On the Moments of Squared Binomial Coefficients

N. D. Gogin* Mika Hirvensalo†

October 15, 2020

Abstract

Explicit recurrent formulas for ordinary and alternated power moments of the squared binomial coefficients are derived in this article. Every such moment proves to be a linear combination of the previous ones via a coefficient list of the relevant Krawtchouk polynomial.

Keywords: Combinatorial identities, Binomial coefficients, Krawtchouk polynomials, MacWilliams duality formula, Umbral variables

1 Introduction

In this article, we study the sums of form

$$\mu_r^{(n)} = \sum_{m=0}^{n} m^r \binom{n}{m}^2 \text{ and } \nu_r^{(n)} = \sum_{m=0}^{n} (-1)^m m^r \binom{n}{m}^2, n \geq 0 \tag{1}$$

and refer them as the r-th order (where $r \geq 0$ and $0^0 = 1$ by convention) ordinary and alternating moments of the squared binomial coefficients.

Such sums emerge very often in different theoretical and applied mathematical areas, for example, see A000984, A002457, A037966, A126869, A100071, and A294486 in Sloane's database OEIS of integer sequences.

Unfortunately, up to the date (February 2020) not many explicit and closed formulas for these sums are known and moreover all such formulas are limited in order by $r \leq 4$, see [2], [3], [6], and A074334.

In the present article we prove two main theorems and their corollaries (in the Sections 3 and 4), providing the explicit recurrent formulas for obtaining the closed forms for the aforesaid moments of any order $r \geq 0$. We also give some examples of applications of these results.

*ngiri@list.ru

†Department of Mathematics and Statistics, University of Turku, Finland. mikhirve@utu.fi

2 Preliminaries

2.1 Hadamard transform, Krawtchouk polynomials and McWilliams duality formula

In this section, we remind to the reader some definitions and results from algebraic coding theory. The proofs of the claims can be found for example in [4].

For any integer $n \geq 1$, let $\mathbb{F}_2^n$ be an n-dimensional vector space over the binary field $\mathbb{F}_2 = \{0, 1\}$ and let V_n be the 2^n-dimensional Euclidean vector space of all functions f: $\mathbb{F}_2^n \to \mathbb{R}$ equipped with the usual scalar product $(f, g) = \sum_{\boldsymbol{v} \in \mathbb{F}_2^n} f(\boldsymbol{v})g(\boldsymbol{v})$

Every additive character of the space $\mathbb{F}_2^n$ can be written in a form $\chi_{\boldsymbol{u}}(\boldsymbol{v}) = \chi_{\boldsymbol{v}}(\boldsymbol{u}) = (-1)^{\boldsymbol{v}\cdot\boldsymbol{u}}$, where the dot product is defined as $\boldsymbol{u} \cdot \boldsymbol{v} = \sum_{i=1}^{n} u_i v_i = |\boldsymbol{u} \cap \boldsymbol{v}|$, where $\boldsymbol{u}$ and $\boldsymbol{v}$ are interpreted (in the obvious way) as subsets of indices $\{1, 2, ..., n\}$. Note that for the vector $\mathbf{1} = (1, 1, \ldots, 1) \in V_n$ the scalar product $(\boldsymbol{v}, \mathbf{1}) = |\boldsymbol{v} \cap \{1, 2, ..., n\}|$ is $|\boldsymbol{v}|$, the Hamming weight of $\boldsymbol{v}$.

It is a well-known fact that the full set of characters forms an orthogonal basis in the space V_n for which

$$(\chi_{\boldsymbol{u}}, \chi_{\boldsymbol{v}}) = 2^n \delta_{\boldsymbol{u},\boldsymbol{v}} = 2^n \begin{cases} 1 & \text{if } \boldsymbol{u} = \boldsymbol{v} \\ 0 & \text{if } \boldsymbol{u} \neq \boldsymbol{v} \end{cases} \tag{2}$$

Definition 1. *For every function $f \in V_n$ we define the Hadamard transform as*

$$\widehat{f}(\boldsymbol{u}) = (f, \chi_{\boldsymbol{u}}) = \sum_{\boldsymbol{v} \in V_n} (-1)^{|\boldsymbol{v} \cap \boldsymbol{u}|} f(\boldsymbol{v}). \tag{3}$$

Note that $(\widehat{f}, \widehat{g}) = 2^n (f, g)$ and $\widehat{\widehat{f}} = 2^n f$ (unitary and involutory properties of Hadamard transform).

