Cellular Automata and Powers of \(p/q\) *

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Abstract

We consider one-dimensional cellular automata \(F_{p,q}\) which multiply numbers by \(p/q\) in base \(pq\) for relatively prime integers \(p\) and \(q\). By studying the structure of traces with respect to \(F_{p,q}\) we show that for \(p \geq 2q - 1\) (and then as a simple corollary for \(p > q > 1\)) there are arbitrarily small finite unions of intervals which contain the fractional parts of the sequence \(\xi(p/q)^n, (n = 0, 1, 2, \ldots)\) for some \(\xi > 0\). To the other direction, by studying the measure theoretical properties of \(F_{p,q}\), we show that for \(p > q > 1\) there are finite unions of intervals approximating the unit interval arbitrarily well which don’t contain the fractional parts of the whole sequence \(\xi(p/q)^n\) for any \(\xi > 0\).

Keywords: distribution modulo 1, Z-numbers, cellular automata, ergodicity, strongly mixing

Introduction

In [11] Weyl proved that for any \(\alpha > 1\) the sequence of numbers \(\{\xi \alpha^i\}, i \in \mathbb{N}\) is uniformly distributed in the interval \([0,1)\) for almost every choice of \(\xi > 0\), where \(\{x\} = x - \lfloor x \rfloor\) is the fractional part of \(x\). In particular, \(\{\xi \alpha^i\} | i \in \mathbb{N}\) is dense in \([0,1)\) for almost every \(\xi > 0\). However, this doesn’t hold for every \(\xi > 0\), and it would be interesting to know what other types of distribution the set \(\{\xi \alpha^i\} | i \in \mathbb{N}\) can exhibit for different choices of \(\xi\).

As a special case of this problem, in [8] Mahler posed the question of whether there exist so-called Z-numbers, i.e. real numbers \(\xi > 0\) such that

\[
\left\{ \xi \left( \frac{3}{2} \right)^i \right\} \in [0, 1/2)
\]

for every \(i \in \mathbb{N}\). We will work with the following generalization of the notion of Z-numbers: let \(p > q > 1\) be relatively prime integers and let \(S \subseteq [0,1)\) be a

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Definition 1.1. Any $A \in w$ the collection of all cylinders over $A$ sequence to be empty if the topology $c_0$ such that $C_A$ is actually the collection of Borel sets of $A$ probability space $(\Omega, \mathcal{F}, \mu)$, where $\mathcal{F}$ is the sigma-algebra generated by $C$ and $\mu : \Sigma(C) \to \mathbb{R}$ is a measure such that $\mu(Cyl(w, i)) = |A|^{-|w|}$ for every $Cyl(w, i) \in C$. Note that $T \subseteq \Sigma(C)$ because $C$ is a countable basis of $T$, so $\Sigma(C)$ is actually the collection of Borel sets of $A$.

1 Preliminaries

For a finite set $A$ (an alphabet) the set $A^\mathbb{Z}$ is called a configuration space and its elements are called configurations. An element $c \in A^\mathbb{Z}$ is a bi-infinite sequence and the element at position $i$ in the sequence is denoted by $c(i)$. A factor of $c$ is any finite sequence $c(i)c(i-1)\ldots c(j)$ where $i, j \in \mathbb{Z}$, and we interpret the sequence to be empty if $j < i$. Any finite sequence $a(1)a(2)\ldots a(n)$ (also the empty sequence, which is denoted by $\lambda$) where $a(i) \in A$ is a word over $A$. The set of all words over $A$ is denoted by $A^*$, and the set of non-empty words is $A^+ = A^* \setminus \{\lambda\}$. The set of words of length $n$ is denoted by $A^n$. For a word $w \in A^*$, $|w|$ denotes its length, i.e. $|w| = n$ if and only if $w \in A^n$.

Definition 1.1. Any $w \in A^+$ and $i \in \mathbb{Z}$ determine a cylinder

$$Cyl_A(w, i) = \{c \in A^\mathbb{Z} \mid c(i)c(i+1)\ldots c(i+|w|−1) = w\}.$$ 

The collection of all cylinders over $A$ is

$$\mathcal{C}_A = \{Cyl_A(w, i) \mid w \in A^+, i \in \mathbb{Z}\}.$$ 

The subscript $A$ is omitted when the used alphabet is clear from the context.

The configuration space $A^\mathbb{Z}$ becomes a topological space when endowed with the topology $T$ generated by $C$. It can be shown that this topology is metrizable, and that a set $S \subseteq A^\mathbb{Z}$ is compact if and only if it is closed. $A^\mathbb{Z}$ can also be endowed with measure theoretical structure: it is known that there is a unique probability space $(A^\mathbb{Z}, \Sigma(C), \mu)$, where $\Sigma(C)$ is the sigma-algebra generated by $C$ and $\mu : \Sigma(C) \to \mathbb{R}$ is a measure such that $\mu(Cyl(w, i)) = |A|^{-|w|}$ for every $Cyl(w, i) \in C$. Note that $T \subseteq \Sigma(C)$ because $C$ is a countable basis of $T$, so $\Sigma(C)$ is actually the collection of Borel sets of $A^\mathbb{Z}$.
Definition 1.2. A one-dimensional cellular automaton (CA) is a 3-tuple \((A, N, f)\), where \(A\) is a finite state set, \(N = (n_1, \ldots, n_m) \in \mathbb{Z}^m\) is a neighborhood vector and \(f : A^n \to A\) is a local rule. A given CA \((A, N, f)\) is customarily identified with a corresponding CA function \(F : A^Z \to A^Z\) defined by

\[F(c)(i) = f(c(i + n_1), \ldots, c(i + n_m))\]

for every \(c \in A^Z\) and \(i \in \mathbb{Z}\).

To every configuration space \(A^Z\) is associated a (left) shift CA \((A, (1), \iota)\), where \(\iota : A \to A\) is the identity function. Put in terms of the CA-function determined by this 3-tuple, the left shift is \(\sigma_A : A^Z \to A^Z\) defined by \(\sigma_A(c)(i) = c(i + 1)\) for every \(c \in A^Z\) and \(i \in \mathbb{Z}\).

