An optimal strongly identifying code in the infinite triangular grid

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Abstract

Assume that $G = (V, E)$ is an undirected graph, and $C \subseteq V$. For every $\mathbf{v} \in V$, we denote by $I(\mathbf{v})$ the set of all elements of C that are within distance one from $\mathbf{v}$. If the sets $I(\mathbf{v}) \setminus \{\mathbf{v}\}$ for $\mathbf{v} \in V$ are all nonempty, and, moreover, the sets $\{I(\mathbf{v}), I(\mathbf{v}) \setminus \{\mathbf{v}\}\}$ for $\mathbf{v} \in V$ are disjoint, then C is called a strongly identifying code. The smallest possible density of a strongly identifying code in the infinite triangular grid is shown to be 6/19.

1 Introduction

Assume that $G = (V, E)$ is an undirected graph with vertex set V and edge set E. A subset $C \subseteq V$ is called a **code** in G, and its elements are called **codewords**.

The distance $d(\mathbf{u}, \mathbf{v})$ between two vertices $\mathbf{u}$ and $\mathbf{v}$ is the number of edges on any shortest path between them.

For all $\mathbf{v} \in V$ we denote

$$I(\mathbf{v}) = \{\mathbf{c} \in C : d(\mathbf{c}, \mathbf{v}) \leq 1\}.$$

If we denote by $B_r(\mathbf{v})$ the ball of radius r with centre $\mathbf{v}$, then $I(\mathbf{v}) = C \cap B_1(\mathbf{v})$.

If all the sets $I(\mathbf{v})$ are nonempty and pairwise different, then C is called an **identifying code**. This concept was introduced in [8] in connection with studying multiprocessor

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architectures. Such an architecture can be viewed as a graph, where each vertex represents a processor, and each edge represents a dedicated link between two processors. Assume that at most one of the processors is malfunctioning. Each of the chosen codewords $\mathbf{c}$ tests the sets $B_1(\mathbf{c})$ and reports YES if it detects a problem and NO otherwise. The fact that C is identifying implies that based on the reports, we can uniquely identify the one malfunctioning processor or tell that everything is fine.

Strongly identifying codes were introduced in [7] (in a more general form); cf. also [14].

Definition 1. *A code C in the graph $G = (V, E)$ is called* **strongly identifying** *if all the sets $I(\mathbf{v}) \setminus \{\mathbf{v}\}$ for $\mathbf{v} \in V$ are nonempty, and, moreover, the sets $\{I(\mathbf{v}), I(\mathbf{v}) \setminus \{\mathbf{v}\}\}$ for $\mathbf{v} \in V$ are disjoint.*

Here the idea is that a malfunctioning processor may or may not be able to send a correct report, and we need a slightly stronger code. Clearly, a strongly identifying code is also identifying.

Strongly identifying codes have also been studied in [9] and [10].

The concept of a **locating-dominating** set introduced by Slater [13] (see also [3]) is closely related to that of identifying codes.

Identifying codes and locating-dominating codes have been extensively studied: see the Internet bibliography [11] maintained by Antoine Lobstein. For results on the triangular grid, see, e.g., [1], [2], [4], [6] and [5].

In the square grid $\mathbb{Z}^2$ it is easy to see that the smallest possible density of a strongly identifying code equals 2/5. Indeed, the code $\{(x, y) \in \mathbb{Z}^2 : x \equiv 1 \text{ or } 3 \pmod 5\}$ is strongly identifying and has density 2/5. The lower bound 2/5 on the density is an immediate corollary of [12, Theorem 14].

In the hexagonal mesh the smallest possible density of a strongly identifying code equals 1/2. Indeed, if we delete all the vertical edges in the hexagonal mesh, and take as codewords all the vertices on every second of the resulting (infinitely many) doubly infinite paths, we clearly get a strongly identifying code with density 1/2. The lower bound on the density is an immediate corollary of [12, Theorem 14].

From now on we consider the infinite triangular grid. The vertex set of the infinite triangular grid T is $V = \{v(i, j) : i, j \in \mathbb{Z}\}$, where

$$v(i, j) = i(1, 0) + j(\frac{1}{2}, \frac{\sqrt{3}}{2}),$$

and two vertices are adjacent if their Euclidean distance is 1. Denote by T_n the set of vertices $v(i, j)$ with $|i| \leq n$ and $|j| \leq n$. The density of a code C in T is defined to be

$$D(C) = \limsup_{n\to\infty} \frac{|C \cap T_n|}{|T_n|}.$$

The smallest density of an identifying code in the infinite triangular grid is 1/4; see [8].

