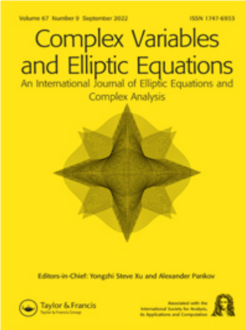

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The Ptolemy–Alhazen problem and quadric surface mirror reflection

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ABSTRACT

We discuss the problem of the reflection of light on spherical and quadric surface mirrors. In the case of spherical mirrors, this problem is known as the Alhazen problem. For the spherical mirror problem, we focus on the reflection property of an ellipse and show that the catacaustic curve of the unit circle follows naturally from the equation obtained from the reflection property of an ellipse. Moreover, we provide an algebraic equation that solves Alhazen's problem for quadric surface mirrors.

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1. Introduction

Alhazen's problem [1, p.1010] is the problem that asks the following: *Given a light source and a spherical mirror, find the point on the mirror where the light will be reflected to the eye of an observer*. This problem was first formulated by Ptolemy in 150 AD and is, therefore, also called the Ptolemy–Alhazen problem. We call the reflection point of this problem the *PA-point*.

This problem is equivalent to solving the following problem for a disk. For given two points $z_1,\ z_2 \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, find $u \in \partial\mathbb{D}$ such that

$$|\angle(z_1, u, 0)| = |\angle(0, u, z_2)|.$$

Many mathematicians and researchers of geometrical optics have investigated this problem. For a short history of this topic, see [1]. Some of the recent studies on this topic, from the point of astrophysics and signal transmission, are [2,3]. The bibliographies of [4,5] include several pointers to the literature.

As above, this problem has a long history, but it has finally been solved algebraically only recently. Elkin [6] found in 1965 an equation of degree 4 solving this problem.

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His strategy was to find the PA-point as the intersection of the unit circle and a circle centred at z_1. In this paper, we will study algebraic equations that solve reflection problems on spherical and quadric surfaces.

In [4], we studied Alhazen's problem and its relation to the triangular ratio metric s_G of a given domain $G \subset \mathbb{C}$ defined as (see, for instance [7–10] and [11] for other studies on the triangular ratio metric)

$$s_G(z_1, z_2) = \sup_{z \in \partial G} \frac{|z_1 - z_2|}{|z_1 - z| + |z - z_2|}, \quad z_1, z_2 \in G. \tag{1}$$

In [4], we discussed an equation that uses the reflection property of an ellipse.

Solutions to Alhazen's problems for quadric surfaces have also found applications in many other fields besides mathematics. Taguchi et al. [12] and Agrawal et al. [13] studied Alhazen's problem for application to a camera with quadric-shaped mirrors. They constructed an equation with six roots that include the PA-points in [13]. In [3], Miller et al. studied an equation solving Alhazen's problem and proposed a fast method for choosing the correct point from the roots. In addition, they mentioned that if their method could be extended to the case of elliptical surfaces, it could be useful for GPS communication, and could also be applied to computer rendering. This motivated us to construct an algebraic equation that yields the PA-point for quadric surfaces.

This paper is organized as follows. In Section 2, we discuss the relation between the equation using the reflection property of an ellipse and the catacaustic curve of the circle. We can also use the properties of the circle of Apollonius to construct an equation that solves Alhazen's problem. This equation is studied in Section 3. In Section 4, we discuss Alhazen's problem for quadric surfaces and provide an equation $F_4 = 0$ that gives the PA-points in Theorem 4.3. This equation is different from the one formulated by Agrawal et al. in [13, Section 2]. In fact, the algebraic equation $F_4 = 0$ is obtained by using the reflective property of an ellipse and a method based on algebraic geometry. Moreover, using Theorem 4.3, we give the calculation method of the triangular ratio metric on conic domains in Theorem 4.5. The application to the calculation of the triangular ratio metric is also discussed in Section 5.

In this paper, several symbolic computation systems are used effectively. For graphics, we use GeoGebra ,[1] dynamic mathematics software, to create Figures 2, 3 and 4, whereas Figures 1 and 5 are drawn using Mathematica[2] and Risa/Asir ,[3] symbolic computation systems, respectively. We also use the techniques of computer algebra such as resultant to obtain some target equations. In particular, we use Risa/Asir to obtain the result of Theorem 4.3, which is difficult to calculate manually.

2. Alhazen's problem on a disk –Solution using ellipses

In this section, we consider Alhazen's problem on the unit disk.

For $z_1,\ z_2 \in \mathbb{D}$, let z be the PA-point on $\partial\mathbb{D}$ with respect to these two points.

2.1. Solution using ellipses

Using the reflective property of an ellipse, the PA-point can be found as the points of tangency of an ellipse with foci $z_1,\ z_2$ and the unit circle (see the left figure in Figure 2).

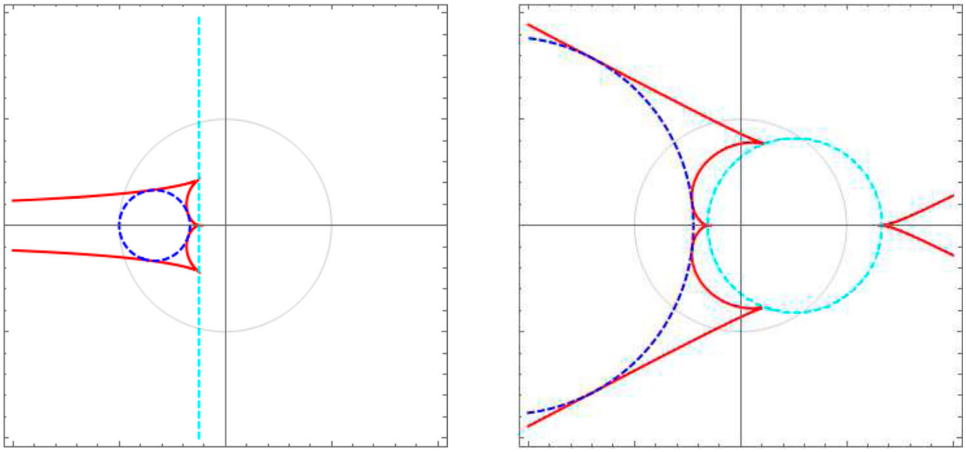

Figure 1. The catacaustics of the unit circle with radiant points $c = 0.5$ (left) and $c = 0.8$ (right). The thick curves indicate the catacaustics. The thick and thin dotted circles represent $E_1(c, z) = 0$ and $E_2(c, z) = 0$, respectively.

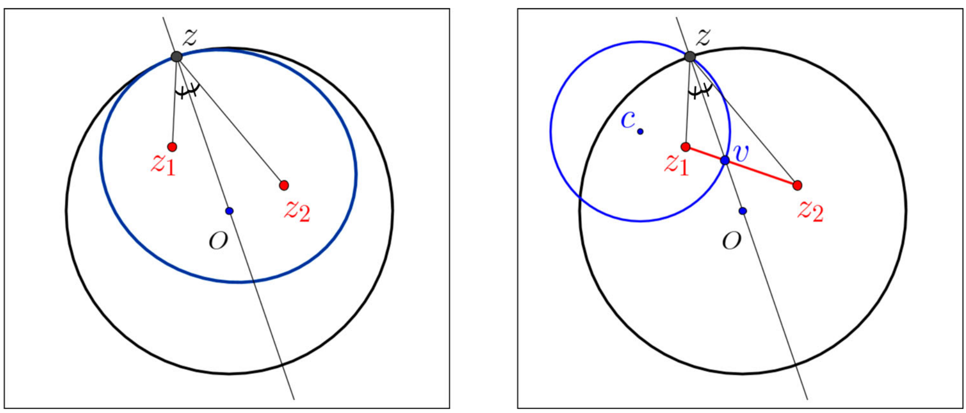

Figure 2. Solution using an ellipse (left). Solution using the circle of Apollonius (right).

Lemma 2.1: (See, e.g. [4, Theorem 1.1])

For $z_1, z_2 \in \mathbb{D}$, the PA-point z is given as a solution of the equation

$$\overline{z_1 z_2} z^4 - (\overline{z_1} + \overline{z_2}) z^3 + (z_1 + z_2) z - z_1 z_2 = 0. \tag{2}$$

This lemma is also valid for the external reflection, i.e. Equation (2) holds for z_1, $z_2 \in \mathbb{C} \setminus \overline{\mathbb{D}}$ if the line segment $[z_1, z_2]$ has no intersection with $\partial\mathbb{D}$. Note that if $[z_1, z_2] \cap \partial\mathbb{D} \neq \emptyset$, the light is blocked by the boundary (mirror) and never reaches the observer. Moreover, in this case, $\sup_{z \in \partial\mathbb{D}} \{|z_1 - z_2| / (|z_1 - z| + |z - z_2|)\} = 1$.

