TURUN YLIOPISTO

On lower bounds of various dominating codes for locating vertices in cubic graphs

Ruishi Qiu

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Self-identifying codes, self-locating dominating codes and solid-locating dominating codes are three subsets of vertices of a graph G to locate vertices. The optimal size of them is denoted by $\gamma^{SID}(G)$,$\gamma^{SLD}(G)$ and $\gamma^{DLD}(G)$. In the master thesis, we mainly discuss their lower bound problem in families of graphs.

In the first section, we briefly describe the background of the study and some related questions.

In the second, third and fourth section, we show some basic definitions, concepts and examples related to self-identifying codes (SID), self-locating dominating codes (SLD) and solid-locating dominating codes (DLD) in rook's graphs.

In the fifth section, we first introduce some known results of lower bounds of open-locating dominating codes in cubic graphs and then in the sixth section we present some new results about the lower bounds of self-identifying codes, self-locating dominating codes and solid-locating dominating codes in cubic graphs.

Keywords: Self-identifying code, locating-dominating code, cubic graph.

Contents

1 Introduction

Locating-dominating codes and identifying codes are two code subsets used to find out where anomalies occur in a network. But both have their limitations. When there are multiple anomalies in the network, they cannot correctly determine the location. In order to ensure that the entire network is working correctly, we need to monitor every location in the network, and a detailed algorithm of a locating method can be found in [2] and [15]. But how to maximize the use of resources and use the fewest detectors to monitor the network needs to be considered, and the smallest cardinality of locating-dominating codes and identifying codes are called optimal. We use γ with qualifiers to denote the smallest number of elements in locating-dominating codes and identifying codes in each network.

Locating-dominating codes are tightly connected to identifying codes, sometimes they were considered in the same paper. The concept of locating-dominating code was first researched by Slater (see [14], [19]). The research on optimal locating-dominating codes later has been deeply studied, for example, Seo and Slater found a lower bound and an upper bound of open-locating dominating codes in trees and infinite grids [1], [17], and later introduced three new locating sets, redundant open-locating-dominating codes (RED:OLD(G)), detection open-locating-dominating codes (DET:OLD(G)) and error open-locating-dominating codes (ERR:OLD(G)) [9], [18]. Moreover, lower bounds of three new locating methods in cubic graphs were found and verified in [5], [16] and [17]. Moreover, recently, lower bounds of redundant locating-dominating codes (RED:LD(G)), detection locating-dominating codes (DET:LD(G)) and error locating-dominating codes (ERR:LD(G)) were determined in [10].

Karpovsky et al. first introduced identifying codes (see [19]) and they have been widely researched, including in the rook's graph [4], in binary Hamming spaces [7] and in Cartesian products [3]. In addition, identifying codes were developed into t-robust 1-identifying codes in [6]. Moreover, there exist research on the complexity of identifying codes in graphs [8]

To overcome the weakness of locating only one irregularity, self-identifying codes were discovered. Self-identifying codes enabled a new s-tolerant identifying collection to check for network failures [12]. Recently, two new locating methods, self-locating dominating codes and solid-locating dominating codes were discovered and a lower bound of them was determined in the rook's graph and the binary Hamming spaces [11], [13].

In this master thesis, because self-locating dominating codes and solid-locating dominating codes are still newly discovered locating methods, we want to find out the lower bounds of self-identifying codes, self-locating dominating codes and solid-locating dominating codes in the cubic graphs using the ideas of lower bounds in previous locating methods such as identifying codes and locating-dominating codes. Our method is similar to a method in self-identifying codes, self-locating dominating codes and solid-locating dominating codes in [13] and to a method of detection locating-dominating codes in [16]. First, from the definition of these locating methods and cubic graphs, we find some restrictive conditions for them in cubic graphs. Then we divide them into multiple situations for discussion and finally get a condensed conclusion.

2 Identifying code

Suppose that we have a simple and undirected graph $G = (V(G), E(G)) = (V, E)$. The set V illustrates all of the positions of vertices in G and the set of E means the connection between two vertices. The open-neighbourhood of a vertex u is denoted by $N(u)$ and it means all of the vertices adjacent to u. In addition, the closed-neighbourhood of vertex u is denoted by $N[u] = \{u\} \cup N(u)$. In addition, a nonempty set C of V, which is a set that has sensors in it, is called a code, and its elements are called codewords. In order to show which sensor is monitoring the vertex u, we use

$$I(u) = N[u] \cap C = I(C; u).$$

Similarly, for a subset U of V, the identifying sets of U is denoted as follows:

$$I(U) = \bigcup_{u \in U} N[u] \cap C = \bigcup_{u \in U} I(C; u) = I(C; U).$$

In this section, we will introduce identifying codes and self-identifying codes. We will also introduce some examples to make it easier to understand relevant concepts.

2.1 Identifying code

First we will introduce some basic concepts of identifying codes.

Definition 2.1. The code $C \subseteq V$ is an identifying code (ID) of G if for any two vertices $u, v \in V$, we have $I(C; u) \neq \emptyset$ and

$$I(C; u) \neq I(C; v).$$

From the definition, it is easy to realize that identifying code works correctly if there is only one irregularity in the network at the same time. In other words, identifying code can locate at most only one irregularity correctly.

For a graph, there can be many identifying codes and the identifying codes with the smallest cardinality $|C|$ are called optimal identifying codes and the cardinality $|C|$ is denoted as $\gamma^{ID}(G)$.

Example 2.2. As showed in Figure 1, $v_4, v_5, v_6 \in C$ and obviously C is an identifying code of the graph G. The code $C = \{v_4, v_5, v_6\}$ is also an optimal

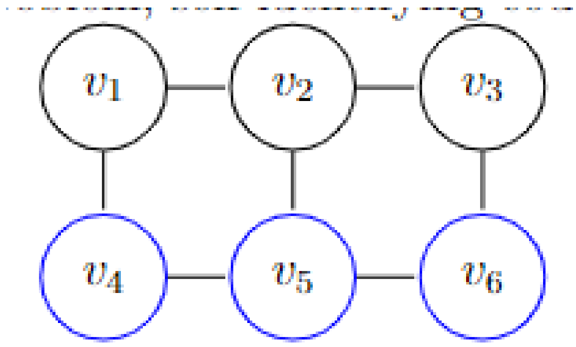

Figure 1: Identifying code in G.

$I(C;v_1) = \{v_4\}$	$I(C;v_2) = \{v_5\}$	$I(C;v_3) = \{v_6\}$
$I(C;v_4) = \{v_4, v_5\}$	$I(C;v_5) = \{v_4, v_5, v_6\}$	$I(C;v_6) = \{v_5, v_6\}$

Table 1: $I(C;v)$ for each vertex in Figure 1.

identifying code of the graph. Indeed if there is only 2 vertices in C, there is only at most 4 possibilities for each vertex,$\{\ \emptyset$, $\{v_x\}$, $\{v_y\}$ and $\{v_x, x_y\}\}$. No matter which two vertices belongs to C, it always has two vertices that have the same neighbourhood in the set C or for some vertex $u \in V$, $I(C;u) = \emptyset$.

Meanwhile if in the network there are two or even more irregularities existing in the network, then from the alarms, we can not determine where is the correct place that needs to be fixed. For instance in Figure 1, if v_1, v_2, v_3 have irregularities, all v_4, v_5, v_6 will alarm and we can not distinguish whether it is the problem of v_5 or v_1, v_2, v_3. It is easy for us to be mistaken which vertices go wrong.

2.2 Self-identifying code

In order to solve the above problem, self-identifying codes were introduced.

