** *p<0.01* *** *p<0.001*

The results in Table 11 imply that maturity is explaining the DLOM but the effect is counterintuitive; the longer the holding period, the smaller the DLOM. The same applies for volatility even though it appears it does not have significant explanatory power. Moreover, market value is also close to being significant, implying it might have some effect in the DLOM. For example, a company that is 100 million euros more valuable would earn a DLOM that is 3 percent smaller. The most obvious is revenue growth; it cannot explain the DLOM.

The nature of the DLOM assumes that the discount is zero when maturity is zero. Therefore, I run regression assuming a zero-intercept. The regression coefficients where the intercepts are coerced to zero are reported in Table 12. The distribution of the model residuals are slightly negatively skewed and leptokurtic (Appendix 6).

Table 12 Company match regressions using all factors and intercept assumed zero

	Model 16		Model 17		Model 18		Model 19	
Predictors	*Estimates*	*P*	*Estimates*	*p*	*Estimates*	*p*	*Estimates*	*p*
Maturity	0.0525	0.100	0.1446 ***	**<0.001**	0.0464	0.142	0.0275	0.371
Volatility	0.5740 ***	**<0.001**			0.5604 ***	**<0.001**	0.6799 ***	**<0.001**
Market value					0.0006	0.071	0.0005	0.119
Revenue growth							0.0012	0.982
Observations	95		95		95		89	
R^2 / R^2 adjusted	0.490 / 0.479		0.132 / 0.123		0.508 / 0.491		0.566 / 0.545	

* *p<0.05* ** *p<0.01* *** *p<0.001*

With volatility, maturity loses its significance and market value remains on the verge of significance. Assuming the intercept zero does not drastically improve the model except for R-squared values. As earlier, I extend my analysis to cover the explanatory power of the option models which exhibit a nonlinear relationship between DLOM, and maturity and volatility. Table 13 shows regression output where the option models are predictors for the DLOM implied by revenue multiples. In models 21 and 22, the distribution of residuals is slightly fat-tailed, and vice versa for model 20. All of the models in Table 13 are negatively skewed. The model residuals are also fat-tailed except for model 20, where the residuals exhibit a slightly platykurtic distribution. (Appendix 6).

Table 13 Company match regressions using option models as predictors

	Model 20		Model 21		Model 22	
Predictors	*Estimates*	*p*	*Estimates*	*p*	*Estimates*	*p*
Longstaff	0.9773 ***	**<0.001**				
Ghaidarov			2.0104 ***	**<0.001**		
Finnerty					4.0819 ***	**<0.001**
Observations	95		95		95	
R^2 / R^2 adjusted	0.412 / 0.406		0.383 / 0.376		0.428 / 0.422	

* *p<0.05* ** *p<0.01* *** *p<0.001*

Longstaff's (1995) lookback put option performs very well in the regression, as the beta coefficient is near 1. If the coefficient was 1, the option model would, generally, equal the market-based DLOM. However, the R-squared should be used with caution for reasons discussed above. Figure 8 shows the regression model plot.