2.2. Number of roots and catacaustic of a circle

A root $u \in \partial\mathbb{D}$ of the equation

$$P_{z_1, z_2}(u) = \overline{z_1 z_2} u^4 - (\overline{z_1} + \overline{z_2})\, u^3 + (z_1 + z_2)\, u - z_1 z_2 = 0$$

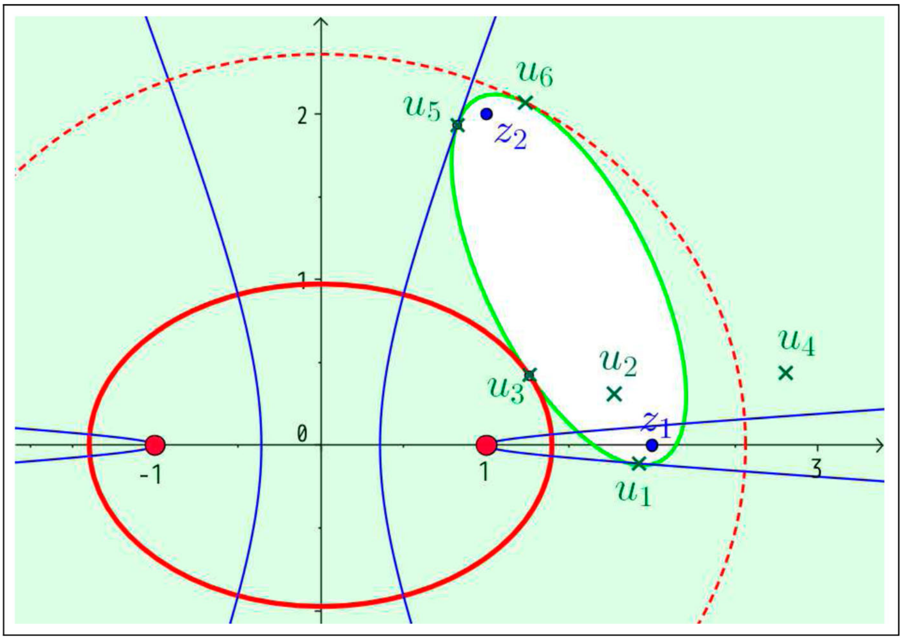

Figure 3. The leaning ellipse indicates C. The points u_3 and u_6 are tangent points of the thick and dotted ellipses with C, respectively. The foci of these thick and dotted ellipses are both $-1, 1$. Here, the points u_1 and u_5 are tangent points of C and the hyperbolas with foci $-1, 1$.

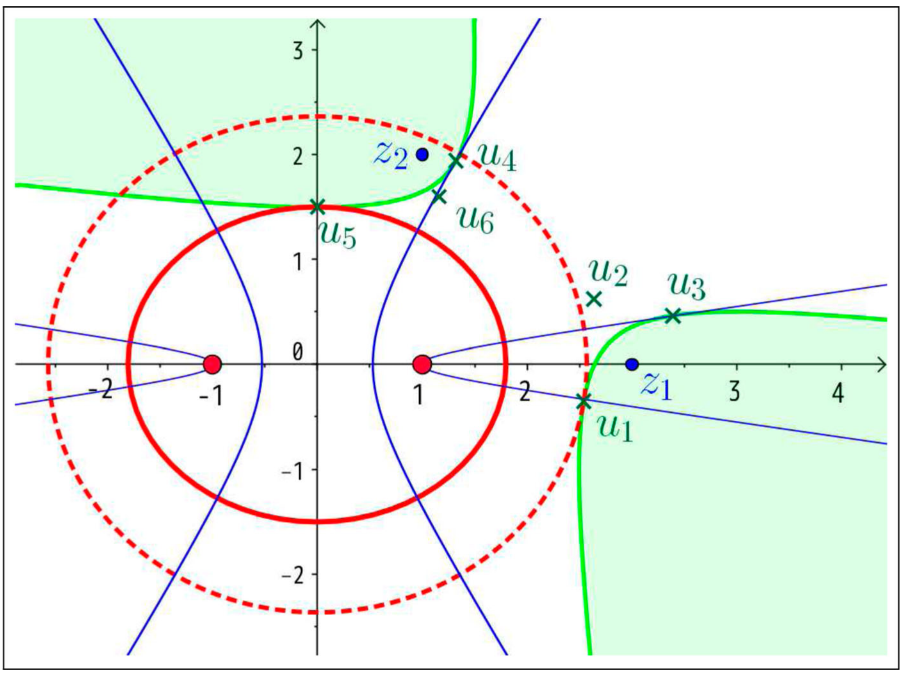

Figure 4. The leaning hyperbola indicates C. The two points u_1 and u_5 are tangent points of the dotted and thick ellipses with C, respectively. The foci of these dotted and thick ellipses are both $-1, 1$. Moreover, the two points u_3 and u_4 are tangent points of C and the hyperbolas with foci $-1, 1$.

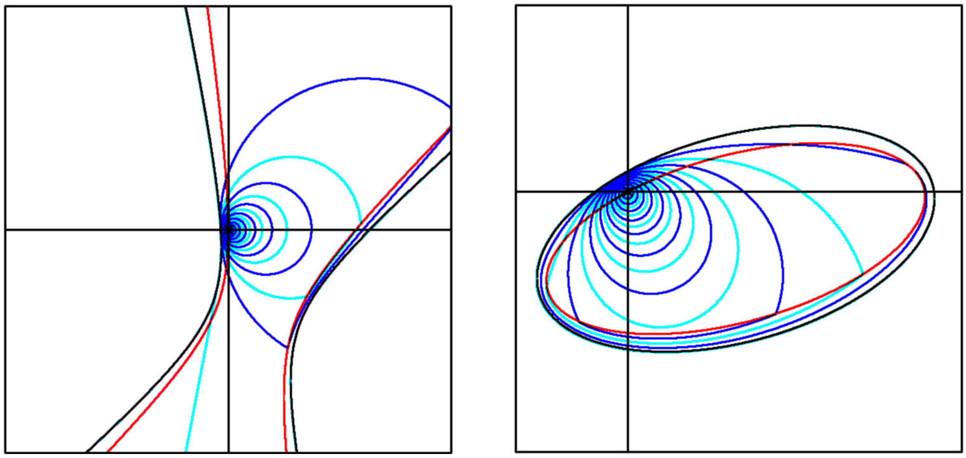

Figure 5. The level sets of $G = \{||z - (-1/2 - 1/2i)| - |z - (1 - i)|| < 4/5\}$ (left) and $G = \{|z - 3/2| + |z - (-1/3 - 1/2i)| < 11/5\}$ (right). Note that each red curve passes through all edge points of the contour curves.

is a PA-point if and only if

$$\mathrm{Re}\left(\overline{z_1 z_2} u^2 - (\overline{z_1} + \overline{z_2})\, u\right) + 1 > 0,$$

as follows from [4, Lemma 3.1]. If $z_1, z_2 \in \mathbb{D}\backslash\{0\}$, then the above inequality holds whenever $u \in \partial\mathbb{D}$; therefore, all the roots of $P_{z_1,z_2} = 0$ that lie on the unit circle, called *unimodular* roots, are PA-points.

If $z_1 = 0$ and $z_2 = |z_2|e^{i\alpha} \neq 0$, then the roots of (2) are 0 and $\pm e^{i\frac{\alpha}{2}}$. In the following, we will assume $z_1, z_2 \in \mathbb{C}\backslash\{0\}$, therefore (2) is a quartic equation.

The equation $P_{z_1,z_2} = 0$ has always at least two distinct unimodular roots [4, Lemma 2.4] and both cannot have multiplicity 2 [4, Lemma 4.1]. Moreover, $P_{z_1,z_2} = 0$ has four simple unimodular roots if $z_1, z_2 \in \mathbb{C}\backslash\mathbb{D}$. If $P_{z_1,z_2} = 0$ has a triple root a and a simple root b, then $|a| = 1$ and $b = -a$ [4, Lemma 4.3]. We characterize in terms of z_1 and z_2 all the possible cases for the number of unimodular roots of $P_{z_1,z_2} = 0$ and their multiplicities, both algebraically and geometrically. Various approaches to particular cases of this problem are scattered through the literature [4,14,15].

We will denote by $D(P)$ the discriminant of the complex polynomial P.

Proposition 2.2: *Let $z_1, z_2 \in \mathbb{C}\backslash\{0\}$ and*

$$P_{z_1,z_2}(u) = \overline{z_1 z_2} u^4 - (\overline{z_1} + \overline{z_2})\, u^3 + (z_1 + z_2)\, u - z_1 z_2.$$

Then

(1) $P_{z_1,z_2} = 0$ *has four simple unimodular roots if and only if* $D(P_{z_1,z_2}) < 0$;
(2) $P_{z_1,z_2} = 0$ *has two simple unimodular roots and two distinct roots off the unit circle if and only if* $D(P_{z_1,z_2}) > 0$;
(3) $P_{z_1,z_2} = 0$ *has at least one multiple root if and only if* $D(P_{z_1,z_2}) = 0$.

