Modelling systemic crises in interbank lending markets

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Ever since the global financial crisis of late 2000s and early 2010s, there has been increased interest in the systemic risk and its measurement. Systemic risk is defined as the risk for severe financial crisis that spreads widely through the interconnected financial markets and has negative spillover effects on the broader economy. One trading network that causes this interconnectedness in the banking sector is the interbank lending market where banks can both lend and borrow short term loans which they use to manage their monetary reserves. For example, the distressed interbank lending markets further escalated the emerging systemic crisis during the late 2000s.

Interbank lending markets and the monetary reserves of the individual banks are modelled with the system of coupled diffusion processes. In the model, banks lend money based on the differences in their monetary reserves and their lending preferences. Under specific assumptions, the total monetary reserves of the whole banking sector follow squared Bessel process where the dimension represents the total growth rate. The growth rate and the lending preference define whether the systemic crises exist in the banking system or not. In general, the banking sector benefits from the increased lending activities and higher growth rate as this decreases the probability of banks to go bankrupt.

So called Mean field model ads some additional assumptions to the more general coupled diffusion model and these assumptions allow the model to be numerically simulated. When the interbank lending activity is high, then the reserves of the individual banks develop almost identically as the differences in the reserve levels diminish. However, this lending activity also causes adverse shocks to spread from one bank to all other banks. Therefore, if the lending activity in the markets is strong but the total growth rate is low, then the interbank lending activity actually increases the probability of severe systemic crisis. Further numerical analysis shows that it is better to increase the size of the banking system by adding new banks to the system rather than by increasing the sizes of the existing banks as the latter option increases the tail risks more than the former option. However, the Coupled banking model framework has many limitations that greatly drive these findings. Thus these limitations should be addressed in the future model development.

Keywords: Interbank lending, Monetary reserves, Systemic risk, Coupled banking model, Mean field model, Coupled diffusion process, Squared Bessel process.

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2000-Luvun lopulla ja 2010-luvun alussa taphtuneen globaalin finanssikriisin jälkeen systeemiriski ja sen mallintaminen ovat herättäneet erityistä kiinnostusta. Systeemiriski määritellään riskinä vakavalle finanssikriisille, joka leviää markkinoiden kautta toisiinsa kytkeytyneiden pankkien välityksellä ja aiheuttaa laskusuhdanteen finanssimarkkinoiden lisäksi myös reaalitaloudessa. Pankkien välisten lainamarkkinoiden kautta pankit hallinnoivat reservivivarojaan hyödyntäen lyhytaikaista anto- ja ottolainausta. Samalla kuitenkin nämä lainamarkkinat luovat pankkien välille riippuvuutta. Esimerkiksi juuri 2000-luvun lopun finanssikriisi kiihtyi ongelmiin joutuneiden pankkien välisten lainamarkkinoiden kautta.

Pankkien välisiä lainamarkkinoita ja pankkien reservejä voidaan mallintaa hyödyntäen toisiinsa kytkeytyneitä stokastisia diffuusioprosesseja. Mallissa pankit lainaavat toisiltaan varoja perustuen pankkien reservitasojen välisiin eroihin ja pankkien omiin lainauspreferensseihin. Tiettyjen oletusten vallitessa pankkisysteemin kokonaisreservit seuraavat tällöin neliöityä Bessel prosessia, jonka dimensio puolestaan kuvaa pannkijärjestelmän kasvuvauhtia. Kasvuvauhti ja lainauspreferenssit yhdessä määrittävät sen, voiko pankkijärjestelmässä syntyä systeemikriisejä ollenkaan. Käytännössä pankit hyötyvät aktiivisista lainamarkkinoista ja korkeasta kasvuvauhdista, sillä nämä alentavat pankkien konkurssitodennäköisyyksiä.

Niin sanotussa Mean field -mallissa tehdään yksinkertaistavia lisäoletuksia kytkeytyneeseen diffuusiomalliin, minkä ansiosta Mean field -mallia voidaan simuloida. Mallissa pankkien reservit kehittyvät lähes identtisesti silloin, kun pankkien väliset lainamarkkinat ovat aktiiviset, sillä aktiivinen lainaaminen tasoittaa eroja pankkien reservien välillä. Tällöin kuitenkin myös vakavat sokit leviävät pankkien välillä tehokkaasti. Jos pankkien väliset lainamarkkinat ovat aktiiviset ja pankkijärjestelmän kasvuvauhti on alhainen, aktiiviset lainamarkkinat itseasiassa lisäävät vakavan systeemikriisin todennäköisyyttä. Simuloimalla voidaan myös näyttää, että pankkijärjestelmän riskit pysyvät alhaisempina, mikäli järjestelmä kasvaa uusien pankkien kautta sen sijaan, että olemassa olevat pankit kasvattaisivat kokoaan. Kytkeytyneiden pankkien mallissa on kuitenkin useita rajoittavia oletuksia, jotka osaltaan johtavat esitettyihin tuloksiin. Mallia tulisikin kehittää niin, että näitä rajoitteita saadaan vähennettyä.

Avainsanat: Pankkien väliset lainamarkkinat, Pankkien reservit, Systeemiriski, Kytkeytyneiden pankkien malli, Mean field -malli, Kytkeytynyt diffuusio prosessi, Neliöity Bessel prosessi.

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1 Introduction

Ever since the global financial crisis that started in the latter half of 2000s and continued until early 2010s, there has been increased interest in the systemic risk and its measurement. Systemic risk is defined as the risk for severe financial crisis that spreads widely in the financial sector and has negative spillover effects on the broader (real) economy too. Such financial crises that adversely affect the broader economy are called systemic crises. For example, the aforementioned global financial crisis was systemic crisis that initially started from fairly limited losses in the US housing markets but then spread and caused global recession. As financial sector has immensely important role in the modern economic system, it is clear that governments have strong incentives to make sure that systemic crises don't happen.

Financial sector is very different compared to the other business sectors as financial institutions actively trade with each other using different and complex instruments. Therefore, adverse shocks that first realize for few institutions can easily spread and contaminate other institutions too. One of these special trading networks is the interbank lending market, where banks can both lend and borrow short term loans which they use to manage their monetary reserves. If bank doesn't have enough reserves available, then it can't meet all its obligations (e.g. deposit outflows) and thus it becomes insolvent. During the early stages of the global financial crisis, interbank lending markets became severely distressed which caused problems for banks that heavily relied on the interbank lending markets. Furthermore, this distress then escalated the emerging systemic crisis.

In this thesis, a model by Fouque and Ichiba (2013) [7] is studied where banks' reserves and interbank lending markets are modelled using coupled stochastic diffusion processes. In the model, banks can either borrow or lend money depending on if they have less or more reserves available than their counter-parties. Under specific (symmetry) assumptions, the total monetary reserves of the whole banking sector follow squared Bessel ($BESQ$) process. The dimension of the $BESQ$ process, that is interpreted as the total growth rate of the banking sector, and the lending activity determine whether the systemic crises exist in the system or not.

The model analysis indicates that the system generally benefits from the increased interbank lending activities. However, using a specific version of the Coupled banking model by Sun (2017) [19], which is called Mean field model, it is numerically shown that the interbank lending activities can actually drive the systemic crises when the total growth in the system is low enough. Finally, the quantitative risk analysis of the Mean field model shows that the banking system should be grown by adding new banks to the system rather than by growing the existing banks since the risks develop more favourable in the former than in the latter case.

In section 2, systemic risk and interbank lending markets are described, and the basics of the quantitative risk measurement are presented. In section 3, theoretical background for Brownian motion and diffusion process are given. In addition, as the first hitting time distributions are important in the Coupled banking model analysis, the theory behind the first hitting time distribution of Brownian motion is summarized. As the modelling of the total reserves applies squared Bessel process, the basic properties of this process are described in section 4. Section 5 includes theoretical analysis for the Coupled banking model and the default and systemic crisis probabilities are evaluated. Section 6 includes simulation study for the Mean field model and the quantitative risk measures for the total system are analysed. Lastly, section 7 concludes this thesis.

2 Financial risk management and interbank lending markets

2.1 Risk management and systemic risk

Although the definition of financial risk itself is not always straightforward, it usually reflects the uncertainty around the future outcomes of financial business activities. Specifically, risk is often related to the possibility of facing adverse outcomes more so than to the possibility of facing favourable outcomes. Furthermore, there are many different types of risks in the financial markets. For example, market risk is typically defined as the risk of change in the value of a financial instrument or portfolio. On the other hand, credit risk is the risk of not receiving promised repayments on investments such as loans. In addition to these two, other commonly mentioned risk types are liquidity risk, operational risk, model risk and underwriting (or insurance) risk.

As financial institutions face different risks in their activities, it is clear that they also need to manage these risks. In practice, financial institution manage risks because they try to decrease the probability of facing adverse events in the future, or at least to limit the impacts of these possible events. In addition, government's (or financial regulator's) have clear incentives to ensure that financial markets don't fail even if severe crises occur. After all, the impacts of severe financial crises (such as the global financial crisis of 2007–2008) don't always limit to just financial markets but the broader economy may suffer too.

For banking industry, much of the regulatory work originates from the Basel committee of Banking Supervision which was set up by the central bank governors of major industrialised countries (G-10) in 1974. The Basel committee does not have any formal supranational legal force, but it formulates broad supervisory standards and guidelines called Basel accords which are then implemented by the local financial authorities. For example, Basel accords include rules regarding minimum capital requirements. In insurance industry, similar set of regulatory rules is called Solvency II framework.

Many of the newer regulatory rules are set up to mitigate systemic risk. Freixas et al. (2015) [9, pp. 13-18] define systemic risk (following definition given by European Central Bank) as the risk of threats to financial stability that impair the functioning of a large part of the financial system with significant adverse effects on the broader economy. Since financial institutions are more intertwined to each other than companies in other business areas generally are, adverse shocks can easily start to spread from one institution to other institutions through their shared network of business activities. As financial industry plays crucial role in modern society, severe impairments in financial markets can produce negative spillovers to the real sector too. Such crisis events are called systemic financial crises.[1] To avoid these systemic crises, regulators need to set up new macro-prudential regulatory rules that aim to manage the stability of the financial markets as a whole.

[1]Since broader economy is not modelled in this thesis, systemic financial crises is simply defined as an event where many or most of the financial institutions face severe financial distress.

2.2 Monetary reserves and interbank lending

In this subsection, monetary reserves and interbank lending markets are introduced following Mishkin et al. (2013) [16, pp. 29, 148, 176-231]. First of, bank's balance sheet can be summarised through following equality:

$$\text{total assets} \;=\; \text{total liabilities} \;+\; \text{capital}. \tag{1}$$

In short, banks obtain funds by borrowing and and by issuing liabilities such as deposits. Deposits consist of time deposits that depositors (customers) have to keep in bank's accounts for minimum periods of time, and sight (demand) deposits that depositors can withdraw at any time they want. Naturally, banks can also borrow money from financial markets by issuing bonds and certificates of deposits.

On the left side of the balance sheet, bank's assets consist of loans that bank has granted to its customers, securities such as government bonds and commercial papers, and net trading assets such as derivatives. On the right side of the balance sheet, capital is defined as the bank's net worth (assets minus liabilities) and it is raised by selling new equity (i.e. bank's stocks) or by keeping old earnings in bank's balance sheets. If the value of the liabilities exceeds the value of the assets, then bank's capital is negative which in practice means that the bank is insolvent. Therefore, capital works as a cushion against the drop in the value of bank's assets.

In addition to loans, securities and net trading assets, monetary reserves also belong to the left side of bank's balance sheet. In practice, banks hold reserves as deposits at the central bank or in their own vaults. Generally, reserves earn very low or even zero interest meaning that banks don't want to hold too much reserves or they will loose profits. Bank's reserves are divided into required and excess reserves. The former is needed since it is regulated that banks need to hold certain percentage (called required reserve ratio) of sight deposits as required reserves. On the other hand, excess reserves are used as cushioning against deposit outflows. In practice, banks constantly need to manage their reserve levels to make sure that excess reserves don't grow too large while simultaneously making sure that their reserves meet the required reserve levels (i.e. to avoid reserve shortfalls). Under severe reserve shortfalls, banks can't meet all their obligations.

In order to manage reserves, banks take part in interbank lending markets where they can either borrow or deposit (lend) funds depending on if they have too little or too much reserves at hand. These interbank deposits can either be demand deposits or short term loans with fixed maturities that generally vary between one day to few weeks. The process of bidding and offering interbank loans creates the market rate of interest which in essence is the price that banks who are borrowing money are paying to banks that are lending money. In addition to borrowing and lending from each others, banks can also borrow from the central bank, which means that central bank can also affect the dynamics of interbank lending markets.

Although interbank lending markets aim to ease the reserve management of financial institutions, they also work as an example of a shared network that can transmit adverse shocks from one financial institution to another. For example, the global financial crisis that started in 2006 from losses in US housing markets was amplified by increased uncertainty in the interbank lending markets. Due to the rising loss rates, banks with excess reserves became more cautious and they were not willing to give interbank loans as easily as before. This distress caused interbank lending rates to increase sharply. Finally, these strains in interbank lending markets forced central banks to provide more liquidity to the markets, but the taken actions were not sufficient and the growing issues eventually triggered the global systemic financial crisis.

2.3 Quantitative risk measurement

Following McNeil et al. (2005) [15, pp. 25–53], this subsection introduces two standard quantitative risk measures called Value-at-Risk (VaR) and Expected Shortfall (ES), and describe some standard modelling methods for the loss distributions.

2.3.1 Risk measures

In general, quantitative risk measures can be used for many different purposes such as to estimate the risk limits for trading portfolios or to estimate the prices of bearing the risk of the insurance policies. Furthermore, quantitative risk measures are used to estimate the capital buffers that individual financial institutions need to hold against the future losses. On more aggregated level, new macro-prudential regulatory rules can give similar capital requirements for the financial system as a whole.

In order to understand quantitative risk measures in mathematical terms, loss L is first defined as the difference between the value V of the portfolio at time t and the future value of the portfolio after the given time horizon Δ, i.e. $L_{t,t+\Delta} = -(V_{t+\Delta} - V_t)$. The distribution $L_{t,t+\Delta}$ is the loss distribution and it is typically assumed to be independent of the time point t. If the time horizon Δ is fixed and $L_{t,t+\Delta}$ is shortly written as L, then the cumulative distribution function for the loss distribution is defined as $F_L(l) = \mathbb{P}(L \leq l)$. The first quantitative risk measure called Value-at-Risk (VaR) is then defined using this cumulative distribution function and predefined confidence level $\alpha \in (0, 1)$.

Definition 1 (Value-at-Riks (VaR))**.** VaR at the confidence level $\alpha \in (0, 1)$ is given by the smallest number l such that the probability that the loss L exceeds l is no larger than $(1 - \alpha)$. More formally, this can be written as

$$VaR_\alpha = \inf\{l \in \mathbb{R} : \mathbb{P}(L > l) \leq 1 - \alpha\} = \inf\{l \in \mathbb{R} : F_L(l) \geq \alpha\}. \tag{2}$$

In probabilistic terms, VaR is the quantile of the loss distribution.

Furthermore, so called mean-VaR measure is normally used for the capital adequacy calculations instead of the regular VaR.

Definition 2 (Mean-VaR)**.** Assuming that $E[L] = \mu$, then mean-VaR is defined as

$$VaR_\alpha^{mean} = VaR_\alpha - \mu. \tag{3}$$

One clear weakness of this VaR measures is the fact that it does not give any information about the severity of the losses that occur with a probability that is less than $1-\alpha$. This problem and some other theoretical and practical weaknesses of VaR (e.g. non-additivity, see Artzner et al. (1999) [1]) have prompted development of other risk measures. One of these is Expected Shortfall (ES) which develops VaR in a sense that it can look further into the tail of the loss distribution. More precisely, ES gives the avarage VaR (i.e. conditional expected loss) over all levels $u \geq \alpha$. In general, both VaR and ES are tail risk measures, as they aim to quantify the extreme (i.e. tail) losses. The definitions and differences between VaR and ES are further illustrated in figure 1 where a random loss distribution and its VaR and ES are plotted.

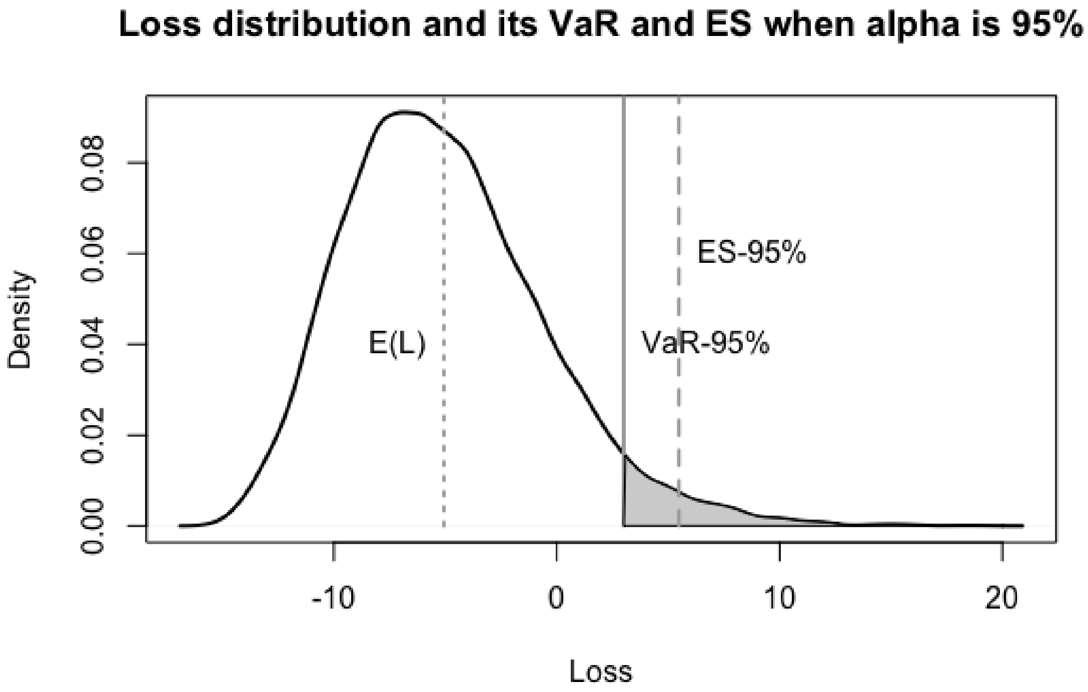

Figure 1: VaR and ES at $\alpha = 0.95$ confidence level for a random loss distribution.

Definition 3 (Expected Shortfall (ES))**.** For loss L with $\mathbb{E}(|L|) < \infty$ and cumulative distribution function F_L, ES at the confidence level $\alpha \in (0,1)$ is defined as

$$ES_\alpha = \frac{1}{1-\alpha} \int_\alpha^1 VaR_u(L)du. \tag{4}$$

Furthermore, if the loss distribution is continuous, then ES can be defined as

$$ES_\alpha = \mathbb{E}(L|L \geq VaR_\alpha). \tag{5}$$

Proof. See McNeil et al. (2005) [15, pp. 45] for the detailed proof of the equation 5. □

2.3.2 Modelling loss distributions

There are different methods that can be used when modelling loss distributions and risk measures. Generally, these methods rely on the assumption that the loss distribution for L_{t+1} can be modelled through risk factors $\mathbf{X}_t = (X_t^1, ..., X_t^n)'$ and loss operator $f_t : \mathbb{R}^n \to \mathbb{R}$. This loss operator essentially maps the risk factor changes into losses, i.e.

$$L_{t+1} = f_t(\Delta\mathbf{X}_{t+1}), \text{ where } \Delta\mathbf{X}_{t+1} = (X_{t+1}^1 - X_t^1, ..., X_{t+1}^n - X_t^n)'. \quad (6)$$

In practice, the choice of risk factors and loss operator is the actual modelling issue. Frequently used risk factors are for example logarithmic prices of financial assets and exchange rates, but many other factors can be used depending on the types of the modelled instruments and markets.

Traditionally, there exists three general classes of methods that can be used when measuring financial risk (especially market risk). The first one of these methods is called variance-covariance method[2]. In this method, it is assumed that the risk factor changes have a multivariate normal distribution with the mean vector $\boldsymbol{\mu}$ and the variance-covariance matrix , meaning that $\Delta\mathbf{X}_{t+1} \sim N_n(\boldsymbol{\mu},)$. The loss operator is often assumed to be linear, i.e. $f_t(\mathbf{x}) = -(c_t + \mathbf{b}_t\mathbf{x})$, which indicates that the loss distribution is

$$L_{t+1} \sim N(-c_t - \mathbf{b}_t'\boldsymbol{\mu}, \mathbf{b}_t'\mathbf{b}_t). \quad (7)$$

Using this distribution, VaR and ES can be easily calculated. In the simplest case, assuming that $n = 1$ (thus variance is σ), $c_t = 0$ and $b_t = -1$, VaR is

$$VaR_\alpha = \mu + \sigma\Phi^{-1}(\alpha), \quad (8)$$

where $\Phi^{-1}(\cdot)$ is the inverse of the standard normal distribution function. It is easy to show (see e.g. McNeil et al. (2005) [15, pp. 45]) that the ES in this case is

$$ES_\alpha = \mu + \sigma\frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}, \quad (9)$$

where $\phi(\cdot)$ is the probability density function for the standard normal distribution.