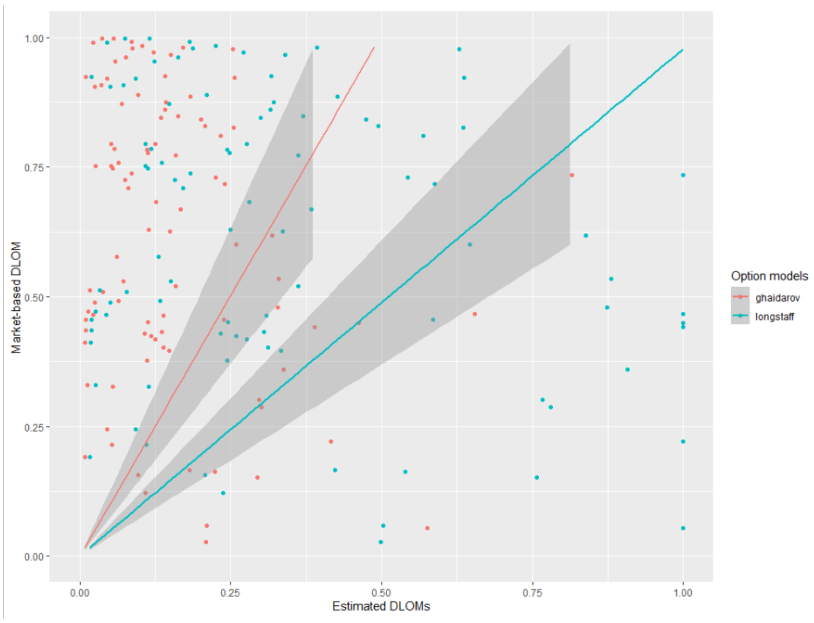

Figure 8 Regressions of matched pairs DLOMs vs option models

The grey area shows the confidence interval of the model predictions with a confidence level of 5 percent. The plotted values of DLOM against the estimated slope clearly indicates that the R-squared is too large as the total sum of squares has grown more than the residual sum of squares due to coercing the intercept to zero. The Longstaff`s (1995) lookback put option slope coefficient is very close to 1. This underlines the earlier results in this thesis that the Longstaff (1995) is the best performer on average. However, many of the estimates of the lookback put option are capped in 100% implying that it does not perform very well with higher maturities and volatilities. Ghaidarov's (2014) forward-starting put seems to understate the majority of the red data points. However, there are clear outliers below the slope, i.e., using the forward-starting put option as a lower bound or an actual estimate could result in major errors. As most of the forward-starting put option data points are above the diagonal, it tends to yield the lower bound for marketability discount in the European market, apart from the few outliers below the diagonal. However, the regression model assumptions are violated in every model in this subsection. For example, the residuals are not normally distributed at any conventional level of alpha (Appendix 6).

In conclusion, the option models can explain the market-based DLOMs relatively well, at least compared to other factors suggested by literature. The matched pairs section completes the results gained from the IPO analysis. Longstaff's (1995) lookback put option seems to work the best among option models. The option models seem to fail at their initially introduced function; being the upper bound for the DLOM. The observed DLOMs are both above and under the option model implied DLOMs. In general, the lookback put option model performs the best while the average-strike put option has the lowest explanatory power among the models.

There is definitely a positive relationship between risk and time, and the DLOM. Market value might also have an impact on the DLOM, as larger companies had a slightly smaller DLOM. Other attributes, such as revenue growth and size did not have significant effect on the DLOM. However, the regressions where the intercepts are zero, market value had the opposite effect. The results suggest that the main drivers for the DLOMs are time and risk, and the relationship could be non-linear as the option models assume. However, linear regressions with time and risk proxies performed very well which is why I refrain from stating that option models, and the non-linear relationship they exhibit, are better suited for estimating the DLOM. Nonetheless, the option models do have explanatory power and with their logical rationale and approach to the DLOM, they can be a great tool for an analyst in determining the DLOM.

4 CONCLUSIONS

This thesis studied the discount for lack of marketability in Europe. The results imply that there is a discount for privately held businesses due to that they cannot be sold instantly. Using IPO method and matching private to public firms the mean (median) DLOM was 32% (55%) and 61% (63%), respectively. The DLOM is driven primarily by risk and time, and secondarily by size. Option models, and the non-linear relationship between the DLOM and the parameters of the models, explain the DLOM, but for practical use, caution is advised. They explain the DLOM quite well on average, but the deviation of single estimations is occasionally quite large.

Internal validity of the methods may be compromised as the DLOM is a complex discount. Simply comparing valuations pre- and post-liquidity may not be enough as the DLOM might have other significant factors, such as lack of control, information asymmetry or a change in company characteristics. These effects are, however, very difficult to control. Hence, drawing conclusions from the results of this thesis is not straightforward and erroneous interpretation of the results is possible. The conclusions of this study are threefold.

