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On vertices contained in all or in no metric basis

Anni Hakanen^{a,*}, Ville Junnila^a, Tero Laihonen^a, Ismael G. Yero^b

^a Department of Mathematics and Statistics, University of Turku, Finland

^b Department of Mathematics, Universidad de Cádiz, Spain

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ABSTRACT

A set $R \subseteq V(G)$ is a *resolving set* of a graph G if for all distinct vertices $v, u \in V(G)$ there exists an element $r \in R$ such that $d(r, v) \neq d(r, u)$. The *metric dimension* $\dim(G)$ of the graph G is the cardinality of a smallest resolving set of G . A resolving set with cardinality $\dim(G)$ is called a *metric basis* of G . We consider vertices that are in all metric bases, and we call them *basis forced vertices*. We give several structural properties of sparse and dense graphs where basis forced vertices are present. In particular, we give bounds for the maximum number of edges in a graph containing basis forced vertices. Our bound is optimal whenever the number of basis forced vertices is even. Moreover, we provide a method of constructing fairly sparse graphs with basis forced vertices. We also study vertices which are in no metric basis in connection to cut-vertices and pendants. Furthermore, we show that deciding whether a vertex is in all metric bases is co-NP-hard, and deciding whether a vertex is in no metric basis is NP-hard.

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1. Introduction

Metric dimension in graphs is a classical invariant that has been studied from a lot of different points of view, starting from pure theoretical aspects and including some applied models mainly arising from computer science, among others. Specifically, in the last decade, there has been an increasing number of investigations on the metric dimension of graphs, and a large number of them are specifically centred on variations of the standard metric dimension concept. The classical metric dimension parameter, although it is nowadays very well studied, strong interest still remains. Indeed, a significant number of works specifically dealing with it appear frequently. For some latest remarkable articles we suggest for instance [6,8,14,17,18,21,23].

For a given connected graph G , the distance $d_G(u, v)$ (the subindex can be removed if it is clear from context) between two vertices $u, v \in V(G)$ is the length of a shortest path between u and v . A set of vertices $R \subseteq V(G)$ is called a *resolving set* for G if for every pair of distinct vertices $u, v \in V(G)$, there exists a vertex $w \in R$ such that $d_G(w, u) \neq d_G(w, v)$. In this sense, it is also said that the set R *resolves* the graph G , and that w *resolves* the vertices u, v (or that u, v are resolved by w). The cardinality of the smallest possible resolving set for G is the *metric dimension* of G , denoted by $\dim(G)$. A resolving set of cardinality $\dim(G)$ is a *metric basis* of G . Such concepts were (independently) introduced in [12,22].

It is natural to ask if a graph contains a unique metric basis, or to ask a broader question, whether a graph has vertices belonging to every metric basis. Such ideas were already investigated in [1,3]. For instance, it was shown in [3], that for all integers k, r with $k \geq 2$ and $0 \leq r \leq k$, there exists a graph G with metric dimension k having r vertices that belong to every metric basis of G . However, no more contributions in this direction, concerning the classical metric dimension invariant, have appeared so far. Such questions are widely studied in other contexts such as dominating sets and stable

* Corresponding author.

E-mail address: anehak@utu.fi (A. Hakanen).

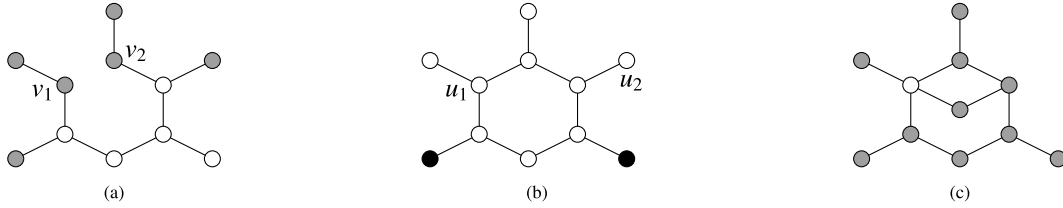


Fig. 1. Void vertices and basis forced vertices are illustrated as white and black vertices, respectively. Grey vertices are in some metric basis but not all.

sets in [5, 11, 19], for example. For some other variants of metric dimension, some related studies have continued. Examples of this are [9, 10], where forced vertices were defined as vertices that are in every ℓ -solid-resolving set or $\{\ell\}$ -resolving set – not only the corresponding metric bases. In this sense, it is now our goal to retrieve such ideas, for the classical metric dimension, and present an exposition of combinatorial results aimed to describe structural properties of graphs containing vertices that either belong to every, or to no metric basis of it. For more formality in our exposition, we shall now name such kind of vertices in a graph G .

Definition 1. A vertex $v \in V(G)$ is a *basis forced vertex* of the graph G if it is contained in every metric basis of G .

In connection with the existent models related to metric dimension in graphs, like that of navigation of robots in networks, the existence of basis forced vertices means that a metric basis cannot be formed without them. In consequence, they need to be included in every possible set of “landmarks” used to uniquely identify the vertices of the graph.

Clearly, all graphs do not have basis forced vertices. For example, the path P_n with $n \in \mathbb{N}(= \{1, 2, \dots\})$ vertices where $n \geq 2$ has two disjoint metric bases (the endpoints), and thus no basis forced vertices. The graph in Fig. 1(b) has two basis forced vertices, which are illustrated as black vertices. This graph has a unique metric basis that consists of only these two vertices. Consequently, the white vertices are in no metric basis. These are the opposite of basis forced vertices, and thus we have the following definition.

Definition 2. A vertex $v \in V(G)$ is a *void vertex* of the graph G if it is in no metric basis of G .

In [2], the set of vertices belonging to all maximum stable sets of a graph is called the *core* of G . In the same spirit, we could call the set of basis forced vertices a core for the metric bases of G and the set of void vertices an anticore for the metric bases of G .

The vertices of any graph can be divided into three categories: basis forced vertices, void vertices and vertices that do not fit in either category, that is, vertices that are in some metric bases but not all. The endpoints of a path are in some metric bases but not all, whereas the rest of the vertices of a path are void vertices. The vertices of a complete graph or a cycle are also all in some bases but not all. If a graph has a unique metric basis (like the graph in Fig. 1(b)), then each of its vertices is either a basis forced vertex or a void vertex. All three types of vertices can be present simultaneously in the same graph. This is the case for the graph later shown in Fig. 2(a).

In [9, 10], forced vertices of ℓ -solid-resolving sets for all ℓ , and $\{\ell\}$ -resolving sets for $\ell \geq 2$ were characterised. These characterisations used the local properties of the vertices and their neighbourhoods. In contrast, it seems that achieving a similar local characterisation for basis forced vertices is a very challenging problem (see Section 5 for issues concerning the algorithmic complexity). For instance, if the graph changes even slightly, the metric bases may look very different even in places that are far away from where the change occurred. This allows to realise that basis forced vertices are very sensitive to changes in the graph. Consider the graph in Fig. 1(a). The metric dimension of this graph is 2, and the vertices that appear in some metric basis are illustrated as grey vertices. When we add the edge v_1v_2 we obtain the graph illustrated in Fig. 1(b). The metric dimension of this graph is also 2. However, this graph has only one metric basis, the elements of which are illustrated as black vertices. Consequently, this graph has two basis forced vertices and ten void vertices. When we add the edge u_1u_2 to this graph, we obtain the graph in Fig. 1(c). The metric dimension of this graph is 3, and there is only one void vertex. The rest of the vertices appear in some metric bases but not all.

We next include some terminology and notations that we shall use throughout our exposition. Whenever two distinct vertices $v, u \in V(G)$ are adjacent, we denote $v \sim u$ (possibly with a subindex \sim_G when such distinction is necessary), and when v and u are not adjacent, we denote $v \not\sim u$. The vertex v is a *pendant* if there exists exactly one element $u \in V(G) \setminus \{v\}$ such that $v \sim u$. Let $S \subseteq V(G)$. The *induced subgraph* $G[S]$ is the graph with the vertex set S and the edge set consisting of the edges of G that are between two elements of S . Let $v \in V(G)$. The graph $G - v$ is obtained from G by removing the vertex v and all edges that have v as an endpoint. The vertex $v \in V(G)$ is a *cut-vertex* of a connected graph G if the graph $G - v$ is disconnected. A connected graph G is *unicyclic* if it contains only one cycle (i.e., the number of edges is equal to the number of vertices in G). From now on, all the graphs considered are connected.

The structure of this article is as follows. In Section 2, we consider cut-vertices and pendants. More precisely, in Section 2.1 we show that cut-vertices are not basis forced vertices, and that in fact most of them are void vertices. In

Section 2.2, we show that pendants are not basis forced vertices in most cases. However, we also give an example of a graph that has pendants as basis forced vertices. In Section 3, we consider sparse and dense graphs and their basis forced vertices. We investigate, what are the sparsest and densest graphs that have basis forced vertices. In Section 4, we give two bounds on the number of basis forced vertices of a graph by using a new colouring method of visualising which pairs of vertices are resolved by which elements of a metric basis. Finally, we consider the algorithmic complexity of deciding whether a vertex is a basis forced or void vertex in Section 5.

2. On cut-vertices and pendants

In this section, we consider cut-vertices and pendants, and establish whether they can be void vertices or basis forced vertices. In the first subsection, we will show that in most cases cut-vertices are void vertices. However, not all cut-vertices are void vertices. For example, the set $\{v_1, v_2\}$ is a metric basis of the graph in Fig. 1(a) and it consists of only cut-vertices. We will show that the cut-vertices that are not void vertices are not basis forced either. In the latter subsection, we consider pendants. We show that pendants are not basis forced vertices in most cases. However, there are some graphs that have pendants as basis forced vertices.

2.1. Cut-vertices

Let us first prove a lemma that we will use in this section and the section on pendants. By components of a graph we mean connected components. The following lemma essentially says that if there are elements of a resolving set in multiple components of $G - v$ for a cut-vertex v , then every pair of vertices that the cut-vertex v resolves is also resolved by other elements of the resolving set. However, the set R is not required to be a resolving set for the following lemma to hold.

Lemma 3. *Let G be a connected graph with a cut-vertex v . Let $R \subseteq V(G)$ be such that there are at least two components in $G - v$ containing elements of R . If $d(v, x) \neq d(v, y)$ for some $x, y \in V(G)$, then there exists an element $r \in R$ ($r \neq v$) such that $d(r, x) \neq d(r, y)$.*

Proof. Let us denote the components of $G - v$ by G_i , $i \in \mathbb{N}$, and let $R_i = R \cap V(G_i)$. Assume that $x, y \in V(G)$ are such that $d(v, x) \neq d(v, y)$. Suppose that for some i we have $R_i \neq \emptyset$ and $x, y \notin V(G_i)$. Now for all $r \in R_i$ we have

$$d(r, x) = d(r, v) + d(v, x) \neq d(r, v) + d(v, y) = d(r, y).$$

Thus, there exists an element $r \in R$ such that $d(r, x) \neq d(r, y)$.