Define now for every $0 \leq r \leq n$ the r-th weight-function $\psi_r(\boldsymbol{v}) = \begin{cases} 1 & \text{if } |\boldsymbol{v}| = r \\ 0 & \text{if } |\boldsymbol{v}| \neq r \end{cases}$ and let $\mathfrak{H}_r^{(n)}$ be the r-th Hamming sphere in $\mathbb{F}_2^n$: $\mathfrak{H}_r^{(n)} = \{\boldsymbol{v} : \psi_r(\boldsymbol{v}) = 1\}$.

Definition 2. *For any $f \in V_n$ the $(n + 1)$-tuple of real numbers*

$\mathfrak{S}(f) = (S_0(f), S_1(f), \ldots S_n(f))$, *where*

$$S_r(f) = (f, \psi_r) = \sum_{\boldsymbol{v} \in \mathfrak{H}_r^{(n)}} f(\boldsymbol{v}) \tag{4}$$

is said to be the weight spectrum of f while $\mathfrak{S}(\widehat{f})$ is referred as a dual weight spectrum of f .

Due to the unitary and involutory properties of the Hadamard transform one has:

$$S_r(\widehat{f}) = (\widehat{f}, \psi_r) = 2^{-n}(\widehat{\widehat{f}}, \widehat{\widehat{\psi}}_r) = (f, \widehat{\psi}_r). \tag{5}$$

Functions $\widehat{\psi}_r(\boldsymbol{v})$ depend obviously only on $|\boldsymbol{v}|$ and this gives rise to the following definition:

Definition 3. *Function*

$$\widehat{\psi}_r(\boldsymbol{v}) = \widehat{\psi}_r(|\boldsymbol{v}|) = K_r^{(n)}(x) = \sum_{i=0}^{r} (-1)^i \binom{n-x}{r-i} \binom{x}{i} \tag{6}$$

where $x = |\boldsymbol{v}|$ is being called as the r-th Krawtchouk polynomial of order n; $0 \le r = \deg K_r^{(n)} \le n$.

Now we can easily rewrite (5) to obtain the famous MacWilliams formula for dual spectrae:

$$S_r(\widehat{f}) = \sum_{i=0}^{n} K_r^{(n)}(i) S_i(f). \tag{7}$$

Here below we list some Krawtchouk polynomials properties which we use in our further considerations:

1. The generating function:

$$(1+t)^{n-x}(1-t)^x = \sum_{r=0}^{\infty} K_r^{(n)}(x) t^r, x \in \mathbb{R} \tag{8}$$

2. Special coefficients:

$$K_r^{(n)}(0) = \binom{n}{r}; \text{ the leading coefficient of } K_r^{(n)} = \frac{(-2)^r}{r!} \tag{9}$$

3. Orthogonality:

$$\sum_{i=0}^{n} \binom{n}{i} K_r^{(n)}(i) K_s^{(n)}(i) = 2^n \binom{n}{r} \delta_{r,s} \tag{10}$$

4. The reciprocity formula:

$$\binom{n}{r} K_s^{(n)}(r) = \binom{n}{s} K_r^{(n)}(s) \tag{11}$$

5. Symmetry:

$$K_r^{(n)}(x) = (-1)^r K_r^{(n)}(n-x) \tag{12}$$

$$K_r^{(n)}(x) = (-1)^x K_{n-r}^{(n)}(x) \text{ for } x \in \{0, 1, ..., n\} \tag{13}$$

6. Special case of the central Krawtchouk polynomials $K_n^{(2n)}$ (see [1]):

$$K_n^{(2n)}(x) = \frac{(-2)^n}{n!} \prod_{q=0}^{n-1} (x - (2q+1)) = (-4)^n \binom{\frac{x-1}{2}}{n} \tag{14}$$

$$K_n^{(2n)}(s) = \begin{cases} 0 & \text{if } s = 2k+1 \\ (-1)^k \binom{2n}{n} \frac{\binom{n}{k}}{\binom{2n}{2k}} & \text{if } s = 2k \end{cases} \tag{15}$$

