Unified picture for spatial, temporal, and channel steering

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Quantum steering describes how local actions on a quantum system can affect another, spacelike separated, quantum state. Lately, quantum steering has been formulated also for timelike scenarios and for quantum channels. We approach all the three scenarios as one using tools from Stinespring dilations of quantum channels. By applying our technique we link all three steering problems one-to-one with the incompatibility of quantum measurements, a result formerly known only for spatial steering. We exploit this connection by showing how measurement uncertainty relations can be used as tight steering inequalities for all three scenarios. Moreover, we show that certain notions of temporal and spatial steering are fully equivalent and prove a hierarchy between temporal steering and macrorealistic hidden variable models.

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I. INTRODUCTION

Quantum steering refers to the possibility of one party, typically called Alice, to affect the quantum state of a spatially separated party, typically called Bob, by making only local measurements on her system and classically communicating the measurement outcome and setting to Bob. Quantum steering formalizes spooky action at a distance [1], and as such it is an entanglement verification method intermediate to trust-based entanglement witnesses and no trust-requiring device-independent scenarios, e.g., Bell inequalities. Steering provides a natural framework for semi-device-independent quantum information protocols [2–4] and a guideline for theoretical and experimental work on both entanglement theory and nonlocality [5–10]. Moreover, steering is known to be closely connected to incompatibility of quantum measurements [11,12]. To be more precise, it has been shown that steering and joint measurability problems are in one-to-one correspondence [13] and that unsteerability of quantum states can be checked through incompatibility breaking properties of quantum channels [14].

Extending the spatial case, steering has recently found its temporal counterpart [15] (see Fig. 1). The idea of temporal steering is to ask whether steeringlike phenomena can happen on a single quantum system, where Alice measures a single particle first and then hands it to Bob. One could argue that some sort of steering effect is easy to reach in such scenarios, because Alice's measurement choice can in principle affect Bob's state, i.e., Alice can signal to Bob. However, signaling can be excluded by using well-chosen input states. The remaining scenarios have found connections to, for example, non-Markovianity [16]. In this work we want to characterize quantum measurements so we do not restrict ourselves to specific input states. Instead, we take an approach where nonsignaling is a feature of the measurement instruments. Physically, these are the scenarios where the original system is first interacting with a probe system in some predefined manner, and then different measurements on the probe system are carried out. We show that all (nontrivial) temporal scenarios can be mapped into our formulation.

State steering has a natural extension to the level of quantum channels through the well-known state-channel isomorphism [17]. This extension is called channel steering, and it investigates the possibility of Alice to affect Bob's end of a broadcast channel from Charlie to Alice and Bob. Technically, channel steering can be seen as a (semi-device-independent) method of verifying the coherence of a channel extension [17].

By now channel steering has been introduced as a theoretical construction, but in this article we show how a certain modification of it provides a powerful framework for all three steering scenarios. Namely, we connect all three steering scenarios one-to-one with the incompatibility of quantum measurements, provide universally applicable steering inequalities through measurement uncertainty relations, show an equivalence between spatial and temporal steering, and prove a hierarchy between temporal steering and macrorealistic hidden variable models.

II. SPATIAL STEERING

Steering scenarios can be seen as processes where an untrusted party (Alice) sends a trusted party (Bob) a state assemblage $\{\rho_{a|x}\}_{a,x}$, where x labels the measurements and a the respective outcomes, satisfying the nonsignaling condition $\sum_a \rho_{a|x} = \sum_a \rho_{a|x'}$ for all x,x'. The nonsignaling property is crucial in our scenarios for reasons to become clear in the following sections. The steerability of a state assemblage is decided by checking the existence of a so-called local hidden state model (see below).