It is easy to check that the code given in Figure 1 is strongly identifying and has density of 6/19. We always denote codewords by black dots. The code is obtained as a doubly periodic tiling and the tile is shown in the figure.

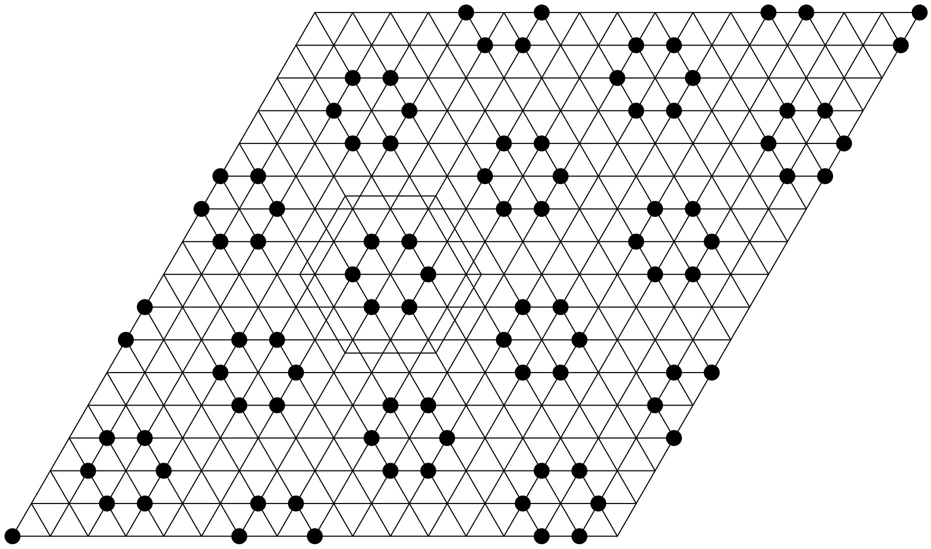

Figure 1: A strongly identifying code with density 6/19.

The purpose of this note is to prove that this code is optimal, i.e., the density of every strongly identifying code in the infinite triangular grid is at least 6/19.

2 The proof

From now on we assume that C is a strongly identifying code in T.

Denote

$$C_i = \{\mathbf{c} \in C : |I(\mathbf{c})| = i\},$$

and

$$C_{\geq j} = \bigcup_{i \geq j} C_i.$$

We also denote

$$N_i = \{\mathbf{v} \notin C : |I(\mathbf{v})| = i\},$$

and

$$N_{\geq j} = \bigcup_{i \geq j} N_i.$$

Trivially, $C_0 = C_1 = \emptyset$.

Following Slater [14] we define the **share** of a codeword $\mathbf{c} \in C$ — which we denote by $s(\mathbf{c})$ — by the formula

$$s(\mathbf{c}) = \sum_{\mathbf{v} \in B_1(\mathbf{c})} \frac{1}{|I(\mathbf{v})|}.$$

We now introduce a voting scheme using which we perform an averaging over the shares of the codewords.

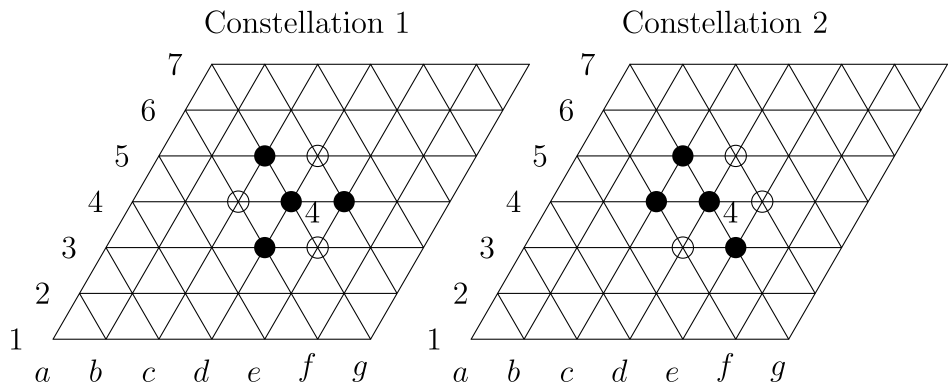

Figure 2: Constellations 1 and 2.