In practice, one needs to parametrize $\boldsymbol{\mu}$ and before the model can be applied. Assuming that the risk factors follow stationary processes, $\boldsymbol{\mu}$ and can be estimated by using the sample means and standard deviations of the historical observations of the risk factors. The parameters calculated this way are unconditional and thus the resulting loss distribution is also unconditional. More advanced methods assume that the historical risk factor data is a multivariate time series meaning that the conditional $\boldsymbol{\mu}$ and can be estimated by using time series models.[3] In this case, the resulting loss distribution is conditional.

Variance-Covariance methods have many known weaknesses. For example, the linear loss operator is often too inaccurate approximation of the true link between

[2]Depending on the source and exact description, this method (or similar methods) can also be called parametric VaR methods.

[3]In short, the parameters are conditional to the past observations.

the risk factors and the loss distribution. Other major weakness of the method is the normality assumption since the actual financial return series tend to be more leptokurtic and heavier-tailed than the Gaussian distribution. This means that the risk is underestimated if normality is assumed. Naturally, normal distribution can be changed to some other distribution that has heavier tail such as multivariate t distribution. However, the use of more complicated loss operators and distributions can lead to a situation where there is no closed form solutions available for the risk measures. In such case, variance-covariance methods need to borrow tools from the second class of the loss distribution methods that are called Monte Carlo methods.

In Monte Carlo -methods, quantitative models for the risk factor changes are first decided and parametrized and then the realizations for the risk factor changes are simulated using these models. The vector of the simulated risk factor realizations is denoted as $\Delta\tilde{\mathbf{X}}^i_{t+1}$. This simulation process is repeated s times, resulting in $\Delta\tilde{\mathbf{X}}^1_{t+1}, ..., \Delta\tilde{\mathbf{X}}^s_{t+1}$. The risk factor realizations are converted to losses through loss operator, i.e. $\tilde{L}^i_{t+1} = f_t(\Delta\tilde{\mathbf{X}}^i_{t+1})$ for $i = 1, ..., s$. Finally, as $s \to \infty$, the loss distribution $F_L(l)$ can be estimated as

$$F_s(l) := \frac{1}{s}\sum_{i=1}^{s}\mathbb{I}(\tilde{L}^i_{t+1} \leq l) \to F_L(l), \tag{10}$$

where $\mathbb{I}(\cdot)$ is the indicator function. Naturally, VaR and ES can be calculated directly as the empirical estimates from the simulated losses $\tilde{L}^1_{t+1}, ..., \tilde{L}^s_{t+1}$.

In practice, Monte Carlo models allow the use of more complicated distributions and loss operators since there is no need to find analytical solutions for the risk measures. The downside of these models is the fact that when the simulation models become big and complicated, then the computational cost becomes quite considerable. This means that it can take a lot of time to estimate the risk measures. Furthermore, the users of the advanced Monte Carlo methods need to have sufficient technical understanding of the simulation algorithms and information technology in general.

The third class of the loss distribution methods are called historical simulation methods. In general, methods in this class are somewhat simplified versions of Monte Carlo methods as historical simulations don't use any complicated simulation algorithms to estimate the risk factor distributions. Therefore, the user only needs to parametrise the loss operator f_t. In fact, the actual historical observations of the risk factors are used directly (as $\Delta\tilde{\mathbf{X}}^i_{t+1}$) and converted to losses. Therefore, the model assumption is that the future loss distribution can directly be estimated based on the historical observations. However, the historical data series need to be fairly large and accurate especially when the rare tail events are evaluated or otherwise the estimated risk measures become inaccurate. However, additional tools like extreme value theory can be used when the extreme tail losses are estimated, although the use of these extreme value theorems also bridges the gap between the historical simulation methods and the variance-covariance (or parametric) methods.

3 Diffusion process

One if not the most common building block when constructing financial models is diffusion process. On a general level, stochastic process $(X_t : t \geq 0)$ is a diffusion if its local dynamics can be approximated by using the following stochastic differential equation:

$$X_{t+\Delta} - X_t = \mu(t, X_t)\Delta t + \sigma(t, X_t)Z_t. \tag{11}$$

Here, the process X_t is driven by the drift function $\mu(t, X_t)$ and diffusion (or volatility) function $\sigma(t, X_t)$ which is multiplied by the independent and normally distributed Gaussian disturbance term Z_t. To understand this process better, Gaussian disturbance term is described through Brownian motion which is introduced in this section along with some basic results for stochastic diffusion processes.

3.1 Brownian motion

3.1.1 Basic properties of Brownian motion

Before conducting any further modelling with the diffusion process, exact definition for the Gaussian disturbance term Z_t is introduced. This is done by defining stochastic process called Brownian motion. The introduction is done following Björk (2020) [2, pp. 43-54].

Definition 4 (Brownian motion)**.** A stochastic process $(B_t : t \geq 0)$ is called a standard (one-dimensional) Brownian motion (Wiener process) if the following conditions hold:

1. $B_0 = 0$.

2. For every pair of disjoint time intervals $[t_1, t_2]$ and $[t_3, t_4]$ where $t_1 < t_2 \leq t_3 < t_4$, the process B_t has independent increments i.e. $B_{t_2} - B_{t_1}$ and $B_{t_4} - B_{t_3}$ are independent stochastic variables.

3. For $t_1 \geq t_2$ the stochastic variable $B_{t_2} - B_{t_1}$ has Gaussian distribution with mean 0 and variance $t_2 - t_1$, i.e. $B_{t_2} - B_{t_1} \sim N(0, t_2 - t_1)$.

4. B_t has continuous trajectories.

Some basic properties can be shown for Brownian motion by fixing two time points, t and $t + \Delta t$, and defining the difference $\Delta B_t = B_{t+\Delta t} - B_t$. Based on the definition 4, it is easy to see that $E[\Delta B_t] = 0$ and $Var[\Delta B_t] = E[(\Delta B_t)^2] = \Delta t$. Furthermore, since $\Delta B_t \sim N(0, \Delta t)$, then based on properties of normal distribution it also holds that $E[(\Delta B_t)^4] = 3(\Delta t)^2$. Lastly, this means that $Var[(\Delta B_t)^2] = E[(\Delta B_t)^4] - (E[(\Delta B_t)^2])^2 = 3(\Delta t)^2 - (\Delta t)^2 = 2(\Delta t)^2$. Therefore, when $E[(\Delta B_t)^2] = \Delta t \to 0$, then the variance $Var[(\Delta B_t)^2]$ will tend to zero much faster than expected value, meaning that $(\Delta B_t)^2$ actually starts to look deterministic. This gives some heuristic justification to the rule which states that

$$\int_0^t (dB_t)^2 = t \iff (dB_t)^2 = dt. \tag{12}$$

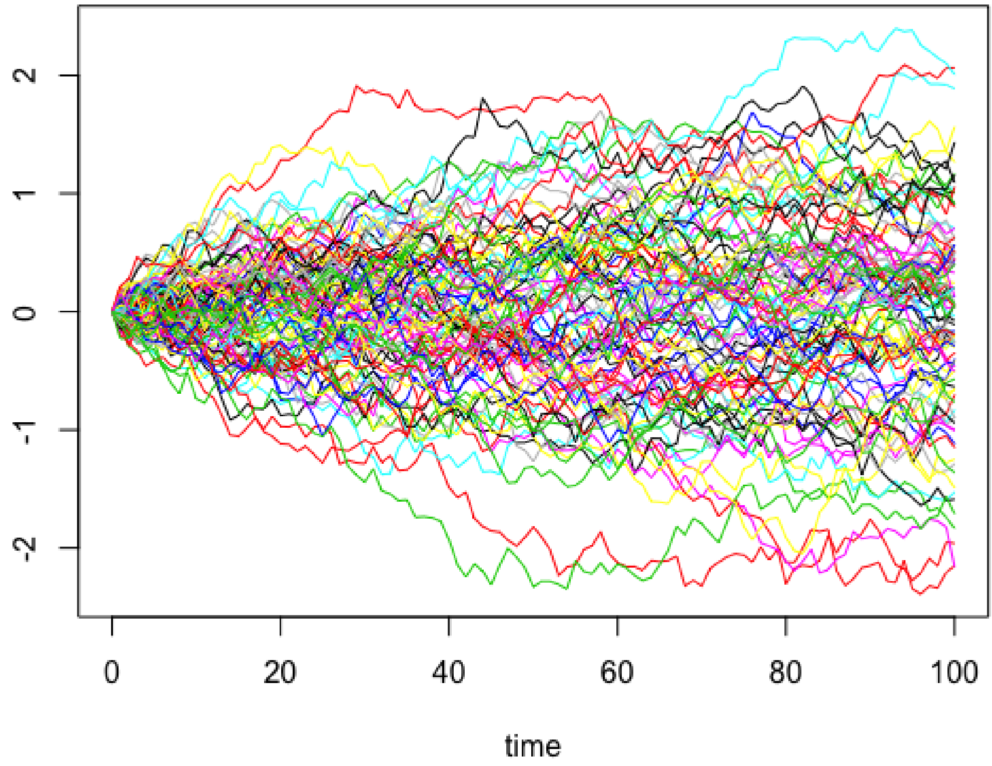

Figure 2: One hundred random trajectories for Brownian motion.

An example of multiple realized trajectories for standard Brownian motion are illustrated in figure 2. As the figure illustrates, the trajectories of Brownian motion are continuous but very kinky. In fact, one could proof that the trajectories are nowhere differentiable.

Theorem 1. *A Brownian motion trajectory is with probability one nowhere differentiable, and it has locally infinite total variation.*

Finally, Brownian motion has some basic transformations that turn out to be standard Brownian motions again.

Proposition 1 (Scaling and inversion laws)**.** *For any $a > 0$, the scaled process defined by*

$$X_t = \frac{1}{\sqrt{a}} B_{at} \text{ for } t \geq 0 \tag{13}$$

and the inverted process defined by

$$Y_0 = 0 \text{ and } Y_t = tB_{1/t} \text{ for } t > 0 \tag{14}$$

are both standard Brownian motion on $[0, \infty)$.

3.1.2 Reflection principle and first hitting time distribution

One often used application in stochastic financial models is to analyse first hitting times and probabilities. In practice, first hitting time is time point τ_a when stochastic process first breaches value a. Following Steele (2001) [18, p.66–69], standard Brownian motion $(B_t : t \geq 0)$ and its first hitting time, defined as $\tau_a = \inf\{t : B_t = a\}$, are analysed. To start with, reflection principle for Brownian motion is defined.

Definition 5 (Reflection principle)**.** If τ_a is a first hitting time for standard Brownian motion $(B_t : t \geq 0)$, then the reflected process $(\tilde{B}_t : t \geq 0)$ can be defined by

$$\tilde{B}_t = \begin{cases} B_t & \text{if t} < \tau_a \\ a - (B_t - a) & \text{if t} \geq \tau_a \end{cases} \tag{15}$$

and $\tilde{B}_t$ is a standard Brownian motion.

One example of a Brownian motion trajectory and its reflection is given in figure 3. It can be observed that the reflected path (after hitting the barrier) is basically a mirror image of the original trajectory.

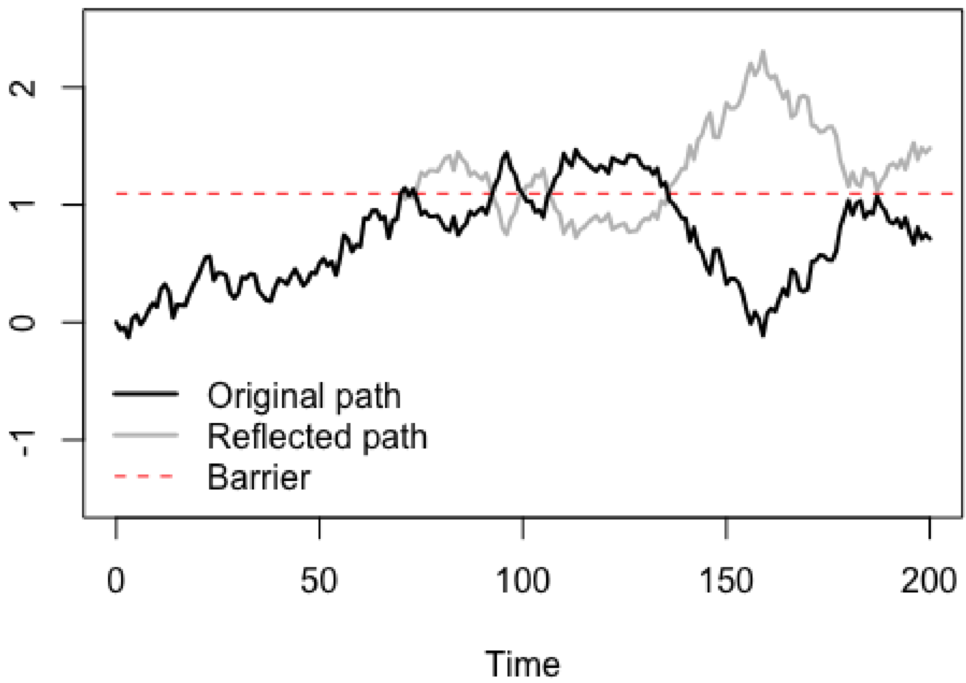

Figure 3: An example of a reflected trajectory for Brownian motion.

Proposition 2. *The process* $(\tilde{B}_t : t \geq 0)$ *is equivalent to process* $(B_t : t \geq 0)$*, which means that all the joint distributions of these processes are equal.*

Based on proposition 2, one can note that if $t \geq \tau_a$ and $B_t > a + x$ where $x \geq 0$, then also $\tilde{B}_t < a + x$ holds. Since both processes are equivalent, this also means that

$$\begin{aligned} \mathbb{P}(\tau_a \leq t, B_t > a + x) &= \mathbb{P}(\tau_a \leq t, \tilde{B}_t < a - x) \\ &= \mathbb{P}(\tau_a \leq t, B_t < a - x). \end{aligned} \tag{16}$$

By introducing maximal process $B_t^* = \max(B_s : 0 \leq s \leq t)$, the last equality in above equation can be illustrated by noting that for all $a \geq 0$ and $x \geq 0$ it holds that

$$\mathbb{P}(B_t^* \geq a, B_t > a + x) = \mathbb{P}(B_t^* \geq a, B_t < a - x) = \mathbb{P}(B_t > a + x), \tag{17}$$

where the last equality holds since $B_t > a + x$ naturally implies that $B_t^* \geq a$.

The equation 17 gives rather nice way to find the distribution for B_t^*. By setting the variable x to 0, the later equality in 17 can be rewritten as $\mathbb{P}(B_t > a) = \mathbb{P}(B_t^* \geq a, B_t < a)$. In addition, it is trivially true that $\mathbb{P}(B_t > a) = \mathbb{P}(B_t^* \geq a, B_t \geq a)$. Therefore, it holds that

$$\begin{aligned} \mathbb{P}(B_t^* \geq a) &= \mathbb{P}(B_t^* \geq a, B_t < a) + \mathbb{P}(B_t^* \geq a, B_t \geq a) \\ &= 2\mathbb{P}(B_t > a) \end{aligned} \tag{18}$$

for all $a \geq 0$. Since by definition the increments of Brownian motion are normally distributed, one can further write that

$$\mathbb{P}(B_t > a) = 1 - \Phi\left(a/\sqrt{t}\right). \tag{19}$$

Therefore, it can be deduced that the cumulative distribution function for maximum process B_t^* is

$$\mathbb{P}(B_t^* \leq a) = 2\Phi\left(a/\sqrt{t}\right) - 1 \tag{20}$$

where equation $\Phi(\bullet)$ is the cumulative density function of standard normal distribution.

Finally, the distributional properties of B_t^* can be translated into first hitting time distribution for barrier $a > 0$. First, by noting that

$$\mathbb{P}(B_t^* < a) = \mathbb{P}(\tau_a > t) = 2\Phi\left(a/\sqrt{t}\right) - 1, \tag{21}$$

then it can be deduced that the cumulative distribution function F_{τ_a} is

$$\mathbb{P}(\tau_a \leq t) = F_{\tau_a}(t) = 2\left(1 - \Phi\left(a/\sqrt{t}\right)\right) = 2\Phi\left(-a/\sqrt{t}\right). \tag{22}$$

Finally, by differentiating this with respect to t, the probability density function f_{τ_a} can be formulated as

$$f_{\tau_a}(t) = \frac{a}{t^{3/2}}\phi\left(\frac{a}{\sqrt{t}}\right), \text{for t} \geq 0, \tag{23}$$

where $\phi(\bullet)$ is the probability density function of standard normal distribution.

3.1.3 Lévy characterization of Brownian motion

First, a heuristic definition for information $\mathcal{F}_t^X$ and definition for $(\mathcal{F}_t^X)$-martingale are given following Björk (2020) [2, p. 45-46].

Definition 6 (The information generated by X)**.** The symbol $\mathcal{F}_t^X$ denotes the information generated by X over the interval $[0, t]$.

- If it is possible to decide whether a given event A has occurred or not based on the trajectory $(X_s : 0 \leq s \leq t)$, then this can be written as $A \in \mathcal{F}_t^X$ which means that A is $\mathcal{F}_t^X$-measurable.
- If the value of a given random variable Z can be completely determined by the observations of the trajectory $(X_s : 0 \leq s \leq t)$ then one can write that $Z \in \mathcal{F}_t^X$.
- If Y is a stochastic process such that $Y_t \in \mathcal{F}_t^X$ for all $t \geq 0$ then one can say that Y is adapted to the filtration $\{\mathcal{F}_t^X\}_{t\geq 0}$.

Definition 7 ($(\mathcal{F}_t^X)$-martingale)**.** A stochastic process X is $(\mathcal{F}_t^X)$-martingale if following conditions hold:

- X is adapted to filtration $\{\mathcal{F}_t^X\}_{t\geq 0}$.
- For all t it holds that $E[|X_t|] < \infty$.
- For all $s \leq t$ it holds that $E[X_t|F_s] = X_s$

In practice, this definition means that the expected future value of X_t is the same as the observed value now. By changing the last condition, so called supermartingale and submartingale can also be defined.

Definition 8 (Supermartingale and submartingale)**.** Given that X is adapted to filtration $\{\mathcal{F}_t^X\}_{t\geq 0}$ and $E[|X_t|] < \infty$, then

- if it holds that $E[X_t|F_s] \leq X_s$ for all $s \leq t$, then this is called supermartingale.
- if it holds that $E[X_t|F_s] \geq X_s$ for all $s \leq t$, then this is called submartingale.

One of the most important notions for martingale theory is stopping time, which intuitively describes a rule that could be used to stop a random process. Following Björk (2020) [2, pp. 530-532], a definition for stopping time and stopped process are given.

Definition 9 (Stopping time)**.** A random variable θ that takes values in $[0, \infty)$ is called a stopping time with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if $\{\theta \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$.

Based on this definition, a stopping time can be characterized by the fact that at any time t one can decide whether θ has occurred or not based upon the information available at t. Furthermore, a bounded stopping time is then defined as $\min(t, \theta)$.

Proposition 3. *Let X be a martingale and let θ be a stopping time with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Then the stopped process X^θ is defined by*

$$X_t^\theta = X_{\min(t,\theta)} \tag{24}$$

and it is a martingale.

A localized version of the martingale property is given following Steele (2001) [18, pp. 103-104].

Definition 10 (Local martingale)**.** If a process X_t is adapted to filtration $\{\mathcal{F}_t^X\}_{t\geq 0}$, then $(X_t : t \geq 0)$ is called a local martingale provided that there is a non-decreasing sequence $\{\theta_k\}$ of stopping times with the property that $\theta_k \to \infty$ with probability one as $k \to \infty$ and such that for each k the process defined by

$$X_t^{\theta_k} = X_{\min(t,\theta_k)} - X_0 \text{ for } t \in [0, \infty) \tag{25}$$

is a martingale with respect to the filtration $\{\mathcal{F}_t^X\}_{t\geq 0}$.

Jeanblanc et al. (2009) [12, pp. 27-30] define predictable quadratic variation for a continuous local martingale M, denoted as $\langle M \rangle = \langle M, M \rangle$, to be equal to the limit in probability of $\sum_i (M_{t_{i+1}^n} - M_{t_i^n})^2$, where $0 = t_0^n < t_1^n ... < t_{p(n)}^n = t$, when $\sup_{0<i\leq p(n)-1}(t_{i+1}^n - t_i^n)$ goes to zero.[4]

Proposition 4 (Quadratic variation of Brownian motion)**.** *For Brownian motion B_t, quadratic variation is defined such that*

$$\langle B \rangle_t = \lim \sum_{i=0}^{p(n)-1} (B_{t_{i+1}^n} - B_{t_i^n})^2 = t \tag{26}$$

So called Lévy characterization of Brownian motion can be given by using the properties described in this subsection. The characterization follows Jeanblanc et al. (2009) [12, p. 30].

