Decidability and Periodicity of Low Complexity Tilings*

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Abstract

We investigate the tiling problem, also known as the domino problem, that asks whether the two-dimensional grid $\mathbb{Z}^2$ can be colored in a way that avoids a given finite collection of forbidden local patterns. The problem is well-known to be undecidable in its full generality. We consider the low complexity setup where the number of allowed local patterns is small. More precisely, suppose we are given at most nm legal rectangular patterns of size $n \times m$, and we want to know whether there exists a coloring of $\mathbb{Z}^2$ containing only legal $n \times m$ patterns. We prove that if such a coloring exists then also a periodic coloring exists. This further implies, using standard arguments, that in this setup there is an algorithm to determine if the given patterns admit at least one coloring of the grid. The results also extend to other convex shapes in place of the rectangle.

*Research supported by the Academy of Finland grant 296018 and ANR project CoCoGro (ANR-16-CE40-0005)
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1 Introduction

The tiling problem, also known as the domino problem, asks whether the two-dimensional grid $\mathbb{Z}^2$ can be colored in a way that avoids a given finite collection of forbidden local patterns. The problem is undecidable in its full generality [1]. The undecidability relies on the fact that there are *aperiodic* systems of forbidden patterns that enforce any valid coloring to be non-periodic [1].

In this paper we consider the low complexity setup where the number of allowed local patterns is small. More precisely, suppose we are given at most nm legal rectangular patterns of size $n \times m$, and we want to know whether there exists a coloring of $\mathbb{Z}^2$ containing only legal $n \times m$ patterns. We prove that if such a coloring exists then also a periodic coloring exists (Corollary 5). This further implies, using standard arguments, that in this setup there is an algorithm to determine if the given patterns admit at least one coloring of the grid (Corollary 6). The results also extend to other convex shapes in place of the rectangle (see Section 6).

We believe the low complexity setting has relevant applications. There are numerous examples of processes in physics, chemistry and biology where macroscopic patterns and regularities arise from simple microscopic interactions. Formation of crystals and quasi-crystals is a good example where physical laws govern locally the attachments of particles to each other. Predicting the structure of the crystal from its chemical composition is a notoriously difficult problem (as already implied by the undecidability of the tiling problem) but if the number of distinct local patterns of particle attachments is sufficiently low, our results indicate that the situation may be easier to handle.

Our work is also motivated by the *Nivat's conjecture* [10], an open problem concerning periodicity in low complexity colorings of the grid. The conjecture claims the following: if a coloring of $\mathbb{Z}^2$ is such that, for some $n, m \in \mathbb{N}$, the number of distinct $n \times m$ patterns is at most nm, then the the coloring is necessarily periodic in some direction. If true, this conjecture directly implies a strong form of our peridicity result: in the low complexity setting, not only a coloring exists that is periodic, but in fact all admitted colorings are periodic. Our contribution to Nivat's conjecture is that we show that under the hypotheses of the conjecture, the coloring must contain arbitrarily large periodic regions (Theorem 4).

2 Preliminaries

To discuss the results in detail we need precise definitions. Let A be a finite alphabet. A coloring $c \in A^{\mathbb{Z}^2}$ of the two-dimensional grid $\mathbb{Z}^2$ with elements of A is called a (two-dimensional) *configuration*. We use the notation $c_{\mathbf{n}}$ for the color $c(\mathbf{n}) \in A$ of cell $\mathbf{n} \in \mathbb{Z}^2$. For any $\mathbf{t} \in \mathbb{Z}^2$, the *translation* $\tau^{\mathbf{t}} : A^{\mathbb{Z}^2} \longrightarrow A^{\mathbb{Z}^2}$ by $\mathbf{t}$ is defined by $\tau^{\mathbf{t}}(c)_{\mathbf{n}} = c_{\mathbf{n}-\mathbf{t}}$, for all $c \in A^{\mathbb{Z}^2}$ and all $\mathbf{n} \in \mathbb{Z}^2$. If $\tau^{\mathbf{t}}(c) = c$ for a non-zero $\mathbf{t} \in \mathbb{Z}^2$, we say that c is *periodic* and that $\mathbf{t}$ is a *vector of periodicity*. If there are two linearly independent vectors of periodicity then c is *two-periodic*, and in this case there are horizontal and vertical vectors of periodicity $(k, 0)$ and $(0, k)$ for some $k > 0$, and consequently a vector of periodicity in every rational direction.

A *finite pattern* is a coloring $p \in A^D$ of some finite domain $D \subset \mathbb{Z}^d$. For a fixed D, we call such p also a *D-pattern*. The set $[p] = \{c \in A^{\mathbb{Z}^2} \mid c|_D = p\}$ of configurations that contain pattern p in domain D is the *cylinder* determined by p. We say that pattern p *appears* in configuration c, or that c *contains* pattern p, if some translate $\tau^{\mathbf{t}}(c)$ of c is in $[p]$. For a fixed finite D, the set of D-patterns that appear in a configuration c is denoted by $\mathrm{Patt}(c, D)$, that is,

$$\mathrm{Patt}(c, D) = \{\tau^{\mathbf{t}}(c)|_D \mid \mathbf{t} \in \mathbb{Z}^2 \}.$$

We say that c has *low complexity* with respect to shape D if $|\mathrm{Patt}(c, D)| \leq |D|$, and we call c a *low complexity configuration* if it has low complexity with respect to some finite D.

