On resolving several objects in the king grid

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The concept of a resolving set was introduced independently by Slater [11] and Harary and Melter [7]. This concept emerges naturally from many diverse areas such as coin weighing problem [1], network discovery and verification [2], and robot navigation [9]. For other recent developments, see [6, 3, 5]. Resolving sets are also related to identifying codes and locating dominating sets which are widely studied — see the list in the web-site [10] for papers on these topics.

Consider a connected, finite, simple, and undirected graph G with vertices V and edges E. Let S be a subset of V. When we think of S as an ordered set $(s_1,\ s_2,\ \ldots,\ s_{|S|})$, we can try to locate an another vertex set X with the distance array

$$\mathcal{D}_S(X) = (d(s_1, X),\ d(s_2, X),\ \ldots,\ d(s_{|S|}, X)),$$

where $d(s_i, X) = \min_{x \in X} d(s_i, x)$ is the shortest distance from s_i to some vertex of X.

Definition 1. The set S is an *ℓ-resolving set* (on *ℓ-set resolving set*) of $G = (V, E)$, where $\ell \leq |V|$, if for every pair of subsets X and Y, with $|X| \leq \ell$ and $|Y| \leq \ell$, we have

$$\mathcal{D}_S(X) \neq \mathcal{D}_S(Y).$$

In other words, an ℓ-resolving set can locate up to ℓ vertices at the same time. Let a surveillance network be modelled by a graph. When we place sensors to the vertices corresponding to the elements of an ℓ-resolving set S, the sensors can locate up to ℓ intruders by sending signals to measure the distance.

Every graph has an ℓ-resolving set for any $\ell \leq |V|$, since V is always such a set. We can simply check which elements of $\mathcal{D}_V(X)$ are 0 and we have located all elements of X. Therefore the existence of resolving sets is not of interest but the size of them is. We denote with $\beta_\ell(G)$ the *ℓ-set-metric dimension* of G, which is the smallest possible cardinality of an ℓ-resolving set of G. An *ℓ-set-metric basis* is an ℓ-resolving set that is of cardinality $\beta_\ell(G)$.

There has been a lot of research on what values $\beta_\ell(G)$ gets with different graphs for $\ell = 1$. For example Khuller, Raghavachari & Rosenfeld proved in 1996 [9] that $\beta_1(G) = 1$ if and only if G is a path and that for a d-dimensional grid graph $\beta_1(G) = d$. Chartrand *et al.* proved in 2000 [6] that for an n-vertex graph $\beta_1(G) = n-1$ if and only if G is a complete graph. They also gave characterisations for all n-vertex graphs with $\beta_1(G) = n-2$. Resolving several objects has been studied recently in [8]. There the two-dimensional grid graph and the binary hypercube are considered.

Let us denote the path of n vertices by P_n. It was shown in [8] that for the Cartesian product $P_m \Box P_n$ of two paths we have

$$\beta_2(P_m \Box P_n) = \min\{m, n\} + 2.$$

Earlier, it was proven by Khuller *et al.* [9] that

$$\beta_1(P_m \Box P_n) = 2.$$

If two vertices x and y are adjacent, we denote $x \sim y$. Let us denote the *strong product* of two graphs $G = (V, E)$ and $H = (V', E')$ by $G \boxtimes H$, that is, it has as the vertex set the Cartesian product $V \times V'$ and there is an edge between (u_1, u_2) and (v_1, v_2) if one of the following three conditions hold: 1) $u_1 = v_1$ and $u_2 \sim v_2$, 2) $u_1 \sim v_1$ and $u_2 = v_2$ or 3) $u_1 \sim v_1$ and $u_2 \sim v_2$. In this paper, we consider the *king grid* $P_m \boxtimes P_n$. This graph has been studied for related topics, see, for example [4].

The king grid is basically a two-dimensional grid graph with diagonal edges in addition to vertical and horizontal ones (see Figure 1). As such, it mimics the movement of the king on a chess board.

The vertices of a king grid can be thought as $\mathbb{N} \times \mathbb{N}$ lattice points. We can give each vertex two coordinates and write the set of vertices of an $m \times n$ king grid as $\{(i, j) \mid i = 1, \dots, m,\ j = 1, \dots, n\}$. Now the distance between the vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ is $d(u, v) = \max\{|u_1 - v_1|,\ |u_2 - v_2|\}$.