Definition 11 (Lévy characterization of Brownian motion)**.** Let B_t be a $\mathbb{R}$-valued continuous process starting from 0 and $\{\mathcal{F}_t\}_{t\geq 0}$ its natural filtration. Then this continuous process B_t is said to be Brownian motion if one of the following equivalent properties is satisfied:

- The processes $(X_t : t \geq 0)$ and $(X_t^2 - t : t \geq 0)$ are $\{\mathcal{F}_t\}_{t\geq 0}$ -local martingales.
- The process $(X_t : t \geq 0)$ is $\{\mathcal{F}_t\}_{t\geq 0}$ -local martingale with $\langle B \rangle_t = t$.

Lastly, it is noted that Jeanblanc et al. (2009) [12, pp. 27-30] define continuous semi-martingale as $\mathbb{R}$-valued process $(X_t : t \geq 0)$ that can be decomposed so that $X_t = M_t + A_t$, where $(M_t : t \geq 0)$ is a continuous local martingale with $M_0 = 0$, and $(A_t : t \geq 0)$ is a continuous adapted process that has locally finite variation.

[4]For quadratic covariation between two continuous local martingales M and N, $\langle M, N \rangle$, it holds that $\langle M, N \rangle = \frac{1}{4}(\langle M + N \rangle - \langle M - N \rangle)$.

3.2 Stochastic differential equations

3.2.1 Itô's formula

Now that Brownian motion and some of its basic properties are described, the diffusion equation (11) can be analysed further. As is noted in Björk (2020) [2, pp. 45], as $\Delta t \to 0$, then the equation (11) becomes following stochastic differential equation:

$$\begin{cases} dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t \\ X_0 = x. \end{cases} \tag{27}$$

Moreover, equation 27 can be expressed equivalently as the following integral equation:

$$X_t = x + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s. \tag{28}$$

Here, the first integral is the standard Rieman integral and the latter stochastic integral is called Itô integral.

Although the exact analysis of stochastic integrals is beyond the scope of this thesis, one of the most important results of stochastic calculus, called Itó's formula (Itó's lemma), is introduced following Björk (2020) [2, pp. 54].

Theorem 2 (Itó's formula)**.** *Assume that process X has a stochastic differential given by*

$$dX_t = \mu_t dt + \sigma_t dB_t \tag{29}$$

where μ and σ are adapted processes, and let f be a $C^{1,2}$-function. Define process Z by $Z_t = f(t, X_t)$, then Z has a stochastic differential given by

$$df(t, X_t) = \left(\frac{\partial f}{\partial t}(t, X_t) + \mu_t \frac{\partial f}{\partial x}(t, X_t) + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2}(t, X_t) \right) dt + \sigma \frac{\partial f}{\partial x}(t, X_t)dB_t \tag{30}$$

Proof. A heuristic proof can be given by noting that Taylor expansion that includes second order terms gives

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{\partial^2 f}{\partial t \partial x}(dt)(dX_t). \tag{31}$$

By definition, $dX_t = \mu_t dt + \sigma_t dB_t$. Therefore

$$(dX_t)^2 = \mu_t^2(dt)^2 + 2\mu_t\sigma_t(dt)(dB_t) + \sigma_t^2(dB_t)^2. \tag{32}$$

Plugging this to the Taylor expansion results to

$$\begin{aligned} df = &\frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}(\mu_t dt + \sigma_t dB_t) \\ &+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\mu_t^2(dt)^2 + 2\mu_t\sigma_t(dt)(dB_t) + \sigma_t^2(dB_t)^2) \\ &+ \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{\partial^2 f}{\partial t \partial x}(dt)(\mu_t dt + \sigma_t dB_t). \end{aligned} \tag{33}$$

When $dt \to 0$, then the term $(dt)^2$ tends to zero much faster. Furthermore, it can be shown that also $(dt)(dB_t)$ tends to zero much faster than dt. These justifications motivate to plug $(dt)^2 = 0$ and $(dt)(dB_t) = 0$ to the equation above. Finally, Itô formula is obtained by plugging the known relation $(dB_t)^2 = dt$ to the equation above. □

3.2.2 Time-homogeneous diffusion

Following Jeanblanc et al. (2009) [12, pp. 270-271], time-homogeneous diffusion is defined as a linear diffusion that is a strong Markov process with continuous paths taking values on interval $I \in [l, r]$ where $l > -\infty$ and $r < \infty$. Then the time homogeneous diffusion (or Itô diffusion) is defined as

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s \tag{34}$$

where $b(\cdot)$ and $\sigma(\cdot)$ are two real valued functions which are Libschitz on the interval I such that $\sigma(x) > 0$ for all x in the interval I. In that case, there exists a unique solution to the diffusion 34 starting at point $x \in (l, r)$ up to the first exit time $\tau_{l,r} = \min(\tau_l, \tau_r)$.

Two useful properties for the time-homogeneous diffusion process are introduced, which are scale function and quadratic variation. Following Jeanblanc et al. (2009) [12, pp. 270-271], scale function is introduced.

Definition 12 (Scale function)**.** Let X be a diffusion on I and $\tau_y = \inf\{t \geq 0 : X_t = y\}$ for $y \in I$. A scale function $s(\cdot)$ is an increasing function from I to $\mathbb{R}$ such that for $x \in [a, b]$

$$\mathbb{P}_x(\tau_a < \tau_b) = \frac{s(x) - s(b)}{s(a) - s(b)}. \tag{35}$$

In addition, if $s(\cdot)$ is scale function, then so is $\alpha s(\cdot) + \beta$ where $\alpha > 0$.

Proposition 5. *The process $(s(X_t), 0 \leq t \leq \tau_{l,r})$ is a local martingale, i.e. $s(X_t)^{\tau_{l,r}}$. The scale function satisfies*

$$\frac{1}{2}\sigma^2(x)s''(x) + b(x)s'(x) = 0. \tag{36}$$

Following Steele (2001) [18, pp. 129-129], quadratic variation for the time-homogenous diffusion process is defined.

Proposition 6 (Quadratic variation of time-homogeneous diffusion process)**.** *Assuming that X_t is time-homogeneous diffusion process defined as in 34, then its quadratic variation is*

$$\langle X_t \rangle = \int_0^t \sigma^2(X_s)ds \tag{37}$$

Finally, assuming that there are two time-homogeneous diffusion processes X_t^1 and X_t^2 with $\sigma_1(\cdot)$ and $\sigma_2(\cdot)$ respectively, then quadratic covariation is

$$\begin{aligned} \langle X_t^1, X_t^2 \rangle &= \frac{1}{4}(\langle X_t^1 + X_t^2 \rangle - \langle X_t^1 - X_t^2 \rangle) \\ &= \frac{1}{4}\left(\int_0^t (\sigma_1(X_s^1) + \sigma_2(X_s^2))^2 ds - \int_0^t (\sigma_1(X_s^1) - \sigma_2(X_s^2))^2 ds\right) \\ &= \frac{1}{4}(4\sigma_1(X_s^1)\sigma_2(X_s^2)) = \int_0^t \sigma_1(X_s^1)\sigma_2(X_s^2)ds. \end{aligned} \tag{38}$$

4 Squared Bessel process

Squared Bessel process is used in many practical applications of financial modelling such as in Cox-Ingersoll-Ross interest rate models and Constant Elasticity Variance models for equity modelling. In this study, squared Bessel process is used to model the total monetary reserves of the banks in financial markets. Therefore, this section introduces definition and some of the main properties of the squared Bessel process. If not stated otherwise, the main sources used in this section are chapter XI of Revuz and Yor (1991) [17, pp. 409-434] and chapter 6 of Jeanblanc et al. (2009) [12, pp. 333-403]. Furthermore, notes by Dufresne (2004) [6, pp. 3–7] are used as a secondary source when the definition and distribution of squared Bessel process in subsections 4.1 and 4.2 are discussed.

4.1 Definition

Assume that $\mathbf{B}_t = (B_t^1, ..., B_t^\delta)$ is δ-dimensional Brownian motion, i.e. $\mathbf{B}_t \sim BM^\delta$. Furthermore, assume that process ρ_t is defined as $\rho_t = ||\mathbf{B}_t|| = \left(\sum_{i=1}^n (B_t^i)^2\right)^{1/2}$ and its square is $\rho_t^2 = \sum_{i=1}^n (B_t^i)^2$. By applying Itô's formula and by noting that $f(x) = x^2$, $f'(x) = 2x$ and $f''(x) = 2$, it follows that

$$d\rho_t^2 = 2\sum_{i=1}^{\delta} B_t^i dB_t^i + \sum_{i=1}^{\delta} dt \tag{39}$$

and therefore

$$\rho_t^2 = \rho_0^2 + 2\sum_{i=1}^{\delta} \int_0^t B_s^i dB_s^i + \delta t. \tag{40}$$

Following this, a new one dimensional process β_t is defined such that

$$\beta_t = \sum_{i=1}^{\delta} \int_0^t \left(\frac{B_s^i}{\rho_s}\right) dB_s^i, \tag{41}$$

where the division by ρ_t causes no problems since for $\delta = 1$ the set $\{t : \rho_t = 0\}$ has Lebesgue measure 0 and for $\delta > 1$, ρ_t is a.s > 0. Importantly, process β_t is also Brownian motion since

$$\langle \beta_t, \beta_t \rangle = \sum_{i=1}^{\delta} \int_0^t \frac{(B_s^i)^2}{\rho_s^2} ds = \int_0^t \frac{\rho_s^2}{\rho_s^2} ds = t. \tag{42}$$

Therefore, the stochastic differential for ρ^2 can be rewritten as

$$\rho_t^2 = \rho_0^2 + 2\int_0^t \rho_s d\beta_s + \delta t, \delta = 1, 2, ... \tag{43}$$

The process ρ_t^2 is further extended for other $\delta \geq 0$ and starting points $y \geq 0$ by considering the following stochastic differential equation:

$$Y_t = y + 2\int_0^t \sqrt{|Y_s|} d\beta_s + \delta t. \tag{44}$$

Using some general theorems (see Revuz and Yor (1991) [17, pp. 409] for further details), it can be shown that this stochastic differential equation has a unique strong solution for any $\delta \geq 0$ and $y \geq 0$. The comparison theorems also show that $Y_t \geq 0$ a.s, meaning that the absolute value in the square root can be discarded.

Definition 13 (The squared Bessel process of dimension δ)**.** For every $y, \delta \geq 0$, the unique strong solution of

$$Y_t = y + 2\int_0^t \sqrt{Y_s} dB_s + \delta t \tag{45}$$

is called the squared Bessel process of dimension δ started at y. The index of the process is $v = \delta/2 - 1$. The shorthand notation for this process is $BESQ^{\delta}(y)$.

Finally, two important properties for the squared Bessel process are introduced, which are additivity property and scaling property.

Theorem 3 (Additivity property of BESQ)**.** *If $Y_t^1 \sim BESQ^{\delta_1}(y_1)$ and $Y_t^2 \sim BESQ^{\delta_2}(y_2)$ are independent, then $Y_t^1 + Y_t^2 \sim BESQ^{\delta_1+\delta_2}(y_1 + y_2)$.*

Proof. For two independent linear BM's B_t^1 and B_t^2, call Y_t^1 and Y_t^2 the corresponding two solutions for (y_1, δ_1) and (y_2, δ_2), and set $Y_t^3 = Y_t^1 + Y_t^2$. Then

$$Y_t^3 = (y_1 + y_2) + 2\int_0^t \left(\sqrt{Y_s^1} dB_s^1 + \sqrt{Y_s^2} dB_s^2\right) + (\delta_1 + \delta_2)t. \tag{46}$$

Now, let B_3 be a third BM independent of B_t^1 and B_t^2, then the process γ_t is defined by

$$\gamma_t = \int_0^t \mathbb{I}(Y_s^3 > 0)\sqrt{\frac{Y_s^1}{Y_s^3}} dB_s^1 + \int_0^t \mathbb{I}(Y_s^3 > 0)\sqrt{\frac{Y_s^2}{Y_s^3}} dB_s^2 + \int_0^t \mathbb{I}(Y_s^3 = 0) dB_s^3, \tag{47}$$

which is linear BM since $\langle \gamma_t, \gamma_t \rangle = t$. Therefore, one can write that

$$Y_t^3 = (y_1 + y_2) + 2\int_0^t \sqrt{Y_s^3} d\gamma_s + (\delta_1 + \delta_2)t \tag{48}$$

which then completes the proof.

□

Theorem 4 (Scaling property of BESQ)**.** *For any $Y_t \sim BESQ^{\delta}(y)$ and $c > 0$, it holds that $cY_{t/c} \sim BESQ^{\delta}(cy)$.*

Proof. Stochastic differential equation for $cY_{t/c}$ can be written as

$$cY_{t/c} = cy + 2\int_0^{t/c} \sqrt{cY_s}\sqrt{c} dB_{s/c} + \delta t. \tag{49}$$

Based on the scaling property of Brownian motion, also $\sqrt{c}B_{t/c}$ is Brownian motion and the result follows from the uniqueness of the solution to this stochastic differential equation. □

4.2 Distribution

To start with, it is assumed that $\rho_t^2 \sim BESQ^\delta(x)$. Then the two sided Laplace transform of the probability density function f_{ρ^2} is[5]

$$\mathcal{L}\{f_t^\delta\}(\lambda) = E[e^{-\lambda \rho_t^2}] = \phi(x, \delta). \tag{50}$$

Now, the additivity property of squared Bessel process implies that $\phi(x_1 + x_2, \delta_1 + \delta_2) = \phi(x_1, \delta_1)\phi(x_2, \delta_2)$ for all $x_1, x_2, \delta_1, \delta_2 \geq 0$. Importantly, this also means that $\phi(x, \delta) = \phi(x, 0)\phi(0, \delta)$. Since $\phi(0, 0) = 1$, it holds that $\phi(x, 0) = \alpha^x$ for some $\alpha > 0$. Similarly, it holds that $\phi(0, \delta) = \beta^\delta$ for some $\beta > 0$. Therefore, one can write that

$$\phi(x, \delta) = \phi(x, 0)\phi(0, \delta) = \alpha^x \beta^\delta. \tag{51}$$

So that α and β can be solved, it is assumed that B_t is one dimensional Brownian motion starting from $\sqrt{x}$, i.e. $B_t \sim BM^1(\sqrt{x})$. Based on this assumption and by setting $\delta = 1$, one can further calculate that

$$\begin{aligned}
\phi(x, 1) &= E[e^{-\lambda B_t^2}] \\
&= \int_{-\infty}^{\infty} e^{-\lambda y^2} \frac{1}{\sqrt{2\pi t}} e^{-\frac{-(\sqrt{x}-y)^2}{2t}} dy \\
&= \frac{e^{-x/2t}}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-(\lambda + \frac{1}{2t})y^2 + \frac{\sqrt{x}y}{t}} dy \\
&= \frac{e^{-x/2t}}{\sqrt{2\pi t}} \sqrt{\frac{\pi}{\lambda + \frac{1}{2t}}} e^{\frac{x}{4t^2(\lambda + \frac{1}{2t})}} \\
&= \frac{1}{(1 + 2\lambda t)^{1/2}} e^{-\lambda x/(1+2\lambda t)}.
\end{aligned} \tag{52}$$

This implies that in general

$$\phi(x, \delta) = \frac{1}{(1 + 2\lambda t)^{\delta/2}} e^{-\lambda x/(1+2\lambda t)}. \tag{53}$$

where

$$\alpha = e^{-\lambda/(1+2\lambda t)} \text{ and } \beta = \frac{1}{(1 + 2\lambda t)^{1/2}}. \tag{54}$$

When $x = 0$ and $\delta > 0$, the exponential disappears which means that

$$\phi(0, \delta) = \frac{1}{(1 + 2\lambda t)^{\delta/2}}. \tag{55}$$

In this case, ρ_2^2 has a gamma distribution[6] with shape parameter $\delta/2$ and scale parameter $2t$ that has probability density function

$$f_t^\delta(0, y) = \frac{y^{\delta/2-1}}{(2t)^{\delta/2}\Gamma(\delta/2)} e^{-y/2t} \mathbb{I}_{\{y>0\}}. \tag{56}$$

[5]For later use, it is noted that the relation between the moment-generating function and the two sided Laplace transform is $M_X(\lambda) = \mathcal{L}\{f_X\}(-\lambda)$

[6]Gamma distribution $\Gamma(k, \theta)$ has moment-generating function $\frac{1}{(1-\lambda\theta)^k}$

When $x, \delta > 0$, the exponential factor in $\phi(x, \delta)$ corresponds to a compound distribution where $Poisson(x/2t)$ is frequency distribution and $\Gamma(1, 2t)$ is severity distribution[7], since

$$e^{-\lambda x/(1+2\lambda t)} = e^{\frac{x}{2t}(M(\lambda)-1)} \text{ where } M(\lambda) = \frac{1}{(1+2\lambda t)}. \tag{57}$$

Furthermore, one can write that

$$\begin{aligned}
\phi(x, \delta) &= M(\lambda)^{\delta/2} e^{\frac{x}{2t}(M(\lambda)-1)} \\
&= e^{-x/2t} M(\lambda)^{\delta/2} \sum_{n=0}^{\infty} \frac{(\frac{x}{2t} M(\lambda))^n}{n!} \\
&= e^{-x/2t} \sum_{n=0}^{\infty} \frac{x^n}{(2t)^n n!} M(\lambda)^{n+\delta/2}.
\end{aligned} \tag{58}$$

Now, it can be shown that the following function

$$f_t^\delta(x, y) = \sum_{n=0}^{\infty} \frac{x^n y^{\delta/2+n-1}}{n!\Gamma(\delta/2+n)(2t)^{\delta/2+2n}} e^{-(x+y)/(2t)} \tag{59}$$

inverts this Laplace transform, i.e.

$$\int_0^\infty e^{-\lambda y} f_t^\delta(x, y) dy = \phi(x, \delta) \tag{60}$$

which means that the function $f_t^\delta(x, y)$ is the probability density function for the squared Bessel process at time t in case when $x, y > 0$. By introducing the modified Bessel function I_v of the first kind of order v:

$$I_v(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{v+2n}}{n!\Gamma(n+v+1)}, v, z \in \mathbb{C}, \tag{61}$$

density $f_t^\delta(x, y)$ can be further written as

$$f_t^\delta(x, y) = \left(\frac{1}{2t}\right) \left(\frac{y}{x}\right)^{v/2} e^{-(x+y)/2t} I_v(\sqrt{xy}/t) \mathbb{I}_{\{y>0\}}, \tag{62}$$

where $v = \delta/2 - 1$ is the index v introduced in the definition of the squared Bessel process.

Lastly, in the case where $\delta = 0$ and $x > 0$, then the Laplace transform $\phi(x, \delta)$ is just

$$\phi(x, 0) = e^{-\lambda x/(1+2\lambda t)} = e^{\frac{x}{2t}(M(\lambda)-1)}. \tag{63}$$

which is the Laplace transform for the aforementioned compound $Poisson(x/2t)$ and Gamma $\Gamma(1, 2t)$ distribution. This also means that the probability for the case

[7] Assume that $X = \sum_{i=1}^{N} Z_i$ where $N \sim Poisson(\theta)$ and Z_i are independently and identically distributed. Then the moment-generating function is $M_X(\lambda) = M_N(\ln(M_z(\lambda))) = e^{\theta(e^{\ln(M_z(\lambda))}-1)} = e^{\theta(M_z(\lambda)-1)}$.

where $\rho_t^2 = 0$ is non-zero since the probability that the $Poisson(x/2t)$ distribution produces zero observation is $e^{-x/2t}$. Therefore, in case when $\delta = 0$, it holds that

$$
\begin{aligned}
f_t^0(x,0) &= e^{-x/2t} \\
f_t^0(x,y) &= \left(\frac{1}{2t}\right)\left(\frac{y}{x}\right)^{-1/2} e^{-(x+y)/2t} I_1(\sqrt{xy}/t)\mathbb{I}_{\{y>0\}}
\end{aligned}
\tag{64}
$$

Finally, the result obtained here can be collected under the following theorem.

Theorem 5 (Probability density function of $BESQ^\delta(x)$)**.** *For $Y_t \sim BESQ^\delta(x)$ and $\delta > 0$, the probability density function is*

$$
f_t^\delta(x,y) = \left(\frac{1}{2t}\right)\left(\frac{y}{x}\right)^{v/2} e^{-(x+y)/2t} I_v(\sqrt{xy}/t), \tag{65}
$$

where $Y_0 = x > 0$, $v = \delta/2 - 1$ and $I_v(\cdot)$ is the Bessel function of index v.

