

ON THE METRIC DIMENSIONS FOR SETS OF VERTICES¹

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Abstract

Resolving sets were originally designed to locate vertices of a graph one at a time. For the purpose of locating multiple vertices of the graph simultaneously, $\{\ell\}$ -resolving sets were recently introduced. In this paper, we present new results regarding the $\{\ell\}$ -resolving sets of a graph. In addition to proving general results, we consider $\{2\}$ -resolving sets in rook's graphs and connect them to block designs. We also introduce the concept of ℓ -solid-resolving sets, which is a natural generalisation of solid-resolving sets. We prove some general bounds and characterisations for ℓ -solid-resolving sets and show how ℓ -solid- and $\{\ell\}$ -resolving sets are connected to each other. In the last part of the paper, we focus on the infinite graph family of flower snarks. We consider the ℓ -solid- and $\{\ell\}$ -metric dimensions of flower snarks. In two proofs regarding flower snarks, we use a new computer-aided reduction-like approach.

Keywords: resolving set, metric dimension, resolving several objects, block design, rook's graph, flower snark.

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1. INTRODUCTION

The graphs we consider are undirected and simple. They are also connected and finite unless otherwise stated. The vertex set of a graph G is denoted by $V(G)$ or simply by V if the graph in question is clear from context. The *distance* between vertices v and u , denoted by $d(v, u)$, is the length of any shortest path between v and u .

Consider a graph G with vertices V . Let $S = \{s_1, \dots, s_k\} \subseteq V$ be nonempty. The *distance array* of vertex $v \in V$ with respect to the set S is defined as $\mathcal{D}_S(v) = (d(s_1, v), \dots, d(s_k, v))$. If no two vertices have the same distance array, the set S is called a *resolving set* of G . This concept was introduced independently by Slater [16] and Harary and Melter [9]. Resolving sets have applications in robot navigation [14] and network discovery and verification [1], for example. For some recent developments, see [6, 12, 13].

Resolving sets can be used to locate vertices of a graph one at a time. Our research focuses on how we can locate multiple vertices simultaneously. To that end, let us define the *distance array* of a vertex set $X \subseteq V$ with respect to $S = \{s_1, \dots, s_k\} \subseteq V$ as

$$\mathcal{D}_S(X) = (d(s_1, X), \dots, d(s_k, X)),$$

where $d(s_i, X) = \min_{x \in X} \{d(s_i, x)\}$ for all $s_i \in S$. For any singleton set $\{v\} \subseteq V$ we naturally have $\mathcal{D}_S(\{v\}) = \mathcal{D}_S(v)$. The following definition was introduced in [15].

Definition 1. Let $\ell \geq 1$ be an integer. The set $S \subseteq V(G)$ is an $\{\ell\}$ -*resolving set* of G , if for all nonempty sets $X, Y \subseteq V(G)$ such that $X \neq Y$, $|X| \leq \ell$ and $|Y| \leq \ell$ we have $\mathcal{D}_S(X) \neq \mathcal{D}_S(Y)$.

When $\ell = 1$, Definition 1 is equivalent to the definition of a resolving set.

Consider the graph H illustrated in Figure 1. The set $R_1 = \{v_2, v_3, v_7\}$ is a $\{1\}$ -resolving set of H . The vertex v_6 and the set $X = \{v_8, v_9\}$ have the same distance array $\mathcal{D}_{R_1}(v_6) = (2, 3, 1) = \mathcal{D}_{R_1}(X)$ with respect to the set R_1 . Thus, the set R_1 cannot distinguish X from v_6 . The set $R_2 = \{v_1, v_2, v_3, v_4, v_8, v_9\}$ is a $\{2\}$ -resolving set of H , and with it we can distinguish X from v_6 . Indeed, we have $\mathcal{D}_{R_2}(v_6) = (3, 2, 3, 2, 2, 2)$ and $\mathcal{D}_{R_2}(X) = (3, 2, 3, 2, 0, 0)$. Moreover, we can uniquely determine the elements of X using $\mathcal{D}_{R_2}(X)$.

We can also distinguish v_6 from X with another type of resolving sets introduced in [7]. The set $S \subseteq V$ is a *solid-resolving set* of a graph G if for all $v \in V$ and nonempty $X \subseteq V$ we have $\mathcal{D}_S(v) \neq \mathcal{D}_S(X)$. For example, the set $S_1 = \{v_1, v_2, v_3, v_7, v_8\}$ is a solid-resolving set of the graph H . Indeed, we have $\mathcal{D}_{S_1}(v_6) = (3, 2, 3, 1, 2)$ and $\mathcal{D}_{S_1}(X) = (3, 2, 3, 1, 0)$. Solid-resolving sets give unique distance arrays to all vertices. However, some sets of vertices with at

least two elements may share the same distance array. Let $Y = \{v_6, v_8\}$. Now $\mathcal{D}_{S_1}(X) = (3, 2, 3, 1, 0) = \mathcal{D}_{S_1}(Y)$, and thus the set S_1 is not a $\{2\}$ -resolving set of H .



Figure 1. The graph H with a $\{1\}$ -resolving set and a $\{2\}$ -resolving set.

The concept of solid-resolving sets can be generalised for larger sets of vertices. Consider again the graph H . We want to be able to distinguish sets with up to two vertices as with a $\{2\}$ -resolving set, but we want to also distinguish sets with up to two vertices from sets with three or more vertices. In other words, the aim is to locate the elements of sets with up to two vertices and detect if a set contains at least three vertices. Our $\{2\}$ -resolving set R_2 can do the former but not the latter; the sets $U = \{v_5, v_7\}$ and $W = \{v_5, v_6, v_7\}$ have the same distance array $\mathcal{D}_{R_2}(U) = (2, 1, 2, 1, 1, 1) = \mathcal{D}_{R_2}(W)$. As a solution to this problem, we now present the following generalisation of solid-resolving sets.

Definition 2. Let $\ell \geq 1$ be an integer. The set $S \subseteq V(G)$ is an ℓ -solid-resolving set of G , if for all distinct nonempty sets $X, Y \subseteq V(G)$ such that $|X| \leq \ell$ we have $\mathcal{D}_S(X) \neq \mathcal{D}_S(Y)$.

When $\ell = 1$, the previous definition is exactly the same as the definition of a solid-resolving set in [7]. The set $S_2 = \{v_1, v_2, v_3, v_4, v_6, v_8, v_9\}$ is a 2-solid-resolving set of H . We can distinguish the sets U and W from each other using S_2 since $\mathcal{D}_{S_2}(U) = (2, 1, 2, 1, 1, 1, 1)$ and $\mathcal{D}_{S_2}(W) = (2, 1, 2, 1, 0, 1, 1)$.

The difference between Definitions 1 and 2 is significant but subtle; the set Y can have any cardinality in Definition 2, but in Definition 1, we have the restriction $|Y| \leq \ell$. If a set S satisfies Definition 2 for some $\ell \geq 1$, then S also satisfies Definition 1 for the same ℓ . However, an $\{\ell\}$ -resolving set is not necessarily an ℓ -solid-resolving set (as we saw in the graph H).

Since $V(G)$ is an ℓ -solid-resolving set of G for any $\ell \in \{1, \dots, |V(G)|\}$, it is clear that an ℓ -solid-resolving set exists for any graph G and any integer $\ell \in \{1, \dots, |V(G)|\}$. Similarly, for any G and $\ell \in \{1, \dots, |V(G)|\}$ the set $V(G)$ is an $\{\ell\}$ -resolving set. Therefore, we focus on determining the minimum cardinality of an ℓ -solid- or $\{\ell\}$ -resolving set of a graph.

The $\{\ell\}$ -metric dimension of G , denoted by $\beta_\ell(G)$, is the minimum cardinality of an $\{\ell\}$ -resolving set of G . An $\{\ell\}$ -resolving set of cardinality $\beta_\ell(G)$ is called an $\{\ell\}$ -metric basis of G . Similarly, the ℓ -solid-metric dimension of G , denoted by $\beta_\ell^s(G)$, is the minimum cardinality of an ℓ -solid-resolving set of G . An ℓ -solid-resolving set of cardinality $\beta_\ell^s(G)$ is called an ℓ -solid-metric basis of G .

We explore the basic properties of ℓ -solid- and $\{\ell\}$ -resolving sets in Section 2. In Section 3, we prove a general lower bound on the ℓ -solid-metric dimension of a graph and characterise the graphs that attain this bound. In Section 4, we consider Cartesian products of graphs. In particular, we consider the rook's graph $K_m \square K_n$, and it turns out that the $\{2\}$ -metric dimension of a rook's graph is connected to combinatorial designs. Finally, in Section 5, we consider the ℓ -solid- and $\{\ell\}$ -metric dimensions of flower snarks. The structure of a flower snark allows us to prove bounds on the 1-solid- and $\{2\}$ -metric dimensions by using a new reduction-like approach. We also point out and correct an error in a proof in [10] regarding the $\{1\}$ -metric dimension of a flower snark.

2. GENERAL RESULTS

2.1. The Connection Between ℓ -Solid- and $\{\ell\}$ -Resolving Sets

The following theorem gives a characterisation for ℓ -solid-resolving sets. Compared to Definition 2, this characterisation provides a significantly easier way to verify that a set is an ℓ -solid-resolving set.

Theorem 3. *Let $S \subseteq V$ and $\ell \geq 1$. The set S is an ℓ -solid-resolving set of G if and only if for all $x \in V$ and nonempty $Y \subseteq V$ such that $x \notin Y$ and $|Y| \leq \ell$ there exists an element $s \in S$ such that*

$$d(s, x) < d(s, Y). \quad (1)$$

Proof. (\Rightarrow) Assume that S does not satisfy (1). There exists a vertex $x \in V$ and a set $Y \subseteq V$ such that $x \notin Y$, $|Y| \leq \ell$ and $d(s, x) \geq d(s, Y)$ for all $s \in S$. Now $\mathcal{D}_S(Y) = \mathcal{D}_S(Y \cup \{x\})$ and S is not an ℓ -solid-resolving set of G by Definition 2.

(\Leftarrow) Assume then that S satisfies (1). Consider nonempty vertex sets $X, Y \subseteq V$ such that $|X| \leq \ell$ and $X \neq Y$. We have the following two cases.

1. $Y \not\subseteq X$: Let $y \in Y \setminus X$. Since S satisfies (1), there exists an element $s \in S$ such that $d(s, y) < d(s, X)$. Now we have $\mathcal{D}_S(X) \neq \mathcal{D}_S(Y)$.
2. $Y \subset X$: Since $X \neq Y$, there exists a vertex $x \in X$ such that $x \notin Y$. Furthermore, we have $|Y| < |X| \leq \ell$. According to (1), we have $d(s, x) < d(s, Y)$ for some $s \in S$, and consequently $\mathcal{D}_S(X) \neq \mathcal{D}_S(Y)$.

Thus, the set S is an ℓ -solid-resolving set of G by Definition 2. ■

Theorem 3 will be very useful throughout the article. This theorem also implies the corresponding result for $\ell = 1$ in [7, Thm 2.2]. A somewhat similar result holds for $\{\ell\}$ -resolving sets as stated in the following lemma. Unlike in Theorem 3, we now have only an implication and not an equivalence.

Lemma 4. *Let $S \subseteq V$ and $\ell \geq 2$. If S is an $\{\ell\}$ -resolving set of G , then for all $x \in V$ and $Y \subseteq V$ such that $x \notin Y$ and $|Y| \leq \ell - 1$ there exists an element $s \in S$ for which we have*

$$d(s, x) < d(s, Y). \quad (2)$$

Proof. Assume that S does not satisfy (2). There exists a vertex $x \in V$ and a set $Y \subseteq V$ such that $x \notin Y$, $|Y| \leq \ell - 1$ and $d(s, x) \geq d(s, Y)$ for all $s \in S$. Now $\mathcal{D}_S(Y) = \mathcal{D}_S(Y \cup \{x\})$ and since $|Y| < |Y \cup \{x\}| \leq \ell$, the set S is not an $\{\ell\}$ -resolving set of G . ■

Now, if S is an $\{\ell + 1\}$ -resolving set of G for some $\ell \geq 1$, then according to Lemma 4 for all $x \in V$ and $Y \subseteq V$ such that $x \notin Y$ and $|Y| \leq \ell$ there exists an element $s \in S$ such that $d(s, x) < d(s, Y)$. According to Theorem 3, the set S is now also an ℓ -solid-resolving set, and the next result is immediate.

Theorem 5. *Let $S \subseteq V$ and $\ell \geq 1$.*

- (i) *If S is an ℓ -solid-resolving set, then it is an $\{\ell\}$ -resolving set of G .*
- (ii) *If S is an $\{\ell + 1\}$ -resolving set, then it is an ℓ -solid-resolving set of G .*

If we know that a set S is an ℓ -solid-resolving set of G , then to prove that the set S is an $\{\ell + 1\}$ -resolving set of G , it is sufficient to check that the distance arrays of vertex sets of cardinality $\ell + 1$ are unique. Indeed, according to Definition 2 the distance arrays $\mathcal{D}_S(X)$, where $|X| \leq \ell$, are unique. The only thing we need to do to prove that S satisfies Definition 1 is to show that no two vertex sets of cardinality $\ell + 1$ have the same distance array with respect to S .