Above we considered the case where for at least one i we have $R_i \neq \emptyset$ and $x, y \notin V(G_i)$. Now we consider the case where no such i exists, i.e., the case where every component that contains elements of R also contains x or y . As there are at least two components with elements of R , the vertices x and y must be in different components. Thus, for some distinct i and j we have $R_i, R_j \neq \emptyset$, $x \in V(G_i)$ and $y \in V(G_j)$. If for some $r \in R_i$ we have $d(r, x) \neq d(r, y)$, then we are done. Suppose that for all $r \in R_i$ we have $d(r, x) = d(r, y)$. Now

$$d(r, v) + d(v, y) = d(r, y) = d(r, x) \leq d(r, v) + d(v, x).$$

Since $d(v, y) \neq d(v, x)$, we have $d(v, y) < d(v, x)$. However, now for all $s \in R_j$, we have

$$d(s, y) \leq d(s, v) + d(v, y) < d(s, v) + d(v, x) = d(s, x).$$

Thus, $d(s, x) \neq d(s, y)$ for all $s \in R_j$. \square

Showing that the cut-vertex v is not in a metric basis is most difficult when there is a path dangling from v , i.e. a path with one end adjacent to v . More precisely, there exists a component G_i of $G - v$ such that $G[V(G_i) \cup \{v\}] \simeq P_n$ for some $n \geq 2$. This path-like structure does not necessarily contain any elements of a metric basis, so we need to be careful when using Lemma 3. To help us differentiate the easy cases from the hard ones, we have the following lemma that already appeared in [22]. We include its proof for completeness of our work, and mainly based on the difficulty of finding Ref. [22].

Lemma 4 ([22, Proposition 2]). *Let v be a cut-vertex of a connected graph G , and let R be a resolving set of G . For each component G_i of $G - v$, we have*

$$V(G_i) \cap R = \emptyset \Rightarrow G[V(G_i) \cup \{v\}] \simeq P_n$$

for some $n \geq 2$. Moreover, there can be only one component G_i such that $V(G_i) \cap R = \emptyset$.

Proof. Let G_i be a component of $G - v$ such that $V(G_i) \cap R = \emptyset$. Assume to the contrary that $G[V(G_i) \cup \{v\}] \not\simeq P_n$ for all $n \geq 2$. Then there exist distinct $x, y \in V(G_i)$ such that $d(v, x) = d(v, y)$. Consequently, $d(r, x) = d(r, v) + d(v, x) = d(r, v) + d(v, y) = d(r, y)$ for all $r \in R$. Then the set R is not a resolving set, a contradiction.

To prove the latter claim, suppose that $V(G_i) \cap R = \emptyset$ and $V(G_j) \cap R = \emptyset$ for some $i \neq j$. Let $v_i \in N(v) \cap V(G_i)$ and $v_j \in N(v) \cap V(G_j)$. Now, we have $d(r, v_i) = d(r, v) + 1 = d(r, v_j)$ for all $r \in R$, a contradiction. \square

The following theorem considers the case where we can use [Lemma 3](#) straight away. In [Theorem 6](#), we consider the case not covered by [Theorem 5](#).

Theorem 5. *Let G be a connected graph. If $v \in V(G)$ is a cut-vertex such that*

1. $G - v$ has at least three components or
2. $G - v$ has two components G_i such that $G[V(G_i) \cup \{v\}] \not\cong P_n$, where $n \geq 2$,

then v is a void vertex.

Proof. Let R be a resolving set of G such that $v \in R$. Let us denote the components of $G - v$ by G_i , $i \in \mathbb{N}$. Due to our assumptions on $G - v$ and [Lemma 4](#), there are at least two components in $G - v$ that contain elements of R .

For all $x, y \in V(G)$ such that $d(v, x) = d(v, y)$ there exists an element $r \in R \setminus \{v\}$ such that $d(r, x) \neq d(r, y)$. Suppose now that $x, y \in V(G)$ are such that $d(v, x) \neq d(v, y)$. According to [Lemma 3](#), for every such pair x, y there exists an element $r \in R \setminus \{v\}$ such that $d(r, x) \neq d(r, y)$. Thus, the set $R \setminus \{v\}$ is a resolving set, and v is not in any metric basis. \square

In many proofs, where we prove that some vertex is not a basis forced vertex, we use a replacing technique. Let R be a metric basis of G . Consider a fixed $r \in R$. There exist some $x, y \in V(G)$ such that $d(r, x) \neq d(r, y)$ and $d(s, x) = d(s, y)$ for all $s \in R \setminus \{r\}$. Otherwise, the set $R \setminus \{r\}$ would be a resolving set that is smaller than the metric basis R . If there exists a vertex $v \in V(G)$ such that $d(v, x) \neq d(v, y)$ for all $x, y \in V(G)$ for which r is the only element of R that resolves them, then we can replace r in R with v and obtain a metric basis of G (i.e. the set $(R \setminus \{r\}) \cup \{v\}$ is a metric basis of G). We denote $R[r \leftarrow v] = (R \setminus \{r\}) \cup \{v\}$. Notice that if $v \in R$, then $R[r \leftarrow v] = R \setminus \{r\}$.

Theorem 6. *Let G be a connected graph. Let $v \in V(G)$ be a cut-vertex such that $G - v$ has only two components G_1 and G_2 . If $G[V(G_i) \cup \{v\}] \cong P_n$ ($n \geq 2$) for $i = 1$ or $i = 2$, then v is not a basis forced vertex.*

Proof. Suppose that $G[V(G_1) \cup \{v\}] \cong P_n$ for some $n \geq 2$ and that R is a metric basis of G such that $v \in R$. Let u be the vertex of G_1 that is furthest away from v in G .

Let us show that $R[v \leftarrow u]$ is a metric basis of G . To that end, let $x, y \in V(G)$ be such that $d(v, x) \neq d(v, y)$ (if $d(v, x) = d(v, y)$, then $d(r, x) \neq d(r, y)$ for some $r \in R \setminus \{v\}$). It is sufficient to show that for every such x and y we also have $d(u, x) \neq d(u, y)$. If $x \in V(G_1)$ or $y \in V(G_1)$ (or both), then we clearly have either $d(u, x) < d(u, y)$ or $d(u, y) < d(u, x)$, since the shortest paths from u to y go through x or vice versa. Suppose that $x, y \in V(G_2) \cup \{v\}$. Now all the shortest paths from u to x and y go through v . Thus, we have

$$d(u, x) = d(u, v) + d(v, x) \neq d(u, v) + d(v, y) = d(u, y). \quad \square$$

According to [Theorem 5](#) and the proof of [Theorem 6](#), a cut-vertex either is not in any metric basis or it can be replaced with a pendant. Thus, the following corollary is immediate.

Corollary 7. *Any finite graph has a metric basis that does not contain any cut-vertices.*

The graph has to be finite for [Corollary 7](#) to hold. For example, the bi-infinite path consists of only cut-vertices, and thus every resolving set contains only cut-vertices.

2.2. Pendants

Consider the graph in [Fig. 1\(b\)](#). This graph has a unique metric basis illustrated as black vertices in the figure. The two pendants that form the unique metric basis are also basis forced vertices. Thus, there does indeed exist graphs that have pendants as basis forced vertices. However, our goal is to prove that in most cases the pendants are not basis forced vertices.

Theorem 8. *Let G be a connected graph with a pendant u . Let v be the cut-vertex adjacent to u . If $G - \{v, u\}$ has at least two components, then u is not a basis forced vertex.*

Proof. Let R be a metric basis of G such that $u \in R$. Let us denote the components of $G - \{v, u\}$ by G_i , $i \in \mathbb{N}$, and let $R_i = R \cap V(G_i)$. Since R is a metric basis, there exists at least one pair of vertices $x, y \in V(G)$ such that u is the only element in R that resolves that pair. In other words, $d(u, x) \neq d(u, y)$ and $d(r, x) = d(r, y)$ for all $r \in R \setminus \{u\}$.

Case 1: Suppose that there exist two components G_i such that $R_i \neq \emptyset$. Let us show that the pair x, y is unique. We will first prove that either $x = u$ or $y = u$. Suppose to the contrary that $x \neq u$ and $y \neq u$. Now

$$d(v, x) = d(u, x) - 1 \neq d(u, y) - 1 = d(v, y).$$

According to [Lemma 3](#) (when applied to the set $R \setminus \{u\}$), the pair x, y is resolved by some $r \in R \setminus \{u\}$, a contradiction. Thus, either $x = u$ or $y = u$.

Suppose without loss of generality that $x = u$. Let us show that y is unique. First of all, we have $y \neq v$, since every $r \in R \setminus \{u\}$ resolves u and v . Thus, $y \in V(G_i)$ for some i . There exists some $j \neq i$ such that $R_j \neq \emptyset$. Since $d(r, y) = d(r, u)$ for all $r \in R_j$, we have $y \in N(v) \cap V(G_i)$. Thus, $d(u, y) = 2$. Suppose that y is not unique, that is, there exists $y' \in N(v) \setminus \{u, y\}$ such that $d(r, y') = d(r, u)$ for all $r \in R \setminus \{u\}$. Now we cannot resolve y and y' with R :

$$d(u, y) = 2 = d(u, y') \text{ and}$$

$$d(r, y) = d(r, u) = d(r, y')$$

for all $r \in R \setminus \{u\}$. Thus, y is unique.

Since all pairs other than u, y are resolved by $R \setminus \{u\}$, the set $R[u \leftarrow y]$ is a metric basis of G . Therefore, u is not a basis forced vertex.

Case 2: Suppose then that there is only one component G_i such that $R_i \neq \emptyset$. Due to Lemma 4, there are only two components G_1 and G_2 . Suppose that $R_2 = \emptyset$. Then $G[V(G_2) \cup \{v\}] \simeq P_n$ for some $n \geq 2$.

Let $w \in V(G_2)$. We will show that $R[u \leftarrow w]$ is a resolving set of G . Let $x, y \in V(G)$ be such that $d(u, x) \neq d(u, y)$. Suppose that $d(v, x) \neq d(v, y)$. According to Lemma 3 there exists an element $r \in R[u \leftarrow w]$ such that $d(r, x) \neq d(r, y)$. Suppose then that $d(v, x) = d(v, y)$. Since we assumed that $d(u, x) \neq d(u, y)$, we now have either $x = u$ or $y = u$.

Suppose without loss of generality that $x = u$. Then $y \in N(v)$. If $y \in V(G_2)$, then $d(w, y) < d(w, u)$, since the shortest path from w to u goes through y . Suppose then that $y \in V(G_1)$. Denote by y' the unique element in $N(v) \cap V(G_2)$. Since R is a metric basis and $d(u, y) = d(u, y')$, we have $d(r, y) \neq d(r, y')$ for some $r \in R \setminus \{u\}$. Since $R_2 = \emptyset$, we have $r \in R_1$. Thus, $d(r, u) = d(r, y') \neq d(r, y)$. Therefore, if $R_2 = \emptyset$, then the set $R[u \leftarrow w]$ is a metric basis for all $w \in V(G_2)$. \square

In the proofs of Theorems 5 and 6, our goal was to replace the cut-vertex with something else or remove it entirely from the resolving set. However, in the following theorem we do the opposite; we replace a pendant with a cut-vertex in a metric basis.

Theorem 9. *Let G be a connected graph such that $G \not\cong P_n$ for all $n \in \mathbb{N}$. If u is a pendant adjacent to a cut-vertex v such that $\deg(v) = 2$, then u is not a basis forced vertex.*

Proof. Suppose that R is a metric basis of G such that $u \in R$. We will show that $R[u \leftarrow v]$ is a resolving set.

Let $x, y \in V(G)$ be such that $d(u, x) \neq d(u, y)$. If $x, y \in V(G) \setminus \{u\}$, then $d(v, x) = d(u, x) - 1 \neq d(u, y) - 1 = d(v, y)$. Suppose that $x = u$ and $d(v, x) = d(v, y)$. Since $G \not\cong P_n$ for all $n \in \mathbb{N}$, there exists an element $r \in R \setminus \{v, u\}$ due to Lemma 4. Since $\deg(v) = 2$ and $y \in N(v)$, the shortest paths from r to u go through y . Thus, $d(r, u) \neq d(r, y)$. \square

Based on the results on the metric bases of trees presented in [4,15,22], we can see that trees do not have any basis forced vertices. Now we can also prove that trees have no basis forced vertices with the results obtained in Sections 2.1 and 2.2. According to Corollary 7 cut-vertices are not basis forced vertices. If the tree in question is not P_2 , then every pendant is adjacent to a cut-vertex that fulfils the requirements of either Theorem 8 or Theorem 9. Thus, neither the pendants of a tree are basis forced vertices.

