What do prospective mathematics teachers mean by “definitions can be proved”?

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The research reported here is part of an ongoing study[3] in which prospective middle school mathematics teachers' conceptions of definition are investigated through their responses to semi-structured interview questions about defining quadrilaterals. Here we present findings from their responses to a subset of the interview questions, with the purpose of understanding what they mean by the expression "definitions can be proved"- an expression commonly referenced, and considered as erroneous in the research literature. Analysis of the responses, through using thematic coding and Toulmin's (1958) scheme, revealed that participants attributed two different meanings to the phrase: (1) proving the claim that a written definition accurately designates an intended concept and (2) proving the concept being defined (erroneous). Based on our findings, we point to a reconsideration of the phenomenon by the research community.

Keywords: Meta-mathematical knowledge, conception of proof, conception of definition, prospective middle school mathematics teachers

Introduction

Most teacher education programs offer college level mathematics courses to strengthen prospective teachers' mathematical preparation. Although these courses provide rich mathematics content and experience of working with definitions and proofs, they do not include explicit information about mathematics at the meta-level (Azrou, 2017). Especially, learning to prove becomes a difficult task for university students (Stylianides & Stylianides, 2009). Previous studies highlight that students at all grade levels experience difficulties related to proofs (e.g., Azrou, 2017; Fiallo & Gutiérrez, 2017), which is most of the time considered as a consequence of an inaccurate understanding of what constitutes a proof (Weber, 2001). Various studies have also informed that prospective teachers lacked an accurate understanding of mathematical definitions (Leikin & Zazkis, 2010; Levenson, 2012). Indeed, that many teachers and students cannot differentiate between definitions and proofs or consider definitions as provable is a robust research finding (Edwards & Ward, 2004; Leikin & Zazkis, 2010; Levenson, 2012).

In this study, we delve deep into prospective teachers' reasoning about if definitions need to be proved or not. In case of occurrence, we investigate prospective teachers' expressions and examples, in order to find out what they mean by the expression "definitions can be/need to be proved." By using Toulmin's (1958) model of arguments, we look for the existence of concrete claims in participants' proof-related attempts, because their responses to interview questions provide the key information on "what is being proved" in their perspectives. By detecting the actual

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“proven”, we aim to find out underlying reasons of using this erroneous expression; which might be a first step in developing proper ways of remediation in teacher education programs.

We also refer to arbitrariness aspect of mathematical definitions (not of definitions, but of concepts) in answering our research question: What meaning do prospective mathematics teachers attribute to “proving/proof of a definition”? Since defining in mathematics is arbitrarily naming concepts (Vinner, 1991), concepts do not possess inherent truth-values. They are neither true nor false (Edwards & Ward, 2008). However, by “arbitrariness” we do not mean that definitions are adopted on a complete random base. Rather, we acknowledge that they are open to intelligent refinements through successive work of mathematicians (Lakatos, 1976). In this study, we position that both the concepts and definitions are “arbitrary”, in the sense we use the word. Concepts are “agreed upon conventions” (Levenson, 2012, p. 209); that is why they are arbitrarily named. On the other hand, definitions can arbitrarily be chosen among multiple equivalent definitions of a concept. We use the former in the case we report here.

Background

Two theoretical foundations were employed in this study. The concept definition-concept image distinction and Toulmin’s (1958) model of arguments are introduced in the next sections.

Concept Definition/Concept Image

Concept image is the collection of all mental representations associated with a particular concept in one’s cognitive structure; while the *concept definition* is the mathematical statement that designate that concept (Tall & Vinner, 1981). Although a concept definition is expected to connote the same meaning to everyone, concept image is specific to the individual and may not be fully compatible with the formal definition. In our study, the distinction between *concept image* and *concept definition* is a theoretical keystone for understanding prospective teachers’ hidden claims underlying their use of the erroneous expression “proving a definition”.

Toulmin’s (1958) Model of Arguments

Toulmin (1958) proposed a schema for describing an argument by identifying three main components. The model describes the connection between the claim (C) - that is desired to be established- and the data (D) - the fact, which can serve as a basis for establishing that claim. For the argument to take the arguer from the data to the claim, another element is defined: The warrant. Warrants can be rules, principles or inference-licenses entitled to “show that, taking these data as a starting point, the step to the original claim or conclusion is an appropriate and legitimate one” (Toulmin, 2003, p.91). The three elements constitute the simplest form of the model. The complete model includes additional elements of backing (B), modal qualifier (Q), and the rebuttal (R). Backing is a further evidence for the connection between data and claim, modal qualifier associates a degree of confidence to the conclusion made; and rebuttal states the conditions under which the conclusion is not valid. However, not all arguments have to contain these latter three elements.