> **Rule 1:** Every codeword in $C_{\geq 4}$ gives 1/3 of a vote to every codeword neighbour in C_2.

In fact, no codeword can have more than one codeword neighbour that belongs to C_2: if $\mathbf{a} \in C_2$ and $\mathbf{b} \in C_2$ were two different codeword neighbours of $\mathbf{c} \in C$, then

$$I(\mathbf{a}) \setminus \{\mathbf{a}\} = \{\mathbf{c}\} = I(\mathbf{b}) \setminus \{\mathbf{b}\},$$

and the code would not be strongly identifying.

> **Rule 2:** Every codeword $\mathbf{c} \in C_{\geq 4}$ gives 1/12 of a vote to every codeword neighbour in C_3 and all their codeword neighbours in C_3 (i.e., if S is the subset of codewords of C_3 referred to above, then each element of S gets 1/12 of a vote from $\mathbf{c}$).

Clearly, if $\mathbf{c} \in C_{\geq 4}$ and $\mathbf{a} \in C_3$ is its codeword neighbour, then $\mathbf{a}$ can have at most one codeword neighbour in C_3.

These two rules describe the voting behaviour of the vertices in $C_{\geq 4}$.

Lemma 1. *If* $\mathbf{c} \in C_{\geq 4}$*, then* $s(\mathbf{c}) \leq 19/6$ *and* $\mathbf{c}$ *gives at most* $19/6 - s(\mathbf{c})$ *votes.*

Proof. Consider first the vertices $\mathbf{c} \in C_4$. There are essentially three different cases.

Assume first that none of the codeword neighbours of $\mathbf{c}$ are adjacent. Without loss of generality, $\mathbf{c}$ is the vertex d4 of Constellation 1 in Figure 2. As we already remarked after introducing Rule 1, at most one of the codewords c5, d3 and e4 is in C_2, and the others are in $C_{\geq 3}$. The vertices c4, d5 and e3 are all in $N_{\geq 3}$, and therefore

$$s(\mathbf{c}) \leq \frac{1}{4} + 5 \cdot \frac{1}{3} + \frac{1}{2} = \frac{29}{12},$$

and $19/6 - s(\mathbf{c}) \geq 9/12$. The number of votes given by $\mathbf{c}$ is clearly at most $1/3 + 4 \cdot 1/12 = 8/12 \leq 9/12$ as claimed.

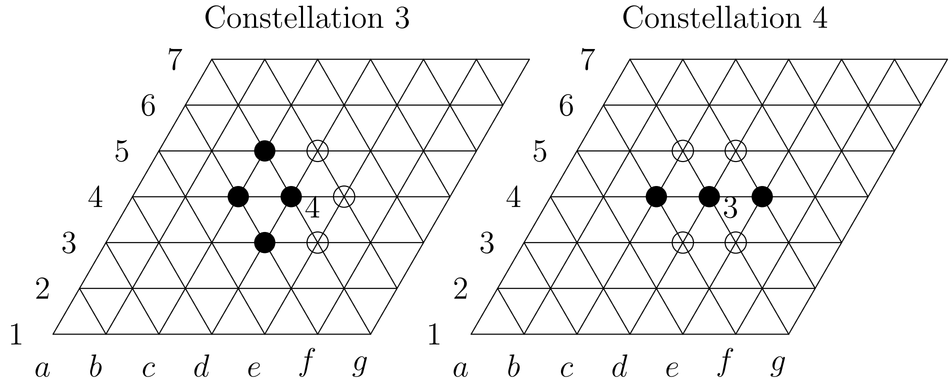

Figure 3: Constellations 3 and 4.