To ease notations, we define the ith *column* for $i \in [1, m]$ as $C_i = \{(i, j) | \ j = 1, \dots, n\}$. A *section* is the union of consecutive columns:

$$C_i^j = \bigcup_{k=i}^{j} C_k.$$

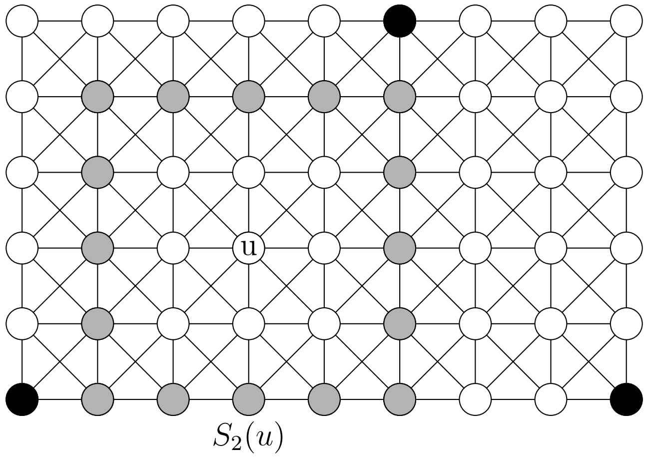


Figure 1: The 9×6 *king grid* $P_9 \boxtimes P_6$. *The black vertices form a* 1*-set-metric basis for the graph.*

We denote with $S_r(u) = \{v \in V \mid d(u,v) = r\}$ the set of vertices that are at the distance of r from the vertex u. Note that if $r \neq r'$, then $S_r(u) \cap S_{r'}(u) = \emptyset$.

For completeness, we first give the following result from [12].

Theorem 1. Let $P_m \boxtimes P_m$ be an $m \times m$ king grid with $2 \leq m$. Then

$$\beta_1(P_m \boxtimes P_m) = 3.$$

Sketch of proof. The greatest distance between any two vertices is $m-1$. Therefore each element of $\mathcal{D}_S(X)$ has m possible values. If $|S| = k$, then there are m^k possible distance arrays. Since $\ell = 1$ no distance array can have more than one zero. Since there are only $m^2 - 1$ acceptable distance arrays of length two but m^2 vertices, we have $\beta_1(P_m \boxtimes P_m) \geq 3$.

If S is a subset of V that contains any three of the graph's corner vertices, it is a 1-resolving set of $P_m \boxtimes P_m$. Therefore $\beta_1(P_m \boxtimes P_m) = 3$. □

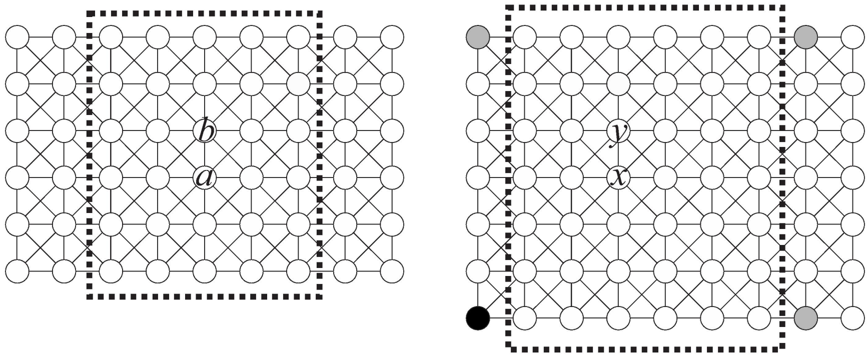
The next result considers the 1-set-metric dimension for any king grid. In [12], Rodríquez-Velázquez *et al.* gave a construction giving $\beta_1(P_n \boxtimes P_m) \leq \lceil \frac{n+m-2}{n-1} \rceil$. They also presented a conjecture that this upper bound is optimal for any integers n and m such that $2 \leq n < m$. This was recently proved by Barragán-Ramírez and Rodríquez-Velázquez in [13]. They used the diameter and bipartiteness of graphs. In this paper, we present a direct and simple proof.

Theorem 2. Let $P_m \boxtimes P_n$ be an $m \times n$ king grid with $2 \leq n < m$. Then

$$\beta_1(P_m \boxtimes P_n) = \left\lceil \frac{n+m-2}{n-1} \right\rceil.$$

Sketch of proof. Assume that S is a 1-resolving set.

Let first n be even. Each $(n-1)\times n$-section C_i^{i+n-2} contains at least one element of S. Indeed, otherwise we would have $\mathcal{D}(a)=\mathcal{D}(b)$ for $a=(i+\frac{n-2}{2},\frac{n}{2})$ and $b=(i+\frac{n-2}{2},\frac{n}{2}+1)$ — for illustration see Figure 2(i). Moreover, in the both ends of the king grid, the $n\times n$-sections have $|C_1^n\cap S|\geq 2$ and $|C_{m-(n-1)}^m\cap S|\geq 2$ (it is easy to see that one element of S is not enough). Let first $m\geq 2n$. Now let us partition the $m\times n$ king grid as follows (see Figure 3). Take first the two $n\times n$-sections at the both ends of the grid and then divide the middle section into as many disjoint $(n-1)\times n$-sections as possible (there are at most $n-2$ leftover columns outside the sections, in the figure there is one column marked by gray vertices). Now the observations above give $|S|\geq 2+2+\lfloor\frac{m-2n}{n-1}\rfloor$, which equals the conjectured lower bound. In the case $n<m<2n$, it is easy to show that $|S|\geq 3$.