For a given CA \(F : A^Z \to A^Z\) and a configuration \(c \in A^Z\) it is often helpful to consider a space-time diagram of \(c\) with respect to \(F\). A space-time diagram is a picture which depicts elements of the sequence \((F^i(c))_{i \in \mathbb{N}}\) (or possibly \((F^i(c))_{i \in \mathbb{Z}}\) in the case when \(F\) is reversible) in such a way that \(F^{i+1}(c)\) is drawn below \(F^i(c)\) for every \(i\). As an example, Figure 1 contains a space-time diagram of \(c = \cdots 01101001 \cdots\) with respect to the left shift on \(A = \{0, 1\}\).

All CA-functions are continuous with respect to \(T\) and commute with the shift.

\[
\begin{array}{cccccccccc}
c & \cdots & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & \cdots \\
\sigma_A(c) & \cdots & 1 & 1 & 0 & 1 & 0 & 0 & 1 & \cdots \\
\sigma^2_A(c) & \cdots & 1 & 0 & 1 & 0 & 0 & 1 & \cdots \\
\end{array}
\]

Figure 1: An example of a space-time diagram.

2 The cellular automata \(G_{p,q}\) and \(F_{p,q}\)

In this section we define auxiliary CA \(G_{p,q}\) for relatively prime \(p, q \geq 2\) and show that they multiply numbers by \(p\) in base \(pq\). Then we use \(G_{p,q}\) in constructing the CA \(F_{p,q}\) which multiply numbers by \(p/q\) in base \(pq\), and cover some basic properties of \(F_{p,q}\).

Let us denote by \(A_n\) the set of digits \(\{0, 1, 2, \ldots, n-1\}\) for \(n \in \mathbb{N}, n > 1\). To perform multiplication using a CA we need be able to represent a nonnegative real number as a configuration in \(A_n^Z\). If \(\xi \geq 0\) is a real number and \(\xi = \sum_{i=-\infty}^{\infty} \xi_i n^i\) is the unique base \(n\) expansion of \(\xi\) such that \(\xi_i \neq n - 1\) for infinitely many \(i < 0\), we define \(\text{config}_n(\xi) \in A_n^Z\) by

\[
\text{config}_n(\xi)(i) = \xi_{-i}
\]

for all \(i \in \mathbb{Z}\). In reverse, whenever \(c \in A_n^Z\) is such that \(c(i) = 0\) for all sufficiently small \(i\), we define

\[
\text{real}_n(c) = \sum_{i=-\infty}^{\infty} c(-i)n^i.
\]
For words $w = w(1)w(2)\ldots w(k) \in A_n^k$ we define analogously

$$\text{real}_n(w) = \sum_{i=1}^{k} w(i) n^{-i}.$$  

Clearly $\text{real}_n(\text{config}_n(\xi)) = \xi$ and $\text{config}_n(\text{real}_n(c)) = c$ for every $\xi \geq 0$ and every $c \in A_n^\infty$ such that $c(i) = 0$ for all sufficiently small $i$ and $c(i) \neq n-1$ for infinitely many $i > 0$.

For relatively prime integers $p, q \geq 2$ let $g_{p,q} : A_{pq} \times A_{pq} \rightarrow A_{pq}$ be defined as follows. Digits $x, y \in A_{pq}$ are represented as $x = x_1q + x_0$ and $y = y_1q + y_0$, where $x_0, y_0 \in A_q$ and $x_1, y_1 \in A_p$; such representations always exist and they are unique. Then

$$g_{p,q}(x, y) = g_{p,q}(x_1q + x_0, y_1q + y_0) = x_0p + y_1.$$

An example in the particular case $(p, q) = (3, 2)$ is given in Figure 2.

<table>
<thead>
<tr>
<th>$x \backslash y$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

Figure 2: The values of $g_{p,q}(x, y)$ in the case $(p, q) = (3, 2)$.

The CA function $G_{p,q} : A_{pq}^\infty \rightarrow A_{pq}^\infty$, $G_{p,q}(c)(i) = g_{p,q}(c(i), c(i+1))$ determined by $(A_{pq}, (0, 1), g_{p,q})$ implements multiplication by $p$ in base $pq$ in the sense of the following lemma.

**Lemma 2.1.** $\text{real}_{pq}(G_{p,q}(\text{config}_{pq}(\xi))) = p\xi$ for all $\xi \geq 0$.

**Proof.** Let $c = \text{config}_{pq}(\xi)$. For every $i \in \mathbb{Z}$, denote by $c(i)_0$ and $c(i)_1$ the natural numbers such that $0 \leq c(i)_0 < q$, $0 \leq c(i)_1 < p$ and $c(i) = c(i)_1q + c(i)_0$. Then

$$\text{real}_{pq}(G_{p,q}(\text{config}_{pq}(\xi))) = \text{real}_{pq}(G_{p,q}(c)) = \sum_{i=-\infty}^{\infty} G_{p,q}(c)(-i)(pq)^i$$

$$= \sum_{i=-\infty}^{\infty} g_{p,q}(c(-i), c(-i+1))(pq)^i = \sum_{i=-\infty}^{\infty} (c(-i)_0p + c(-i + 1)_1)(pq)^i$$

$$= \sum_{i=-\infty}^{\infty} (c(-i)_0p(pq)^i + c(-i + 1)pq(pq)^{i-1})$$

$$= \sum_{i=-\infty}^{\infty} (c(-i)_1q + c(-i)_0)(pq)^i = p \text{real}_{pq}(c) = p \text{real}_{pq}(\text{config}_{pq}(\xi)) = p\xi.$$  