Moreover, in case (3) *all the roots of* $P_{z_1,z_2} = 0$ *are unimodular and* $P_{z_1,z_2} = 0$ *has either a double root and two simple roots, or a triple root* v *and* $-v$ *as a simple root.*

Proof: Denote $s = z_1 + z_2 = s_1 + is_2$ and $p = z_1 z_2 = p_1 + ip_2$, where s_1, s_2 and p_1, p_2 are real numbers. Then $P_{z_1,z_2}(u) = \overline{p}u^4 - \bar{s}u^3 + su - p$.

As in [14], using the substitution $u = (1 + it)/(1 - it)$ we see that (2) turns into an algebraic equation with real coefficients. We have

$$P_{z_1,z_2}\left(\frac{1+it}{1-it}\right) = \frac{(-2i)}{(1-it)^4} Q_{z_1,z_2}(t), \quad t \in \mathbb{C} \setminus \{-i\}$$

where

$$Q_{z_1,z_2}(t) = \left(s_2 + p_2\right) t^4 + 2\left(s_1 + 2p_1\right) t^3 - 6p_2 t^2 + 2\left(s_1 - 2p_1\right) t - \left(s_2 - p_2\right).$$

There is a one-to-one correspondence between the unimodular zeros of $P_{z_1,z_2} = 0$ different from (-1) and the real roots of $Q_{z_1,z_2} = 0$, as $P_{z_1,z_2}(e^{i\varphi}) = 0$ if and only if $Q_{z_1,z_2}(\tan \frac{\varphi}{2}) = 0$, where $\varphi \in (-\pi, \pi)$.

Similarly, we consider the reciprocal polynomials $P^*_{z_1,z_2}$ and $Q^*_{z_1,z_2}$ of P_{z_1,z_2} and Q_{z_1,z_2}, respectively:

$$P^*_{z_1,z_2}(u) = -pu^4 + su^3 - \bar{s}u + \overline{p},$$
$$Q^*_{z_1,z_2}(t) = -\left(s_2 - p_2\right) t^4 + 2\left(s_1 - 2p_1\right) t^3 - 6p_2 t^2 + 2\left(s_1 + 2p_1\right) t + \left(s_2 + p_2\right).$$

We have $P^*_{z_1,z_2}((t+i)/(t-i)) = (-2i)/(t-i)^4 Q^*_{z_1,z_2}(t)$, $t \in \mathbb{C}\setminus\{i\}$. There is a one-to-one correspondence between the unimodular zeros of $P^*_{z_1,z_2} = 0$ different from 1 and the real roots of $Q^*_{z_1,z_2} = 0$, as $P_{z_1,z_2}(e^{i\varphi}) = 0$ if and only if $Q^*_{z_1,z_2}(\cot \frac{\varphi}{2}) = 0$, where $\varphi \in (0, 2\pi)$.

Note that $Q^*_{z_1,z_2}$ is a quartic polynomial if and only if $s_2 - p_2 \neq 0$, which is equivalent to $P_{z_1,z_2}(1) \neq 0$.

We discuss the number of real roots of $Q_{z_1,z_2} = 0$ and $Q^*_{z_1,z_2} = 0$. It is known that the discriminant of a quartic equation with real coefficients is positive if and only if the equation has four simple roots that are either all real or two pairs of complex conjugates (see, e.g. [16]).

Case 1. Assume that $s_2 + p_2 \neq 0$. Then $P_{z_1,z_2}(-1) \neq 0$ and Q_{z_1,z_2} is a quartic polynomial. The discriminant of the polynomial P_{z_1,z_2} is the real number

$$D(P_{z_1,z_2}) = \frac{1}{27}\left(4I^3 - J^2\right),$$

where $I = -12|p|^2 + 3|s|^2$ and $J = -27(s^2\overline{p} - \bar{s}^2 p)$, hence

$$\begin{aligned} D\left(P_{z_1,z_2}\right) &= 4\left(|s|^2 - 4\left|p\right|^2\right)^3 - 27\left(s^2\overline{p} - \bar{s}^2 p\right)^2 \\ &= 4\left|s\right|^6 + 6\left|s\right|^4\left|p\right|^2 + 192\left|s\right|^2\left|p\right|^4 - 256\left|p\right|^6 - 54\mathrm{Re}\left(\bar{s}^4 p^2\right). \end{aligned} \tag{3}$$

The above formula appears in [14]. Using the properties of the discriminant of a binary form as a projective invariant [17, Definition 2.2] it follows that $D(Q_{z_1,z_2}) = -64D(P_{z_1,z_2})$, see also [14].

Since $P_{z_1,z_2} = 0$ has at least two distinct unimodular roots, $Q_{z_1,z_2} = 0$ has at least two distinct real roots. The discriminant $D(Q_{z_1,z_2}) = 0$ if and only $Q_{z_1,z_2} = 0$ has multiple roots. $D(Q_{z_1,z_2}) > 0$ if and only if all the roots of $Q_{z_1,z_2} = 0$ are real, since the equation $Q_{z_1,z_2} = 0$ can have at most 2 non-real roots. Then $D(Q_{z_1,z_2}) < 0$ if and only if $Q_{z_1,z_2} = 0$ has two distinct real roots and two conjugated non-real roots. For the relation between the roots of a quartic equation and the discriminant, see also [16]. We obtain the following:

(i) $P_{z_1,z_2} = 0$ has four simple unimodular roots if and only if all the roots of $Q_{z_1,z_2} = 0$ are real, which is equivalent to $D(Q_{z_1,z_2}) > 0$, i.e. to $D(P_{z_1,z_2}) < 0$.
(ii) $P_{z_1,z_2} = 0$ has two simple unimodular roots and two distinct roots off the unit circle if and only if $Q_{z_1,z_2} = 0$ has two distinct real roots and two non-real roots, which is equivalent to $D(Q_{z_1,z_2}) < 0$, i.e. to $D(P_{z_1,z_2}) > 0$.
(iii) $P_{z_1,z_2} = 0$ has at least one multiple root if and only if $D(P_{z_1,z_2}) = 0$. Assuming that $P_{z_1,z_2} = 0$ has at least one multiple root the claim is obtained by [4, Lemmas 4.2 and 4.3].

Case 2. Assume that $s_2 - p_2 \neq 0$. Then $P_{z_1,z_2}(1) = P^*_{z_1,z_2}(1) \neq 0$ and $Q^*_{z_1,z_2}$ is a quartic polynomial.

We have $D(P_{z_1,z_2}) = D(P^*_{z_1,z_2})$ and the discriminants of $P^*_{z_1,z_2}$ and $Q^*_{z_1,z_2}$ are related by $D(Q^*_{z_1,z_2}) = -64D(P^*_{z_1,z_2})$.

The discussion continues as in Case 1, replacing P_{z_1,z_2} by $P^*_{z_1,z_2}$ and Q_{z_1,z_2} by $Q^*_{z_1,z_2}$.

Case 3. The remaining case $s_2 = p_2 = 0$. We can assume that s and p are real numbers. In this case $P_{z_1,z_2}(u) = (u-1)(u+1)(pu^2 - su + p)$. The polynomial P_{z_1,z_2} has only real coefficients and $D(P_{z_1,z_2}) = 4(s^2 - 4p^2)^3$. The roots of $P_{z_1,z_2} = 0$ are 1, (-1) and the roots of $R(u) = pu^2 - su + p$.

If $D(P_{z_1,z_2}) > 0$, i.e. $|s| > 2|p|$, then the roots of $R = 0$ are $(s \pm \sqrt{s^2 - 4p^2})/(2p)$ and cannot be unimodular.

If $|s| < 2|p|$, then the roots $(s \pm i\sqrt{4p^2 - s^2})/(2p)$ of $R = 0$ are unimodular.

If $|s| = 2|p|$, then the double root $s/(2p)$ of $R = 0$ belongs to $\{\pm 1\}$.

Claims (1), (2) and (3) follow straightforward in this case. Moreover, if $D(P_{z_1,z_2}) = 0$, then P_{z_1,z_2} has a triple root and one simple root. ■

Using Proposition 2.2, we get a purely algebraic proof of [4, Proposition 4.5], whose original proof used Complex Analysis arguments and results on self-inversive polynomials.