In *spatial* steering the state assemblage originates from spacelike separated local measurements on one party and is hence naturally nonsignaling. Formally, consider a bipartite system described by a quantum state ρ_{AB}. When Alice performs measurements described by positive operator valued measures (POVMs) $\{A_{a|x}\}_{a,x}$ (i.e., $A_{a|x} \geqslant 0$ and $\sum_a A_{a|x} = \mathbb{1}$

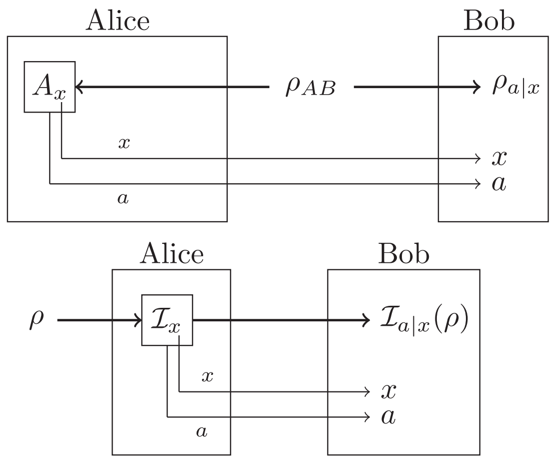

FIG. 1. Spatial steering (top): Alice and Bob share a bipartite state ρ_{AB}, Alice measures A_x and classically communicates the measurement setting (x) and result (a) to Bob. The (non-normalized) postmeasurement state assemblage Bob receives is given as $\rho_{a|x} = \mathrm{tr}_A[(A_{a|x} \otimes \mathbb{1})\rho_{AB}]$. Temporal steering (bottom): Alice applies an instrument $\mathcal{I}_x$ on a single-system state ρ and classically communicates the measurement setting (x) and result (a) to Bob, together with the (non-normalized) output state $\mathcal{I}_{a|x}(\rho)$.

for all a,x) on her system, Bob is left with a non-normalized state assemblage

$$\rho_{a|x} := \mathrm{tr}_A[(A_{a|x} \otimes \mathbb{1})\rho_{AB}]. \tag{1}$$

Here, $\mathbb{1}$ is the identity operator on Bob's system. The setup is called (spatially) unsteerable if Bob can recover his state assemblage from a local state ensemble (or state preparator) $\{p(\lambda),\sigma_\lambda\}_\lambda$ together with additional information about Alice's choice of measurement x and obtained outcome a by means of classical postprocessing, i.e., if for every a,x

$$\rho_{a|x} = \sum_\lambda p(\lambda)p(a|x,\lambda)\sigma_\lambda \tag{2}$$

and steerable otherwise. Here $p(\lambda) \geqslant 0$ is the probability that Bob's state σ_λ occurs and $p(a|x,\lambda) \geqslant 0$ are conditional probabilities so that $\sum_a p(a|x,\lambda) = 1$ for each x, λ. The right-hand side of Eq. (2) is called, when existing, a local hidden state (LHS) model for the assemblage $\{\rho_{a|x}\}_{a,x}$.

III. TEMPORAL STEERING

For temporal steering one needs the concept of quantum instruments. Quantum instruments are collections of completely positive maps which sum up to a completely positive trace preserving (cptp) map, i.e., to a quantum channel. Physically, instruments describe the state transformation caused by a measurement, and one can think of them as a generalization of the projection postulate to the case of POVMs. For a POVM $\{A_a\}_a$ the most typical instrument is the von Neumann–Lüders instrument $\mathcal{I}_a^L(\rho) = \sqrt{A_a}\rho\sqrt{A_a}$, and all possible instruments compatible with $\{A_a\}_a$ are the ones which code the measurement outcome probabilities into the postmeasurement state, i.e., $\mathrm{tr}[\mathcal{I}_a(\rho)] = \mathrm{tr}[A_a\rho]$ for all ρ. It can be shown [18] that any instrument implementing $\{A_a\}_a$ can be described by the quantum channels $\{\Lambda_a\}_a$ from Alice to Bob applied to the von Neumann–Lüders instrument via $\mathcal{I}_a(\rho) = \Lambda_a[\mathcal{I}_a^L(\rho)]$.