(i) (ii)

Figure 2: (i) The section C_i^{i+n-2} illustrated for $n=6$ and the vertices a and b, (ii) The section C_i^{i+n-2} for $n=7$ and the vertex $(i-1,1)$ is the black vertex.

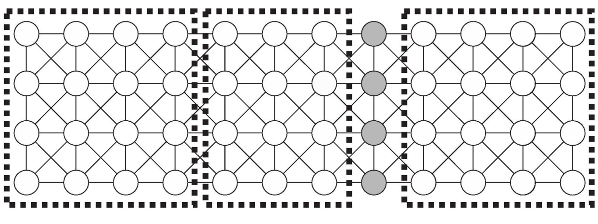


Figure 3: The partition for $n=4$ and $m=12$.

The case for n odd goes similarly. Just notice that now the $(n-1)\times n$-section C_i^{i+n-2} can be empty of the elements of S, but in that case the neighbouring columns C_{i-1} on the left and C_{i+n-1} on the right contain both at least two elements of S (see Figure 2(ii)). Indeed, suppose that the section C_i^{i+n-2} is empty. This implies that $(i-1,1)$ belongs to S, since otherwise $\mathcal{D}(x) = \mathcal{D}(y)$ for $x = (i + \frac{n-3}{2}, \frac{n+1}{2})$ and $y = (i + \frac{n-3}{2}, \frac{n+1}{2} + 1)$. In the same way, the vertex $(i-1,n)$ belongs to S. Similarly we can show that $(i+n-1,1)$ and $(i+m-1,n)$ belong to S. Therefore, it is easy to show that in average there is at least one element of S in each of the $(n-1)\times n$-section. For more details, see [14].

□

When $\ell = 2$, the vertices at the frame of the king grid can "hide" behind its neighbour closer to the center. Therefore all vertices at the frame of the grid must be in any 2-resolving set, and it turns out that this condition is sufficient.

Theorem 3. Let $P_m \boxtimes P_n$ be an $m \times n$ king grid with $2 \leq n \leq m$. Then

$$\beta_2(P_m \boxtimes P_n) = 2m + 2n - 4.$$

Sketch of proof. Let $u = (u_1,\ u_2)$ be a vertex at the frame of the graph i.e. $u_1 \in \{1, m\}$ or $u_2 \in \{1, n\}$.

Assume that $u_2 = 1$. Let $v = (v_1, v_2) \neq u$ and $u' = (u_1,\ 2)$. Now $d(u,v) = \max\{|u_1 - v_1|,\ |1 - v_2|\}$ and $d(u',v) = \max\{|u_1 - v_1|,\ |2 - v_2|\}$. If $v_2 = 1$, then $|u_1 - v_1| \geq 1$, since $v \neq u$. Now $d(u,v) = d(u',v)$. If $v_2 \geq 2$, then $|2 - v_2| < |1 - v_2|$ and therefore $d(u,v) \geq d(u',v)$.

Let S be a 2-resolving set of $P_m \boxtimes P_n$ and assume that $u \notin S$. Consider two vertex sets $A = \{u'\}$ and $B = \{u, u'\}$. Now $\mathcal{D}_S(A) = \mathcal{D}_S(B)$ since no vertex of S can be closer to u than u' as we saw above. Therefore $u \in S$.

The other cases are handled similarly, namely $u_1 = 1$, $u_1 = m$, and $u_2 = n$.

This shows that all vertices at the frame of the graph must be included in the resolving set. With some effort one can show that these vertices indeed form a 2-resolving set [14]. □

When $\ell \geq 3$, we cannot leave any vertex out of the ℓ-resolving set. If we do, we can always find two sets of vertices that have the same distance array.

Theorem 4. Let $P_m \boxtimes P_n$ be an $m \times n$ king grid with $2 \leq n \leq m$. Then

$$\beta_{\geq 3}(P_m \boxtimes P_n) = mn.$$

Proof. In Theorem 3 we saw that the vertices at the frame of the graph must be included in any 2-resolving set. Therefore they must also be in any 3-resolving set. If $n = 2$ or $m = 2$, all vertices are at the frame of the graph and the claim holds.

Let S be a 3-resolving set of $P_m \boxtimes P_n$ where $2 < n \leq m$. Assume that $u = (u_1, u_2) \notin S$ where $u_1 \in [2, m-1]$ and $u_2 \in [2, n-1]$. Let $v = (u_1 - 1, u_2)$ and $w = (u_1 + 1, u_2)$.