Corollary 2.3: *Let $z_1, z_2 \in \mathbb{C}\backslash\{0\}$. Denote*

$$P_{z_1,z_2}(u) = \overline{z_1 z_2} u^4 - (\overline{z_1} + \overline{z_2})\, u^3 + (z_1 + z_2)\, u - z_1 z_2,$$
$$E_1(z_1, z_2) = |z_1 + z_2| - |z_1 z_2| \quad \text{and} \quad E_2(z_1, z_2) = |z_1 + z_2| - 2\,|z_1 z_2|\,.$$

Assume that $\mathrm{Im}((z_1 + 1)(z_2 + 1)) \neq 0$.

(a) *If $E_2(z_1, z_2) > 0$, then $P_{z_1,z_2} = 0$ has two distinct unimodular roots and two roots off the unit circle.*
(b) *If $E_1(z_1, z_2) < 0$, then $P_{z_1,z_2} = 0$ has four distinct unimodular roots.*
(c) *If $P_{z_1,z_2} = 0$ has four distinct unimodular roots, then $E_2(z_1, z_2) < 0$.*

(d) *If* $P_{z_1,z_2} = 0$ *has two distinct unimodular roots and two roots off the unit circle, then* $E_1(z_1, z_2) > 0$.

Proof: Denote $s = z_1 + z_2$ and $p = z_1 z_2$.

(a) From (3), we have

$$D\left(P_{z_1,z_2}\right) = 4\left(|s|^2 - 4\left|p\right|^2\right)^3 + 108\left(\mathrm{Im}\left(s^2\overline{p}\right)\right)^2. \tag{4}$$

If $E_2(z_1, z_2) > 0$, then $|s|^2 - 4|p|^2 > 0$, hence $D(P_{z_1,z_2}) > 0$ and by Proposition 2.2 (2) $P_{z_1,z_2} = 0$ has two distinct unimodular roots and two roots off the unit circle.

(b) We may write

$$D(P_{z_1,z_2}) = 4\left(|s|^2 - \left|p\right|^2\right)\left(|s|^2 + 8\left|p\right|^2\right)^2 - 54\left(|s|^4\left|p\right|^2 + \mathrm{Re}\left(s^4\overline{p}^2\right)\right).$$

Note that $|s|^4|p|^2 = |\overline{s}^4 p^2| \geq -\mathrm{Re}(s^4\overline{p}^2)$. If $E_1(z_1, z_2) < 0$, then $|s|^2 - |p|^2 < 0$, therefore $D(P_{z_1,z_2}) < 0$ and by Proposition 2.2 (1) $P_{z_1,z_2} = 0$ has four distinct unimodular roots.

(c) Assume that $P_{z_1,z_2} = 0$ has four distinct unimodular roots. By Proposition 2.2 (1) $D(P_{z_1,z_2}) < 0$, but

$$4\left(E_2(z_1, z_2)\right)^2(|s| + 2\left|p\right|)^3 = D(P_{z_1,z_2}) - 108\left(\mathrm{Im}(s^2\overline{p})\right)^2 < 0,$$

therefore, $E_2(z_1, z_2) < 0$.

(d) Assume that $P_{z_1,z_2} = 0$ has two distinct unimodular roots and two roots off the unit circle. By Proposition 2.2 (b) $D(P_{z_1,z_2}) > 0$, but

$$4E_1\left(z_1, z_2\right)\left(|s| + \left|p\right|\right)\left(|s|^2 + 8\left|p\right|^2\right)^2 = D\left(P_{z_1,z_2}\right) + 54\left(|s|^4\left|p\right|^2 + \mathrm{Re}(s^4\overline{p}^2)\right) > 0;$$

therefore, $E_1(z_1, z_2) > 0$. ■

Remark 2.4: The domain $E_1(z_1, z) < 0$ is the interior of the circle $E_1(z_1, z) = 0$ if $0 < |z_1| < 1$, the exterior of the circle $E_1(z_1, z) = 0$ if $|z_1| > 1$ and a half-plane not containing the origin for $|z_1| = 1$. The domain $E_2(z_1, z) > 0$ is the exterior of the circle $E_2(z_1, z) = 0$ if $|z_1| < \frac{1}{2}$, the interior of the circle $E_1(z_1, z) = 0$ if $|z_1| > \frac{1}{2}$ and a half-plane not containing the origin for $|z_1| = \frac{1}{2}$.

Next, we will give geometrical interpretations of Proposition 2.2 and [4, Proposition 4.5] relying on the catacaustic of the unit circle with radiant point z_1. Recall that the catacaustic of a curve is the envelope of the family of the reflected rays by that curve, for a light source at a given point, called the radiant point. In [15], Drexler and Gander studied the equation $Q_{z_1,z_2}(t) = 0$ [14, Equation (9)] for $z_1, z_2 \in \mathbb{D}$, assuming without loss of generality that $z_1 = c \in (-1, 0)$. They determined the parametrical equations of a curve, called separatrix, bounding the regions that correspond to the cases when the polynomial equation $Q_{z_1,z_2} = 0$ has two simple roots, respectively, four simple roots. If z_2 is on the separatrix, then $Q_{z_1,z_2} = 0$ has only real roots, either one double root and two simple roots (if z_2 is a regular point) or a triple root and one simple root (if z_2 is a cusp). In [15], the separatrix was represented for six particular values of the parameter c and highlighted through computer and optical experiments. One can see that for a fixed $z_1 = c \in (-1, 0)$ the separatrix from

[15] is the catacaustic of the unit circle with the radiant point z_1, denoted in the following by K_{z_1}.

Our graphical experiments have shown that the (generalized) circle given by $E_1(z_1, z) := |z_1 + z| - |z_1 z| = 0$ is tangent to the catacaustic K_{z_1} and that the (generalized) circle given by $E_2(z_1, z) := |z_1 + z| - 2|z_1 z| = 0$ intersects K_{z_1} exactly at the cusps of K_{z_1} (see, Figure 1). In the following, we will explain the results of these experiments using the connection between the discriminant of the polynomial P_{z_1,z_2} and the resultant $\mathrm{resul}_u(P_{z_1,z_2}, P'_{z_1,z_2})$.

The equation of the reflected ray for the incident ray passing through z_1, with the reflection point $u = e^{i\varphi}$, is

$$\begin{vmatrix} z & \overline{z} & 1 \\ u & 1/u & 1 \\ \overline{z_1}u^2 & z_1/u^2 & 1 \end{vmatrix} = 0,$$

which is equivalent to

$$F_{z_1}(z, \overline{z}, u) := (u - z_1)z + u^3\,(\overline{z_1}u - 1)\,\overline{z} - u(\overline{z_1}u^2 - z_1) = 0.$$

Note that $F_{z_1}(z, \overline{z}, u) = P_{z_1,z}(u)$ for all $u, z_1, z \in \mathbb{C}$.

The envelope of the family of curves $F_{z_1}(z, \overline{z}, u) = 0$, $u \in \partial\mathbb{D}$ is the catacaustic of the circle, with radiant point z_1, which will be denoted by K_{z_1}. The standard equations of K_{z_1} are

$$\begin{cases} F_{z_1}(z, \overline{z}, u) = 0, \\ \dfrac{\partial}{\partial u} F_{z_1}(z, \overline{z}, u) = 0, \end{cases} \quad u \in \partial\mathbb{D}. \tag{5}$$

Solving (5) as a linear system with respect to z and $\overline{z}$, it follows that

$$z = \frac{u(\overline{z_1}^2 u^3 - 3\,|z_1|^2\,u + 2z_1)}{3\overline{z_1}u^2 - 2(2\,|z_1|^2 + 1)u + 3z_1} \quad \text{and} \quad \overline{z} = \frac{2\overline{z_1}u^3 - 3\,|z_1|^2\,u^2 + z_1^2}{u^2\left(3\overline{z_1}u^2 - 2(2\,|z_1|^2 + 1)u + 3z_1\right)}.$$

The parametric equations of the catacaustic, obtained from the above equalities for $u = e^{i\varphi}$, are

$$\begin{cases} x = \dfrac{1}{2}\dfrac{\mathrm{Re}(\overline{z_1}^2 u^3 - 3\,|z_1|^2\,u + 2z_1)}{3\mathrm{Re}\,(\overline{z_1}u) - (2\,|z_1|^2 + 1)}, \\ y = \dfrac{1}{2}\dfrac{\mathrm{Im}(\overline{z_1}^2 u^3 - 3\,|z_1|^2\,u + 2z_1)}{3\mathrm{Re}\,(\overline{z_1}u) - (2\,|z_1|^2 + 1)}, \end{cases} \quad \text{where } u = e^{i\varphi} \quad \text{and} \quad z = x + iy.$$

Due to rotation invariance, we may assume that z_1 is a positive number. In the case when $z_1 = c$ is a positive real number, we get the well-known parametric equations of the catacaustic of the circle with a radiant point on the positive real axis [18]:

$$\begin{cases} x = \dfrac{c}{2}\dfrac{c\cos 3\varphi - 3c\cos\varphi + 2}{3c\cos\varphi - (2c^2 + 1)}, \\ y = \dfrac{c}{2}\dfrac{c\sin 3\varphi - 3c\sin\varphi}{3c\cos\varphi - (2c^2 + 1)}. \end{cases}$$

System (5) is equivalent to $\mathrm{resul}_u(F_{z_1}(z, \overline{z}, u), \frac{\partial}{\partial u} F_{z_1}(z, \overline{z}, u)) = 0$, which is an implicit equation of the catacaustic K_{z_1}.