Given that proof is a mathematical argument produced with the purpose of convincing oneself and others of the truth of a mathematical statement (Fiallo & Gutiérrez, 2017), we apply Toulmin’s model on prospective teachers’ productions of exemplary proofs, in order to identify the actual claims they aim to prove, while expressing it as “proving a definition”. Since our focus is on identifying any existing claims in participants’ examples, we use the simplest form of the model, consisting only of the three elements data (D), claim (C) and warrant (W).

Method

Basic qualitative research methods were used in this study.

Context and Participants of the Study

Participants of the study were six senior (4th-year) prospective middle school mathematics teachers in a four-year teacher education program. The program, prepared around 40 mathematics teachers each year to teach at the grade levels from 5 to 8, by offering college-level mathematics courses mostly in the first two years and concentrating more on the teaching-related courses in the last two years. Participants were selected based on their active participation in the educational courses, their inclination to express and discuss mathematical ideas and the variation in their knowledge of mathematics, as observed in the teaching related courses by the authors. All six participants volunteered to participate in the study as an out-of-class activity. Since the data were collected through the end of the academic year, they had nearly completed all the courses in the program. Although the program included the study of undergraduate level mathematics courses (offered by the Mathematics Department) in which definitions and proofs played important roles, students had not been offered any specific information about meta-mathematical constructs of definitions and proofs in these courses. Also, it may worth to highlight that a detailed chapter on the geometry terms, especially the hierarchical way of defining quadrilaterals were covered in the mathematics teaching methods course that participants took in their third year.

Data Collection and Analysis

In semi-structured interviews conducted by the first author in one-to-one settings, prospective teachers responded to a broad range of verbal and task-based questions aimed at revealing their understanding of mathematical definitions. After completing an initial open-ended task about defining quadrilaterals (participants were asked to propose a sequence for introducing quadrilaterals, by supplying their own definitions), they were asked questions such as “What is a mathematical definition for you?” and “Why do we state definitions in mathematics?” One of the questions asked participants to explain their thinking about the relationship between definitions and proofs. They were explicitly asked to indicate if definitions need/have proofs or not, and explain their reasoning. In case of accepting definitions as provable, they were requested to give an example. Participants’ responses were analyzed through thematic coding procedure (Braun & Clarke, 2006) and Toulmin’s (1958) model of arguments was used to describe their examples.

Findings

Analyses of prospective teachers’ explanations and examples resulted in two different interpretations of the expression “proving a definition”. In particular, prospective teachers considered “proving a definition” as (1) justifying the claim that a written definition accurately designates an intended concept (which would be an appropriate action in the discipline of mathematics) and (2) justifying the concept being defined (which remains ambiguous in meaning).

Table 1 presents a summary of each participant's thinking about the phrase, as inferred from their verbal explanations and concrete examples, which were mostly different from what they explicitly said. A representative quotation from each participant is given in order to reveal their use of words in their conversations. Two separate rows are used for the participant (P5) who displayed both type of interpretations.

PST	Sample Wording PST Used	Meaning Inferred from Further Explanations (through thematic coding)	Meaning Inferred from Example Case (through Toulmin's scheme)
P1	"I am proving the triangle (definition)."	Proving the claim	Proving the claim
P2	"I proved the truth of this definition."	Proving the claim	Proving the claim
P3	"Definitions should be proved."	Proving the claim	(Did not provide)
P4	"I could not understand what you mean?"	Proving the claim (if such a thing exists)	Proving the claim
P5	"By ..., we can prove definitions."	Proving the claim	Proving the claim
P5	"After the shape square has been proved."	Proving the concept	(Did not provide)
P6	"If I am defining a concept, it has nothing to do with proving."	Proving the concept (non-existent)	(Not applicable)

Table 1: Meaning attributed to the phrase of "proving a definition" in prospective teachers' (PST) explanations and examples

The two types of interpretation resulting from participants' responses are described in the following sections. In the reporting of quotations and examples, brackets are used for indicating the authors' insertions, and square brackets are used either for indicating excluded parts of the interview (with ellipsis: […]) or for specifying the components of the participant's arguments (i.e., [data], [claim], and [warrant]).