Let us then consider a general graph G . Suppose that G has a cut-vertex v . If there is a path- or tree-like structure attached to v , then there are no basis forced vertices in this structure. More precisely, let G_i be a component of $G - v$ such that G_i is isomorphic to a tree or a path of length at least 2. According to Corollary 7 and Theorems 8 and 9, G_i does not contain basis forced vertices.

Suppose that there is a pendant u adjacent to v . If there is a path- or tree-like structure attached to v , then the graph fulfils the conditions of Theorem 8 or 9 and u is not a basis forced vertex. Thus, if u is a basis forced vertex, then every edge $\{v, x\} \in E(G)$ where $x \in V(G) \setminus \{u\}$ is along a cycle. However, all pendants of the graph in Fig. 1(b) satisfy this condition, but only two of them are basis forced vertices and the rest are void vertices.

3. Sparse and dense graphs

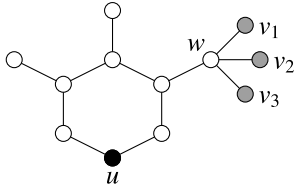
We describe a graph as sparse if it has few edges compared to the number of vertices. Conversely, a graph is dense if it has many edges compared to the number of vertices. We want to find out what restrictions the existence of basis forced vertices places on the number of edges.

3.1. Sparse graphs

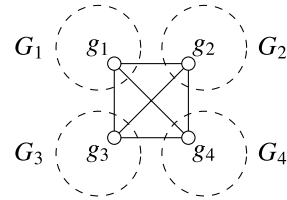
As we saw in Section 2, trees do not have any basis forced vertices. However, the graph in Fig. 1(b) has two basis forced vertices. Thus, the sparsest graphs that have basis forced vertices are unicyclic graphs.

The basis forced vertices of the graph in Fig. 1(b) are pendants. A unicyclic graph can also have a basis forced vertex along the cycle; the vertex u of the graph in Fig. 2(a) is such a vertex. All metric bases of this graph are of the form $\{u, v_i, v_j\}$, where $i, j \in \{1, 2, 3\}$ and $i \neq j$. Both of these example graphs have only one or two basis forced vertices. Indeed, later in Section 3.1.1 we obtain a result which states that a unicyclic graph can have at most two basis forced vertices.

We want to construct connected graphs with, say, k basis forced vertices and as few edges as possible. In this endeavour, the following construction will be useful. The general idea and the structure of the end-result of this construction is



(a) A unicyclic graph with a basis forced vertex and metric dimension 3.



(b) An example of the idea behind the construction in Theorem 10.

Fig. 2.

illustrated in Fig. 2(b). The graph W constructed in Theorem 10 is in fact a rooted product graph as defined in [7]. The metric dimension of rooted product graphs was considered in [16]. In order to explain how the basis forced vertices of a rooted product graph are related to those of its components, we give in Theorem 10 the metric basis of the rooted product graph with the aid of metric bases of the components. It also reveals (as illustrated in Example 11) that there can be basis forced vertices even if the components do not have any basis forced vertices.

Recall that $\dim(G) = 1$ if and only if $G = P_n$ for some $n \in \mathbb{N}$ [4,15]. Thus, the resolving sets of the components G_i in the following theorem have at least two elements.

Theorem 10. Let G_i be a connected graph such that $G_i \not\cong P_n$ where $i \in \{1, \dots, k\}$ and $n \in \mathbb{N}$, and let all G_i be vertex disjoint. Let g_i be a fixed element of $V(G_i)$ such that g_i is in some metric basis of G_i (i.e. g_i is not a void vertex). Let W be the graph with

$$V(W) = \bigcup_{i=1}^k V(G_i) \quad \text{and} \quad E(W) = \{\{g_i, g_j\} \mid i \neq j\} \cup \bigcup_{i=1}^k E(G_i),$$

i.e. the graph we obtain by connecting every g_i with one another. A set $R \subseteq V(W)$ is a metric basis of W if and only if

$$R = \bigcup_{i=1}^k R_i \setminus \{g_i\}$$

where R_i is a metric basis of G_i that contains g_i for all $i \in \{1, \dots, k\}$.

Proof. Let $R \subseteq V(W)$ be a resolving set of W . Since the vertices g_i are cut-vertices of W and $G_i \not\cong P_n$ for all i and $n \in \mathbb{N}$, we have $S_i = R \cap V(G_i) \neq \emptyset$ for all i due to Lemma 4.

Let us show that the set $S'_i = S_i \cup \{g_i\}$ is a resolving set of G_i . Suppose to the contrary that S'_i is not a resolving set of G_i . Then there exist $x, y \in V(G_i)$ such that $d(s', x) = d(s', y)$ for all $s' \in S'_i$. However, now the set R is not a resolving set of W ; we have $d(s_i, x) = d(s_i, y)$ for all $s_i \in S_i$ and

$$d(s_j, x) = d(s_j, g_i) + d(g_i, x) = d(s_j, g_i) + d(g_i, y) = d(s_j, y)$$

for all $s_j \in S_j$ where $j \neq i$.

Let R_i be a resolving set of G_i that contains g_i for all $i \in \{1, \dots, k\}$. Let

$$R = \bigcup_{i=1}^k R_i \setminus \{g_i\}.$$

Let us show that R is a resolving set of W by showing that we can resolve every pair $x, y \in V(W)$. We divide the proof into two cases:

1. $x, y \in V(G_i)$ for some $i \in \{1, \dots, k\}$: Since R_i is a resolving set of G_i , there exists some $r_i \in R_i$ such that $d(r_i, x) \neq d(r_i, y)$. If $r_i \neq g_i$, then $r_i \in R$ and we are done. Suppose that $r_i = g_i$. Now we can resolve the pair x, y with any $r_j \in R_j$ where $j \neq i$. Indeed, we have $d(r_j, x) = d(r_j, g_i) + d(g_i, x) \neq d(r_j, g_i) + d(g_i, y) = d(r_j, y)$ for all $r_j \in R_j$.
2. $x \in V(G_i)$ and $y \in V(G_j)$ for some $i \neq j$: In this case, we use Lemma 3. Each vertex g_i is a cut-vertex in W . Suppose that $d(g_i, x) = d(g_i, y)$. Now, $d(g_j, x) = d(g_i, x) + 1 \neq d(g_i, y) - 1 = d(g_j, y)$. Similarly, if $d(g_j, x) = d(g_j, y)$, then $d(g_i, x) \neq d(g_i, y)$. The graph $W - g_i$ has two components, neither of which is isomorphic to a path, since $G_i \not\cong P_n$ for all i and $n \in \mathbb{N}$. Thus, both components of $W - g_i$ (and similarly $W - g_j$) contain elements of R according to Lemma 4. Thus, we can use Lemma 3 for either g_i or g_j and obtain that for some $r \in R$ we have $d(r, x) \neq d(r, y)$. \square

Notice that $\dim(W) = \sum_{i=1}^k \dim(G_i) - k$ follows from Theorem 10. This was also shown in [16]. According to Theorem 10 the basis forced vertices of the components G_i are also basis forced vertices of the graph W as long as the vertex g_i is not a basis forced vertex. (Indeed, the vertex g_i is a void vertex of W .)

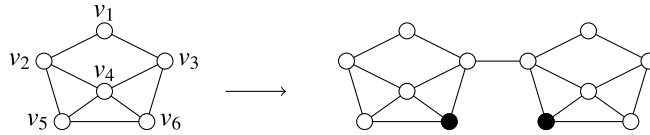


Fig. 3. An example where we construct a graph with 2 basis forced vertices from a graph that has no basis forced vertices.

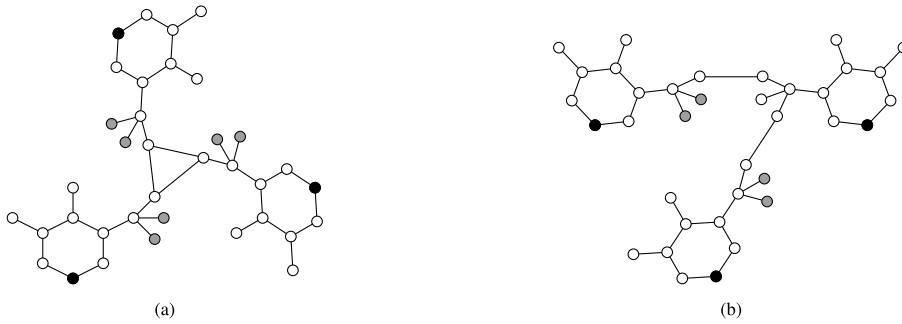


Fig. 4. An example of the two ways we can use Theorem 10.

Example 11. Consider the graph G in Fig. 3. This graph has only three metric bases: $\{v_2, v_5\}$, $\{v_3, v_6\}$ and $\{v_5, v_6\}$. Let us construct a new graph W by using Theorem 10 and two copies of G . Let us choose $g_1 = v_3$ and $g_2 = v_2$. Since there is only one metric basis that contains v_3 and one metric basis that contains v_2 , the only metric basis of W is $\{v_5^1, v_6^2\}$, where the superscripts indicate in which copy of G the vertex is. Thus, W is a graph with a unique metric basis and metric dimension 2. Consequently, the graph W contains two basis forced vertices.

In general, there are two ways to use Theorem 10. If we want to use k graphs, we can connect them all in one go or iterate the use of Theorem 10. In the first option, the vertices g_i induce a clique in the constructed graph. In the latter option, we first use Theorem 10 on two graphs, then choose a new g_1 from the resulting graph and connect another graph to that. We can also use a combination of these two methods. The following example further demonstrates the difference of these two options.

Example 12. Consider the graph G in Fig. 2(a). Let us combine three of these graphs by using Theorem 10. Let us indicate with a superscript from which copy of the graph G the vertices are from. We choose $g_1 = v_1^1$, $g_2 = v_1^2$ and $g_3 = v_1^3$. The constructed graph W_1 is illustrated in Fig. 4(a). There are two metric bases of G that contain v_1 : $\{u, v_1, v_2\}$ and $\{u, v_1, v_3\}$. Thus, the metric bases of W_1 are of the form $\{u^1, v_j^1, u^2, v_k^2, u^3, v_l^3\}$ where $j, k, l \in \{2, 3\}$. Thus, the metric dimension of W_1 is 6, and it has three basis forced vertices.

Let us then iterate Theorem 10. Let us first use Theorem 10 on two copies of G . We again choose $g_1 = v_1^1$ and $g_2 = v_1^2$. The metric bases of the resulting graph W_2 are of the form $\{u^1, v_j^1, u^2, v_k^2\}$ where $j, k \in \{2, 3\}$. Let us then use Theorem 10 again, but this time on the graph W_2 and another copy of G . We choose $g_1 = v_3^2$ and $g_2 = v_3^3$. The resulting graph W_3 is illustrated in Fig. 4(b). The only metric bases of W_2 that contain v_3^2 are of the form $\{u^1, v_j^1, u^2, v_3^2\}$ where $j \in \{2, 3\}$. Thus, the metric bases of W_3 are of the form $\{u^1, v_j^1, u^2, u^3, v_l^3\}$ where $j, l \in \{2, 3\}$. Now, we have constructed a graph with metric dimension 5 and three basis forced vertices.