Since $F_{z_1}(z,\overline{z},u) = P_{z_1,z}(u)$ for all $u, z_1, z \in \mathbb{C}$, using the connection between the discriminant of a polynomial $f(x)$ and the resultant $\text{resul}_x(f,f')$, we see that

$$D\left(P_{z_1,z}\right) = \frac{1}{\overline{\overline{z_1 z}}}\ \text{resul}_u\left(F_{z_1}(z,\overline{z},u), \frac{\partial}{\partial u}F_{z_1}(z,\overline{z},u)\right).$$

It follows that a simpler implicit equation of the catacaustic K_{z_1} is

$$\left(K_{z_1}\right):D\left(P_{z_1,z}\right) = 0.$$

Using formula (4) with $s = z_1 + z$ and $p = z_1 z$, we get

$$D\left(P_{z_1,z}\right) = 4\Big(|z+z_1|^2 - 4|z_1|^2|z|^2\Big)^3 + 108\Big(\text{Im}\big((z+z_1)^2\,\overline{z_1 z}\big)\Big)^2. \tag{6}$$

Therefore, an implicit equation of the catacaustic of the circle with the radiant point z_1 is

$$4\Big(|z+z_1|^2 - 4|z_1|^2|z|^2\Big)^3 + 108\Big(\text{Im}\big((z+z_1)^2\,\overline{z_1 z}\big)\Big)^2 = 0.$$

Note that we may write

$$\begin{aligned} D\left(P_{z_1,z}\right) &= 4\big(|z+z_1|^2 - |z_1|^2|z|^2\big)\big(|z+z_1|^2 + 8|z_1|^2|z|^2\big)^2 \\ &\quad - 54\Big(|z+z_1|^4|z_1|^2|z|^2 + \text{Re}\big((z+z_1)^4\,\overline{z_1}^2\overline{z}^2\big)\Big). \end{aligned}$$

In conclusion, $z_2 \in K_{z_1}$ if and only if $D(P_{z_1,z_2}) = 0$. By Proposition 2.2 (3), if $D(P_{z_1,z_2}) = 0$, then all the roots u of $P_{z_1,z_2} = 0$ are unimodular and $P_{z_1,z_2} = 0$ has either a double root and two simple roots, or a triple root and one simple root. The latter case occurs if and only if z_2 is a cusp of the catacaustic of the circle with radiant point z_1, as shown in [19] in the case where $z_1 = c$ is a positive number, see also [4, Lemma 4.3].

If z_2 belongs to a connected component of the complement of the catacaustic $D(P_{z_1,z}) = 0$, we have either $D(P_{z_1,z_2}) > 0$ (hence $P_{z_1,z_2} = 0$ has two simple unimodular roots and two distinct roots off the unit circle) or $D(P_{z_1,z_2}) < 0$ (hence $P_{z_1,z_2} = 0$ has four simple unimodular roots).

Assuming that $z_1 \neq 0$, using Corollary 2.3 we easily identify the regions $D(z_1,z) > 0$ and $D(z_1,z) < 0$, since

$$\begin{aligned} &0 \in \{z \in \mathbb{C} : E_2(z_1,z) > 0\} \subset \{z \in \mathbb{C} : D(z_1,z) < 0\} \\ \text{and}\quad &(-z_1) \in \{z \in \mathbb{C} : E_2(z_1,z) < 0\} \subset \{z \in \mathbb{C} : D(z_1,z) > 0\}. \end{aligned}$$

Remark 2.5: If $|z_1| > 1$, then the catacaustic K_{z_1} is a Jordan curve contained in the closed unit disk. If $|z_1| > 1$ and $|z_2| > 1$ it follows that $D(P_{z_1,z_2}) < 0$; hence, the equation $P_{z_1,z_2}(u) = 0$ has four simple unimodular roots, as it is proved geometrically in [4, Proposition 3.8 (i)].

3. Alhazen's problem and the circle of Apollonius

The circle of Apollonius is the locus of points that have a constant ratio of distances from two given points. Figure 2 illustrates the two geometric ideas to find the PA-point on the unit circle, the ellipse method (on the left) and the angle bisection property of the circle of Apollonius (on the right).

Lemma 3.1: *Assume that the segment* $[z_1, z_2]$ *is either in* $\mathbb{D}$*, or in* $\mathbb{C} \setminus \overline{\mathbb{D}}$*, and that* $0, z_1, z_2$ *are not collinear.*

Then, $z \in \partial\mathbb{D}$ *is a PA-point if and only if there exists some* $t \in (0,1)$ *such that* $z = \pm(tz_2 + (1-t)z_1)/|tz_2 + (1-t)z_1|$ *and* z *belongs to the circle of Apollonius with respect to the two points* z_1, z_2 *and the ratio* t*.*

Proof: Sufficiency follows from the angle bisector property for the circle of Apollonius.

Necessity: Let $z \in \partial\mathbb{D}$ be a PA-point. The line L_z joining z to the origin bisects the angle $\angle(z_1, z, z_2)$. If $|z_1| = |z_2|$ and L_z is the perpendicular bisector of the segment $[z_1, z_2]$, the claim follows with $t = 1/2$. This corresponds to the case in which the circle of Apollonius is a straight line.

In the remaining case, the intersection between L_z and the segment $[z_1, z_2]$ is a point $v = tz_2 + (1-t)z_1$ for some $t \in (0,1) \setminus \{1/2\}$. Let u be the harmonic conjugate of v with respect to z_1 and z_2. Then z belongs to the circle of diameter $[u, v]$, which is the circle of Apollonius with respect to the two points z_1, z_2 and the ratio t. ∎

Proposition 3.2: *Assume that the segment* $[z_1, z_2]$ *is either in* $\mathbb{D}$*, or in* $\mathbb{C} \setminus \overline{\mathbb{D}}$*, and that* $0, z_1, z_2$ *are not collinear. Then the point* $z \in \partial\mathbb{D}$ *is a PA-point if and only if* z *is written as* $z = \pm(tz_2 + (1-t)z_1)/|tz_2 + (1-t)z_1|$ *for some* $t \in (0,1)$ *such that*

$$(1-2t)^2\Big((1-t)^2|z_1|^2 + t^2|z_2|^2 + 2t(1-t)\mathrm{Re}\,(z_1\overline{z_2})\Big) = \Big((1-t)^2|z_1|^2 - t^2|z_2|^2\Big)^2. \tag{7}$$

Proof: By Lemma 3.1, $z \in \partial\mathbb{D}$ is a PA-point if and only if there exists some $t \in (0,1)$ such that $z = \pm(tz_2 + (1-t)z_1)/|tz_2 + (1-t)z_1|$ belongs to the circle of Apollonius with respect to the two points z_1, z_2 and the ratio t.

For $|z_1| = |z_2|$ and $t = 1/2$, (7) holds, and z is a PA-point.

We will consider other cases. Let v be the internal division point of the segment $[z_1, z_2]$ in the ratio $t : (1-t)$ and u the external division point of $[z_1, z_2]$ in the same ratio, i.e.

$$v = tz_2 + (1-t)z_1, \qquad u = \frac{-tz_2 + (1-t)z_1}{1-2t}.$$

The corresponding circle of Apollonius is given by $|\zeta - (u+v)/2| = |(u+v)/2c - v|$, which is equivalent to

$$|z|^2 - \mathrm{Re}\left((u+v)\overline{z}\right) + \mathrm{Re}\,(\overline{u}v) = 0. \tag{8}$$

By (8), $z = \pm v/|v|$ is on the corresponding circle of Apollonius if and only if

$$\mathrm{Re}\,(\overline{u}v)\left(1 \mp \frac{1}{|v|}\right) = \pm|v| - 1.$$

As the segment $[z_1, z_2]$ does not intersect the unit circle, we have $|v| - 1 \neq 0$. Then $z = \pm v/|v|$ is on the circle of Apollonius with diameter $|u - v|$ if and only if $\mathrm{Re}\,(\overline{u}v) = \pm|v|$, i.e.