Definition 2.3. A code $C \subseteq V$ is a self-identifying code (SID) of G if the code C is an identifying code of G and for all $u \in V$ and $U \subseteq V$ such that $|U| \geq 2$ it has

$$I(C;u) \neq I(C;U)$$

For a graph, there can be many self-identifying codes and the self-identifying codes with the smallest cardinality $|C|$ are called optimal self-identifying codes and the

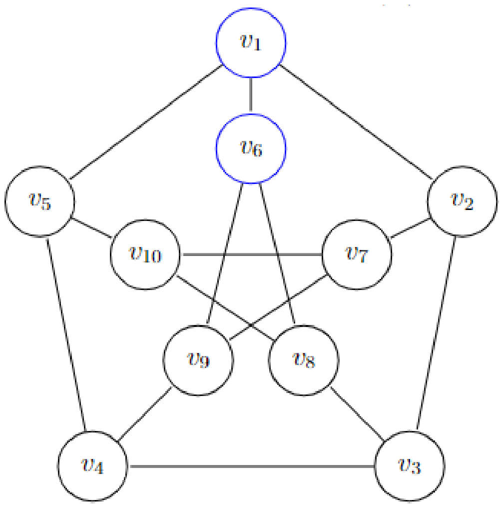

Figure 2: The Petersen graph.

cardinality $|C|$ is denoted as $\gamma^{SID}(G)$.

Example 2.4. In the Petersen graph in Figure 2, by the definition, we summarize all of the vertex neighbourhoods in C and get the Table 2, which implies that the codes with black contour form a self-identifying code of the graph G. In addition, C is also a minimal self-identifying code of the graph. Let us now verify that.

First suppose that in the figure 2, we remove any vertex from the set C(there are two cases),

(i) Vertex which we removed is the neighbourhood of v_1 or v_6, for example v_5 is removed now, then $I(C; v_1) = \{v_2\}$ and it obviously conflicts to the definition.

(ii) Vertex which removed is not the neighbourhood of any of them. For instance, we remove v_3 from the set C. This makes that $I(C; v_2) = \{v_2, v_7\}$. It is easy to see that it violates the Definition 2.3.

Hence, C is a minimal self-identifying code in graph G,

Next we discuss why 8 is also the minimum size. Now suppose that 7 is the minimum size of SID in the Peterson graph. We have already seen above that if two vertices are adjacent and both of them are not in C, we can not remove any other vertex. So the 3 vertices not in C should not be adjacent to each other. Hence now in Figure 2, first we select v_1, then v_2, v_5 and v_6 are not in C. For other 6 vertices, no matter which one we select, there always exists vertex that does not satisfy the definition of SID. For example, now $v_1, v_3 \notin C$, this gives $I(C; v_2) = \{v_2, v_7\}$, $I(C; v_7) = \{v_2, v_7, v_9, v_{10}\}$ and $I(C; v_2) \subseteq I(C; v_7)$. Hence it is impossible to find even two vertices not in C that are not adjacent in Figure 2.

So C is also optimal, so we get $\gamma^{SID}(G) = 8$.

$I(C; v_1) = \{v_2, v_5\}$	$I(C; v_2) = \{v_2, v_7, v_3\}$
$I(C; v_3) = \{v_2, v_3, v_4, v_8\}$	$I(C; v_4) = \{v_3, v_4, v_5, v_9\}$
$I(C; v_5) = \{v_4, v_5, v_{10}\}$	$I(C; v_6) = \{v_8, v_9\}$
$I(C; v_7) = \{v_2, v_7, v_9, v_{10}\}$	$I(C; v_8) = \{v_3, v_8, v_{10}\}$
$I(C; v_9) = \{v_4, v_7, v_9\}$	$I(C; v_{10}) = \{v_5, v_7, v_8, v_{10}\}$

Table 2: $I(C; v)$ for each vertex in Figure 2.

Theorem 2.5.([11][12][13]) For self-identifying codes, there are three equivalent conditions for it.

(i) For all $u \in V$ and $U \subseteq V$ such that $|U| \geq 2$ we have

$$I(C; u) \neq I(C; U).$$

(ii) For all distinct $u, v \in V$, we have $I(C; u) \setminus I(C; v) \neq \emptyset$.

(iii) For all $u \in V$ we have $I(C; u) \neq \emptyset$ and

$$\bigcap_{c \in I(C;u)} N[c] = \{u\}.$$

Proof. We only show that the last claim follows from (i) and (ii). For (iii) suppose that set C is a self-identifying code, but there exist u, v such that $\bigcap_{c \in I(C;u)} N[c] = \{u, v\}$. So we get $I(C; u) \subseteq I(C; v)$ which means we select $U = \{u, v\}$ and $I(C; v) = I(C; U)$ (a contradiction with definition of SID). Hence (iii) follows from (i) and (ii). $\square$

3 Locating-dominating code

When a detector itself can report not only 0 or 1 to detect its neighbour, but also the sensor itself can distinguish whether the problem occurred on itself and report 2. This means that when irregularity occured on the neighbour of a detector, it sends 1 and on itself 2, otherwise 0. This provides a new method of locating vertices.

3.1 Locating-dominating code

Locating-dominating codes are one of the earliest codes for locating vertices and first we will talk about it.

Definition 3.1. The code $C \subseteq V$ is locating-dominating code (LD) of G if for any two vertices u,$v \in V \setminus C$, we have $I\left(C;u\right) \neq \emptyset$ and

$$I\left(C;u\right) \neq I\left(C;v\right).$$

A locating-dominating code C with the smallest possible cardinality $|C|$ is called optimal and for this LD, $|C|$ is denoted as $\gamma^{LD}\left(G\right)$.

Example 3.2. In Figure 3, v_2 and v_5 form a locating-dominating code C of the graph G, since $I\left(C;v_1\right) = \{v_2\}$, $I\left(C;v_3\right) = \{v_5\}$ and $I\left(C;v_4\right) = \{v_2,v_5\}$. Meanwhile, C is also the optimal locating-dominating code of G. Because there are 5 vertices, if only one vertex belongs to C, obviously other four vertices can not have different neighbourhoods in C.

3.2 Self-locating dominating code and solid-locating dominating code

Self-locating dominating codes and solid-locating dominating codes are two new locating methods recently developed from locating-dominating codes.

Definition 3.3. A code $C \subseteq V$ is self-locating-dominating (SLD) of G if for any vertex $u \in V \setminus C$, we have $I\left(C;u\right) \neq \emptyset$ and

$$\bigcap_{c \in I(C;u)} N[c] = \{u\}.$$

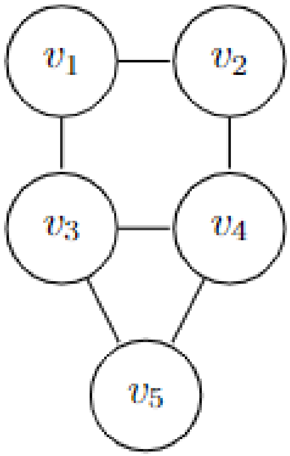

Figure 3: The graph G of examples 3.2 and 3.5.

A self-locating-dominating code C of the graph G with the smallest possible cardinality is called optimal and for such SLD, $|C|$ is denoted as $\gamma^{SLD}(G)$.

Definition 3.4. A code $C \subseteq V$ is solid-locating dominating (DLD) of G if for any two vertices $u, v \in V \setminus C$, we have

$$I(C; u) \setminus I(C; v) \neq \emptyset.$$

A solid-locating-dominating code C of a graph G with the smallest possible cardinality is called optimal and the number $|C|$ of such DLD is denoted as $\gamma^{DLD}(G)$.

Example 3.5. In Figure 3, $\{v_1, v_2, v_3, v_5\}$ is not only a self-locating dominating code C of the graph G, but also a solid-locating dominating code. If for a self-locating dominating code, there are only three codewords in set C, by this graph symmetry, there are three cases, $\{v_1, v_2, v_3\}$, $\{v_1, v_3, v_5\}$ or $\{v_1, v_4, v_5\}$. No matter which subset with cardinality of 3 is chosen, it can not satisfy definition of self-locating dominating code. Hence it is optimal self-locating dominating code but not optimal solid-locating dominating code. For the code set $\{v_2, v_3, v_4\}$, we have Table 3 as below.

$$I(C; v_1) = \{v_2, v_3\} \quad I(C; v_5) = \{v_3, v_4\}$$

Table 3: Optimal solid-locating dominating code of figure 3.