The graph W_3 has one edge less than the graph W_1 . If we use the construction on more graphs, then this difference will only grow. Indeed, if we used m copies of the graph G , then the first method would produce $m(m - 1)$ edges to the resulting graph in addition to the edges already present in G . Thus, the resulting graph would have $12m + m(m - 1)$ edges. However, if we iterate the construction, then each step adds only one edge in addition to those present in G . If we iterate the construction on m copies of G , we will use Theorem 10 $m - 1$ times, and the resulting graph has only $12m + m - 1$ edges.

We have now demonstrated a way to construct a graph that has k basis forced vertices, $n = 12k$ vertices and $n + k - 1$ edges. So far, we have not found any sparser graph with k basis forced vertices.

3.1.1. An upper bound for the number of basis forced vertices of unicyclic graphs

According to the terminology given in [20], a unicyclic graph of type 1 is a unicyclic graph G of maximum degree 3 and such that every vertex of maximum degree belongs to the unique cycle of G . Any other unicyclic graph is called there as of type 2. It was proved in [20] that a unicyclic graph of type 1 with unique cycle C_n has metric dimension 2 when n is odd, and otherwise its metric dimension is between 2 and 3. We notice the following straightforward observation (see [20]).

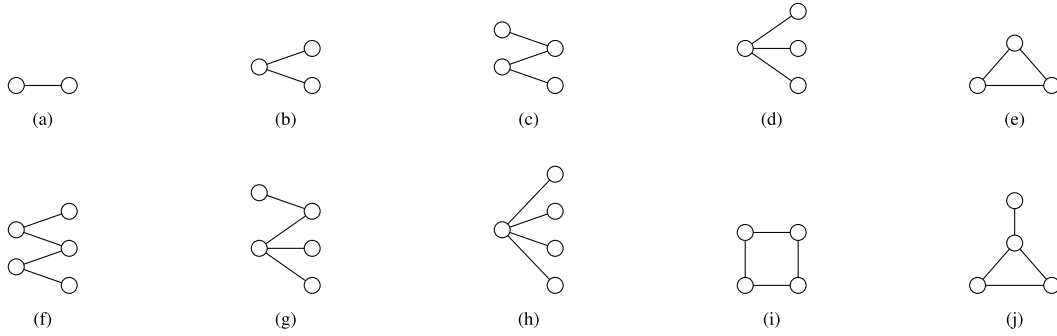


Fig. 5. Nonisomorphic pieces of edge deletion patterns with which we can construct all nonisomorphic edge deletion patterns with up to 4 edges.

Remark 13. If G is a unicyclic graph of type 1 with unique cycle $C = v_0v_1 \cdots v_{r-1}v_0$, $r \geq 3$, then the set $S = \{v_i, v_{i+1}, v_{i+\lfloor r/2 \rfloor}\}$ (operations with subindex are done modulo r) is a resolving set for G for all $i \in \{0, \dots, r - 1\}$.

As a consequence of the remark above, if G is a unicyclic graph of type 1 with metric dimension 3, then G does not contain basis forced vertices. Thus, a unicyclic graph of type 1 can have at most two basis forced vertices. We next consider unicyclic graphs of type 2. From [20], lower and upper bounds for the metric dimension of such unicyclic graphs are known. However, from [21], we can easier proceed with our deduction. For a unicyclic graph G , let $b(G)$ be the number of vertices on the cycle that have something other than one pendant or path attached to it outside of the cycle. Furthermore, let

$$L(G) = \sum_{v \in V(G), \ell(v) > 1} (\ell(v) - 1),$$

where $\ell(v)$ is the number of pendants and paths attached to v . The following result was proved in [21].

Theorem 14 ([21]). *If G is a unicyclic graph, then $\dim(G)$ equals $L(G) + \max\{2 - b(G), 0\}$ or $L(G) + \max\{2 - b(G), 0\} + 1$.*

The vertices that contribute to the value of $L(G)$ are pendants or vertices of a path attached to some vertex v (but the paths do not include v) with $\ell(v) > 1$. For example, for the vertex w in Fig. 2(a) we have $\ell(w) = 3$. The vertices of a path attached to v are not basis forced vertices according to Theorem 6 and (for the endvertex of a path) Theorem 9. If v is such that $\ell(v) > 1$, then $\deg(v) \geq 3$, since G is a unicyclic graph. Now, the pendants attached to v are not basis forced vertices according to Theorem 8. Therefore, the vertices that contribute to the value of $L(G)$ are not basis forced vertices. Thus, since $b(G) \geq 0$, we have $0 \leq \max\{2 - b(G), 0\} \leq 2$, and the two possible values for $\dim(G)$, from the theorem above, leads to observe that G could have between 0 and 3 basis forced vertices. However, if there are 3 basis forced vertices in G , then it must happen that $\dim(G) = L(G) + \max\{2 - b(G), 0\} + 1$ with $b(G) = 0$. But then it follows that every vertex of G has degree at most 3 and every vertex of degree 3 belongs to the unique cycle of G . Consequently, G is a unicyclic graph of type 1 (according to [20]), and we have seen that such graphs can have at most 2 basis forced vertices. Thus, we have the following corollary.

Corollary 15. *If G is a unicyclic graph, then G has at most 2 basis forced vertices.*

Unicyclic graphs with 1 and 2 basis forced vertices are illustrated in Figs. 2(a) and 1(b), respectively.

3.2. Dense graphs

We want to find out, how many edges a graph can have and still contain basis forced vertices. We will show that a graph with n vertices and some basis forced vertices can have at most $\frac{n(n-1)}{2} - 4$ edges. We will also show that the graph attaining this bound is unique for each n . To obtain these results, we will consider graphs that can be constructed from K_n by removing up to four edges. The relevant edge deletion patterns (i.e. components of the complement graph) are illustrated in Fig. 5.

Distinct vertices $v, u \in V(G)$ are *true twins* if $N[v] = N[u]$, and *false twins* if $N(v) = N(u)$. We simply say that two vertices are twins if they are true or false twins but it does not matter which. As the graphs we consider are quite dense, the graphs will most likely contain twins. The following lemma follows directly from Corollary 2.4 of [13], although an independent proof would not be too difficult either.

Lemma 16. *Let G be a connected graph, and let $T \subseteq V(G)$ be such that all its elements are true twins with one another or all its elements are false twins with one another. If $|T| \geq 2$, then $|R \cap T| \geq |T| - 1$ for all resolving sets R of G .*



Fig. 6. Example illustrations of the graphs considered in Lemmas 18 and 19. The graphs may have any (nonempty) structure inside the dashed area.

Notice that if some resolving set contains only $|T| - 1$ elements of the set T , then it does not matter which twin is left out of the resolving set. Indeed, for all vertices $x, y \in T$ and $z \in V(G) \setminus T$ we have $d(x, z) = d(y, z)$.

Lemma 17. *If G is a connected graph with a universal vertex v , then the vertex v is not a basis forced vertex.*

Proof. Let R be a metric basis of G such that $v \in R$. Then the set $R \setminus \{v\}$ is not a resolving set. There exist distinct $x, y \in V(G)$ such that $d(v, x) \neq d(v, y)$ and $d(r, x) = d(r, y)$ for all $r \in R \setminus \{v\}$. Since v is universal, we have either $x = v$ or $y = v$. Suppose without loss of generality that $x = v$. Now, $d(r, y) = 1$ for all $r \in R$, and the vertex y is unique. Consequently, the set $R[v \leftarrow y]$ is a resolving set. Thus, the vertex v is not a basis forced vertex. \square

We like to describe the graphs we consider by giving the complement graphs, since the complement graph is often far simpler in structure. The components of the complement graph also give us a natural way to partition the vertices. Consider a graph G such that the complement graph \bar{G} has two components: \bar{G}_1 and \bar{G}_2 . All distinct vertices $x \in V(\bar{G}_1)$ and $y \in V(\bar{G}_2)$ are adjacent in G . Thus, $d_G(x, y) = 1$. We cannot resolve any pair of vertices in $V(\bar{G}_1)$ with a vertex in $V(\bar{G}_2)$ in G . In order to find a metric basis of G , we need to make sure that each component of \bar{G} contains enough elements of a metric basis to resolve the pairs within that component. In addition to that, we need to be able to resolve pairs of vertices that are in different components of \bar{G} .

Lemma 18. *Let G be a connected graph. If \bar{G}_i is a component of \bar{G} such that $\bar{G}_i \simeq K_n$ for some $n \geq 2$, then no vertex $v \in V(\bar{G}_i)$ is a basis forced vertex of G .*

Proof. One example of the graph G is illustrated in Fig. 6(a).

For simplicity, let $i = 1$ and let us denote $V(\bar{G}_1) = \{v_1, \dots, v_n\}$. Since $\bar{G}_1 \simeq K_n$, we have $v_i \not\sim_G v_j$ for all $i \neq j$ and $v_i \sim_G u$ for all i and $u \in V(G) \setminus V(\bar{G}_1)$. Thus, $N_G(v_i) = N_G(v_j)$ for all $i \neq j$ and the vertices v_i are false twins in G . According to Lemma 16, every resolving set of G contains at least $n - 1$ vertices v_i .

If there exists a metric basis R such that $v_1 \notin R$, then none of the vertices v_i are basis forced vertices. Indeed, the set $R[v_1 \leftarrow v_i]$ is also a metric basis of G as the vertices v_i are twins with each other.

Suppose then that R is a metric basis of G such that $v_i \in R$ for all $i \in \{1, \dots, k\}$. We will show that such metric basis cannot in fact exist by proving that the set $R \setminus \{v_i\}$ is a resolving set of G for all $i \in \{1, \dots, k\}$. Let us consider a fixed v_i . In what follows, the distances we consider are distances in G , and we omit the subscript G from the notation of distance. Since $d(v_i, u) = 1$ if and only if $u \notin V(\bar{G}_1)$, the vertex v_i can only resolve pairs where one or both vertices are some v_j , $j \in \{1, \dots, n\}$. However, all of these pairs can be resolved with other elements of R :

- Consider the vertices v_j and v_k where $j \neq k$. Suppose without loss of generality that $v_j \neq v_i$. Now, $v_j \in R \setminus \{v_i\}$ and $d(v_j, v_j) \neq d(v_j, v_k)$.
- Consider the vertex v_i and some $u \notin V(\bar{G}_1)$. Let v_j be such that $j \neq i$. Now $v_j \in R \setminus \{v_i\}$ and $d(v_j, v_i) = 2 \neq 1 = d(v_j, u)$.
- Consider some vertex v_j , where $j \neq i$, and some $u \notin V(\bar{G}_1)$. Now, $v_j \in R \setminus \{v_i\}$ and $d(v_j, v_j) \neq d(v_j, u)$.

In conclusion, the set $R \setminus \{v_i\}$ is a resolving set for all v_i . This contradicts the assumption that the set R is a metric basis of G . Consequently, for every $v_i \in V(\bar{G}_1)$ there exists a metric basis R such that $v_i \notin R$, and thus the vertex v_i is not a basis forced vertex. \square

Lemma 19. *Let G be a connected graph. If \bar{G}_i is a component of \bar{G} such that $\bar{G}_i \simeq K_{1,n}$ for some $n \geq 2$, then no vertex $v \in V(\bar{G}_i)$ is a basis forced vertex of G .*

Proof. One example of the graph G is illustrated in Fig. 6(b).

Let $\bar{G}_i \simeq K_{1,n}$ for some $n \geq 2$, and denote $V(\bar{G}_1) = \{v_0, \dots, v_n\}$, where $\deg_{\bar{G}}(v_0) = n$ and $\deg_{\bar{G}}(v_i) = 1$ for all $i \neq 0$. Since G is connected, $V(G) \setminus V(\bar{G}_1) \neq \emptyset$. The vertices v_i where $i \neq 0$ are true twins in G . According to Lemma 16 each resolving set of G contains at least $n - 1$ vertices v_i , $i \neq 0$.