Assume that there is a vertex $s = (s_1, s_2) \in S$ such that $d(s, u) < d(s, v)$ and $d(s, u) < d(s, w)$.

- If $s_1 = u_1$, then $d(s, u) < d(s, v)$ implies that $|s_2 - u_2| \leq |s_1 - u_1| = 0$ and therefore $s_2 = u_2$. Now $s = u$ but this is a contradiction, since $u \notin S$.

- If $s_1 < u_1$, then $s_1 - u_1 < 0$ and therefore $|s_1 - u_1 + 1| < |s_1 - u_1|$. In fact $|s_1 - u_1 + 1| = |s_1 - u_1| - 1$. Since $d(s, u) < d(s, v)$,

  $$\max\{|s_1 - u_1|,\ |s_2 - u_2|\} < \max\{|s_1 - u_1 + 1|,\ |s_2 - u_2|\}.$$

  Now $|s_2 - u_2| < |s_1 - u_1|$, because otherwise $d(s, u) = |s_2 - u_2| = d(s, v)$. But now $|s_2 - u_2| \leq |s_1 - u_1| - 1 = |s_1 - u_1 + 1|$, and therefore $d(s, u) = |s_1 - u_1| > |s_1 - u_1 + 1| = d(s, v)$, which is a contradiction.

- If $s_1 > u_1$, we can just replace v and $|s_1 - u_1 + 1|$ with w and $|s_1 - u_1 - 1|$ in the previous case.

Therefore, every $s \in S$ is as close or closer to either v or w than u. But now $\mathcal{D}_S(A) = \mathcal{D}_S(B)$, where $A = \{v, w\}$ and $B = \{u, v, w\}$. Therefore S cannot be a 3-resolving set if it does not include all vertices of the graph. $\square$

References

[1] N. Alon, D.N. Kozlov and V.H. Vu. The geometry of coinweighing problem, in Proc. 37th Annual Symposium on Foundation of Computer Science, FOCS96, IEEE, 1998, pp. 524–532.

[2] Z. Beerliova, F. Eberhard, T. Erlebach, A. Hall, M. Hoffmann, M. Mihal'ák, and L. S. Ram. Network discovery and verification. Graph-theoretic concepts in computer science, Lecture Notes in Comput. Sci., 3787, pp. 127–138, Springer, Berlin, 2005.

[3] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, and M. L. Puertas. On the metric dimension of infinite graphs. Discrete Appl. Math., 160, pp. 2618–2626, 2012.

[4] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, and M. L. Puertas. Locating-dominating codes: Bounds and extremal cardinalities. Discrete Appl. Math., 220, pp. 38–45, 2013.

[5] J. Cáceres, C. Hernando, M. Mora, I.M. Pelayo, M. L. Puertas, C. Seara, and D.R. Wood. On the metric dimension of Cartesian products of graphs. SIAM J. Discrete Math., 21, pp. 423–441, 2007.

[6] G. Chartrand, L. Eroh, M. A. Johnson, and O. R. Oellermann. Resolvability in graphs and the metric dimension of a graph. Discrete Appl. Math., 105, 99–113, 2000.

[7] F. Harary and R. Melter. On the metric dimension of a graph. Ars Combinatoria, 2, pp. 191–195, 1976.

[8] T. Laihonen. The metric dimension for resolving several objects. Information Processing Letters, 116, pp. 694–700, 2016.

[9] S. Khuller, B. Raghavachari, and A. Rosenfeld. Landmarks in graphs. Discrete Appl. Math., 70, pp. 217–229, 1996.

[10] A. Lobstein. Watching systems, identifying, locating-dominating and discrminating codes in graphs, a bibliography. Published electronically at `http://perso.telecom-paristech.fr/~lobstein/debutBIBidetlocdom.pdf`.

[11] P. Slater. Leaves of trees. Proceedings of the Sixth Southeastern Conference on Combinatorics, Graph Theory, and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1975), pp. 549–559. Congressus Numerantium, No. XIV, Utilitas Math., Winnipeg, Man., 1975.

[12] J. A. Rodríguez-Velázquez, D. Kuziak, I. G. Yero, and J. M. Sigarreta. The metric dimension of strong product graphs. Carpathian J. Math., 31 (2015), No. 2, pp. 261–268.

[13] G. A. Barragán-Ramírez and J. A. Rodríquez-Velázquez. The Local Metric Dimension of Strong Product Graphs. Graphs and Combinatorics, 32 (2016), pp. 1263–1278.

[14] A. Hakanen. Resolving sets and resolving several objects in the finite king grid. Master's thesis, University of Turku, in preparation, 2017.