□
we also define $G_{p,q}(w)$ for words $w = w(1)w(2)\ldots w(|w|)$ such that $|w| \geq 2$:

$$G_{p,q}(w) = u = u(1)\ldots u(|w| - 1) \in A_{pq}^{[w]-1},$$

where $u(i) = g_{p,q}(w(i), w(i + 1))$ for $1 \leq i \leq |w| - 1$. Inductively it is possible to define $G_{p,q}(w)$ for every $t > 0$ and word $w$ such that $|w| \geq t + 1$:

$$G^t_{p,q}(w) = G_{p,q}(G^{t-1}_{p,q}(w)) \in A_{pq}^{[w]-t}.$$

Clearly the shift CA $\sigma_{A_{pq}}$ multiplies by $pq$ in base $pq$ and its inverse divides by $pq$. This combined with Lemma 2.1 shows that the composition $F_{p,q} = \sigma_{A_{pq}} \circ G_{p,q} \circ G_{p,q}$ implements multiplication by $p/q$ in base $pq$. The value of $F_{p,q}(c)(i)$ is given by the local rule $f_{p,q}$ defined as follows:

$$F_{p,q}(c)(i) = f_{p,q}(c(i - 1), c(i), c(i + 1)) = g_{p,q}(g_{p,q}(c(i - 1), c(i)), g_{p,q}(c(i), c(i + 1))).$$

The CA function $F_{p,q}$ is reversible: if $c \in A_{pq}^Z$ is a configuration with a finite number of non-zero coordinates, then

$$F_{p,q}(F_{q,p}(c)) = F_{q,p}(F_{p,q}(\text{config}_{pq}(\text{real}_{pq}(c)))) \\ \text{L}^{=1} \text{config}_{pq}((p/q)(q/p) \text{real}_{pq}(c)) = c.$$ 

Since $F_{p,q} \circ F_{q,p}$ is continuous and agrees with the identity function on a dense set, it follows that $F_{p,q}(F_{q,p}(c)) = c$ for all configurations $c \in A_{pq}^Z$. We will denote the inverse of $F_{p,q}$ interchangeably by $F_{q,p}$ and $F_{p,q}^{-1}$.

As for $G_{p,q}$, we define $F_{p,q}(w)$ for words $w = w(1)w(2)\ldots w(|w|)$ such that $|w| \geq 3$:

$$F_{p,q}(w) = u = u(1)\ldots u(|w| - 2) \in A_{pq}^{[w]-2},$$

where $u(i) = f_{p,q}(w(i), w(i + 1), w(i + 2))$ for $1 \leq i \leq |w| - 2$, and $F^t_{p,q}(w)$ for every $t > 0$ and word $w$ such that $|w| \geq 2t + 1$:

$$F^t_{p,q}(w) = F_{p,q}(F^{t-1}_{p,q}(w)) \in A_{pq}^{[w]-2t}$$

(see an example in Figure 3).

By the definition of $F_{p,q}$, for every $c \in A_{pq}^Z$ and every $i \in \mathbb{Z}$ the value of $F_{p,q}(c)(i)$ is uniquely determined by $c(i - 1), c(i)$ and $c(i + 1)$, the three nearest digits above in the space-time diagram. Proposition 2.5 gives similarly that each digit in the space-time diagram is determined by the three nearest digits to the right (see Figure 4). Its proof is broken down into the following sequence of lemmas.

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**Figure 3:** Iterated application of $F_{p,q}$ on $w$ for $(p, q) = (3, 2)$ and $w = 3434205$. 

<table>
<thead>
<tr>
<th>$w$</th>
<th>$F^3_{3,2}(w)$</th>
<th>$F^2_{3,2}(w)$</th>
<th>$F^1_{3,2}(w)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 4 3 4 2 0 5</td>
<td>3 5 3 3 1</td>
<td>5 2 1</td>
<td>0</td>
</tr>
</tbody>
</table>
Proof. For every $x, a, y, w$ and $\sigma^i_{A_{pq}}(c)$ (mod $q$).

Lemma 2.3. If $g_{p,q}(x, a) \equiv g_{p,q}(y, a) \pmod{q}$, then $x \equiv y \pmod{q}$.

Proof. Let $x = x_1 q + x_0$, $y = y_1 q + y_0$, $a = a_1 q + a_0$. Then
$$g_{p,q}(x, a) \equiv g_{p,q}(y, a) \pmod{q} \iff x_0 p + a_1 \equiv y_0 p + a_1 \pmod{q} \iff x_0 = y_0 \iff x \equiv y \pmod{q}.$$

Lemma 2.4. If $f_{p,q}(x, a, y) = f_{p,q}(z, a, w)$, then $x \equiv z \pmod{q}$.

Proof.
$$f_{p,q}(x, a, y) = f_{p,q}(z, a, w) \implies g_{p,q}(g_{p,q}(x, a), g_{p,q}(y, a)) = g_{p,q}(g_{p,q}(z, a), g_{p,q}(a, w)) \overset{L2}{\implies} g_{p,q}(x, a) \equiv g_{p,q}(z, a) \pmod{q} \overset{L2}{\implies} x \equiv z \pmod{q}.$$

Proposition 2.5. For every $c \in A_{pq}^Z$ and for all $k, i \in \mathbb{Z}$, the value of $F_{p,q}^k(c)(i)$ is uniquely determined by the values of $F_{p,q}^{k-1}(c)(i + 1)$, $F_{p,q}^k(c)(i + 1)$ and $F_{p,q}^{k+1}(c)(i + 1)$.

Proof. Denote $e = \sigma^i_{A_{pq}}(F_{p,q}^k(c))$. It suffices to show that $e(0)$ is uniquely determined by $F_{q,p}(e)(1)$, $e(1)$ and $F_{p,q}(e)(1)$. Since $F_{p,q}(e)(1) = f_{p,q}(e(0), e(1), e(2))$, by Lemma 2.4 $e(1)$ and $F_{p,q}(e)(1)$ determine the value of $e(0)$ modulo $q$ (see Figure 2.5, left). Similarly, because $F_{q,p}(e)(1) = f_{q,p}(e(0), e(1), e(2))$, by the same lemma $e(1)$ and $F_{q,p}(e)(1)$ determine the value of $e(0)$ modulo $p$ (Fig. 2.5, middle). In total, $F_{q,p}(e)(1)$, $e(1)$ and $F_{p,q}(e)(1)$ determine the value of $e(0)$ both modulo $q$ and modulo $p$ (Fig. 2.5, right). Because $e(0) \in A_{pq}$, the value of $e(0)$ is uniquely determined.
Figure 5: The proof of Proposition 2.5 (here \((p, q) = (3, 2)\)).