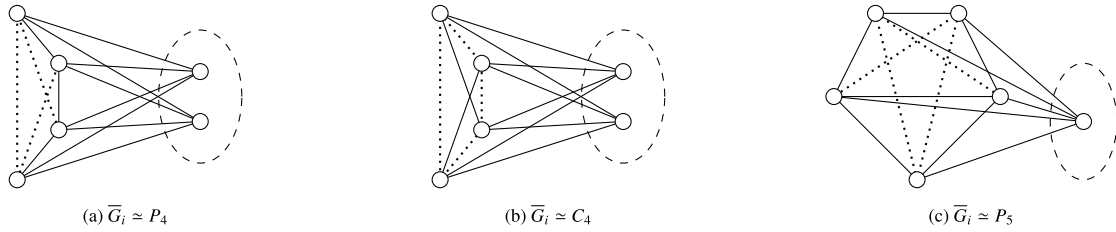


Fig. 7. Example illustrations of the graphs considered in Lemmas 20–22. The graphs may have any (nonempty) structure inside the dashed area.

We will first show that no v_i , where $i \neq 0$, is a basis forced vertex of G . Suppose that R is a resolving set of G such that $v_i \in R$ for all $i \neq 0$. Consider a fixed v_i . Let us show that the set $R[v_i \leftarrow v_0]$ is a resolving set of G . In what follows, all distances that we consider are distances in G . Consider the pairs of vertices that v_i resolves. Let $x, y \in V(G)$ be such that $d(v_i, x) \neq d(v_i, y)$. We have $d(v_i, u) = 1$ if and only if $u \neq v_i$ and $u \neq v_0$. Consequently, $x \in \{v_i, v_0\}$ or $y \in \{v_i, v_0\}$. Suppose that $x = v_0$. Then v_0 resolves x and y no matter what y is. Suppose then that $x = v_i$. If $d(v_0, y) \neq d(v_0, v_i)$, then we are done. Suppose that $d(v_0, y) = d(v_0, v_i) = 2$. Since $d(v_0, u) = 2$ if and only if $u = v_j$ for some $j \in \{1, \dots, n\}$, we have $y = v_j$ for some $j \in \{1, \dots, n\}$. However, we also have that $y \neq v_i$, and thus $y \in R[v_i \leftarrow v_0]$ and y itself resolves v_i and y . Thus, $R[v_i \leftarrow v_0]$ is a resolving set of G and v_i is not a basis forced vertex of G .

Let us then show that v_0 is not a basis forced vertex. Let R be a metric basis of G such that $v_0 \in R$. Due to what we have already shown above, we may assume that $v_i \notin R$ for some $i \in \{1, \dots, n\}$ and $v_j \in R$ for all $j \neq i$. Let us show that the set $R[v_0 \leftarrow v_i]$ is a metric basis of G .

Since R is a metric basis, the set $R \setminus \{v_0\}$ is not a resolving set. Suppose that the pair $x, y \in V(G)$ is not resolved by any element of $R \setminus \{v_0\}$. We will show that then either $x = v_i$ or $y = v_i$. It is clear that $x, y \notin R \setminus \{v_0\}$. If $x, y \notin V(\bar{G}_i)$, then $d(v_0, x) = d(v_0, y) = 1$ and the pair x, y is not resolved by any element of R , which contradicts the assumption that the set R is a metric basis of G . For all $j \neq i$ we have $d(v_j, v_0) = 2$ and $d(v_j, u) = 1$ for all $u \in V(G) \setminus \{v_0\}$. If $x = v_0$ or $y = v_0$, then v_j (where $j \neq i$) resolves x and y , which contradicts the assumption that no element of $R \setminus \{v_0\}$ resolves x and y , since we have $v_j \in R$ for all $j \neq i$. Thus, we have $x \neq v_0$ and $y \neq v_0$, and the only option we now have left is that either $x = v_i$ or $y = v_i$. Consequently, the set $R[v_0 \leftarrow v_i]$ is a metric basis of G and v_0 is not a basis forced vertex of G . \square

Lemma 20. *Let G be a connected graph with at least 5 vertices. If \bar{G}_i is a component of \bar{G} such that $\bar{G}_i \simeq P_4$, then the vertices $v \in V(\bar{G}_i)$ are not basis forced vertices of G .*

Proof. One example of the graph G is illustrated in Fig. 7(a).

Since the path P_4 is self-complementary, we have $G[V(\bar{G}_i)] \simeq P_4$. Let us denote the elements of $V(\bar{G}_i)$ by v_i so that $v_1v_2v_3v_4$ is a path in G . Since G has at least 5 vertices, $V(G) \setminus V(\bar{G}_i) \neq \emptyset$. Consequently, $d(v_1, v_4) = 2$.

In order to resolve v_i and v_j , where $i \neq j$, we need at least two elements of $V(\bar{G}_i)$ in each resolving set. Indeed, we have $d(u, v_i) = 1 = d(u, v_j)$ for all $u \in V(G) \setminus V(\bar{G}_i)$, so the only vertices that possibly resolve v_i and v_j are the elements of $V(\bar{G}_i)$. However, for every element $v \in V(\bar{G}_i)$ there are at least two vertices v_i at the same distance from it. For example, we have $d(v_1, v_3) = 2 = d(v_1, v_4)$ and $d(v_2, v_1) = 1 = d(v_2, v_3)$. Thus, each resolving set of G contains at least two elements of $V(\bar{G}_i)$.

Let us then show that two elements of $V(\bar{G}_i)$ are enough. The elements of $V(\bar{G}_i)$ can only resolve pairs of vertices where one or both vertices are in $V(\bar{G}_i)$. Let us show that all such pairs are resolved by v_1 and v_2 . Consider distinct $x, y \in V(G)$. Suppose that $x, y \in V(\bar{G}_i)$. If $x = v_1$ or $x = v_2$, then x and y are resolved by x itself (similarly for y). Let $x = v_3$ and $y = v_4$. Now, $d(v_2, v_3) = 1 \neq 2 = d(v_2, v_4)$, and thus x and y are resolved. Suppose then that $x \in V(\bar{G}_i)$ and $y \in V(G) \setminus V(\bar{G}_i)$. Now, $d(v_1, y) = 1 = d(v_2, y)$. However, $d(v_1, x) \neq 1$ when $x \neq v_2$ and $d(v_2, x) \neq 1$ when $x = v_2$. In conclusion, it is enough for a resolving set to contain two elements of $V(\bar{G}_i)$; v_1 and v_2 , or v_3 and v_4 by symmetry. \square

Lemma 21. *Let G be a connected graph with at least 5 vertices. If \bar{G}_i is a component of \bar{G} such that $\bar{G}_i \simeq C_4$, then the vertices $v \in V(\bar{G}_i)$ are not basis forced vertices of G .*

Proof. Let us denote the vertices of \bar{G}_i by v_1, \dots, v_4 so that $v_1v_2v_3v_4v_1$ is a cycle in \bar{G} . Now v_1 and v_3 are adjacent in G , and v_2 and v_4 are adjacent in G (see Fig. 7(b)). The vertices v_1 and v_3 are true twins, and so are v_2 and v_4 . According to Lemma 16 each resolving set contains v_1 or v_3 (and v_2 or v_4).

Let us show that a metric basis of G contains at most two vertices of $V(\bar{G}_i)$. No element of $V(\bar{G}_i)$ can resolve x and y if both $x \notin V(\bar{G}_i)$ and $y \notin V(\bar{G}_i)$. Thus, it is sufficient to consider distinct $x, y \in V(G)$ such that $x \in V(\bar{G}_i)$. Let us show that we can resolve every such pair x, y with v_1 and v_2 . Clearly, if $x \in \{v_1, v_2\}$ or $y \in \{v_1, v_2\}$, then x and y are resolved. Suppose then that $x = v_3$. Now, $d(v_2, x) = 2$. The only vertices that are at distance 2 from v_2 are v_1 and v_3 . Thus, if we had $d(v_2, x) = d(v_2, y)$, then $y = v_1$ and $d(v_1, y) \neq d(v_1, x)$. Similarly, if $x = v_4$ and $d(v_1, x) = 2 = d(v_1, y)$, then $y = v_2$ and

$d(v_2, y) \neq d(v_2, x)$. In conclusion, there exists a metric basis R of G such that $R \cap V(\bar{G}_i) = \{v_1, v_2\}$. By symmetry, there also exists a metric basis R of G such that $R \cap V(\bar{G}_i) = \{v_3, v_4\}$. Thus, none of the elements of $V(\bar{G}_i)$ are basis forced vertices. \square

The following lemma is used in this section for $k = 2$. We will, however, use the following lemma for other values of k later in Section 4.

Lemma 22. *Let k be an even positive integer. Let G be a connected graph such that $\bar{G} \simeq P_5 \cup \dots \cup P_5 \cup \bar{K}_m$ where P_5 appears $\frac{k}{2}$ times and $m \geq 1$. The graph G has k basis forced vertices and $\dim(G) = k + m - 1$.*

Proof. The graph G for $k = 2$ and $m = 1$ is illustrated in Fig. 7(c).

Let us denote the copies of P_5 by P_5^i where $i \in \{1, \dots, \frac{k}{2}\}$. We denote the vertices of each P_5^i by v_1^i, \dots, v_5^i where the vertices are numbered in the order natural for paths. We further denote the vertices of \bar{K}_m by u_j .

It is clear that in order to resolve all pairs $x, y \in V(P_5^i)$ in G , we need at least two elements of $V(P_5^i)$. The vertices u_j are twins with one another. Thus, according to Lemma 16, there exists at most one u_j such that $u_j \notin R$ for each resolving set R . Therefore, $\dim(G) \geq 2 \cdot \frac{k}{2} + m - 1 = k + m - 1$.

Let us consider how we can resolve the elements of $V(P_5^i)$ in G . One can note that the only two element subsets of $V(P_5^i)$ that resolve $V(P_5^i)$ are $\{v_1^i, v_3^i\}, \{v_1^i, v_5^i\}, \{v_2^i, v_3^i\}, \{v_2^i, v_4^i\}, \{v_3^i, v_4^i\}$ and $\{v_3^i, v_5^i\}$.

Let $R \subseteq V(G)$ be such that $R \cap V(P_5^i) = \{v_2^i, v_4^i\}$ for all i and $R \cap V(\bar{K}_m) = V(\bar{K}_m) \setminus \{u_j\}$ for some j . Let us show that R is a resolving set of G . To that end, consider distinct vertices $x, y \in V(G)$. If $x, y \in V(P_5^i)$ for some i , then the set R resolves x and y by the observation above. Suppose that $x \in V(P_5^i)$ and $y \notin V(P_5^i)$ for some i . Now, $d(v_2^i, y) = d(v_4^i, y) = 1$. One of the distances $d(v_2^i, x)$ and $d(v_4^i, x)$ is 0 or 2 for every $x \in V(P_5^i)$. Thus, we have $d(v_2^i, x) \neq d(v_2^i, y)$ or $d(v_4^i, x) \neq d(v_4^i, y)$. Suppose finally that $x, y \notin V(P_5^i)$ for all i . In other words, $x, y \in V(\bar{K}_m)$. Since $u_j \notin R$ for only one vertex $u_j \in V(\bar{K}_m)$, the vertices x and y are resolved by some element of R . Therefore, the set R is a resolving set of G . Since $|R| = 2 \cdot \frac{k}{2} + m - 1$, the set R is a metric basis of G and $\dim(G) = k + m - 1$.