When $x = 0$, then the density is

$$
f_t^\delta(0,y) = \frac{y^{\delta/2-1}}{(2t)^{\delta/2}\Gamma(\delta/2)} e^{-y/2t}, \tag{66}
$$

which means that $Y_t \sim \Gamma(\delta/2, 2t)$.

When $\delta = 0$, then the probability density that $Y_t = 0$ is

$$
f_t^0(x,0) = e^{-x/2t}. \tag{67}
$$

Based on e.g. Delbaen and Shirakawa (2002) [5, pp. 90–91], there is a convenient connection between the squared Bessel process and the non-central chi-squared distribution. The non-central chi-squared distribution, denoted as $V \sim \chi^2(k, \Lambda)$, is defined so that $V = \sum_{i=1}^k Z_i^2$, where Z_i are independently distributed normal random variables, i.e. $Z_i \sim N(\mu_i, 1)$. The parameter k is the degree of freedom parameter and $\Lambda = \sum_i^k \mu_i^2$ is the non-centrality parameter.

Lemma 1. *For $Y_t \sim BESQ^\delta(x)$, where $x \geq 0$ and $\delta \geq 0$, it holds that*

$$
Y_t \overset{d}{=} tV, \tag{68}
$$

where $V \sim \chi^2(\delta, \frac{x}{t})$.

Proof. The Laplace transform for $V \sim \chi^2(\delta, \frac{x}{t})$ is

$$
\mathbb{E}\left[e^{-\lambda V}\right] = \frac{e^{-\frac{\lambda}{1+2\lambda}\frac{x}{t}}}{(1+2\lambda)^{\delta/2}}. \tag{69}
$$

As was shown before, the Laplace transform for the squared Bessel process, i.e. $BESQ^\delta(x)$, is $\phi(x,\delta)$. Therefore, it holds that

$$
\phi(x,\delta) = \frac{e^{\frac{-\lambda x}{1+2\lambda t}}}{(1+2\lambda t)^{\delta/2}} = \frac{e^{\frac{-\lambda t}{1+2\lambda t}\frac{x}{t}}}{(1+2\lambda t)^{\delta/2}} = \mathbb{E}\left[e^{-\lambda t V}\right], \tag{70}
$$

As the Laplace transforms for both random variables are equal, this concludes the proof. □

4.3 Trajectories

In many modelling applications, the behaviour of the process trajectories play an important role, as for example the process behaviour around zero or the long-term convergence of the process can tell a lot about the properties of the underlying phenomenon. For squared Bessel process, the dimension δ is the key variable that defines the behaviour of its trajectories. This fact is illustrated in figure 4 where example paths for squared Bessel process are plotted using different dimensions.

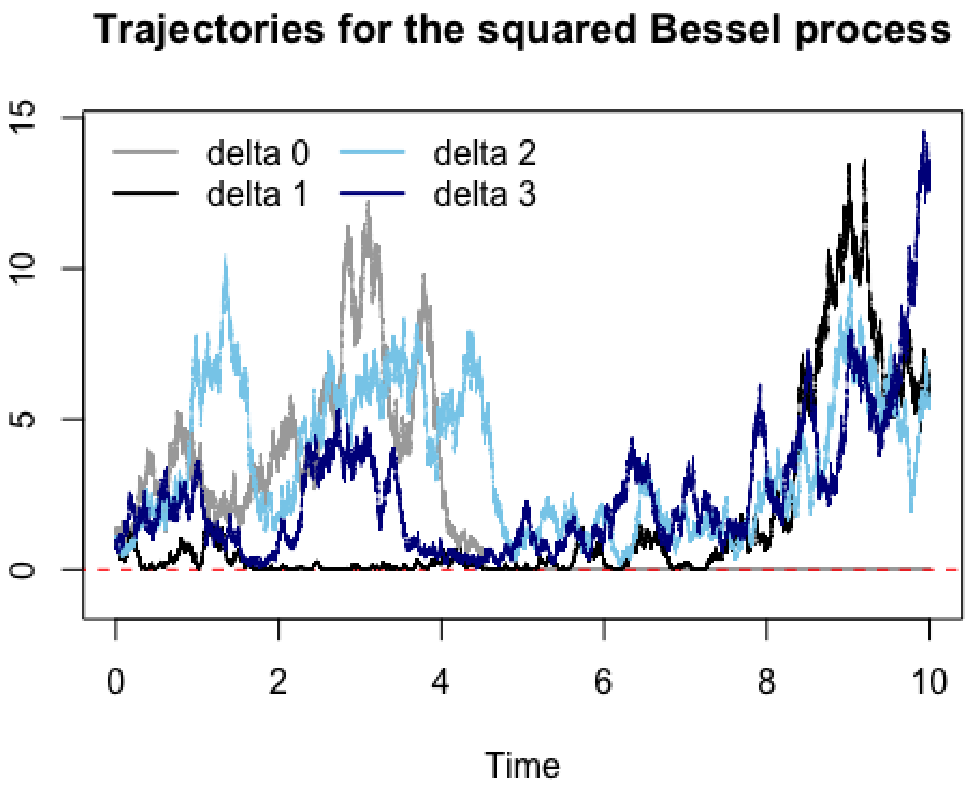

Figure 4: Four random squared Bessel process trajectories for different dimensions. When $\delta = 0$, once process hits zero it thereafter remains at zero level. When $\delta = 1$, the process hits zero multiple times, but instantly reflects away from zero point. When $\delta = 2$ and $\delta = 3$, then the process never hits zero.

The analysis of the the trajectories requires the use of scale functions for squared Bessel process. Assuming that Y_t is a squared Bessel process with dimension δ, then $s(Y_t)^\tau$ is a local martingale where τ is the first hitting time of 0. The scale functions for the different dimensions are introduced in the following proposition.

Proposition 7. *Let Y_t be a squared Bessel process with dimension δ, then the scale* *functions are*

$$\begin{aligned} &s(x) = x^{1-\delta/2} \text{ for } 0 \leq \delta < 2,\\ &s(x) = \ln(x) \text{ for } \delta = 2 \text{ and}\\ &s(x) = -x^{1-\delta/2} \text{ for } \delta > 2. \end{aligned} \tag{71}$$

Proof. Based on proposition 5 of subsection 3.2.2, one needs to show that condition

$$\frac{1}{2}\sigma^2(x)s''(x) + b(x)s'(x) = 0. \tag{72}$$

holds for the scale functions. For squared Bessel process, $\sigma(x) = 2\sqrt{x}$, $\sigma^2(x) = 4x$, $(1/2)\sigma^2(x) = 2x$ and $b(x) = \delta$. Now, when $0 \leq \delta < 2$, then $s'(x) = (1-\delta/2)x^{-\delta/2}$ and $s''(x) = -(\delta/2)(1-\delta/2)x^{-\delta/2-1}$. Thus left side of 72 becomes

$$\begin{aligned} &-2x(\delta/2)(1-\delta/2)x^{-\delta/2-1} + \delta(1-\delta/2)x^{-\delta/2}\\ =&-\delta(1-\delta/2)x^{-\delta/2} + \delta(1-\delta/2)x^{-\delta/2}\\ =&0, \end{aligned} \tag{73}$$

which means that the condition holds. Similarly, when $\delta > 2$, then $s'(x) = -(1-\delta/2)x^{-\delta/2}$, $s''(x) = (\delta/2)(1-\delta/2)x^{-\delta/2-1}$ and condition 72 holds. Lastly, when $\delta = 2$, then $s'(x) = 1/x$ and $s''(x) = -1/(x^2)$. Thus left side of 72 becomes

$$-\frac{4x}{2x^2} + \frac{2}{x} = -\frac{2}{x} + \frac{2}{x} = 0. \tag{74}$$

meaning that condition 72 holds. □

In order to understand the behaviour of the squared Bessel trajectories around zero, local time formula is introduced following Jeanblanc et al. (2009) [12, pp. 223].

Theorem 6 (Local time formula for a continuous semi-martingale)**.** *If X is a continuous semi-martingale, then local time L_t^x at x satisfies*

$$L_t^x(X) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^t \mathbb{I}(x \leq X_t < x+\epsilon)d\langle X_s\rangle, \tag{75}$$

and it holds that

$$L_t^x(X) - L_t^{x-}(X) = 2\int_0^t \mathbb{I}(X_s = x)dX_s \tag{76}$$

Proposition 8. *Let Y_t be a δ-dimensional squared Bessel process. For $\delta = 0$, the point 0 is absorbing (i.e. the process remains zero after hitting that level for the first time) and for $0 < \delta < 2$, the process is reflected instantaneously.*

Proof. In case if $\delta = 0$ the point is reached a.s. Naturally, the point is absorbing as the process remains at zero level (note that $dY_t = \delta dt + 2\sqrt{Y_t}dB_t = 0 + 0 = 0$ when $\delta = 0$ and $Y_t = 0$). The proof for case $0 < \delta < 2$ requires the use of the fact that squared Bessel process is a semi-martingale for $0 < \delta < 2$. From the theory of the local times, it can be shown that $L_t^{0-}(Y) = 0$ and then

$$L_t^0(Y) = 2\delta \int_0^t \mathbb{I}(Y_s = 0)ds. \tag{77}$$

Thereafter, it needs to be shown that since $d\langle Y_t \rangle = 4Y_t dt$, then the occupational time formula leads to

$$\begin{aligned} t \geq \int_0^t \mathbb{I}(Y_s = 0)ds &= \int_0^t \mathbb{I}(Y_s = 0)(4Y_s)^{-1} d\langle Y_s \rangle \\ &= \int_0^\infty (4a)^{-1} L_t^a(Y) da. \end{aligned} \tag{78}$$

The local time at $x = 0$ needs to be identically equal to zero, i.e. $L_t^0 \equiv 0$, since otherwise the integral on the right-hand side is not convergent (see e.g. Revuz and Yor (1991) [17, pp. 412] for further details). □

Finally, the properties of squared Bessel process trajectories are collected under one theorem. Although the exact details of the proof for the theorem are beyond the scope of this thesis, it is noted that the proof applies the convergence theorems for local martingales (as $s(Y_t)^\tau$ is a local martingale) and proposition 8 (see Revuz and Yor (1991) [17, pp. 409-434] for further details).

Theorem 7 (Trajectories of the squared Bessel process)**.** *Let* $Y_t \sim BESQ^\delta(Y_0)$. *Then,*

1. *if* $\delta = 0$, *then* Y_t *hits* 0 *at some time and the point is absorbing.*
2. *if* $0 < \delta < 2$, *then* Y_t *hits zero at arbitrary times but the point is instantly reflecting. Also,* $\limsup_{t \to \infty} Y_t = \infty$.
3. *if* $\delta = 2$, *then* Y_t *is strictly positive at all times and* $\limsup_{t\to\infty} Y_t = \infty$ *and* $\liminf_{t\to\infty} Y_t = 0$.
4. *if* $\delta > 2$, *then* Y_t *is strictly positive at all times and* $Y_t \to \infty$ *as* $t \to \infty$.

4.4 First hitting time distribution

Following Göing-Jaeschke and Yor (2003) [10], a general description of the steps required for deriving the first hitting time distribution of 0 for squared Bessel process is given. As discussed in subsection 4.3, squared Bessel process can only hit zero when $0 \leq \delta < 2$. The probability density function for the first hitting time distribution is introduced in Makarov and Glew (2009) [14, pp. 3]. In order to find the first hitting time distribution, (non-squared) Bessel process needs to be introduced first. Assuming that $Y_t \sim BESQ^\delta(x)$, then the δ-dimensional Bessel process $R_t = \sqrt{Y_t}$ is the solution to the following differential equation

$$dR_t = \left(\frac{\delta - 1}{2} \frac{dt}{R_t} \right) + dB_t, \ R_0 = r = \sqrt{x}. \tag{79}$$

To find the first hitting time distribution, Bessel process R_t is set to start from 0 at time 0 and its dimension is set as $\delta = v > 0$. It can be shown that it holds for $l = \sup\{t > 0, R_t = 1\}$ that

$$l \stackrel{d}{=} \frac{1}{2Z_v} \tag{80}$$

where $\mathbb{P}(Z_v \in dt) = (t^{v-1}e^{-t})/\Gamma(v)dt$ and $t > 0$.

In addition, time reversed Bessel process $\hat{R}_t$ is introduced. This process starts from 1 at time 0 and its dimension is $\delta = 2(1 - v)$. The first time that this process hits zero is $\hat{\tau} = \inf\{t > 0; \hat{R}_t = 0\}$ and it can be shown that the following relation holds between the original and time reversed Bessel processes:

$$\left(\hat{R}_{\hat{\tau}-u}; u \leq \hat{\tau}\right) \stackrel{d}{=} (R_u; u < l)\,. \tag{81}$$

Therefore, it also holds that

$$\hat{\tau} \stackrel{d}{=} \frac{1}{2Z_v}. \tag{82}$$

Now, based on the scaling property of the squared Bessel process, one may write that

$$\hat{\tau} \stackrel{d}{=} \frac{x}{2Z_v}. \tag{83}$$

Finally, it can be concluded that since Z_v follows Gamma distribution with parameter $v = 1 - \delta/2$, it holds that

$$\mathbb{P}(\hat{\tau} \in dt) \stackrel{d}{=} \frac{1}{t\Gamma(v)} \left(\frac{x}{2t}\right)^v e^{-x/2t} dt \tag{84}$$

which is the probability density function for the first hitting time distribution of the squared Bessel process.

5 Modelling monetary reserves using coupled diffusion processes

In this section, a model by Fouque and Ichiba (2013) [7] is described and analysed. In the model, the monetary reserves of individual banks are modelled through coupled diffusions processes. The coupling represents interbank lending activities where banks borrow and lend money from each other.

5.1 Diffusion processes for individual banks

Assume that bank i has monetary reserve X_t^i and all the banks in the financial markets are represented by a vector of monetary reserves $\mathbf{X}_t := \big(X_t^1, ..., X_t^N, 0 \leq t < \infty\big)$. Furthermore, the dynamics of the monetary reserves for individual bank i are represented by the following diffusion:

$$\begin{aligned} X_t^i = X_0^i + \int_0^t \left[\delta_i + \sum_{j=1}^{N} (X_u^j - X_u^i) p_{i,j}\left(\mathbf{X}_u\right)\right] du \\ + 2\int_0^t \sqrt{X_u^i} dB_u^i. \end{aligned} \tag{85}$$

The assumptions required for the system are that

1. the vector of starting values for individual monetary reserves is $(X_1(0), ..., X_N(0)) \in [0, \infty)^N$,
2. the vector $(B_t^1, ..., B_t^N, 0 \leq t < \infty)$ is standard N-dimensional Brownian motion,
3. $\delta_i \geq 0$ for $i = 1, ..., N$,
4. the function $p_{i,j} : [0, \infty)^N \to [0, 1]$ is bounded α-Hölder continuous on compact sets in $(0, \infty)^N$ for some $\alpha \in (0, 1], 1 \leq i, j \leq N$.

The diffusion process 85 and its assumptions construct simple banking system where interbank lending is allowed. In the system, each bank reserves money with a drift term $\delta_i \geq 0$ (called growth rate) which is taken from bank's profits that come from its business activities such as investment operations and money lending. Second drift term $(X_u^j - X_u^i)p_{i,j}\left(\mathbf{X}_u\right)$ arises from the overnight short term lending and it is driven by the difference in the monetary reserves of bank i and j, i.e. $X_t^j - X_t^i$, which is multiplied by the lending preference $0 \leq p_{i,j}\left(\mathbf{X}_u\right) \leq 1, 1 \leq i, j \leq N$. If bank i has bigger reserves than bank j, i.e. $X_t^i > X_t^j$, then money flows from bank i to j and vice versa. Bank i can lend or borrow money from all other banks in the system, which is reflected by the sum term $\sum_{j=1}^{N}$. Lastly, monetary reserves are affected by the independent shock term B_u^i and the effect of this shock term increases when the size of bank's reserves increases.

The lending preference function $p_{i,j}(\cdot)$ plays an important role in driving the dynamics in the system as it describes how willingly banks lend and borrow money with each other based on the market conditions (i.e. the current reserve levels of all banks in the system). Basically, the model works so that at each time point t bank i actively seeks for lending and borrowing opportunities and asks every other bank in the markets whether they could lend money from it ($X_t^i > X_t^j$) or borrow money to it ($X_t^i < X_t^j$). The lending preference then defines how large part of the difference $X_t^i - X_t^j$ will flow between the banks. If $p_{i,j}(\cdot) = 0$, then there is no monetary flow between the banks j and i. If $p_{i,j}(\cdot) = 1$, then the whole difference $X_t^i - X_t^j$ will flow from the bank with bigger reserves to the bank with smaller reserves. If $0 < p_{i,j}(\cdot) < 1$, then the difference will only flow partially between the banks. Banks can have asymmetric lending preferences, which means that $p_{i,j}(\cdot) \neq p_{j,i}(\cdot)$. In practice, this allows bank to behave differently depending on if it is the one who is actively seeking for lending and borrowing opportunities or if it is the one being asked to take part in transactions.

If bank's reserves hit zero, i.e. $X_t^i = 0$, bank i is then in bankruptcy. However, this default state is (usually) temporary since defaulted bank immediately receives money from other banks or from external bailouts, meaning that it can instantly recover. Other possible way to interpret this instant salvation is to consider that a new but identical bank is immediately created after the old bank has defaulted. Either way, this property means that the total number of banks will remain the same and that bankruptcies don't bring any real consequences to the system. Furthermore, it is assumed that bank's reserves can be interpreted as an approximation of its size, which means that bigger bank's are assumed to have larger reserves too.

Right away, it is clear that this simple model has some shortcomings which should be kept in mind when the model is analysed. For example, since assets and liabilities from bank's balance sheet are not modelled, there is no real obligation to repay any interbank debts. This also means that bank is not harmed even if it keeps very low reserves, although in real world low reserves usually imply that bank has difficulties to meet its obligations and that its reserves may even be below the required reserve levels. Furthermore, bank's growth rate δ_i is not affected by the size of its reserves which means that bigger banks are not growing any faster than smaller banks. Consequently, bigger size is not a direct competitive advantage in the model, although in the real world size usually brings some advantages against smaller competitors. It also overly simple to assume that growth rate δ_i is deterministic and non-negative as bank's operating results are stochastic (i.e. one can not predict the exact result in advance) and sometimes banks need to reduce their reserves due to the realized losses from their business activities.

5.2 Diffusion process for the total reserves

The dynamics of the total monetary reserves in the banking system can be modelled by summing all the reserves of individual banks together, i.e. $Y_t = \sum_{i=1}^{N} X_t^i$. The dynamic equation for total monetary reserve is then

$$\begin{aligned} Y_t = Y_0 + &\int_0^t \left[\sum_{i=1}^{N} \delta_i + \sum_{i=1}^{N} \sum_{j=1}^{N} (X_u^j - X_u^i) p_{i,j} (\mathbf{X}_u) \right] du \\ &+ 2 \int_0^t \sqrt{Y_u} d\tilde{B}_u. \end{aligned} \tag{86}$$

The stochastic integral part of equation 86 is reached by introducing new Brownian motion $\tilde{B}_t$ which is effectively set so that $\int_0^t \sqrt{Y_u} d\tilde{B}_u = \int_0^t \sum_{i=1}^{N} \sqrt{X_u^i} dB_u^i$.[8] Since δ_i is the growth rate for individual bank, then the sum $\sum_{i=1}^{N} \delta_i$ naturally represents the total growth rate of the whole banking system.

By assuming that the lending preferences $p_{i,j}(\cdot)$ are symmetric, i.e. $p_{i,j}(\cdot) = p_{j,i}(\cdot)$, then the equation 86 can be significantly simplified. When $\mathbf{x} \in \mathbb{R}^N$, it holds that

$$\begin{aligned} \sum_{i=1}^{N} \sum_{j=1}^{N} (x_j - x_i) p_{i,j}(\mathbf{x}) &= \sum_{i<j}^{N} (x_j - x_i) p_{i,j}(\mathbf{x}) + \sum_{j<i}^{N} (x_j - x_i) p_{i,j}(\mathbf{x}) \\ &+ \sum_{i,j=1}^{N} (x_i - x_j) p_{i,j}(\mathbf{x}) \\ &= \sum_{i<j}^{N} (x_j - x_i) p_{i,j}(\mathbf{x}) - \sum_{i<j}^{N} (x_j - x_i) p_{i,j}(\mathbf{x}) \\ &= 0. \end{aligned}$$

Therefore, equation 86 reduces to

$$Y_t = Y_0 + \delta_\Sigma t + 2 \int_0^t \sqrt{Y_u} d\tilde{B}_u, \tag{87}$$

where $\delta_\Sigma t$ is the total growth of the whole banking system during time interval $[0, t]$, i.e. $\delta_\Sigma t := \int_0^t \sum_{i=1}^{N} \delta_i$. Importantly, 87 also holds when there is no interbank lending, i.e. $p(\cdot)_{l_i,j} = 0$.