Conjecture (Maurice Nivat 1997 [10])**.** *Let $c \in A^{\mathbb{Z}^2}$ be a two-dimensional configuration. If c has low complexity with respect to some rectangle $D = \{1, \ldots, n\} \times \{1, \ldots, m\}$ then c is periodic.*

The analogous claim in dimensions higher than two fails, as does an analogous claim in two dimensions for many other shapes than rectangles [5].

2.1 Algebraic concepts

Kari and Szabados introduced in [9] an algebraic approach to study low complexity configurations. The present paper heavily relies on this technique. In this approach we replace the colors in A by distinct integers, so that we assume $A \subseteq \mathbb{Z}$. We then express a configuration $c \in A^{\mathbb{Z}^2}$ as a formal power series $c(x, y)$ over two variables x and y in which the coefficient of monomial $x^i y^j$ is $c_{i,j}$, for all $i, j \in \mathbb{Z}$. Note that the exponents of the variables range from $-\infty$ to $+\infty$. In the following also polynomials may have negative powers of variables so all polynomials considered are actually Laurent polynomials. Let us denote by $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ and $\mathbb{Z}[[x^{\pm 1}, y^{\pm 1}]]$ the sets of such polynomials and power series, respectively. We call a power series $c \in \mathbb{Z}[[x^{\pm 1}, y^{\pm 1}]]$ *finitary* if its coefficients take only finitely many different values. Since we color the grid using finitely many colors, configurations are identified with finitary power series.

Multiplying a configuration $c \in \mathbb{Z}[[x^{\pm 1}, y^{\pm 1}]]$ by a monomial corresponds to translating it, and the periodicity of the configuration by vector $\mathbf{t} = (n, m)$ is then equivalent to $(x^n y^m - 1)c = 0$, the zero power series. More generally, we say that polynomial $f \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ *annihilates* power series c if the formal product fc is the zero power series. Note that variables x and y in our power series and polynomials are treated only as “position indicators”: in this work we never plug in any values to the variables.

The set of polynomials that annihilates a power series is a Laurent polynomial ideal, and is denoted by

$$\mathrm{Ann}(c) = \{f \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}] \mid fc = 0\}.$$

It was observed in [9] that if a configuration has low complexity with respect to some shape D then it is annihilated by some non-zero polynomial $f \neq 0$.

Lemma 1 ([9])**.** *Let $c \in \mathbb{Z}[[x^{\pm 1}, y^{\pm 1}]]$ be a low complexity configuration. Then $Ann(c)$ contains a non-zero polynomial.*

One of the main results of [9] states that if a configuration c is annihilated by a non-zero polynomial then it has annihilators of particularly nice form:

Theorem 2 ([9])**.** *Let $c \in \mathbb{Z}[[x^{\pm 1}, y^{\pm 1}]]$ be a configuration (a finitary power series) annihilated by some non-zero polynomial. Then there exist pairwise linearly independent $(i_1, j_1), \ldots, (i_m, j_m) \in \mathbb{Z}^2$ such that*

$$(x^{i_1} y^{j_1} - 1) \cdots (x^{i_m} y^{j_m} - 1) \in Ann(c).$$

Note that both Lemma 1 and Theorem 2 were proved in [9] for configurations $c \in A^{\mathbb{Z}^d}$ in arbitrary dimension d. In this work we only deal with two-dimensional configurations, so above we stated these results for $d = 2$.

If $X \subseteq A^{\mathbb{Z}^2}$ is a set of configurations, we denote by $\mathrm{Ann}(X)$ the set of Laurent polynomials that annihilate all elements of X. We call $\mathrm{Ann}(X)$ the annihilator ideal of X.

2.2 Dynamical systems concepts

Cylinders $[p]$ are a base of a compact topology on $A^{\mathbb{Z}^2}$, namely the product of discrete topologies on A. See, for example, the first few pages of [6]. The topology is equivalently defined by a metric on $A^{\mathbb{Z}^2}$ where two configurations are close to each other if they agree with each other on a large region around cell $\mathbf{0}$.

A subset X of $A^{\mathbb{Z}^2}$ is a *subshift* if it is closed in the topology and closed under translations. Equivalently, every configuration c that is not in X contains a finite pattern p that prevents it from being in X: no configuration that contains p is in X. We can then as well define subshifts using forbidden patterns: for a set P of finite patterns, define

$$X_P = \{c \in A^{\mathbb{Z}^2} \mid \forall \mathbf{t} \in \mathbb{Z}^2 \ \forall p \in P \ : \ \tau^{\mathbf{t}}(c) \notin [p] \ \},$$

the set of configurations that avoid all patterns in P. Set X_P is a subshift, and every subshift is X_P for some P. If $X = X_P$ for some finite P then X is a *subshift of finite type* (SFT).

The *tiling problem* (aka the domino problem) is the decision problem that asks whether a given SFT is empty, that is, whether there exists a configuration avoiding a given finite collection P of forbidden finite patterns. Usually this question is asked in terms of so-called Wang tiles, but our formulation is equivalent. The tiling problem is undecidable [1]. An SFT is called *aperiodic* if it is non-empty but does not contain any periodic configurations. Aperiodic SFTs exist [1], and in fact they must exist because of the undecidability of the tiling problem [13]. We recall the reason for this fact in the proof of Corollary 6.