If there exists an DLD in Figure 3 with cardinality of 2, such as $\{v_1, v_3\}$, $\{v_1, v_4\}$

or $\{v_1, v_5\}$, there always exists two vertices not in C, such that $I(C;u) \setminus I(C;v) = \emptyset$.Hence, $\{v_2, v_3, v_4\}$ is the optimal DLD in Figure 3. From here, we get the idea that if a set C is a self-locating dominating code, then it must also be a solid-locating dominating code and $\gamma^{DLD}(G) \leq \gamma^{SLD}(G)$.

The paper [13] provides a rather constructive result and later we always use these two theorems below to find the lower bounds of SLD and DLD in cubic graphs.

Theorem 3.6.([13]) If $G = (V, E)$ is a connected graph with $n \geq 2$, the code $C \subseteq V$ is self-locating-dominating if and only if for all distinct $u \in V \setminus C$ and $v \in V$ we have

$$I(C;u) \setminus I(C;v) \neq \emptyset.$$

Proof. We prove here only the only-if-side of the claim. Suppose that there exist two vertices $u \in V \setminus C$ and $v \in V$ and $I(C;u) \setminus I(C;v) = \emptyset$. This means for codewords in $I(C;u)$, we have

$$\{u, v\} \subseteq \bigcap_{c \in I(C;u)} N[c].$$

Hence, it is a contradiction to the definition of SLD. □

Theorem 3.7.([13]) Let $G = (V, E)$ be a connected graph on at least two vertices. A code $C \subseteq V$ is solid-locating-dominating if and only if for all $u \in V \setminus C$ we have $I(C;u) \neq \emptyset$ and

$$\left(\bigcap_{c \in I(C;u)} N[c] \right) \setminus C = \{u\}.$$

Proof. We prove here only the only-if-side of the claim. Suppose that there exists $u \in V \setminus C$ such that

$$\left(\bigcap_{c \in I(C;u)} N[c] \right) \setminus C = \{u, v\}.$$

This implies that $I(C;u) \setminus I(C;v) = \emptyset$ (contradict to definition). □

By definition of SID, SLD and DLD, we get the following corollary.

Corollary 3.8. If the set C is a self-identifying code, then it must also be a self-locating dominating code and $\gamma^{SLD}(G) \leq \gamma^{SID}(G)$ and if the set C is a self-locating dominating code, then it must also be a solid-locating dominating code and $\gamma^{DLD}(G) \leq \gamma^{SLD}(G)$.

4 Some known results about locating-dominating code in rook's graphs

First, we will introduce some basic concepts of the rook's graph. Rook's graph can be considered as a rook move on a chess board and the Cartesian product of graphs of completed graphs G_1 and G_2. Suppose the chess board has x columns and y rows and there are two graphs $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$, then the rook's graph constructed by G_1 and G_2 are $G_1 \times G_2 = (V(G_1) \times V(G_2), E)$. Two vertices $u = (x_1, y_1)$ and $v = (x_2, y_2)$ are connected if and only if $x_1 = x_2$ and $y_1y_2 \in E(G_2)$ or $y_1 = y_2$ and $x_1x_2 \in E(G_1)$, such as Figure 4 is G_1, Figure 5 is G_2 and 6 is the rook's graph formed by Figures 4 and 5. The K row and H column of the graph is denoted by $R_k = \{(x_i,\ y_k) \mid i{=}1,\ 2,\ 3,...,\ n\}$ and $C_h = \{(x_h,\ y_j) \mid j{=}1,\ 2,\ 3,...,\ m\}$. Then we focus on the lower bound of five locating-dominating codes ID, SID, LD, SLD and DLD in rooks graphs.

For the size of $\gamma^{ID}(G)$ and $\gamma^{SID}(G)$ in the rook's graph is found in [3], [4] and [11]:

$$\gamma^{ID}(G) = \begin{cases} m + \lfloor \frac{n}{2} \rfloor, & m \leq \frac{3n}{2}, \\ 2m - n, & m \geq \frac{3n}{2}, \end{cases}$$

$$\gamma^{SID}(G) = 2m, m \geq n,$$

and for locating dominating code in [13]:

$$\gamma^{LD}(G) = \begin{cases} m - 1, & m \geq 2n, \\ \lceil \frac{2n+2m}{3} - 1 \rceil, & 2n - 1 \geq m \geq n. \end{cases}$$

As for SLD and DLD we have the following results.

Theorem 4.1.([13])

$$\gamma^{SLD}(G) = \begin{cases} m, & m \geq 2n, or\ n = 1, \\ 2n, & 2n \geq m > n \geq 2, \\ 2n - 1, & m = n > 2, \\ 4, & n = m = 2. \end{cases}$$

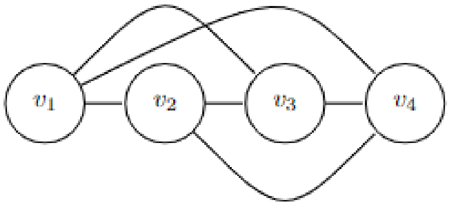

Figure 4: $G_1 = K_4$.

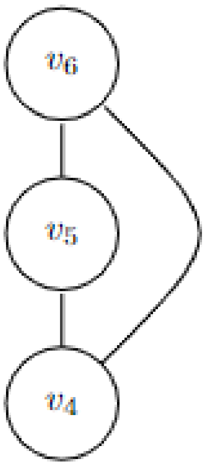

Figure 5: $G_2 = K_3$.

Proof. First for a vertex $v = (x_i, y_j)$ in a graph if $|I(v)| = 1$, then the set S obviously can not be a SLD.

(i) Then if $|I(v)| = 2$ and they are not in the same rows or columns, then $I(v) = \{(x_i,\ y_{j_1}),\ (x_{i_1},y_j)\}$ $(i_1 \neq i,\ j \neq j_1)$, there must be another vertex $u = (x_{i_1}, y_{j_1})$ such that $I(S; v) \setminus I(S; u) = \emptyset$.

(ii) If $|I(v)| \geq 2$, and $I(v)$ in the same column or row, then it is easy to see that $N(v) \cap S$ belongs to C_i or R_j.

(iii) If $|I(v)| \geq 3$, and $I(v)$ are not all in the same column or column, suppose $I(v) = \{(x_i, y_{j_1}), (x_i, y_{j_2}), (x_{i_1}, y_j)\}$, v is the only vertex in the intersection of these three three vertices.

From above, we get the idea that for each vertex that does not belong to S, it needs at least three codeword neighbours and they should not in the same column or row. In other words, in order to make each vertex not in S to have $|I(v)| \geq 3$, in each row and column there must be at least one codeword.

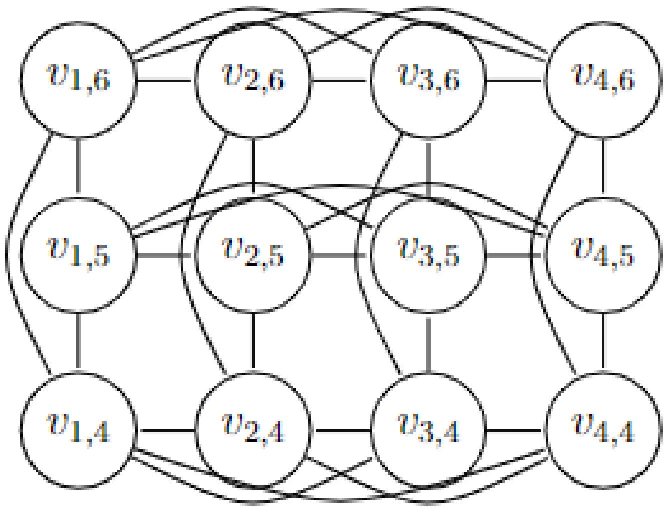

Figure 6: Rook's graph formed by G_1 and G_2.

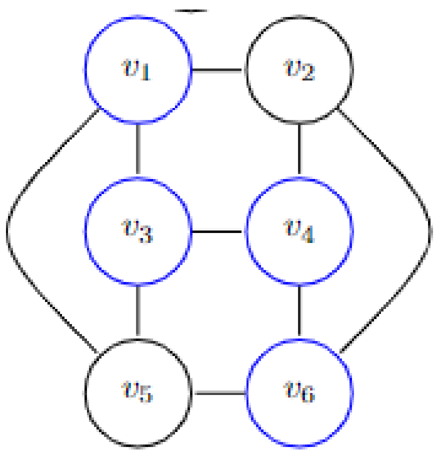

Figure 7: Optimal SLD in $K_2 \times K_3$.

