On Kolmogorov quotients

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Abstract

Every topological space has a Kolmogorov quotient that is obtained by identifying points if they are contained in exactly the same open sets. In this survey, we look at the relationship between topological spaces and their Kolmogorov quotients. In most natural examples of spaces, the Kolmogorov quotient is homeomorphic to the original space. A non-trivial relationship occurs, for example, in the case of pseudometric spaces, where the Kolmogorov quotient is a metric space. The author's interest in the subject was sparked by study of abstract model theory, specifically the paper [1] by X. Caicedo, where Kolmogorov quotients are used in a topological proof of Lindström's theorem. We omit the proofs in this extended abstract; a full version [2] with detailed proofs is in preparation.

1 Introduction

Given a topological space X, we obtain its *Kolmogorov quotient* $X/\equiv$ by identifying points x and y if and only if they have exactly the same open neighbourhoods. Such points are topologically indistinguishable; there is no sequence of operations on open sets that would give a set A such that $x \in A$ and $y \notin A$. Nothing topologically important to the space X is lost in identifying these points.

The resulting space is a T_0-*space*: a space where all points are topologically distinguishable. Most topological spaces of interest are T_0. A T_0-space is, arguably, aesthetically more pleasing than a space that is not T_0. In a

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T_0-space, every point serves a purpose. When studying the topology of X, there seems to be no reason to keep useless, superfluous points around.

The construction of the Kolmogorov quotient is simple, intuitive, and can be carried out for any topological space. If a mathematician comes across a space that is not naturally T_0, the unnecessary points can be left out from the space right at the beginning and the original space forgotten. Perhaps for this reason, the construction is not even mentioned in most textbooks on topology, and where it is mentioned, this is done very briefly, and proofs are generally omitted.

However, there are situations where it is inconvenient if a space is T_0. Such a situation occurs when one is interested in refinements of the topology: the more points there are in X, the more choices there are for refinements. The same is true for subspaces, though the loss here is not so dramatic: for each subspace $S \subseteq X$ that we lose, $X/\equiv$ retains a subspace homeomorphic to $S/\equiv$. Still, if one is interested in the specific points of the space, one might not wish to clump them together in equivalence classes.

Removing the T_0-property from a space can generate new properties for topological spaces. Given a property P (for example, the Hausdorff separation axiom T_2) of a T_0-space we obtain a new property P' by defining: a space X has the property P' if and only if $X/\equiv$ has the property P. Generally the arising property is interesting in itself and admits a more direct definition. In a similar vein, given a structure S (for example, a metric) on a T_0-space we can define: a space X has the structure S' if and only if $X/\equiv$ has the property S.

This survey is not about T_0-spaces, but focuses rather on the relationship between spaces and their Kolmogorov quotients. It appears that no comprehensive treatment on the matter has been published, and as stated before, standard textbooks often omit the construction entirely. As our sources don't usually give proofs, it seems unnecessary to cite each theorem individually. Various results presented here can be found without proofs in [1] and [3]. The notes in [4] contain some proofs. We present the results in a more general form when possible.

2 Kolmogorov quotients

Given a topological space X and a subset $A \subseteq X$, we write A^c for the complement $X \setminus A$ and $\overline{A}$ for the closure of A. We denote the Borel algebra of X by Σ_X and the collection of (not necessarily open) neighbourhoods of $x \in X$ by $\mathcal{N}(x)$.

Let X be a topological space. We define an equivalence relation $\equiv \subseteq X^2$

by letting $x \equiv y$ if and only if every open neighbourhood of x is an open neighbourhood of y and vice versa. If $x \equiv y$, we say that the points x and y are *topologically indistinguishable*. Otherwise they are *topologically distinguishable*, and we write $x \not\equiv y$. A space where all pairs of distinct points are topologically distinguishable is called a *T_0-space* or a *Kolmogorov space*. Most spaces studied by mathematicians are T_0.

Example 2.1. A space with the trivial topology is not T_0, unless it has less than two points.

Example 2.2. All Hausdorff spaces are T_0. This includes all discrete spaces and the space $\mathbb{R}$ with the euclidean topology.

Example 2.3. Let $X = \{0, 1\}$ and $\tau = \{\emptyset, \{1\}, \{0, 1\}\}$. The *Sierpiński space* (X, τ) is T_0 but not Hausdorff.

Example 2.4. The product of $\mathbb{R}$ with the euclidean topology and $\mathbb{R}$ with the trivial topology is not T_0: indeed, the points $(1, 0)$ and $(1, 1)$ are topologically indistinguishable.

We will see more examples later. In the meanwhile, the following lemma should provide intuition into topological indistinguishability.