Convergence of a sequence $c^{(1)}, c^{(2)}, \ldots$ of configurations to a configuration c in our topology has the following simple meaning: For every cell $\mathbf{n} \in \mathbb{Z}^2$ we must have $c^{(i)}_{\mathbf{n}} = c_{\mathbf{n}}$ for all sufficiently large i. As usual, we denote then $c = \lim_{i\to\infty} c^{(i)}$. Note that if all $c^{(i)}$ are in a subshift X, so is the limit. Compactness of space $A^{\mathbb{Z}^2}$ means that every sequence has a converging subsequence. In the proof of Theorem 3 in Section 4 we frequently use this fact and extract converging subsequences from sequences of configurations.

The *orbit* of configuration c is the set $\mathcal{O}(c) = \{\tau^{\mathbf{t}}(c) \mid \mathbf{t} \in \mathbb{Z}^2 \ \}$ that contains all translates of c. The *orbit closure* $\overline{\mathcal{O}(c)}$ of c is the topological closure of the orbit $\mathcal{O}(c)$. It is a subshift, and in fact it is the intersection of all subshifts that contain c. The orbit closure $\overline{\mathcal{O}(c)}$ can hence be called the subshift generated by c. In terms of finite patters, $c' \in \overline{\mathcal{O}(c)}$ if and only if every finite pattern that appears in c' appears also in c.

A configuration c is called *uniformly recurrent* if for every $c' \in \overline{\mathcal{O}(c)}$ we have $\overline{\mathcal{O}(c')} = \overline{\mathcal{O}(c)}$. This is equivalent to $\overline{\mathcal{O}(c)}$ being a *minimal subshift* in the sense that it has no proper non-empty subshifts inside it. A classical result by Birkhoff [3] implies that every non-empty subshift contains a minimal subshift, so there is a uniformly recurrent configuration in every non-empty subshift.

We use the notation $\langle \mathbf{x}, \mathbf{y} \rangle$ for the inner product of vectors $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^2$. For a nonzero vector $\mathbf{u} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ we denote

$$H_{\mathbf{u}} = \{\mathbf{x} \in \mathbb{Z}^2 \mid \langle \mathbf{x}, \mathbf{u} \rangle < 0\}$$

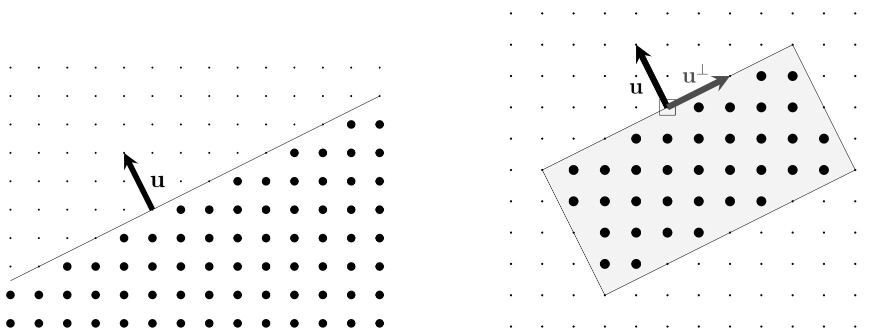
for the discrete *half plane* in direction $\mathbf{u}$. See Figure 1(a) for an illustration. A subshift X is *deterministic* in direction $\mathbf{u}$ if for all $c, c' \in X$

$$c|_{H_{\mathbf{u}}} = c'|_{H_{\mathbf{u}}} \Longrightarrow c = c',$$

that is, if the contents of a configuration in the half plane $H_{\mathbf{u}}$ uniquely determines the contents in the rest of the cells. Note that it is enough to verify that the value $c_{\mathbf{0}}$ on the boundary of the half plane is uniquely determined — the rest follows by translation invariance of X. Moreover, by compactness, determinism in direction $\mathbf{u}$ implies that there is a finite number k such that already the contents of a configuration in the discrete box

$$B_{\mathbf{u}}^k = \{\mathbf{x} \in \mathbb{Z}^2 \mid -k < \langle \mathbf{x}, \mathbf{u} \rangle < 0 \text{ and } -k < \langle \mathbf{x}, \mathbf{u}^\perp \rangle < k\}$$

are enough to uniquely determine the contents in cell $\mathbf{0}$, where we denote by $\mathbf{u}^\perp$ a vector that is orthogonal to $\mathbf{u}$ and has the same length as $\mathbf{u}$, e.g., $(n, m)^\perp = (m, -n)$. See Figure 1(b) for an illustration.



(a) The discrete half plane $H_{\mathbf{u}}$

(b) The discrete box $B_{\mathbf{u}}^k$ with $k = 10$.

Figure 1: Discrete regions determined by vector $\mathbf{u} = (-1, 2)$.

If X is deterministic in directions $\mathbf{u}$ and $-\mathbf{u}$ we say that $\mathbf{u}$ is a direction of *two-sided* determinism. If X is deterministic in direction $\mathbf{u}$ but not in direction $-\mathbf{u}$ we say that $\mathbf{u}$ is a direction of *one-sided* determinism. Directions of two-sided determinism correspond to directions of expansivity in the symbolic dynamics literature. If X is not deterministic in direction $\mathbf{u}$ we call $\mathbf{u}$ a *direction of non-determinism.* Finally, note that the concept of determinism in direction $\mathbf{u}$ only depends on the orientation of vector $\mathbf{u}$ and not on its magnitude.