Assume then that $\mathbf{c}$ is like d4 in Constellation 2 in Figure 2. Then $\mathrm{c4}, \mathrm{c5} \in C_{\geq 3}$, and at least one of the vertices c4 and c5 is in $C_{\geq 4}$ (otherwise $I(\mathrm{c4}) = I(\mathrm{c5}) = \{\mathrm{c4}, \mathrm{c5}, \mathrm{d4}\}$). If $\mathrm{e3} \in C_2$, then $\mathrm{e4} \in N_{\geq 3}$ (otherwise $I(\mathrm{e3}) = I(\mathrm{e4}) = \{\mathrm{e3}, \mathrm{d4}\}$); and for the same reason if $\mathrm{e4} \in N_2$, then $\mathrm{e3} \in C_{\geq 3}$. Anyway,

$$s(\mathbf{c}) \leq 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{3} + 2 \cdot \frac{1}{2} = \frac{5}{2},$$

and $19/6 - s(\mathbf{c}) \geq 2/3$. The number of votes given by $\mathbf{c}$ is clearly at most 2/3.

Assume then that $\mathbf{c}$ is like the vertex d4 in Constellation 3 in Figure 3. It is possible that $\mathrm{e4} \in N_1$. However, at least one of the codewords d3 and c5 belongs to $C_{\geq 4}$: if both of them were in C_3, then $I(\mathrm{d3}) \setminus \{\mathrm{d3}\} = I(\mathrm{c5}) \setminus \{\mathrm{c5}\}$, contradicting the fact that C is a strongly identifying code. This implies that

$$s(\mathbf{c}) \leq 3 \cdot \frac{1}{4} + \frac{1}{3} + 2 \cdot \frac{1}{2} + 1 = \frac{37}{12},$$

and $19/6 - s(\mathbf{c}) \geq 1/12$, but it also implies that $\mathbf{c}$ gives at most 1/12 of a vote: if neither c5 nor d3 is in C_3, then $\mathbf{c}$ gives no votes at all; if c5, say, is in C_3, then it gets 1/12 of a vote from $\mathbf{c}$, but neither of the codeword neighbours of c5 is in C_3.

If $\mathbf{c} \in C_{\geq 5}$, then trivial calculations show that in all cases $s(\mathbf{c}) \leq 5/2$ and that $\mathbf{c}$ gives at most 2/3 of a vote. □

The final three voting rules tell how the vertices of C_3 vote.

Rule 3: If a codeword $\mathbf{c} \in C_3$ has a codeword neighbour $\mathbf{a} \in C_2$, then $\mathbf{c}$ gives 1/3 of a vote to $\mathbf{a}$, if $\mathbf{c}$ and its two codeword neighbours are collinear, and 1/4 of a vote, otherwise.

Rule 4: Assume that $\mathbf{c} \in C_3$ and that $\mathbf{c}$ has a codeword neighbour that belongs to C_2 or that $\mathbf{c}$ and its two codewords neighbours are collinear. Then $\mathbf{c}$ gives 1/12 of a vote to every codeword neighbour in C_3 and all their codeword neighbours in $C_3 \setminus \{\mathbf{c}\}$.

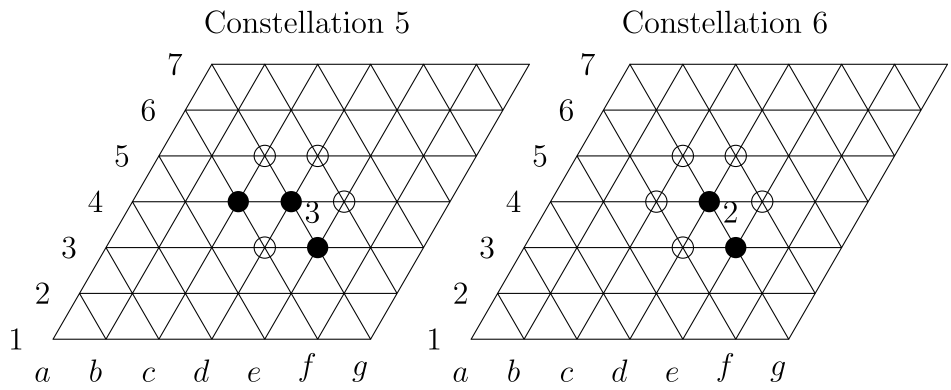

Figure 4: Constellations 5 and 6.

Rule 5: Assume that **c** and its two codeword neighbours all belong to C_3 but are not collinear. If the share of **c** is at most 37/12 and the share of exactly one of its codeword neighbours is bigger than 19/6, then that codeword gets 1/12 of a vote from **c**.

