APOLLONIAN METRIC, UNIFORMITY AND GROMOV HYPERBOLICITY

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ABSTRACT. The main purpose of this paper is to investigate the properties of a mapping which is required to be roughly bilipschitz with respect to the Apollonian metric (roughly Apollonian bilipschitz) of its domain. We prove that under these mappings the uniformity, φ -uniformity and δ -hyperbolicity (in the sense of Gromov with respect to quasihyperbolic metric) of proper domains of \mathbb{R}^n are invariant. As applications, we give four equivalent conditions for a quasiconformal mapping which is defined on a uniform domain to be roughly Apollonian bilipschitz, and we conclude that φ -uniformity is invariant under quasimöbius mappings.

1. INTRODUCTION AND MAIN RESULTS

In geometric function theory, one mainly investigates the interplay between analytic properties of mappings and geometric properties of sets and domains. A key question is how to measure the distance between two points x, y in a proper subdomain $G \subset \mathbb{R}^n$. Instead of using distance functions which measure the position of the points with respect to each other, such as Euclidean and chordal metrics, it is more useful to take into account also the position of the points with respect to the boundary of the domain. Many authors have used this idea to define metrics of hyperbolic type and to study the geometries defined by these metrics in domains. Some examples are the quasihyperbolic metric, Apollonian metric, the distance ratio metric, Seittenranta's metric, see [1, 6, 7, 9, 16, 28]. In particular, the quasihyperbolic metric has become a basic tool in geometric function theory and it has many important applications [6, 21].

Suppose that we are given a domain $G \subset \mathbb{R}^n$ and two metrics m_1 and m_2 on it. It is natural to study whether or not these metrics are comparable in some sense. It turns out that the comparison properties

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of metrics imply geometric properties of the domain: this idea was used by Gehring and Osgood [7] to characterise so called uniform domains, by Gehring and Hag [5] to study quasidisks, by Vuorinen [28] to define φ -uniform domains, by Hästö [8] to study comparison properties of so called Apollonian metric. Seittenranta [16] defined a Möbius invariant metric on subdomains of \mathbb{R}^n and, comparing this metric to Ferrand's metric, defined a Möbius invariant class of domains. In the general case, we could call domains with such a comparison property (m_1, m_2) uniform domains. Uniform domains and quasidisks form classes of domains, which have been studied by many authors. In spite of all this work, there are many pairs of function theoretically interesting metrics m_1, m_2 , for which practically nothing is known about (m_1, m_2) -uniform domains.

One of the key features of hyperbolic type metrics is the Gromov hyperbolicity property. It is well-known that the Gehring-Osgood jmetric and the quasihyperbolic metric of uniform domains are Gromov hyperbolic. We note that Hästö in [10] proved that the j-metric is always Gromov hyperbolic, but the *j*-metric is Gromov hyperbolic if and only if G has exactly one boundary point. In fact, in \mathbb{R}^n , many results in quasiconformal mappings can be explained through negative curvature, or "Gromov hyperbolicity". It would be interesting to know, what the precise relationship between the higher dimensional quasiconformal theory and the work of Gromov is. On the other hand, Gromov hyperbolicity for metric spaces is a coarse notion of negative curvature which yields a very satisfactory theory. It is natural to consider the properties of coarse maps with respect to the hyperbolic type metrics and the geometry of domains. In the spirit of this motivation, we mainly study a class of mappings which are roughly bilipschitz with respect to the Apollonian metric in \mathbb{R}^n .

In fact, the study of Apollonian metric and the so called Apollonian bilipschitz mapping, (i.e., bilipschitz mapping with respect to Apollonian metric) has been largely motivated and considered by questions about Apollonian isometries, which in turn was a continuation of work by Beardon [1], Gehring and Hag [6], Hästö and his collaborators [8, 9, 13, 11]. In order to make this paper more readable, we review some notations from [28] and [8].