First, the European market is not, not surprisingly, very efficient. Longstaff's (1995) lookback put option performs relatively well even though it relies on the perfect market timing of investors. Brooks (2016) argued that the decomposing of the lookback put option leads to two components where the other component involves investor skill especially with higher volatilities. The lookback put outperforms other option models in the European market, suggesting investors have extra skill. Investor skill can stem from the inefficiency of the market, for example, in the form of asymmetrical information. However, the option models have been in analysts' use for almost 50 years which might explain why investors recognize and price the opportunity cost implied by holding a privately held stock. Still, it seems that none of the option models has established superiority over other models as there is not a clear winner among the put option models in robustly explaining the DLOMs due to the large deviation in the occasional observation of DLOM.

Second, the difference between my evidence and other European evidence seems quite irrational. Interestingly, earlier research in Europe shows smaller DLOMs compared to the US. As the consensus in research is that the US public stock market is more liquid than its European counterpart, these results seem odd. This thesis suggests a greater gap in liquidity between private and public equity in Europe than in the US. After all, marketability discount on private equity reflects the inability to execute trades with privately held stock in relation to public equity instead of being an indicator of market liquidity as a whole, at least with the methods generally used in research of the marketability discount. In conclusion, with the more illiquid public stock market and a greater gap between the liquidity of private and public equity suggest that European private equity is far less marketable implying a poorly developed exit market for private equity investors. Furthermore, investors seem to price it in by adding a discount for lack of marketability.

Third, the option models do not seem to work very well in the European market. I draw this conclusion leaning on the imperfections of the European market. The aforementioned information asymmetry, illiquidity and weak efficiency at best, of the market violate the assumptions of the underlying option theory. Furthermore, as the option models are driven by maturity and volatility, they may fail to capture all the elements affecting the DLOM. Maturity covers the trade-off between time and money, but volatility has numerous different factors, such as risk, growth and profitability of the firms, to account for. However, more thorough assessment of the parameters of the option models could yield more accurate results. The option models provide, however, an approximation for the DLOM and the upper and lower bounds indicate a starting point for an analyst determining appropriate DLOMs.

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APPENDICES

Appendix 1 Controlling for market value in option model regressions.

	DLOM		DLOM		DLOM	
Predictors	*Estimates*	*p*	*Estimates*	*p*	*Estimates*	*p*
Longstaff	0.7475 ***	**<0.001**				
Market value	0.0001	0.467	0.0001	0.410	0.0001	0.432
Ghaidarov			1.4799 ***	**<0.001**		
Finnerty					2.9602 ***	**<0.001**
Observations	35		35		34	
R^2 / R^2 adjusted	0.763 / 0.749		0.683 / 0.664		0.763 / 0.748	

* *p<0.05*   ** *p<0.01*   *** *p<0.001*

Appendix 2 MV/EBITDA matched pairs DLOM data descriptive statistics

	DLOM MVEBITDA	*Maturity*	*Private company revenue*	*Public match revenue*	*Volatility*
Mean	0.492	2.177	8268.9	762.938	0.362
Std.Dev	0.332	2.153	22781.137	943.327	0.205
Min	0.045	0.247	7.6	1.698	0.199
Q1	0.253	0.494	146.2	146.209	0.236
Median	0.482	1.286	276.2	206.208	0.27
Q3	0.577	2.831	777.1	759.29	0.389
Max	0.996	5.633	68964.7	2701.579	0.805
Skewness	0.369	0.605	2.067	1.016	1.138
SE.Skewness	0.717	0.717	0.717	0.717	0.717
Kurtosis	-1.33	-1.426	2.611	-0.63	-0.3
N.Valid	9	9	9	9	9
Pct.Valid	100	100	100	100	100

Appendix 3 Correlation table for private company deals that were matched

	DLOM	*Maturity*	*Volatility*	*Market value*	*Revenue growth*	*Longstaff*	*Finnerty*	*Ghaidarov*
DLOM		-0.227*	-0.128	-0.148	0.082	-0.245*	-0.227*	-0.210*
Maturity	-0.227*		-0.037	-0.010	0.047	0.652***	0.702***	0.732***
Volatility	-0.128	-0.037		-0.188	0.042	0.483***	0.451***	0.420***
Market value	-0.148	-0.010	-0.188		-0.373***	-0.152	-0.150	-0.140
Revenue growth	0.082	0.047	0.042	-0.373***		0.087	0.077	0.059
Longstaff	-0.245*	0.652***	0.483***	-0.152	0.087		0.978***	0.937***
Finnerty	-0.227*	0.702***	0.451***	-0.150	0.077	0.978***		0.987***
Ghaidarov	-0.210*	0.732***	0.420***	-0.140	0.059	0.937***	0.987***	

Computed correlation used pearson-method with pairwise-deletion.