In temporal steering one is interested in state assemblages $\{\rho_{a|x}^{\mathrm{temp}}\}_{a,x}$ which are given by the actions of a set of quantum instruments $\{\mathcal{I}_{a|x}\}_{a,x}$ on a single system state ρ_A. The steerability of this assemblage is decided by checking the existence of a LHS model, i.e., the scenario is temporally nonsteerable if

$$\rho_{a|x}^{\mathrm{temp}} := \mathcal{I}_{a|x}(\rho_A) = \sum_\lambda p(\lambda)p(a|x,\lambda)\sigma_\lambda \tag{3}$$

and steerable otherwise. In temporal steering the nonsignaling condition is not a built-in feature. However, as some input states lead to steering trivially, it makes sense to talk about temporal steering only in the case of nonsignaling assemblages. Finally, note that sometimes temporal state assemblages are defined through an instrument and an additional time evolution. As a concatenation of an instrument and a channel is an instrument, we do not write the channel explicitly to our state assemblages.

IV. MAIN TECHNIQUE

As our main technique we use the Stinespring dilation of quantum channels. In textbook quantum mechanics any quantum channel Λ on a finite-dimensional system is given through the representation $\Lambda(\rho) = \mathrm{tr}_E[U(\rho_0 \otimes \rho)U^\dagger]$, where U is a unitary operator on the total space of the system and an environment E, and ρ_0 is a quantum state of the environment [19]. This type of representation is, however, not the only way to dilate a channel. It appears that a slightly modified version of Stinespring dilation is better tailored for our purposes. Namely, instead of using a unitary operator on the system and its environment, we define an isometry $V : \mathcal{H} \to \mathcal{A} \otimes \mathcal{K}$, where $\mathcal{H}$ and $\mathcal{K}$ are the Hilbert spaces of the input and output systems and $\mathcal{A}$ is the Hilbert space of a dummy system. For a channel given in the Kraus form $\Lambda(\rho) = \sum_{k=1}^r K_k\rho K_k^\dagger$, the isometry V can be constructed as $V|\psi\rangle = \sum_{k=1}^r |\varphi_k\rangle \otimes K_k|\psi\rangle$ for all $|\psi\rangle$, with $\{|\varphi_k\rangle\}_{k=1}^r$ being an orthonormal basis of the dummy system. With this isometry the dilation simply reads $\Lambda(\rho) = \mathrm{tr}_{\mathcal{A}}[V\rho V^\dagger]$. Note that this dilation does not have a specific initial state on the environment and, hence, in order to make a clear distinction between the textbook unitary dilation and our isometric dilation we talk about a dummy system instead of an environment.

We are specifically interested in sets of instruments $\{\mathcal{I}_{a|x}\}_{a,x}$ which do not allow signaling, i.e., which have the same total channel $\Lambda := \sum_a \mathcal{I}_{a|x} = \sum_a \mathcal{I}_{a|x'}$ for every x,x'. Nonsignaling instruments are related to observables on the dummy space of their total channel Λ [20,21]. Namely, the actions of nonsignaling instruments $\{\mathcal{I}_{a|x}\}_{a,x}$ can be written as actions of a set of POVMs $\{\tilde{A}_{a|x}\}_{a,x}$ on the dummy system, i.e.,

$$\mathcal{I}_{a|x}(\rho) = \mathrm{tr}_{\mathcal{A}}[(\tilde{A}_{a|x} \otimes \mathbb{1})V\rho V^\dagger]. \tag{4}$$

Note that in general the dummy POVMs $\{\tilde{A}_{a|x}\}_{a,x}$ do not coincide with the POVMs $\{A_{a|x}\}_{a,x}$ one measures on the actual system. Note, moreover, that the isometry V is constructed from Λ and, due to nonsignaling, does not depend on x. Hence, nonsignaling instruments can be implemented using a predefined interaction with a probe system, as described in the Introduction.

In what follows, we will mainly concentrate on minimal dummy systems, i.e., minimal Stinespring dilations. The minimality means that r is the smallest possible dimension, in which case the Kraus operators of the total channel are linearly independent. In this case (for a given total channel) the correspondence between the dummy POVMs and the instruments they define is one-to-one [20,21]. Namely, we have that

$$\mathcal{I}_{a|x}(\rho) = \sum_{k,l=1}^{r} \langle\varphi_l|\tilde{A}_{a|x}|\varphi_k\rangle K_k\rho K_l^\dagger, \tag{5}$$

from where the matrix elements $\langle\varphi_l|\tilde{A}_{a|x}|\varphi_k\rangle$ of the dummy POVMs can be computed.