2.2. Forced Vertices

A vertex $v \in V(G)$ is called a *forced vertex* of an $\{\ell\}$ -resolving set (similarly ℓ -solid-resolving set) of G if it must be included in any $\{\ell\}$ -resolving set of G . In other words, no subset of $V(G) \setminus \{v\}$ is an $\{\ell\}$ -resolving set of G . The graph we are considering is often clear from the context, and we may refer to a forced vertex of that graph by saying simply that the vertex is forced for an ℓ -solid- or $\{\ell\}$ -resolving set. The number of forced vertices of an ℓ -solid- or $\{\ell\}$ -resolving set gives us an immediate lower bound on the corresponding metric dimension.

The concept of forced vertices was first introduced in [8], where the forced vertices of $\{\ell\}$ -resolving sets were partially characterised. As was pointed out in [4], the set $V \setminus \{v\}$ is a $\{1\}$ -resolving set of a nontrivial connected graph G for all

$v \in V$. Thus, no such graph has forced vertices for a $\{1\}$ -resolving set. In [7], the forced vertices of 1-solid-resolving sets were fully characterised. In this section, we prove characterisations for ℓ -solid- and $\{\ell\}$ -resolving sets for all ℓ .

We denote by $N(v)$ the *open neighbourhood* of vertex v which is defined as $N(v) = \{u \in V \mid d(v, u) = 1\}$. The *closed neighbourhood* of a vertex $v \in V$ is $N[v] = N(v) \cup \{v\}$ and the closed neighbourhood of a vertex set U is $N[U] = \bigcup_{u \in U} N[u]$.

Theorem 6. *Let $\ell \geq 1$. A vertex $v \in V$ is a forced vertex of an ℓ -solid-resolving set of G if and only if there exists a set $U \subseteq V$ such that $v \notin U$, $|U| \leq \ell$ and $N(v) \subseteq N[U]$.*

Proof. (\Leftarrow) Assume that v and U are as described. The shortest path from any $s \in V \setminus \{v\}$ to v goes through $N(v)$. Since $N(v) \subseteq N[U]$, we have $d(s, v) \geq d(s, U)$ for all $s \in V \setminus \{v\}$. Thus, $\mathcal{D}_S(U) = \mathcal{D}_S(U \cup \{v\})$ for all subsets $S \subseteq V \setminus \{v\}$.

(\Rightarrow) Assume then that $v \in V$ and for all $U \subseteq V$ such that $v \notin U$ and $|U| \leq \ell$ we have $N(v) \not\subseteq N[U]$. Now there exists a vertex $w \in N(v) \setminus N[U]$, and we have $d(w, v) < d(w, U)$. Since $d(x, x) < d(x, Y)$ for any $x \in V$ and $Y \subseteq V \setminus \{x\}$, the set $V \setminus \{v\}$ satisfies (1) and is thus an ℓ -solid-resolving set of G , which contradicts the fact that v is forced for an ℓ -solid-resolving set. ■

According to Theorem 5 an $\{\ell\}$ -resolving set, where $\ell \geq 2$, is always an $(\ell - 1)$ -solid-resolving set of the graph in question. Thus, if a vertex is forced for $(\ell - 1)$ -solid-resolving sets of a graph, then it is also forced for the $\{\ell\}$ -resolving sets of the same graph. The following theorem characterises all forced vertices of an $\{\ell\}$ -resolving set of a graph, and shows that the forced vertices of $\{\ell\}$ -resolving sets are in fact *exactly the same* as those of $(\ell - 1)$ -solid-resolving sets.

Theorem 7. *Let $\ell \geq 2$. A vertex $v \in V$ is a forced vertex of an $\{\ell\}$ -resolving set of G if and only if there exists a set $U \subseteq V$ such that $v \notin U$, $|U| \leq \ell - 1$ and $N(v) \subseteq N[U]$.*

Proof. (\Leftarrow) Clear by Theorems 5 and 6.

(\Rightarrow) Assume then that $v \in V$ and that for all $U \subseteq V$ such that $v \notin U$ and $|U| \leq \ell - 1$ we have $N(v) \not\subseteq N[U]$. We will show that the set $S = V \setminus \{v\}$ is an $\{\ell\}$ -resolving set of G by showing how to determine the elements of a vertex set X when the distance array $\mathcal{D}_S(X)$ is known. Consider a nonempty set $X \subseteq V$, where $|X| \leq \ell$, and let $\mathcal{D}_S(X)$ be known. We can easily determine the elements of $X' = X \cap S$ by considering the zeros in the distance array $\mathcal{D}_S(X)$. If $|X'| = \ell$, then $X = X'$ and we have uniquely determined all elements of X . Otherwise, we still need to determine whether v is in X since it is the only vertex of the graph that is not in S . Since $|X'| \leq \ell - 1$ and $v \notin X'$, there exists a vertex $w \in N(v) \setminus N[X']$ according to our assumption. Now, $d(w, v) < d(w, X')$ and $d(w, X) = d(w, v)$ if and only if $v \in X$. ■

To illustrate the previous theorems, consider again the graph H in Figure 1. Since $N(v_1) = \{v_2, v_3, v_4\}$ and $N(v_3) = \{v_1, v_2, v_4\}$, we have $N(v_1) \subseteq N[v_3]$ and $N(v_3) \subseteq N[v_1]$. By Theorems 6 and 7, the vertices v_1 and v_3 are forced vertices of 1-solid- and $\{2\}$ -resolving sets of H .

Consider then any connected graph G . If $\deg(v) \leq \ell$ for some vertex v and integer $\ell \geq 1$, then $N(v) \subseteq N[N(v)]$ and v is forced for ℓ -solid- and $\{\ell + 1\}$ -resolving sets of G by Theorems 6 and 7. In particular, if G is a tree, then a vertex v is forced for ℓ -solid- and $\{\ell + 1\}$ -resolving sets if and only if $\deg(v) \leq \ell$. In [8], it was shown that the forced vertices of an $\{\ell\}$ -resolving set of a tree indeed form an $\{\ell\}$ -resolving set, when $\ell \geq 2$. Since any $\{\ell + 1\}$ -resolving set is an ℓ -resolving set and the forced vertices of these two types of resolving sets are exactly the same, the ℓ -solid-resolving sets of a tree consist of only the corresponding forced vertices. Thus, for any ℓ we can construct trees that have nontrivial ℓ -solid- and $\{\ell\}$ -resolving sets.

3. BOUNDS AND CHARACTERISATIONS

For the $\{1\}$ -metric dimension of a graph there is the obvious lower bound $\beta_1(G) \geq 1$. This lower bound is attained if and only if $G = P_n$ [4, 14]. In this section, we prove a lower bound on the ℓ -solid-metric dimension of a graph and characterise the graphs attaining that bound. The lower bound $\beta_1^s(G) \geq 2$ on the 1-solid-metric dimension of a graph was shown in [7]. The following theorem generalises this lower bound for ℓ -solid-metric dimensions where $\ell \geq 2$.

Theorem 8. *Let G be a graph with n vertices. When $1 \leq \ell \leq n - 1$, we have $\beta_\ell^s(G) \geq \ell + 1$.*

Proof. Let $S \subseteq V$ such that $1 \leq |S| \leq \ell$. Since $\ell \leq n - 1$, there exists at least one vertex v which is not in S . Now, $\mathcal{D}_S(S) = (0, \dots, 0) = \mathcal{D}_S(S \cup \{v\})$, and S is not an ℓ -solid-resolving set of G according to Definition 2. ■

The following theorem characterises the graphs attaining the bound of Theorem 8.

Theorem 9. *Let G be a connected graph with n vertices and let $2 \leq \ell \leq n - 1$. We have*

$$\beta_\ell^s(G) = \ell + 1 \text{ if and only if } n = \ell + 1 \text{ or } G = K_{1, \ell + 1}.$$

Proof. If $n = \ell + 1$, on the one hand, $\beta_\ell^s(G) \leq n = \ell + 1$ and on the other hand $\beta_\ell^s(G) > \ell$, and thus $\beta_\ell^s(G) = \ell + 1$. Also, by Theorem 2.9 of [8], the star $K_{1, \ell + 1}$ with $\ell + 2$ vertices satisfies $\beta_{\ell + 1}(K_{1, \ell + 1}) = \ell + 1$. Therefore $\ell < \beta_\ell^s(K_{1, \ell + 1}) \leq \beta_{\ell + 1}(K_{1, \ell + 1}) = \ell + 1$, and thus $\beta_\ell^s(K_{1, \ell + 1}) = \ell + 1$.

Conversely, suppose that G is a connected graph such that $|V| = n \geq \ell + 2$ and $\beta_\ell^s(G) = \ell + 1$ and let $S \subseteq V$ be an ℓ -solid-resolving set with $\ell + 1$ vertices. The following properties hold.

1. The set S is independent: Suppose to the contrary that there exist $s_1, s_2 \in S$ such that $d(s_1, s_2) = 1$. Since $|V| \geq \ell + 2$, there exists $u \in V \setminus S$ that satisfies $d(u, s_1) \geq 1 = d(s_2, s_1)$. Now, $d(v, u) \geq d(v, S \setminus \{s_1\})$ for all $v \in S$, and since $|S \setminus \{s_1\}| = \ell$, the set S is not an ℓ -solid-resolving set of G according to (1), when $x = u$ and $Y = S \setminus \{s_1\}$.

2. We have $\deg(s) = 1$ for every $s \in S$: Denote $S = \{s_1, \dots, s_{\ell+1}\}$. Since G is connected and S is independent, each s_i has a neighbour in $V \setminus S$, say $v_i \in N(s_i)$ for $i = 1, \dots, \ell + 1$. Suppose to the contrary that $\deg(s_i) \geq 2$ for some i . Assume without loss of generality that $\deg(s_1) \geq 2$. There exists a vertex $v'_1 \in N(s_1)$, $v'_1 \neq v_1$. Let $A = \{v_1, \dots, v_\ell\}$. Since S is an ℓ -solid-resolving set of G , according to Theorem 3 we must have $d(s_i, v'_1) < d(s_i, A)$ for some $i \in \{1, \dots, \ell + 1\}$. However, we have $d(s_i, A) = 1$ for all $i \in \{1, \dots, \ell\}$, and thus $d(s_{\ell+1}, v'_1) < d(s_{\ell+1}, A)$. Specifically, we have $d(s_{\ell+1}, v'_1) < d(s_{\ell+1}, v_1)$. Similarly, for v_1 and $B = \{v'_1, v_2, \dots, v_\ell\}$ we have $d(s_{\ell+1}, v_1) < d(s_{\ell+1}, B)$, and specifically $d(s_{\ell+1}, v_1) < d(s_{\ell+1}, v'_1)$, a contradiction. Thus, $\deg(s) = 1$ for all $s \in S$.

We now consider two cases.

Case 1. There exists $u \in V \setminus S$ and two different vertices $s_1, s_2 \in S$ such that $d(u, s_1) = d(u, s_2) = 1$. If $|V \setminus S| \geq 2$, then let $v \in V \setminus S$ be such that $v \neq u$. Let $X = (S \setminus \{s_1, s_2\}) \cup \{u\}$ and $Y = X \cup \{v\}$. We obtain that $\mathcal{D}_S(X) = \mathcal{D}_S(Y) = (1, 1, 0, \dots, 0)$, a contradiction. This means that, in this case, $V \setminus S = \{u\}$, and since u is not a forced vertex, $\deg(u) \geq \ell + 1$, and thus u is a neighbour of every vertex in S . Finally, $G = K_{1, \ell+1}$ because S is independent.

Case 2. Every vertex in $V \setminus S$ has at most one neighbour in S . As seen above, we know that every vertex in S has exactly one neighbour in $V \setminus S$. We denote $S = \{s_1, \dots, s_{\ell+1}\}$ and $A = \{v_1, \dots, v_{\ell+1}\}$ ($|A| = \ell + 1$) where v_i is the unique neighbour of s_i , for $1 \leq i \leq \ell + 1$, and note that $\ell + 1 \geq 3$. The following properties hold.

(a) The set A is independent. Suppose to the contrary that, say, v_1 and v_2 are neighbours. Thus, $d(v_1, v_2) = 2$. Define the sets $X = (S \setminus \{s_1, s_2\}) \cup \{v_1\}$ and $Y = X \cup \{v_3\}$. Clearly $\mathcal{D}_S(X) = \mathcal{D}_S(Y) = (1, 2, 0, \dots, 0)$, a contradiction.

(b) No pair of vertices of A has a common neighbour. Suppose to the contrary (without loss of generality) that there exists $w \in V$ that satisfies $d(v_1, w) = d(v_2, w) = 1$. Then $\deg(w) \geq 2$ and $w \notin S$. Moreover $w \notin A$, because A is independent. Let $X = (S \setminus \{s_1, s_2\}) \cup \{w\}$ and $Y = X \cup \{v_3\}$. Then $\mathcal{D}_S(X) = \mathcal{D}_S(Y) = (2, 2, 0, \dots, 0)$, a contradiction.

Note that every $v_i \in A$ has at least ℓ neighbours ($\ell \geq 2$) in $V \setminus S$, say $\{v_{i,j} \mid 1 \leq j \leq \ell\}$, because it is not forced. The last property gives that $v_{i,j} \neq v_{i',j'}$ for $(i, j) \neq (i', j')$.