Suppose then that R is a resolving set of G . Let us denote $S = R \cap V(P_5^i)$. Suppose that $|S| = 2$ and $S \neq \{v_2^i, v_4^i\}$ for some i . (In other words, the intersection $R \cap V(P_5^i)$ is one of the five listed earlier that are not $\{v_2^i, v_4^i\}$.) By making a 5×5 -table of distances, it is easy to see that there exists a vertex $x \in V(P_5^i)$ such that $d(s, x) = 1$ for all $s \in S$ if and only if $S \neq \{v_2^i, v_4^i\}$. Thus, since $S \neq \{v_2^i, v_4^i\}$, we have $d(s, x) = d(s, u_j)$ for all $s \in S$ and $u_j \in V(\bar{K}_m)$. Now $u_j \in R$ for all j , since R is a resolving set of G and $d(u_l, u_j) = 1 = d(u_l, x)$ for all $l \neq j$. We have $|R| \geq 2 \cdot \frac{k}{2} + m = k + m$, since $|R \cap V(P_5^i)| \geq 2$ for all $j \in \{1, \dots, \frac{k}{2}\}$ as we pointed out earlier. Consequently, R is not a metric basis of G .

In conclusion, a set $R \subseteq V(G)$ is a metric basis of G if and only if $R \cap V(P_5^i) = \{v_2^i, v_4^i\}$ for all $i \in \{1, \dots, \frac{k}{2}\}$ and $R \cap V(\bar{K}_m) = V(\bar{K}_m) \setminus \{u_j\}$ for some $j \in \{1, \dots, m\}$. Consequently, each v_2^i and v_4^i is a basis forced vertex of G and G has k basis forced vertices in total. \square

Now we are ready for the main result in this section.

Theorem 23. *If G is a connected graph with $n \geq 6$ vertices and at least one basis forced vertex, then*

$$|E(G)| \leq \frac{n(n-1)}{2} - 4.$$

If $|E(G)| = \frac{n(n-1)}{2} - 4$, then $\bar{G} \simeq P_5 \cup \bar{K}_{n-5}$ and G has two basis forced vertices.

Proof. Let us consider G as constructed from K_n by removing edges.

Suppose that G is obtained from K_n by removing up to three edges. Then the graph \bar{G} can be presented as a union of some of the following graphs (see Fig. 5(a-e)): $K_2, K_{1,2}, P_4, K_{1,3}, K_3$ and \bar{K}_{n-k} for some k . Lemmas 17, 18, 19 and 20 cover all possible cases, and in each case the graph G does not have any basis forced vertices.

We have now shown that if $|E(G)| \geq \frac{n(n-1)}{2} - 3$, then the graph G does not have any basis forced vertices. Consequently, if G has basis forced vertices, then $|E(G)| \leq \frac{n(n-1)}{2} - 4$.

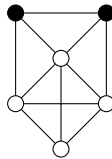
Let us then prove the latter claim. Let G be a graph that is obtained from K_n by removing four edges. Suppose that \bar{G} can be presented as the union of some of the following graphs (see Fig. 5(a-e,h,i)): $K_2, K_{1,2}, P_4, K_{1,3}, K_3, K_{1,4}, C_4$ and \bar{K}_{n-k} for some k . As before, we can use Lemmas 17, 18, 19, 20 and now 21 also, and obtain that in all cases the graph G does not have any basis forced vertices. Thus, if G has basis forced vertices, then there are only three possibilities for the edge deletion pattern: the graphs in Fig. 5(f,g,j). Let us consider each case separately.

Suppose that \bar{G} is the disjoint union of the graph in Fig. 5(g) and \bar{K}_{n-5} . Let us denote the elements as in Fig. 8(a), that is, the isolated vertices are denoted by u_i and the five other vertices are denoted by v_i . One can note that the sets $V(G) \setminus \{v_2, v_3, v_5\}$ and $V(G) \setminus \{v_1, v_4, u_i\}$ are metric bases for all i . Thus, the graph G does not have any basis forced vertices.

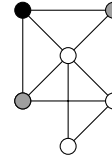
Suppose that \bar{G} is the disjoint union of the graph in Fig. 5(j) and \bar{K}_{n-4} . Let us denote the elements of $V(\bar{G}_i)$ by v_1, \dots, v_4 as in Fig. 8(b). It is clear that in order to resolve $V(\bar{G}_i)$ in G , we need at least two elements of $V(\bar{G}_i)$. One can note that the sets $\{v_1, v_3\}$ and $\{v_2, v_4\}$ resolve $V(\bar{G}_i)$ in G . Both of these options also resolve every pair $x \in V(\bar{G}_i)$ and $y \in V(G) \setminus V(\bar{G}_i)$. Indeed, for all $x \in V(\bar{G}_i)$ one of the distances $d(x, v_1)$ and $d(x, v_3)$ is 2 or 0 (and similarly for v_2 and v_4). Thus, the graph G does not have any basis forced vertices.



Fig. 8. Example illustrations of the two special cases of the proof of Theorem 23. The graphs may have any (nonempty) structure inside the dashed area.



(a) The unique graph with 6 vertices that attains the bound in Theorem 23



(b) A graph with one basis forced vertex and $\frac{n(n-1)}{2} - 5$ edges.

Fig. 9.

Thus, if G has basis forced vertices, then $\bar{G} \simeq P_5 \cup \bar{K}_{n-5}$. The graph G is illustrated in Fig. 9(a). According to Lemma 22, this graph does indeed have exactly two basis forced vertices. \square

It is a bit odd that the densest graphs that have basis forced vertices have two of them. If a graph G has one basis forced vertex, then $|E(G)| \leq \frac{n(n-1)}{2} - 5$. A graph that attains this bound is illustrated in Fig. 9(b). The black vertex is the basis forced vertex. The metric dimension of this graph is 2. It has two metric bases; in addition to the black vertex we must include either of the grey vertices.

If a graph has three or more basis forced vertices, then the densest graph that we have found has $\frac{n(n-1)}{2} - 7$ edges. It seems that in general the bound in Theorem 23 leaves room for improvement. Indeed, in Theorem 27 we will show a more general bound on the number of edges that depends also on the number of basis forced vertices that the graph contains.

4. The colour graph G_R

Let G be a graph and $R \subseteq V(G)$. Let $r \in R$. We denote

$$\mathcal{U}_R(r) = \{\{x, y\} \in V(G)^2 \mid d(r, x) \neq d(r, y) \text{ and } \forall t \in R \setminus \{r\}: d(t, x) = d(t, y)\}.$$

The set $\mathcal{U}_R(r)$ consists of the pairs of vertices for which r is the unique element in R that resolves the pairs.

We denote by G_R the graph with the same vertex set as G and the edge set

$$\bigcup_{r \in R} \mathcal{U}_R(r).$$

Each $r \in R$ is assigned a colour, and we colour the edges in G_R given by $\mathcal{U}_R(r)$ with the colour associated with r .

Example 24. Consider again the familiar graph G illustrated in Fig. 10(a). Let us construct the graph G_R with respect to the resolving set $R = \{r_1, r_2\}$. We have $\mathcal{U}_R(r_1) = \{\{r_1, v_1\}, \{r_1, v_3\}, \{v_1, v_3\}, \{v_2, v_4\}\}$ and $\mathcal{U}_R(r_2) = \{\{r_2, v_1\}, \{r_2, v_2\}, \{v_1, v_2\}, \{v_3, v_4\}\}$. When we assign black to r_1 and ‘dashed’ to r_2 , we can visualise the graph G_R as in Fig. 10(b).

Let us explore some basic properties of G_R . If there is no edge between x and y in G_R , then either R does not resolve x and y or there are at least two elements in R that resolve x and y . If R is a resolving set, then only the latter is possible.

Since both $r \in R$ and $s \in R$ resolve the pair r, s , the edge $\{r, s\}$ is not present in G_R . Consequently, the set R is independent in G_R . Moreover, if there does exist an edge incident to $r \in R$ in G_R , then that edge has the colour associated with r , since r resolves all pairs where r itself is included.

If the colour associated with some $r \in R$ is not present in G_R , then every pair that r resolves is resolved by some other element of R . Thus, the set $R \setminus \{r\}$ is a resolving set of G . Consequently, if R is a metric basis of G , then the graph G_R has at least one edge of the colour associated with each $r \in R$. In other words, the set $\mathcal{U}_R(r)$ is nonempty for all $r \in R$.



Fig. 10. The graph \$G\$ with the resolving set \$R = \{r_1, r_2\}\$ and the colour graph \$G_R\$.

Lemma 25. Let \$G\$ be a graph. The following properties hold.

- (i) A colour that appears in a cycle of \$G_R\$ appears at least twice in that cycle.
- (ii) Let \$R\$ be a resolving set of \$G\$ and \$x, y, z \in V(G)\$. If the edges \$\{x, y\}\$ and \$\{x, z\}\$ have the same colour in \$G_R\$, then the edge \$\{y, z\}\$ also has the same colour in \$G_R\$.
- (iii) If \$b \in V(G)\$ is a basis forced vertex of \$G\$ and \$R\$ is a metric basis of \$G\$, then the graph \$G_R\$ has at least two edges of the colour associated with \$b\$.
- (iv) If \$b \in V(G)\$ is a basis forced vertex of \$G\$ and \$R\$ is a metric basis of \$G\$, then the graph \$G_R\$ has at least one edge \$\{x, y\}\$, \$x, y \in V \setminus R\$, of the colour associated with \$b\$.

Proof. (i) Suppose to the contrary that the edge \$\{v_1, v_k\}\$ is the only edge of its colour (associated with \$r \in R\$) in the cycle \$v_1 v_2 v_3 \dots v_k v_1\$, \$k \ge 3\$. Since the colour of the edge \$\{v_i, v_{i+1}\}\$ where \$i \in \{1, \dots, k-1\}\$ is not the colour associated with \$r\$, we have \$d(r, v_i) = d(r, v_{i+1})\$ for all \$i \in \{1, \dots, k-1\}\$. Consequently, \$d(r, v_1) = d(r, v_k)\$ and \$\{v_1, v_k\} \notin \mathcal{U}_R(r)\$, a contradiction.

(ii) For all \$s \in R \setminus \{r\}\$ we have \$d(s, y) = d(s, x) = d(s, z)\$. As \$R\$ is a resolving set of \$G\$, we have \$d(r, y) \neq d(r, z)\$. Thus, \$\{y, z\} \in \mathcal{U}_R(r)\$.

(iii) Suppose \$\mathcal{U}_R(b) = \{u, v\}\$. Since every pair \$r, s \in R\$ is resolved by both \$r\$ and \$s\$, we can assume that \$v \notin R\$. The vertex \$v\$ resolves the pair \$\{u, v\}\$ and all other pairs are resolved by the elements of \$R \setminus \{b\}\$. Thus, the set \$R[b \leftarrow v]\$ is a metric basis of \$G\$, a contradiction.

(iv) Suppose to the contrary that no such edge exists. According to (iii) the graph \$G_R\$ has at least two edges of the colour associated with \$b\$. Let these edges be \$\{x_1, x_2\}\$ and \$\{y_1, y_2\}\$. Due to our assumption, both of these edges have at least one endpoint in \$R\$. If there is an edge incident to \$r \in R\$ in \$G_R\$, then that edge has the colour associated with \$r\$, since \$r\$ resolves every pair \$r, x\$, where \$x \in V(G)\$. Thus, we may assume that \$x_1 = b = y_1\$. Since every pair \$r, s \in R\$ is resolved by both \$r\$ and \$s\$, we have \$x_2 \notin R\$ and \$y_2 \notin R\$. Now, the edge \$\{x_2, y_2\}\$ has the colour associated with \$b\$ due to (ii), a contradiction. \$\square\$

Suppose that \$R\$ is a resolving set of \$G\$. Due to Lemma 25 (i) and (ii), all cliques of \$G_R\$ are monochromatic. The subgraph of \$G_R\$ that consists of only the edges of the colour associated with \$r \in R\$ is a disjoint union of cliques.