V. MINIMAL DILATION FOR A STATE ASSEMBLAGE

A crucial concept for our study is joint measurability. A set of POVMs $\{A_{a|x}\}_{a,x}$ is called jointly measurable if there exists a common POVM $\{G_\lambda\}_\lambda$ from which the original POVMs can be postprocessed, i.e., if for every a,x

$$A_{a|x} = \sum_\lambda p(a|x,\lambda)G_\lambda \tag{6}$$

and incompatible otherwise. Here $p(\cdot|x,\lambda)$ is a probability distribution for every x,λ.

Because of the one-to-one connection between dummy POVMs and instruments, we see that a set $\{\tilde{A}_{a|x}\}_{a,x}$ of dummy POVMs is jointly measurable if and only if the instruments $\{\mathcal{I}_{a|x}\}_{a,x}$ (defined through the minimal dilation) have a common refinement, i.e., for every a,x one has

$$\mathcal{I}_{a|x} = \sum_\lambda p(a|x,\lambda)\mathcal{I}_\lambda. \tag{7}$$

To relate this connection to spatial and temporal steering, note that any state ensemble $\{p(\lambda),\rho_\lambda\}_\lambda$, where $\sum_\lambda p(\lambda) = 1$, is an output of a state preparator, i.e., an instrument with a trivial input space $\mathbb{C}$. Even though using a one-dimensional Hilbert space might sound unconventional, it appears to be useful for our purposes, as any nonsignaling state assemblage $\{\rho_{a|x}\}_{a,x}$ corresponds to a nonsignaling set $\{\mathcal{I}_{a|x}\}_{a,x}$ of state preparators through the minimal Stinespring dilation as [see Eq. (4)]

$$\rho_{a|x} = \mathcal{I}_{a|x}(|1\rangle\langle 1|) = \mathrm{tr}_{\mathcal{A}}[(\tilde{A}_{a|x}\otimes\mathbb{1})|\psi\rangle\langle\psi|], \tag{8}$$

where $|1\rangle$ is a complex number with norm 1 and $|\psi\rangle := V|1\rangle$ is a unit vector on the compound system. As the dummy POVMs $\{\tilde{A}_{a|x}\}_{a,x}$ are unique for a given minimal dilation, and as the state preparator corresponding to a LHS model has the same total channel as state preparators associated to the assemblage, we arrive at our first Observation [see also Eq. (7)].

Observation 1. Any nonsignaling state assemblage $\{\rho_{a|x}\}_{a,x}$ is unsteerable if and only if the associated observables $\{\tilde{A}_{a|x}\}_{a,x}$ on the minimal dilation of the corresponding state preparator are jointly measurable.

In order to make Observation 1 more concrete, consider a state assemblage given by a set of state preparators $\{\mathcal{I}_{a|x}\}_{a,x}$ through Eq. (8). The state of the total system $V|1\rangle\langle 1|V^\dagger$ is clearly a purification of $\rho_B := \sum_a \rho_{a|x}$. One possible choice of this purification is the canonical one $|\psi\rangle = (\mathbb{I}\otimes\rho_B^{1/2})|\psi^+\rangle$, where $|\psi^+\rangle = \sum_i |ii\rangle$ is a non-normalized singlet state written in the eigenbasis of ρ_B. For this choice Eq. (8) reads

$$\rho_{a|x} = \mathrm{tr}_{\mathcal{A}}[(\tilde{A}_{a|x}\otimes\mathbb{1})|\psi\rangle\langle\psi|] = \rho_B^{1/2}\tilde{A}_{a|x}^T\rho_B^{1/2}, \tag{9}$$

where the transpose is taken in the eigenbasis of ρ_B. Hence, the dummy POVMs whose joint measurability solves spatial and temporal steerability are given as $\tilde{A}_{a|x} = \rho_B^{-1/2}\rho_{a|x}^T\rho_B^{-1/2}$.