Assume, without loss of generality, that $d(v_{1,1}, s_{\ell+1}) = \min\{d(v_{1,j}, s_{\ell+1}) \mid 1 \leq j \leq \ell\}$ and let $X = \{v_{1,1}, v_{2,1}, \dots, v_{\ell,1}\}$. Then for all $s_i \in S$, where $i \neq \ell + 1$, we have $d(s_i, X) = 2 \leq d(s_i, v_{1,2})$ since A and S are both independent. Furthermore, since $d(v_{1,1}, s_{\ell+1}) \leq d(v_{1,2}, s_{\ell+1})$, we have $d(s_{\ell+1}, X) \leq d(s_{\ell+1}, v_{1,1}) \leq d(s_{\ell+1}, v_{1,2})$. Thus, $d(s_i, v_{1,2}) \geq d(s_i, X)$ for all $s_i \in S$, and S is not an ℓ -solid-resolving set of G by Theorem 3, a contradiction. ■

Notice that the number of graphs that attain the lower bound $\beta_\ell^s(G) \geq \ell + 1$ is infinite when $\ell = 1$ and finite when $\ell \geq 2$. Corresponding results for $\{\ell\}$ -resolving sets can be found in [8].

Let us then consider infinite graphs, that is, graphs with infinitely many vertices. In [2], it was shown that an infinite graph may have finite or infinite $\{1\}$ -metric dimension. We will show that the $\{\ell\}$ -metric dimension, where $\ell \geq 2$, is infinite for any infinite graph. Moreover, the ℓ -solid-metric dimension of any infinite graph is infinite. To prove these results, we will consider doubly resolving sets.

Definition 10 [3]. Let G be a graph with $|V(G)| \geq 2$. Two vertices $v, w \in V(G)$ are *doubly resolved* by $x, y \in V(G)$ if $d(v, x) - d(w, x) \neq d(v, y) - d(w, y)$. A set of vertices $S \subseteq V(G)$ doubly resolves G , and S is a *doubly resolving set*, if every pair of distinct vertices $v, w \in V(G)$ is doubly resolved by two vertices in S .

In [7], it was shown that a 1-solid-resolving set of G is a doubly resolving set of G . According to Theorem 5 any $\{\ell\}$ -resolving set, where $\ell \geq 2$, and ℓ -solid-resolving set is a 1-solid-resolving set. The following result is now immediate.

Corollary 11. *If $S \subseteq V(G)$ is an $\{\ell\}$ -resolving set ($\ell \geq 2$) or an ℓ -solid-resolving set of G ($\ell \geq 1$), then S is a doubly resolving set of G .*

Lemma 12 [2]. *If G is an infinite graph, then any doubly resolving set of G is infinite.*

The following corollary is now immediate due to Corollary 11 and Lemma 12.

Corollary 13. *If G is an infinite graph, then $\beta_\ell(G) = \infty$, when $\ell \geq 2$, and $\beta_\ell^s(G) = \infty$, when $\ell \geq 1$.*

4. ON CARTESIAN PRODUCTS OF GRAPHS

The *Cartesian product* of the graphs G and H is the graph $G \square H$ with the vertex set $\{av \mid a \in V(G), v \in V(H)\}$. Distinct vertices $av, bu \in V(G \square H)$ are adjacent

if $a = b$ and $v \in N_H(u)$, or $a \in N_G(b)$ and $v = u$. We have $d_{G \square H}(av, bu) = d_G(a, b) + d_H(v, u)$. To simplify notations, we may denote V instead of $V(G \square H)$ and omit the subscript $G \square H$ from the distance function. The *projection* of $X \subseteq V$ onto G is the set $\{x_1 \in V(G) \mid x_1 x_2 \in X\}$. Similarly, the projection of $X \subseteq V$ onto H is the set $\{x_2 \in V(H) \mid x_1 x_2 \in X\}$.

Theorem 14. *Let G and H be nontrivial connected graphs and $\ell \geq 1$.*

1. *If S is an ℓ -solid-resolving set of $G \square H$, then the projection of S onto G (respectively onto H) is an ℓ -solid-resolving of G (respectively of H).*
2. *If T is an ℓ -solid-resolving set of G and U is an ℓ -solid-resolving set of H , then $T \times U$ is an ℓ -solid-resolving of $G \square H$.*
3. *We have $\max\{\beta_\ell^s(G), \beta_\ell^s(H)\} \leq \beta_\ell^s(G \square H) \leq \beta_\ell^s(G) \cdot \beta_\ell^s(H)$.*

Proof. 1. Let $a \in V(G)$ and $Y \subseteq V(G)$, $|Y| \leq \ell$, and let $h_0 \in V(H)$ be a fixed vertex. Let $ah_0 \in V(G \square H)$ and $Y_0 = Y \times \{h_0\}$. Now $|Y_0| \leq \ell$ and there exists $s = g_s h_s \in S$ such that

$$\begin{aligned} d_G(g_s, a) + d_H(h_s, h_0) &= d(g_s h_s, ah_0) < d(g_s h_s, Y_0) \\ &= \min\{d_G(g_s, y) + d_H(h_s, h_0) \mid y \in Y\} \\ &= \min\{d_G(g_s, y) \mid y \in Y\} + d_H(h_s, h_0). \end{aligned}$$

Therefore, $d_G(g_s, a) < \min\{d_G(g_s, y) \mid y \in Y\} = d_G(g_s, Y)$, as desired.

2. Let $ab \in V(G \square H)$ and $Y \subseteq V(G \square H)$ such that $|Y| \leq \ell$. Then the projections Y_G and Y_H of Y onto G and H , respectively, satisfy $|Y_G|, |Y_H| \leq \ell$. Therefore, there exist $t \in T$ and $u \in U$ such that $d_G(t, a) < d_G(t, Y_G) = \min\{d_G(t, y_g) \mid y_g \in Y_G\}$ and $d_H(u, b) < d_H(u, Y_H) = \min\{d_H(u, y_h) \mid y_h \in Y_H\}$.

Note that $\min\{d_G(t, y_g) \mid y_g \in Y_G\} + \min\{d_H(u, y_h) \mid y_h \in Y_H\} \leq \min\{d_G(t, \alpha) + d_H(u, \beta) \mid \alpha\beta \in Y\} = \min\{d(tu, \alpha\beta) \mid \alpha\beta \in Y\} = d(tu, Y)$.

Finally, $d(tu, ab) = d_G(t, a) + d_H(u, b) < \min\{d_G(t, y_g) \mid y_g \in Y_G\} + \min\{d_H(u, y_h) \mid y_h \in Y_H\} \leq d(tu, Y)$, as desired.

3. The lower bound follows from 1. and the upper bound follows from 2. ■

Notice that in 2., it would be sufficient that the set U satisfies the condition (1) with equality, that is, for all $x \in V(H)$ and nonempty $Y \subseteq V(H)$ such that $x \notin Y$ and $|Y| \leq \ell$ there exists $u \in U$ such that $d_H(u, x) \leq d_H(u, Y)$.

Theorem 15. *Let G and H be nontrivial connected graphs and $\ell \geq 2$.*

1. *If S is an $\{\ell\}$ -resolving set of $G \square H$, then the projection of S onto G (respectively onto H) is an $\{\ell\}$ -resolving of G (respectively of H).*
2. *If S is an $\{\ell\}$ -resolving set of G (respectively of H) and S' is an ℓ -solid-resolving set of H (respectively of G), then $S \times S'$ (respectively $S' \times S$) is a $\{\ell\}$ -resolving set of $G \square H$.*

3. We have $\max\{\beta_\ell(G), \beta_\ell(H)\} \leq \beta_\ell(G \square H) \leq \min\{\beta_\ell(G) \cdot \beta_\ell^s(H), \beta_\ell^s(G) \cdot \beta_\ell(H)\}$.

Proof. 1. Let S be an $\{\ell\}$ -resolving set of $G \square H$, and X and Y be subsets of $V(G)$ such that $X \neq Y$, $1 \leq |X| \leq \ell$ and $1 \leq |Y| \leq \ell$. Define $X_0 = X \times \{h_0\}$ and $Y_0 = Y \times \{h_0\}$, where $h_0 \in H$. Clearly, we have $X_0 \neq Y_0$, $|X| = |X_0|$ and $|Y| = |Y_0|$. Hence, there exists a vertex $s = g_s h_s \in S$ such that $d(s, X_0) \neq d(s, Y_0)$. Therefore, as $d(s, X_0) = d_G(g_s, X) + d_H(h_s, h_0)$ and $d(s, Y_0) = d_G(g_s, Y) + d_H(h_s, h_0)$, we obtain that $d_G(g_s, X) \neq d_G(g_s, Y)$. Thus, the projection of S onto G is an $\{\ell\}$ -resolving set of G . Analogously, it can be shown that the projection of S onto H is an $\{\ell\}$ -resolving set of H .

2. Let S be an $\{\ell\}$ -resolving set of G and S' be an ℓ -solid-resolving set of H . Assume that $X, Y \subseteq V(G \square H)$ are such that $X \neq Y$, $1 \leq |X| \leq \ell$ and $1 \leq |Y| \leq \ell$. Denote $X = \{g_1 h_1, \dots, g_k h_k\}$ and $Y = \{g'_1 h'_1, \dots, g'_{k'} h'_{k'}\}$, where $k = |X|$, $k' = |Y|$, $g_i, g'_i \in V(G)$ and $h_i, h'_i \in V(H)$. Further denote $X_G = \{g_1, \dots, g_k\}$ and $Y_G = \{g'_1, \dots, g'_{k'}\}$, and $X_H = \{h_1, \dots, h_k\}$ and $Y_H = \{h'_1, \dots, h'_{k'}\}$. The proof now divides into the following two cases.

- Suppose that $X_G \neq Y_G$. Now there exists a vertex $s \in S$ such that $d_G(s, X_G) \neq d_G(s, Y_G)$. Without loss of generality, we may assume that $d_G(s, g_1) = d_G(s, X_G) < d_G(s, Y_G)$. Observe that by the condition (1) there exists $s' \in S'$ such that $d_H(s', h_1) < d_H(s', h)$ for any $h \in Y_H \setminus \{h_1\}$ since $|Y_H \setminus \{h_1\}| \leq \ell$; we agree that if $Y_H \setminus \{h_1\} = \emptyset$, then any $s' \in S'$ meets the required (empty) condition (similar agreement is also made in the case with $X_G = Y_G$). Thus, we have a vertex $s' \in S$ satisfying $d_H(s', h_1) = d_H(s', Y_H)$. Therefore, we obtain that $d(ss', X) \leq d(ss', g_1 h_1) = d_G(s, g_1) + d_H(s', h_1) < d_G(s, Y_G) + d_H(s', Y_H) \leq d(ss', Y)$.

- Suppose that $X_G = Y_G$. Since $X \neq Y$, we have $X \Delta Y = (X \setminus Y) \cup (Y \setminus X) \neq \emptyset$ and, without loss of generality, we may assume that $g_1 h_1 \in X \Delta Y$. By the condition (2), there exists $s \in S$ such that $d_G(s, g_1) < d_G(s, g)$ for any $g \in Y_G \setminus \{g_1\}$ since $|Y_G \setminus \{g_1\}| \leq \ell - 1$. Analogously, by (1), there exists $s' \in S'$ such that $d_H(s', h_1) < d_H(s', h')$ for any $h' \in Y_H \setminus \{h_1\}$ since $|Y_H \setminus \{h_1\}| \leq \ell$. For any $g'_i h'_i \in Y$ we have $g'_i \neq g_1$ or $h'_i \neq h_1$ since $g_1 h_1 \notin Y$. Now $d(ss', X) \leq d(ss', g_1 h_1) = d_G(s, g_1) + d_H(s', h_1) < d_G(s, g'_i) + d_H(s', h'_i) = d(ss', g'_i h'_i)$ for any $g'_i h'_i \in Y$. Hence, we have shown that $d(ss', X) < d(ss', Y)$.

Thus, $S \times S'$ is an $\{\ell\}$ -resolving set of $G \square H$. The other claim can be proven analogously.

3. The lower bound follows from 1. and the upper bound follows from 2. ■

4.1. The Rook's Graph $K_m \square K_n$

The graph $K_m \square K_n$ can be illustrated as a grid, see Figure 2. A *column* of $K_m \square K_n$ is the set $\{vu \mid u \in V(K_n)\}$ for some fixed $v \in V(K_m)$. Similarly, a *row* of $K_m \square K_n$ is the set $\{vu \mid v \in V(K_m)\}$ for some fixed $u \in V(K_n)$. Two vertices

are adjacent if and only if they are on the same row or column. Moreover, if two distinct vertices x and y are on different rows and columns, we have $d(x, y) = 2$.

Consider any $K_m \square K_n$ where $m, n \geq 2$. Let x, y and z be distinct vertices such that x and y are on the same column, and x and z are on the same row. Any neighbour of x is in the closed neighbourhood of either y or z . Thus, we have $N(x) \subseteq N[\{y, z\}]$. According to Theorems 6 and 7, x is a forced vertex for ℓ -solid-resolving sets when $\ell \geq 2$ and $\{\ell\}$ -resolving sets when $\ell \geq 3$. Consequently, $\beta_\ell^s(K_m \square K_n) = mn$ for all $\ell \geq 2$ and $\beta_\ell(K_m \square K_n) = mn$ for all $\ell \geq 3$.