3 Traces of configurations

For \(\xi \geq 0\) we are interested in the values of \(\{\xi(p/q)^i\}\) as \(i\) ranges over \(\mathbb{N}\). In terms of the configuration \(\text{config}_{pq}(\xi)\) these correspond to the tails of the configurations \(F^i_{p,q}(\text{config}_{pq}(\xi))\), i.e. to the digits \(F^i_{p,q}(\text{config}_{pq}(\xi))(j)\) for \(j > 0\). Partial information on the tails is preserved in the traces of a configuration. In this section we study traces with respect to \(F_{p,q}\) to prove in the case \(p \geq 2q - 1\) the existence of small sets \(S\) such that \(Z_{p/q}(S)\) is non-empty, and then as a corollary for all \(p > q > 1\).

**Definition 3.1.** For any \(k \in \mathbb{Z}\), the \(k\)-trace of a configuration \(c \in A^\infty_{pq}\) (with respect to \(F_{p,q}\)) is the sequence

\[
\text{Tr}_{p,q}^k(c) = (F^k_{p,q}(c)(n))_{n \in \mathbb{Z}}.
\]

In the special case \(k = 1\), we denote \(\text{Tr}_{p,q}(c) = \text{Tr}_{p,q}(c)\).

A \(k\)-trace of \(c\) is simply the sequence of digits in the \(k\)-th column of the space-time diagram of \(c\) with respect to \(F_{p,q}\) (see Figure 6).

<table>
<thead>
<tr>
<th>(F_{3,2}^{-2}(c))</th>
<th>5</th>
<th>4</th>
<th>0</th>
<th>1</th>
<th>5</th>
<th>3</th>
<th>4</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>(F_{3,2}^{-1}(c))</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>(c)</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td>0</td>
<td>5</td>
<td>...</td>
</tr>
<tr>
<td>(F_{3,2}(c))</td>
<td>5</td>
<td>3</td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>...</td>
</tr>
<tr>
<td>(F_{3,2}^2(c))</td>
<td>5</td>
<td>2</td>
<td>5</td>
<td>2</td>
<td>1</td>
<td>5</td>
<td>1</td>
<td>...</td>
</tr>
</tbody>
</table>

Figure 6: A trace of a configuration.

**Definition 3.2.** The set of allowed words of \(\text{Tr}_{p,q}\) is

\[
L(p, q) = \{w \in A^\infty_{pq} | w \text{ is a factor of } \text{Tr}_{p,q}(c) \text{ for some } c \in A^\infty_{pq}\},
\]

i.e. the set of words that can appear in the columns of space-time diagrams with respect to \(F_{p,q}\).

The following is a reformulation of Proposition 2.5 in terms of traces (see Figure 7).
The proof is by induction. The case $k=1$ follows from the fact that $\text{Tr}_{p,q}(c,1)(0) = c(1)$. Next assume that the claim holds for some $k > 0$ and consider the values of $\text{Tr}_{p,q}(c,k+1)(i)$ for $-k \leq i \leq k$. By Proposition 2.5 these determine $\text{Tr}_{p,q}(c,k)(i)$ for $-(k-1) \leq i \leq (k-1)$, which in turn determine $c(j)$ for $1 \leq j \leq k$ by the induction hypothesis. The value of $c(k+1)$ is determined by $\text{Tr}_{p,q}(c,k+1)(0) = c(k+1)$. \qed

Next we prove an important restriction on the words in the set $L(q, p)$ when $p \geq 2q - 1$. Note that the words in $L(q, p)$ are mirror images of the words in $L(p, q)$ (traces with respect to $F_{p,q}$ are read "from bottom to top").

**Lemma 3.4.** Let $p > q \geq 2$ be relatively prime such that $p \geq 2q - 1$, and for every $d \in A_q$ let $k_d \in A_p$ and $j_d \in A_q$ be the unique elements such that $k_dq \equiv d \pmod{p}$ and $k_dq = j_dp + d$. If $wab \in L(q, p)$ for some $w \in A_{pq}^*$, $a, b \in A_{pq}$ and $a \equiv k_d \pmod{p}$, then $b \equiv j_d \pmod{q}$.

**Proof.** From $wab \in L(q, p)$ it follows that $b = f_{q,p}(x, a, y)$ for some $x, y \in A_{pq}$. Let us write $a = a_1 p + a_0$, $y = y_1 p + y_0$, $g_{q,p}(x, a) = z = z_1 p + z_0$ and $g_{q,p}(a, y) = w = w_1 p + w_0$, where $a_0, y_0, z_0, w_0 \in A_p$ and $a_1, y_1, z_1, w_1 \in A_q$. Here $a_0 = k_d$ because $a \equiv k_d \pmod{p}$ and $w_1 = j_d$ because $g_{q,p}(a, y) = k_d q + y_1 = j_dp + (d + y_1)$ and $d + y_1 \leq (q - 1) + (q - 1) < p$. Now

$$f_{q,p}(x, a, y) = g_{q,p}(g_{q,p}(x, a), g_{q,p}(a, y)) = g_{q,p}(z, w) = z_0 q + j_d,$$

and thus $b \equiv j_d \pmod{q}$. \qed

Based on the previous lemma, we define a special set of digits

$$D_{p,q} = \{ a \in A_{pq} \mid a \equiv k_d \pmod{p} \text{ for some } d \in A_q \},$$

where the digits $k_d$ are as above.