Interpretation I: "Proving a definition" as "proving the claim that a written definition designates the intended concept"

Four of the six prospective teachers (P1, P2, P3, and P5) indicated that definitions "can be proved" (P1) or "need to be proved" (P3). Although their wording did not reflect the existence of an explicit claim to be proved (e.g., "proving the triangle (definition)" (P1); "I will give a definition and prove it." (P2)), their explanations and examples revealed that what they considered provable was an actual claim. In particular, they were talking about proving that a written definition truly reflects the image of the intended concept in their minds. Following scripts from P1 and P2 illustrate the case.

Researcher: The proof you thought of there… What exactly is that proof of?

P1: Of the triangle (definition), in fact.

Researcher: How is that? Could you open this up a little?

P1: Of the triangle (definition). It is about the shape of the triangle. I mean, in our minds there is a shape about… the shape of the triangle and we are trying to make this definition fit to it. With the proof, we check if it fits the shape or not.

Researcher: Is there a relationship between definition and proof?

P2: Yes. In order to prove that the definition we create is true, we do proofs. I mean I create a definition; but to what extent is that true, when it holds? Maybe under some conditions it does not hold. For proving this, proofs are written.

Their examples were based on proving concrete claims as well. They were basically comparing a written definition with the corresponding concept image in their minds to evaluate their congruence.

Researcher: Now, what exactly is that you try to prove here? […] Can you give me an example?

P2: I will give a definition and prove it. Hmm… Let me take the parallelogram. Opposite sides need to be equal in length and parallel (reads the definition she wrote in a previous task), I say. I will prove this. (Draws the figure that satisfies the given conditions.) [data] Actually, by drawing this (points to the figure she drew) [warrant: the figure fits into her concept image] I proved it [claim].

Researcher: What is that you proved here?

P2: Properties of the parallelogram. I try if it does hold for the given definition. I do some trials and then I see that it holds for this definition. And I proved the truth of this definition, I mean.

Figure 1 presents P2's example case of "proving" by using Toulmin's (1958) model arguments.

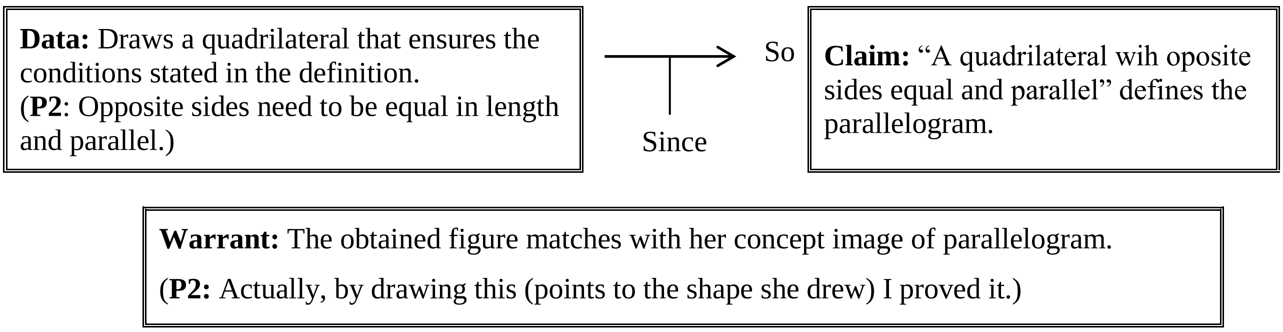

Figure 1: An argument schema for P2's example

On the other hand, the idea of "proof of a definition" did not make any sense to P4 first. However, when she thought over it by the help of an example case, she ended up with the same interpretation as the previous participants:

P4: Let me think about this. Here, I had written something for rectangle (in a previous task). There I have defined rectangle as the parallelogram with a right angle. Now, am I required to prove that this definitely defines the rectangle?

Her exemplary proof attempt provided an accurate representation of the participants' thinking.

P4: I know what a parallelogram is (Draws a parallelogram-angles very close to 90°). [data] Now, I should have drawn a real rectangle. It is OK, if I do not. One of the angles is 90 degrees [data], I started here (marks one of the angles with the perpendicularity symbol). I know that in a parallelogram opposite sides are parallel. Then, these (two adjacent angles) add up to 180 degrees. This is also 90 (degrees). […] Then, its all interior angles are 90 degrees. Opposite sides are parallel, equal and so forth… It satisfies all the properties of rectangle. [warrant][…] I mean, I can prove that this definition is rectangle. [claim]

Interpretation II: "Proving a definition" as "proving the concept being defined"

Two of the prospective teachers perceived "proving a definition" differently. They maintained the odd wording of "proving a concept" in their explanations (e.g., "proving the shape square" (P5)). However, their approaches to this idea were different from each other's. P5 thought that it was a possible action to "prove a concept". Although she could not elaborate much on this idea of her, since she demonstrated two different meanings at the same time (both proving the claim and proving the concept) it was evident that she was talking about an issue different than proving the claim in the following dialog:

Researcher: Is there a relationship between definition and proof?