The 1-solid- and $\{1\}$ -metric dimensions of $K_m \square K_n$ were considered in [7] and [3], respectively. Thus, the only ℓ -solid- or $\{\ell\}$ -metric dimension of $K_m \square K_n$ yet to be determined is the $\{2\}$ -metric dimension. In what follows, we show a characterisation for the $\{2\}$ -resolving sets of $K_m \square K_n$. As it turns out, this characterisation provides us an exciting connection between combinatorial designs and $\{2\}$ -resolving sets of $K_m \square K_n$.

A *quadruple* of $K_m \square K_n$ is the set $\{av, au, bv, bu\}$ where $a, b \in V(K_m)$ and $v, u \in V(K_n)$ are distinct. For example, in $K_7 \square K_7$ illustrated in Figure 2, the set $\{v_1u_1, v_1u_3, v_4u_1, v_4u_3\}$ is a quadruple, and we can see that these four vertices lie on the corners of a rectangle.

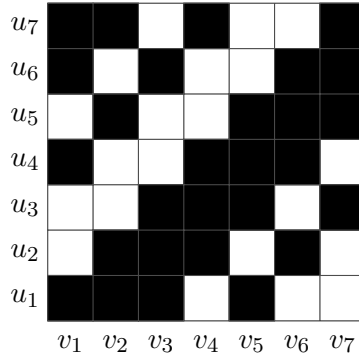


Figure 2. The graph $K_7 \square K_7$, where $v_i \in V(K_7)$ and $u_i \in V(K_7)$. The black squares form a $\{2\}$ -resolving set of $K_7 \square K_7$.

Lemma 16. *Let $m, n \geq 2$. If the set S is a $\{2\}$ -resolving set of $K_m \square K_n$, then each quadruple contains at least one element of S .*

Proof. Let $Q = \{av, au, bv, bu\} \subseteq V(K_m \square K_n)$ be a quadruple that does not contain any elements of S . Let us denote $X = \{av, bu\}$ and $Y = \{au, bv\}$. Since $N[X] = N[Y]$, we have $d(s, X) = 1$ if and only if $d(s, Y) = 1$ for all $s \in S$. Consequently, $\mathcal{D}_S(X) = \mathcal{D}_S(Y)$ and S is not a $\{2\}$ -resolving set of $K_m \square K_n$, a contradiction. ■

In the following theorem, we show that there are two types of $\{2\}$ -resolving sets of $K_m \square K_n$.

Theorem 17. *Let $m, n \geq 2$. If the set S is a $\{2\}$ -resolving set of $K_m \square K_n$, then*

1. *the set $\{v\} \cup (V \setminus N(v))$ is a subset of S for some $v \in V$ or*
2. *each row and column contains at least two elements of S and each quadruple contains at least one element of S .*

Proof. Suppose first that for some $a \in V(K_m)$ the column $C = \{au \mid u \in V(K_n)\}$ does not contain elements of S . Let $av \in C$ and $bv \in V \setminus C$ for some $v \in V(K_n)$. Since the column C does not contain any elements of S , we have $N(av) \cap S \subseteq N[bv]$. Consequently, $d(s, av) \geq d(s, bv)$ for all $s \in S$, and the set S is not a $\{2\}$ -resolving set of $K_m \square K_n$ according to Lemma 4. Thus, if S is a $\{2\}$ -resolving set of $K_m \square K_n$, then each column (and row, by symmetry) contains at least one element of S .

Suppose then that $C \cap S = \{au\}$. Let $b \in V(K_m) \setminus \{a\}$ and $t \in V(K_n) \setminus \{u\}$. Consider the sets $X = \{bu, bt\}$ and $Y = \{bu, at\}$. For some $cv \in S$, we have $d(cv, X) \neq d(cv, Y)$. As bu is in both X and Y , we have $d(cv, bt) \neq d(cv, at)$. Since at and bt are on the same row, cv is either on the column C or the column $D = \{bw \mid w \in V(K_n)\}$. The only element of S in C is au . However, the element bu is in both X and Y , and we have $d(au, X) = d(au, Y) = 1$. Thus, cv must be in D . The column D contains the element bu , and thus $d(cv, X) = d(cv, Y) = 1$ if $cv \neq bt$. Therefore, we have $cv = bt$ and $bt \in S$. Since this holds for all $b \neq a$ and $t \neq u$, we have that $w \in S$ for all $w \in V \setminus N(au)$.

In conclusion, if S is a $\{2\}$ -resolving set of $K_m \square K_n$ and some column (or row) contains only one element of S , the set $\{v\} \cup (V \setminus N(v))$ is a subset of S for some $v \in V$. If each row and column contains at least two elements of S , each quadruple contains at least one element of S according to Lemma 16. ■

If $\{v\} \cup (V \setminus N(v))$ is a *proper* subset of S for some $v \in V$, then S is a $\{2\}$ -resolving set of $K_m \square K_n$. The proof is straightforward but quite technical. The set S contains almost all vertices of the graph. When the graph $K_m \square K_n$ is sufficiently large, the condition 2. of Theorem 17 has potential to produce significantly smaller $\{2\}$ -resolving sets. To show that a set satisfying 2. is a $\{2\}$ -resolving set of $K_m \square K_n$, we need the following lemma.

Lemma 18. *Let $m \geq n \geq 6$ and $S \subseteq V(K_m \square K_n)$. If each quadruple contains at least one element of S , then there exists at most one row and one column that contain at most two elements of S .*

Proof. Suppose to the contrary that there exist some $r, t \in V(K_n)$, $r \neq t$, such that the rows $R = \{vr \mid v \in V(K_m)\}$ and $T = \{vt \mid v \in V(K_m)\}$ both contain at most two elements of S . Consider the two rows as partitioned into pairs $\{vr, vt\}$,

where $v \in V(K_m)$. The rows R and T contain at most four elements of S in total. However, we have $m \geq 6$ pairs, and thus there are at least two pairs, say $\{ar, at\}$ and $\{br, bt\}$, that do not contain an element of S . Now the quadruple $\{ar, at, br, bt\}$ does not contain an element of S , a contradiction. The claim holds for columns by symmetry. ■

Theorem 19. *Let $m \geq n \geq 6$ and $S \subseteq V(K_m \square K_n)$. If each row and column contains at least two elements of S and each quadruple contains at least one element of S , then the set S is a $\{2\}$ -resolving set of $K_m \square K_n$.*

Proof. To prove that S is a 1-solid-resolving set, it suffices to check that (1) holds for any $x \in V \setminus S$. To that end, let $x \in V \setminus S$ and $y \in V$, $y \neq x$. Both the row and column that contain x also contain at least two elements of S . The closed neighbourhood of y contains all these four elements if and only if $y = x$. Thus, for any $x \in V \setminus S$ and $y \in V$ there exists $s \in S$ such that $d(s, x) < d(s, y)$. According to Theorem 3, the set S is a 1-solid-resolving set of $K_m \square K_n$.

Let us then consider distinct sets $X, Y \subseteq V(K_m \square K_n)$ such that $|X| = |Y| = 2$. If for some $x \in X \setminus Y$ and $y \in Y \setminus X$ we have $\{x, y\} \cap S \neq \emptyset$, then clearly $\mathcal{D}_S(X) \neq \mathcal{D}_S(Y)$.

Suppose that for some $x \in X \setminus Y$ and $y \in Y \setminus X$ we have $\{x, y\} \cap S = \emptyset$. According to Lemma 18 at least one of x and y has three elements of S on its row or column. Assume without loss of generality that x is on the row R and R contains at least three elements of S . If $Y \cap R = \emptyset$, then for at least one $s \in S \cap R$ we have $d(s, X) = 1 < d(s, Y)$. Thus, $\mathcal{D}_S(X) \neq \mathcal{D}_S(Y)$.

Suppose that $Y \cap R \neq \emptyset$, and let $y_1 \in Y \cap R$. Since $x \notin Y$, the vertex y_1 cannot be on the same column as x . The column C that contains x contains at least two elements of S , say $c_1, c_2 \in C \cap S$. We have $d(c_1, x) = d(c_2, x) = 1$ and $d(c_1, y_1) = d(c_2, y_1) = 2$. Let $y_2 \in Y$, $y_2 \neq y_1$. If $y_2 \notin C$, then $d(c_1, y_2) = 2$ or $d(c_2, y_2) = 2$, and thus $\mathcal{D}_S(X) \neq \mathcal{D}_S(Y)$.

Suppose $y_2 \in C$. Only one of y_1 and y_2 can be in S . Suppose $y_1 \in S$. If $y_1 \notin X$, then $\mathcal{D}_S(X) \neq \mathcal{D}_S(Y)$. Suppose $y_1 \in X$. The row T that contains y_2 also contains at least two elements of S , say $t_1, t_2 \in T \cap S$. Since $y_2 \notin S$, $t_1 \neq y_2$ and $t_2 \neq y_2$, and thus $t_1, t_2 \notin C$. Now $d(t_1, X) = 2$ or $d(t_2, X) = 2$ since only one of t_1 and t_2 can be on the same column as y_1 . Thus, $\mathcal{D}_S(X) \neq \mathcal{D}_S(Y)$. Similarly, if $y_2 \in S$, we can prove that there is a vertex $s \in S$ in the same column as y_1 such that $d(s, y_1) = 1 < d(s, X)$.

Suppose $y_1 \notin S$ and $y_2 \notin S$. If the element $x' \in X \setminus \{x\}$ is in the intersection of the column containing y_1 and the row containing y_2 , the elements x, x', y_1 and y_2 form a quadruple. According to our assumption one of these elements is in S , and consequently $\mathcal{D}_S(X) \neq \mathcal{D}_S(Y)$. If x' is not on the same column as y_1 or on the same row as y_2 (both of which contain two elements of S), we clearly have $\mathcal{D}_S(X) \neq \mathcal{D}_S(Y)$. ■

According to Theorem 19, the set illustrated as black squares in Figure 2 is a $\{2\}$ -resolving set of $K_7 \square K_7$. The following theorem can be used to obtain a lower bound on the $\{2\}$ -metric dimension of $K_m \square K_n$. Indeed, the left side of Equation (3) decreases as the size of the $\{2\}$ -resolving set S increases. Thus, this gives a lower bound on $|S|$.

Theorem 20. *Let $m \geq n \geq 2$. If S is a $\{2\}$ -resolving set of $K_m \square K_n$, and q and r are integers such that $|S| = qm + r$ with $0 \leq r < m$, then*

$$r \binom{n - (q + 1)}{2} + (m - r) \binom{n - q}{2} \leq \binom{n}{2}. \quad (3)$$

Proof. Assume first that S is an arbitrary subset of $V(K_m \square K_n)$ and q and r are integers such that $|S| = qm + r$ with $0 \leq r < m$. Denote the columns of $K_m \square K_n$ by C_1, \dots, C_m . For $i = 1, \dots, m$, let x_i be the number of elements of S in the column C_i , i.e., $x_i = |S \cap C_i|$. Using this notation, each column C_i contains $\binom{n - x_i}{2}$ pairs of vertices not belonging to S . Furthermore, the number of such pairs of vertices over all the columns is equal to

$$\sum_{i=1}^m \binom{n - x_i}{2}. \quad (4)$$

Assume that the set S' gives the minimum value of the sum (4) among the sets with $|S'|$ elements. Let us then show that no column of S' contains less than q elements. Suppose to the contrary that there is a column C_i with $|S' \cap C_i| = k_1 < q$. Since $|S'| = qm + r$, there exists a column C_j with $|S' \cap C_j| = k_2 \geq q + 1$. Now we have

$$\begin{aligned} \binom{k_1}{2} + \binom{k_2}{2} &= \binom{k_1}{2} + \binom{k_2 - 1}{2} + (k_2 - 1) > \binom{k_1}{2} + k_1 + \binom{k_2 - 1}{2} \\ &= \binom{k_1 + 1}{2} + \binom{k_2 - 1}{2}. \end{aligned}$$

Hence, the elements of S' in the columns C_i and C_j can be redistributed to obtain a set with the same number of elements as S' and with a smaller sum (4) (a contradiction). Similarly, it can be shown that no column contains at least $q + 2$ elements of S' . Indeed, if such a column, say C_i with $|S' \cap C_i| = k_1 \geq q + 2$, exists, then there is a column C_j with $|S' \cap C_j| = k_2 \leq q$ and as above we have

$$\begin{aligned} \binom{k_1}{2} + \binom{k_2}{2} &= \binom{k_1 - 1}{2} + (k_1 - 1) + \binom{k_2}{2} > \binom{k_1 - 1}{2} + \binom{k_2}{2} + k_2 \\ &= \binom{k_1 - 1}{2} + \binom{k_2 + 1}{2} \end{aligned}$$

leading to a contradiction. Hence, we may assume that each column contains at least q and at most $q + 1$ elements of S' . Therefore, as $|S'| = |S| = qm + r$, there exist r columns containing $q + 1$ elements and $m - r$ columns containing q elements of S' . Thus, we obtain that

$$\sum_{i=1}^m \binom{n - x_i}{2} \geq r \binom{n - (q + 1)}{2} + (m - r) \binom{n - q}{2}.$$

Observe that the right side of this inequality decreases as the number of elements of S increases.