7. List of the first seven Krawtchouk polymomials:

$$\begin{aligned}
K_0^{(n)}(x) &= 1, \quad K_1^{(n)}(x) = n - 2x, \quad K_2^{(n)}(x) = \binom{n}{2} - 2nx + 2x^2, \\
K_3^{(n)}(x) &= \binom{n}{3} + \frac{1}{6}\left(-6n^2 + 6n - 4\right) x + 2nx^2 - \frac{4x^3}{3}, \\
K_4^{(n)}(x) &= \binom{n}{4} + \frac{1}{24}\left(-8n^3 + 24n^2 - 32n\right) x + \frac{1}{24} \\
&+ \left(24n^2 - 24n + 32\right) x^2 - \frac{4nx^3}{3} + \frac{2x^4}{3}, \\
K_5^{(n)}(x) &= \binom{n}{5} + \frac{1}{120}\left(-10n^4 + 60n^3 - 150n^2 + 100n - 48\right) x \\
&+ \frac{1}{120}\left(40n^3 - 120n^2 + 240n\right) x^2 \\
&+ \frac{1}{120}\left(-80n^2 + 80n - 160\right) x^3 + \frac{2nx^4}{3} - \frac{4x^5}{15}
\end{aligned}$$

$$
\begin{aligned}
K_6^{(n)}(x) &= \binom{n}{6} + \frac{1}{720}\left(-12n^5+120n^4-500n^3+840n^2-736n\right)x \\
&+ \frac{1}{720}\left(60n^4-360n^3+1140n^2-840n+736\right)x^2 \\
&+ \frac{1}{720}\left(-160n^3+480n^2-1280n\right)x^3 \\
&+ \frac{1}{720}\left(240n^2-240n+640\right)x^4 - \frac{4nx^5}{15} + \frac{4x^6}{45}
\end{aligned}
$$

2.2 Auxiliary lemmata

We will use these results in the sequel, but they may have independent combinatorial interest, as well.

Lemma 1. *For any nonnegative integer d, $d \geq 1$ let $g(\boldsymbol{v}) = \binom{d}{|\boldsymbol{v}|}$, where $\boldsymbol{v} \in V_n$. Then*

$$\widehat{g}(\boldsymbol{u}) = K_d^{(n+d)}(|\boldsymbol{u}|), \quad \boldsymbol{u} \in V_n \tag{16}$$

Proof. From formula (3) one gets

$$\widehat{g}(\boldsymbol{u}) = \sum_{\boldsymbol{v}\in V_n} (-1)^{(\boldsymbol{v},\boldsymbol{u})}\binom{d}{|\boldsymbol{v}|} = 2\times \sum_{|\boldsymbol{u}\cap\boldsymbol{v}|\equiv 0 \pmod 2} \binom{d}{|\boldsymbol{v}|} - \sum_{\boldsymbol{v}\in V_n}\binom{d}{|\boldsymbol{v}|} \tag{17}$$

and after some cumbersome but straightforward combinatorial calculations the second sum in (17) proves to be equal to $\binom{n+d}{d}$ while the first one equals $\frac{1}{2}\left(\binom{n+d}{d} + \sum_{r=0}^{n}\binom{d}{d-r}K_r^{(n)}(|\boldsymbol{u}|)\right)$. Substituting these expressions in (17) we get $\widehat{g}(\boldsymbol{u}) = \sum_{r=0}^{n}\binom{d}{d-r}K_r^{(n)}(|\boldsymbol{u}|) = K_d^{(n+d)}(|\boldsymbol{u}|)$. □

Lemma 2. *For every integer $k, 0 \leq k \leq d$ the following identity is valid:*

$$\sum_{i=0}^{n} K_i^{(n)}(k)K_d^{(n+d)}(i) = 2^n\binom{d}{k} \tag{18}$$

Proof. Applying the McWilliams duality formula (7) and the reciprocity formula (11) we get:

$$
\begin{aligned}
S_k(g) &= \binom{d}{k}\cdot\binom{n}{k} = 2^{-n}S_k(\widehat{g}) = 2^{-n}\sum_{i=0}^{n}K_k^{(n)}(i)S_i(\widehat{g}) \\
&= 2^{-n}\sum_{i=0}^{n}K_k^{(n)}(i)\binom{n}{i}K_d^{(n+d)}(i) = 2^{-n}\binom{n}{k}\sum_{i=0}^{n}K_i^{(n)}(k){K_d}^{(n+d)}(i),
\end{aligned}
$$

i.e.

$$\binom{d}{k}\binom{n}{k} = 2^{-n}\binom{n}{k}\sum_{i=0}^{n} K_i^{(n)}(k)K_d^{(n+d)}(i)$$

which evidently implies equality (18). □

Corollary 1.