3 Our results

Our first main new technical result is the following:

Theorem 3. *Let c be a two-dimensional configuration that has a non-trivial annihilator. Then $\overline{\mathcal{O}(c)}$ contains a configuration c' such that $\overline{\mathcal{O}(c')}$ has no direction of one-sided determinism.*

From this result, using a technique by Cyr and Kra [7], we then obtain the second main result:

Theorem 4. *Let c be a two-dimensional configuration that has low complexity with respect to a rectangle. Then $\overline{\mathcal{O}(c)}$ contains a periodic configuration.*

These two theorems are proved in Sections 4 and 5, respectively. But let us first demonstrate how these results imply relevant corollaries. First we consider SFTs defined in terms of allowed rectangular patterns. Let $D = \{1, \ldots, n\} \times \{1, \ldots, m\}$ for some $m, n \in \mathbb{N}$, and let $P \subseteq A^D$ be a set of D-patterns over alphabet A. Define $X = X_{A^D \setminus P} = \{x \in A^{\mathbb{Z}^2} \mid \text{Patt}(c, D) \subseteq P\}$, the set of configurations whose D-patterns are among P.

Corollary 5. *With the notations above, if $|P| \leq nm$ and $X \neq \emptyset$ then X contains a periodic configuration.*

Proof. Let $c \in X$ be arbitrary. By Theorem 4 then, $\overline{\mathcal{O}(c)} \subseteq X$ contains a periodic configuration. $\square$

Corollary 6. *With the notations above, there is an algorithm to determine whether $X \neq \emptyset$ for a given P of cardinality $|P| \leq nm$.*

Proof. This is a classical argumentation by H. Wang [13]: there is a semi-algorithm to test if a given SFT is empty, and there is a semi-algorithm to test if a given SFT contains a periodic configuration. Since X is an SFT, we can execute both these semi-algorithms on X. By Corollary 5, if $X \neq \emptyset$ then X contains a periodic configuration. Hence, exactly one of these two semi-algorithms will return a positive answer. $\square$

The next corollary solves Nivat's conjecture for uniformly recurrent configurations.

Corollary 7. *A uniformly recurrent configuration c that has low complexity with respect to a rectangle is periodic.*

Proof. Because c has low complexity with respect to a rectangle then by Theorem 4 there is a periodic configuration $c' \in \overline{\mathcal{O}(c)}$. All elements in $\overline{\mathcal{O}(c')}$ are periodic. Because c is uniformly recurrent we have $\overline{\mathcal{O}(c)} = \overline{\mathcal{O}(c')}$, which implies that all elements of $\overline{\mathcal{O}(c)}$, including c itself, are periodic. $\square$

In Section 6 we briefly argue that all our results remain true if the $m \times n$ rectangle is replaced by any convex discrete shape.

4 Removing one-sided determinism

In this section we prove Theorem 3 by showing how we can "remove" one-sided directions of determinism from subshifts with annihilators.

Let c be a configuration over alphabet $A \subseteq \mathbb{Z}$ that has a non-trivial annihilator. By Theorem 2 it has then an annihilator $\phi_1 \cdots \phi_m$ where each ϕ_i is of the form

$$\phi_i = x^{n_i} y^{m_i} - 1 \text{ for some } \mathbf{v}_i = (n_i, m_i) \in \mathbb{Z}^2. \tag{1}$$

Moreover, vectors $\mathbf{v}_i$ can be chosen pairwise linearly independent, that is, in different directions. We may assume $m \geq 1$.

Denote $X = \overline{\mathcal{O}(c)}$, the subshift generated by c. A polynomial that annihilates c annihilates all elements of X, because they only have local patterns that already appear in c. It is easy to see that X can only be non-deterministic in a direction that is perpendicular to one of the directions $\mathbf{v}_i$ of the polynomials ϕ_i:

Proposition 8. *Let c be a configuration annihilated by $\phi_1 \cdots \phi_m$ where each ϕ_i is of the form (1). Let $\mathbf{u} \in \mathbb{Z}^2$ be a direction that is not perpendicular to $\mathbf{v}_i$ for any $i \in \{1, \ldots, m\}$. Then $X = \overline{\mathcal{O}(c)}$ is deterministic in direction $\mathbf{u}$.*

Proof. Suppose X is not deterministic in direction $\mathbf{u}$. By definition, there exist $d, e \in X$ such that $d \neq e$ but $d|_{H_\mathbf{u}} = e|_{H_\mathbf{u}}$. Denote $\Delta = d - e$. Because $\Delta \neq 0$ but $\phi_1 \cdots \phi_m \cdot \Delta = 0$, for some i we have $\phi_1 \cdots \phi_{i-1} \cdot \Delta \neq 0$ and $\phi_1 \cdots \phi_i \cdot \Delta = 0$. Denote $\Delta' = \phi_1 \cdots \phi_{i-1} \cdot \Delta$. Because $\phi_i \cdot \Delta' = 0$, configuration Δ' is periodic in direction $\mathbf{v}_i$. But because Δ is zero in the half plane $H_\mathbf{u}$, also Δ'

is zero in some translate $H' = H_{\mathbf{u}} - \mathbf{t}$ of the half plane. Since the periodicity vector $\mathbf{v}_i$ of Δ' is not perpendicular to $\mathbf{u}$, the periodicity transmits the values 0 from the region H' to the entire $\mathbb{Z}^2$. Hence $\Delta' = 0$, a contradiction.

□

Let $\mathbf{u} \in \mathbb{Z}^2$ be a one-sided direction of determinism of X. In other words, $\mathbf{u}$ is a direction of determinism but $-\mathbf{u}$ is not. By the proposition above, $\mathbf{u}$ is perpendicular to some $\mathbf{v}_i$. Without loss of generality, we may assume $i = 1$. We denote $\phi = \phi_1$ and $\mathbf{v} = \mathbf{v}_1$.