$$\left(\mathrm{Re}\,(\overline{u}v)\right)^2 = |v|^2.$$

Inserting here the formulas of u and v, we get (7). ■

Remark 3.3: Equation (7) can also be written as follows:

$$\begin{aligned} T(z_1, z_2) = &\left((|z_1|^2 - |z_2|^2)^2 - 4|z_1 - z_2|^2\right)t^4 \\ &- 4\left(|z_1|^2(|z_1|^2 - |z_2|^2) - 2|z_1 - z_2|^2 - |z_1|^2 + |z_2|^2\right)t^3 \\ &+ \left(2|z_1|^2(3|z_1|^2 - |z_2|^2) - 8|z_1|^2 + 4|z_2|^2 - 5|z_1 - z_2|^2\right)t^2 \\ &- \left(|z_1|^2(4|z_1|^2 - 5) + |z_2|^2 - |z_1 - z_2|^2\right)t + |z_1|^2(|z_1|^2 - 1) = 0. \end{aligned}$$

Therefore, if $z_1, z_2 \in \mathbb{D}$, the equation $T(z_1, z_2) = 0$ has at least two real roots t_1, t_2 satisfying $0 < t_1 \leq 1/2 \leq t_2 < 1$ since the following hold

$$T(z_1, z_2)|_{t=0} = z_1\overline{z_1}(z_1\overline{z_1} - 1) < 0, \quad T(z_1, z_2)|_{t=1} = z_2\overline{z_2}(z_2\overline{z_2} - 1) < 0,$$
$$\text{and} \quad T(z_1, z_2)|_{t=\frac{1}{2}} = \frac{1}{16}(z_1\overline{z_1} - z_2\overline{z_2})^2 \geq 0.$$

So, the PA-point that attains the supremum in the definition of the triangular ratio metric $s_{\mathbb{D}}(z_1, z_2)$ is on the shorter of the two arcs connecting $e^{i\arg(z_1)}$ and $e^{i\arg(z_2)}$.

4. Alhazen's problem on a conic domain

In this section, we consider Alhazen's problem for quadric surfaces. Since the section of a quadric surface (conicoid) by each plane is a quadratic curve (conic), we will consider this problem in a planar domain whose boundary is a conic.

Problem 4.1: For two points $z_1,\ z_2 \in \mathbb{C}$ and a conic domain D, find the PA-points on ∂D.

Since it is difficult to extend the solution method using the circle of Apollonius, we will apply the solution method using ellipses to this problem.

Here, we consider similarity transformations of the form $A(z) = \alpha z + \beta$, where $\alpha,\ \beta \in \mathbb{C}$. Then A maps each subdomain D of $\mathbb{C}$ onto $A(D)$ which is a translated, rotated, and rescaled version of D. The similarity transformation

$$A(z) = \frac{2}{z_1 - z_2}z - \frac{z_1 + z_2}{z_1 - z_2}$$

sends the points z_1 and z_2 to 1 and -1, respectively, and maps conics to conics. So, instead of Problem 4.1, we just need to solve the following problem.

Problem 4.2: For two points $z_1,\ z_2 \in \mathbb{C}$ and a conic domain D, let C be the boundary of $A(D)$, where $A(z) = 2/(z_1 - z_2)z - (z_1 + z_2)/(z_1 - z_2)$. Then, find $u \in C$ such that

$$|\angle(-1, u, 0)| = |\angle(0, u, 1)|.$$

Let $C(=\partial A(D))$ be a conic given by

$$C\ :\ c(z)=\overline{a}z^2+pz\overline{z}+a\overline{z}^2+\overline{b}z+b\overline{z}+q=0\quad (a,b\in\mathbb{C},\ p,q\in\mathbb{R}), \tag{9}$$

and E the ellipse with foci 1 and -1

$$E\ :\ |z-1|+|z+1|=r\quad (r>2). \tag{10}$$

The ellipse E is also expressed as

$$e(z)=z^2+(2-2r_2)z\overline{z}+\overline{z}^2+r_2^2-2r_2=0, \tag{11}$$

where $r_2:=r^2/2\,(>2)$.

Note that (11) is obtained by squaring both sides of Equation (10), so it includes the hyperbola $||z-1|-|z+1||=r$ as well as the ellipse E.

From now on, we assume that C is a non-degenerate conic, i.e. C is an ellipse, a hyperbola, or a parabola. A conic (9) can be classified as follows. For $p^2-4a\overline{a}<0$, C represents a hyperbola or its degenerate form; for $p^2-4a\overline{a}>0$, C represents an ellipse or its degenerate form; and for $p^2-4a\overline{a}=0$, C represents a parabola or its degenerate form. For a more detailed classification of conics C, see, e.g. [20, Lemma 3].

The following result is an extension of Lemma 2.1. Since the boundary of the domain is extended from the unit circle to a conic, the technique of the proof of Lemma 2.1, i.e. Theorem 1.1 in [4], is not available. However, it is difficult to calculate it manually and directly. Here we use the Risa/Asir, the symbolic computation system, to process the equation.

Theorem 4.3: *Let C be a conic given by $c(z)=\overline{a}z^2+pz\overline{z}+a\overline{z}^2+\overline{b}z+b\overline{z}+q=0$ $(a,b\in\mathbb{C},\ p,q\in\mathbb{R})$, and E the ellipse given by $e(z)=z^2+(2-2r_2)z\overline{z}+\overline{z}^2+r_2^2-2r_2=0$. Suppose E does not coincide with C for some r, and has a point of tangency on C for some r. The point of tangency is given by the solution of $F_4=W_6z^6+W_5z^5+W_4z^4+W_3z^3+W_2z^2+W_1z+W_0=0$, where*

$$
\begin{aligned}
W_6 &= 4\overline{a}(\overline{b}bp-\overline{b}^2a-\overline{a}b^2)(p^2-4\overline{a}a),\\
W_5 &= -2\Big[bp^5-\overline{b}(a+\overline{a})p^4-b(4\overline{a}q+8\overline{a}a+\overline{b}^2)p^3+\overline{b}\big(8\overline{a}aq+8\overline{a}a^2+(8\overline{a}^2+\overline{b}^2)a\\
&\quad -3\overline{a}b^2\big)p^2+4\overline{a}b(4\overline{a}aq+4\overline{a}a^2+4\overline{b}^2a+\overline{a}b^2)p-4a\overline{a}\overline{b}\big(8\overline{a}aq+4\overline{a}a^2\\
&\quad +(4\overline{a}^2+3\overline{b}^2)a+3\overline{a}b^2\big)\Big],\\
W_4 &= -\Big[p^6-\big((4a+4\overline{a})q+a^2+10\overline{a}a-9b^2+\overline{a}^2+\overline{b}^2\big)p^4-2b\overline{b}(2q+11a+\overline{a})p^3\\
&\quad +\Big(16\overline{a}aq^2+4\big(8\overline{a}a^2+(8\overline{a}^2+2\overline{b}^2)a-5\overline{a}b^2\big)q+8\overline{a}a^3+(32\overline{a}^2+14\overline{b}^2)a^2\\
&\quad +(-42\overline{a}b^2+8\overline{a}^3+18\overline{b}^2\overline{a})a+(-6\overline{a}^2-5\overline{b}^2)b^2\Big)p^2+2b\overline{b}\big(48\overline{a}aq+44\overline{a}a^2\\
&\quad +(4\overline{a}^2+7\overline{b}^2)a+\overline{a}b^2\big)p-64\overline{a}^2a^2q^2-16a^2\overline{a}(4\overline{a}a+4\overline{a}^2+7\overline{b}^2)q-16\overline{a}^2a^4-(32\overline{a}^3\\
&\quad +56\overline{b}^2\overline{a})a^3+(24\overline{a}^2b^2-16\overline{a}^4-56\overline{b}^2\overline{a}^2-9\overline{b}^4)a^2+(24\overline{a}^3-6\overline{b}^2\overline{a})b^2a+3\overline{a}^2b^4\Big],
\end{aligned}
$$