Lemma 2. *Assume that* $\mathbf{c} \in C_3$ *and none of its neighbours belongs to* C_2*, and that* **c** *gives votes. Then* $s(\mathbf{c}) \leq 19/6$ *and* **c** *gives at most* $19/6 - s(\mathbf{c})$ *votes.*

Proof. The case when **c** gives votes according to Rule 5 is trivial; so assume that **c** does not give any votes according to Rule 5. Then **c** gives votes according to Rule 4, and hence **c** and its two codeword neighbours $\mathbf{a} \in C_{\geq 3}$ and $\mathbf{b} \in C_{\geq 3}$ are collinear. Without loss of generality, **c** is the vertex d4 in Constellation 4 in Figure 3. Then at least one of the vertices d3 and c5 belongs to $N_{\geq 3}$; likewise at least one of the vertices e3 and d5 belongs to $N_{\geq 3}$. Consequently,

$$s(\mathbf{c}) \leq 5 \cdot \frac{1}{3} + 2 \cdot \frac{1}{2} = \frac{8}{3},$$

and $19/6 - s(\mathbf{c}) \geq 1/2$. According to Rule 4, **c** gives 1/12 of a vote to at most four vertices, and the claim is clear. □

Lemma 3. *Assume that* $\mathbf{c} \in C_3$ *has a neighbour* $\mathbf{a} \in C_2$*. Then* $s(\mathbf{c}) \leq 19/6$ *and* **c** *gives at most* $19/6 - s(\mathbf{c})$ *votes.*

Proof. Assume first that **c** and its two codeword neighbours are collinear. Without loss of generality, **c** is again the vertex d4 in Constellation 4 and c4 belongs to C_2 (and then obviously e4 is in $C_{\geq 3}$). Then both d3 and c5 belong to $N_{\geq 3}$; and at least one of the vertices e3 and d5 belongs to $N_{\geq 3}$. But again we see that $s(\mathbf{c}) \leq 8/3$, and $19/6 - s(\mathbf{c}) \geq 1/2$, and the total number of votes given by **c** is at most $1/3 + 2 \cdot 1/12 = 1/2$.

Assume second that **c** and its two codeword neighbours are not collinear. Without loss of generality **c** is the vertex d4 in Constellation 5 in Figure 4, and c4 belongs to C_2. Again, e3 is in $C_{\geq 3}$. Because C is strongly identifying, we know that $I(\text{c4}) \setminus \{c4\} \neq I(\text{d5})$,

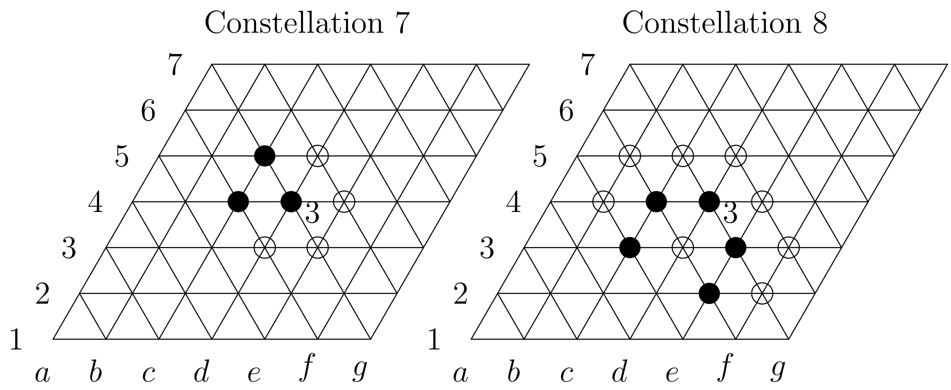

Figure 5: Constellations 7 and 8.

and therefore d5 $\in N_{\geq 2}$. Because $I(\mathrm{c5}) \neq I(\mathrm{c4})$, we know that c5 $\in N_{\geq 3}$; and because $I(\mathrm{d3}) \neq I(\mathrm{d4})$, we know that d3 $\in N_{\geq 4}$. All in all,

$$s(\mathbf{c}) \leq \frac{1}{4} + 3 \cdot \frac{1}{3} + 3 \cdot \frac{1}{2} = \frac{11}{4},$$

and $19/6 - s(\mathbf{c}) \geq 5/12$. According to Rules 3 and 4 the vertex **c** gives 1/4 of a vote to c4 and 1/12 of a vote to at most two vertices, and hence at most 5/12 of a vote altogether. □