For the first case, when $m \geq 2n$, because each row needs at least one code word, hence $\gamma^{SLD}(G) \geq m$. Now $S = \{(x_i, y_j)\,|i - j \equiv 0 \pmod{m}\}$ has m vertices, this implies that every row has at least one codeword, and because $m \geq 2n$, which means each column, there are at least two codewords. This implies that for each non-codeword, at least it has three codeword neighbours not all in a same row or column.

Then when $2n \geq m > n \geq 2$, if for SLD set S, $|S| \leq (2n-1)$, then $(2n-1)$ codewords means one column will have only one codeword and by $m > n$, at least two rows with only one codeword. Hence for the two vertices in the column with only

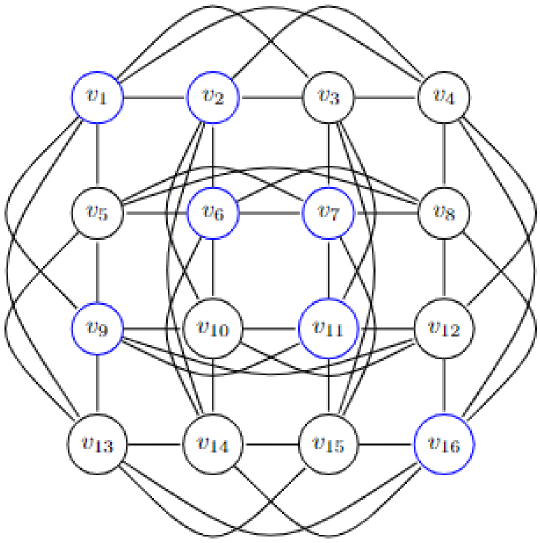

Figure 8: Optimal SLD in $K_4 \times K_4$.

one codeword and rows with only one codeword, at least one of them $u = (x_i, y_{j_1})$ and $v = (x_i, y_{j_2})$ is a non-codeword. Now suppose that $u \notin S$, then $|I(u)| = 2$ and by above (i), we get if $|S| \leq 2n - 1$, S can not be a SLD. Hence for $2n \geq m > n \geq 2$,

$$\gamma^{SLD}(G) \geq 2n.$$

We omit the proof of the upper bound, but the code S in Figure 5 ($K_2 \times K_3$) attains the lower bound $2n$ when $n = 3$.

For the third case, let $m = n > 2$ and suppose set S is an SLD with cardinality of at most $(2n - 2)$. It means that two rows and columns has only one codeword or for one row (column) with no codeword. Hence there exists a vertex u such that $|I(u)| = 2$ and by above (i) and (ii), S can not be an SLD. Hence for $m = n > 2$,

$$\gamma^{SLD}(G) \geq 2n - 1.$$

We omit the proof of the upper bound, but the code S in Figure 8($K_4 \times K_4$) is an optimal SLD with cardinality $2n - 1$, when $n = 4$.

For the last case, when $m = n = 2$, if there is only three vertices in S, then for the non-codeword u, it has only two codeword neighbours v and w and the intersection of v and w has another vertex $x \in S$, which shows that $(I(u) \cap S) \setminus (I(x) \cap S) = \emptyset$.

□

Theorem 4.2.([13])

$$\gamma^{DLD}(G)=\begin{cases} m, & m\geq 2n\geq 4, or\ n=2,\\ 2n, & 2n>m>n\geq 2,\\ 2n-1, & m=n>2,\\ m, & m>n=1.\end{cases}$$

Proof. Suppose that $|S|<m-1$, then we know that at least one row R_i that has no vertices. Now for vertex $u=(x_1,y_{R_i})$, $I(u)\cap S\subseteq C_{x_1}$. This implies for other non-codeword vertex $v\in C_{x_1}$, $(I(v)\cap S)\setminus(I(u)\cap S)=\emptyset$. Hence we get a conclusion that if if there is a column or row without codeword, then S can not be an DLD of size less than $m(n-1)$. In addition, by Corollary 3.8, $\gamma^{DLD}(G)\leq\gamma^{SLD}(G)$. For $m\geq 2n\geq 4$, we get $\gamma^{DLD}(G)\geq m$.

When $2n>m>n\geq 2$, suppose that set S is DLD with cardinality $2n-1$, which means there are at least three rows and one column with only one codeword. Let us illustrates the rows with only one codeword $\{R_1,\ R_2,...,\ R_p\}$ and the columns with one code word $\{C_1,\ C_2,...,\ C_q\}$ and codewords on them are showed as $\left(x_{h_j},y_j\right)(x_i,y_{s_i})$ $(1\leq i\leq q, 1\leq j\leq p)$, respectively. Now let a codeword $u=(x_k,y_{s_k})$ $(1\leq k\leq q)$ is in the intersection of column with one codeword and row with only one codeword. Then at least there exist one vertex $v=(x_1,y_{s_k})\notin S$. Now $I(v)\cap S=\{(x_1,y_{s_1}),(x_k,y_{s_k})\}$ and $\bigcap\limits_{s\in I(v)\cap S}N[s]=\{v,(x_k,y_{s_1})\}$. Because column C_k has only one codeword and both v and $(x_{s_k},\ y_1)$ are not in S, thus S can not be a DLD set.

From above we get for each vertex $v=(x_i,y_j)$ in the intersection of columns (C_i) and rows (R_j) and $I(v)=\{(x_i,y_{s_i}),\left(x_{h_j},y_j\right)\}$, in order to satisfy DLD definition, $(x_{h_j},\ y_{s_i})$ must belong to S. Now let letter a be the number of different rows that codewords $(x_i,\ y_{s_i})$ occupy and b showed the number of different columns codeword $(x_{h_j},\ y_j)$ occupy. Thus each intersection of them $(x_{h_j},\ y_{s_i})$ must belong to S. Meanwhile when there are columns or rows have codewords more than 2, such as c, then it is easy to see that there will be $(c-2)$ rows or columns with one codeword. Hence there are at least $(3+q+(b-2)\,a)$ rows with one codeword and at least $(1 + p + (b-2)a)$ columns with one codeword. Then we get an equality

$$p + q \geq (3 + q + (b-2)a) + (1 + p + (b-2)a),$$

which implies

$$p + q \geq p + q + 2,$$

which is a contradiction. Hence, when $2n > m > n \geq 2$, $\gamma^{DLD}(G) = 2n$.

Then for case $m = n > 2$, suppose $|S| \leq 2n - 2$, it means at least 2 rows and 2 columns with one codeword ($p, q \geq 2$). Meanwhile, $b, c \geq 1$, so we get the same equation as above. Hence when $m = n \geq 2$, $\gamma^{DLD}(G) = 2n - 1$.

When $n = 2$, if $|S| < m$, then there at least one row without codeword (R_j). Now suppose $v = (x_{h_i}, y_j)$ and we select a non-codeword $u = (x_{h_i}, y_{j_2})$ in the same column of v. It is easy to see that $I(S; u) \setminus I(S; v) = \emptyset$. This means $|S| = m$.

When $m > n = 1$, it is easy to see that the graph is complete graph, all of the vertices are connected to each other. If $|S| \leq m - 2$, there are at least two non-codewords and they have the same neighbours, which means S is not a DLD set.

□

5 Lower bound of fault-tolerant OLD in cubic graph

5.1 Cubic graph

This master thesis mainly focus on cubic graphs later. So first we will introduce some basic concepts about cubic graphs and show some known results about other locating methods in cubic graphs.

In a graph G, if all vertices in it has three neighbours, then this graph is called a cubic graph (also called a 3-regular graph). Let us denote that the number of edges incident with v by $deg\,(v)$ and the whole set of edges of the graph by E. By the handshaking lemma, the sum of the degrees must equal twice of the whole edges E,

$$\sum_{v \in V} deg(v) = 2\,|E|\,.$$

In a cubic graph, we get

$$3|V| = 2\,|E|\,.$$

So it must have an even number of vertices and there are many famous cubic graphs, such as the Petersen graph, the Heawood graph and the Möbius–Kantor graph as showed in Figure 2 and 9.