Noting that joint measurability is invariant under transposition, we can reproduce the known result [13] for spatial steering stating that a state assemblage $\{\rho_{a|x}\}_{a,x}$ is unsteerable if and only if the so-called Bob's steering equivalent observables defined as $B_{a|x} := \rho_B^{-1/2}\rho_{a|x}\rho_B^{-1/2}$ are jointly measurable.

It is worth mentioning that Observation 1 can also be used to reproduce a known example of the connection between temporal steering and joint measurability for scenarios using Lüders instruments and a maximally mixed input state [22]. The result of the article states that a set of observables is nonjointly measurable if and only if it can be used for temporal steering. Whereas this claim works perfectly for the maximally mixed input state, it is worth noting that, for example, a typical joint measurement scenario with orthogonal noisy qubit observables $A^\eta_{\pm1|x} := \frac{1}{2}(\mathbb{1}\pm\eta\vec{x}\cdot\vec{\sigma})$, where $0<\eta\leqslant 1$ is the noise parameter, leads to signaling assemblages with any other input state than the maximally mixed one. Hence, even jointly measurable observables, i.e., $\eta\leqslant\frac{1}{\sqrt{3}}$ [23], can lead to temporal steering in the state-dependent framework, providing a counterexample for the general claim in [22].

For scenarios including the maximally mixed input state and Lüders instruments, one sees that our approach gives the transposed versions of Alice's measurements as dummy POVMs. Hence, one sees that the claims made in [22] for the specific input state and instruments can be reproduced using our method.

VI. CHANNEL STEERING

In channel steering [17] one is interested in an assemblage of instruments $\{\mathcal{I}_{a|x}\}_{a,x}$ instead of states. This assemblage originates from a process where Charlie sends quantum states to Bob through a quantum channel $\Lambda^{C\to B}$ which possibly entangles some of the states to an environment (Alice) (see Fig. 2). The task is to decide if the entanglement between Alice and Bob is strong enough to allow Alice to steer Bob's outputs of the channel. Mathematically this means that one takes a channel extension $\Lambda^{C\to A\otimes B}$ of the channel $\Lambda^{C\to B}$ and defines an instrument assemblage through

$$\mathcal{I}_{a|x}(\rho) = \mathrm{tr}_A[(A_{a|x}\otimes\mathbb{1})\Lambda^{C\to A\otimes B}(\rho)]. \tag{10}$$

Note that here the assemblage is nonsignaling by definition. The unsteerability of this instrument assemblage is defined as the existence of a common instrument $\mathcal{I}_\lambda$ and postprocessings $p(a|x,\lambda)$ such that

$$\mathcal{I}_{a|x} = \sum_\lambda p(a|x,\lambda)\mathcal{I}_\lambda. \tag{11}$$

Noticing that Eq. (11) and Eq. (7) are identical and using a minimal dummy system instead of a generic extension in Eq. (10) we arrive to the following Observation:

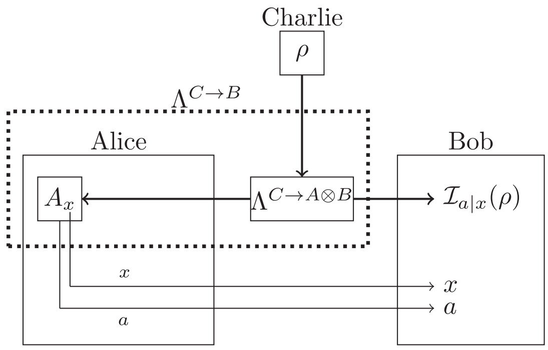

FIG. 2. Channel steering: The setup is similar to the spatial steering scenario, but in the channel case the shared state is prepared by Charlie via the broadcast channel $\Lambda^{C\to A\otimes B}$. The operations enclosed in the dotted line are then viewed by Bob as instruments which have the total channel $\Lambda^{C\to B}$. The main difference to spatial steering is that here Bob's task is to build a local (instrument) model for all possible input states, see Eq. (11).