Lemma 2.5. *Let X be a topological space and $x, y \in X$. The following statements are equivalent:*

(i) *$x \equiv y$;*
(ii) *$\mathcal{N}(x) = \mathcal{N}(y)$;*
(iii) *x and y are contained in the same basic open sets;*
(iv) *x and y are contained in the same subbasic open sets;*
(v) *x and y are contained in the same open sets;*
(vi) *x and y are contained in the same closed sets;*
(vii) *$\overline{\{x\}} = \overline{\{y\}}$;*
(viii) *x and y are contained in the same Borel sets;*
(ix) *a filter or net that converges to x, converges also to y, and vice versa;*
(x) *a filter or net that has x as a cluster point, has also y as a cluster point, and vice versa.*

Example 2.6. Let $U_m = \{n \in \mathbb{N} \mid m \text{ divides } n\}$ for all $m \in \mathbb{Z}_+$. Then $\mathcal{S} = \{\mathbb{N}\} \cup \{U_p \mid p \text{ is a prime}\}$ is a subbasis of a topology on $\mathbb{N}$. By lemma 2.5, $x \equiv y$ if and only if x and y have the same prime factors.

Given a topological space X, we denote by $\eta(x)$ the equivalence class of $x \in X$ with respect to $\equiv$, that is, $\eta(x) = \{y \in X \mid y \equiv x\}$. The following theorem gives a simple formula for the equivalence classes.

Theorem 2.7. *Let (X, τ) be a topological space. For all $x \in X$,*

$$\eta(x) = \overline{\{x\}} \cap \bigcap_{\substack{U \in \tau \\ x \in U}} U = \bigcap_{\substack{B \in \Sigma_X \\ x \in B}} B.$$

Corollary 2.8. *For all $x \in X$,*

$$\eta(x) \subseteq \bigcap_{U \in \mathcal{N}(x)} U.$$

Corollary 2.9. *For all $x \in X$, $\eta(x) \subseteq \overline{\{x\}}$.*

Given a topological space X, we define $X/{\equiv}$ as the topological space, where the space as a set is the set of equivalence classes under $\equiv$, and the topology is the finest such topology that the quotient map $\eta\colon X \to X/{\equiv}$ that maps each element $x \in X$ to its equivalence class $\eta(x)$ is continuous. In other words, the open sets of $X/{\equiv}$ are precisely those sets whose preimage under η is open in X. We call the space $X/{\equiv}$ the *Kolmogorov quotient* of X.

Clearly the Kolmogorov quotient is always a Kolmogorov space. A space is T_0 if and only if it is homeomorphic to the Kolmogorov quotient of itself.

The continuity of η already lets us know some things about the relationship between X and $X/{\equiv}$; for example, if $A \subseteq X$ is compact, then so is $\eta(A)$.

Example 2.10. Take the set $X = \{1, 2, 3, 4\}$ with the clopen basis $\{\{1, 2\}, \{3, 4\}\}$. The Kolmogorov quotient $X/{\equiv}$ is the two-element set $\{\eta(1), \eta(3)\} = \{\{1, 2\}, \{3, 4\}\}$ with the discrete topology.

Example 2.11. The Kolmogorov quotient of any nonempty set with the trivial topology is a space consisting of a single point.

Example 2.12. Let $p \geq 1$. Let L^p be the set of all measurable functions f from a measure space (S, Σ, μ) to $\mathbb{R}$ such that

$$\int_S |f|^p \,\mathrm{d}\mu < \infty.$$

Denote

$$\|f\|_p = \left(\int_S |f|^p \,\mathrm{d}\mu \right)^{\frac{1}{p}}.$$

The map $f \mapsto \|f\|_p$ is a *seminorm*: there are functions f other than the zero function for which $\|f\|_p = 0$, but all other properties of a norm are satisfied. In the Kolmogorov quotient $\mathcal{L}^p = L^p/{\equiv}$, this seminorm becomes a norm. The spaces $\mathcal{L}^p$ are important in analysis and measure theory ([5]).

Example 2.13. A discrete version of example 2.12 is obtained by taking the measure space $\mathbb{N}$ with the counting measure i.e. the measure of a subset of $\mathbb{N}$ is its cardinality. In this case, the space consists of sequences converging to 0, and

$$\|(x_n)\|_p = \left(\sum_{n=0}^{\infty} |x_n|^p \right)^{\frac{1}{p}}.$$

Based on the quotient map $\eta\colon X \to X/{\equiv}$, we define two maps $\eta^{\to}\colon \Sigma_X \to \Sigma_{X/\equiv}$ and $\eta^{\leftarrow}\colon \Sigma_{X/\equiv} \to \mathcal{P}(X)$ as follows:

$$\eta^{\to}(B) = \eta(B) = \{\eta(x) \mid x \in B\},$$

and

$$\eta^{\leftarrow}(B') = \eta^{-1}(B') = \{x \in X \mid \eta(x) \in B'\}$$

for all $B \in \Sigma_X$ and $B' \in \Sigma_{X/\equiv}$.