Lemma 17 states that universal vertices are not basis forced vertices. We can obtain the same result using Lemma 25 (iv): if \$u\$ is a universal vertex of \$G\$ and \$R\$ is a metric basis of \$G\$ that contains \$u\$, then \$d(u, x) = d(u, y)\$ for all \$x, y \in V(G) \setminus R\$ and (iv) cannot be satisfied for \$u\$.

Theorem 26. If \$G\$ is a graph with \$n\$ vertices and \$k > 0\$ basis forced vertices, then \$k \le n - \dim(G) - 1\$. Moreover, we have \$k \le \frac{n-1}{2}\$.

Proof. Let \$R\$ be a metric basis of \$G\$. Denote \$U = V \setminus R\$ and \$m = |U|\$. Due to Lemma 25 (iv), \$G_R[U]\$ contains at least \$k\$ differently coloured edges. Let us consider \$k\$ differently coloured edges in \$G_R[U]\$. If \$m \le k\$, then some (or all) of these edges form a cycle that contradicts Lemma 25 (i). Thus, \$m \ge k + 1\$ and \$k \le m - 1 = n - \dim(G) - 1\$. Since \$k \le \dim(G)\$, we also have \$k \le \frac{n-1}{2}\$. \$\square\$

We have not found a graph that attains this bound. However, for the graph \$G\$ described in Lemma 22 we have \$k = n - \dim(G) - 2\$ when \$k = 2\$. Indeed, let \$G\$ be such that \$\bar{G} \simeq P_5 \cup \bar{K}_{n-5}\$. According to Lemma 22, the graph \$G\$ has two basis forced vertices and \$\dim(G) = n - 4\$.

Theorem 27. Let \$G\$ be a connected graph with \$n \ge 3\$ vertices. If \$G\$ has \$k > 0\$ basis forced vertices, then

$$|E(G)| \leq \frac{n(n-1)}{2} - 2k.$$

Proof. Let us denote the basis forced vertices of \$G\$ by \$v_1, \dots, v_k\$. Due to Lemma 17, none of the vertices \$v_i\$ are universal vertices. Thus, for each \$v_i\$ there exists an \$x \in V(G)\$ such that the edge \$v_i x\$ is an edge of \$\bar{G}\$.

Let \$R\$ be a metric basis of \$G\$. Due to Lemma 25 (iv), for each \$v_i\$ there exist distinct \$x, y \in V(G) \setminus R\$ such that \$d(v_i, x) \neq d(v_i, y)\$ and \$d(r, x) = d(r, y)\$ for all \$r \in R \setminus \{v_i\}\$. Thus, at least one of the edges \$v_i x\$ and \$v_i y\$ is an edge in \$\bar{G}\$. Therefore, every basis forced vertex \$v_i\$ is not adjacent to some vertex \$u\$ that is not a basis forced vertex.

Suppose that $v_i u$ is the only edge in \bar{G} from v_i to $V(G) \setminus R$. Now, $d(v_i, v) = 1$ for all $v \in V(G) \setminus R$ such that $v \neq u$.

As we stated earlier, according to Lemma 25 (iv) there exist distinct $x, y \in V(G) \setminus R$ such that $d(v_i, x) \neq d(v_i, y)$ and $d(r, x) = d(r, y)$ for all $r \in R \setminus \{v_i\}$. Now, either $x = u$ or $y = u$, since otherwise we have $d(v_i, x) = 1 = d(v_i, y)$, a contradiction.

Suppose without loss of generality $x = u$. Then y is unique due to Lemma 25 (ii) and the fact that $d(v_i, v) = 1$ for all $v \in V(G) \setminus R, v \neq u$. According to Lemma 25 (iii) the graph G_R has at least two edges of the colour associated with v_i . The edge uy is one such edge. However, the edge uy is the only edge within $V(G) \setminus R$ of the colour associated with v_i . Since the set R is independent in G_R , there exists a vertex $w \in V(G) \setminus R$ such that the edge $v_i w$ has the colour associated with v_i in G_R .

Suppose that $w \in \{u, y\}$. Then according to Lemma 25 (ii) both edges $v_i u$ and $v_i y$ have the colour associated with v_i in G_R . Now, the three edges $uy, v_i u$ and $v_i y$ are the only edges of the colour associated with v_i in G_R . Otherwise, we have a contradiction due to Lemma 25 (ii) and the fact that $d(v_i, v) = 1$ for all $v \in V(G) \setminus R, v \neq u$. Let us show that now v_i is not a basis forced vertex. If the set $R[v_i \leftarrow u]$ is a metric basis, then we are done. Hence, suppose that the set $R[v_i \leftarrow u]$ is not a metric basis of G . The only pairs not resolved by $R \setminus \{v_i\}$ are $\{u, y\}, \{v_i, u\}$ and $\{v_i, y\}$. Since u clearly resolves the first two pairs, we have $d(u, v_i) = d(u, y)$. However, now $d(y, u) \neq d(y, v_i) = 1$, and the set $R[v_i \leftarrow y]$ is a metric basis of G . Thus, $R[v_i \leftarrow u]$ or $R[v_i \leftarrow y]$ is a metric basis of G , and v_i is not a basis forced vertex.

Suppose then that $w \notin \{u, y\}$. Now the vertex w is unique. Otherwise, we again have a contradiction due to Lemma 25 (ii) and the fact that $d(v_i, v) = 1$ for all $v \in V(G) \setminus R, v \neq u$. Let us show that now v_i is not a basis forced vertex. The set $R \setminus \{v_i\}$ resolves all but two pairs of vertices: v_i, w and u, y . Now, if some vertex $z \neq v_i$ resolves both of these pairs, then the set $R[v_i \leftarrow z]$ is a metric basis of G and v_i is not a basis forced vertex. Let us show that such a vertex z exists.

If $d(w, u) \neq d(w, y)$, then the vertex w resolves the two aforementioned pairs and $R[v_i \leftarrow w]$ is a metric basis of G . Suppose that $d(w, u) = d(w, y)$. Since $d(v_i, w) = 1 = d(v_i, y)$, we have $d(w, y) \leq d(w, v_i) + d(v_i, y) = 2$. If $d(w, u) = d(w, y) = 1$, then $d(u, v_i) \neq d(u, w)$ and the set $R[v_i \leftarrow u]$ is a metric basis of G . If $d(w, u) = d(w, y) = 2$, then $d(y, v_i) \neq d(y, w)$ and the set $R[v_i \leftarrow y]$ is a metric basis of G . In all cases, the set $R[v_i \leftarrow z]$ is a metric basis of G for some $z \in \{w, u, y\}$, and thus the vertex v_i is not a basis forced vertex of G .

In conclusion, for each basis forced vertex v_i there exist at least two edges from v_i to $V(G) \setminus R$ in \bar{G} . Since G has k basis forced vertices, we have $|E(G)| \leq \frac{n(n-1)}{2} - 2k$. \square

Let G be a graph such that $\bar{G} \simeq P_5 \cup \dots \cup P_5 \cup \bar{K}_m$, where P_5 appears $\frac{k}{2}$ times in the disjoint graph union. According to Lemma 22 the graph G has k basis forced vertices, two in each complement of P_5 . The following corollary is now immediate.

Corollary 28. For every even positive integer k there exists a graph G with k basis forced vertices and

$$|E(G)| = \frac{n(n-1)}{2} - 2k$$

where n is the number of vertices of G .

The bound in Theorem 27 does not seem attainable for odd k . However, we have found a graph family whose members have an odd (≥ 3) number of basis forced vertices and $\frac{n(n-1)}{2} - 2k - 1$ edges. Let H be the graph we obtain by attaching a pendant to the middle vertex of the path $v_1 \dots v_7$. If G is a graph such that $\bar{G} \simeq H \cup \bar{K}_m$, then the graph G has $\frac{n(n-1)}{2} - 2 \cdot 3 - 1$ edges and three basis forced vertices: v_2, v_4 and v_6 (this can be shown using similar techniques as in the proof of Lemma 22). Moreover, it is possible to show that the graph G for which $\bar{G} \simeq H \cup P_5 \cup \dots \cup P_5 \cup \bar{K}_m$ where P_5 appears $\frac{k-3}{2}$ times has three basis forced vertices in $V(H)$ and two basis forced vertices in each $V(P_5)$. All in all, the graph G has $3 + 2 \cdot \frac{k-3}{2} = k$ basis forced vertices and $\frac{n(n-1)}{2} - 7 - 4 \cdot \frac{k-3}{2} = \frac{n(n-1)}{2} - 2k - 1$ edges.

5. Algorithmic complexity

In this section, we consider the algorithmic complexity of determining whether a given vertex is a basis forced or void vertex of a metric basis. In particular, we show that the first problem is co-NP-hard and the latter problem is NP-hard. The proofs are based on a polynomial-time reduction from the well-known 3-SAT problem. Previously, in [15], it has been shown that given an arbitrary graph $G = (V, E)$ and an integer k , it is NP-complete to decide whether $\dim(G) \leq k$. The reductions of the proofs of this section are inspired by the one presented in [15]. In what follows, this reduction is briefly recapped.

For the 3-SAT problem, denote the set of variables by $X = \{x_1, x_2, \dots, x_n\}$ and the set of literals by $U = \{x_1, x_2, \dots, x_n, \bar{x}_1, \bar{x}_2, \dots, \bar{x}_n\}$, where \bar{x}_i denotes the negation of the variable x_i . Let F be an instance of the 3-SAT problem; more precisely, let F be a formula $F = C_1 \wedge C_2 \wedge \dots \wedge C_m$, where each clause C_j contains exactly three literals, i.e., each clause is of the form $C_j = u_{j,1} \vee u_{j,2} \vee u_{j,3}$, where $u_{j,1}, u_{j,2}, u_{j,3} \in U$. Based on the given formula F , we form a graph $G = (V, E)$ as follows:

- For each variable $x_i \in X$, we construct a variable gadget of x_i with vertices $a_{i,1}, a_{i,2}, b_{i,1}, b_{i,2}, T_i$ and F_i and edges as given in Fig. 11(a).



Fig. 11.

- For each clause $C_j = u_{j,1} \vee u_{j,2} \vee u_{j,3}$, we construct a clause gadget C_j with vertices $c_{j,1}, c_{j,2}, c_{j,3}, c_{j,4}$, and $c_{j,5}$ and edges as given in Fig. 11(b). Moreover, if $u_{j,k} = x_i$ (where $k \in \{1, 2, 3\}$), then $c_{j,3}$ is adjacent to F_i , else $u_{j,k} = \bar{x}_i$ and $c_{j,3}$ is adjacent to T_i . In addition, $c_{j,3}$ is adjacent to both T_i and F_i for all variables x_i not occurring in C_j and $c_{j,1}$ is adjacent to both T_i and F_i for all variables x_i occurring in C_j .

It is clear that the graph G can be constructed in polynomial time. In [15], it is shown that the formula F is satisfiable if and only if $\dim(G) = n + m$. This implies that the problem of deciding whether $\dim(G) \leq k$ is NP-complete.

Inspired by the previous reduction, we show in the following theorem that determining whether a given vertex is a basis forced one or a void one are algorithmically difficult.

Theorem 29. Let G be a graph and u be a vertex of G .