Observation 2. An instrument assemblage $\{\mathcal{I}_{a|x}\}_{a,x}$ defined through a minimal dilation $\Lambda^{C\to A\otimes B}$ is unsteerable if and only if the associated dummy POVMs $\{\tilde{A}_{a|x}\}_{a,x}$ are jointly measurable.

There exists a former result [24] reporting a one-to-one connection between joint measurability of measurements $\{A_{a|x}\}_{a,x}$ on any dilation (or extension) of the total channel and the nonsteerability of the instrument assemblage they define. This result, however, can be proven false. Namely, while it is true that compatible measurements will not lead to channel steering no matter which dilation (or extension) is used, the other direction is not true in general. Take, for example, any instrument assemblage $\{\mathcal{I}_{a|x}\}_{a,x}$ defined through linearly dependent Kraus operators $K_1 = \frac{1}{\sqrt{2}}U$, $K_2 = \frac{1}{\sqrt{2}}U$ of some unitary channel $\Lambda_U(\rho) = U\rho U^\dagger$. The instrument assemblage is given by

$$\mathcal{I}_{a|x}(\rho) = \frac{1}{2}\sum_{k,l=1}^{2}\langle\varphi_l|\tilde{A}_{a|x}|\varphi_k\rangle U\rho U^\dagger. \tag{12}$$

Hence defining $p(a|x,\lambda) = \frac{1}{2}\sum_{k,l}\langle\varphi_l|\tilde{A}_{a|x}|\varphi_k\rangle$ (which is a probability distribution as $\{\tilde{A}_{a|x}\}_{a,x}$ is a POVM), $\Lambda_\lambda = \Lambda_U$, and the hidden variable space to be trivial, one sees that the setup is unsteerable for compatible as well as for incompatible sets $\{\tilde{A}_{a|x}\}_{a,x}$ of POVMs.

To relate our result to the above example, note that as the *minimal* dilation of the channel Λ_U is one dimensional, observables in this space are always jointly measurable and hence the instrument assemblage is nonsteerable.

VII. STEERING INEQUALITIES FROM INCOMPATIBILITY

As various joint measurement uncertainty relations have been analytically characterized [23,25–31], our Observation 1 and Observation 2 open up a possibility to use them as steering inequalities for all three scenarios. As an example, take the simplest case of two two-valued qubit observables $A_{\pm|x} = \frac{1}{2}(\mathbb{1} \pm \vec{a}_x \cdot \vec{\sigma})$, $x = 1,2$. These observables are jointly measurable [23] if and only if $\|\vec{a}_1 + \vec{a}_2\| + \|\vec{a}_1 - \vec{a}_2\| \leqslant 2$. This inequality is universally applicable to all three steering scenarios and gives an "if and only if" condition for each of them. Inserting the measurements $\{A_{\pm|x}\}_{x=1,2}$ as the dummy POVMs to, for example, Eq. (5) gives instruments for which channel steering can directly be decided. We are ready to state our next Observation.

Observation 3. Joint measurement uncertainty relations can be used as steering inequalities for spatial, temporal, and channel steering.

VIII. EQUIVALENCE BETWEEN TEMPORAL AND SPATIAL STEERING

Applying the Stinespring dilation to a set of nonsignaling instruments $\{\mathcal{I}_{a|x}\}_{a,x}$ shows that the temporal steering scenario they define can be mapped into the spatial steering scenario, see Eq. (4). The question remains which spatial scenarios can be reached by these instruments as the mapping is in general not injective.