Appendix 4 Correlation table for pre-IPO deal data

	DLOM	*Maturity*	*Volatility*	*Market value*	*Revenue growth*	*Longstaff*	*Finnerty*	*Ghaidarov*
DLOM		0.469**	-0.278	-0.201	0.190	0.292	0.313	0.197
Maturity	0.469**		-0.210	0.075	0.199	0.543***	0.579***	0.531***
Volatility	-0.278	-0.210		-0.248	0.004	0.470**	0.541***	0.621***
Market value	-0.201	0.075	-0.248		-0.073	-0.287	-0.330*	-0.268
Revenue growth	0.190	0.199	0.004	-0.073		0.286	0.259	0.248
Longstaff	0.292	0.543***	0.470**	-0.287	0.286		0.931***	0.880***
Finnerty	0.313	0.579***	0.541***	-0.330*	0.259	0.931***		0.986***
Ghaidarov	0.197	0.531***	0.621***	-0.268	0.248	0.880***	0.986***	

Computed correlation used pearson-method with pairwise-deletion.

Appendix 5 IPO regression model diagnostics

Table 5 regression model diagnostics

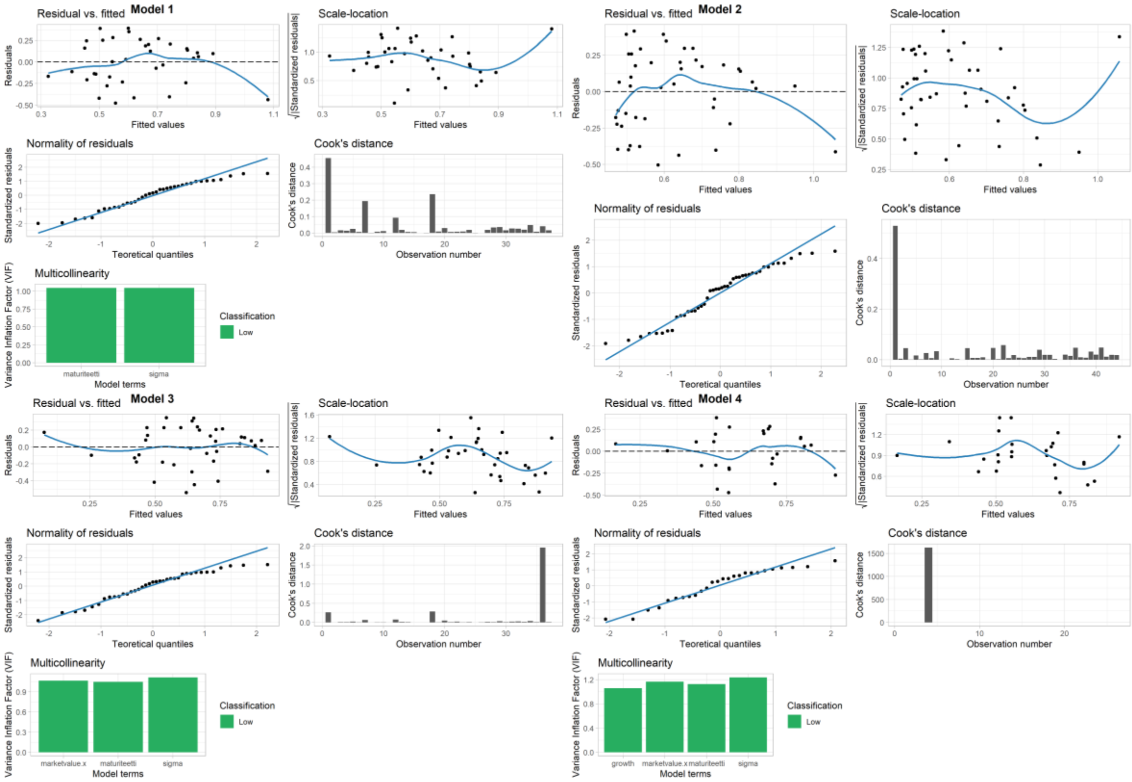


Table 6 regression model diagnostics

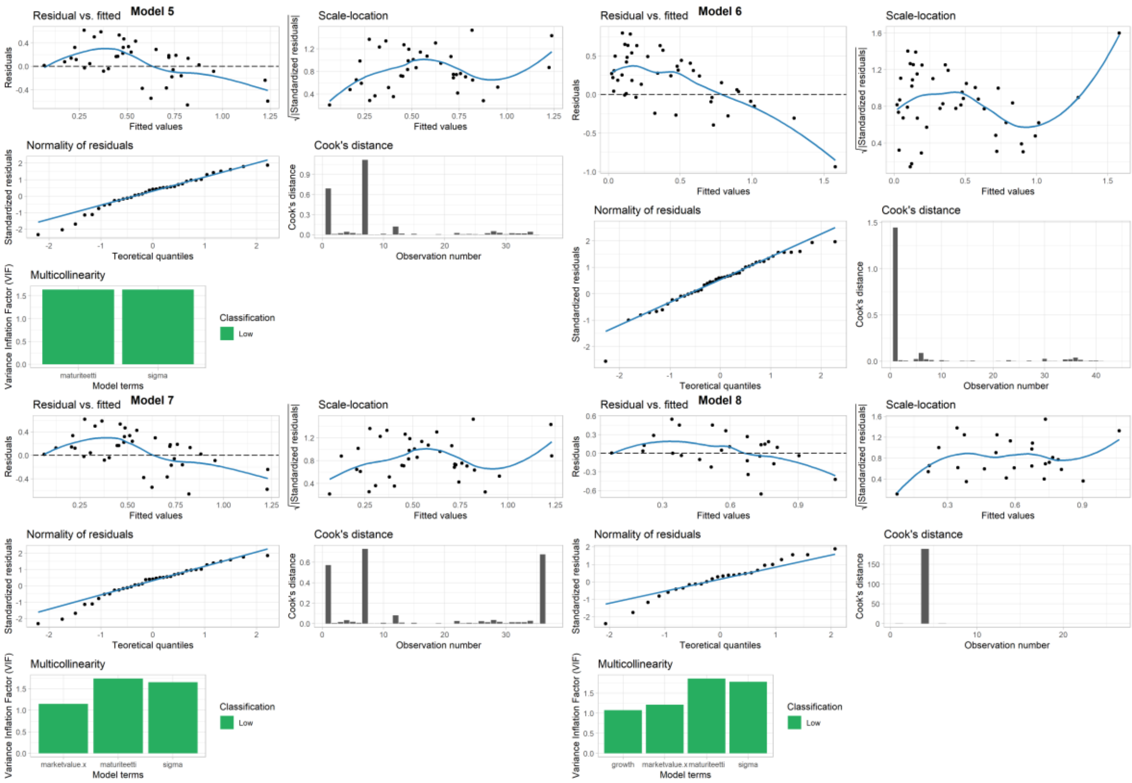
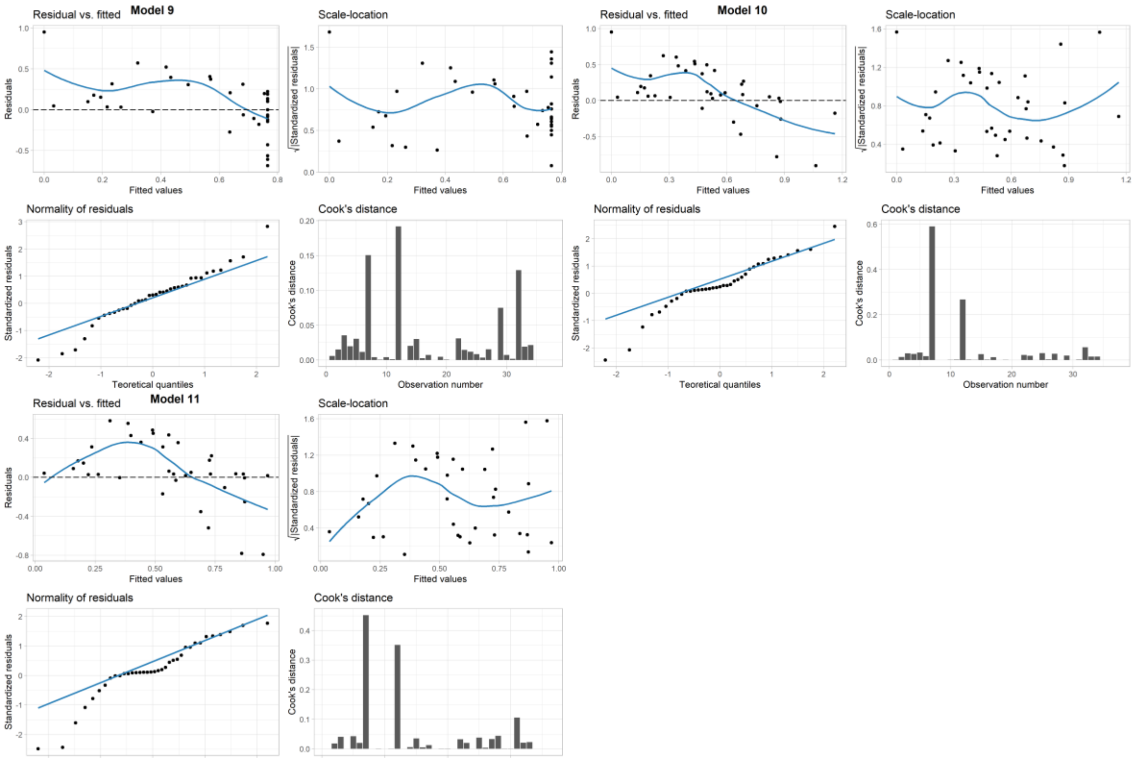