Equation 87 shows that under the symmetric lending preferences, interbank lending activities don't affect the development of the total monetary reserves and that

[8] The exact proof for this equality is beyond the scope of this thesis, but the backbone for this proof is given in theorem 3.4.2 of Karatzas and Shreve (1991) [13, p. 170]. This theorem is also introduced in appendix B of this thesis. Furthermore, one practical justification for this equality (under symmetric lending preferences) is seen in figure 9, where the total reserves are simulated by using the dynamics in 86 directly and by simulating each bank in the system individually based on dynamics in 85. The final (loss) distributions for the total reserves are very similar in both cases which implies that the equality indeed holds.

the reserves follow squared Bessel process of dimension δ_Σ. Therefore, the development of the total monetary reserves can be described by the total growth rate δ_Σ and the trajectories of the squared Bessel process (theorem 7).

- If $\delta_\Sigma = 0$, then the total monetary reserves will almost certainly reach to value zero in a finite time. Since the total growth rate is zero, the banking system will stop existing when the total monetary reserves reach value zero (no external bailouts).

- If $0 < \delta_\Sigma < 2$, then the banking system will grow forever. However, the banking system will almost certainly face a severe financial crises at some finite point in the future where the total monetary reserves reach the zero level. The zero point is instantly reflecting meaning that the reserves will instantly start to grow again (external bailouts).

- If $\delta_\Sigma = 2$, then the total reserves will never reach zero point, i,e, $\mathbb{P}(Y_t > 0) = 1$. In addition, the system will grow forever, but it almost certainly faces severe financial crises where the monetary reserves almost breach the zero level, i.e. $\mathbb{P}(\inf_{0 \leq t < \infty} Y_t = 0) = 1$.

- If $\delta_\Sigma > 2$, then the total reserves will never reach zero and the reserves will grow to infinity.

5.3 Existence of systemic crisis

As the real economy is not modelled within this coupled banking system, systemic crisis is defined as a situation where multiple banks default simultaneously. A subset of risky banks is denoted as $(l_1, ..., l_k) \subset \{1, ..., N\}$ where $k \in \{1, ..., N\}$. Furthermore, it is assumed that the lending preferences $p_{i,j}(\cdot)$ are restricted to some range that is close to zero. More precisely, it is assumed that lending preference $p_{i,j}(\cdot)$ and the growth rates $\delta_{l_i}, ..., \delta_{l_k}$ satisfy

$$\sup_{\mathbf{x} \in [0,\infty)^N} |x_{l_i} - x_j| \cdot p_{l_i,j}(\mathbf{x}) < 2c_0 := \frac{1}{k(N-1)} \left(2 - \sum_{i=1}^{k} \delta_{l_i} \right) \tag{88}$$

for $1 \leq i \leq k, 1 \leq j \leq N$. Under this assumption, it can be shown that the banking system almost surely faces systemic crisis where multiple banks are broke simultaneously.

Proposition 9. *Under the additional assumption 88, banks $(l_1, ..., l_k)$ are simultaneously in default at some time $t \in (0, \infty)$ almost surely, i.e.*

$$\mathbb{P}(X_t^{l_1}, ..., X_t^{l_k} = 0 \text{ for some } t \in (0, \infty)) = 1. \tag{89}$$

Proof. This proof applies the comparison theorem of Ikeda and Watanabe (1977) [11], which is introduced in appendix A. By summing the monetary reserves of individual banks $(l_1, ..., l_k)$ together, the process of total monetary reserves Y_t^k for this subset of banks is

$$dY_t^k = \sum_{i=1}^{k} \left[\delta_{l_i} + \sum_{j=1}^{N} (X_t^j - X_t^{l_i}) p_{l_i,j} (\mathbf{X}_t) \right] dt + 2\sqrt{Y_t^k} d\tilde{B}_t^k, \tag{90}$$

where new Brownian motion $\tilde{B}_t^k$ is once again set so that $\int_0^t \sqrt{Y_u^k} d\tilde{B}_u^k$ $= \int_0^t \sum_{i=1}^k \sqrt{X_u^{l_i}} dB_u^{l_i}$. Based on the assumption 88 , it can be noted that

$$\begin{aligned} \bar{\delta} &:= \sum_{i=1}^{k} \delta_{l_i} + \sup_{\mathbf{x} \in [0,\infty)^N} \left| \sum_{i=1}^{k} \sum_{j=1}^{N} (x_j - x_{l_i}) \cdot p_{l_i,j}(\mathbf{x}) \right| \\ &< \sum_{i=1}^{k} \delta_{l_i} + 2c_0 k(N-1) = 2. \end{aligned} \tag{91}$$

It follows from the comparison theorem that the total monetary reserves Y_t^k for the subset $(l_1, ..., l_k)$ of banks is less than or equal to the squared Bessel process $\tilde{Y}_t^k$ of dimension $\bar{\delta} < 2$ with the same initial value $Y_0^k = \tilde{Y}_0^k$, i.e. $\tilde{Y}_t^k \sim BESQ^{\bar{\delta}}(\tilde{Y}_0^k)$. Since it has been shown that the squared Bessel process with dimension that is less than two will almost surely reach zero at a finite time, and since it was shown that $\bar{\delta} < 2$, this means that also the total reserves Y_t^k reach zero level at some finite time point. During such event, all the banks are in default simultaneously which proofs the proposition 9.

□

In practise, additional condition 88 restricts the banks with larger reserves from lending money to banks with smaller reserves which therefore leads to a situation where these distressed banks can go bankrupt simultaneously. Therefore, restrictions on the interbank lending preferences can create multiple defaults. However, interbank lending can also be encouraged so that the banks with larger reserves are forced to lend enough money for distressed banks which then avoid bankruptcies. In fact, under the following restriction

$$\inf_{\mathbf{x} \in [0,\infty)^N} \sum_{j=1}^{k} \sum_{i=1}^{N} (x_j - x_{l_i}) \cdot p_{l_i,j}(\mathbf{x}) \geq 2c_0 k, \tag{92}$$

the subset $(l_1, ..., l_k)$ of banks will never default simultaneously.

Proposition 10. *Under the additional assumption 92, it holds banks $(l_1, ..., l_k)$ will almost surely avoid multiple defaults, i.e.*

$$\mathbb{P}(X_t^{l_1}, ..., X_t^{l_k} = 0 \textit{ for some } t \in (0, \infty)) = 0. \tag{93}$$

Proof. This proof follows similar steps as the proof for proposition 9, but now the dimension $\bar{\delta}$ for the corresponding squared Bessel process is shown to be limited from the lower end, i.e. $\bar{\delta} \geq 2$. □

5.4 Probability of systemic crisis

The time when the total monetary reserves of banks $(l_1, ..., l_k)$ first hit zero is interpreted as a time where systemic crisis occurs, and this time point τ is defined as

$$\tau = \inf\{t \geq 0 : Y_t^k = 0\}, \text{ where } Y_t^k = \sum_{i=1}^{k} X_t^{l_i}. \tag{94}$$

Although the general analytical solution for the first hitting time distribution is not known for the Coupled banking system, the probabilities can still be evaluated by using the known properties of the squared Bessel process.

By assuming banking system where interbank lending is not allowed, i.e. $p(\cdot)_{l_i,j} = 0$, the total monetary reserves for the subset of banks follow squared Bessel process, i.e. $Y_t^k \sim BESQ^{\delta_{\Sigma_k}}(Y_0^k)$, where $\delta_{\Sigma_k} = \sum_{i=1}^k \delta_{l_i}$ is the total growth rate for this subset of banks. As was shown in subsection 4.4, the first hitting time probability is then

$$\mathbb{P}(\tau_k \leq t) = \int_0^t \frac{1}{s\Gamma(v)} \left(\frac{Y_0^k}{2s}\right)^v \exp\left(-\frac{Y_0^k}{2s}\right) ds =: F\left(Y_0^k, v, t\right), \tag{95}$$

where $v = 1 - \delta_{\Sigma_k}/2$ and $t > 0$. Therefore, the function $F(\cdot)$ gives the probability for the systemic crises event where the subset of banks with no interbank lending default simultaneously before (or at) time point t. However, it should be noted that this only holds when $\delta_{\Sigma_k} < 2$ since otherwise squared Bessel process is always strictly positive.

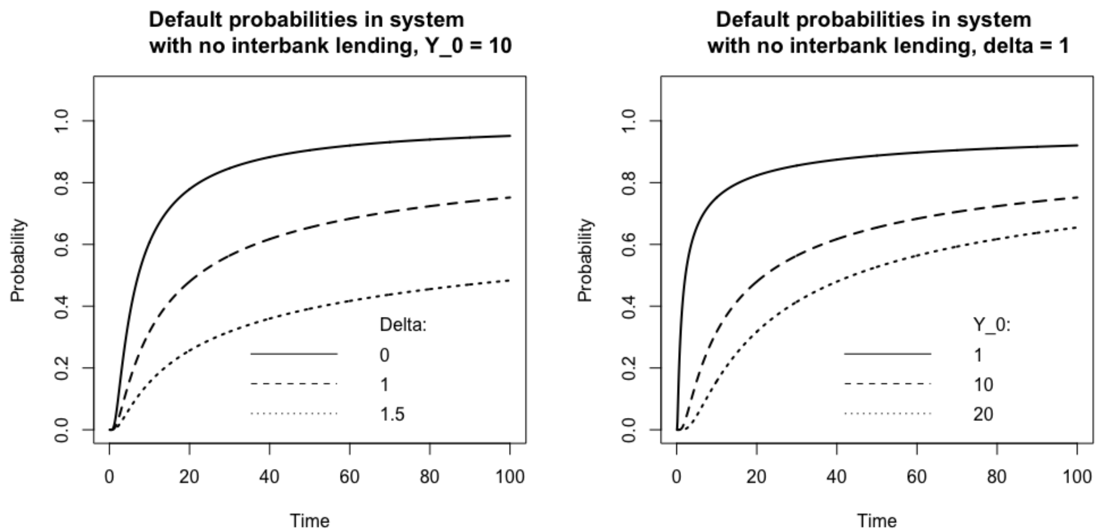

Figure 5: Default probabilities in banking system with no interbank lending, $Y_0^k \in \{1, 10, 20\}$ and $\delta_{\Sigma_k} \in \{0, 1, 1.5\}$.

System with no interbank lending is analysed in figure 5, where it can be observed that the default probabilities increase when the observed time interval increases and it is almost certain that systemic crisis occurs if low enough growth rate and initial reserves are given. Furthermore, it can be observed that the default probabilities decrease when the total growth rate δ_{Σ_k} increases (left). This is very intuitive finding since higher returns from banking operations will naturally decrease the default probabilities. Similarly, it can be observed that the default probabilities decrease when the initial monetary reserves Y_0^k increase (right). Again, this is intuitive finding since larger reserves work as a buffer against the bankruptcies.

Although the exact analysis is much more complicated in general case where interbank lending is allowed, especially when the lending preferences are not symmetric, it is still possible to evaluate the default probabilities by defining upper and lower limits for the interbank lending activities. The lower limit for the total growth rates and interbank lending activities is

$$\underline{\delta} := \delta_{\Sigma_k} + \inf_{\mathbf{x}\in[0,\infty)^N} \sum_{i=1}^{k} \sum_{j=1}^{N} (x_j - x_{l_i}) \cdot p_{l_i,j}(\mathbf{x}) \tag{96}$$

and the upper limit is

$$\bar{\delta} := \delta_{\Sigma_k} + \sup_{\mathbf{x}\in[0,\infty)^n} \left| \sum_{i=1}^{k} \sum_{j=1}^{N} (x_j - x_{l_i}) \cdot p_{l_i,j}(\mathbf{x}) \right| . \tag{97}$$

Using these limits, a upper limit process $\bar{Y}_t^k \sim BESQ^{\bar{\delta}}(Y_0^k)$ and lower limit process $\underline{Y}_t^k \sim BESQ^{\underline{\delta}}(Y_0^k)$ are created. Naturally, these processes also have their own default times which are defined as $\bar{\tau}_k = \inf\{t \geq 0 : \bar{Y}_t^k = 0\}$ and $\underline{\tau}_k = \inf\{t \geq 0 : \underline{Y}_t^k = 0\}$.

By applying the comparison theorem A, it can be shown that the default time probability for the system with interbank lending is limited between the default time probabilities for the upper and lower limit processes, i.e.

$$\mathbb{P}(\bar{\tau}_k \leq t) \leq \mathbb{P}(\tau_k \leq t) \leq \mathbb{P}(\underline{\tau}_k \leq t). \tag{98}$$

This result together with the findings from figure 5 imply that by increasing the interbank lending activities in the banking system, the default probabilities decrease, and by decreasing the lending activities, the default probabilities increase.

5.5 Number of defaulting banks

In many practical applications, it is actually more interesting to evaluate the number of the defaults that occur in the banking system rather than to evaluate the total default probabilities. Therefore, the process D_t is introduced which calculates the number of occurred defaults in the banking system before (or at) time point $t \geq 0$ and is defined as

$$D_t = \sum_{i=1}^{N} \mathbb{I}\left(\min_{0\leq s\leq t} X_s^i = 0 \right), \tag{99}$$

where $\mathbb{I}(A)$ is the indicator function which returns value 1 if A is true and 0 otherwise.

Once again, a banking system with no interbank lending is considered as a simple example. As was shown in 95, function $F(\cdot)$ gives default probability for the subset

$(l_1, ..., l_k)$ of banks in a system without interbank lending. Therefore, $F(X_0^i, v_i, t)$, where $v_i = 1 - \delta_i/2$, gives the default probability for individual bank i (as if $k = 1$) and $1 - F(X_0^i, v_i, t)$ gives the probability that the bank i will survive. The probability that k number of banks go bankrupt and $N - k$ survive (before or at the time point t) can be calculated by considering all possible choices of $(l_1, ..., l_k)$, i.e.

$$\mathbb{P}(D_t = k) = \sum_{1 \leq l_1 < ... < l_k \leq N} \left(\prod_{j=1}^{k} (F(X_0^{l_j}, v_j, t) \right) \left(\prod_{i \notin (l_1, ..., l_k)} (1 - (F(X_0^i, v_i, t)) \right). \tag{100}$$

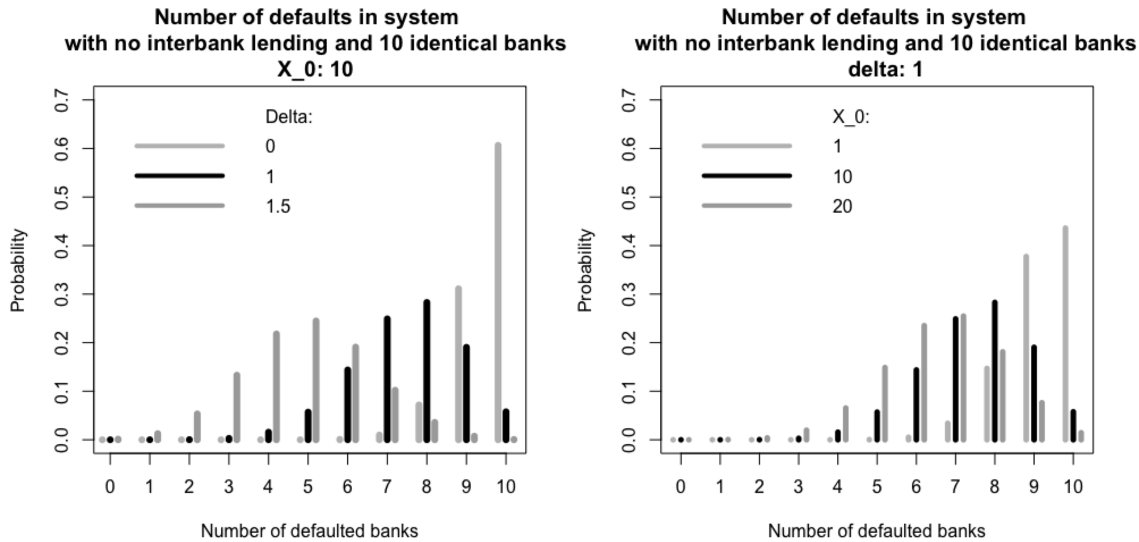

Figure 6: Theoretical probabilities that k number of banks default before (or at) time point $t = 100$ in a system with no interbank lending and 10 identical banks, $\delta_i \in \{0, 1, 1.5\}$ and $X_0^i \in \{1.5, 3, 4.5\}$.

The system of ten banks is numerically analysed in figure 6, where it is assumed that all banks have identical growth rates, identical initial monetary reserves at $t = 0$, and that interbank lending is not allowed. The results follow similar patterns as observed before. When either growth rate or initial reserves are increased (decreased), the default probability for individual bank decreases (increases) and therefore it is more likely that fewer banks will default. However, default for individual bank only occurs if the growth rate of the bank is below 2.

Once again, the exact analysis of the general case where interbank lending is allowed is fairly difficult and requires numerical methods. However, it is implied (see e.g. 98) that by increasing (decreasing) the interbank lending activities in the system, the total default probability for the subset of banks will decrease (increase). Therefore, it seems reasonable to state that interbank lending works as a stabilizing force in the banking system that seemingly decreases the probability of individual banks to go bankrupt and thus decreases the overall risk of systemic crisis. However, in section 6.1, it is shown that interbank lending can actually drive systemic crises in some specific situations.

6 Measuring risk in Mean field banking model

In this section, a model is studied where bank's monetary reserves are set to reverse to the average level of the reserves in the system. Moreover, the model works as a simple example of the coupled banking system introduced in section 5 with symmetric lending preferences. The model is analysed using simple simulation methods for which the codes are provided in appendix C.[9] In subsection 6.1, this model, called Mean field model, and some of its basic properties are introduced following Sun (2017) [19]. In subsection 6.2, the total loss distribution of the whole banking sector is modelled using the Mean Field model and tail risk measures (VaR and ES) are numerically estimated.

6.1 Mean field model and systemic crisis

As mentioned, the Mean field model belongs to the family of the Coupled banking models introduced in section 5. Specifically, this Mean field model describes the monetary reserves X_t^i of bank i using constant growth rate δ and fixed normalized lending preference $\alpha/N \leq 1$ where N refers to the total number of banks in the banking system. The stochastic differential equation for bank i is

$$\begin{aligned} dX_t^i &= \left(\frac{\alpha}{N} \sum_{j=1}^{N} (X_t^j - X_t^i) + \delta \right) dt + 2\sqrt{X_t^i} dB_t^i \\ &= \left(\alpha(\bar{X}_t - X_t^i) + \delta \right) dt + 2\sqrt{X_t^i} dB_t^i, \ i = 1, ..., N, \end{aligned} \tag{101}$$

where B_t^i is a standard uncorrelated Brownian motion and $\bar{X}_t = \sum_{j=1}^{N} X_t^j / N$ is the averaged value of the reserves at time point t. For simplicity, it is assumed that all banks have same initial reserves X_0 available at time $t = 0$ which means that all banks are identical. Furthermore, X_t^i is reverting to the mean reserves $\bar{X}_t$ with the mean reversion rate α.[10]

The model name "Mean field" refers to the Mean field game theory. In general, Mean field game theory studies the strategic decision making between N-number of small agents in very large populations (when $N \to \infty$). The goal of these games is to minimize the target cost functions $J^1(a^1, ..., a^N), ..., J^N(a^1, ..., a^N)$ where $(a^1, ..., a^N)$ represents the actions taken by the players in the game (see e.g. Carmona and Delarue (2018) [3] for further information regarding the Mean field games). Although this thesis does not focus on the game side of interbank lending models, it is noted that Sun (2017) [19] analyses the model introduced in this section as a Mean field game, and the latter equation in the formula 101 represents the lending and borrowing in the Mean field form.[11]

[9] Simple Euler scheme is applied. The process is simulated using **R**

[10] Mean reversion means that stochastic variable tends to converge to its average level over time.

[11] The Mean field game set-up of this model ads central bank to the banking system. In this set-up, the equation 101 is denoted as $dX_t^i = \left(\alpha(\bar{X}_t - X_t^i) + \delta + a_t^i\right) dt + 2\sqrt{X_t^i} dB_t^i$, where a_t^i is the strategy taken by bank i at time point t. Each bank chooses this strategy independently to optimize its lending and borrowing rates to/from the central bank at each time point t.

The stochastic differential equation for the monetary reserves of the total banking system, i.e. $Y_t = \sum_{i=1}^{N} X_t^i$, is

$$dY_t = N\delta dt + 2\sqrt{Y_t}d\tilde{B}_t. \tag{102}$$

where $\tilde{B}_t$ is a standard Brownian motion in some extension probability space. This means that the total monetary reserves follow squared Bessel process, i.e. $Y_t \sim BESQ^{N\delta}(Y_0)$ where $Y_0 = NX_0$. In addition, it is important to note that the lending preference parameter α is not present in the stochastic differential equation for the total monetary reserves.