To answer this, take a nonsignaling state assemblage $\{\rho_{a|x}\}_{a,x}$ with a d-dimensional support. Notice that this state assemblage can be prepared through spatial steering using a purification of the total state $\rho_B := \sum_a \rho_{a|x}$ [32–34] [see also Eq. (9)]. Hence, we need an isometry V which has such purification in its range. One possible choice is the set of Kraus operators $K_k = |k\rangle\langle k|$, where $\{|k\rangle\}_{k=1}^d$ is the eigenbasis of ρ_B. Taking the input state $|\psi\rangle := \sum_{i=1}^d \sqrt{\lambda_i}|i\rangle$, where the numbers $\lambda_i > 0$ are the eigenvalues of the state ρ_B, and the observables $\tilde{A}_{a|x} := \rho_B^{-1/2}\rho_{a|x}^T\rho_B^{-1/2}$, where the transpose is taken in the eigenbasis of ρ_B, we get through the minimal dilation of the channel $\Lambda(\rho) := \sum_k K_k\rho K_k^\dagger$ the desired state assemblage

$$\mathcal{I}_{a|x}(|\psi\rangle\langle\psi|) = \sum_{k,l=1}^{d}\langle l|\tilde{A}_{a|x}|k\rangle K_k|\psi\rangle\langle\psi|K_l^\dagger = \rho_{a|x}. \tag{13}$$

With this, we are ready to state the next Observation:

Observation 4. Temporal and spatial steering are fully equivalent problems in that temporal steering can be embedded into the spatial scenario and the two can produce exactly the same assemblages. Moreover, any nonsignaling state assemblage on a d-level system can be reproduced with nonsignaling instruments acting on a d-level system.

The above Observation has two crucial consequences. First, for nontrivial instances of temporal steering the restriction to nonsignaling instruments is actually not a restriction at all. Second, Observation 4 allows one to prove a hierarchy between temporal steering and macrorealistic hidden variable models (see below).

IX. TEMPORAL STEERING AND MACROREALISM

We now proceed to show that steering has an analogous role in the temporal scenario to that of the spatial case. Namely, whereas spatially nonsteerable correlations are a proper subset of local correlations, we show that temporally nonsteerable correlations are a proper subset of macrorealistic correlations.

To do so, recall that the probabilities in a sequential measurement scenario (consisting here of two different time steps) are said to have a macrorealistic hidden variable model if they can be written in the form $\mathrm{tr}[\mathcal{I}_{a|x}(\rho)B_{b|y}] = \sum_\lambda p(\lambda)p(a|x,\lambda)p(b|y,\lambda)$, where $p(\cdot), p(\cdot|x,\lambda)$ and $p(\cdot|y,\lambda)$ are probability distributions for all x,y and λ [35]. Provided that one uses nonsignaling instruments, the left-hand side of the above equation can be written in the distributed scenario simply as $\mathrm{tr}[(\tilde{A}_{a|x} \otimes B_{b|y})V\rho V^\dagger]$. As the nonsignaling condition is automatically satisfied for a given total channel, our question boils down to finding an isometry V and a state ρ such that the state $V\rho V^\dagger$ is steerable but local. As an example, consider the Kraus operators $K_0 = |0\rangle\langle 0| + |1\rangle\langle 1|$ and $K_1 = |0\rangle\langle 2| + |1\rangle\langle 3|$. Now the state $\rho := \lambda|\psi\rangle\langle\psi| + (1-\lambda)\frac{1}{4}\mathbb{1}_4$, where $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |3\rangle)$, maps to the isotropic state $V\rho V^\dagger = \lambda|\psi^+\rangle\langle\psi^+| + (1-\lambda)\frac{1}{4}\mathbb{1}_4$. Isotropic states are steerable but local for projective measurements with $\frac{1}{2} < \lambda \leqslant \frac{1}{K_G(3)}$, where $K_G(3)$ is a Grothendieck constant and $\frac{1}{K_G(3)} \geqslant 0.6829$ [8,36,37].