5.2 Three types of OLD

Before discussing about the open-locating dominating codes in detail, first we will introduce the basic definition and concepts of open-locating dominating codes.

A set $S \subseteq V$ is called k-distinguished if for any two vertices $u, v \in V$, $|N\,(v) \cap S \setminus N\,(u) \cap S| + |N\,(u) \cap S \setminus N\,(v) \cap S| \geq k$.

A set $S \subseteq V$ is called $k^{\#}$-distinguished if for any two vertices $u, v \in V$, $|N\,(v) \cap S \setminus N\,(u) \cap S| \geq k$ or $|N\,(u) \cap S \setminus N\,(v) \cap S| \geq k$.

Definition 5.1. A set S is called an open-locating dominating code (OLD(G)) if $N\,(v) \cap S \neq \emptyset$ for all $v \in V$ and for every vertex $u, v \in V$

$$N\,(u) \cap S \neq N\,(v) \cap S.$$

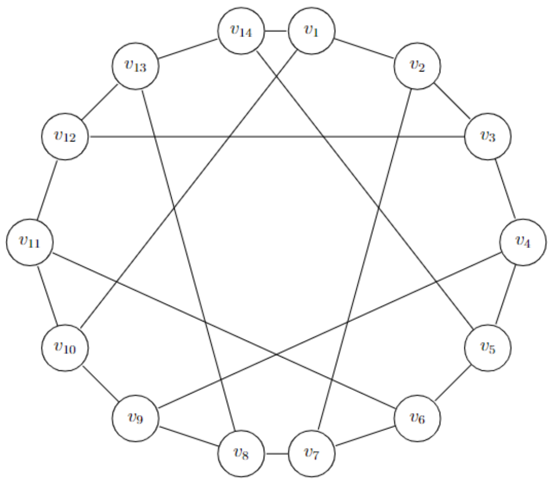

Figure 9: Heawood graph.

By the definition, a set S is an OLD if and only if every pair of vertices is 1-distinguished. OLD(G) usually denotes the optimal open-locating dominating code in a graph G and in infinite graphs, OLD%(G) represents the minimum density.

Lower bounds of several fault-tolerant OLD in cubic graphs has been widely researched. The lower bound of OLD in cubic graphs has been proved to be OLD(G) $\geq \frac{n}{2}$. When a detector has a problem and cannot transmit information to the control point, we still want to know exactly which detector has irregularity. At this time we require the redundant OLD-set. Later in order to overcome another kind of failure, device detection capability failed but is still transmitting, Slater introduced a new set - DET:OLD(G)-set and determined that under certain conditions DET:OLD(G) $\geq \frac{6n}{7}$. In addition, there is a fault-tolerant OLD that allows all transmission errors, such as the value from detector is wrong. This type of OLD set is called an error OLD-set, denoted as ERR:OLD(G). Then we would present more definitions about OLD(G), RED:OLD(G), DET:OLD(G) and ERR:OLD(G).

Theorem 5.2.([17]) If G is a cubic graph with n vertices, then

$$OLD\,(G) \geq \frac{n}{2}.$$

Theorem 5.3.([17]) In the hexagonal grid, OLD%(G) $= \frac{1}{2}$.

Proof. Because the infinite hexagonal grid is symmetrical, we can divide the infinite hexagonal grid into each small area and treat each hexagon as a rectangle, as shown

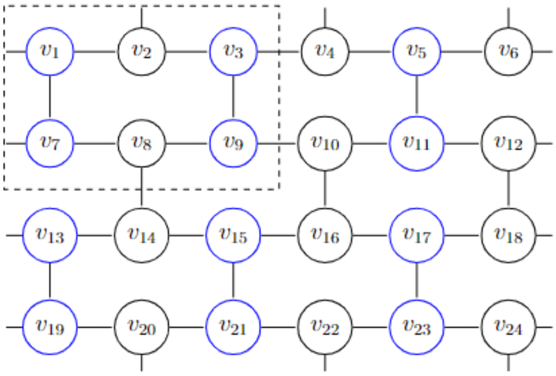

Figure 10: OLD%(G) $= \frac{1}{2}$ for the hexagonal grid.

in Figure 10. In total, the vertices in the figure are divided into blue and black. The blue vertices ensure that each vertex has unique and nonempty $N(v) \cap S$. Similarly it can also considered as black vertices uniquely located and dominated blue ones. This means that OLD%(G) $\leq \frac{1}{2}$. By theorem 5.2, we get that OLD%(G) $\geq \frac{1}{2}$.

Hence for hexagonal grid, OLD%(G) $= \frac{1}{2}$. □

Definition 5.4.A set S is called a redundant open-locating-dominating (RED:OLD) code if S is a open-locating-dominating code and for all vertex $v \in S$, $S - \{v\}$ is also an open-locating-dominating code. In infinite graph, RED:OLD%(G) represents the minimum density.

Theorem 5.5.([9], [20]) For a RED:OLD, it must hold for all vertices that, at least they should be 2-dominated and for each pair of them are 2-distinguished.

Proof. Suppose that S is a RED:OLD, now we select one vertex $u \in S$ and delete it, by definition 5.1 and 5.4, $S - \{u\}$ is still a RED:OLD and is 1-distinguished for all vertices. Hence for all the vertices that are dominated by $u \in S$, they are at least 2-dominated and each pair are 2-distinguished.

□

Theorem 5.6.([16]) If a graph G is a cubic graph and C_4-free, then the whole vertex set V is a RED:OLD-set.

Proof. Select $u, v \in V$ and because the graph is C_4-free, if distance between u and v is 1, at least, u and v have two different neighbours to each other respectively as

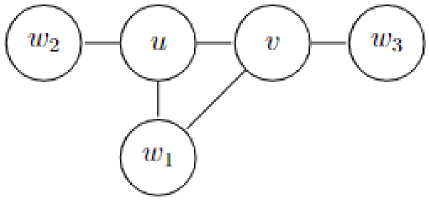

Figure 11: $d(u,v) = 1$.

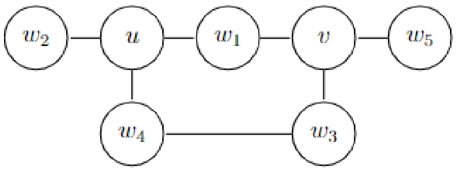

Figure 12: $d(u,v) = 2$.

showed in Figure 11. At least, u and v have two different neighbours to each other which means they are 2-distinguished. In Figure 12, we can see it is almost the same case for $d(u,v) = 2$. As for the case when $d(u,v) \geq 3$, obviously they have at least 6 different neighbours to each other. Hence, by Theorem 5.5, we get V is a RED:OLD-set of cubic graph G which is C_4-free.

$\square$

Theorem 5.7.([18]) If a graph is a r-regular graph, then RED:OLD(G) $\geq \frac{2n}{r}$ and for the infinite hexagonal grid, RED:OLD%(G) $= \frac{2}{3}$.

Proof. Let G is a r-regular graph and S is RED:OLD(G), by Theorem 5.5 for each vertex $u \in V$, $|N(u) \cap S| \geq 2$. Hence RED:OLD(G) $\geq \frac{2n}{r}$.

Now suppose G is a infinite hexagonal grid as showed in Figure 13. Let vertex set in the line x that satisfy $x \equiv 0 \; or \; 1 \pmod 3$ belong to S. Now for any vertex $u \in V$, $N(u) \cap S = \{v, w\}$. We can not find another vertex y such that $N(y) \cap S = \{v, w\}$. Hence at least v or w is distinguished from u to y. Similarly, there exist a vertex that distinguished from y to u. So RED:OLD%(G) $\leq \frac{2}{3}$ and by Theorem 5.7, we get RED:OLD%(G) $= \frac{2}{3}$.