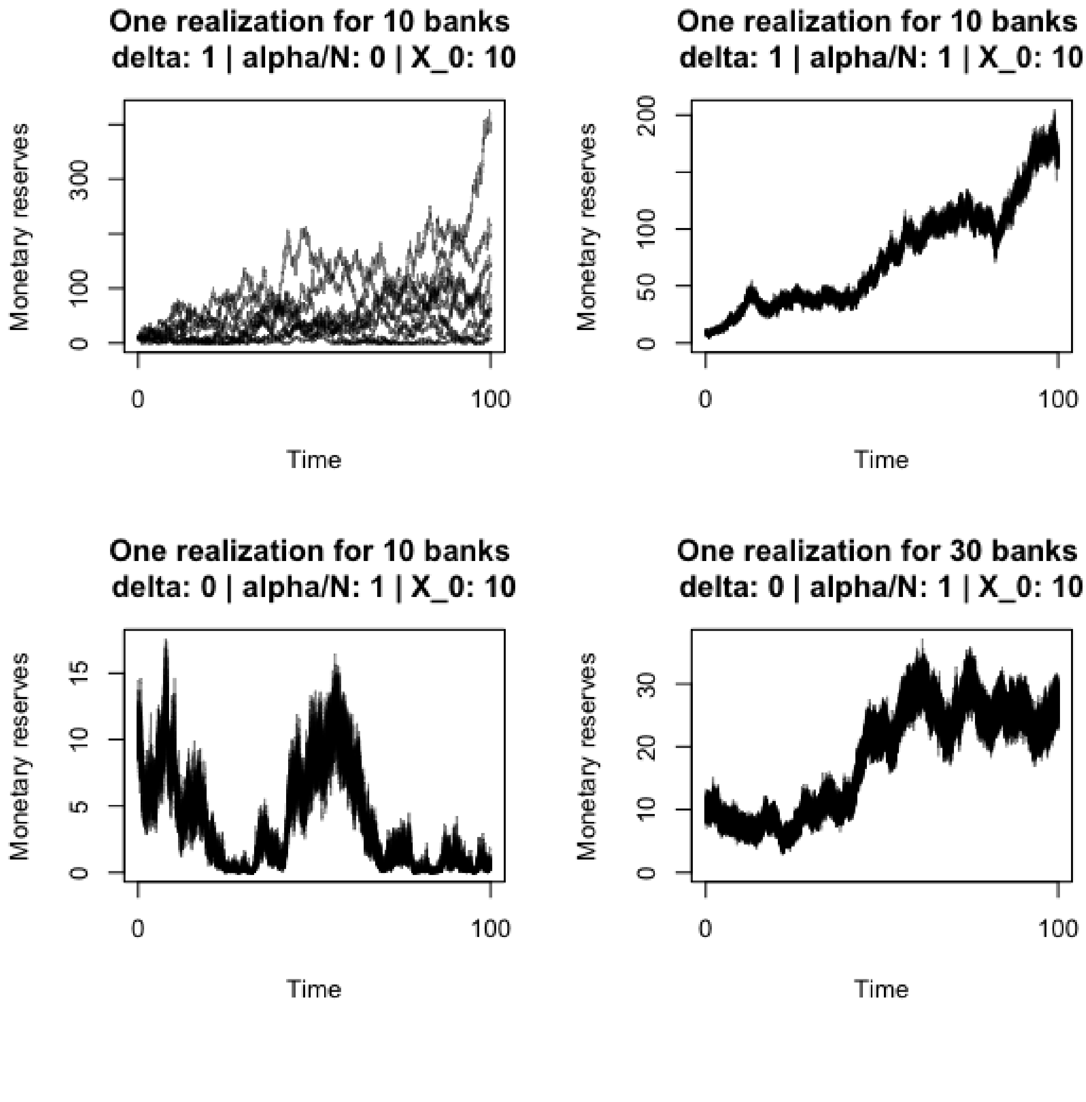

Figure 7: Example trajectories for banking system with different parameters.

The behaviour of the Mean field system is further illustrated in figure 7, where four different trajectories for the banking system are drawn using different parameter sets. When interbank lending preference α is zero, then the monetary reserves develop independently meaning that some banks triumph while other banks even face defaults despite the fact that the growth rate is positive (top-left). When full interbank lending (i.e. $\alpha/N = 1$) is applied, then the monetary reserves for individual banks develop almost identically (i.e. grouped development) and positive growth rate $\delta > 2/N$ causes monetary reserves to grow for all banks (top-right). When δ drops to zero, i.e. $\delta < 2/N$, then the growth of the grouped reserves is weak and the banking system may even face systemic crises where reserves for all banks are close to zero (bottom-right). However, by increasing the total number of banks in the system, the development of the reserves gets seemingly stronger even though growth rate is still zero (bottom-left).

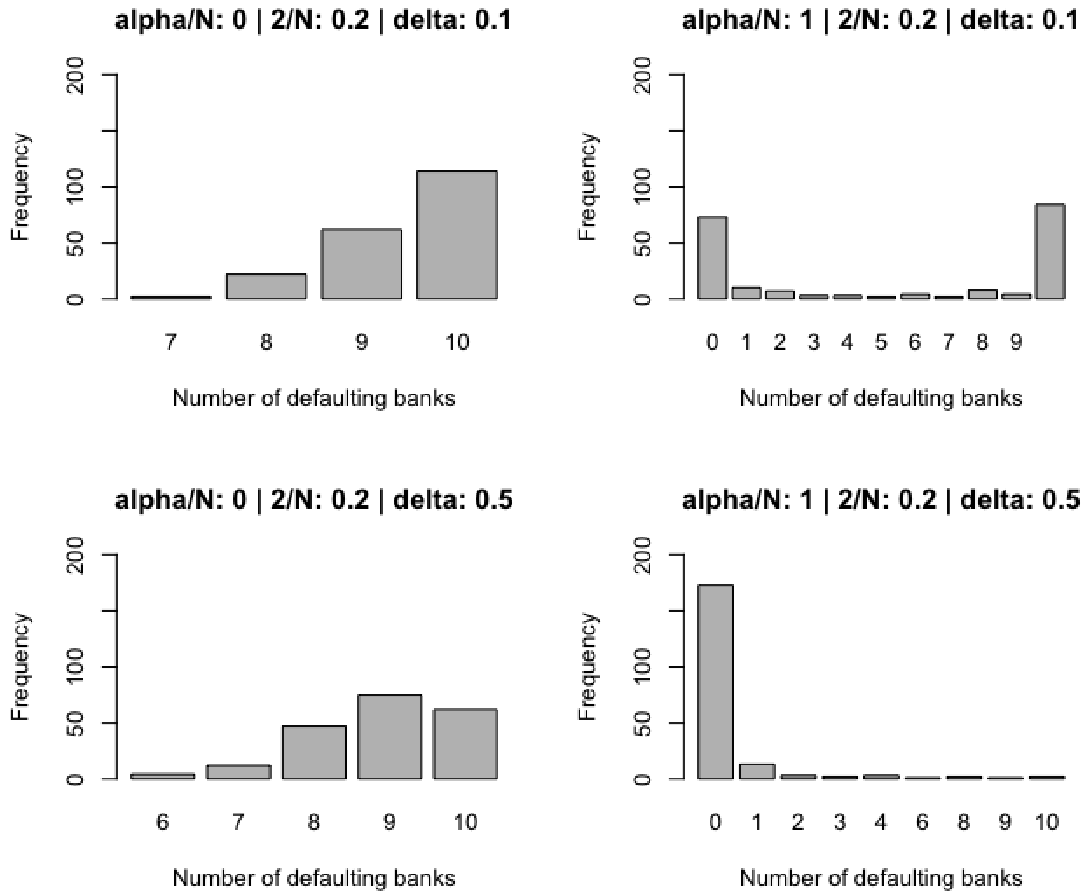

Figure 8: System of 10 banks analysed by simulating 200 scenarios until $t = 100$ starting from $X_0 = 10$. If bank faces default (i.e. its reserves reach zero level) during time interval $(0, 100]$, then the bank is counted as defaulting bank.

Naturally, the behaviour of the trajectories can be explained by the same properties that were discussed in section 5.2. If $\delta > 2/N$, then the total reserves Y_t will never reach zero. In case if $\delta = 2/N$, then the total reserves will diminish to almost zero almost surely at some point in the future. However, the system will always survive since $\mathbb{P}(Y_t > 0 \text{ for } t \in [0, \infty)) = 1$. In case if $0 < \delta < 2/N$, all banks will likely default in the future and the total reserves reach zero almost surely at some point in the future, but the system will instantly reflect away from this crisis state (due to external bailouts). Finally, in case if $\delta = 0$, then the total reserves will reach zero in some finite time and remain there. This means that all banks will default and then remain as defaulted almost surely.

Based on the behaviour of the trajectories, it can be seen that the stability of this banking system is hugely affected by the total number of banks in the system N and growth rate δ. More precisely, when $\delta > 2/N$, then interbank lending creates stability, but when $0 \leq \delta \leq 2/N$, then interbank lending actually creates systemic risk. This finding is illustrated in figure 8, where the banking system is analysed by simulating independent scenarios and by counting the number of defaulting banks per each scenario. Clearly, most of the banks default when both interbank lending and growth rate are low (top-left), but when growth rate alone is increased, then the number of defaulting banks gets lower (bottom-left).

Interestingly, when $\delta < 2/N$ and full interbank lending is applied, then it is very likely that either zero banks or all banks will default (top-right, figure 8). Due to these low growth rates, banks are weak against adverse shocks and these shocks spread from one bank to other banks through interbank lending activities. Therefore, interbank lending can actually drive systemic risk if the total growth rate in the banking system is low enough. Naturally, when the growth rate is increased to $\delta > 2/N$, then the system becomes stronger against adverse shocks (bottom-right, figure 8). In this case, if one bank faces adverse shock, it can borrow money from other banks that are likely thriving and thus it will likely survive too. In conclusion, active interbank lending markets alone don't ensure that the banking system is safe, but large enough growth rates (i.e. operating results) are also needed.

6.2 VaR and ES for the total monetary reserves

As was discussed in subsection 2.3.1, common way of measuring risk is to simulate loss distributions and calculate risk measures such as Value-at-Risk or Expected Shortfall. In this subsection, VaR, mean-VaR and ES measures are estimated for the total banking system. To make the analysis simple enough, it is assumed that all the banks in the Mean field system are identical. Therefore, loss distribution for the total monetary reserves is simply defined as a change between initial monetary reserves $Y_0 = NX_0$ and current monetary reserves at time point t, i.e. $L_t = -(Y_t - Y_0)$.

As the total monetary reserves follow squared Bessel process $Y_t \sim BESQ^{N\delta}(Y_0)$ and based on the analysis conducted in the subsection 4.2, the total monetary reserves at time t follow non-central chi-squared distribution that is scaled by the time point t, i.e.

$$Y_t \sim t\chi^2(N\delta, \frac{Y_0}{t}), \tag{103}$$

where $N\delta \geq 0$, $Y_0 \geq 0$ and $t > 0$. For the loss distribution L_t, it is easy to see that L_t follows non-central chi-squared distribution that is scaled with the negative time point $-t$ and displaced using the starting reserves Y_0, i.e.

$$L_t = -(Y_t - Y_0) \sim Y_0 - t\chi^2(N\delta, \frac{Y_0}{t}). \tag{104}$$

Furthermore, given the theory of the loss operators described in section 2, the loss operator $f_t : \mathbb{R} \to \mathbb{R}$ takes one-dimensional linear form here. More specifically, $f_t(x) = -(c_t + b_t x)$ where $c_t = -Y_0$, $b_t = t$ and the only risk factor follows the non-central chi-squared distribution $\chi^2(N\delta, Y_0/t)$.

Unfortunately, there is no closed form solution for the quantile function (i.e. inverse cumulative distribution function) of the non-central chi-squared distribution which means that there isn't closed form solutions for VaR and ES measures either. However, as mentioned in subsection 2.3.2, Monte Carlo methods can be applied when there is no closed form solutions available. In this case, simulations for the loss distribution are conducted using two methods. The first method simulates total monetary reserves using 103 directly. This is computationally very efficient method and returns results that very closely follow theoretical distributions. The second method is to simulate trajectories for individual banks according to 101 and then to aggregate the total monetary reserves, but this is computationally much more demanding method.

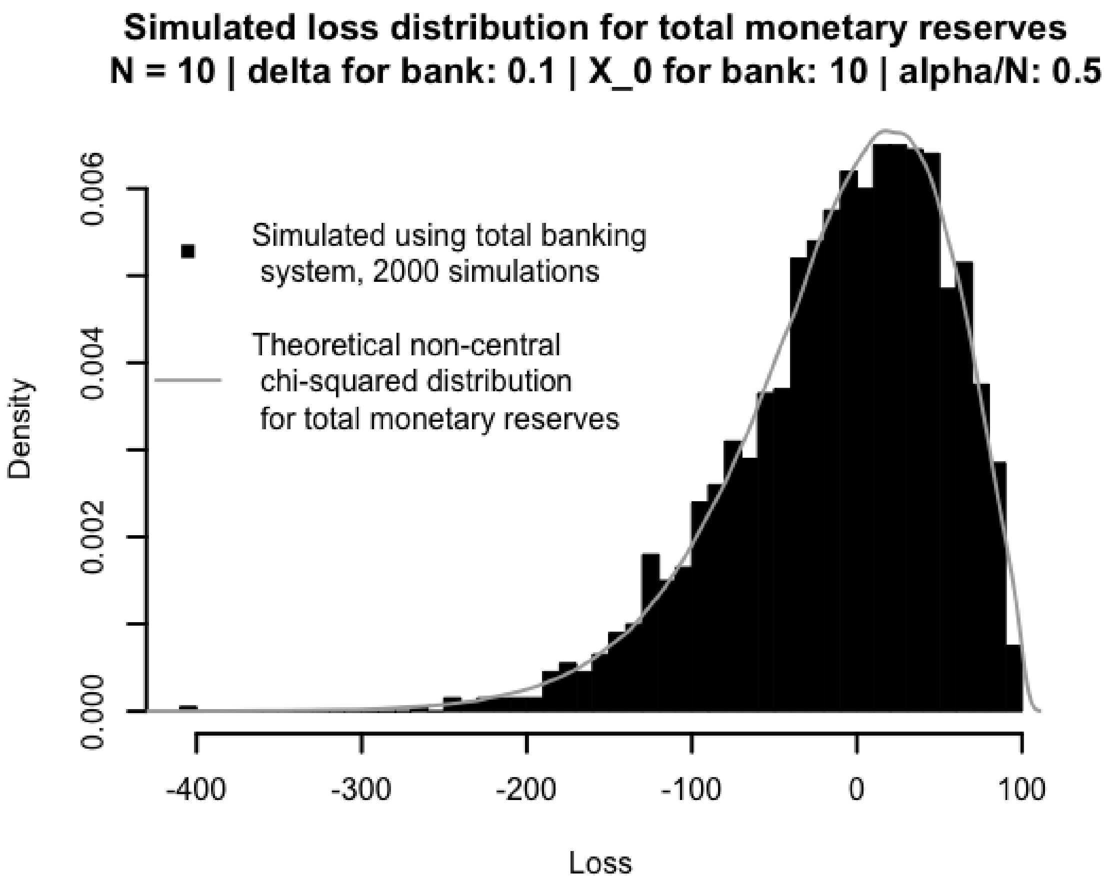

Figure 9: Loss distribution for the total monetary reserves at time point $t = 10$ created by simulating the non-central chi-squared distribution and simulating the trajectories for individual banks.

In figure 9, total monetary reserves at time point $t = 10$ are simulated using both these methods. The simulation error in the first method is in practice almost negligible, although also the second method creates loss distribution that follows the theoretical distribution fairly closely.[12] Using these simulation methods, it is analysed how the risk measures change when the growth rate δ, number of banks N, initial monetary reserves X_0 and interbank lending preference α are changed. First three of these tests can be conducted by using the non-central chi-squared distribution directly and the last test is conducted by simulating the trajectories for individual banks.

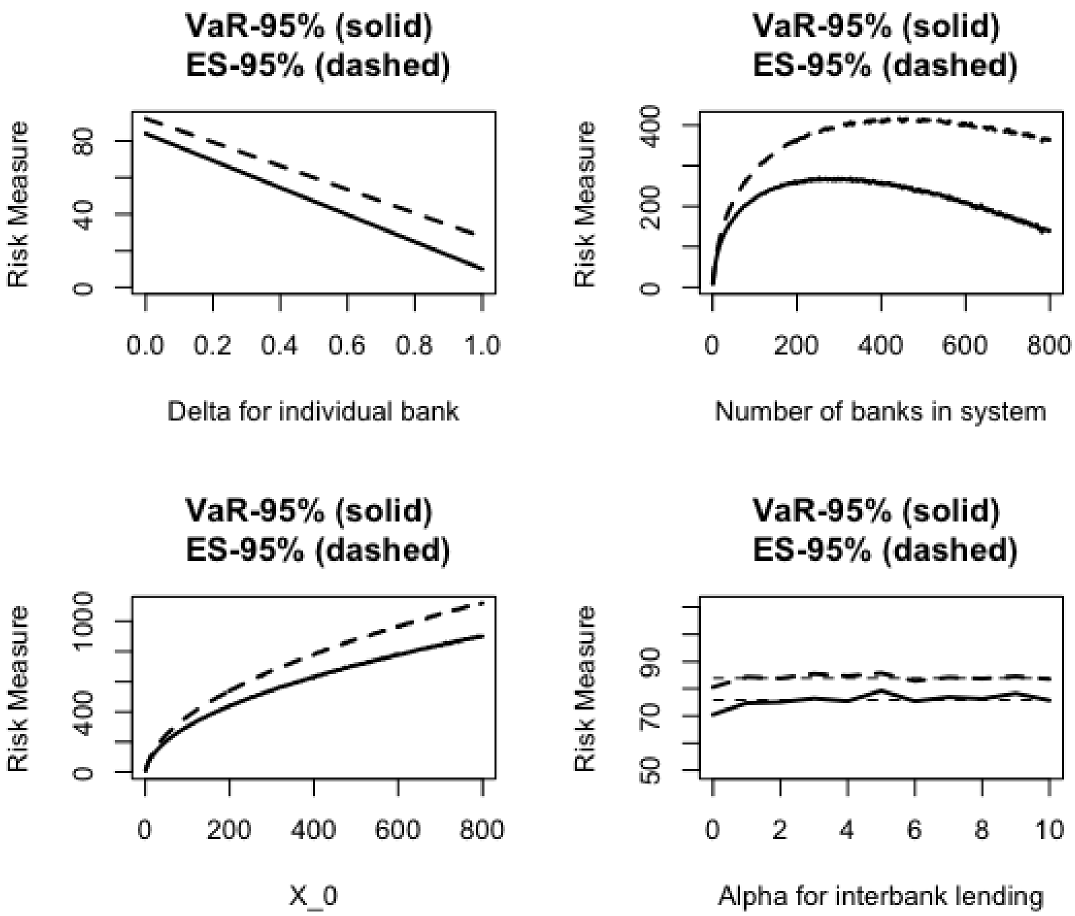

Figure 10: Analysing changes in the risk measures (VaR and ES on 95% confidence level) for the total monetary reserves when individual variables are changed. The base level of variables is $N = 10$, $\alpha = 5$, $\delta = 0.1$, $t = 10$ and $X_0 = 10$.

Based on the results in figure 10 (bottom-right), the interbank lending preference has no effect on the risk measures as was expected based on the theoretical model analysis. The variation seen in the risk measures happens due to the simulation error as the total number of simulation rounds needs to remain low when banks are

[12]Simulation error is defined here as the difference between the theoretical outcomes and the simulated outcomes. The non-central chi square distribution is simulated by using **rchisq** function in base **R**. Repeating this simulation multiple times (e.g. 1 million times), one can quickly create (almost) the exact non-central chi-squared distribution.

simulated individually. Furthermore, when the growth rate δ is increased (top-left), then the banks essentially have better operating results which means that they can also endure adverse shocks better. As the risk is measured unsymmetrical through the tail loss events, it is clear that the risk measures decrease too.

Despite the fact that the total initial reserves ($Y_0 = NX_0$) develop similarly when N or X_0 is altered, still the risk measures develop very differently. When the initial reserves are increased, then the risk grows as the potential tail losses grow too (figure 10, bottom-left). However, when the number of banks in the system is increased, then the risk measures initially grow, but after a certain threshold, the risk measures start to decrease instead (figure 10, top-right). This happens because the total growth rate $N\delta$ also increases when N is increased. Since higher growth rate makes the banking system stronger, the overall risk starts to decrease once the growth rate becomes high enough.