**Example 3.5.** Consider the case $p = 3$ and $q = 2$. Then $A_q = \{0, 1\}$ and $D_{3,2} = \{0, 2, 3, 5\}$ consists of the elements of $A_q$ which are congruent to either $k_0 = 0$ or $k_1 = 2 \pmod{3}$. 

\begin{table}[h]
\centering
\begin{tabular}{c|c c c}
$c$ & 2 & 3 & 5 & 1 \\
\hline
$F_{3,2}(c)$ & 1 & & & \\
$F_{3,2}^2(c)$ & & 0 & 2 & \\
$F_{3,2}^3(c)$ & & 4 & 3 & 3 \\

\end{tabular}
\caption{A trace determining part of the configuration.}
\end{table}
Lemma 3.6. If \( p \geq 2q - 1 \), then \( |L(p,q) \cap D_{p,q}^n| \leq q^{n+1} \) for every \( n > 0 \).

Proof. The proof is by induction. The case \( n = 1 \) is clear because \( |D_{p,q}| = q^2 \). Next assume that the claim holds for some \( n > 0 \). It is sufficient to compute an upper bound for \( |L(q,p) \cap D_{p,q}^{n+1}| \), because the words in \( L(q,p) \) are mirror images of the words in \( L(q,p) \). If \( v \in L(q,p) \cap D_{p,q}^{n+1} \), by the previous lemma it can be written in the form \( v = wab \), where \( a \equiv k_d \) (mod \( p \)) and \( b \equiv j_d \) (mod \( q \)) for some \( d \in A_q \). Because \( wa \in L(q,p) \cap D_{p,q}^n \), by the induction hypothesis there are at most \( q^{n+1} \) different choices for the word \( wa \). Let us fix \( wa \) and \( d \in A_q \) such that \( a \equiv k_d \) (mod \( p \)). To prove the claim, it is enough to show that there are at most \( q \) choices for the digit \( b \).

Let us assume to the contrary that there are distinct digits \( b_1, b_2, \ldots, b_{q+1} \in D_{p,q} \) such that \( wab_i \in L(q,p) \cap D_{p,q}^{n+1} \) whenever \( 1 \leq i \leq q + 1 \). For every \( i \) the congruence \( b_i \equiv k_d \) (mod \( p \)) holds for some \( d_i \in A_q \). By pigeonhole principle we may assume that \( d_1 = d_2 \) and therefore \( b_1 \equiv d_1 \equiv b_2 \) (mod \( p \)). Because \( wab_1, wab_2 \in L(q,p) \cap D_{p,q}^{n+1} \), we also have \( b_1 \equiv d_2 \equiv b_2 \) (mod \( q \)). Because \( b_1, b_2 \in A_{pq} \) are congruent both modulo \( p \) and modulo \( q \), they are equal, contradicting the distinctness of \( b_1, b_2, \ldots, b_{q+1} \).

As in the introduction, for relatively prime \( p > q > 1 \) and any \( S \subseteq [0,1) \) we denote

\[
Z_{p/q}(S) = \left\{ \xi > 0 \mid \left( \xi \left( \frac{p}{q} \right)^i \right) \in S \text{ for every } i \in \mathbb{N} \right\}.
\]

In [1] it was proved that if \( p, q > 1 \) are relatively prime integers such that \( p > q^2 \), then for every \( \epsilon > 0 \) there exists a finite union of intervals \( J_{p,q,\epsilon} \) of total length at most \( \epsilon \) such that \( Z_{p/q}(J_{p,q,\epsilon}) \neq \emptyset \). We extend this result to the case \( p > q > 1 \), which in particular covers \( p/q = 3/2 \). The following theorem by Akiyama, Frougny and Sakarovitch is needed.

Theorem 3.7 (Akiyama, Frougny, Sakarovitch [2]). If \( p \geq 2q - 1 \), then \( Z_{p/q}(Y_{p,q}) \neq \emptyset \), where

\[
Y_{p,q} = \bigcup_{d \in A_q} \left[ \frac{1}{p} k_d, \frac{1}{p} (k_d + 1) \right]
\]

and \( k_d \in A_p \) are as in Lemma 3.4.

Corollary 3.8. If \( p \geq 2q - 1 \), then \( Z_{p/q}(X_{p,q}) \neq \emptyset \), where

\[
X_{p,q} = \bigcup_{a \in D_{p,q}} \left[ \frac{1}{pq} a, \frac{1}{pq} (a + 1) \right].
\]

Proof. If \( \xi \in Z_{p/q}(Y_{p,q}) \), then \( \xi/q \in Z_{p/q}(X_{p,q}) \).

Theorem 3.9. If \( p \geq 2q - 1 \) and \( k > 0 \), then there exists a finite union of intervals \( I_{p,q,k} \) of total length at most \( (q/p)^k \) such that \( Z_{p/q}(I_{p,q,k}) \neq \emptyset \).

Proof. Let \( k > 0 \) be fixed and choose any \( \xi' \in Z_{p/q}(X_{p,q}) \), where \( X_{p,q} \) is the set in the previous corollary. Let \( \xi = \xi'(pq)^{-k-1}(p/q)^{k-1} \) and denote \( c = \text{config}_{pq}(\xi) \). Based on \( c \) we define a collection of words

\[
W = \{ w = e(1)e(2)\ldots e(k) \mid e = F_{p,q}^n(c) \text{ for some } n \in \mathbb{N} \}.
\]
The set $W$ determines a finite union of intervals

$$I_{p,q,k} = \bigcup_{w \in W} \left[ \text{real}_{pq}(w), \text{real}_{pq}(w) + (pq)^{-k} \right]$$

and $\xi \in Z_{p/q}(I_{p,q,k})$ by the definition of $W$. Each interval in $I_{p,q,k}$ has length $(pq)^{-k}$, so to prove that the total length of $I_{p,q,k}$ is at most $(q/p)^k$ it is sufficient to show that $|W| \leq q^{2k}$.