$$(-2)^n\sum_{i=0}^{n}\binom{\frac{i-1}{2}}{n}K_i^{(n)}(k) = \binom{n}{k} \tag{19}$$

Proof. With $d = n$ in formula (18) we get $\sum_{i=0}^{n} K_i^{(n)}(k)K_n^{(2n)}(i) = 2^n\binom{n}{k}$. But according to formula (14), one has $K_n^{(2n)}(i) = (-4)^n\binom{\frac{i-1}{2}}{n}$. □

Lemma 3.

$$\sum_{m=0}^{n}\binom{n}{m}^2 K_j^{(n)}(m) = \binom{n}{j}K_n^{(2n)}(j). \tag{20}$$

for every nonnegative integer j.

Proof. Let us find a scalar product of functions $\binom{n}{|\boldsymbol{v}|}$ and $K_j^{(n)}(|\boldsymbol{v}|)$, for some $j, 0 \le j \le n$, applying formula (19) :

$$\begin{aligned}
\sum_{\boldsymbol{v}\in V_n}\binom{n}{|\boldsymbol{v}|}K_j^{(n)}(|\boldsymbol{v}|) &= \sum_{m=0}^{n}\binom{n}{m}^2 K_j^{(n)}(m) \\
&= (-2)^n\sum_{i=0}^{n}\binom{\frac{i-1}{2}}{n}\left(K_i^{(n)}, K_j^{(n)}\right) \\
&= (-2)^n\sum_{i=0}^{n}\binom{\frac{i-1}{2}}{n}2^n\binom{n}{j}\delta_{i,j}
\end{aligned}$$

hence

$$\sum_{m=0}^{n}\binom{n}{m}^2 K_j^{(n)}(m) = (-4)^n\binom{n}{j}\binom{\frac{j-1}{2}}{n} = \binom{n}{j}K_n^{(2n)}(j).$$

□

Remark 1. *For $j = 0$ we get from (20) the classical identity*

$$\mu_0^{(n)} = \sum_{m=0}^{n}\binom{n}{m}^2 = \binom{2n}{n}. \tag{21}$$

3 Recurrent formula for the moments $\mu_j^{(n)}$ and its applications

For a fixed nonnegative integer j let $\left(\kappa_r^{(n,j)}\right)_{0\le r\le j}$ be a coefficient list of the polynomial $K_j^{(n)}$ and let $\boldsymbol{\mu} = (\mu^s)_{s\ge 0}$ be an *umbral* variable with a rule $\mu^s \rightleftarrows \mu_s^{(n)}, s \ge 0$.

Remark 2. *Descriptions of umbral calculus can be found in many sources. We recommend the reader to consult [8], for instance. A more developed and formalized treatises are available at [5] and [7].*

In this article, we need only the elementary notion of umbral variable as a linear functional $\to$ *on* $\mathbb{C}[[\mu]]$ *(the formal power series over* μ*) defined as* $\to(\mu^s) = \mu_s^{(n)}$, $s \ge 0$.

Theorem 1.

$$K_j^{(n)}(\boldsymbol{\mu}) = \binom{n}{j} K_n^{(2n)}(j). \tag{22}$$

Proof. Applying formula (20) the proof is obvious from the following simple computation:

$$\binom{n}{j} K_n^{(2n)}(j) = \sum_{m=0}^{n} \binom{n}{m}^2 K_j^{(n)}(m) = \sum_{m=0}^{n} \binom{n}{m}^2 \sum_{r=0}^{j} \kappa_r^{(n,j)} m^r$$
$$= \sum_{r=0}^{j} \kappa_r^{(n,j)} \sum_{m=0}^{n} \binom{n}{m}^2 m^r = \sum_{r=0}^{j} \kappa_r^{(n,j)} \mu_r^{(n)} = K_j^{(n)}(\boldsymbol{\mu})$$

□

Corollary 2. *(Recurrent formula for* $\mu_j^{(n)}$*)*

$$\frac{(-2)^j}{j!}\mu_j^{(n)} = \binom{n}{j} K_n^{(2n)}(j) - \sum_{r=0}^{j-1} \kappa_r^{(n,j)} \mu_r^{(n)}, \quad j \ge 1, \mu_0^{(n)} = \binom{2n}{n}. \tag{23}$$

Proof. Since the leading coefficient of the polynomial $K_j^{(n)}$ is equal to $\frac{(-2)^j}{j!}$ formula (23) follows immediately from formula (22) . □

The following examples below illustrate applications of formula (23) for moments $\mu_j^{(n)}$:

Each row of the table below shows the initial segment, $0 \leq n \leq 6$, of the sequence of moments $\mu_j^{(n)}$ for $0 \leq j \leq 6$. The last column carries the OEIS-number (if any) of the corresponding sequence.