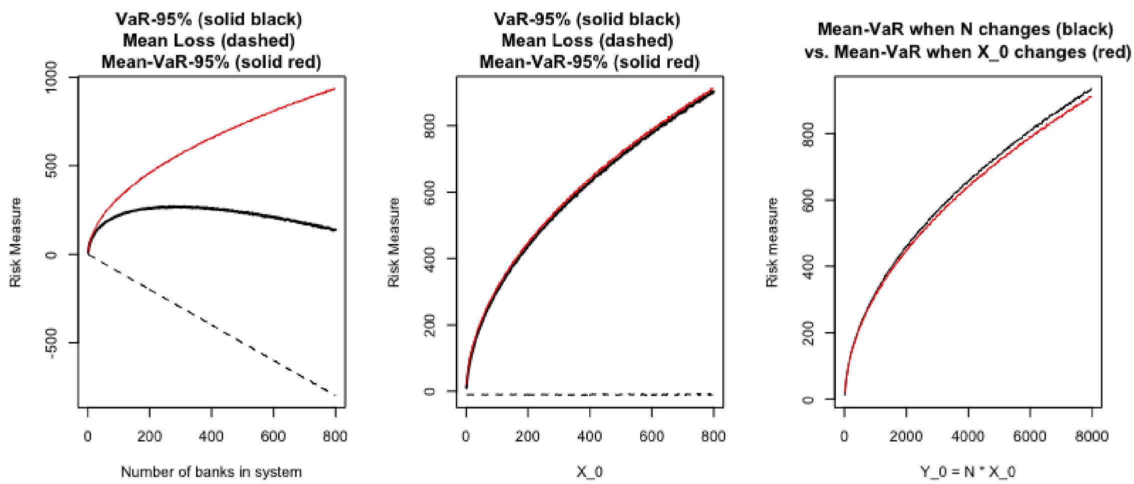

Figure 11: Comparing changes in mean-VaR measures on 95% confidence level when N and X_0 are changed. The base level of variables is $N = 10$, $\alpha = 5$, $\delta = 0.1$, $t = 10$ and $X_0 = 10$.

In figure 11, N and X_0 are analysed again so that the mean losses and mean-VaR measures are plotted with the standard VaR measures. Although the mean-VaR measures develop similarly in both cases (right graph), still the mean losses and VaR measures develop very differently (left and centre graphs). When the number of banks is increased, then the mean loss starts to decrease (negative loss is interpreted as profit) which indicates that the system overall has lower risk for severe losses compared to the case where initial reserves are increased instead.

The results obtained here clearly indicate that from a macro-prudential point of view it is better (i.e. less risky) to have more banks in the markets than to have fewer but larger banks. However, the crude model assumptions clearly drive these results, as it is assumed in the model that any new bank can instantly add more growth to the system whereas existing banks have constant growth rates. Yet real world banking markets don't work like that as operating results (i.e. growth rates) are not constant or deterministic and usually larger banks have at least some competitive advantages against the smaller banks. Therefore, one potential improvement to the model is to replace the constant growth rate component with a stochastic growth rate component that also depends on the sizes of the banks. This stochastic component would capture the nature of the uncertain markets and operating results better and make it possible to directly link the growth rate to bank's size.

7 Conclusion

Interbank lending markets and reserve management have crucial role in the risk management of the banking system. For example, the distressed interbank lending markets further escalated the emerging systemic crisis during the late 2000s. As proposed by e.g. Fouque and Ichiba (2013) [7], interbank lending markets can be modelled with a system of coupled diffusion processes, and under specific (symmetry) assumptions the total monetary reserves of the whole banking sector follow squared Bessel process. Furthermore, the dimension of this $BESQ$ process is then interpreted as the total growth rate and together with the lending preference, these two factors define whether the systemic crises exist in the banking system or not. In general, the banking sector benefits from the increased lending activities (and higher growth rate) as this decreases the probability of individual banks to go bankrupt.

Somewhat simplified version of the Coupled banking model was proposed by Sun (2017) [19] and this model is called Mean field model. In the Mean field model, it is assumed that the reserves of the individual banks revert to the average level of the reserves and that the speed of this reversion is defined by the constant lending preference parameter. Based on the numerical simulations, it is shown that the monetary reserves of individual banks develop almost identically when the interbank lending preference is strong. This happens because each bank constantly compares its reserves to the reserves of the other banks and acts in interbank lending markets based on the differences in the reserve levels. Therefore, the interbank lending activity reduces the differences in the reserves of the individual banks. However, this active lending also causes the adverse shocks to spread from one bank to all the other banks in the markets. Furthermore, if the total growth in the system is low, then the banks are fairly vulnerable to these widespread shocks. Therefore, the interbank lending activity can actually increase the probability of severe systemic crises if the total growth rate in the banking system is low enough.

In the Mean field model, the loss distribution of the total monetary reserves follows non-central chi-squared distribution. Quantitative risk analysis shows that larger initial reserves (i.e. larger banks) lead to larger tail losses and risk whereas higher growth rate decreases the potential tail losses and risk. However, when the size of the banking system is increased by adding new banks to the system, then the tail risks first grow slowly, but after a certain threshold the risks start to decrease instead. This finding indicates that from a macro-prudential point of view it is better to increase the size of the banking system by adding new banks to the system rather than by increasing the sizes of the existing banks as the former alternative creates less risk than the latter alternative.

The Coupled banking model has many limiting assumptions that drive these aforementioned findings. Specifically, it is assumed that the growth rates of the individual banks are constant, that the banks don't gain any competitive edge when their sizes grow, that any new bank can instantly add more growth to the system (i.e. increase the total growth rate), and that bank's growth rate is not linked to its size (i.e. its growth rate remains the same even if its reserves grow). These crude assumptions drive towards the aforementioned conclusion that from a macro-prudential point of view it is better to have more banks than to have larger banks.

Therefore, one potential future improvement to the model is to replace the constant growth rate with a stochastic growth rate component. This component would capture the uncertain nature of the markets better and directly link the growth rate to bank's size.

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A Comparison theorem for solutions of stochastic differential equations

A Comparison theorem for solutions of stochastic differential equations by Ikeda and Watanabe (1977) [11] is introduced here.

Theorem 8 (Comparison theorem of Ikeda and Watanabe)**.** *Given*

- *a real continuous function $\sigma_t(x)$ defined on $x \in \mathbb{R}$ and $t \geq 0$ such that*

$$|\sigma_t(x) - \sigma_t(y)| \leq \rho(|x-y|), x, y \in \mathbb{R}, t \geq 0, \tag{105}$$

  where $\rho(\cdot)$ is an increasing function on $[0,\infty)$ such that $\rho(0) = 0$ and $\int_{0+} \rho(z)^{-2} dz = \infty$,

- *real continuous functions $b_t^1(x)$ and $b_t^2(x)$ defined on $x \in \mathbb{R}$ and $t \geq 0$ such that*

$$b_t^1(x) < b_t^2(x),\ \ t \geq 0, x \in \mathbb{R}, \tag{106}$$

- *two continuous processes X_t^1 and X_t^2, and a one-dimensional standard Brownian motion B_t,*

- *two well measurable processes β_t^1 and β_t^2.*

Assuming that they satisfy the following conditions with probability one:

$$\begin{cases} X_t^i - X_0^i = \int_0^t \sigma_s(X_s^i) dB_s + \int_0^t \beta_s^i ds,\ \ i = 1,2 \\ X_0^1 \leq X_0^2 \\ \beta_t^1 \leq b_t^1(X_t^1),\ \ \textit{for all}\ t \geq 0 \\ \beta_t^2 \geq b_t^2(X_t^2),\ \ \textit{for all}\ t \geq 0. \end{cases} \tag{107}$$

Then it holds with probability one that

$$X_t^1 \leq X_t^2,\ \ \textit{for all}\ t \geq 0. \tag{108}$$

Furthermore, if the pathwise uniqueness of solutions holds for at least one of the following stochastic differential equations

$$dX_t = \sigma_t(X_t) dB_t + b_t^i(X_t) dt,\ \ i = 1,2. \tag{109}$$

then the same conclusion 108 holds if property 106 is weakened to

$$b_t^1(x) \leq b_t^2(x),\ \ t \geq 0, x \in \mathbb{R}, \tag{110}$$

Proof. See Ikeda and Watanabe (1977) [11, p. 619—622] for complete details of the theorem 8 and its proof. □

B Extension probability space

Probability space is denoted as $(\Omega, \mathcal{F}, P)$, where Ω is the sample space which is the set of all the possible outcomes, $\mathcal{F}$ is the set of events from the sample space, and P is the probability function that assigns each event in the event space with a probability between 0 and 1. In probability theory, stochastic process is a collection of the random variables $(X_t, 0 \leq t < \infty)$ defined on $(\Omega, \mathcal{F}, P)$ and sample path is defined as function $t \to X_t(\omega), t \geq 0$ for a fixed sample point $\omega \in \Omega$. Furthermore, filtered probability space is denoted as $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. The following theorem is from Karatzas and Shreve (1991) [13, p. 170] (theorem 3.4.2).

Theorem 9. *Suppose* $\mathbf{M}_t = ((M_t^1, ..., M_T^d), \mathcal{F}_t, 0 \leq t < \infty)$ *is defined on* $(\Omega, \mathcal{F}, P)$ *so that* M^i *is a continuous local martingale for* $i = 1, .., d$ *and that the cross variation* $\langle M^i, M^j \rangle_t(\omega)$ *is an absolute continuous function of* t *for every* ω *almost surely with respect to probability measure* P*. Then there is an extension probability space* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ *of* $(\Omega, \mathcal{F}, P)$ *on which is defined a* d*-dimensional Brownian motion* $\mathbf{B}_t = ((B_t^1, ..., B_t^d), \tilde{\mathcal{F}}_t, 0 \leq t < \infty)$ *and a matrix* $\left((X_t^{(i,k)})_{i,k=1}^d, \tilde{\mathcal{F}}_t, 0 \leq t < \infty\right)$ *of a measurable adapted process with*

$$\tilde{P}\left(\int_0^t (X_s^{(i,k)})^2 ds < \infty\right) = 1,\ 1 \leq i, k \leq d,\ 0 \leq t < \infty, \tag{111}$$

such that there are, almost surely with respect to probability measure $\tilde{P}$*, representations*

$$M_t^i = \sum_{k=1}^{d} \int_0^t X_s^{(i,k)} dB_s^k,\ 1 \leq i \leq d,\ 0 \leq t < \infty, \tag{112}$$

and

$$\langle M^i, M^j \rangle_t = \sum_{k=1}^{d} \int_0^t X_s^{(i,k)} X_s^{(j,k)} ds,\ 1 \leq i, j \leq d,\ 0 \leq t < \infty. \tag{113}$$

Proof. See Karatzas and Shreve (1991) [13, pp. 170-172] for the full proof of this theorem. □

The theorem 9 provides a theoretical backbone for the equality $\int_0^t \sqrt{Y_u} d\tilde{B}_u = \int_0^t \sum_{i=1}^N \sqrt{X_u^i} dB_u^i$, which is an important detail in the Coupled banking model framework as this equality makes it possible to model the total reserves by using squared Bessel process. A sketch of the proof for this equality can be given by noting that the monetary reserves of banks $i = 1, ..., N$, i.e. $\mathbf{X}_t = ((X_t^1, ..., X_t^N, \mathcal{F}_t, 0 \leq t < \infty)$, are on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ and their dynamics are defined as in equation 85, where $(B_t^1, ..., B_t^N, \mathcal{F}_t, 0 \leq t < \infty)$ is standard N-dimensional Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. When the dynamics for the total monetary reserves, i.e. $Y_t = \sum_{i=1}^N X_t^i$, are formulated, an extension probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t\geq 0}, \tilde{P})$ is introduced on which a 1-dimensional Brownian motion $(\tilde{B}_t, \tilde{\mathcal{F}}_t, 0 \leq t \leq \infty)$ is defined. Finally, one can show that based on theorem 9 and properties of $\mathbf{X}_t$ and Y_t, the equality indeed holds.