Let k be such that the contents of the discrete box $B = B_{\mathbf{u}}^k$ determine the content of cell $\mathbf{0}$, that is, for $d, e \in X$

$$d|_B = e|_B \Longrightarrow d_{\mathbf{0}} = e_{\mathbf{0}}. \tag{2}$$

As pointed out in Section 2.2, any sufficiently large k can be used. We can choose k so that $k > |\langle \mathbf{u}^\perp, \mathbf{v} \rangle|$. To shorten notations, let us also denote $H = H_{-\mathbf{u}}$.

Lemma 9. *For any $d, e \in X$ such that $\phi d = \phi e$ holds:*

$$d|_B = e|_B \Longrightarrow d|_H = e|_H.$$

Proof. Let $d, e \in X$ be such that $\phi d = \phi e$ and $d|_B = e|_B$. Denote $\Delta = d - e$. Then $\phi\Delta = 0$ and $\Delta|_B = 0$. Property $\phi\Delta = 0$ means that Δ has periodicity vector $\mathbf{v}$, so this periodicity transmits values 0 from the region B to the strip

$$S = \bigcup_{i \in \mathbb{Z}} (B + i\mathbf{v}) = \{\mathbf{x} \in \mathbb{Z}^2 \mid -k < \langle \mathbf{x}, \mathbf{u} \rangle < 0\}.$$

See Figure 2 for an illustration of the regions H, B and S. As $\Delta|_S = 0$, we have that $d|_S = e|_S$. Applying (2) on suitable translates of d and e allows us to conclude that $d|_H = e|_H$.

□

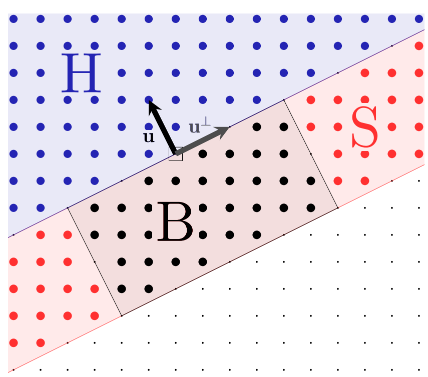


Figure 2: Discrete regions $H = H_{-\mathbf{u}}$, $B = B_{\mathbf{u}}^k$ and S in the proof of Lemma 9. In the illustration $\mathbf{u} = (-1, 2)$ and $k = 10$.

A reason to prove the lemma above is the following corollary, stating that X can only contain a bounded number of configurations that have the same product with ϕ:

Corollary 10. *Let $c_1, \ldots, c_n \in X$ be pairwise distinct. If $\phi c_1 = \cdots = \phi c_n$ then $n \le |A|^{|B|}$.*

Proof. Let $H' = H - \mathbf{t}$, for $\mathbf{t} \in \mathbb{Z}^2$, be a translate of the half plane $H = H_{-\mathbf{u}}$ such that $c_1, \ldots, c_n$ are pairwise different in H'. Consider the translated configurations $d_i = \tau^{\mathbf{t}}(c_i)$. We have that $d_i \in X$ are pairwise different in H and $\phi d_1 = \cdots = \phi d_n$. By Lemma 9, configurations d_i must be pairwise different in domain B. There are only $|A|^{|B|}$ different patterns in domain B.

□

Let $c_1, \ldots, c_n \in X$ be pairwise distinct such that $\phi c_1 = \cdots = \phi c_n$, with n as large as possible. By Corollary 10 such configurations exist. Let us repeatedly translate the configurations c_i by $\tau^{\mathbf{u}}$ and take a limit: by compactness there exists $n_1 < n_2 < n_3 \ldots$ such that

$$d_i = \lim_{j \to \infty} \tau^{n_j \mathbf{u}}(c_i)$$

exists for all $i \in \{1, \ldots, n\}$. Configurations $d_i \in X$ inherit the following properties from c_i:

Lemma 11. *Let $d_1, \ldots, d_n$ be defined as above. Then*

(a) $\phi d_1 = \cdots = \phi d_n$, and

(b) Configurations d_i are pairwise different in translated discrete boxes $B' = B - \mathbf{t}$ for all $\mathbf{t} \in \mathbb{Z}^2$.

Proof. Let $i_1, i_2 \in \{1, \ldots, n\}$ be arbitrary, $i_1 \neq i_2$.

(a) Because $\phi c_{i_1} = \phi c_{i_2}$ we have, for any $n \in \mathbb{N}$,

$$\phi \tau^{n\mathbf{u}}(c_{i_1}) = \tau^{n\mathbf{u}}(\phi c_{i_1}) = \tau^{n\mathbf{u}}(\phi c_{i_2}) = \phi \tau^{n\mathbf{u}}(c_{i_2}).$$

Function $c \mapsto \phi c$ is continuous in the topology so

$$\phi d_{i_1} = \phi \lim_{j \to \infty} \tau^{n_j \mathbf{u}}(c_{i_1}) = \lim_{j \to \infty} \phi \tau^{n_j \mathbf{u}}(c_{i_1}) = \lim_{j \to \infty} \phi \tau^{n_j \mathbf{u}}(c_{i_2}) = \phi \lim_{j \to \infty} \tau^{n_j \mathbf{u}}(c_{i_2}) = \phi d_{i_2}.$$