By the definition of $X_{p,q}$, $\text{Tr}_{p,q}(\text{config}_{pq}(\xi'))(i) \in D_{p,q}$ for every $i \geq 0$. For the $k$-trace of $c$

$$\text{Tr}_{p,q}(c,k)(i) = \text{Tr}_{p,q}(\text{config}_{pq}(\xi'(pq)^{(k-1)}(p/q)^{k-1}), k)(i)$$

$$= \text{Tr}_{p,q}(\text{config}_{pq}(\xi'))(i) = \text{Tr}_{p,q}(\text{config}_{pq}(\xi'))(i + (k-1))$$

for every $i \in \mathbb{N}$, from which it follows that $\text{Tr}_{p,q}(c,k)(i) \in D_{p,q}$ for every $i \geq -(k-1)$. Thus, the words in the set

$$V = \{ \text{Tr}_{p,q}(F^n_{p,q}(c), k)(-(k-1)) \ldots \text{Tr}_{p,q}(F^n_{p,q}(c), k)(k-1) \mid n \in \mathbb{N} \}$$

also belong to $L(p,q) \cap D_{p,q}^{2k-1}$, and by Corollary 3.3 and Lemma 3.6

$$|W| \leq |V| \leq |L(p,q) \cap D_{p,q}^{2k-1}| \leq q^{2k}.$$

\[\square\]

**Remark 3.10.** The set $I_{p,q,k}$ constructed in the proof of the previous theorem is a union of $q^{2k}$ intervals, each of which is of length $(pq)^{-k}$.

**Corollary 3.11.** If $p > q > 1$ and $\epsilon > 0$, then there exists a finite union of intervals $J_{p,q,\epsilon}$ of total length at most $\epsilon$ such that $Z_{p/q}(J_{p,q,\epsilon}) \neq \emptyset$.

**Proof.** Choose some $n > 0$ such that $p^n \geq 2q^n - 1$. Then by the previous theorem there exists a finite union of intervals $I_0$ of total length at most $\eta = \epsilon(p-1)/(p^n-1)$ such that $Z_{p^n/q^n}(I_0) \neq \emptyset$. For $0 < i < n$ define inductively

$$I_i = \left\{ \left\{ \frac{p^i}{q} \right\} \in [0,1) \ \big| \ \xi \geq 0 \text{ and } \{\xi\} \in I_{i-1} \right\},$$

each of which is a finite union of intervals of total length at most $p^i\eta$. Then $J_{p,q,\epsilon} = \bigcup_{i=0}^{n-1} I_i$ is a finite union of intervals of total length at most

$$\sum_{i=0}^{n-1} (p^i)\eta = \frac{p^n - 1}{p - 1}\eta = \epsilon$$

and $Z_{p/q}(J_{p,q,\epsilon}) \supseteq Z_{p^n/q^n}(I_0) \neq \emptyset$. \[\square\]

### 4 Ergodicity of $F_{p,q}$

In this section we study the measure theoretical properties of $F_{p,q}$ to prove the existence of large sets $S$ such that $Z_{p/q}(S)$ is empty.
Definition 4.1. A CA function $F : \mathbb{A}^\mathbb{Z} \to \mathbb{A}^\mathbb{Z}$ is measure preserving if
\[ \mu(F^{-1}(S)) = \mu(S) \] for every $S \in \Sigma(C)$.

Definition 4.2. A measure preserving CA function $F : \mathbb{A}^\mathbb{Z} \to \mathbb{A}^\mathbb{Z}$ is ergodic if for every $S \in \Sigma(C)$ with $F^{-1}(S) = S$ either $\mu(S) = 0$ or $\mu(S) = 1$.

The next lemma is a special case of a well known measure theoretical result (see e.g. Theorem 2.18 in [9]):

Lemma 4.3. For every $S \in \Sigma(C)$ and $\epsilon > 0$ there is an open set $U \subseteq \mathbb{A}^\mathbb{Z}$ such that $S \subseteq U$ and $\mu(U \setminus S) < \epsilon$.

Lemma 4.4. If $F : \mathbb{A}^\mathbb{Z} \to \mathbb{A}^\mathbb{Z}$ is an ergodic CA, then for every $\epsilon > 0$ there is a finite collection of cylinders $\{U_i\}_{i \in I}$ such that $\mu(\bigcup_{i \in I} U_i) < \epsilon$ and
\[
\left\{ c \in \mathbb{A}^\mathbb{Z} \mid F^t(c) \in \bigcup_{i \in I} U_i \text{ for some } t \in \mathbb{N} \right\} = \mathbb{A}^\mathbb{Z}.
\]

Proof. Let $C \in C$ be such that $0 < \mu(C) < \epsilon/2$. By continuity of $F$, $B = \bigcup_{t \in \mathbb{N}} F^{-t}(C)$ is open and $\mu(B) = 1$ by ergodicity of $F$ (see Theorem 1.5 in [10]). Equivalently, $B' = \mathbb{A}^\mathbb{Z} \setminus B$ is closed (and compact) and $\mu(B') = 0$. Let $V$ be an open set such that $B' \subseteq V$ and $\mu(V) < \epsilon/2$: such a set exists by Lemma 4.3. Because $\mathcal{C}$ is a basis of $\mathcal{T}$, there is a collection of cylinders $\{V_i\}_{i \in I}$ such that $V = \bigcup_{i \in I} V_i$. By compactness of $B'$ there is a finite set $I' \subseteq I$ such that $B' \subseteq \bigcup_{i \in I'} V_i$. Now $\{U_i\}_{i \in I} = \{C\} \cup \{V_i\}_{i \in I'}$ is a finite collection of cylinders such that $\mu(\bigcup_{i \in I} U_i) < \epsilon$ and
\[
\left\{ c \in \mathbb{A}^\mathbb{Z} \mid F^t(c) \in \bigcup_{i \in I} U_i \text{ for some } t \in \mathbb{N} \right\} \supseteq B \cup \bigcup_{i \in I'} V_i \supseteq B \cup B' = \mathbb{A}^\mathbb{Z}.
\]

To apply this lemma in our setup, we need to show that $F_{p,q}$ is ergodic for $p > q > 1$. In fact, it turns out that a stronger result holds.

Definition 4.5. A measure preserving CA function $F : \mathbb{A}^\mathbb{Z} \to \mathbb{A}^\mathbb{Z}$ is strongly mixing if
\[
\lim_{t \to \infty} \mu(F^{-t}(U) \cap V) = \mu(U)\mu(V)
\]
for every $U, V \in \Sigma(C)$.