$\square$

Definition 5.8. A set S is called detection open-locating-dominating (DET:OLD)

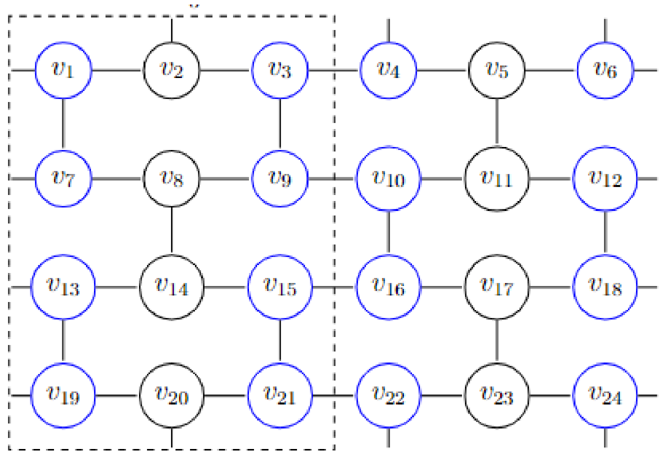

Figure 13: RED:OLD%(G) $= \frac{2}{3}$ for the hexagonal grid.

if every vertex is at least 2-dominated and each pair of them is $2^{\#}$-distinguished. DET:OLD%(G) denotes the minimum density of the subset in infinite graph.

Observation. By definitions 5.5 and 5.8, we get that if a set S is DET:OLD(G), it must also be a RED:OLD. So for r-regular graph, DET:OLD(G)$\geq$ RED:OLD(G) $\geq \frac{2n}{r}$.

Theorem 5.9.([18]) For the infinite hexagonal grid, DET:OLD%(G) $= \frac{6}{7}$.

Proof. (i) First assume that u is not in S, as illustrated in Figure 14 (a small part of the infinite hexagonal grid, vertex with black circle illustrated $v \in S$). Now u and w are two vertices that belong to $V \setminus S$. Now $N(u) \cap S = \{v_4, v_6\}$. It is clear that for the vertex that is adjacent to u, such as v_4, v_6 and w, $|N(u) \cap S \setminus N(w) \cap S| \geq 2$. For another vertex v that is not adjacent to u and not adjacent to v_4 and v_6, $|N(u) \cap S \setminus N(v) \cap S| = \{v_4, v_6\}$. Symmetrically, v also has two distinct neighbors. If v is adjacent to v_4 or v_6, such as v_2, v_7, vertices like v_2, v_7 has three neighbours in S, so for these vertices $|N(v_2) \cap S \setminus N(u) \cap S| = 2$.

(ii) If $u \in S$, it has two cases. The first case is $|N(u) \cap S| = 2$. Suppose $N(u) \cap S = \{v_1, v_2\}$, this means that for all vertices that are not adjacent to v_1 and v_2, they are $2^{\#}$-distinguished with u. For vertices in $N(v_1) \cup N(v_2)$, such as v_3 ($v_3 \neq u$), because $|N(v_3)| = 3$, this makes u is $2^{\#}$-distinguished with v_3. For the second situation, when $|N(u) \cap S| = 3$, no matter which vertex v we select, it always has $|N(u) \cap S \setminus N(v) \cap S| \geq 2$. In this part, there are 14 vertices and 12

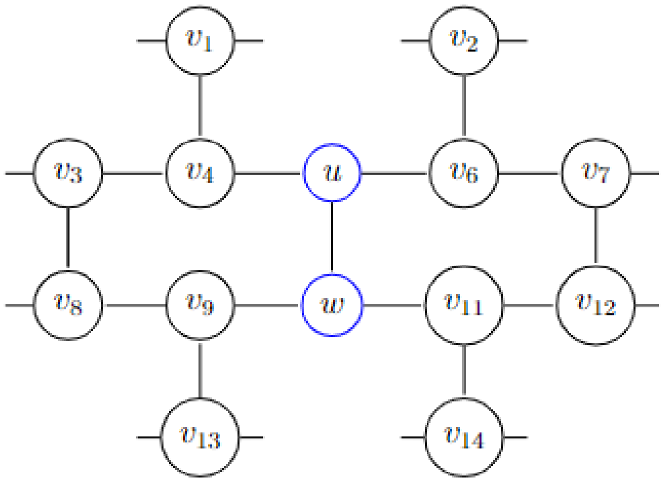

Figure 14: A small part of the infinite hexagonal.

vertices in S.

Hence, we get that for a infinite hexagonal grid, DET:OLD%(G) $\leq \frac{6}{7}$.

Now suppose $u \notin S$ and $u =$ (0,0), the three vertices adjacent to u is (1,0), (-1,0), (0,1) and they must be 2-dominated by set S, which means the six vertices (1,-1), (2,0), (-1,-1), (-2,0), (-1,1) and (1,1) must belong to S. The distance between u and these vertices is 2. In addition, as for (0,1) and (1,2), N((0,1)) = {(0,0), (-1,1), (1,1)}, because $u \notin S$ and (1,1) $\in N$((1,2)), so in order to satisfy DET:OLD, (0,2) and (2,2) must belong to S, which means vertices with a distance of 4 must belong to S as well. So DET:OLD%(G) $\geq \frac{6}{7}$.

Now we complete the proof that in hexagonal grid, DET:OLD%(G) $= \frac{6}{7}$. □

Observation. Let $N^k(v)$ denote the set of vertices x with a path of length k from x to v. If set S is an DET:OLD and $u \notin S$, $N^2(v) \subseteq S$ and $N^4(v) \subseteq S$.

Theorem 5.10.([16]) A cubic graph has an DET:OLD if and only if it is a C_4-free graph. For a C_4-free cubic graph, we have DET:OLD(G) $\geq \frac{6n}{7}$.

Proof. Suppose that there is a cycle of length 4 in the cubic graph G and the four vertices are u, v, w, x. Because $N(u) \cap N(w) = \{v, x\}$, this means $\mid N(u) \cap S \setminus N(w) \cap S \mid < 2$ and $\mid N(w) \cap S \setminus N(u) \cap S \mid < 2$.

In addition, if there is no cycle of length 4 in the graph, suppose u is adjacent to v, then let $N(u) = \{v, w, x\}$. Because G is C_4-free, v can not adjacent to either

w or x and we have $\mid N(u) \cap S \setminus N(w) \cap S \mid \geq 2$. If u is not adjacent to v, let $N(u)$ $= \{w,\ x,\ y\}$ and v can not adjacent to more then one of them. Since if v connect to more than two of them, for example, w and x, then u, v, w and x form a cycle of length 4.

By the observation, we find that if the set S is an DET:OLD and vertex $u \notin S$, vertices at distances 2 and 4 must be in S. The proof of lower bound of DET:OLD in C_4-free cubic graph is similar with proof of Theorem 5.9. The main idea is that when $u \notin S$, we need to assign at least 6 vertices in S to u to guarantee that the set S conforms to the definition.

(i) When $N(u) = \{v,\ w,\ x\} \subseteq S$, because graph is C_4-free, it is possible that there exist an edge from v to w. Now $N(v) = \{u,\ w,\ v_1\}$, $N(w) = \{u,\ v,\ w_1\}$, $N(x)$ $= \{u,\ x_1,\ x_2\}$. If y is adjacent to v_1, y must be in S, since by observation $y \in N^4(u)$ and $u \notin S$. Meanwhile for the $N(y)$, y_1 is also in S, because $y_1 \in N^4(u)$ and $u \notin S$. Hence we get $N^2(v_1) \cap (V(G) \setminus S) = \{u\}$. The same goes for w_1 and its neighbour. So we assign v_1 and w_1 to u. Now for x_1 and x_2, the first case is that if x_1 is adjacent a vertex $z \notin S$, then we consider $N(x_2)$. $N(x_2) = \{x,\ x_3,\ x_4\}$ and by $x_3, x_4 \in N^4(z)$, we get x_3 and x_4 must be in S. So $N(x_2) \subseteq S$ and $N^2(x_2) \cap (V(G) \setminus S) = \{u\}$. We can assign x_2 to u as well. Now for vertex $u \notin S$, it at least needs 6 vertices $\{v,\ w,$ $x,\ v_1,\ w_1,\ x_2\} \subseteq S$ to ensure S is an DET:OLD. The second case is that $N(x_1) \subseteq S$, then it is clear by observation, $N^2(x_1) \cap (V(G) \setminus S) = \{u\}$ and we can assign x_1 to u. The Figure 15 illustrated how it goes. If there is no edge between vertices v, w, x, then we can directly assign $\{v, w, x\}$ to u and for the left three vertices, using the same method as above, we can assign one vertex from $N(v) \cap S$, one from $N(w) \cap S$ and one from $N(x) \cap S$. Then we still can assign 6 vertices to u.