Lemma 4. *If* $\mathbf{c} \in C_2$*, then* **c** *gets at least* $s(\mathbf{c}) - 19/6$ *votes.*

Proof. Without loss of generality, assume that $s(\mathbf{c}) > 19/6$ and that **c** is the vertex d4 in Constellation 6 in Figure 4. Because C is identifying, we know that $I(\mathrm{d3}) \neq I(\mathrm{d4})$ and hence d3 $\in N_{\geq 3}$; and similarly, e3 $\in C_{\geq 3}$ and e4 $\in N_{\geq 3}$. At most one of the vertices c4, c5 and d5 belongs to N_1 and at least two of them belong to $N_{\geq 2}$. Hence

$$s(\mathbf{c}) \leq 3 \cdot \frac{1}{3} + 3 \cdot \frac{1}{2} + 1 = \frac{7}{2}.$$

Therefore, if e3 gives 1/3 of a vote to d4, then the claim certainly holds. By Rules 1 and 3 this is true, unless e3 belongs to C_3 and its remaining codeword neighbour is either e2 or f3. These are symmetrical cases, so assume that e2 is in C and f2 and f3 are non-codewords. Then $I(\mathrm{d3}) \neq I(\mathrm{e3})$ implies that d3 is in $N_{\geq 4}$. Hence $s(\mathbf{c}) \leq 41/12$, and $s(\mathbf{c}) - 19/6 \leq 1/4$, and the claim is again valid, because e3 now gives 1/4 of a vote to d4 by Rule 3. □

Lemma 5. *If* $\mathbf{c} \in C_3$ *and* $s(\mathbf{c}) > 19/6$*, then* **c** *gets at least* $s(\mathbf{c}) - 19/6$ *votes.*

Proof. Because $\mathbf{c} \in C_3$ and $s(\mathbf{c}) > 19/6$, we know that **c** and its two codeword neighbours cannot be collinear, and without loss of generality **c** is the vertex d4 in Constellation 5 or in Constellation 7.

Consider first Constellation 7. Here, c4 and c5 both belong to $C_{\geq 4}$ and at most one of the vertices e3 and e4 can belong to N_1. Hence $s(\mathbf{c}) \leq 10/3$, and it suffices to show that d4 gets at least 1/6 of a vote. But, indeed, both c4 and c5 give 1/12 of a vote to d4 by Rule 2.

Assume therefore that $\mathbf{c}$ is the vertex d4 in Constellation 5. The fact that $s(\mathbf{c}) > 19/6$ implies that d5 belongs to N_1. Because $I(\mathrm{c4}) \setminus \{c4\} \neq I(\mathrm{d5})$, we know that c4 $\in C_{\geq 3}$ and similarly e3 $\in C_{\geq 3}$. Because $I(\mathrm{d3}) \neq I(\mathrm{d4})$, we know that d3 $\in N_{\geq 4}$. Therefore

$$s(\mathbf{c}) \leq \frac{1}{4} + 3 \cdot \frac{1}{3} + 2 \cdot \frac{1}{2} + 1 = \frac{13}{4}.$$

The fact that $s(\mathbf{c}) > 19/6$ now implies that c5 and e4 are both in N_2. Therefore b5 and f3 are both non-codewords. The claim is clearly true if d4 gets at least 1/12 of a vote. If c4 or e3 is in $C_{\geq 4}$, then this is true by Rule 2, so we can assume that they both belong to C_3. If b4 is in C, then c4 gives 1/12 of a vote to d4 by Rule 4, so we can assume that b4 is not in C; similarly, we can assume that f2 is not in C. But then c3 and e2 are both in C and we have Constellation 8 in Figure 5.

If c3 is in C_2, then c4 gives 1/12 of a vote to d4 by Rule 4; so assume that c3 is in $C_{\geq 3}$. If c3 is in $C_{\geq 4}$, then c3 gives 1/12 of a vote to d4 by Rule 2; so assume that c3 is in C_3.

If b3 is in C, then $s(\mathrm{c3}) < 19/6$ and $s(\mathrm{c4}) \leq 37/12$ and therefore c4 gives 1/12 of a vote to d4 by Rule 5; so assume that b3 is not in C.