$\mu_0^{(n)}$:	1	2	6	20	70	252	924	...	(*A*000984)
$\mu_1^{(n)}$:	0	1	6	30	140	630	2772	...	(*A*002457)
$\mu_2^{(n)}$:	0	1	8	54	320	1750	9072	...	(*A*037966)
$\mu_3^{(n)}$:	0	1	12	108	800	5250	31752	...	(*A*037972)
$\mu_4^{(n)}$:	0	1	20	234	2144	16750	117432	...	(*A*074334)
$\mu_5^{(n)}$:	0	1	36	540	6080	56250	455112	...	
$\mu_6^{(n)}$:	0	1	68	1314	18080	197350	1836792	...	

To justify the above table, we present the following: First, for $j = 0$ we have $\mu_0^{(n)} = \binom{2n}{n}$

For $j = 1$ get

$$\frac{(-2)^1}{1!}\mu_1^{(n)} = n \cdot 0 - n \cdot \binom{2n}{n}, \text{which implies that } \mu_1^{(n)} = \frac{n}{2}\binom{2n}{n}, \quad n \geq 0,$$

For $j = 2$:

$$\begin{aligned} \frac{(-2)^2}{2!}\mu_2^{(n)} &= -\binom{n}{2}\binom{2n}{n}\frac{4n}{2n(2n-1)} - \left(\binom{n}{2}\binom{2n}{n} - 2n \cdot \frac{n}{2}\binom{2n}{n}\right) \\ &\Rightarrow \quad \mu_2^{(n)} = \frac{n^3}{2(2n-1)}\binom{2n}{n}. \end{aligned}$$

Since $\mu_0^{(n)}$, $\mu_1^{(n)}$ are already known, $K_n^{(2n)}(2) = -\binom{2n}{n}\frac{1}{2n-1}$ and the coefficient list of the polynomial $K_2^{(n)}$ is $\{\binom{n}{2}, -2n, 2\}$

Following the same directions, for $j = 3$, $j = 4$, $j = 5$ and $j = 6$ we get

$$\begin{aligned} \mu_3^{(n)} &= \frac{n^3(n+1)}{4(2n-1)}\binom{2n}{n}, \\ \mu_4^{(n)} &= \frac{n^3(n^3+n^2-3n-1)}{4(2n-1)(2n-3)}\binom{2n}{n}, \\ \mu_5^{(n)} &= \frac{n^4(n^3+3n^2-3n-5)}{8(2n-1)(2n-3)}\binom{2n}{n}, \\ \mu_6^{(n)} &= \frac{n^3\left(n^6+3n^5-13n^4-15n^3+30n^2+8n-2\right)}{8(2n-1)(2n-3)(2n-5)}\binom{2n}{n}. \end{aligned}$$

(Here we omit rather cumbersome though elementary computations.)

4 Recurrent formula for the alternating moments $\nu_j^{(n)}$ and their applications

The case of the alternated moments is in general similar to the previous one but is more subtle:

Let $\boldsymbol{\nu} = (\nu^s)_{s\geq 0}$ be an umbral variable with a rule $\nu^s \rightleftarrows \nu_s^{(n)}, s \geq 0$.

Theorem 2.

$$K_j^{(n)}(\boldsymbol{\nu}) = \binom{n}{j} K_n^{(2n)}(n-j). \tag{24}$$

Proof. First of all we need to prove the following analogue of formula (20):

$$\sum_{m=0}^{n} (-1)^m \binom{n}{m}^2 K_j^{(n)}(m) = \binom{n}{j} K_n^{(2n)}(n-j). \tag{25}$$

The proof of (25) is clear from the following simple computations:

$$\begin{aligned}\binom{n}{j} K_n^{(2n)}(n-j) &= \binom{n}{n-j} K_n^{(2n)}(n-j) = \sum_{m=0}^{n} \binom{n}{m}^2 K_{n-j}^{(n)}(m) \\ &= \sum_{m=0}^{n} (-1)^m \binom{n}{m}^2 K_j^{(n)}(m)\end{aligned}$$