(b) Let $B' = B - \mathbf{t}$ for some $\mathbf{t} \in \mathbb{Z}^2$. Suppose $d_{i_1}|_{B'} = d_{i_2}|_{B'}$. By the definition of convergence, for all sufficiently large j we have $\tau^{n_j \mathbf{u}}(c_{i_1})|_{B'} = \tau^{n_j \mathbf{u}}(c_{i_2})|_{B'}$. This is equivalent to $\tau^{n_j \mathbf{u} + \mathbf{t}}(c_{i_1})|_B = \tau^{n_j \mathbf{u} + \mathbf{t}}(c_{i_2})|_B$. By Lemma 9 then also $\tau^{n_j \mathbf{u} + \mathbf{t}}(c_{i_1})|_H = \tau^{n_j \mathbf{u} + \mathbf{t}}(c_{i_2})|_H$ where $H = H_{-\mathbf{u}}$. This means that for all sufficiently large j the configurations c_{i_1} and c_{i_2} are identical in the domain $H - n_j \mathbf{u} - \mathbf{t}$. But these domains cover the whole $\mathbb{Z}^2$ as $j \longrightarrow \infty$ so that $c_{i_1} = c_{i_2}$, a contradiction.

□

Now we pick one of the configurations d_i and consider its orbit closure. Choose $d = d_1$ and set $Y = \overline{\mathcal{O}(d)}$. Then $Y \subseteq X$. Any direction of determinism in X is also a direction of determinism in Y. Indeed, this is trivially true for any subset of X. But, in addition, we have the following:

Lemma 12. *Subshift Y is deterministic in direction $-\mathbf{u}$.*

Proof. Suppose the contrary: there exist configurations $x, y \in Y$ such that $x \neq y$ but $x|_H = y|_H$ where, as usual, $H = H_{-\mathbf{u}}$. In the following we construct $n+1$ configurations in X that have the same product with ϕ, which contradicts the choice of n as the maximum number of such configurations.

By the definition of Y all elements of Y are limits of sequences of translates of $d = d_1$, that is, there are translations $\tau_1, \tau_2, \ldots$ such that $x = \lim_{i \to \infty} \tau_i(d)$, and translations $\sigma_1, \sigma_2, \ldots$ such that $y = \lim_{i \to \infty} \sigma_i(d)$. Apply the translations $\tau_1, \tau_2, \ldots$ on configurations $d_1, \ldots, d_n$, and take jointly converging subsequences: by compactness there are $k_1 < k_2 < \ldots$ such that

$$e_i = \lim_{j \to \infty} \tau_{k_j}(d_i)$$

exists for all $i \in \{1, \ldots, n\}$. Here, clearly, $e_1 = x$.

Let us prove that $e_1, \ldots, e_n$ and y are $n+1$ configurations that (i) have the same product with ϕ, and (ii) are pairwise distinct. This contradict the choice of n as the maximum number of such configurations, and thus completes the proof.

(i) First, $\phi x = \phi y$: Because $x|_H = y|_H$ we have $\phi x|_{H-\mathbf{t}} = \phi y|_{H-\mathbf{t}}$ for some $\mathbf{t} \in \mathbb{Z}^2$. Consider $c' = \tau^{\mathbf{t}}(\phi x - \phi y)$, so that $c'|_H = 0$. As $\phi_2 \cdots \phi_m$ annihilates ϕx and ϕy, it also annihilates c'. An application of Proposition 8 on configuration c' in place of c shows that $\overline{\mathcal{O}(c')}$ is deterministic in direction $-\mathbf{u}$. (Note that $-\mathbf{u}$ is not perpendicular to $\mathbf{v}_j$ for any $j \neq 1$, because $\mathbf{v}_1$ and $\mathbf{v}_j$ are not parallel and $-\mathbf{u}$ is perpendicular to $\mathbf{v}_1$.) Due to the determinism, $c'|_H = 0$ implies that $c' = 0$, that is, $\phi x = \phi y$.

Second, $\phi e_{i_1} = \phi e_{i_2}$ for all $i_1, i_2 \in \{1, \ldots, n\}$: By Lemma 11 we know that $\phi d_{i_1} = \phi d_{i_2}$. By continuity of the function $c \mapsto \phi c$ we then have

$$\begin{array}{rl} \phi e_{i_1} = \phi \lim_{j\to\infty} \tau_{k_j}(d_{i_1}) = \lim_{j\to\infty} \phi\tau_{k_j}(d_{i_1}) = & \lim_{j\to\infty} \tau_{k_j}(\phi d_{i_1}) \\ & \| \\ \phi e_{i_2} = \phi \lim_{j\to\infty} \tau_{k_j}(d_{i_2}) = \lim_{j\to\infty} \phi\tau_{k_j}(d_{i_2}) = & \lim_{j\to\infty} \tau_{k_j}(\phi d_{i_2}) \end{array}$$

Because $e_1 = x$, we have shown that $e_1, \ldots, e_n$ and y all have the same product with ϕ.

(ii) Pairwise distinctness: First, y and $e_1 = x$ are distinct by the initial choice of x and y. Next, let $i_1, i_2 \in \{1, \ldots, n\}$ be such that $i_1 \neq i_2$. Let $\mathbf{t} \in \mathbb{Z}^2$ be arbitrary and consider the translated discrete box $B' = B - \mathbf{t}$. By Lemma 11(b) we have $\tau_{k_j}(d_{i_1})|_{B'} \neq \tau_{k_j}(d_{i_2})|_{B'}$ for all $j \in \mathbb{N}$, so taking the limit as $j \longrightarrow \infty$ gives $e_{i_1}|_{B'} \neq e_{i_2}|_{B'}$. This proves that $e_{i_1} \neq e_{i_2}$. Moreover, by taking $\mathbf{t}$ such that $B' \subseteq H$ we see that $y|_{B'} = x|_{B'} = e_1|_{B'} \neq e_i|_{B'}$ for $i \geq 2$, so that y is also distinct from all e_i with $i \geq 2$.