However, considering only projective measurements does not cover all possible instruments compatible with the total channel. In order to cover this more general scenario, we recall that all possible instrument assemblages $\{\mathcal{I}_{a|x}\}_{a,x}$ compatible with a channel Λ are given by the minimal Stinespring dilation:

$$\{\mathcal{I}_{a|x}(\cdot)\}_{a,x} = \{\mathrm{tr}_A[(\tilde{A}_{a|x} \otimes \mathbb{1})V(\cdot)V^\dagger]|\{\tilde{A}_{a|x}\}_a \text{ is a POVM}\}. \tag{14}$$

To provide the desired example, we use a known steerable qutrit-qutrit state which is local for POVMs [38] as our target state $V\rho V^\dagger$. The state reads

$$\begin{aligned}\tilde{\rho} := \tfrac{1}{9}[a|\varphi^-\rangle\langle\varphi^-| + (3-a)\tfrac{1}{2}\mathbb{1} \otimes |2\rangle\langle 2| \\ + 2a|2\rangle\langle 2| \otimes \tfrac{1}{2}\mathbb{1} + (6-2a)|22\rangle\langle 22|],\end{aligned} \tag{15}$$

where $|\varphi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, $\mathbb{1} = |0\rangle\langle 0| + |1\rangle\langle 1|$, and $0 < a \leqslant \frac{3}{2}$. To reach this state we can use a channel acting on $\mathbb{C}^7$ defined through the Kraus operators

$$K_0 = |1\rangle\langle 0| + |2\rangle\langle 2|, \tag{16}$$

$$K_1 = -|0\rangle\langle 1| + |2\rangle\langle 3|, \tag{17}$$

$$K_2 = |0\rangle\langle 4| + |1\rangle\langle 5| + |2\rangle\langle 6|. \tag{18}$$

Now the state

$$\begin{aligned}\rho := \tfrac{1}{9}[a|\psi\rangle\langle\psi| + (3-a)\tfrac{1}{2}(|2\rangle\langle 2| + |3\rangle\langle 3|) \\ + a(|4\rangle\langle 4| + |5\rangle\langle 5|) + (6-2a)|6\rangle\langle 6|],\end{aligned} \tag{19}$$

where $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, maps to the state $\tilde{\rho}$ on the minimal dilation space, hence completing the example.

As the (nontight) inclusion of temporally nonsteerable correlations to the set of macrorealistic correlations follows from their definitions, we can formulate:

Observation 5. The set of temporally nonsteerable correlations is a proper subset of macrorealistic correlations.

The above Observation shows that there exists instances of temporal steering where a certain steerable channel-state pair can never lead to nonmacrorealistic behavior, no matter what (nonsignaling) measurements (compatible with the channel) are performed on the first party.

X. CONCLUSIONS

In this work, we have approached spatial, temporal, and channel steering through a modified version of the well-known Stinespring dilation. We have demonstrated the power of our approach by showing that incompatibility of quantum measurements is one-to-one connected to quantum steering in all three scenarios. In addition, we have shown how measurement uncertainty relations can be used as universal steering inequalities through this connection.

In contrast to the formerly known connections between spatial steering and joint measurability [11–14], the current approach is not limited to incompatibility. Using the Stinespring approach, we have mapped temporal steering into a framework where nonsignaling is a built-in state-independent feature. Moreover, we have shown an equivalence between temporal and spatial steering and shown that temporally unsteerable correlations are a proper subset of nonmacrorealistic correlations. For future works it would be interesting to investigate other possible connections between temporal and spatial correlations, e.g., investigate if our approach can be used to translate such concepts as entanglement in a meaningful way to the temporal scenario, and to see if our approach can be related to the recent works [39,40] comparing spatial and temporal scenarios.

Note added. Recently, we became aware of the work of Ref. [41], which independently proved a hierarchy between temporal steering and macrorealism.

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[1] The fact that steering is the essence of the EPR argument, was, to our knowledge, first noted by E. Schrödinger in a letter to A. Einstein (13/07/1935): "... All the others told me that there is no incredible magic in the sense that the system in America gives q = 6 if I perform in the European system nothing or a certain action (you see, we put emphasis on spatial separation), while it gives q = 5 if I perform another action; but I only repeated myself: It does not have to be so bad in order to be silly. I can, by maltreating the European system, steer the American system deliberately into a state where either q is sharp, or into a state which is certainly not of this class, for example where p is sharp. This is also magic!" see also, K. v. Meyenn, *Eine Entdeckung von ganz außerordentlicher Tragweite* (Springer, Berlin, 2011), p. 551.

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