Because $s(\mathbf{c}) > 19/6$, we know that d2 is not in C. But then c2 must be in C and c3 gives 1/12 of a vote to d4 by Rule 4. □

Theorem 1. *The density of a strongly identifying code in the infinite triangular grid is at least* 6/19.

Proof. Assume that C is a strongly identifying code in the infinite triangular grid. Let $n \geq 5$ be fixed, and consider the set T_n.

Consider now the voting process described above. For all codewords $\mathbf{c}$, define $m(\mathbf{c})$ as the total number of votes given minus the total number of votes received.

We first check that for all $\mathbf{c} \in C$ we have

$$s(\mathbf{c}) + m(\mathbf{c}) \leq \frac{19}{6}. \quad (1)$$

There are several cases to consider:

- If $\mathbf{c} \in C_{\geq 4}$, then $\mathbf{c}$ gives at most $19/6 - s(\mathbf{c})$ votes by Lemma 1, and the sum of $s(\mathbf{c})$ and the number of votes given by $\mathbf{c}$ is at most 19/6.

- If $\mathbf{c} \in C_2$, then $\mathbf{c}$ does not give any votes and by Lemma 4, the number of votes received by $\mathbf{c}$ is at least $s(\mathbf{c}) - 19/6$, and again (1) holds.

- If $\mathbf{c} \in C_3$ **gives votes**, then by Lemmas 2 and 3, $\mathbf{c}$ gives at most $19/6 - s(\mathbf{c})$ votes, and (1) holds.

- Finally, assume that $\mathbf{c} \in C_3$ **does not give any votes**. If $s(\mathbf{c}) \le 19/6$, then (1) trivially holds. If $s(\mathbf{c}) > 19/6$, then by Lemma 5, $\mathbf{c}$ gets at least $s(\mathbf{c}) - 19/6$ votes and again (1) holds.

Consider the sum

$$\sum_{\mathbf{c}\in C\cap T_n} (s(\mathbf{c}) + m(\mathbf{c})).$$

Except for the votes received from codewords not in T_n and votes given to codewords not in T_n, the number of votes given by the codewords in T_n is the same as the number of votes received by the codewords in T_n. From the voting rules we immediately see that if a codeword gives votes to another, their distance is at most 2. Consequently,

$$\sum_{\mathbf{c}\in C\cap T_n} m(\mathbf{c}) \ge -4|T_{n+2} \setminus T_n| = -4(16n + 24), \tag{2}$$

where $4|T_{n+2} \setminus T_n|$ is an upper bound (cf. Lemmas 1, 2 and 3) on the total number of votes received by the codewords in T_n from the codewords not in T_n.

On the other hand, if we consider the sum $\sum_{\mathbf{c}\in C\cap T_n} s(\mathbf{c})$, then every vertex $\mathbf{v} \in T_{n-1}$ with $|I(\mathbf{v})| = i$ contributes the summand $1/i$ to $s(\mathbf{c})$ for all the i codewords $\mathbf{c}$ within distance one from $\mathbf{v}$ (and these codewords $\mathbf{c}$ all belong to T_n). Hence

$$\begin{aligned}\sum_{\mathbf{c}\in C\cap T_n} s(\mathbf{c}) &\ge |T_{n-1}| \\ &= |T_n| - 8n. \end{aligned} \tag{3}$$

From (1) we see that $s(\mathbf{c}) + m(\mathbf{c}) \le 19/6$ for all $\mathbf{c} \in C \cap T_n$ and therefore

$$\sum_{\mathbf{c}\in C\cap T_n} (s(\mathbf{c}) + m(\mathbf{c})) \le \frac{19}{6}|C \cap T_n|. \tag{4}$$

Using (2), (3) and (4) we now see that

$$|T_n| - 8n - 4(16n + 24) \le \sum_{\mathbf{c}\in C\cap T_n} (s(\mathbf{c}) + m(\mathbf{c})) \le \frac{19}{6}|C \cap T_n|,$$

i.e.,

$$|T_n| - (72n + 96) \le \frac{19}{6}|C \cap T_n|,$$

i.e.,

$$\frac{|C \cap T_n|}{|T_n|} \ge \frac{6}{19} - \frac{6(72n + 96)}{19(2n + 1)^2},$$

from which we see that the claim is true. □

So we have proved:

Theorem 2. *The smallest possible density of a strongly identifying code in the infinite triangular grid is* $6/19$. □

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