We will prove that $F_{p,q}$ is strongly mixing. For the statement of the following lemmas, we define a function $\text{int} : A^k_{pq} \to \mathbb{N}$ by
\[
\text{int}(w(1)w(2)\ldots w(k)) = \sum_{i=0}^{k-1} w(k-i)(pq)^i,
\]
i.e. $\text{int}(w)$ is the integer having $w$ as a base $pq$ representation.

Lemma 4.6. Let $w_1, w_2 \in A^k_{pq}$ for some $k \geq 2$ and let $t > 0$ be a natural number. Then
1. $\text{int}(w_1) < q^t \implies \text{int}(G_{p,q}(w_1)) < q^{t-1}$ and
2. \( \text{int}(w_2) \equiv \text{int}(w_1) + q^t \pmod{(pq)^k} \implies \text{int}(G_{p,q}(w_2)) \equiv \text{int}(G_{p,q}(w_1)) + q^{t-1} \pmod{(pq)^{k-1}}. \)

Proof. Let \( c_i \in A_{pq}^k \) \((i = 1, 2)\) be such that \( c_i(-(k-1))c_i(-(k-1)+1) \ldots c_i(0) = w_i \) and \( c_i(j) = 0 \) for \( j < -(k-1) \) and \( j > 0 \). From this definition of \( c_i \) it follows that \( \text{int}(w_i) = \text{real}_{pq}(c_i) \). Denote \( c_i = G_{p,q}(c_i) \). We have

\[
\sum_{j=-(k-1)}^{\infty} e_i(-j)(pq)^j = \text{real}_{pq}(c_i) = p \text{real}_{pq}(c_i) = p \text{int}(w_i)
\]

and

\[
\text{int}(G_{p,q}(w_i)) = \text{int}(e_i(-(k-1)) \ldots e_i(-1))
\]

\[
= \sum_{j=1}^{k-1} e_i(-(j))(pq)^{j-1} \equiv \lfloor \text{int}(w_i)/q \rfloor \pmod{(pq)^{k-1}}.
\]

Also note that \( \text{int}(G_{p,q}(w_i)) < (pq)^{k-1} \).

For the proof of the first part, assume that \( \text{int}(w_1) < q^t \). Combining this with the observations above yields \( \text{int}(G_{p,q}(w_1)) \leq \lfloor \text{int}(w_1)/q \rfloor < q^{t-1} \).

For the proof of the second part, assume that \( \text{int}(w_2) \equiv \text{int}(w_1) + q^t \pmod{(pq)^k} \). Then there exists \( n \in \mathbb{Z} \) such that \( \text{int}(w_2) = \text{int}(w_1) + q^t + n(pq)^k \) and

\[
\text{int}(G_{p,q}(w_2)) \equiv \lfloor \text{int}(w_2)/q \rfloor \equiv \lfloor \text{int}(w_1)/q \rfloor + q^{t-1} + np(pq)^k \equiv \lfloor \text{int}(w_1)/q \rfloor + q^{t-1} \equiv \text{int}(G_{p,q}(w_1)) + q^{t-1} \pmod{(pq)^{k-1}}.
\]

\[\square\]

Lemma 4.7. Let \( t > 0 \) and \( w_1, w_2 \in A_{pq}^k \) for some \( k \geq 2t + 1 \). Then

1. \( \text{int}(w_1) < q^{2t} \implies \text{int}(F^t_{p,q}(w_1)) = 0 \) and

2. \( \text{int}(w_2) \equiv \text{int}(w_1) + q^{2t} \pmod{(pq)^k} \implies \text{int}(F^t_{p,q}(w_2)) \equiv \text{int}(F^t_{p,q}(w_1)) + 1 \pmod{(pq)^{k-2t}}. \)

Proof. First note that \( F_{p,q}(w) = G_{p,q}^2(w) \) for every \( w \in A_{pq}^\ast \) such that \( |w| \geq 3 \), because \( F_{p,q} = \sigma_{A_{pq}}^{-1} \circ G_{p,q} \circ G_{p,q} \). Then both claims follow by repeated application of the previous lemma.

\[\square\]

The content of Lemma 4.7 is as follows. Assume that \( \{w_i\}_{i=0}^{(pq)^k-1} \) is the enumeration of all the words in \( A_{pq}^k \) in the lexicographical order, meaning that \( w_0 = 00 \ldots 00, w_1 = 00 \ldots 01, w_2 = 00 \ldots 02 \) and so on. Then let \( i \) run through all the integers between 0 and \( (pq)^k - 1 \). For the first \( q^{2t} \) values of \( i \) we have \( F_{p,q}^t(w_i) = 00 \ldots 00 \), for the next \( q^{2t} \) values of \( i \) we have \( F_{p,q}^t(w_i) = 00 \ldots 01 \), and for the following \( q^{2t} \) values of \( i \) we have \( F_{p,q}^t(w_i) = 00 \ldots 02 \). Eventually, as \( i \) is incremented from \( q^{2t}(pq)^{k-2t} - 1 \) to \( q^{2t}(pq)^{k-2t} \), the word \( F_{p,q}^t(w_i) \) loops from \( (pq - 1)(pq - 1) \ldots (pq - 1) \) back to \( 00 \ldots 00 \).

Theorem 4.8. If \( p > q > 1 \), then \( F_{p,q} \) is strongly mixing.
Proof. Firstly, because $F_{p,q}$ is surjective, the fact that $F_{p,q}$ is measure preserving follows from Theorem 5.4 in [5]. Then, by Theorem 1.17 in [10] it is sufficient to verify the condition
\[
\lim_{t \to \infty} \mu(F_{p,q}^{-t}(C_1) \cap C_2) = \mu(C_1) \mu(C_2)
\]
for every $C_1, C_2 \in \mathcal{C}$. Without loss of generality we may consider cylinders $C_1 = \text{Cyl}(v_1,0)$ and $C_2 = \text{Cyl}(v_2,i)$. Denote $l_1 = |v_1|$, $l_2 = |v_2|$ and let $t \geq i + l_2$ be a natural number.