Now we proceed as in Theorem (1):

$$\begin{aligned}&\binom{n}{j} K_n^{(2n)}(n-j) = \sum_{m=0}^{n} (-1)^m \binom{n}{m}^2 K_j^{(n)}(m) \\ = &\sum_{m=0}^{n} (-1)^m \binom{n}{m}^2 \sum_{r=0}^{j} \kappa_r^{(n,j)} m^r \\ = &\sum_{r=0}^{j} \kappa_r^{(n,j)} \sum_{m=0}^{n} (-1)^m \binom{n}{m}^2 m^r = \sum_{r=0}^{j} \kappa_r^{(n,j)} \nu_r^{(n)} = K_j^{(n)}(\boldsymbol{\nu})\end{aligned}$$

□

Corollary 3. *(Recurrent formula for $\nu_j^{(n)}$)*

$$\frac{(-2)^j}{j!}\nu_j^{(n)} = \binom{n}{j} K_n^{(2n)}(n-j) - \sum_{r=0}^{j-1} \kappa_r^{(n,j)} \nu_r^{(n)}, \quad j \geq 1, \tag{26}$$

$$\nu_0^{(n)} = \begin{cases} (-1)^{\frac{n}{2}} \binom{n}{n/2} & \text{if } n \equiv 0 \ (\text{mod } 2) \\ 0 & \text{if } n \equiv 1 \ (\text{mod } 2) \end{cases}. \tag{27}$$

Proof. For $j = 0$ we evidently have from formula (15):

$$\nu_0^{(n)} = \sum_{m=0}^{n} (-1)^m \binom{n}{m}^2 = K_n^{(2n)}(n) = \begin{cases} (-1)^{\frac{n}{2}} \binom{n}{n/2} & \text{if } n \equiv 0 \ (\text{mod } 2) \\ 0 & \text{if } n \equiv 1 \ (\text{mod } 2) \end{cases} \tag{28}$$

The rest of the proof is the same as for formula (23). □

The examples below illustrate applications of formula (26) for moments $\nu_j^{(n)}$:

Each row of the table below shows the initial segment, $0 \leq n \leq 6$, of the sequence of moments $\nu_j^{(n)}$ for $0 \leq j \leq 6$. The OEIS does not contain sequences which coincide exactly with any row of this table, so the last column shows the OEIS-number of the sequence which seems to be the "closest" one related to the corresponding row.

$\nu_0^{(n)}$:	1	0	−2	0	6	0	−20	...	(cf. A126869)
$\nu_1^{(n)}$:	0	−1	−2	6	12	−30	−60	...	(cf. A100071)
$\nu_2^{(n)}$:	0	−1	0	18	0	−150	0	...	(cf. A294486)
$\nu_3^{(n)}$:	0	−1	4	36	−96	−450	1080	...	
$\nu_4^{(n)}$:	0	−1	12	54	−480	−750	7560	...	
$\nu_5^{(n)}$:	0	−1	28	36	−1728	1350	35640	...	
$\nu_6^{(n)}$:	0	−1	60	−162	−5280	20250	128520	...	

Now applying formula (26) we find explicit formulas for $\nu_j^{(n)}$ for initial values of j.

For $j = 0$ we already know that

$$\nu_0^{(n)} = \begin{cases} (-1)^{\frac{n}{2}} \binom{n}{n/2} & \text{if } n \equiv 0 \ (\text{mod } 2) \\ 0 & \text{if } n \equiv 1 \ (\text{mod } 2) \end{cases}.$$

For $j=1$ we find that for $n \geq 1$

$$\begin{aligned}\nu_1^{(n)} &= -\frac{1}{2}\Big(nK_n^{(2n)}(n-1) - nK_n^{(2n)}(n)\Big)\\ &= \frac{n}{2}\begin{cases} (-1)^{\frac{n}{2}}\binom{n}{n/2} & \text{if } n \equiv 0 \pmod 2\\ -(-1)^{\frac{n-1}{2}}\binom{2n}{n}\frac{\binom{n}{(n-1)/2}}{\binom{2n}{n-1}} & \text{if } n \equiv 1 \pmod 2\end{cases},\end{aligned}$$

from where we get finally

$$\nu_1^{(n)} = \frac{n}{2}\begin{cases} (-1)^{\frac{n}{2}}\binom{n}{n/2} & \text{if } n \equiv 0 \pmod 2\\ (-1)^{\frac{n+1}{2}}\frac{n+1}{n}\binom{n}{(n+1)/2} & \text{if } n \equiv 1 \pmod 2\end{cases}. \tag{29}$$