Assume then that S is a $\{2\}$ -resolving set of $K_m \square K_n$ (instead of being arbitrary). Now, due to Lemma 16, no two columns have two same rows without elements of S . This implies (by the pigeon hole principle) that

$$r \binom{n - (q + 1)}{2} + (m - r) \binom{n - q}{2} \leq \binom{n}{2}.$$

Thus, the claim follows. ■

The conditions of Theorem 19 can also be interpreted as a certain type of design as explained in the following remark. For more on combinatorial designs, see [5] (specifically, parts I and IV).

Remark 21. Let X be a set with n elements and \mathcal{B} be a collection of m subsets called *blocks* of X such that (i) any block has at most $n - 2$ elements, (ii) each element of X is included in at most $m - 2$ blocks and (iii) any pair of elements of X is included in at most one block of \mathcal{B} . Each block of \mathcal{B} represents a column of $K_m \square K_n$; more precisely, the elements of a block correspond to the elements of a column not belonging to S . Observe that maximizing the total number of elements in the blocks of \mathcal{B} minimizes the corresponding $\{2\}$ -resolving set S of $K_m \square K_n$. Although the designs satisfying (i), (ii) and (iii) have not earlier been studied, some usual designs work nicely for our purposes.

- Let $n = m = 7$ and $X = \{1, \dots, 7\}$. A collection $\mathcal{B}_1 = \{\{1, 2, 4\}, \{1, 3, 7\}, \{1, 5, 6\}, \{2, 3, 5\}, \{2, 6, 7\}, \{3, 4, 6\}, \{4, 5, 7\}\}$ is a (balanced incomplete block) design such that each block has 3 elements, each element is included in 3 blocks and any pair of elements of X is included in exactly one block of \mathcal{B}_1 . When we interpret \mathcal{B}_1 as explained above, we obtain a $\{2\}$ -resolving set of $K_7 \square K_7$ with 28 elements (see Figure 2). Moreover, by Theorem 20, no smaller $\{2\}$ -resolving set exists. Hence, we have $\beta_2(K_7 \square K_7) = 28$.

- Let $n = 10$, $m = 12$ and $X = \{1, \dots, 10\}$. A collection $\mathcal{B}_2 = \{\{1, 2, 3, 4\}, \{1, 5, 6, 7\}, \{1, 8, 9, 10\}, \{2, 5, 8\}, \{2, 6, 9\}, \{2, 7, 10\}, \{3, 5, 10\}, \{3, 6, 8\}, \{3, 7, 9\}, \{4, 5, 9\}, \{4, 6, 10\}, \{4, 7, 8\}\}$ is a (pairwise balanced) design such that each block has 3

or 4 elements, each element is included in 3 or 4 blocks and any pair of elements of X is included in exactly one block of \mathcal{B}_2 . Hence, we obtain a $\{2\}$ -resolving set of $K_{10} \square K_{12}$ with 81 elements. Therefore, by Theorem 20, we have $\beta_2(K_{10} \square K_{12}) = 81$.

Analogously, any $\{2\}$ -resolving set S of $K_m \square K_n$ can be interpreted as a certain type of design. Indeed, construct a design with m blocks each formed by the elements of a column not belonging to S . By Lemma 16, each such design satisfies the previous condition (iii) and some other minor constraints depending on whether 1. or 2. of Theorem 17 holds.

5. FLOWER SNARKS

Flower snarks were first introduced by Isaacs in [11]. Flower snarks were one of the first infinite graph families of 3-regular graphs proven to have no proper 3-edge-coloring. In [10], flower snarks were shown to have a constant $\{1\}$ -metric dimension. Let us define flower snarks with the following construction.

Construction. Let $n = 2k + 1$ be an odd integer, $n \geq 5$.

1. First we draw n copies of the star $K_{1,3}$. We denote by $T_i = \{a_i, b_i, c_i, d_i\}$ the vertices of the i th star, where the leaves of the star are a_i, c_i and d_i .
2. We connect the vertices a_i by drawing the cycle $a_1 a_2 \cdots a_n a_1$.
3. We connect the remaining leaves of the stars by drawing the cycle $c_1 c_2 \cdots c_n d_1 d_2 \cdots d_n c_1$.

The resulting graph is the flower snark J_n with $4n$ vertices.

Probably the most common way to draw a flower snark is illustrated in Figure 3(a) for J_5 . The graph J_5 (and all flower snarks in general) can be drawn as in Figure 3(b). From this figure it is easy to see that the graph has many automorphisms and that the vertices c_i and d_i do not have any essential differences.

Any shortest path from $v \in T_i$ to $u \in T_j$ can be divided into three parts; the parts inside T_i and T_j , and the part from T_i to T_j . The part from T_i to T_j is usually the obvious, except for c_1 and c_{k+2} (and isomorphic cases). For example, one shortest path between b_1 and b_4 in J_5 is $b_1 a_1 a_5 a_4 b_4$. However, the unique shortest path between c_1 and c_4 is $c_1 c_2 c_3 c_4$.

In [10], it was shown that $\beta_1(J_n) = 3$ when $n \geq 5$. However, the proof for the upper bound $\beta_1(J_n) \leq 3$ is erroneous. The authors claim that the set $W = \{c_1, d_1, d_k\}$ is a resolving set of J_n since all vertices have unique distance arrays with respect to W . However, we have $\mathcal{D}_W(a_1) = (2, 2, k + 1) = \mathcal{D}_W(b_n)$ and $\mathcal{D}_W(a_k) = (k + 1, k + 1, 2) = \mathcal{D}_W(b_{k+1})$. Thus, the set W is not a resolving set of J_n . Despite this, their result holds. We can replace d_k with d_{k+1} in W ,

after which it is straightforward to correct the proof and verify that the new set is indeed a resolving set of J_n .

Our goal is to determine the ℓ -solid- and $\{\ell\}$ -metric dimensions of flower snarks. To that end, we first consider the forced vertices of flower snarks. Consider any flower snark J_n . Since $n \geq 5$, J_n is a 3-regular graph of girth at least 5. Now, for all $v \in V$ and $U \subseteq V$, $v \notin U$, if $N(v) \subseteq N[U]$, then the set U has at least three elements. Thus, no vertex of J_n is forced for $\{\ell_1\}$ -resolving sets or ℓ_2 -solid-resolving sets where $\ell_1 \leq 3$ and $\ell_2 \leq 2$. For all other ℓ -solid- and $\{\ell\}$ -resolving sets all vertices are forced vertices; for all $v \in V$ we can choose $U = N(v)$, and we naturally have $N(v) \subseteq N[U]$. Thus, we have the following theorem.

Theorem 22. *Let n be an odd integer, $n \geq 5$. We have $\beta_\ell(J_n) = 4n$ when $\ell \geq 4$ and $\beta_\ell^s(J_n) = 4n$ when $\ell \geq 3$.*

As for the remaining metric dimensions, we begin by considering $\{3\}$ -resolving sets since, quite surprisingly, the difficulty of the proofs increases as the value of ℓ decreases.

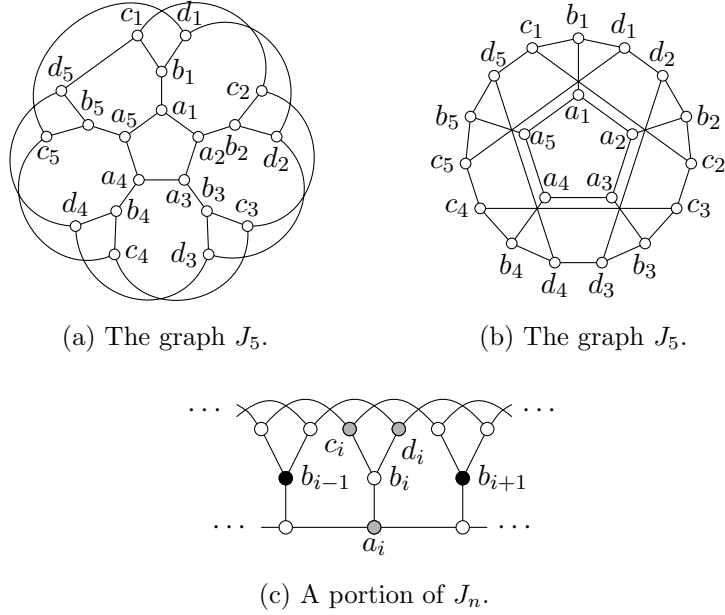


Figure 3.

5.1. The $\{3\}$ -Metric Dimension of J_n

We begin by proving two technical lemmas. In these lemmas, we consider certain sets of vertices with at most three elements. Any $\{3\}$ -resolving set should be able

to distinguish these sets from each other. However, as we will see, there are very few vertices able to do that. In Figure 3(c), we have illustrated a part of a flower snark, which will help in visualising the sets of vertices discussed in the lemmas. Notice that if $i = 1$, then $b_{i-1} = b_n$, and if $i = n$, then $b_{i+1} = b_1$.

Lemma 23. *Let $i \in \{1, \dots, n\}$ and*

$$B = \{b_{i-1}, b_{i+1}\}, \quad X = B \cup \{a_i\}, \quad Y = B \cup \{c_i\}, \quad Z = B \cup \{d_i\}.$$

We have

- (i) $d(s, X) \neq d(s, B)$ if and only if $s \in \{a_i, b_i\}$,
- (ii) $d(s, Y) \neq d(s, B)$ if and only if $s \in \{c_i, b_i\}$,
- (iii) $d(s, Z) \neq d(s, B)$ if and only if $s \in \{d_i, b_i\}$.

Proof. Let $v \in V \setminus T_i$ and $u \in T_i$. Any shortest path $v - u$ goes through either T_{i-1} or T_{i+1} . Thus, either $d(v, u) \geq d(v, b_{i-1})$ or $d(v, u) \geq d(v, b_{i+1})$, and we have $d(v, X) = d(v, Y) = d(v, Z) = \min\{d(v, b_{i-1}), d(v, b_{i+1})\} = d(v, B)$.

Consider then the elements of T_i . The distances from each element $s \in T_i$ to each of the sets B, X, Y and Z are presented in the following table.

s	$d(s, B)$	$d(s, X)$	$d(s, Y)$	$d(s, Z)$
b_i	3	1	1	1
a_i	2	0	2	2
c_i	2	2	0	2
d_i	2	2	2	0

■

Lemma 24. *Let $i \in \{1, \dots, n\}$ and*

$$X = \{a_i, b_{i-1}, b_{i+1}\}, \quad Y = \{c_i, b_{i-1}, b_{i+1}\}, \quad Z = \{d_i, b_{i-1}, b_{i+1}\}.$$

We have

- (i) $d(s, X) \neq d(s, Y)$ if and only if $s \in \{a_i, c_i\}$,
- (ii) $d(s, X) \neq d(s, Z)$ if and only if $s \in \{a_i, d_i\}$,
- (iii) $d(s, Y) \neq d(s, Z)$ if and only if $s \in \{c_i, d_i\}$.

Proof. Follows from the proof of Lemma 23. ■

In the following theorem, the exact values of $\beta_3(J_n)$ are determined for all $n \geq 5$.

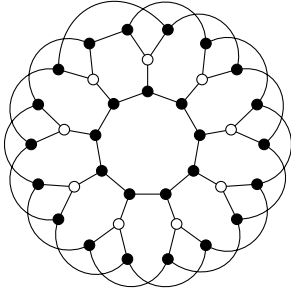
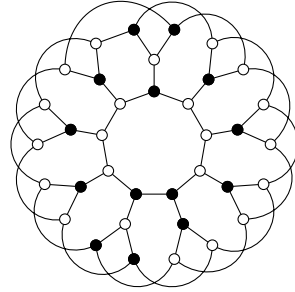
Theorem 25. *Let $n \geq 5$ be an odd integer. We have $\beta_3(J_n) = 3n$.*

Proof. $\beta_3(J_n) \geq 3n$: Let S be a $\{3\}$ -resolving set of J_n . Any set of two vertices of T_i contains an element of S by Lemmas 23 and 24. Thus, $|S \cap T_i| \geq 3$ for all $i = 1, \dots, n$, and the lower bound $\beta_3(J_n) \geq 3n$ follows.

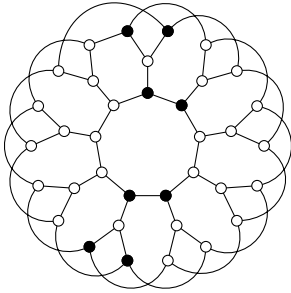
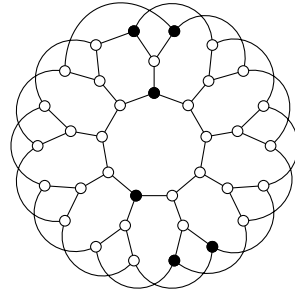
$\beta_3(J_n) \leq 3n$: Let $S = V \setminus \{b_i \mid i = 1, \dots, n\}$ (see Figure 4(a)) and let $X \subseteq V$ such that $|X| \leq 3$. We will prove that S is a $\{3\}$ -resolving set of J_n by showing how to determine the elements of X when we know the distance array $\mathcal{D}_S(X)$.

If for some $s \in S$ we have $d(s, X) = 0$, then clearly $s \in X$. Thus, if $\mathcal{D}_S(X)$ has three zeros, we have found all elements of X since $|X| \leq 3$.