C R script

```
## R SCRIPT

## Iiro Marttila

##################################################################################

## VaR and ES illustrated

par(mfrow=c(1,1))

set.seed(1234)
L <-rnbinom(10000,10,0.5)-15
alpha <-0.95
EL <-mean(L)
VaR <-quantile(L,alpha)
ES <-mean(L[VaR<=L])

den <-density(L)
plot(den,lty=1,lwd=2,main=paste0("Loss distribution and its VaR and ES when alpha is ", 100*alpha,"%"),
     xlab="Loss", ylab="Density", col="black")
lines(y=c(0,1),x=c(EL,EL),col="darkgray",lwd=2,lty=3)
text(y=0.04,x=EL,labels="E(L)",pos=2)
lines(y=c(0,1),x=c(VaR,VaR),col="darkgray",lwd=2)
text(y=0.04,x=VaR,labels=paste0("VaR-",round(100*(alpha),0),"%"),pos=4)
polygon(c(den$x[den$x >= VaR ], VaR),
        c(den$y[den$x >= VaR ], 0),
        col = "lightgray",
        border = 1)
lines(y=c(0,1),x=c(ES,ES),col="darkgray",lwd=2,lty=2)
text(y=0.06,x=ES,labels=paste0("ES-",round(100*(alpha),0),"%"),pos=4)



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## Brownian motion - trajectories

par(mfrow=c(1,1))

#Time
t <-0:100
#Sigma
sig <-1/(length(t)-1)
#Number of simulated paths
nsim <- 100

#Simulate paths
set.seed(123)
X <- matrix(rnorm(n = nsim * (length(t) - 1), sd = sqrt(sig)), nsim, length(t) -
              1)
X <- cbind(rep(0, nsim), t(apply(X, 1, cumsum)))

#Plot paths with random colors
plot(t, X[1, ], xlab = "time", ylab = "", ylim = range(X), type = "l")
apply(X[2:nsim, ], 1, function(x, t) lines(t, x, col=round(runif(1,0,100),0)), t = t)

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## Reflection principle - example

par(mfrow=c(1,1))

## set reflection point
m <-1.096
## set time
t <- 0:200
## set variance
sig2 <- 0.01
## set simulation seed
set.seed(1)
## first, simulate a set of random deviates
x <- rnorm(n = length(t) - 1, sd = sqrt(sig2))
## now compute their cumulative sum
x <- c(0, cumsum(x))
## reflect brownian motion
x_refl <-2*m-x
## start reflection from first passage point
x_refl[1:(which(x>m)[1]-1)] <-x[1:(which(x>m)[1]-1)]

plot(t, x_refl, type = "l", ylim = c(-1.5, 2.5),lwd=2, col="gray", ylab="",xlab="Time")
lines(x=c(0,2000),y=c(m,m),lty=2,col="red")
lines(t,x,lty=1,lwd=2,col="black")
legend("bottomleft",ncol=1,lty=c(1,1,2),col=c("black","gray","red"),lwd=c(2,2,1),
       legend=c("Original path", "Reflected path", "Barrier"), bty='n')


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## Simulating squared Bessel process trajectories

par(mfrow=c(1,1))
```

```
# Squared Bessel process function
Bessel_path <-function(X_0,delta,t_val,time_steps){
  dt <-t_val/time_steps
  X_t <-X_0
  for(i in 1:time_steps){
    dX_t <-delta*dt+2*sqrt(abs(X_t[i]))*rnorm(1)*sqrt(dt)
    X_t[i+1] <-X_t[i] + dX_t
    if(X_t[i+1] < 0){X_t[i+1] <-0}
  }
  return(X_t)
}

## Simulate using different deltas
set.seed(537128)
path1 <-Bessel_path(X_0=1,delta=0,t_val=10,time_steps=100000)

set.seed(18)
path2 <-Bessel_path(X_0=1,delta=1,t_val=10,time_steps=100000)

set.seed(6)
path3 <-Bessel_path(X_0=1,delta=2,t_val=10,time_steps=100000)

set.seed(10)
path4 <-Bessel_path(X_0=1,delta=3,t_val=10,time_steps=100000)

## Plot
plot(y=path1,x=seq(0,10,10/100000), main="Trajectories for the squared Bessel process", xlab="Time",
     type='l',col="darkgray",  ylim=c(-1,max(c(path1,path2,path3,path4))),lwd=1, ylab="")
lines(y=c(0,0),x=c(-100,100000),lwd=1,lty=2,col="red")
lines(y=path2,x=seq(0,10,10/100000), col="black",lwd=0.5)
lines(y=path3,x=seq(0,10,10/100000), col="skyblue",lwd=0.5)
lines(y=path4,x=seq(0,10,10/100000), col="darkblue",lwd=0.5)
legend("topleft",ncol=2,legend=c("delta 0","delta 1","delta 2","delta 3"),lty=rep(1,4),
       col=c("darkgray","black","skyblue","darkblue"),bty='n',lwd=rep(2,4))


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## Estimating default time probabilities when there is no lending money

par(mfrow=c(1,2))

## probability density function
integrant <-function(s){(1/(s*gamma(v)))*(((x_k0)/(2*s))^(v))*exp(-(x_k0)/(2*s))}
t <-seq(0.1,100,0.1)

## Change in delta
x_k0 <-10
deltas <-c(0,1,1.5)

for(ind in 1:length(deltas)){

  delta <-deltas[ind]
  v <-1-delta/2
  for(i in 1:length(t)){
    t_val <-t[i]
    prob <-integrate(integrant,lower=0,upper=t_val)$value
    #1-integrate(integrant,lower=t_val,upper=Inf)$value
    if(i==1){probs <-prob}
    if(i!=1){probs[i] <-prob}
  }
  if(ind==1){plot(x=t,y=probs,type='l',col=ind,ylim=c(0,1.1), lwd=2,
                  xlab="Time", ylab="Probability",
                  main=paste("Default probabilities in system
                             with no interbank lending, Y_0 =",x_k0))}
  if(ind!=1){lines(x=t,y=probs,col="black",lty=ind,lwd=2)}
  if(ind==length(deltas)){legend("bottomright",legend=c("Delta:",deltas),col=c(NA,rep("black",3)),
                                 lty=c(NA,1,2,3), bty='n')}
  }

## Change in X_k(0)
delta <-1
x_k0s <-c(1,10,20)

for(ind in 1:length(x_k0s)){

  x_k0 <-x_k0s[ind]
  v <-1-delta/2
  for(i in 1:length(t)){
    t_val <-t[i]
    prob <-integrate(integrant,lower=0,upper=t_val)$value
    #1-integrate(integrant,lower=t_val,upper=Inf)$value
    if(i==1){probs <-prob}
    if(i!=1){probs[i] <-prob}
  }
  if(ind==1){plot(x=t,y=probs,type='l',col=ind,ylim=c(0,1.1), lwd=2,
                  xlab="Time", ylab="Probability",
                  main=paste("Default probabilities in system
                             with no interbank lending, delta =",delta))}
  if(ind!=1){lines(x=t,y=probs,col="black",lty=ind,lwd=2)}
  if(ind==length(x_k0s)){legend("bottomright",legend=c("Y_0:",x_k0s),col=c(NA,rep("black",3)),
                                lty=c(NA,1,2,3), bty='n')}
}
```

```
##########################################################################################

## Analyse number of defaults in theory

par(mfrow=c(1,2))

# delta varies
deltas <-c(0,1,1.5)
x_k0 <-10
t_val <-100
banks <-10

all_probss <-matrix(NA,ncol=banks+1,nrow=length(deltas))
for(i in 1:length(deltas)){
  delta <-deltas[i]
  v <-((4-delta)/2)-1
  prob_def <-integrate(integrant,lower=0,upper=t_val)$value
  prob_sur <-1-integrate(integrant,lower=0,upper=t_val)$value
  probss <-numeric(0)
  for(k in 0:banks){
    probss[k+1] <-sum(rep((prob_def^k)*prob_sur^(banks-k),ncol(combn(1:banks,k))))
    names(probss)[k+1] <-k
  }
  all_probss[i,] <-probss
}
plot(x=(0:10-0.2),y=all_probss[1,],type="h",col="gray", ylim=c(0,0.7), lwd=6, xaxt='n',
     xlab="Number of defaulted banks", ylab="Probability",xlim=c(-0.01,10.1)
     ,main=paste("Number of defaults in system
                 with no interbank lending and",banks, "identical banks
                 X_0:",x_k0)
     )
lines(x=0:10,y=all_probss[2,],lty=1, col="black", type='h', lwd=6)
lines(x=(0:10+0.2),y=all_probss[3,],lty=1, col="darkgray",type='h', lwd=6)
axis(1,labels=(0:banks),at=(0:banks))
legend("topleft",
       legend=c("Delta:",deltas),
       ncol=1,
       lty=c(NA,1,1,1), col=c(NA,"gray","black","darkgray"), bty='n', lwd=c(NA,4,4,4))
rowSums(all_probss)


# X_0 varies
x_k0s <-c(1,10,20)
delta <-1
t_val <-100
banks <-10

all_probss <-matrix(NA,ncol=(banks+1),nrow=length(x_k0s))
for(i in 1:length(x_k0s)){
  x_k0 <-x_k0s[i]
  v <-((4-delta)/2)-1
  prob_def <-integrate(integrant,lower=0,upper=t_val)$value
  prob_sur <-1-integrate(integrant,lower=0,upper=t_val)$value
  probss <-numeric(0)
  for(k in 0:banks){
    probss[k+1] <-sum(rep((prob_def^k)*prob_sur^(banks-k),ncol(combn(1:banks,k))))
    names(probss)[k+1] <-k
  }
  all_probss[i,] <-probss
}
plot(x=(0:10-0.2),y=all_probss[1,],type="h",col="gray", ylim=c(0,0.7), lwd=5, xaxt='n',
     xlab="Number of defaulted banks", ylab="Probability",xlim=c(-0.1,10.1)
     ,main=paste("Number of defaults in system
                 with no interbank lending and",banks, "identical banks
                 delta:",delta)
     )
lines(x=0:10,y=all_probss[2,],lty=1, col="black", type='h', lwd=5)
lines(x=(0:10+0.2),y=all_probss[3,],lty=1, col="darkgray",type='h', lwd=5)
axis(1,labels=(0:banks),at=(0:banks))
legend("topleft",
       legend=c("X_0:",x_k0s),
       ncol=1,
       lty=c(NA,1,1,1), col=c(NA,"gray","black","darkgray"), bty='n', lwd=c(NA,4,4,4))
rowSums(all_probss)


##########################################################################################

## Trajectories in Mean field banking model

## Parameter set 1

par(mfrow=c(2,2))

N <-10
alpha <-0
delta <-1
time <-100
steps <-10000
simulations <-1
X_0 <-10

X_matrix <-matrix(NA,nrow=(steps+1),ncol=N)
```

```
colnames(X_matrix) <-1:N
X_matrix[1,] <-rep(X_0,N)
dt <-time/steps
set.seed(1234)
for(i in 1:steps){
  for(n in 1:N){
    dX <-((alpha/N)*sum(X_matrix[i,]-X_matrix[i,n])+delta)*dt+2*sqrt(X_matrix[i,n])*rnorm(1)*sqrt(dt)
    X_matrix[i+1,n] <-X_matrix[i,n] + dX
    if(X_matrix[i+1,n] < 0){X_matrix[i+1,n] <-0}
  }
}
plot(X_matrix[,1],type='l',col=rgb(0,0,0,alpha=0.5),ylim=c(0,max(X_matrix)),xaxt='n',ylab="Monetary reserves", xlab="Time",
     main=paste("One realization for", N, "banks
                delta:", delta, "| alpha/N:", alpha/N, "| X_0:", X_0))
for(n in 2:N){
  lines(X_matrix[,n],col=rgb(0,0,0,alpha=0.5))
}
axis(1,at=c(1,nrow(X_matrix)),labels=c(0,time))

## Parameter set 2

N <-10
alpha <-N #meanin alpha/N = 1
delta <-1
time <-100
steps <-10000
simulations <-1
X_0 <-10

X_matrix <-matrix(NA,nrow=(steps+1),ncol=N)
colnames(X_matrix) <-1:N
X_matrix[1,] <-rep(X_0,N)
dt <-time/steps
set.seed(144)
for(i in 1:steps){
  for(n in 1:N){
    dX <-((alpha/N)*sum(X_matrix[i,]-X_matrix[i,n])+delta)*dt+2*sqrt(X_matrix[i,n])*rnorm(1)*sqrt(dt)
    X_matrix[i+1,n] <-X_matrix[i,n] + dX
    if(X_matrix[i+1,n] < 0){X_matrix[i+1,n] <-0}
  }
}
plot(X_matrix[,1],type='l',col=rgb(0,0,0,alpha=0.5),ylim=c(0,max(X_matrix)),xaxt='n',ylab="Monetary reserves", xlab="Time",
     main=paste("One realization for", N, "banks
                delta:", delta, "| alpha/N:", alpha/N, "| X_0:", X_0))
for(n in 2:N){
  lines(X_matrix[,n],col=rgb(0,0,0,alpha=0.5))
}
axis(1,at=c(1,nrow(X_matrix)),labels=c(0,time))

## Parameter set 3

N <-10
alpha <-N #meanin alpha/N = 1
delta <-0
time <-100
steps <-10000
simulations <-1
X_0 <-10

X_matrix <-matrix(NA,nrow=(steps+1),ncol=N)
colnames(X_matrix) <-1:N
X_matrix[1,] <-rep(X_0,N)
dt <-time/steps
set.seed(44126)
for(i in 1:steps){
  for(n in 1:N){
    dX <-((alpha/N)*sum(X_matrix[i,]-X_matrix[i,n])+delta)*dt+2*sqrt(X_matrix[i,n])*rnorm(1)*sqrt(dt)
    X_matrix[i+1,n] <-X_matrix[i,n] + dX
    if(X_matrix[i+1,n] < 0){X_matrix[i+1,n] <-0}
  }
}
plot(X_matrix[,1],type='l',col=rgb(0,0,0,alpha=0.5),ylim=c(0,max(X_matrix)),xaxt='n',ylab="Monetary reserves", xlab="Time",
     main=paste("One realization for", N, "banks
                delta:", delta, "| alpha/N:", alpha/N, "| X_0:", X_0))
for(n in 2:N){
  lines(X_matrix[,n],col=rgb(0,0,0,alpha=0.5))
}
axis(1,at=c(1,nrow(X_matrix)),labels=c(0,time))


## Parameter set 4

N <-30
alpha <-N #meanin alpha/N = 1
delta <-0
time <-100
steps <-10000
simulations <-1
X_0 <-10

X_matrix <-matrix(NA,nrow=(steps+1),ncol=N)
colnames(X_matrix) <-1:N
```

```r
X_matrix[1,] <-rep(X_0,N)
dt <-time/steps
set.seed(100)
for(i in 1:steps){
  for(n in 1:N){
    dX <-((alpha/N)*sum(X_matrix[i,]-X_matrix[i,n])+delta)*dt+2*sqrt(X_matrix[i,n])*rnorm(1)*sqrt(dt)
    X_matrix[i+1,n] <-X_matrix[i,n] + dX
    if(X_matrix[i+1,n] < 0){X_matrix[i+1,n] <-0}
  }
}
plot(X_matrix[,1],type='l',col=rgb(0,0,0,alpha=0.5),ylim=c(0,max(X_matrix)),xaxt='n',ylab="Monetary reserves", xlab="Time",
      main=paste("One realization for", N, "banks
                  delta:", delta, "| alpha/N:", alpha/N, "| X_0:", X_0))
for(n in 2:N){
  lines(X_matrix[,n],col=rgb(0,0,0,alpha=0.5))
}
axis(1,at=c(1,nrow(X_matrix)),labels=c(0,time))

###################################################################################

## Calculating number of defaults in the mean fiel banking system

par(mfrow=c(2,2))

## Parameter set 1

N <-10
alpha <-0
delta <-0.1
time <-100
steps <-1000
simulations <-200
X_0 <-10

set.seed(1234)
defaults_dist <-numeric(0)
for(s in 1:simulations){
  X_matrix <-matrix(NA,nrow=(steps+1),ncol=N)
  colnames(X_matrix) <-1:N
  X_matrix[1,] <-rep(X_0,N)
  dt <-time/steps
  for(i in 1:steps){
    for(n in 1:N){
      dX <-((alpha/N)*sum(X_matrix[i,]-X_matrix[i,n])+delta)*dt+2*sqrt(X_matrix[i,n])*rnorm(1)*sqrt(dt)
      X_matrix[i+1,n] <-X_matrix[i,n] + dX
      if(X_matrix[i+1,n] < 0){X_matrix[i+1,n] <-0}
    }
  }
  defaults <-sum(apply(X_matrix,2,min)<=0)
  defaults_dist <-c(defaults_dist,defaults)
}
defaults_dist <-table(defaults_dist)
barplot(defaults_dist,main=paste0("alpha/N: ",alpha/N, " | 2/N: ",2/N," | delta: ",delta), xlab="Number of defaulting banks"
        , ylab="Frequency", ylim=c(0,simulations))

## Parameter set 2

N <-10
alpha <-N
delta <-0.1
time <-100
steps <-1000
simulations <-200
X_0 <-10

set.seed(111)
defaults_dist <-numeric(0)
for(s in 1:simulations){
  X_matrix <-matrix(NA,nrow=(steps+1),ncol=N)
  colnames(X_matrix) <-1:N
  X_matrix[1,] <-rep(X_0,N)
  dt <-time/steps
  for(i in 1:steps){
    for(n in 1:N){
      dX <-((alpha/N)*sum(X_matrix[i,]-X_matrix[i,n])+delta)*dt+2*sqrt(X_matrix[i,n])*rnorm(1)*sqrt(dt)
      X_matrix[i+1,n] <-X_matrix[i,n] + dX
      if(X_matrix[i+1,n] < 0){X_matrix[i+1,n] <-0}
    }
  }
  defaults <-sum(apply(X_matrix,2,min)<=0)
  defaults_dist <-c(defaults_dist,defaults)
}
defaults_dist <-table(defaults_dist)
barplot(defaults_dist,main=paste0("alpha/N: ",alpha/N, " | 2/N: ",2/N," | delta: ",delta), xlab="Number of defaulting banks"
        , ylab="Frequency", ylim=c(0,simulations))


## Parameter set 3

N <-10
alpha <-0
delta <-0.5
```

```
time <-100
steps <-1000
simulations <-200
X_0 <-10

set.seed(111)
defaults_dist <-numeric(0)
for(s in 1:simulations){
  X_matrix <-matrix(NA,nrow=(steps+1),ncol=N)
  colnames(X_matrix) <-1:N
  X_matrix[1,] <-rep(X_0,N)
  dt <-time/steps
  #set.seed(100)
  for(i in 1:steps){
    for(n in 1:N){
      dX <-((alpha/N)*sum(X_matrix[i,]-X_matrix[i,n])+delta)*dt+2*sqrt(X_matrix[i,n])*rnorm(1)*sqrt(dt)
      X_matrix[i+1,n] <-X_matrix[i,n] + dX
      if(X_matrix[i+1,n] < 0){X_matrix[i+1,n] <-0}
    }
  }
  defaults <-sum(apply(X_matrix,2,min)<=0)
  defaults_dist <-c(defaults_dist,defaults)
}
defaults_dist <-table(defaults_dist)
barplot(defaults_dist,main=paste0("alpha/N: ",alpha/N, " | 2/N: ",2/N," | delta: ",delta), xlab="Number of defaulting banks"
        , ylab="Frequency", ylim=c(0,simulations))


## Parameter set 4

N <-10
alpha <-N
delta <-0.5
time <-100
steps <-1000
simulations <-200
X_0 <-10

set.seed(124)
defaults_dist <-numeric(0)
for(s in 1:simulations){
  X_matrix <-matrix(NA,nrow=(steps+1),ncol=N)
  colnames(X_matrix) <-1:N
  X_matrix[1,] <-rep(X_0,N)
  dt <-time/steps
  #set.seed(100)
  for(i in 1:steps){
    for(n in 1:N){
      dX <-((alpha/N)*sum(X_matrix[i,]-X_matrix[i,n])+delta)*dt+2*sqrt(X_matrix[i,n])*rnorm(1)*sqrt(dt)
      X_matrix[i+1,n] <-X_matrix[i,n] + dX
      if(X_matrix[i+1,n] < 0){X_matrix[i+1,n] <-0}
    }
  }
  defaults <-sum(apply(X_matrix,2,min)<=0)
  defaults_dist <-c(defaults_dist,defaults)
}
defaults_dist <-table(defaults_dist)
barplot(defaults_dist,main=paste0("alpha/N: ",alpha/N, " | 2/N: ",2/N," | delta: ",delta), xlab="Number of defaulting banks"
        , ylab="Frequency", ylim=c(0,simulations))


##################################################################################

## Create loss distribution function for Mean field banking model

bessel_loss_dist <-function(N,alpha,delta,time,steps,simulations,X_0){

  loss_dist <-numeric(0)
  for(s in 1:simulations){
    X_matrix <-matrix(NA,nrow=(steps+1),ncol=N)
    colnames(X_matrix) <-1:N
    X_matrix[1,] <-rep(X_0,N)
    dt <-time/steps
    for(i in 1:steps){
      for(n in 1:N){
        dX <-((alpha/N)*sum(X_matrix[i,]-X_matrix[i,n])+delta)*dt+2*sqrt(abs(X_matrix[i,n]))*rnorm(1)*sqrt(dt)
        X_matrix[i+1,n] <-X_matrix[i,n] + dX
        if(X_matrix[i+1,n] < 0){X_matrix[i+1,n] <-0}
      }
    }
    X_t <-rowSums(X_matrix)[nrow(X_matrix)]
    loss_dist <- c(loss_dist,-(X_t-X_0*N))
  }
  return(loss_dist)
}

## Check process against theoretical non-central chi-square distribution

par(mfrow=c(1,1))

N <-10 #number of banks
alpha <-5 #interbank lending, 10 banks -> alpha/N = 0.5 <1
delta <-0.1 #per bank
```

```
time <-10
steps <-100
simulations <-2000
X_0 <-10 #per bank

set.seed(111)
loss_dist1 <-bessel_loss_dist(N=N,alpha=alpha,delta=delta,time=time,steps=steps,simulations=simulations,X_0=X_0)

X_t <-time*rchisq(n=1000000, df=(delta*N), ncp = ((X_0*N)/time))
loss_dist2 <- -(X_t-X_0*N)

hist(loss_dist1,main=paste("Simulated loss distribution for total monetary reserves \n",
                           "N =",N,"| delta for bank:",delta,"| X_0 for bank:",X_0, "| alpha/N:",round(alpha/N,1)),
     xlab="Loss",col="black",lwd=2,freq=FALSE,breaks=40)
lines(density(loss_dist2),col="darkgray",lwd=2,lty=1)
legend("topleft",legend=c(paste("Simulated using total banking \n system,",simulations,"simulations"),
                          "Theoretical non-central \n chi-squared distribution \n for total monetary reserves"),
       lty=c(NA,1),pch=c(15,NA),
       col=c("black","darkgray"),lwd=c(2,2),bty='n')

##################################################################################

## Analys how changes in variables affect VaR and ES for total monetary reserves
## in Mean field model

par(mfrow=c(2,2))
var_alpha <-0.95

# 1. Change of growth rate delta

N <-10 #number of banks
alpha <-5 #interbank lending
delta <-0.1 #per bank
time <-10
X_0 <-10 #per bank

deltas <-range(0,1,0.01) #different deltas
vars <- es <-numeric(0)
for(delta in deltas){
  X_t <-time*rchisq(n=200000, df=(delta*N), ncp = ((X_0*N)/time))
  loss_dist <- -(X_t-X_0*N)
  var <-quantile(loss_dist,var_alpha)
  es <-c(es,mean(loss_dist[loss_dist>=var]))
  vars <-c(vars,var)
}
plot(y=vars,x=deltas,type='l',col="black",lwd=2,ylim=c(0,max(c(vars,es))),ylab="Risk Measure",xlab="Delta for individual bank",
     main=paste0("VaR-",round(100*var_alpha,0),"% (solid) \n ES-",round(100*var_alpha,0),"% (dashed)"))
lines(y=es,x=deltas,col="black",lwd=2,lty=2)



# 2. Change of number of banks

N <-10 #number of banks
alpha <-5 #interbank lending
delta <-0.1 #per bank
time <-10
X_0 <-10 #per bank

Ns <-seq(1,800,2) #number of banks
vars <- es <-numeric(0)
for(N in Ns){
  X_t <-time*rchisq(n=200000, df=(delta*N), ncp = ((X_0*N)/time))
  loss_dist <- -(X_t-X_0*N)
  var <-quantile(loss_dist,var_alpha)
  es <-c(es,mean(loss_dist[loss_dist>=var]))
  vars <-c(vars,var)
}
plot(y=vars,x=Ns,type='l',col="black",lwd=2,ylim=c(0,max(c(vars,es))),ylab="Risk Measure",xlab="Number of banks in system",
     main=paste0("VaR-",round(100*var_alpha,0),"% (solid) \n ES-",round(100*var_alpha,0),"% (dashed)"))
lines(y=es,x=Ns,col="black",lwd=2,lty=2)




# 3. Change of start value X_0

N <-10 #number of banks
alpha <-5 #interbank lending
delta <-0.1 #per bank
time <-10
X_0 <-10 #per bank


X_0s <-seq(1,800,2)
vars <- es <-numeric(0)
for(X_0 in X_0s){
  X_t <-time*rchisq(n=200000, df=(delta*N), ncp = ((X_0*N)/time))
  loss_dist <- -(X_t-X_0*N)
  var <-quantile(loss_dist,var_alpha)
  es <-c(es,mean(loss_dist[loss_dist>=var]))
  vars <-c(vars,var)
}
```

```r
plot(y=vars,x=X_0s,type='l',col="black",lwd=2,ylim=c(0,max(c(vars,es))),ylab="Risk Measure",xlab="X_0",
     main=paste0("VaR-",round(100*var_alpha,0),"% (solid) \n ES-",round(100*var_alpha,0),"% (dashed)"))
lines(y=es,x=X_0s,col="black",lwd=2,lty=2)



# 4. Change of alpha -> simulating whole banking system

N <-10 #number of banks
alpha <-5 #interbank lending
delta <-0.1 #per bank
time <-10
steps <-100
simulations <-2000
X_0 <-10 #per bank


alphas <-seq(0,N,1) #maximum alpha/N can be 1
vars <- es <-numeric(0)
set.seed(767)
for(alpha in alphas){
  loss_dist <- bessel_loss_dist(N=N,alpha=alpha,delta=delta,time=time,steps=steps,simulations=simulations,X_0=X_0)
  var <-quantile(loss_dist,var_alpha)
  es <-c(es,mean(loss_dist[loss_dist>=var]))
  vars <-c(vars,var)
}
plot(y=vars,x=alphas,type='l',col="black",lwd=2,ylim=c(0.7*min(c(vars,es)),1.3*max(c(vars,es))),ylab="Risk Measure",
     xlab="Alpha for interbank lending",
     main=paste0("VaR-",round(100*var_alpha,0),"% (solid) \n ES-",round(100*var_alpha,0),"% (dashed)"))
lines(y=es,x=alphas,col="black",lwd=2,lty=2)
lines(y=c(mean(vars),mean(vars)),x=alphas[c(1,length(alphas))],col="black",lwd=1,lty=2)
lines(y=c(mean(es),mean(es)),x=alphas[c(1,length(alphas))],col="black",lwd=1,lty=2)


#####################################################################################

## Analyse VaRs, mean losses and mean-VaRs for change of N and X_0 in mean fiel model

par(mfrow=c(1,3))

# Change of number of banks

N <-10 #number of banks
alpha <-5 #interbank lending
delta <-0.1 #per bank
time <-10
X_0 <-10 #per bank

Ns <-seq(1,800,2) #number of banks
vars <- means <- mean_vars <-numeric(0)
for(N in Ns){
  X_t <-time*rchisq(n=200000, df=(delta*N), ncp = ((X_0*N)/time))
  loss_dist <- -(X_t-X_0*N)
  mu <-mean(loss_dist)
  var <-quantile(loss_dist,var_alpha)
  mean_var <-var-mu
  means <-c(means,mu)
  vars <-c(vars,var)
  mean_vars <-c(mean_vars,mean_var)
}
plot(y=vars,x=Ns,type='l',col="black",lwd=2,ylim=range(c(mean_vars,vars,means)),ylab="Risk Measure",
     xlab="Number of banks in system",
     main=paste0("VaR-",round(100*var_alpha,0),"% (solid black) \n",
                 "Mean Loss (dashed) \n",
                 "Mean-VaR-",round(100*var_alpha,0),"% (solid red)"))
lines(y=means, x=Ns, col="black", lty=2)
lines(y=mean_vars, x=Ns, col="red",lty=1)
mean_vars1 <-mean_vars



# Change of start value X_0

N <-10 #number of banks
alpha <-5 #interbank lending
delta <-0.1 #per bank
time <-10
X_0 <-10 #per bank


X_0s <-seq(1,800,2)
vars <- means <- mean_vars <-numeric(0)
for(X_0 in X_0s){
  X_t <-time*rchisq(n=200000, df=(delta*N), ncp = ((X_0*N)/time))
  loss_dist <- -(X_t-X_0*N)
  mu <-mean(loss_dist)
  var <-quantile(loss_dist,var_alpha)
  mean_var <-var-mu
  means <-c(means,mu)
  vars <-c(vars,var)
  mean_vars <-c(mean_vars,mean_var)
}
plot(y=vars,x=X_0s,type='l',col="black",lwd=2,ylim=range(c(mean_vars,vars,means)),ylab="Risk Measure",xlab="X_0",
     main=paste0("VaR-",round(100*var_alpha,0),"% (solid black) \n",
                 "Mean Loss (dashed) \n",
```

```
                 "Mean-VaR-",round(100*var_alpha,0),"% (solid red)"))
lines(y=means, x=X_0s, col="black", lty=2)
lines(y=mean_vars, x=X_0s, col="red",lty=1)

#check the difference in mean vars
diff_mean_var <-(mean_vars1-mean_vars)/mean_vars1
summary(diff_mean_var)

plot(y=mean_vars1,x=(N*X_0s),type='l',xlab="Y_0 = N * X_0", ylab="Risk measure",
     main="Mean-VaR when N changes (black) \nvs. Mean-VaR when X_0 changes (red)")
lines(y=mean_vars,x=(N*X_0s),col="red")
```