For $j=2$:

$$\begin{aligned}\nu_2^{(n)} &= \frac{1}{2}\Bigg(\binom{n}{2}K_n^{(2n)}(n-2) - \Bigg(\binom{n}{2}K_n^{(2n)}(n) - 2n\\ &\qquad \Big(-\frac{n}{2}(K_n^{(2n)}(n-1) - K_n^{(2n)}(n))\Big)\Bigg)\Bigg)\end{aligned}$$

from where after simplifications (omitted here) we get :

$$\nu_2^{(n)} = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 2\\ (-1)^{\frac{n+1}{2}}\binom{n+1}{2}\binom{n}{(n+1)/2} & \text{if } n \equiv 1 \pmod 2\end{cases}. \tag{30}$$

For $j=3$, $j=4$, $j=5$, and $j=6$ one has:

$$\nu_3^{(n)} = \begin{cases} (-1)^{\frac{n+2}{2}}\frac{n^3}{4}\binom{n}{n/2} & \text{if } n \equiv 0 \pmod 2\\ \frac{(-1)^{\frac{n+1}{2}}n(n+1)^2}{4}\binom{n}{(n+1)/2} & \text{if } n \equiv 1 \pmod 2\end{cases}. \tag{31}$$

$$\nu_4^{(n)} = \begin{cases} (-1)^{\frac{n+2}{2}}\frac{n^3(n+1)}{4}\binom{n}{n/2} & \text{if } n \equiv 0 \pmod 2\\ \frac{(-1)^{\frac{n+1}{2}}n^2(n+1)}{2}\binom{n}{(n+1)/2} & \text{if } n \equiv 1 \pmod 2\end{cases}. \tag{32}$$

$$\nu_5^{(n)} = \begin{cases} (-1)^{\frac{n+2}{2}}\frac{n^4(n+5)}{8}\binom{n}{n/2} & \text{if } n \equiv 0 \pmod 2\\ \frac{(-1)^{\frac{n-1}{2}}n(n+1)^2(n^2-4n+1)}{8}\binom{n}{(n+1)/2} & \text{if } n \equiv 1 \pmod 2\end{cases}. \tag{33}$$

$$\nu_6^{(n)} = \begin{cases} (-1)^{\frac{n+2}{2}}\frac{n^3(n+1)(3n-1)}{8}\binom{n}{n/2} & \text{if } n \equiv 0 \pmod 2\\ \frac{(-1)^{\frac{n-1}{2}}n^2(n+1)(n^3+n^2-9n+3)}{8}\binom{n}{(n+1)/2} & \text{if } n \equiv 1 \pmod 2\end{cases}. \tag{34}$$

References

[1] Gogin N., Hirvensalo M.: *Recurrent Construction of MacWilliams and Chebyshev Matrices*. Fundamenta Informaticae 116:1–4, pp. 93–110 (2012). DOI: 10.3233/FI-2012-671

[2] Gould H. W.: *Combinatorial Identities: A Standardized Set of Tables Listing 500 Binomial Coefficient Summations*. Morgantown Printing and Binding Co (1972).

[3] Graham R.L., Knuth D. E., Patashnik O.: *Concrete Mathematics*. Addison-Wesley (1994).

[4] MacWilliams F.J., Sloane N. J. A.: *The Theory of Error-Correcting Codes*. North Holland (1977).

[5] Licciardi, S.: *Umbral Calculus, A Different Mathematical Language*. `https://arxiv.org/abs/1803.03108` (2018) (retrieved February 15, 2020).

[6] Riordan J.: *Combinatorial Identities*. John Wiley & Sons Inc. (1968).

[7] Nigel Ray: *Universal Constructions in Umbral Calculus*. In: Bruce E. Sagan, Richard P. Stanley (Eds.): Mathematical Essays in honor of Gian-Carlo Rota, pp. 343–357. Springer (1996).

[8] Roman, S.M., Rota G.-C.: *The Umbral Calculus*. Advances in Mathematics 27, pp. 95–188 (1978).