Theorem 2.14. *The map $\eta^{\to}$ is an isomorphism between the Boolean algebras Σ_X and $\Sigma_{X/\equiv}$.*

Corollary 2.15. *The quotient map η is open, i.e. if $A \subseteq X$ is open, then $\eta^{\to}(A)$ is open.*

Corollary 2.16. *The quotient map η is closed, i.e. if $A \subseteq X$ is closed, then $\eta^{\to}(A)$ is closed.*

Lemma 2.17. *Let X and Y be topological spaces and $f\colon X \to Y$ continuous. If $x_1 \equiv x_2$ for some $x_1, x_2 \in X$, then $f(x_1) \equiv f(x_2)$.*

Theorem 2.18. *Let $\eta_X\colon X \to X/{\equiv}$ and $\eta_Y\colon Y \to Y/{\equiv}$ be the quotient maps and $f\colon X \to Y$ an arbitrary continuous map. Then there exists a continuous map $f_{\equiv}\colon X/{\equiv} \to Y/{\equiv}$ such that the diagram below commutes.*

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle \eta_X}\big\downarrow & & \big\downarrow{\scriptstyle \eta_Y} \\ X/{\equiv} & \xrightarrow{\;f_{\equiv}\;} & Y/{\equiv} \end{array}$$

Choosing a representative from each equivalence class gives the following theorem, which states that all topological properties of the Kolmogorov quotient of X hold also in a dense subspace of X. If there are infinitely many equivalence classes, then the axiom of choice is required.

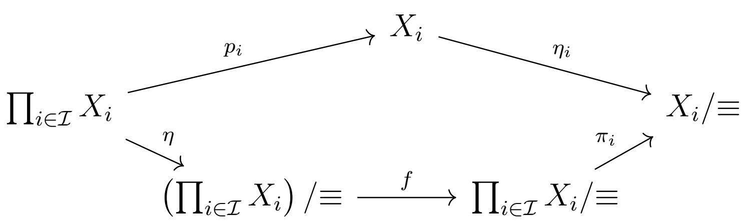
Theorem 2.19. *The space $X/{\equiv}$ is homeomorphic to a dense subspace of X.*

The Kolmogorov quotient may have fewer subspaces than the original space. The following theorem tells that the quotients of the lost subspaces are still subspaces of $X/\equiv$, up to homeomorphism.

Theorem 2.20. *Let X be a topological space and S a subspace of X. Then the space $S/\equiv$ is homeomorphic to some subspace of $X/\equiv$.*

Sketch of proof. Let $\eta\colon X \to X/\equiv$ and $\eta_S\colon S \to S/\equiv$ be the quotient maps. Let $f\colon S/\equiv \to \eta(S)$, $f(\eta_S(x)) = \eta(x)$ for all $\eta_S(x) \in S/\equiv$. The map f is a homeomorphism when $\eta(S)$ is considered as a subspace of $X/\equiv$. □

Theorem 2.21. *Let $\mathcal{I}$ be a set and $(X_i)_{i\in\mathcal{I}}$ a sequence of topological spaces. The spaces $\left(\prod_{i\in\mathcal{I}} X_i\right)/\equiv$ and $\prod_{i\in\mathcal{I}} X_i/\equiv$ are homeomorphic.*



Sketch of proof. Let η be the quotient map from $\prod_{i\in\mathcal{I}} X_i$ to $\left(\prod_{i\in\mathcal{I}} X_i\right)/\equiv$, and let η_i be the quotient map from X_i to $X_i/\equiv$ for all $i \in \mathcal{I}$. Define a map $f\colon \left(\prod_{i\in\mathcal{I}} X_i\right)/\equiv \to \prod_{i\in\mathcal{I}} X_i/\equiv$ from the condition $f(\eta(z))(i) = \eta_i(z(i))$ for all $i \in \mathcal{I}$ and all $z \in \prod_{i\in\mathcal{I}} X_i$. The diagram above should commute. The maps p_i and π_i are the canonical projections. The map f is a homeomorphism. □

3 Properties of spaces compared to properties of their Kolmogorov quotients

The *separation axioms* are properties a topological space can have that guarantee the existence of disjoint neighbourhoods in various situations. The separation axioms are ordered so that T_i implies T_j whenever $i \geq j$. There is also another set of analogous properties called the *regularity axioms* such that $T_i = R_{i-1} \wedge T_0$. In other words, a space satisfies T_i if and only if it is a Kolmogorov quotient of a space that satisfies R_{i-1}. Table 1 shows the connection. Some authors require normal and regular spaces to be Hausdorff; we do not.