Consider an arbitrary word $w \in A_{pq}^{2t+l_1}$ and its decomposition $w = w_1w_2w_3$, where $w_1 \in A_{pq}^{t+l_1}$, $w_2 \in A_{pq}^l$ and $w_3 \in A_{pq}^{t+l_1-i-l_2}$. The following conditions may or may not be satisfied by $w$ (see Figure 8):

1. $F_{p,q}^t(w) = v_1$
2. $w_2 = v_2$.

![Figure 8: Relations between the words $v_1$, $v_2$ and $w_1w_2w_3$.](image)

Note that if $w$ satisfies condition (1), then $F_{p,q}^t(\text{Cyl}(w,-t)) \subseteq C_1$, and otherwise $F_{p,q}^t(\text{Cyl}(w,-t)) \cap C_1 = \emptyset$. Also, if $w$ satisfies condition (2), then $\text{Cyl}(w,-t) \subseteq C_2$, and otherwise $\text{Cyl}(w,-t) \cap C_2 = \emptyset$. Let $W_t \subseteq A_{pq}^{2t+l_1}$ be the collection of those words $w$ that satisfy both conditions. It follows that

\[
\mu(F_{p,q}^{-t}(C_1) \cap C_2) = \mu \left( \bigcup_{w \in W_t} \text{Cyl}(w,-t) \right) = |W_t|(pq)^{-2t-l_1}.
\]

Next, we estimate the number of words $w = w_1w_2w_3$ in $W_t$. In any case, to satisfy condition (2), $w_2$ must equal $v_2$. Then, for any of the $(pq)^{i+l_1}$ choices of $w_1$, the number of choices for $w_3$ that satisfy condition (1) is between $(pq)^{i+l_1-i-l_2}/(pq)^{i} - q^{2t}$ and $(pq)^{i+l_1-i-l_2}/(pq)^{i} + q^{2t}$ by Lemma 4.7 (and the paragraph following it). Thus,

\[
((pq)^{i-l_2} - q^{2t}) (pq)^{i+i}(pq)^{-2t-l_1} \leq \mu(F_{p,q}^{-t}(C_1) \cap C_2) \leq ((pq)^{i-l_2} + q^{2t}) (pq)^{i+i}(pq)^{-2t-l_1},
\]

and as $t$ tends to infinity,

\[
\lim_{t \to \infty} \mu(F_{p,q}^{-t}(C_1) \cap C_2) = (pq)^{-l_2} \mu(C_1) \mu(C_2).
\]

\[\square\]

13
Theorem 4.9. If $p > q > 1$ and $\epsilon > 0$, then there exists a finite union of intervals $K_{p,q,\epsilon}$ of total length at least $1 - \epsilon$ such that $Z_{p/q}(K_{p,q,\epsilon}) = \emptyset$.

Proof. The previous theorem implies that $F_{p,q}$ is ergodic: if $S \in \Sigma(C)$ is such that $F_{p,q}^{-t}(S) = S$, then

$$\mu(S) = \lim_{t \to \infty} \mu(F_{p,q}^{-t}(S) \cap S) = \mu(S)\mu(S),$$

which means that $\mu(S) = 0$ or $\mu(S) = 1$.

Since $F_{p,q}$ is ergodic, by Lemma 4.4 there is a finite collection of cylinders $\{U_i\}_{i \in I}$ such that $\mu(\bigcup_{i \in I} U_i) < \epsilon$ and

$$\left\{ c \in \mathbb{A}_{pq}^{\mathbb{Z}} \mid F_{p,q}^t(c) \in \bigcup_{i \in I} U_i \text{ for some } t \in \mathbb{N} \right\} = \mathbb{A}_{pq}^{\mathbb{Z}}.$$

Without loss of generality we may assume that for every $i \in I$, $U_i = \text{Cyl}(w_i, 1)$ and $w_i \in \mathbb{A}_{pq}^k$ for a fixed $k > 0$. Consider the collection of words $W = \mathbb{A}_{pq}^k \setminus \{w_i\}_{i \in I}$ and define

$$K_{p,q,\epsilon} = \bigcup_{v \in W} \left[ \text{real}_{pq}(v), \text{real}_{pq}(v) + (pq)^{-k} \right].$$

The set $K_{p,q,\epsilon}$ has total length

$$\frac{|W|}{(pq)^k} = 1 - \frac{|I|}{(pq)^k} = 1 - \mu\left( \bigcup_{i \in I} U_i \right) \geq 1 - \epsilon.$$

Now let $\xi > 0$ be arbitrary and denote $c = \text{config}_{pq}(\xi)$. There exists a $t \in \mathbb{N}$ such that $F_{p,q}(c) \in \bigcup_{i \in I} U_i$, and equivalently, $F_{p,q}^t(c) \notin \bigcup_{v \in W}(\text{Cyl}(v, 1))$. This means that $\{\xi(p/q)^t\} \notin K_{p,q,\epsilon}$, and therefore $Z_{p/q}(K_{p,q,\epsilon}) = \emptyset$. \hfill $\square$

5 Conclusions

We have shown in Theorem 3.9 and Corollary 3.11 that for $p > q > 1$ and $\epsilon > 0$ there exists a finite union of intervals $J_{p,q,\epsilon}$ of total length at most $\epsilon$ such that $Z_{p/q}(J_{p,q,\epsilon}) \neq \emptyset$. Moreover, by following the proof of this result, it is possible (at least in principle) to explicitly construct the set $J_{p,q,\epsilon}$ for any given $\epsilon$. We have also shown in Theorem 4.9 that for $p > q > 1$ and $\epsilon > 0$ there exists a finite union of intervals $K_{p,q,\epsilon}$ of total length at least $1 - \epsilon$ such that $Z_{p/q}(K_{p,q,\epsilon}) = \emptyset$. The proof of this theorem is non-constructive.

Problem 5.1. Assume that $p > q > 1$. Is it possible to construct explicitly for every $\epsilon > 0$ a finite union of intervals $S$ such that the total length of $S$ is at least $1 - \epsilon$ and $Z_{p/q}(S) = \emptyset$?

References