□

The following proposition captures the result established above.

Proposition 13. *Let c be a configuration with a non-trivial annihilator. If $\mathbf{u}$ is a one-sided direction of determinism in $\overline{\mathcal{O}(c)}$ then there is a configuration $d \in \overline{\mathcal{O}(c)}$ such that $\mathbf{u}$ is a two-sided direction of determinism in $\overline{\mathcal{O}(d)}$.* □

Now we are ready to prove Theorem 3.

Proof of Theorem 3. Let c be a two-dimensional configuration that has a non-trivial annihilator. Every non-empty subshift contains a minimal subshift [3], and hence there is a uniformly recurrent configuration $c' \in \overline{\mathcal{O}(c)}$. If $\overline{\mathcal{O}(c')}$ has a one-sided direction of determinism $\mathbf{u}$, we can apply Proposition 13 on c' and find $d \in \overline{\mathcal{O}(c')}$ such that $\mathbf{u}$ is a two-sided direction of determinism in $\overline{\mathcal{O}(d)}$. But because c' is uniformly recurrent, $\overline{\mathcal{O}(d)} = \overline{\mathcal{O}(c')}$, a contradiction.

□

5 Periodicity in low complexity subshifts

In this section we prove Theorem 4. Every non-empty subshift contains a uniformly recurrent configuration, so we can safely assume that c is uniformly recurrent.

Our proof of Theorem 4 splits in two cases based on Theorem 3: either $\overline{O(c)}$ is deterministic in all directions or for some $\mathbf{u}$ it is non-deterministic in both directions $\mathbf{u}$ and $-\mathbf{u}$. The first case is handled by the following well-known corollary from a theorem of Boyle and Lind [4]:

Proposition 14. *A configuration c is two-periodic if and only if $\overline{O(c)}$ is deterministic in all directions.* $\square$

For the second case we apply the technique by Cyr and Kra [7]. This technique was also used in [11] to address Nivat's conjecture. The result that we read from [7, 11], although it is not explicitly stated in this form, is the following:

Proposition 15. *Let c be a two-dimensional uniformly recurrent configuration that has low complexity with respect to a rectangle. If for some $\mathbf{u}$ both $\mathbf{u}$ and $-\mathbf{u}$ are directions of non-determinism in $\overline{\mathcal{O}(c)}$ then c is periodic in a direction perpendicular to $\mathbf{u}$.*

Let us prove this proposition using lemmas from [11]. We first recall some definitions, adjusted to our terminology. Let $D \subseteq \mathbb{Z}^2$ be non-empty and let $\mathbf{u} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$. The *edge* $E_{\mathbf{u}}(D)$ of D in direction $\mathbf{u}$ consists of the cells in D that are furthest in the direction $\mathbf{u}$:

$$E_{\mathbf{u}}(D) = \{\mathbf{v} \in D \mid \forall \mathbf{x} \in D\ \langle \mathbf{x}, \mathbf{u} \rangle \leq \langle \mathbf{v}, \mathbf{u} \rangle\}.$$

We call D *convex* if $D = C \cap \mathbb{Z}^2$ for a convex subset $C \subseteq \mathbb{R}^2$ of the real plane. For $D, E \subseteq \mathbb{Z}^2$ we say that D *fits* in E if $D + \mathbf{t} \subseteq E$ for some $\mathbf{t} \in \mathbb{Z}^2$.

A (closed) *stripe* of width k perpendicular to $\mathbf{u}$ is the set

$$S^k_{\mathbf{u}} = \{\mathbf{x} \in \mathbb{Z}^2 \mid -k < \langle \mathbf{x}, \mathbf{u} \rangle \leq 0\}.$$

Consider the stripe $S = S^k_{\mathbf{u}}$. Clearly its edge $E_{\mathbf{u}}(S)$ in direction $\mathbf{u}$ is the discrete line $\mathbb{Z}^2 \cap L$ where $L \subseteq \mathbb{R}^2$ is the real line through $\mathbf{0}$ that is perpendicular to $\mathbf{u}$. The *interior* S° of S is $S \setminus E_{\mathbf{u}}(S)$, that is, $S^\circ = \{\mathbf{x} \in \mathbb{Z}^2 \mid -k < \langle \mathbf{x}, \mathbf{u} \rangle < 0\}$.

A central concept from [7, 11] is the following. Let c be a configuration and let $\mathbf{u} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ be a direction. Recall that $\mathrm{Patt}(c, D)$ denotes the set of D-patterns that c contains. A finite discrete convex set $D \subseteq \mathbb{Z}^2$ is called $\mathbf{u}$-*balanced in c* if the following three conditions are satisfied, where we denote $E = E_{\mathbf{u}}(D)$ for the edge of D in direction $\mathbf{u}$:

(i) $|\mathrm{Patt}(c, D)| \leq |D|$,

(ii) $|\mathrm{Patt}(c, D \setminus E)| < |\mathrm{Patt}(c, D)| + |E|$, and

(iii) $|D \cap L| \geq |E| - 1$ for every line L perpendicular to $\mathbf{u}$ such that $D \cap L \neq \emptyset$.