P5: I cannot say absolutely there is, but I think should be. […] After the shape square has been proved, it must have fit to its definition. Otherwise, if we do not know what is the thing that we call square, without proving this, we cannot make the definition.

Researcher: What is it that we prove here?

P5: Which shapes we call "square"? How does the square come into existence?

Researcher: Can you give me an example of this?

P5: I do not know. Now… I can't find.

Immediately after, when she was asked if definitions were provable or not, she demonstrated the same understanding of "proving a claim", similar to what the previous participants did:

P5: We draw the multiple shapes of what we do (define), I mean by looking for counterexamples, we can prove definitions.

Her example also supported that she was proving a claim (whether a given definition of square would actually define square or not), although she relied on empirical reasoning in her argument.

P5: Let me think with square again. More than one person draws its definition [data], because it may not represent the same thing to everyone. We check if it does represent the same thing to everyone. If one person draw a thing that is different from what we try to explain [warrant], then that means the definition we have is not correct or not clear, erroneous [claim]. In this way, I think we can prove it.

Unlike P5, P6 seemed to be aware of the fact that "defining was arbitrarily naming concepts" and hence definitions (concepts) needed no justification.

P6: Of course there may be (a relationship in between), but if I am trying to name something, if it is something like a term… You see, here when I am trying to define the trapezoid, I am not proving the properties of the trapezoid […] Because I am just giving it a name.

As we consider that they are the concepts which are arbitrary, rather than the defining statements (in our case), we name this second type of interpretation with the phrase "proving the concept being defined". Both P5 and P6 seem to be thinking about proving "why concepts exist in mathematics as they are". While P6 correctly rejects this kind of thinking about definitions, P5 seem to consider it as a necessity. Also, P5's erroneous understanding may still be residing in other participants' minds, as prospective teachers may not be aware of arbitrariness aspect of definitions.

Discussion and Implications

Findings of the study provide insights into participating prospective teachers' conceptions of proof. Although at the first glance they seem to be trying to prove a non-claim, existence of a real claim in their arguments reveals that they have an implicit (because they do not say so) insight about what needs a proof in mathematics. This is an unexpected finding and a positive outcome for teacher education programs compared to previous research findings, because no such claims were proposed by the participants of other studies who communicated that definitions could be proved (Levenson, 2012) or who could not distinguish between a theorem and a definition (Edwards & Ward, 2004; Leikin & Zazkis, 2010). However, this finding does not necessarily mean that participants are also sure of what cannot be proved in mathematics. P5's explanations displayed that one of the things she tried to prove was a claim (Interpretation I), while the other was not (Interpretation II). The same might be the case for all of the participants of the study, except P6; but might have remained uncovered in our interviews, because no participants other than P6 mentioned the arbitrariness aspect of defining concepts in their responses. They did not reveal any thinking about if concepts were provable or not, as P5 and P6 did. This addresses that while interviewing prospective teachers, handling the nature of proofs and definitions concurrently and from multiple aspects might provide a more complete picture of their meta-mathematical knowledge. Otherwise we might end up with unrealistic judgments of prospective teachers' knowledge and understandings.

On the other hand, our observation that most of the prospective teachers attributed the same acceptable meaning (Interpretation I) to the principally imperfect phrase of "proving a definition," have important implications about the common practice of using the words "proof" and "proving" imprecisely. Besides not questioning the misuse of the word "proving" in the question we directed to them (P4 did only); most of the participants consistently used unclear wordings such as "proving the triangle" (P1) in their explanations. Also, the inconsistency between what they do (or think) and how they talk about it was striking; which would probably have a negative influence on their future students' learning of mathematics at the meta-level. Previous studies acknowledge the need for discussing notions of definition and proof in teacher education programs, along with the other meta-mathematical constructs such as assumptions and axioms, and the interrelationships among them

(Levenson, 2012). Based on the findings of our study, we want to point out to the importance of using the meta-mathematical terms "proof" and "definition" rigorously within such discussions.

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