Assume that $\mathcal{D}_S(X)$ has at most two zeros. We need to determine whether $b_i \in X$ for any $i \in \{1, \dots, n\}$. Consider any T_i . If $d(s, X) \geq 2$ for some $s \in T_i \cap S$, then clearly $b_i \notin X$. If $d(s, X) \leq 1$ for all $s \in T_i \cap S$, then $b_i \in X$. Indeed, assume to the contrary that $b_i \notin X$. There is an element of X in $N(s) \setminus \{b_i\}$ for every $s \in T_i \cap S$ such that $d(s, X) = 1$. However, all neighbours of s other than b_i are also in S . Since $N(s) \cap N(s') = \{b_i\}$ for all distinct $s, s' \in T_i \cap S$, there must be at least three zeros in the distance array $\mathcal{D}_S(X)$, a contradiction. Therefore, when $\mathcal{D}_S(X)$ has at most two zeros $b_i \in X$ if and only if $d(s, X) \leq 1$ for all $s \in T_i \cap S$. ■

(a) A $\{3\}$ -resolving set.

(b) A 2-solid-resolving set.

(c) A $\{2\}$ -resolving set.

(d) A 1-solid-resolving set.

Figure 4. Optimal $\{\ell\}$ - and ℓ -solid-resolving sets of J_9 .

5.2. The 2-Solid-Metric Dimension of J_n

Let S be a 2-solid-resolving set of J_n . For any distinct sets $X, Y \subseteq V$ such that $|X| = 2$ and $|Y| \geq 3$, we have $\mathcal{D}_S(X) \neq \mathcal{D}_S(Y)$. In particular, Lemma 23 holds for S . Thus, either $b_i \in S$ or $\{a_i, c_i, d_i\} \subseteq S$. This observation gives us the obvious lower bound $\beta_2^s(J_n) \geq n$. However, as we will show in Theorem 28, the 2-solid-metric dimension of J_n is $n + 5$. In order to obtain the lower bound $\beta_2^s(J_n) \geq n + 5$, we need the following two lemmas. These lemmas tell us, how many vertices a_i, c_i and d_i a 2-solid-resolving set must contain.

Recall that we denote $n = 2k + 1$, where k is an integer.

Lemma 26. *Denote $A = \{a_i \mid i = 1, \dots, n\}$. If a vertex set S is a 2-solid-resolving set of J_n , then there can be at most $k - 1$ consecutive elements of A that are not elements of S . Consequently, S must contain at least three elements of A .*

Proof. Assume to the contrary that there are k or more consecutive elements of A that are not in S . Without loss of generality, we can assume that $\{a_i \mid i = k + 2, \dots, n\} \cap S = \emptyset$. We will show that the set S is not a 2-solid-resolving set as it does not satisfy (1). To that end, let us consider the vertex a_n and the set $X = \{c_n, a_1\}$. For all b_i we have $d(b_i, a_n) = d(b_i, c_n)$, and thus $d(b_i, X) \leq d(b_i, a_n)$. For all c_i we have $d(c_i, c_n) \leq d(a_i, a_n) + 2 = d(c_i, a_n)$. Similarly, we have $d(d_i, c_n) \leq d(d_i, a_n)$ for all d_i . Since $d(c_i, X) \leq d(c_i, c_n)$ and $d(d_i, X) \leq d(d_i, d_n)$, we have $d(c_i, X) \leq d(c_i, a_n)$ and $d(d_i, X) \leq d(d_i, a_n)$ for all $i \in \{1, \dots, n\}$. Let $a_j \in A \cap S$. Since $1 \leq j \leq k + 1$, we have $d(a_j, a_1) \leq d(a_j, a_n)$, and thus $d(a_j, X) \leq d(a_j, a_n)$. Consequently, the set S does not satisfy (1) for a_n and X , and is not a 2-solid-resolving set of J_n according to Theorem 3.

Consequently, A contains at least three elements of S . ■

We denote the cycle $c_1 c_2 \cdots c_n d_1 d_2 \cdots d_n c_1$ by C .

Lemma 27. *If a vertex set S is a 2-solid-resolving set of J_n , then there can be at most k consecutive vertices of C that are not in S . Consequently, S must contain at least four elements of C .*

Proof. Assume to the contrary that there are at least $k + 1$ consecutive vertices of C that are not in S . Without loss of generality, assume that $\{c_i \mid i = 1, \dots, k + 1\} \cap S = \emptyset$.

Consider the vertex c_1 and the set $X = \{d_1, d_n\}$. We will show that $d(s, c_1) \geq d(s, X)$ for all $s \in S$. For all a_i , we have $d(a_i, c_1) = d(a_i, a_1) + 2 = d(a_i, d_1)$. Consequently, $d(a_i, X) \leq d(a_i, c_1)$ for all $i \in \{1, \dots, n\}$. Similarly, we have $d(b_i, c_1) = d(a_i, a_1) + 1 = d(b_i, d_1)$ and $d(b_i, X) \leq d(b_i, c_1)$ for all $i \in \{1, \dots, n\}$. Consider then a vertex d_j . If $1 \leq j \leq k$, then $d(d_j, c_1) = d(d_j, d_1) + 2$. If $k + 1 \leq j \leq n$, then $d(d_j, c_1) = d(d_j, d_n) + 1$. Thus, $d(d_j, X) < d(d_j, c_1)$ for all

$j \in \{1, \dots, n\}$. Similarly, for all c_j where $j \in \{k+3, \dots, n\}$ we have $d(c_j, c_1) = d(c_j, d_1) + 2$. Finally, since $d(c_{k+2}, c_1) = k+1 = d(c_{k+2}, d_n)$, we have $d(c_j, X) \leq d(c_j, c_1)$ for all $j \in \{k+2, \dots, n\}$. Now the set S is not a 2-solid-resolving set of J_n since it does not satisfy (1) for c_1 and X .

Consequently, C contains at least four elements of S . ■

Theorem 28. *Let $n \geq 5$ be an odd integer. We have $\beta_2^s(J_n) = n + 5$.*

Proof. $\beta_2^s(J_n) \geq n + 5$: Assume that S is a 2-solid-resolving set of J_n with at most $n+4$ elements. Recall that according to Lemma 23 we have either $b_i \in S$ or $\{a_i, c_i, d_i\} \subseteq S$ for all $i \in \{1, \dots, n\}$. According to Lemma 27 the set S contains at least four elements of C . Since $|S| \leq n+4$, the set S has exactly four elements of C due to Lemma 23(i). Now, if $c_i \notin S$ or $d_i \notin S$, then $b_i \in S$ and $a_i \notin S$ since otherwise S would have more than $n+4$ elements. If c_i and d_i are both in S , we have either $b_i \in S$ or $a_i \in S$. Since S contains four elements of C , there can be at most two elements a_i in S . Now, according to Lemma 26 the set S is not a 2-solid-resolving set of J_n .

$\beta_2^s(J_n) \leq n + 5$: Let

$$S = \{a_1, c_1, d_1, a_{k+1}, a_{k+2}, c_{k+2}, d_{k+2}\} \cup \{b_i \mid i \in \{1, \dots, n\}, i \neq 1, k+2\}.$$

See Figure 4(b) for an example of this set. We have $|S| = 7 + n - 2 = n + 5$. We will show that S satisfies (1) for $\ell = 2$, and is thus a 2-solid-resolving set of J_n . Clearly, for all $s \in S$ and $X \subseteq V$ such that $s \notin X$ we have $d(s, s) < d(s, X)$. Consider then the vertices that are not in S . We divide the study by the types of the vertices in J_n .

a_i : Assume that $2 \leq i \leq k$, the other case where $k+3 \leq i \leq n$ goes similarly. Since $a_i \notin S$, we have $b_i \in S$. Let $X \subseteq V$, $|X| \leq 2$ and $a_i \notin X$. If $X \cap T_i = \emptyset$, then $d(b_i, a_i) < d(b_i, X)$. Assume then that $X \cap T_i \neq \emptyset$. Observe that $d(a_1, a_i) < d(a_1, X \cap T_i)$ and $d(a_{k+1}, a_i) < d(a_{k+1}, X \cap T_i)$. If $X \subseteq T_i$, then $d(a_1, a_i) < d(a_1, X)$ and $d(a_{k+1}, a_i) < d(a_{k+1}, X)$. Suppose then that $|X \cap T_i| = 1$ and $x \in X \setminus T_i$. If $d(a_1, x) \leq d(a_1, a_i)$ and $d(a_{k+1}, x) \leq d(a_{k+1}, a_i)$, then $d(a_1, a_{k+1}) \leq d(a_1, x) + d(x, a_{k+1}) \leq d(a_1, a_i) + d(a_i, a_{k+1})$. Since the path $a_1 a_2 \dots a_{k+1}$ is the unique shortest path between a_1 and a_{k+1} , we have $x = a_j$ for some $j \in \{1, \dots, k+1\}$, $j \neq i$. Consequently, either $d(a_1, a_i) < d(a_1, x)$ or $d(a_{k+1}, a_i) < d(a_{k+1}, x)$. Thus, either $d(a_1, a_i) < d(a_1, X)$ or $d(a_{k+1}, a_i) < d(a_{k+1}, X)$.

b_i : Since $b_i \notin S$, either $i = 1$ or $i = k+2$. Consider the case where $i = 1$ (the case where $i = k+2$ goes similarly). Let $X \subseteq V$, $|X| \leq 2$ and $b_1 \notin X$. If S does not satisfy (1), then $d(a_1, X)$, $d(c_1, X)$ and $d(d_1, X)$ are all at most 1. However, now each of the sets $\{a_1, a_2, a_n\}$, $\{c_1, c_2, d_n\}$ and $\{d_1, d_2, c_n\}$ must contain at least

one element of X . Since these sets do not intersect, the set X has at least three elements, a contradiction.

c_i, d_i : Consider the vertex c_i where $2 \leq i \leq k+1$ (the other cases go similarly). Let $X \subseteq V$, $|X| \leq 2$ and $c_i \notin X$. Assume that $d(s, X) \leq d(s, c_i)$ for all $s \in S$. Since $d(b_i, X) \leq d(b_i, c_i)$, we have $X \cap \{b_i, d_i, a_i\} \neq \emptyset$. However, for all $v \in \{b_i, d_i, a_i\}$ we have $d(c_1, c_i) < d(c_1, v)$ and $d(c_{k+2}, c_i) < d(c_{k+2}, v)$. Thus, X must have an element x such that $d(c_1, x) \leq d(c_1, c_i)$ and $d(c_{k+2}, x) \leq d(c_{k+2}, c_i)$.

The path $c_1 c_2 \dots c_{k+1} c_{k+2}$ is the unique shortest path between c_1 and c_{k+2} . Naturally, for all c_j , where $j \neq i$, we have either $d(c_1, c_i) < d(c_1, c_j)$ or $d(c_{k+2}, c_i) < d(c_{k+2}, c_j)$. For all other vertices $v \notin \{c_1, \dots, c_{k+2}\}$, we have $d(c_1, v) + d(v, c_{k+2}) > d(c_1, c_{k+2}) = d(c_1, c_i) + d(c_i, c_{k+2})$, and thus $d(c_1, c_i) < d(c_1, v)$ or $d(c_{k+2}, c_i) < d(c_{k+2}, v)$. Thus, there is no such vertex x that $d(c_1, x) \leq d(c_1, c_i)$ and $d(c_{k+2}, x) \leq d(c_{k+2}, c_i)$. ■

5.3. The $\{2\}$ -Metric Dimension of J_n

As we have seen in the two previous sections, the $\{3\}$ - and 2-solid-metric dimensions of J_n are dependent on n . However, we will see in Theorem 29 that the $\{2\}$ -metric dimension is at most eight for any J_n .

Our computer calculations have shown that $\beta_2(J_5) = 7$, and $S = \{a_1, a_3, b_2, b_4, c_1, c_3, d_1\}$, for example, is a $\{2\}$ -metric basis of J_5 . Our calculations have also shown that $\beta_2(J_n) = 8$ when $7 \leq n \leq 19$. We will prove the upper bound $\beta_2(J_n) \leq 8$ in the following theorem, and we conjecture that the lower bound $\beta_2(J_n) \geq 8$ holds for all $n \geq 7$.

The proof of the following theorem is surprisingly difficult with traditional methods of comparing distance arrays. To show the upper bound $\beta_2(J_n) \leq 8$ we will construct a $\{2\}$ -resolving set of J_n with eight elements. We have verified with a computer that the set we provide is indeed a $\{2\}$ -resolving set of J_n when $7 \leq n \leq 19$. To show the claim for $n \geq 21$ we use a reduction-like approach. We will show that if the set was not a $\{2\}$ -resolving set of J_n then it would not be a $\{2\}$ -resolving set of J_{n-2} . The idea behind the proof is that if we carefully remove two stars T_i from J_n and add necessary edges (for example, in Figure 3(c), we can remove the star T_i and connect the stars T_{i-1} and T_{i+1}), we obtain J_{n-2} , and the distances in J_n and J_{n-2} are highly dependent on each other.

Theorem 29. *Let $n = 2k + 1 \geq 7$. We have $\beta_2(J_n) \leq 8$.*

Proof. Denote

$$\begin{aligned} I &= T_n \cup T_1 \cup T_2 \cup T_3, & J &= T_k \cup T_{k+1} \cup T_{k+2} \cup T_{k+3}, \\ I' &= I \cup T_{n-1} \cup T_4, & J' &= J \cup T_{k-1} \cup T_{k+4}, \\ S_I &= \{a_1, c_1, d_1, a_2\}, & S_J &= \{a_{k+1}, a_{k+2}, c_{k+2}, d_{k+2}\}. \end{aligned}$$

Let $S = S_I \cup S_J$ (see Figure 4(c)). We will show that the set S is a $\{2\}$ -resolving set of J_n . It is easy to check with a computer that the set S is a $\{2\}$ -resolving set when $7 \leq n \leq 19$.