(ii) When $N(u) = \{w,\ x\} \subseteq S$, $v \notin S$ and $N(v) = \{y,\ z\} \subseteq S$, C_4-free Graph shows that w and x can not be adjacent to either y or z. Now let us denote $N(x)$ $= \{u,\ x_1,\ x_2\}$, $N(w) = \{u,\ w_1,\ w_2\}$, $N(y) = \{v,\ y_1,\ y_2\}$, $N(z) = \{v,\ z_1,\ z_2\}$. If suppose $x_1 = w_1$, vertices $\{x,\ x_1,\ w,\ u\}$ form a cycle of length 4 (contradiction). In addition if $x_1 = y_1$, then $N^4(u) = \{v\}$ and by observation v should be in S (contradiction). So all of the eight vertices $\{w_1,\ w_2,\ x_1,\ x_2,\ y_1,\ y_2,\ z_1,\ z_2\}$ are distinct. Now let $x_3 \in N(x_1)$ and $x_4 \in N(x_3)$, then x_3 and $x_4 \in S$, since by

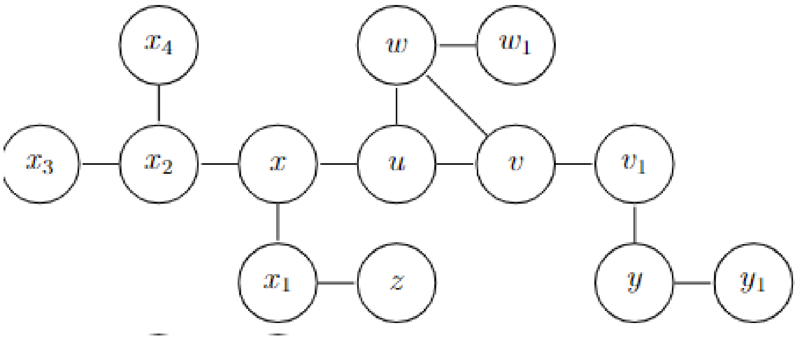

Figure 15: Graph of case (i) in proof of Theorem 5.10.

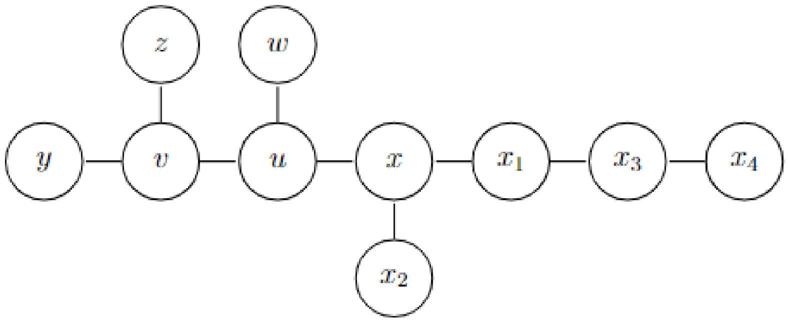

Figure 16: Graph of case (ii) in proof of Theorem 5.10.

observation, $N^4(u) = \{x_3\}$ and $N^4(v) = \{x_4\}$. Hence we get $N^2(x_1) \cap (V(G) \setminus S) = \{u\}$. By symmetry $N^2(x_2) \cap (V(G) \setminus S) = \{u\}$, $N^2(w_1) \cap (V(G) \setminus S) = \{u\}$ and $N^2(w_2) \cap (V(G) \setminus S) = \{u\}$. Hence we can assign $\{w, w_1, w_2, x, x_1, x_2\}$ to vertex u as showed in Figure 16. Correspondingly, $\{y, y_1, y_2, z, z_1, z_2\}$ is for v. Then we get the inequality

$$| S | \geq 6 | V \setminus S | .$$

In total, DET:OLD(G) for a C_4-free cubic graph with order n is $\geq \frac{6n}{7}$. □

Definition 5.11. A set S is ERR:OLD(G) of graph G if and only if for each vertex u, $N(u) \cap S \geq 3$ and any two vertices $u, v \in V$,

$$(N(u) \cap S \setminus N(v) \cap S) \cup (N(v) \cap S \setminus N(u) \cap S) \geq 3.$$

For results on ERR:OLD(G) and DET:OLD(G), see [16].

6 Lower bounds for optimal SID, SLD and DLD in cubic graphs

In order to find out the lower bounds for SID, SLD and DLD in cubic graphs, we should think about how to allocate as many non-codewords as possible to each codeword and ensure that the requirements of each locating method are met. Here we show some new results about SID, SLD and DLD. In this section, vertices with black contour means that vertices belongs to C and blue contour denotes that vertices are not in C.

6.1 Self-identifying code (SID)

Considering the definition of self-identifying codes, we have carried out the possible situation from the most extreme concerning $N(v) \cap C$, the ideal situation gradually increase to the achievable situation. In this part, C denotes a self-identifying code set of a graph G and V is the whole vertices set of G.

Theorem 6.1. Let a cubic graph G on n vertices be such that it admits a self-identifying code. We have

$$\gamma^{SID}(G) \geq \frac{2n}{3}.$$

Proof. Let $G = (V, E)$ be a cubic graph with $|V| = n$. Assume that $C \subseteq$ V is a self-identifying code in G. We have $|N(v) \cap C| \geq 2$ for every $v \in$ V. Indeed, if $|N(v) \cap C| = 1$, say $N(v) \cap C = \{c\}$, then $I(c) \setminus I(v) = \emptyset$, which is not allowed. Similarly, if $|N(v) \cap C|$ =0, then $N(w) \setminus N(v) = \emptyset$ for any $w \in N(v)$.

Let us now calculate the number N of pairs (c, v) where $c \in C$ and $v \in V$ such that $d(c, v) = 1$. Since each codeword of C has three neighbours and, and as we saw above, $|N(v) \cap C| \geq 2$ for every $v \in V$, this gives us the inequality

$$3|C| = N = \sum_{v \in V} |N(v) \cap C| \geq 2|V|,$$

and we get

$$|C| \geq \frac{2|V|}{3}.$$

□

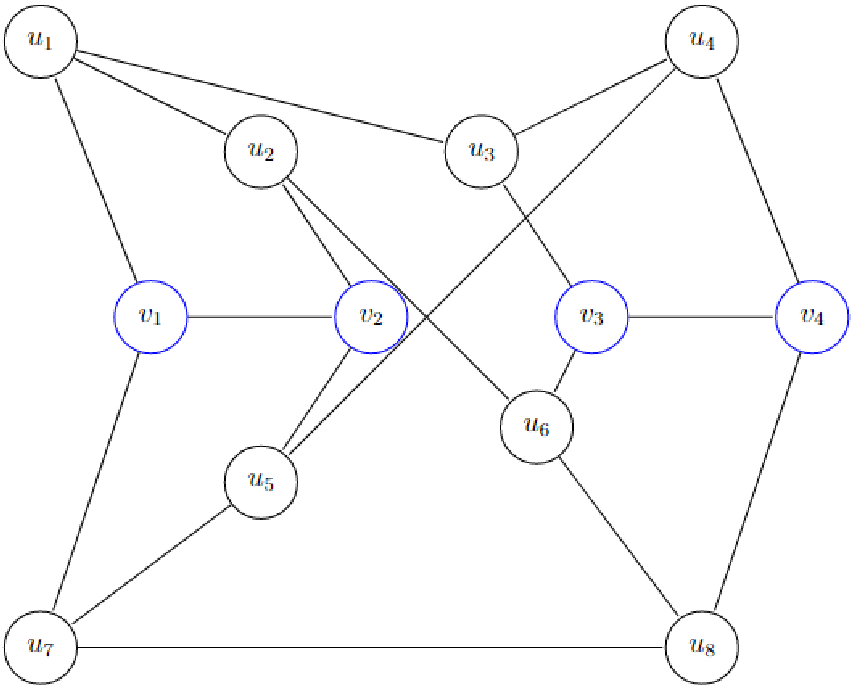

Figure 17: An optimal self-identifying code for a cubic graph with $n = 12$.