A topological space X is *symmetric* if for all pairs of topologically distinguishable points $x, y \in X$, there are open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

Table 1: The connection between separation and regularity axioms

$X/\equiv$	X
Kolmogorov (T_0)	topological space
Fréchet (T_1)	symmetric (R_0)
Hausdorff (T_2)	preregular (R_1)
regular Hausdorff (T_3)	regular (R_2)
Tychonoff ($T_{3.5}$)	completely regular ($R_{2.5}$)
normal Hausdorff (T_4)	normal regular (R_3)
completely normal Hausdorff (T_5)	completely normal regular (R_4)
perfectly normal Hausdorff (T_6)	perfectly normal regular (R_5)

Theorem 3.1. *If X is symmetric, then $\eta(x) = \overline{\{x\}}$ for all $x \in X$.*

A topological space X is *preregular* if for all pairs of topologically distinguishable points $x, y \in X$, there are open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Theorem 3.2. *If K_1 and K_2 are disjoint compact subsets of a preregular topological space X and do not have disjoint open neighbourhoods, then there exist $x_1 \in K_1$ and $x_2 \in K_2$ such that $x_1 \equiv x_2$.*

A *pseudometric on X* is a function $d\colon X^2 \to \mathbb{R}$ that satisfies all the properties of a metric, except it is possible that $d(x, y) = 0$ even if $x \neq y$. A pseudometric determines a topology in the same way a metric does. The resulting topological space is called a *pseudometric space* and can be denoted by (X, d).

Example 3.3. Pseudometrics can be used in the context of cellular automata. Given a finite set A, let $A^{\mathbb{Z}}$ denote the set of functions from $\mathbb{Z}$ to A. For $x \in A^{\mathbb{Z}}$, we write x_j for $x(j)$. Also, for n, $k \in \mathbb{Z}$, let $[n, k]$ denote the set of integers m such that $n \leq m \leq k$. Finally, for sequences $(a_n)_{n=0}^{\infty}$ of natural numbers, denote

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \left(\sup_{m\geq n} a_m \right).$$

Then

$$d_B(x, y) = \limsup_{l\to\infty} \frac{|\{j \in [-l, l] \mid x_j \neq y_j\}|}{2l + 1}$$

is the *Besicovitch pseudometric* on $A^{\mathbb{Z}}$, and

$$d_W(x, y) = \limsup_{l\to\infty} \max_{k\in\mathbb{Z}} \frac{|\{j \in [k + 1, k + l] \mid x_j \neq y_j\}|}{2l + 1}$$

is the *Weyl pseudometric* on $A^{\mathbb{Z}}$. The topologies induced by these pseudometrics have some advantages to the standard approach, where A is given the discrete topology and $A^{\mathbb{Z}}$ the product topology; for example, the class of continuous functions from $A^{\mathbb{Z}}$ to itself is larger [6].

Theorem 3.4. *Let (X, d) be a pseudometric space. Then $d^*\colon (X/{\equiv})^2 \to \mathbb{R}$, $d^*(\eta(x), \eta(y)) = d(x, y)$ for all $x, y \in X$, is a metric on $X/{\equiv}$ that determines the same topology as the quotient map.*

The space $(X/{\equiv}, d^*)$ is called the *metric identification of* (X, d).

We mentioned seminorms in example 2.12. A (semi)norm on V induces a (pseudo)metric on V by defining $d(x, y) = \|x - y\|$ for all $x, y \in X$. The resulting topological space is called a *(semi)normed vector space* and can be denoted by $(V, \|\cdot\|)$.

Theorem 3.5. *Let $(V, \|\cdot\|)$ be a seminormed vector space. Then $(V/{\equiv}, \|\cdot\|^*)$ is a normed vector space, where*

$$\begin{aligned} \lambda\eta(x) &= \eta(\lambda x) && \text{for all } \lambda \in K, x \in V, \\ \eta(x) + \eta(y) &= \eta(x + y) && \text{for all } x, y \in V, \end{aligned}$$

and

$$\|\eta(x)\|^* = \|x\| \qquad \text{for all } x \in V.$$

Furthermore, $\|\cdot\|^*$ determines the same topology as the quotient map.

A space is *Alexandrov-discrete* if all intersections of open sets are open. All finite spaces are Alexandrov-discrete, as is the space of natural numbers with a basis consisting of the sets $V_n = \{m \in \mathbb{N} \mid m \geq n\}$.

Theorem 3.6 ([7]). *If X is an Alexandrov-discrete space, then η is a homotopy equivalence.*

The proof of theorem 3.6 uses the axiom of choice.

References

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