Assume to the contrary that the set S is not a $\{2\}$ -resolving set of J_n , where $n \geq 21$, and that the set S is a $\{2\}$ -resolving set of J_{n-2} . We denote the distance arrays in J_n by \mathcal{D}_S^n and the distance arrays in J_{n-2} by \mathcal{D}_S^{n-2} . Consider nonempty sets $X, Y \subseteq V(J_n)$ such that $|X| \leq 2$, $|Y| \leq 2$, $X \neq Y$ and $\mathcal{D}_S^n(X) = \mathcal{D}_S^n(Y)$. It is easy to see that if $\mathcal{D}_{S_I}^n(X)$ contains at least one distance that is at most 2, then we have $X \cap I' \neq \emptyset$. Furthermore, if all distances in $\mathcal{D}_{S_I}^n(X)$ are at least 3, then we have $X \cap I = \emptyset$. The same holds for S_J , J and J' by symmetry.

If both $\mathcal{D}_{S_I}^n(X)$ and $\mathcal{D}_{S_J}^n(X)$ contain at least one distance that is at most 2, we have $X \cap I' \neq \emptyset$ and $X \cap J' \neq \emptyset$. Since $\mathcal{D}_S^n(X) = \mathcal{D}_S^n(Y)$, we have $Y \cap I' \neq \emptyset$ and $Y \cap J' \neq \emptyset$. We may think of J_{n-2} as being obtained from J_n by removing two stars from opposite sides of J_n such that they are halfway between I and J . Let $X', Y' \subseteq V(J_{n-2})$ consist of vertices that are in exactly the same positions as the elements of X and Y with respect to S_I and S_J . Since $n - 2 \geq 19$, we have $d_{J_{n-2}}(s, v) \leq 5 \leq k - 4 \leq d_{J_{n-2}}(s, u)$ for all $s \in S_I$, $v \in I'$ and $u \in J'$ (similarly for $s \in S_J$, $v \in J'$ and $u \in I'$). Thus, $\mathcal{D}_{S_I}^{n-2}(X' \cap I') = \mathcal{D}_{S_I}^{n-2}(X')$ and $\mathcal{D}_{S_J}^{n-2}(X' \cap J') = \mathcal{D}_{S_J}^{n-2}(X')$, and the same also holds for Y' . Now we have $\mathcal{D}_S^{n-2}(X') = \mathcal{D}_S^n(X) = \mathcal{D}_S^n(Y) = \mathcal{D}_S^{n-2}(Y')$. However, $X \neq Y$ implies that $X' \neq Y'$, and since S is a $\{2\}$ -resolving set of J_{n-2} , we must have $\mathcal{D}_S^{n-2}(X') \neq \mathcal{D}_S^{n-2}(Y')$, a contradiction.

Assume then that all distances in $\mathcal{D}_{S_I}^n(X)$ are at least 3 (the case where this holds for $\mathcal{D}_{S_J}^n(X)$ goes similarly). Now, we have $X \cap I = \emptyset$ and $Y \cap I = \emptyset$. We may think of J_{n-2} as being obtained from J_n by removing the stars T_3 and T_n . Let $X', Y' \subseteq V(J_{n-2})$ consist of vertices that are in exactly the same positions as the elements of X and Y with respect to S_J . Now, we have $(X' \cup Y') \cap (T_1 \cup T_2) = \emptyset$, and thus

$$\begin{aligned} \mathcal{D}_{S_I}^{n-2}(X') &= \mathcal{D}_{S_I}^n(X) - (1, 1, 1, 1), & \mathcal{D}_{S_J}^{n-2}(X') &= \mathcal{D}_{S_J}^n(X), \\ \mathcal{D}_{S_I}^{n-2}(Y') &= \mathcal{D}_{S_I}^n(Y) - (1, 1, 1, 1), & \mathcal{D}_{S_J}^{n-2}(Y') &= \mathcal{D}_{S_J}^n(Y). \end{aligned}$$

Consequently, $\mathcal{D}_S^{n-2}(X') = \mathcal{D}_S^{n-2}(Y')$ if and only if $\mathcal{D}_S^n(X) = \mathcal{D}_S^n(Y)$. Since S is a $\{2\}$ -resolving set of J_{n-2} and $X' \neq Y'$, we have $\mathcal{D}_S^{n-2}(X') \neq \mathcal{D}_S^{n-2}(Y')$, a contradiction. \blacksquare

5.4. The 1-Solid-Metric Dimension of J_n

We begin the section by giving an upper bound on $\beta_1^s(J_n)$ for all $n \geq 5$.

Theorem 30. *Let $n = 2k + 1 \geq 5$. We have $\beta_1^s(J_n) \leq 6$.*

Proof. Let $S = \{a_1, a_{k+2}, c_1, d_1, c_{k+1}, d_{k+1}\}$ (see Figure 4(d)). We will show that the set S is a 1-solid-resolving set of J_n by proving that S satisfies (1). We divide the proof by the types of the vertices of J_n .

a_i : Assume that $i \in \{2, \dots, k+1\}$. The vertex a_i is along some shortest path from a_1 to c_{k+1} . If there exists a vertex $v \in V \setminus \{a_i\}$ such that $d(a_1, v) \leq d(a_1, a_i)$ and $d(c_{k+1}, v) \leq d(c_{k+1}, a_i)$, then v is also along a shortest path from a_1 to c_{k+1} . Moreover, we have $d(a_1, v) = d(a_1, a_i)$ and $d(c_{k+1}, v) = d(c_{k+1}, a_i)$, and thus $v \in \{b_{i-1}, c_{i-2}\}$. Similarly, if $d(d_{k+1}, v) \leq d(d_{k+1}, a_i)$, then $v \in \{b_{i-1}, d_{i-2}\}$. Thus, we have $v = b_{i-1}$. However, we clearly have $d(a_{k+2}, a_i) < d(a_{k+2}, b_{i-1})$. Thus, (1) is satisfied for all a_i where $i \in \{2, \dots, k+1\}$. The case where $i \in \{k+3, \dots, n\}$ goes similarly (look at the shortest paths from a_{k+2} to c_1 and d_1).

b_i : Assume that $i \in \{1, \dots, k+1\}$. By the argument above, the only vertex $v \in V \setminus \{b_i\}$ that is at the same distance from a_1 , c_{k+1} and d_{k+1} as b_i is a_{i+1} . However, since $d(c_1, b_i) < d(c_1, a_{i+1})$ for all $i \in \{1, \dots, k+1\}$, the condition (1) is satisfied for all b_i where $i \in \{1, \dots, k+1\}$. Similarly, we can prove that (1) holds for all b_i where $i \in \{k+2, \dots, n\}$ by looking at the shortest paths from a_{k+2} to c_1 and d_1 .

c_i, d_i : Each c_i and d_i is along one of the four unique shortest paths: $c_1 - c_{k+1}$, $c_{k+1} - d_1$, $d_1 - d_{k+1}$ and $d_{k+1} - c_1$. Thus, (1) is satisfied for all c_i and d_i . ■

Let P be a shortest path between u and v in J_n . We denote $\rho_n(u, v) = t - 1$, where t is the number of stars that intersect with P . Thus, $\rho_n(u, v)$ is the distance P traverses in order to get from the star that contains u to the star that contains v . The distance $d(u, v)$ could now be written as $d(u, v) = \rho_n(u, v) + r$, where r is the distance that P traverses inside the stars that contain u and v .

To determine the exact 1-solid-metric dimension of J_n we still need to prove the lower bound $\beta_1^s(J_n) \geq 6$. Computer calculations have shown this lower bound to hold for $5 \leq n \leq 39$. The idea behind the proof of the following theorem is to prove that if for some J_n we have $\beta_1^s(J_n) \leq 5$, then we also have $\beta_1^s(J_{n-2}) \leq 5$. To that end, we assume that the set S , $|S| = 5$, is a 1-solid-resolving set of J_n . We then construct J_{n-2} from J_n by removing the stars T_1 and T_{k+1} and adding necessary edges (see Figure 5). As long as the stars close to the stars that were removed did not contain any elements of S the distances from the elements of S to other vertices behave well and predictably after the removal of the two stars. Then we can construct a 1-solid-resolving set of J_{n-2} from S , and we reach a contradiction to the lower bound shown with a computer.

Theorem 31. *Let $n = 2k + 1 \geq 5$. We have $\beta_1^s(J_n) = 6$.*

Proof. Due to Theorem 30, it suffices to show the lower bound $\beta_1^s(J_n) \geq 6$. We showed this lower bound for $n \leq 39$ by an exhaustive search with a computer.

To prove the claim for all $n \geq 41$ we will show that if for some $n \geq 41$ we have $\beta_1^s(J_n) \leq 5$, then we also have $\beta_1^s(J_{n-2}) \leq 5$.

Let $n = 2k + 1 \geq 41$ and let S be a 1-solid-resolving set of J_n such that $|S| = 5$. Throughout the proof, we will refer to Figure 5, where the flower snarks are smaller than what the proof requires for technical reasons. Consider the set $\{T_i \mid i \in \{1, 2, k-1, k, k+1, k+2, n-1, n\}\}$ (illustrated with a gray background in Figure 5 for J_{21}) and its isomorphic images. There are n such sets and each $s \in S$ is in eight of these sets. Since $|S| = 5$, at least one of these sets does not contain any elements of S if $n > 8 \cdot 5 = 40$. Since $n \geq 41$, we can assume that the stars T_i where $i \in \{1, 2, k-1, k, k+1, k+2, n-1, n\}$ do not contain elements of S . Let $m = n - 2 = 2l + 1$. We denote by T_i^n a star in J_n and by T_i^m a star in J_m .

Let $\alpha : V(J_n) \rightarrow V(J_m)$ be a surjection such that

$$\alpha(x_i) = \begin{cases} x_1, & \text{if } i = n, \\ x_i, & \text{if } i \in \{1, \dots, k\}, \\ x_{i-1}, & \text{otherwise,} \end{cases}$$

where $x_i \in \{a_i, b_i, c_i, d_i\}$. The image of x_i is the same type as x_i , that is, if $x_i = a_i$, then $\alpha(x_i) = a_j$ for some $j \in \{1, \dots, m\}$ and similarly for $x_i = b_i, c_i, d_i$. The preimages of x_1 and x_{l+1} are $\alpha^{-1}(x_1) = \{x_1, x_n\}$ and $\alpha^{-1}(x_{l+1}) = \{x_k, x_{k+1}\}$ (illustrated as black vertices in Figure 5). For all other $x_i \in V(J_m)$ the preimages are unique.

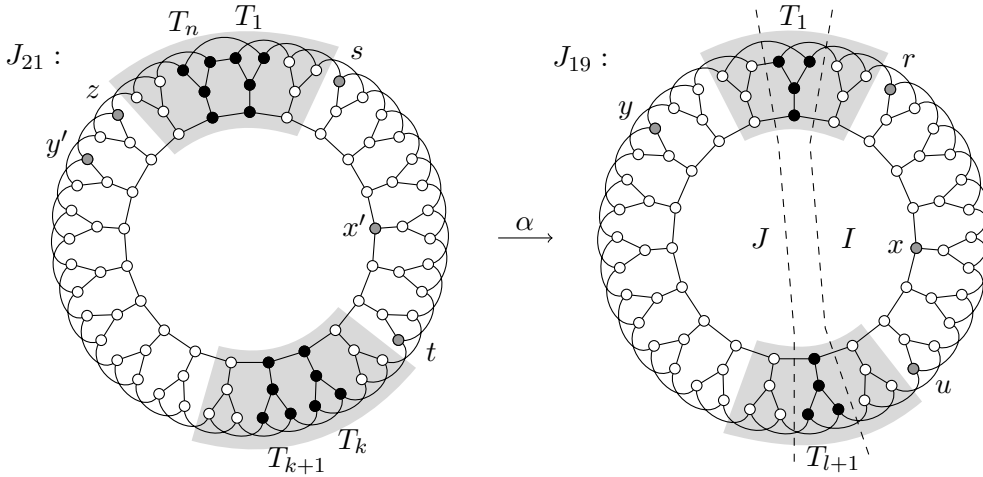


Figure 5. An example of the last case of the proof of Theorem 31 where $n = 21$ and $m = 19$.

Let $R = \{\alpha(s) \mid s \in S\}$. Since the stars T_i^n , where $i \in \{1, 2, k-1, k, k+1, k+2, n-1, n\}$, do not contain elements of S , the vertices $\alpha(T_i^n)$ in J_m are not in R ,

and $|R| = |S|$. In other words, the stars T_j^m , where $j \in \{1, 2, l, l+1, l+2, m\}$, do not contain elements of R . We denote

$$I = \bigcup_{i=2}^l T_i^m \quad \text{and} \quad J = \bigcup_{i=l+2}^m T_i^m$$

(see Figure 5). Let $r \in R$ and $v \in V(J_m)$. Let $s = \alpha^{-1}(r)$ and v' be a preimage of v . Since the stars T_i^m where $i \in \{1, 2, l, l+1, l+2, m\}$ do not contain elements of R , the shortest paths from r to v are closely related to the shortest paths from s to v' .