$$
\begin{aligned}
W_3 = -2\Big[&2bp^5 - \overline{b}(q+4a)p^4 - b\big((14a+2\overline{a})q + 2a^2 + 12\overline{a}a - 8b^2 + 2\overline{a}^2 + \overline{b}^2\big)p^3 \\
&+ \overline{b}\big(4aq^2 + (20a^2 + 20\overline{a}a - 7b^2)q + 4a^3 + 24\overline{a}a^2 + (-24b^2 + 4\overline{a}^2 + 3\overline{b}^2)a - \overline{a}b^2\big)p^2 \\
&+ b\Big(32\overline{a}aq^2 + \big(56\overline{a}a^2 + (8\overline{a}^2 + 26\overline{b}^2)a - 6\overline{a}b^2\big)q + 8\overline{a}a^3 + (16\overline{a}^2 + 26\overline{b}^2)a^2 \\
&+ (-18\overline{a}b^2 + 8\overline{a}^3 + 2\overline{b}^2\overline{a})a + (-6\overline{a}^2 - \overline{b}^2)b^2\Big)p - \overline{b}\Big(80\overline{a}a^2q^2 + 4a\big(20\overline{a}a^2 \\
&+ (16\overline{a}^2 + 6\overline{b}^2)a - 3\overline{a}b^2\big)q + 16\overline{a}a^4 + (32\overline{a}^2 + 12\overline{b}^2)a^3 + (-28\overline{a}b^2 + 16\overline{a}^3 + 16\overline{b}^2\overline{a})a^2 \\
&+ (-24\overline{a}^2 - \overline{b}^2)b^2a - \overline{a}b^4\Big)\Big], \\
W_2 = 2\Big[&\big((3a-\overline{a})q - 3b^2\big)p^4 + b\overline{b}(3q + 9a + \overline{a})p^3 + \Big(-2(6a^2 + 2\overline{a}a - b^2)q^2 - \big(4a^3 \\
&+ 16\overline{a}a^2 + (-27b^2 - 4\overline{a}^2 + 9\overline{b}^2)a + 4\overline{a}b^2\big)q + 3(b^2 - 3\overline{b}^2)a^2 + (9\overline{a}b^2 - \overline{b}^2\overline{a})a - 7b^4 \\
&+ 2\overline{a}^2b^2\Big)p^2 - b\overline{b}\big(28aq^2 + (60a^2 - 5b^2)q + 8a^3 + 20\overline{a}a^2 - (19b^2 - 12\overline{a}^2)a - 5\overline{a}b^2\big)p \\
&+ 32\overline{a}a^2q^3 + 4a\big(12\overline{a}a^2 + (4\overline{a}^2 + 11\overline{b}^2)a - 3\overline{a}b^2\big)q^2 + \big(16\overline{a}a^4 + (16\overline{a}^2 + 44\overline{b}^2)a^3 \\
&+ (-32\overline{a}b^2 + 40\overline{b}^2\overline{a})a^2 - (12\overline{a}^2 + 9\overline{b}^2)b^2a + \overline{a}b^4\big)q + 8\overline{b}^2a^4 + (-4\overline{a}b^2 + 20\overline{b}^2\overline{a})a^3 \\
&+ \big(-(4\overline{a}^2 + 15\overline{b}^2)b^2 + 12\overline{b}^2\overline{a}^2 + 3\overline{b}^4\big)a^2 + (5\overline{a}b^4 - 15\overline{b}^2\overline{a}b^2)a + 2\overline{a}^2b^4\Big], \\
W_1 = 2\Big[&2b\big(q^2 + (3a-\overline{a})q - b^2\big)p^3 - \overline{b}\big(8aq^2 + (12a^2 - 4\overline{a}a + b^2)q - 6b^2a - 2\overline{a}b^2\big)p^2 \\
&- b\Big(8aq^3 + (32a^2 - 8\overline{a}a - 2b^2)q^2 + \big(8a^3 + 8\overline{a}a^2 + (-20b^2 - 2\overline{b}^2)a + 2\overline{a}b^2\big)q \\
&+ (-2b^2 + 6\overline{b}^2)a^2 + (-2\overline{a}b^2 + 6\overline{b}^2\overline{a})a + 3b^4 - \overline{b}^2b^2\Big)p + \overline{b}\Big(32a^2q^3 \\
&+ 4a(12a^2 + 4\overline{a}a - 3b^2)q^2 + \big(16a^4 + 16\overline{a}a^3 + (-32b^2 + 8\overline{b}^2)a^2 - 12\overline{a}b^2a + b^4\big)q \\
&+ (-4b^2 + 4\overline{b}^2)a^3 + (-4\overline{a}b^2 + 4\overline{b}^2\overline{a})a^2 + (5b^4 - 3\overline{b}^2b^2)a + 2\overline{a}b^4\Big)\Big], \\
W_0 = {}&q^2p^4 - 2\overline{b}bqp^3 + \big(-8aq^3 + (-8a^2 + 2b^2)q^2 + (6b^2 + 2\overline{b}^2)aq - b^4 + \overline{b}^2b^2\big)p^2 \\
&+ 2b\overline{b}\big(4aq^2 + (-4a^2 - b^2)q + (b^2 - \overline{b}^2)a\big)p + 16a^2q^4 + 8a(4a^2 - b^2)q^3 \\
&+ \big(16a^4 + (-32b^2 + 8\overline{b}^2)a^2 + b^4\big)q^2 - 2a\big((4b^2 - 4\overline{b}^2)a^2 - 5b^4 + 3\overline{b}^2b^2\big)q \\
&+ (b-\overline{b})(b+\overline{b})\big((b^2 - \overline{b}^2)a^2 - b^4\big).
\end{aligned}
$$

Proof: Note that, if $a = 0$, the conic C is a circle and we can apply Lemma 2.1. Therefore, we assume that $a \neq 0$.

From the assumption of Theorem 4.3, there is an intersection point of E and C. Eliminating $\overline{z}$ from $c(z) = 0$ and $e(z) = 0$, we have the following quartic equation of z variable,

$$
\begin{aligned}
S(z) = {}&\big(4\overline{a}ar_2^2 + 2((a+\overline{a})p - 4\overline{a}a)r_2 + (p - a - \overline{a})^2\big)z^4 \\
&+ \big(4\overline{b}ar_2^2 + 2(\overline{b}p + (b - 4\overline{b})a + \overline{a}b)r_2 + 2(b - \overline{b})(p - a - \overline{a})\big)z^3
\end{aligned}
$$

$$+ \big(2apr_2^3 + (p^2 - 6ap + 4aq + 2a^2 - 2\bar{a}a)r_2^2 - 2(p^2 - (q+2a)p + 4aq + 2a^2$$
$$- 2\bar{a}a - \bar{b}b)r_2 - 2qp + (2a + 2\bar{a})q + b^2 - 2\bar{b}b + \bar{b}^2\big)z^2$$
$$+ \big(2bar_2^3 + 2(bp + (-3b - \bar{b})a)r_2^2 - 2(2bp - bq + (-2b - 2\bar{b})a)r_2 - 2(b - \bar{b})q\big)z$$
$$+ a^2r_2^4 - 4a^2r_2^3 + (-2aq + 4a^2 + b^2)r_2^2 + (4aq - 2b^2)r_2 + q^2 = 0.$$

As there is a point of tangency of two curves C and E, the equation $S(z) = 0$ must have multiple roots. This condition is equivalent to the requirement that the system of equations

$$S(z) = 0, \quad \text{and} \quad S'(z) = 0 \tag{12}$$

has a common root.

Here, we remark that the leading coefficient of $S(z) = 0$ as z variable is not constant zero. In fact, if

$$4\bar{a}ar_2^2 + 2\big((a + \bar{a})p - 4\bar{a}a\big)r_2 + (p - a - \bar{a})^2 \equiv 0$$

then $a = p = 0$ holds, and $C: \ c(z) = \bar{b}z + b\bar{z} + q = 0$ degenerates to a line. Moreover, the leading term $a^2r_2^4$ of $S(z) = 0$ as r_2 variable does not vanish by the assumption.

Now, eliminating r_2 from (12) by calculating

$$\mathrm{resul}_{r_2}(S(z), S'(z)) = 0, \tag{13}$$

we have $a^2F_1F_2F_3^2F_4 = 0$, where

$$F_1(z) = (p - a - \bar{a})z^2 + (b - \bar{b})z - q,$$
$$F_2(z) = (p + a + \bar{a})z^2 + (b + \bar{b})z + q,$$
$$F_3(z) = (4a^2 - 4\bar{a}a)z^4 - 4\bar{b}az^3 + (p^2 - 4aq - 4a^2)z^2 + 2bpz + b^2.$$

Here, we use Risa/Asir, a symbolic computation system, for computing the resultant in (13). (See, e.g. [21] for details on the relationship between the resultant and the solution of a system of equations.)

We need to examine the properties of factors $F_1, \ldots, F_4$.

- The factor $a \neq 0$ from the assumption. (If $a = 0$, C is a circle.)
- The equation $F_1 = 0$ is obtained from substituting $-z$ for $\bar{z}$ in $c(z) = 0$. Therefore $F_1 = 0$ gives the intersection points of C and the imaginary axis.
- The equation $F_2 = 0$ is obtained from substituting z for $\bar{z}$ in $c(z) = 0$. Therefore $F_2 = 0$ gives the intersection points of C and the real axis.
- The solutions of $F_3 = 0$ give the condition that there exist multiple roots for r_2. In fact, we have the following,

$$\mathrm{resul}_{r_2}\left(S, \frac{\partial}{\partial r_2}S\right)$$
$$= 16a^4(z-1)^2(z+1)^2(\bar{a}z^2 + (-p + \bar{b})z + q + a - b)$$
$$\times (\bar{a}z^2 + (p + \bar{b})z + q + a + b)((p^2 - 4\bar{a}a)z^2 + (2bp - 4\bar{b}a)z - 4aq + b^2)^2$$

$$\times\,((4a^2-4\bar{a}a)z^4-4\bar{b}az^3+(p^2-4aq-4a^2)z^2+2bpz+b^2)^2.$$

Therefore, the equality $aF_1F_2F_3=0$ does not give the condition that the equation $S=0$ has multiple roots. Hence, the equation $F_4=0$ gives the condition that the equation $S=0$ has multiple roots and includes a point of tangency as its solution. ■

Remark 4.4: In the above proof, the conic C is assumed not to be the circle. In the case that $a=0$ and $p\neq 0$, C represents a circle. Moreover, we can set $p=1$ without loss of generality, and if $|b|^2>q$, $C: c(z)=z\bar{z}+\bar{b}z+b\bar{z}+q=0$ represents a circle of radius $\sqrt{|b|^2-q}$ with centre b.