- Deciding whether u is a basis forced vertex of G is a co-NP-hard problem.
- Deciding whether u is a void vertex of G is an NP-hard problem.

Proof. In order to prove the first claim (i), we show that the problem of deciding whether a given 3-SAT formula is *not satisfiable* – a co-NP-complete problem – can be reduced in polynomial time to the problem of determining if a given vertex is basis forced one of a graph. For the second claim (ii), we similarly prove that the problem of deciding whether a given 3-SAT formula is *satisfiable* – an NP-complete problem – can be reduced in polynomial time to the problem of determining if a given vertex is basis void one of a graph.

Using the notation introduced above, let F be an instance of the 3-SAT problem. Based on the given formula F , we form a graph $G' = (V', E')$ as follows:

- Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two disjoint copies of the graph G constructed in the reduction of [15] (with respect to F). Denote the vertices of the graphs by $a_{i,1}^k, a_{i,2}^k, b_{i,1}^k, b_{i,2}^k, T_i^k, F_i^k, c_{j,1}^k, c_{j,2}^k, c_{j,3}^k, c_{j,4}^k$ and $c_{j,5}^k$, where $k \in \{1, 2\}$.
- Furthermore, an edge is added from each $c_{j,1}^1$ and $c_{j,3}^1$ to T_i^1 and F_i^1 for all $i \in \{1, \dots, n\}$. Analogously, an edge is added from each $c_{j,1}^2$ and $c_{j,3}^2$ to T_i^2 and F_i^2 for all $i \in \{1, \dots, n\}$.
- Finally, add a vertex w such that it is adjacent to $c_{j,3}^k$ for all $k \in \{1, 2\}$ and $j \in \{1, \dots, m\}$.

It is immediate that the graph $G' = (V', E') = (V_1 \cup V_2 \cup \{w\}, E')$ can be constructed in polynomial time. Before diving into the proofs of (i) and (ii), we need to discuss some preliminary results.

Let R be a resolving set of G' . We first present the following simple observations, which mimic the ones given in Lemmas A.2 and A.3 of [15]:

- Observe that for $k \in \{1, 2\}$ and $i \in \{1, \dots, n\}$, at least one of the vertices $a_{i,1}^k, a_{i,2}^k, b_{i,1}^k$ and $b_{i,2}^k$ belongs to the resolving set R . Indeed, suppose to the contrary that $R \cap \{a_{i,1}^k, a_{i,2}^k, b_{i,1}^k, b_{i,2}^k\} = \emptyset$. This implies that $a_{i,1}^k$ and $a_{i,2}^k$ are not resolved (a contradiction) since the corresponding variable gadget is connected to the rest of the graph only through the vertices T_i^k and F_i^k .
- Observe that for $k \in \{1, 2\}$ and $j \in \{1, \dots, m\}$, at least one of the vertices $c_{j,4}^k$ and $c_{j,5}^k$ belongs to the resolving set R . Indeed, if $R \cap \{c_{j,4}^k, c_{j,5}^k\} = \emptyset$, then $c_{j,4}^k$ and $c_{j,5}^k$ are clearly not resolved (a contradiction).

By the observations, we immediately obtain that the metric dimension of G' is at least $2n + 2m$. In what follows, we show that $\dim(G') = 2n + 2m$ if and only if the formula F is satisfiable.

Let us first show that if the formula F is satisfiable, then the metric dimension $\dim(G') = 2n + 2m$. Let A be a satisfiable truth assignment of F . Construct then a set C as follows: C consists of all the vertices $c_{j,4}^k$, where $k \in \{1, 2\}$ and $j \in \{1, \dots, m\}$, and if the assignment of x_i is *true* in A , then $a_{i,1}^1$ and $a_{i,1}^2$ belong to C , and otherwise $b_{i,1}^1$ and $b_{i,1}^2$ are in C . We immediately notice that C contains exactly $2n + 2m$ vertices. In what follows, we show that C is a resolving set of G' . For this purpose, we first observe that all the pairs $x, y \in V'$ of distinct vertices except $c_{j,1}^k$ and $c_{j,3}^k$ are resolved by some element of C :

- Suppose first that $x = c_{j,l}^k$ with $l \in \{1, \dots, 5\}$ and y is any other vertex not in the same clause gadget as x . Then $d(c_{j,4}^k, x) \leq 2$ and $d(c_{j,4}^k, y) > 2$, and we are immediately done. Furthermore, if y belongs to the same clause gadget as x , then some element of C clearly resolves x and y unless the vertices are $c_{j,1}^k$ and $c_{j,3}^k$.

- Suppose then that $x = w$. Now the distance of x to each vertex of C is exactly three. Clearly, this is not the case for any other vertex y and we are done.
- Finally, suppose that x belongs to some variable gadget. Without loss of generality, we may assume that x belongs to the copy of the gadget of x_1 in G_1 and that $a_{1,1}^1 \in C$. Now it is immediate that if y is in the same variable gadget as x , then either $a_{1,1}^1$ or every $c_{j,4}^2$ resolves x and y . By the previous cases, we are also immediately done if y does not belong to any variable gadget. Hence, we may assume that y belongs to some variable gadget other than the one of x_1 . Now, if $x \neq b_{1,2}^1$, then $d(a_{1,1}^1, x) \leq 2$ and $d(a_{1,1}^1, y) > 2$. Furthermore, if $x = b_{1,2}^1$ and $d(a_{1,1}^1, x) = d(a_{1,1}^1, y)$, then $d(a_{1,1}^1, x) = d(a_{1,1}^1, y) = 3$ and y is equal to T_i^k or F_i^k for some k and i . This further implies that there exists a vertex $c \in C$ in the same variable gadget as y such that $d(c, y) \leq 2$ and $d(c, x) > 2$. Hence, we are done.

Thus, it is enough to consider the pairs of vertices $x = c_{j,1}^k$ and $y = c_{j,3}^k$. It is immediate that the distance of $c_{j,1}^k$ to each vertex of C in the variable gadgets corresponding to $u_{j,1}, u_{j,2}$ and $u_{j,3}$ is equal to 2. However, by the fact that A is a satisfiable truth assignment of F and the construction of C , the distance of $c_{j,3}^k$ to some vertex of C in the variable gadgets corresponding to $u_{j,1}, u_{j,2}$ and $u_{j,3}$ is equal to 3. Thus, in conclusion, C is a resolving set of G' and $\dim(G') = 2n + 2m$.

Let us then show that if the metric dimension of G' is $2n + 2m$, then the formula F is satisfiable. Let C be a resolving set of G' with $2n + 2m$ vertices. Due to the observations (a) and (b), we know that for each $k \in \{1, 2\}$ and $i \in \{1, \dots, n\}$ exactly one of the vertices $a_{i,1}^k, a_{i,2}^k, b_{i,1}^k$ and $b_{i,2}^k$ belongs to C and for each $k \in \{1, 2\}$ and $j \in \{1, \dots, m\}$ exactly one of the vertices $c_{j,4}^k$ and $c_{j,5}^k$ belongs to C . Form then a truth assignment A of F as follows: if $a_{i,1}^1$ or $a_{i,2}^1$ belongs to C , then set the variable x_i to be *true*, else ($b_{i,1}^1$ or $b_{i,2}^1$ belongs to C and) set x_i to be *false*. In what follows, we show that the truth assignment A satisfies the formula F . Suppose to the contrary that a clause C_j is not satisfied by A . This implies that the distance of $c_{j,1}^1$ and $c_{j,3}^1$ to each vertex of C in the variable gadgets corresponding to $u_{j,1}, u_{j,2}$ and $u_{j,3}$ is equal to 2. Furthermore, it is straightforward to verify that the distance of $c_{j,1}^1$ and $c_{j,3}^1$ to each vertex of C in other variable gadgets is also equal to 2 and that their distances are equal to 4 to any other vertices of C (in the clause gadgets). Thus, they are not resolved by any element of C and a contradiction follows. Thus, the truth assignment A satisfies the formula F .

Let us next show that $\dim(G') \leq 2n + 2m + 1$ regardless of the existence of a satisfiable truth assignment for the formula F . For this purpose, let C be a set consisting of w and all the vertices $a_{i,1}^k$ and $c_{j,4}^k$, where $k \in \{1, 2\}, i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$. Clearly, the cardinality of C is equal to $2n + 2m + 1$. As above, it can be shown that a pair of distinct vertices is resolved even without taking into account the vertex w unless the pair is $c_{j,1}^k$ and $c_{j,3}^k$. However, it is immediate that they are resolved since $d(w, c_{j,1}^k) = 3 \neq 1 = d(w, c_{j,3}^k)$. Therefore, the set C is a resolving set of G' and $\dim(G') \leq 2n + 2m + 1$.

(i) Now we are ready to prove that w is a basis forced vertex of G' if and only if the formula F is not satisfiable. Observe first that if w is a basis forced vertex of G' , then by the observations (a) and (b), we obtain that $\dim(G') \geq 2n + 2m + 1$. Therefore, we have $\dim(G') = 2n + 2m + 1$ and F is not satisfiable. For the other direction, assume that F is not satisfiable. Hence, we have $\dim(G') \neq 2n + 2m$ implying $\dim(G') = 2n + 2m + 1$. Suppose to the contrary that w is not a basis forced vertex of G' and there exists a metric basis C of G' with $2n + 2m + 1$ vertices such that $w \notin C$. By the observations (a) and (b), we obtain that either G_1 or G_2 contains exactly $n + m$ vertices of C ; without loss of generality, we may assume that G_1 is such a graph. Notice that $c_{j,1}^1$ and $c_{j,3}^1$ are resolved for all $j \in \{1, \dots, m\}$ since C is a metric basis of G' . As above, it can now be shown that F is satisfiable (a contradiction). Therefore, w is a basis forced vertex of G' . Thus, there exists a polynomial-time reduction of the complement of the 3-SAT problem to the problem of deciding whether a given vertex is a basis forced vertex. Hence, the studied problem is co-NP-hard.

(ii) Let us then show that w is a void vertex of G' if and only if the formula F is satisfiable. Observe first that if F is satisfiable, then $\dim(G') = 2n + 2m$. Hence, if C is any metric basis of G' (with cardinality $2n + 2m$), then w does not belong to C by the observations (a) and (b). Therefore, w is a void vertex of G' . For the other direction, assume that w is a void vertex of G' . Recall that F is satisfiable if and only if $\dim(G') = 2n + 2m$. Suppose to the contrary that F is not satisfiable, i.e., $\dim(G') \neq 2n + 2m$. This implies that $\dim(G') = 2n + 2m + 1$. Let C be a metric basis of G' (with cardinality $2n + 2m + 1$ and $w \notin C$). As above, we obtain that G_1 or G_2 contains exactly $n + m$ vertices of C ; without loss of generality, we may assume that G_1 is such a graph. Analogously, as in the case (i), we can show that F is satisfiable since $w \notin C$ (a contradiction). Hence, if w is a void vertex, then F is satisfiable. Thus, there exists a polynomial-time reduction of the 3-SAT problem to the problem of deciding whether a given vertex is a void vertex. Hence, the studied problem is NP-hard. \square

6. Future works

Here are some open problems related to the questions in this paper.

- Find graphs with k basis forced vertices but fewer edges than in our construction in Section 3.1.
- Find a characterisation of graphs with k basis forced vertices and metric dimension k .
- We have not found any graph that attains the bound $k \leq n - \dim(G) - 1$ established in Theorem 26. Further research is needed to see whether there indeed exists a graph that attains this bound or if we can improve this bound. (Note that we have $k = n - \dim(G) - 2$ when $k = 2$ for the graph described in Lemma 22.)

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