Let us denote by d_n and d_m the distances in J_n and J_m , respectively. If $r, v \in I$ or $r, v \in J$, then we have $d_m(r, v) = d_n(s, v')$. If $r \in I$ and $v \in J$, or $r \in J$ and $v \in I$, then $d_m(r, v) = d_n(s, v') - 1$. If $v \in T_1^m$ or $v \in T_{l+1}^m$, then $d_m(r, v) = d_n(s, v')$ or $d_m(r, v) = d_n(s, v') - 1$ depending on which of the preimages of v the vertex v' is. Indeed, let $\alpha^{-1}(v) = \{x_1, x_n\}$. If $r \in I$, then $d_m(r, v) = d_n(s, x_1) = d_n(s, x_n) - 1$. If $r \in J$, then $d_m(r, v) = d_n(s, x_n) = d_n(s, x_1) - 1$.

Let $x, y \in V(J_m)$ be distinct. In what follows, we will show that the set R satisfies (1). Suppose first that $x, y \in T_1^m$ or $x, y \in T_{l+1}^m$. Due to symmetry, it suffices to show that when $x, y \in T_1^m$ there exists an element $r \in R$ such that $d_m(r, x) < d_m(r, y)$. Let x' and y' be the preimages of x and y that are in T_1^n . Since S is a 1-solid-resolving set of J_n , there exists some $s \in S$ such that $d_n(s, x') < d_n(s, y')$. Now, we have $d_m(\alpha(s), x) = d_n(s, x')$ if and only if $d_m(\alpha(s), y) = d_n(s, y')$. Consequently, $d_m(\alpha(s), x) < d_m(\alpha(s), y)$.

Suppose then that $x \in T_1^m$ and $y \in T_{l+1}^m$ (the case where $x \in T_{l+1}^m$ and $y \in T_1^m$ goes similarly). Assume to the contrary that there does not exist any $r \in R$ such that $d_m(r, x) < d_m(r, y)$. We have $d_m(r, x) \geq d_m(r, y)$ for all $r \in R$. Let x_1 and x_n be the preimages of x that are in the stars T_1^n and T_n^n , respectively. Similarly, let y_k and y_{k+1} be the preimages of y in the stars T_k^n and T_{k+1}^n , respectively. Let $s \in S$. If $\alpha(s) \in I$, then we have $d_n(s, x_1) = d_m(\alpha(s), x) \geq d_m(\alpha(s), y) = d_n(s, y_k)$. Since $d_n(s, x_n) = d_n(s, x_1) + 1$ and $d_n(s, y_{k+1}) = d_n(s, y_k) + 1$, we have $d_n(s, x_n) \geq d_n(s, y_{k+1})$. If $\alpha(s) \in J$, then $d_n(s, x_n) = d_m(\alpha(s), x) \geq d_m(\alpha(s), y) = d_n(s, y_{k+1})$. Thus, we have $d_n(s, x_n) \geq d_n(s, y_{k+1})$ for all $s \in S$, a contradiction. Therefore, there must exist some $r \in R$ such that $d_m(r, x) < d_m(r, y)$.

Suppose that $x \in T_1^m \cup T_{l+1}^m$ and $y \notin T_1^m \cup T_{l+1}^m$. Assume that $x \in T_1^m$ (the case where $x \in T_{l+1}^m$ follows by symmetry). We denote $y' = \alpha^{-1}(y)$ and $\alpha^{-1}(x) = \{x_1, x_n\}$, where $x_1 \in T_1^n$ and $x_n \in T_n^n$. Suppose that $y \in I$ (the case where $y \in J$ goes similarly). Assume to the contrary that $d_m(v, x) \geq d_m(v, y)$ for all $v \in R$. Let $r, u \in R$ be such that $r \in I$ and $u \in J$. We denote $s = \alpha^{-1}(r)$ and $t = \alpha^{-1}(u)$. Now we have $d_n(s, y') = d_m(r, y)$, $d_n(t, y') = d_m(u, y) + 1$, $d_m(r, x) = d_n(s, x_1)$ and $d_m(u, x) = d_n(t, x_1) - 1$. Since $d_m(v, x) \geq$

$d_m(v, y)$ for all $v \in R$, we have $d_n(s, x_1) = d_m(r, x) \geq d_m(r, y) = d_n(s, y')$ and $d_n(t, x_1) = d_m(u, x) + 1 \geq d_m(u, y) + 1 = d_n(t, y')$. Thus, for all $v' \in S$ we have $d_n(v', x_1) \geq d_n(v', y')$ and the set S does not satisfy (1), a contradiction. Thus, for some $v \in R$ we have $d_m(v, x) < d_m(v, y)$. Similarly, if $d_m(v, y) \geq d_m(v, x)$ for all $v \in R$, then $d_n(s, y') = d_m(r, y) \geq d_m(r, x) = d_n(s, x_1)$ and $d_n(t, y') = d_m(u, y) + 1 \geq d_m(u, x) + 1 = d_n(t, x_1)$. Consequently, for all $v' \in S$ we have $d_n(v', y') \geq d_n(v', x_1)$ and the set S does not satisfy (1), a contradiction. Thus, we also have $d_m(v, y) < d_m(v, x)$ for some $v \in R$.

Finally, assume that $x, y \notin T_1^m \cup T_{l+1}^m$. Let us denote $x' = \alpha^{-1}(x)$ and $y' = \alpha^{-1}(y)$. Assume that $x, y \in I$. Let $s \in S$ be such that $d_n(s, x') < d_n(s, y')$. Denote $r = \alpha(s)$. If $r \in I$, then $d_m(r, x) = d_n(s, x')$ and $d_m(r, y) = d_n(s, y')$. If $r \in J$, then $d_m(r, x) = d_n(s, x') - 1$ and $d_m(r, y) = d_n(s, y') - 1$. In both cases we have $d_m(r, x) < d_m(r, y)$. Thus, the set R satisfies (1) for any $x, y \in I$. The case where $x, y \in J$ goes similarly.

Suppose that $x \in I$ and $y \in J$. There is at least one star between the stars that contain x and y . We have the following two cases.

Case 1. There is exactly one star between x and y . Since $x \in I$ and $y \in J$, the star between x and y is either T_1^m or T_{l+1}^m . Thus, there are two stars between x' and y' . Suppose that T_1^m is the star between x and y , and $x \in T_2^m$ and $y \in T_m^m$. We have $x' \in T_2^n$ and $y' \in T_{n-1}^n$. Let $x_1 \in T_1^n$ and $y_n \in T_n^n$ be such that they are the same type as x' and y' , respectively. By 'same type' we mean that if $x' = c_2$, for example, then $x_1 = c_1$. Since the stars T_i^n where $i \in \{1, 2, k-1, k, k+1, k+2, n-1, n\}$ do not contain any elements of S , the vertex $s \in S$ is on the same side as x' (that is, $\alpha(s) \in I$) if and only if we have $d_n(s, x') < d_n(s, y')$. Similarly, s is on the same side as y' if and only if $d_n(s, y') < d_n(s, x')$.

Assume that for all $v \in R$ we have $d_m(v, x) \geq d_m(v, y)$. Since S is a 1-solid-resolving set of J_n , there exist vertices $s, t \in S$ such that $d_n(s, x') < d_n(s, y')$ and $d_n(t, y') < d_n(t, x')$. According to our previous observation, s is on the same side as x' and t is on the same side as y' . Denote $r = \alpha(s)$. Since $d_m(r, x) \geq d_m(r, y)$, we have $d_n(s, y') - 1 \geq d_n(s, x') = d_m(r, x) \geq d_m(r, y) = d_n(s, y') - 1$. Consequently, $d_m(r, x) = d_m(r, y)$ and $d_n(s, y') = d_n(s, x') + 1$. Thus, we have $d_n(s, x_1) = d_n(s, x') + 1 = d_n(s, y')$. Since the stars T_i^n , where $i \in \{1, 2, k-1, k, k+1, k+2, n-1, n\}$, do not contain any elements of S , all shortest paths from t to x_1 go through the star that contains y' . Since the star T_n^n is between T_1^n and the star that contains y' , we have $d_n(t, y') \leq d_n(t, x_1)$. Thus, the set S does not satisfy (1) for x_1 and y' , a contradiction. Similarly, if $d_m(v, y) \geq d_m(v, x)$ for all $v \in R$, the set S does not satisfy (1) for y_n and x' .

Case 2. There are at least two stars between x and y . Now, there are at least three stars between x' and y' . Let $s \in S$ be such that $d_n(s, x') < d_n(s, y')$, and denote $r = \alpha(s)$. As $d_m(r, y) \geq d_n(s, y') - 1$, we have $d_m(r, x) \leq d_m(r, y)$. If

$d_m(r, x) < d_m(r, y)$, then we are done. Suppose that $d_m(r, x) = d_m(r, y)$. We have $r \in I$ since otherwise $d_m(r, x) = d_n(s, x') - 1 < d_n(s, y') - 1 = d_m(r, y) - 1$. Since $d_m(r, x) = d_n(s, x')$ and $d_m(r, y) = d_n(s, y') - 1$, we have $d_n(s, x') = d_n(s, y') - 1$. Clearly, there does not exist a shortest path from r to x that goes through the star that contains y . Since there are at least two stars between x and y , there does not exist a shortest path from r to y that goes through the star that contains x . Indeed, otherwise we would have $d_m(r, x) \leq \rho_m(r, x) + 2 < \rho_m(r, y) \leq d_m(r, y)$. Thus, the shortest paths $r - x$ and $r - y$ can coincide with each other only in the star that contains r .

Let $z \in V(J_n)$ be the unique vertex that is the same type as y' (i.e., a_i, b_i, c_i or d_i), is in a star next to y' and for which $d_n(s, z) = d_n(s, x')$ holds (see Figure 5). The vertex z is indeed unique since the first two conditions reduce the options to two and the third condition uniquely determines z as n in odd. Since S is a 1-solid-resolving set of J_n , there exists a $t \in S$ such that $d_n(t, x') < d_n(t, z)$. If the vertex t is in the same star as y' or z , then $\rho_n(t, x') \geq 3$ since $\rho_n(y', x') \geq 4$. However, now $d_n(t, z) \leq 3 \leq \rho_n(t, x') \leq d_n(t, x')$. Thus, t is not in the same star as y' or z , and we have

$$\begin{aligned} d_n(t, y') - 1 &\leq d_n(t, z) \leq d_n(t, y') + 1, \\ d_n(t, z) - 1 &\leq d_n(t, y') \leq d_n(t, z) + 1. \end{aligned}$$

If $d_n(t, y') < d_n(t, x')$, then $d_n(t, z) \leq d_n(t, y') + 1 \leq d_n(t, x')$, a contradiction. Thus, we have $d_n(t, y') \geq d_n(t, x')$.

We denote $u = \alpha(t)$. If $u \in J$, then $d_m(u, y) = d_n(t, y')$ and $d_m(u, x) = d_n(t, x') - 1$. Consequently, $d_m(u, x) \leq d_n(t, y') - 1 < d_m(u, y)$ and (1) is satisfied for x and y .

Suppose then that $u \in I$. Now, $d_m(u, y) = d_n(t, y') - 1$ and $d_m(u, x) = d_n(t, x')$. If $d_m(u, x) < d_m(u, y)$, then (1) is again satisfied. Assume that $d_m(u, x) \geq d_m(u, y)$. Since there are at least two stars between x and y , a shortest path $u - y$ cannot go through the star that contains x . Consequently, there is no shortest path $t - y'$ that goes through the star that contains x' . If there is a shortest path $t - y'$ that goes through the star that contains z , then we have $d_n(t, y') = d_n(t, z) + 1$ and

$$d_m(u, y) = d_n(t, y') - 1 = d_n(t, z) > d_n(t, x') = d_m(u, x).$$

Thus, u satisfies (1) for x and y . Suppose then that there is no shortest path $t - y'$ that goes through the star that contains z . The shortest paths $t - y'$ and $s - y'$ can coincide only in the star that contains y' . Consequently, the shortest paths $u - y$ and $r - y$ in J_m can coincide only in the star that contains y . As we have seen before, the shortest paths $r - x$ and $r - y$ can coincide only in the star that contains r , and the shortest paths $u - y$ do not go through the star that

contains x . Thus, the shortest paths $u - y$, $u - x$, $r - x$ and $r - y$ can coincide only in the stars that contain x , y , r or u (see Figure 5 for an example of this situation). Consequently, we have

$$\begin{aligned} d_m(r, x) + d_m(u, x) &\leq \rho_m(r, x) + 2 + \rho_m(u, x) + 2 = \rho_m(r, u) + 4 \\ &\leq l - 4 + 4 = l, \\ d_m(r, y) + d_m(u, y) &\geq \rho_m(r, y) + \rho_m(u, y) = m - \rho_m(r, u) \geq l + 5. \end{aligned}$$

Thus, $d_m(r, x) + d_m(u, x) < d_m(r, y) + d_m(u, y)$. Since $d_m(r, x) = d_m(r, y)$, we have $d_m(u, x) < d_m(u, y)$. Using similar arguments we can show that there exists some $u' \in R$ such that $d_m(u', y) < d_m(u', x)$. ■

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