Table 7 regression model diagnostics



IPO study regression model residuals' skewness, kurtosis and normality test

Models	Skewness	Kurtosis	Shapiro-Wilk test for normality (p-values)
Model 1	-0.32	-1.08	0.1008
Model 2	-0.29	-1.16	0.0331
Model 3	-0.55	-0.68	0.0909
Model 4	-0.41	-0.93	0.3373
Model 5	-0.47	-0.31	0.3220
Model 6	-0.67	0.71	0.1879
Model 7	-0.48	-0.31	0.294
Model 8	-0.37	0.03	0.8140
Model 9	-0.13	0.59	0.3498
Model 10	-0.58	0.92	0.0828
Model 11	-0.85	0.79	0.0076

Appendix 6 Matched pairs regression model diagnostics

Table 11 regression model diagnostics

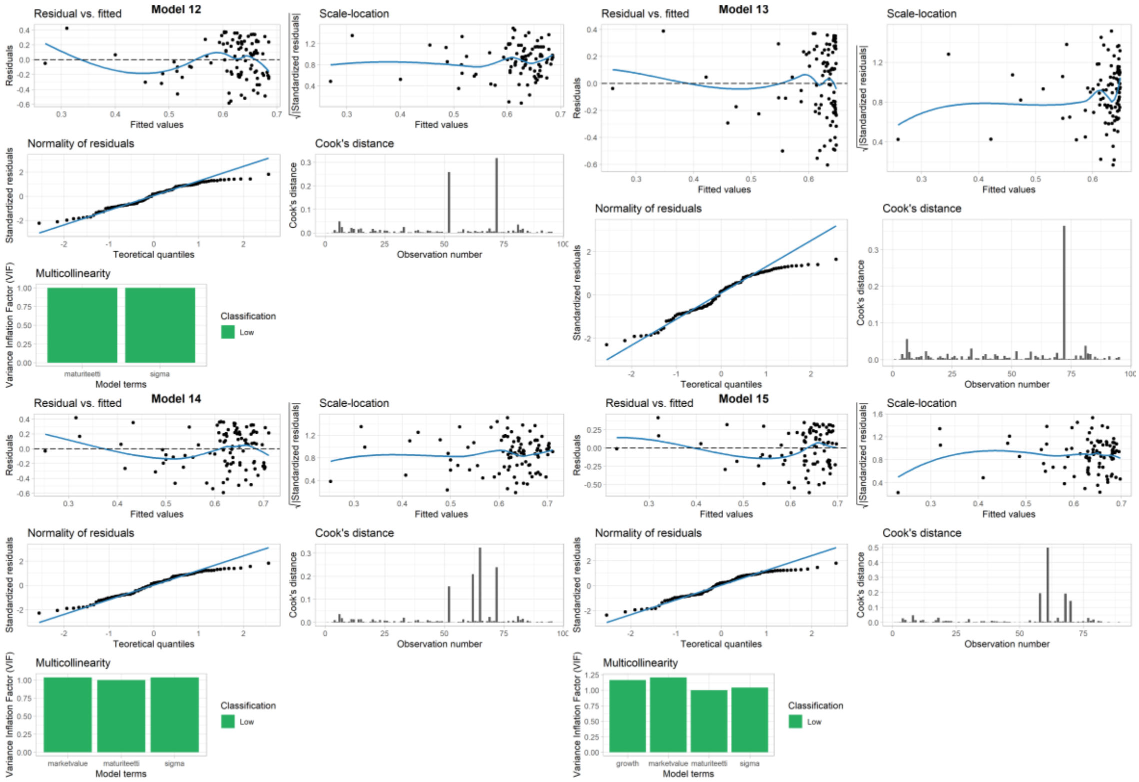


Table 12 regression model diagnostics

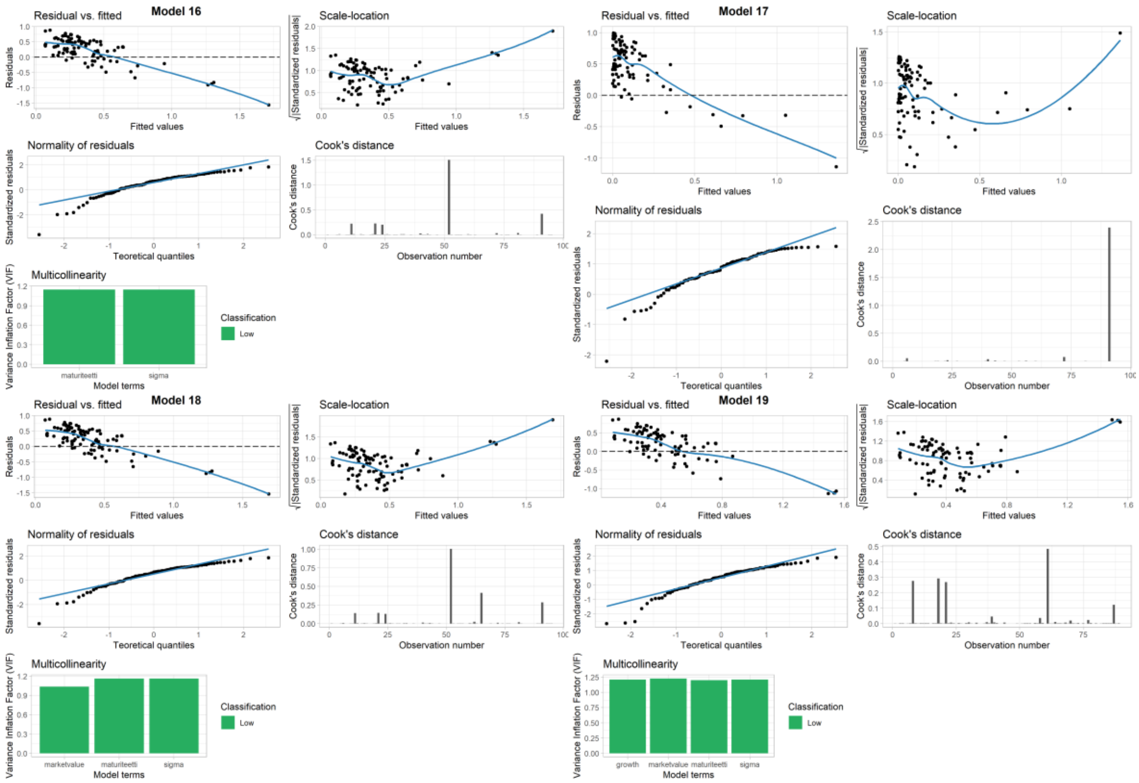
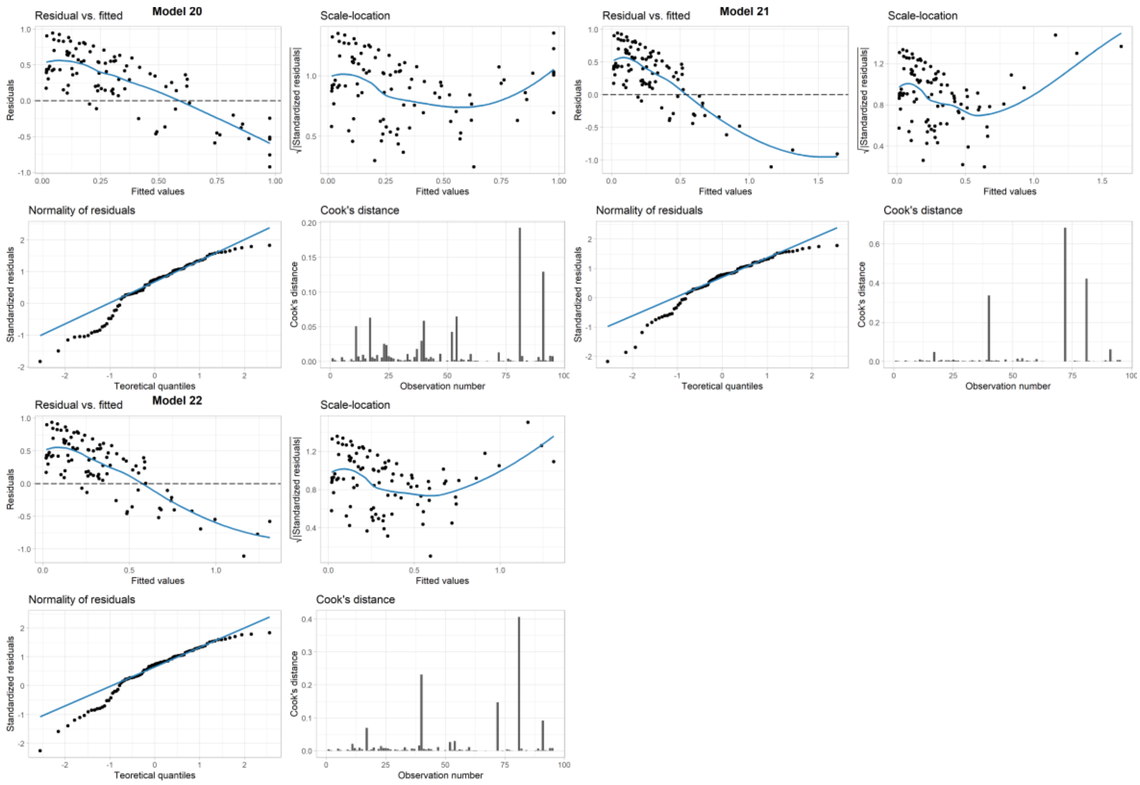


Table 13 regression model diagnostics



Matched pairs regression model residuals' skewness, kurtosis and normality test

Models	Skewness	Kurtosis	Shapiro-Wilk test for normality (p-values)
Model 12	-0.32	-0.93	0.0029
Model 13	-0.36	-0.93	<0.001
Model 14	-0.34	-0.92	0.0033
Model 15	-0.43	-0.85	<0.001
Model 16	-1.44	2.90	<0.001
Model 17	-1.28	2.53	<0.001
Model 18	-1.28	2.37	<0.001
Model 19	-1.15	1.63	<0.001
Model 20	-0.73	-0.23	<0.001
Model 21	-0.99	0.74	<0.001
Model 22	-0.83	0.23	<0.001