Substituting $a=0$ and $p=1$ for F_4, we have

$$F_4(0,b,1,q,z)=(2bz+b^2+1)\big((\bar{b}^2-1)z^4+(2\bar{b}q+(2\bar{b}^2-4)b)z^3+(6\bar{b}bq-6b^2)z^2$$
$$+(2bq^2+2\bar{b}b^2q-4b^3)z+(b^2+1)q^2-2\bar{b}bq-b^4+\bar{b}^2b^2\big)=0.$$

The first factor represents a single point. If the second factor is transformed by the similarity transformation $z=sw-b$ ($s^2=|b|^2-q$), the equation that gives the PA-point for the unit circle is obtained as follows:

$$(|b|^2-q)^2\big((\bar{b}^2-1)w^4-2\bar{b}\sqrt{|b|^2-q}\,w^3+2b\sqrt{|b|^2-q}\,w-b^2+1\big)=0. \tag{14}$$

Then, by the transformation $z=sw-b$, the foci 1 and -1 correspond to $(1+b)/s$ and $(-1+b)/s$, respectively. For these two points, Equation (2) of the PA-point is

$$(\bar{b}^2-1)z^4-2\bar{b}\sqrt{|b|^2-q}\,z^3+2b\sqrt{|b|^2-q}\,z-b^2+1=0.$$

The above equation coincides with (14).

Thus, the equation $F_4=0$ is also valid for the case that C is a circle.

Theorem 4.5: *If the segment* $[-1,1]$ *does not intersect with C, the point* $z\in C$ *such that the sum* $|z-1|+|z+1|$ *is minimal is given as a root of the equation* $F_4=0$ *of degree 6.*

The following examples show how to find the reflection points.

Example 4.6: Let $D=\{z\in\mathbb{C} : |z-2|+|z-(1+2i)|>\sqrt{6}\}$. The PA-point $u\in\partial D$ can be found by the following procedure.

The boundary $C=\partial D$ is the ellipse written as

$$c(z)=(-3+4i)z^2-14z\bar{z}+(-3-4i)\bar{z}^2+(38-20i)z+(38+20i)\bar{z}-71=0.$$

In this case, the equation $F_4=0$ is given by

$$(924-1232i)z^6+(-15308+7432i)z^5+(81677+2608i)z^4+(-189086-106196i)z^3$$
$$+(185621+278356i)z^2+(-37976-281192i)z+(-29632+97824i)=0,$$

and its roots are

$$u_1 \approx 1.923740 - 0.117041i, \quad u_2 \approx 1.772166 + 0.309916i, \quad u_3 \approx 1.259144 + 0.426617i,$$
$$u_4 \approx 2.808489 + 0.435057i, \quad u_5 \approx 0.825548 + 1.934592i, \quad u_6 \approx 1.235845 + 2.067480i.$$

It is easy to see that four roots u_1, u_3, u_5, and u_6 are in C, and the function $|z-1|+|z+1|$ attains its minimum at the point $u_6 \in C$ (see Figure 3 for details). Note that the case $D = \{|z-2| + |z-(1+2i)| < \sqrt{6}\}$ can be calculated exactly in the same way.

Example 4.7: Let D be the region given by $\{z \in \mathbb{C} : ||z-3| - |z-(1+2i)|| > \sqrt{5}\}$. The PA-point $u \in \partial D$ can be found by the following procedure.

The boundary $C = \partial D$ is the hyperbolic curve defined by

$$c(z) = 8iz^2 - 4z\bar{z} - 8i\bar{z}^2 + (24-36i)z + (24+36i)\bar{z} - 99 = 0.$$

In this case, the equation $F_4 = 0$ is given by

$$6048z^6 + (-66960 - 34992i)z^5 + (212760 + 346428i)z^4 + (47268 - 1215900i)z^3$$
$$+ (-1363032 + 1675647i)z^2 + (2156652 - 408726i)z + (-850176 - 550557i) = 0,$$

and its roots are

$$u_1 \approx 2.542018 - 0.357669i, \quad u_2 \approx 2.645886 + 0.629896i, \quad u_3 \approx 3.393387 + 0.463604i,$$
$$u_4 \approx 1.323205 + 1.940610i, \quad u_5 \approx 1.5i, \quad u_6 \approx 1.166931 + 1.609271i.$$

It is easy to check that four roots u_1, u_3, u_4, and u_5 are in C, and the function $|z-1| + |z+1|$ attains its minimum at the point $u_5 \in C$ (see Figure 4 for details). Note that the case $D = \{||z-3| - |z-(1+2i)|| < \sqrt{5}\}$ can be calculated just the same way.

5. Triangular ratio metric on conic domains

5.1. The procedure for calculating the triangular ratio distance

For two points $z_1, z_2 \in \mathbb{C}$ and the domain D whose boundary is given by a conic Γ, the triangular ratio metric

$$s_D(z_1, z_2) = \sup_{z \in \Gamma} \frac{|z_1 - z_2|}{|z_1 - z| + |z - z_2|}$$

is obtained as follows.

(1) Let $C: \; c(z) = \bar{a}z^2 + pz\bar{z} + a\bar{z}^2 + \bar{b}z + b\bar{z} + q = 0$ be the conic given by $A(\Gamma)$, where A is the similarity transformation $A(z) = 2/(z_1 - z_2)z - (z_1 + z_2)/(z_1 - z_2)$.
(2) For C, solve the equation $F_4(z) = 0$.
(3) Find the points $\zeta \in C$ for which the minimum $\min_{F_4(z)=0}\{|z+1| + |z-1|\}$ is attained.
(4) Then, we have $s_C(1,-1) = 2/(|1-\zeta| + |\zeta+1|)$.
(5) Because a similarity transformation preserves the quotient of distances, we have

$$s_D(z_1, z_2) = \sup_{z \in \Gamma} \frac{|z_1 - z_2|}{|z_1 - z| + |z - z_2|} = s_C(1, -1).$$

5.2. Examples of level sets

For a given domain $G \subset \mathbb{C}$ and $0 < t < 1$, the set $B_s(z,t) = \{\zeta \in G : s_G(z,\zeta) < t\}$ is called the contour domain of the level t.

Here, we will draw the level sets by solving the equation $F_4 = 0$. The used algorithm is the same as [4, p.145 Algorithm].

Example 5.1: The left figure of Figure 5 indicates the level sets $B_s(0,t) = \{\zeta \in G : s_G(0,\zeta) < t\}$ for $t = 0.05, 0.1, \ldots, 0.95$, and 1 and the hyperbolic domain $G = \{||z - (-1/2 - 1/2i)| - |z - (1-i)|| < 4/5\}$. It seems that the edges of each contour curve are located on the set

$$\left\{\big||z - (-1/2 - 1/2i)| - |z - (1-i)|\big| = \sqrt{1/2}\right\}.$$

The right figure indicates the level sets for the elliptic domain $G = \{|z - 3/2| + |z - (-1/3 - 1/2i)| < 11/5\}$. It seems that the edges of each contour curve are located on the set

$$\left\{|z - 3/2| + |z - (-1/3 - 1/2i)| = (9 + \sqrt{13})/6\right\}.$$

The above example leads to the following conjecture.

Conjecture 5.2: Let G be the domain defined by $\{|z - f_1| + |z - f_2| < r\}$ or $\{||z - f_1| - |z - f_2|| < r\}$, and $B_s(z,t) = \{\zeta \in G : s_G(z,\zeta) < t\}$. Using a similarity transformation, we can set the centre point z to 0. Then, the edge points of $\partial B_s(0,t)$ are on the conic $|z - f_1| + |z - f_2| = |f_1| + |f_2|$ or $||z - f_1| - |z - f_2|| = ||f_1| - |f_2||$ if $G = \{|z - f_1| + |z - f_2| < r\}$ or $\{||z - f_1| - |z - f_2|| < r\}$, respectively.

Notes

1. https://www.geogebra.org/.
2. https://www.wolfram.com/.
3. http://www.math.kobe-u.ac.jp/Asir/asir.html (Kobe Distribution).

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