The first condition states that c has low complexity with respect to shape D. The second condition implies that there are fewer than $|E|$ different $(D \setminus E)$-patterns in c that can be extended in more than one way into a D-pattern of c. The last condition states that the edge E is nearly the shortest among the parallel cuts across D.

Lemma 16 (Lemma 2 in [11])**.** *Let c be a two-dimensional configuration that has low complexity with respect to a rectangle, and let $\mathbf{u} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$. Then c admits a $\mathbf{u}$-balanced or a $(-\mathbf{u})$-balanced set $D \subseteq \mathbb{Z}^2$.*

A crucial observation in [7] connects balanced sets and non-determinism to periodicity. This leads to the following statement.

Lemma 17 (Lemma 4 in [11])**.** *Let d be a two-dimensional configuration and let $\mathbf{u} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ be such that d admits a $\mathbf{u}$-balanced set $D \subseteq \mathbb{Z}^2$. Assume there is a configuration $e \in \overline{\mathcal{O}(d)}$ and a stripe $S = S^k_{\mathbf{u}}$ perpendicular to $\mathbf{u}$ such that D fits in S and $d|_{S^\circ} = e|_{S^\circ}$ but $d|_S \neq e|_S$. Then d is periodic in direction perpendicular to $\mathbf{u}$.*

With these we can prove Proposition 15.

Proof of Proposition 15. Let c be a two-dimensional uniformly recurrent configuration that has low complexity with respect to a rectangle. Let $\mathbf{u}$ be such that both $\mathbf{u}$ and $-\mathbf{u}$ are directions of non-determinism in $\overline{\mathcal{O}(c)}$. By Lemma 16 configuration c admits a $\mathbf{u}$-balanced or a $(-\mathbf{u})$-balanced set $D \subseteq \mathbb{Z}^2$. Without loss of generality, assume that D is $\mathbf{u}$-balanced in c. As $\overline{\mathcal{O}(c)}$ is non-deterministic in direction $\mathbf{u}$, there are configurations $d, e \in \overline{\mathcal{O}(c)}$ such that $d|_{H_\mathbf{u}} = e|_{H_\mathbf{u}}$ but $d_\mathbf{0} \neq e_\mathbf{0}$. Because c is uniformly recurrent, exactly same finite patterns appear in d as in c. This means that D is $\mathbf{u}$-balanced also in d. From the uniform recurrence of c we also get that $e \in \overline{\mathcal{O}(d)}$. Pick any k large enough so that D fits in the stripe $S = S_\mathbf{u}^k$. Because $\mathbf{0} \in S$ and $S^\circ \subseteq H_\mathbf{u}$, the conditions in Lemma 17 are met. By the lemma, configuration d is $\mathbf{p}$-periodic for some $\mathbf{p}$ that is perpendicular to $\mathbf{u}$. Because d has the same finite patterns as c, it follows that c cannot contain a pattern that breaks period $\mathbf{p}$. So c is also $\mathbf{p}$-periodic.

□

Now Theorem 4 follows from Propositions 14 and 15, using Theorem 3 and the fact that every subshift contains a uniformly recurrent configuration.

Proof of Theorem 4. Let c be a two-dimensional configuration that has low complexity with respect to a rectangle. Replacing c by a uniformly recurrent element of $\overline{\mathcal{O}(c)}$, we may assume that c is uniformly recurrent. Since c is a low-complexity configuration, by Lemma 1 it has a non-trivial annihilator. By Theorem 3 there exists $c' \in \overline{\mathcal{O}(c)}$ such that $\overline{\mathcal{O}(c')}$ has no direction of one-sided determinism. If all directions are deterministic in $\overline{\mathcal{O}(c')}$, it follows from Proposition 14 that c' is two-periodic. Otherwise there is a direction $\mathbf{u}$ such that both $\mathbf{u}$ and $-\mathbf{u}$ are directions of non-determinism in $\overline{\mathcal{O}(c')}$. Now it follows from Proposition 15 that c' is periodic. □

6 Conclusions

We have demonstrated how the low local complexity assumption enforces global regularities in the admitted configurations, yielding algorithmic decidability results. The results were proved in full details for low complexity configurations with respect to an arbitrary rectangle. The reader can easily verify that the fact that the considered shape is a rectangle is not used in any proofs presented here, and the only quoted result that uses this fact is Lemma 16. A minor modification in the proof of Lemma 16 presented in [11] yields that the lemma remains true for any two-dimensional configuration that has low complexity with respect to any convex shape. We conclude that also all our results remain true if we use any convex discrete shape in place of a rectangle.

If the considered shape is not convex the situation becomes more difficult. Theorem 4 is not true for an arbitrary shape in place of the rectangle but all counter examples we know are based on periodic sublattices [5, 8]. For example, even lattice cells may form a configuration that is horizontally but not vertically periodic while the odd cells may have a vertical but no horizontal period. Such a non-periodic configuration may be uniformly recurrent and have low complexity with respect to a scatted shape D that only sees cells of equal parity. It remains an interesting direction of future study to determine if a sublattice structure is the only way to contradict Theorem 4 for arbitrary shapes. We conjecture that Corollaries 5 and 6 hold for arbitrary shapes, that is, that there does not exist a two-dimensional low complexity aperiodic SFT. A special case of this is the recently solved periodic cluster tiling problem [2, 12].

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