Example 6.2. In Figure 17, there are 12 vertices and $v_1, v_2, v_3, v_4 \in V \setminus C$. Each vertex has different neighbors in the set C and can be found in Table 4. All of vertices $u, v \in V$ satisfy the condition $I(C; u) \setminus I(C; v) \neq \emptyset$. Thus $S = \{u_1, u_2, \ldots, u_8\}$ is an SID of the graph G and attains the lower bound.

$I(C; v_1) = \{u_1, u_7\}$	$I(C; v_2) = \{u_2, u_5\}$
$I(C; v_3) = \{u_3, u_6\}$	$I(C; v_4) = \{u_4, u_8\}$
$I(C; u_1) = \{u_1, u_2, u_3\}$	$I(C; u_2) = \{u_1, u_2, u_6\}$
$I(C; u_3) = \{u_1, u_3, u_4\}$	$I(C; u_4) = \{u_3, u_4, u_5\}$
$I(C; u_5) = \{u_4, u_5, u_7\}$	$I(C; u_6) = \{u_2, u_6, u_8\}$
$I(C; u_7) = \{u_5, u_7, u_8\}$	$I(C; u_8) = \{u_6, u_7, u_8\}$

Table 4: Sets $I(C; v)$ for each vertex $v \in V$ in Figure 17.

6.2 Self-locating-dominating code (SLD)

In cubic graph, a graph always has a self-locating-dominating code and solid-locating-dominating code. In order to get the lower bounds of them in cubic graph, we briefly separate each vertex not in C in few cases and get the theorem.

Theorem 6.3. Let a cubic graph G on n vertices, then we have

$$\gamma^{SLD}(G) \geq \frac{2n}{5}.$$

Proof. Let $G = (V, E)$ be a cubic graph with $|V| = n$. Assume that $C \subseteq V$ is a self-locating dominating code in G. We have $|N(v) \cap C| \geq 2$ for every $v \in V \setminus C$. Indeed, if $|N(v) \cap C| = 1$, say $N(v) \cap C = \{c\}$, then $I(v) \setminus I(c) = \emptyset$, which is not allowed. Similarly, if $|N(v) \cap C| = 0$, then $N(w) \setminus N(v) = \emptyset$ for any $w \in N(v)$.

Let us now calculate the number of pairs (c, v) where $c \in C$ and $v \in V \setminus C$ such that $d(c, v) = 1$. Since each codeword of C has at most 3 non-codeword neighbours, and as we saw above, $|N(v) \cap C| \geq 2$ for every $v \in V \setminus C$, this gives us that

$$3|C| \geq \sum_{v \in V \setminus C} |N(v) \cap C| \geq 2(|V| - |C|),$$

and we get

$$|C| \geq \frac{2|V|}{5}.$$

□

Example 6.4. In Figure 18, $n = 10$, the code attains the lower bound of SLD in cubic graphs. For each vertex, its codeword neighbours are showed as in Table 5 below.

$I(C; v_1) = \{u_1, u_2\}$	$I(C; v_2) = \{u_1, u_3\}$
$I(C; v_3) = \{u_1, u_4\}$	$I(C; v_4) = \{u_2, u_3\}$
$I(C; v_5) = \{u_3, u_4\}$	$I(C; v_6) = \{u_2, u_4\}$
$I(C; u_1) = \{u_1\}$	$I(C; u_2) = \{u_2\}$
$I(C; u_3) = \{u_3\}$	$I(C; u_4) = \{u_4\}$

Table 5: Sets $I(C; v)$ for each vertex $v \in V$ in Figure 18.

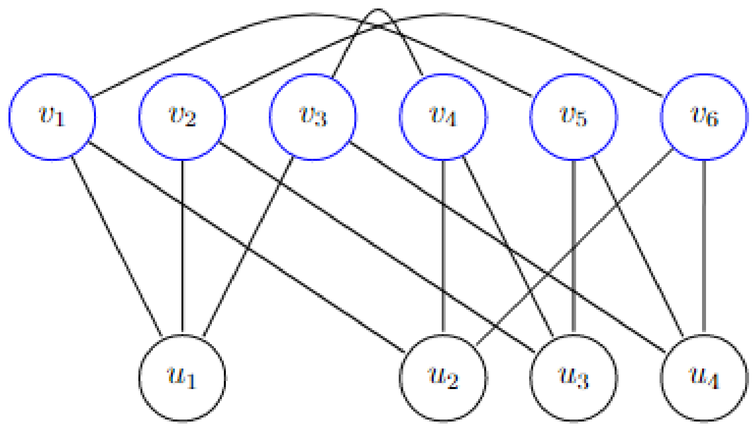

Figure 18: An optimal self-locating-dominating code for a cubic graph with $n = 10$.

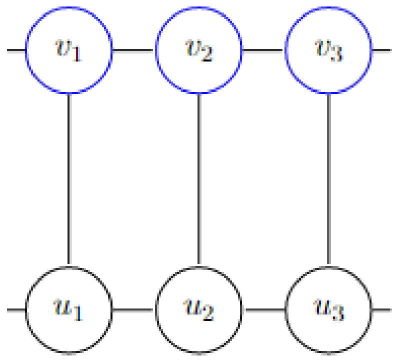

Figure 19: A new type in DLD.

6.3 Solid-locating-dominating code (DLD)

By corollary, we have $\gamma^{DLD}(G) \le \gamma^{SLD}(G)$ and $\gamma^{SLD}(G) \ge \frac{2n}{5}$, Even though SLD is more demanding codes, we show that also

$$\gamma^{DLD}(G) \ge \frac{2n}{5}.$$

Let us compare with SLD and DLD in a cubic graph. The main difference is that for DLD in a cubic graph, it is possible that for some non-codeword, it may have only one codeword neighbour, as showed in Figure 19 (Because in DLD, the comparison of adjacent nodes is limited to non-codewords). Thus we get the following theorem.

Theorem 6.5. Let a cubic graph G on n vertices, then we have

$$\gamma^{DLD}(G) \ge \frac{2n}{5}.$$

Proof. Let $G = (V, E)$ be a cubic graph with $|V| = n$. Assume that $C \subseteq V$ is a solid-locating dominating code in G. We have $|N(v) \cap C| \geq 1$ for every $v \in V \setminus C$. Indeed, if $|N(v) \cap C| = 1$, say $N(v) \cap C = \{c\}$, then $I(c) \cap (V \setminus C) = \{v\}$. While if $|N(v) \cap C| = 0$, then $N(w) \setminus N(v) = \emptyset$ for any $w \in V \setminus C$.

Let us now calculate the pairs (c, v) where $c \in C$ and $v \in V \setminus C$ and $N(v) \cap C = \{c\}$, denote the corresponding subsets by C_1 and V_1, such that $d(c, v) = 1$ and it is obvious that $|C_1| = |V_1|$. This gives us that

(i) If there exists $v \in V \setminus C$ and $|N(v) \cap C| = 1$, then we get

$$3|C| - 2|C_1| \geq \sum_{v \in V \setminus C} |N(v) \cap C| \geq 2(|V - C|) - |V_1|,$$

$$5|C| \geq 2|V| + |C_1|,$$
$$|C| \geq \frac{2|V| + |C_1|}{5}.$$

And because $|C_1| > 0$, for this case, $|C| > \frac{2n}{5}$.

(ii) If for all vertices $v \in V \setminus S$, $|N(v) \cap C| \geq 2$. Then we get

$$3|C| \geq \sum_{v \in V \setminus C} |N(v) \cap C| \geq 2(|V| - |C|),$$

$$|C| \geq \frac{2n}{5}.$$

Hence from above two cases, we get

$$|C| \geq \frac{2|V|}{5}.$$

$\square$

Now we find that $\{u_1, u_2, u_3, u_4\}$ in Figure 18 is not only the optimal self-locating-dominating code but also the optimal solid-locating dominating code that attains the lower bound.

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