We will consider domains (open connected non-empty sets) G in the Möbius space $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$. The Apollonian metric is defined by

$$\alpha_G(x,y) := \log \sup_{a,b \in \partial G} |a,y,x,b| \qquad \text{ where } |a,y,x,b| = \frac{|a-x||b-y|}{|a-y||b-x|},$$

for $x, y \in G \subsetneq \mathbb{R}^n$ with understanding that if $a = \infty$ then we set |a-x|/|a-y| = 1 and similarly for b. It is in fact a metric if ∂G is not contained in a hyperplane or sphere, as was noted by [1, Theorem 1.1].

In the paper [6] Gehring and Hag proved that a quasi-disk is invariant under a quasiconformal mapping which is also Apollonian bilipschitz. Along this line, Hästö [8] introduced A-uniform domains: A domain $G \subsetneq \mathbb{R}^n$ is said to be A-uniform with constant A_1 if for some constant $A_1 > 0$ and for every $x, y \in G$, we have $k_G(x, y) \leq A_1 \alpha_G(x, y)$, where $k_G(x, y)$ is the quasihyperbolic metric (for definition see Subsection 2.2) between x and y in G. A domain $G \subsetneq \mathbb{R}^n$ is said to be A-uniform if it is A-uniform with some constant $A_1 < \infty$. In particular, he proved the following result:

Theorem 1.1. ([8, Theorem 1.8]) Let $G \subsetneq \mathbb{R}^n$ be A-uniform and let $f: G \to G' \subsetneq \mathbb{R}^n$ be an Apollonian bilipschitz mapping. The following conditions are equivalent:

- (1) G' is A-uniform;
- (2) f is quasiconformal in G.

We note that Hästö in [8, Proposition 6.6] proved that a domain G is A-uniform if and only if G is L-quasi-isotropic (for definition see Subsection 2.19) and α_G is quasiconvex. So in this paper, we first complement Theorem 1.1 in the following way.

Theorem 1.2. Let $G \subsetneq \mathbb{R}^n$ be A-uniform and let $f : G \to G' \subsetneq \mathbb{R}^n$ be an Apollonian bilipschitz mapping. Then the following conditions are equivalent:

- (1) G' is A-uniform;
- (2) G' is L-quasi-isotropic;
- (3) f is quasiconformal in G;
- (4) f is quasimobius in G.

Furthermore, it follows from [8, Example 4.4 and Proposition 6.6] that the class of A-uniform domains is a proper subset of the class of uniform domains (see Subsection 2.2 for the definition) and thus a proper subset of the class of φ -uniform domains (for definition see Subsection 2.2). It is a natural question to consider whether or not there is an analogous result for uniform or φ -uniform domains as stated in Theorem 1.1. In particular,

are uniform or φ -uniform domains preserved by an Apollonian bilipschitz mapping which is also quasiconformal?

The main purpose of this paper is to deal with this question and we obtain that the uniformity, φ -uniformity and δ -hyperbolicity (in the sense of Gromov with respect to quasihyperbolic metric, for definition see Subsection 2.6) of proper domains of \mathbb{R}^n are invariant under roughly Apollonian bilipschitz mappings (see Subsection 2.11 for the definition) as follows.

Theorem 1.3. Let $G \subsetneq \mathbb{R}^n$ be a domain and let $f : G \to G' \subsetneq \mathbb{R}^n$ be an (M, C)-roughly Apollonian bilipschitz mapping. Then we have

- (1) If G is c-uniform, then G' is c_1 -uniform with c_1 depending only on c, n, C and M;
- If G is φ-uniform, then G' is φ'-uniform with φ' depending only on φ, n, C and M;
- (3) If G is δ-hyperbolic, then G' is δ'-hyperbolic with δ' depending only on δ, n, C and M.

We remark that in Theorem 1.3 the quasiconformality for the maps is not needed. Next, as an application of Theorem 1.3 we shall demonstrate four equivalence conditions for a quasiconformal mapping which is defined on a uniform domain to be roughly Apollonian bilipschitz.

Theorem 1.4. Let $G \subsetneq \mathbb{R}^n$ be a uniform domain and let $f : G \to G' \subsetneq \mathbb{R}^n$ be a quasiconformal mapping. Then the following conditions are equivalent:

- (1) f is a roughly Apollonian bilipschitz mapping in G;
- (2) G' is uniform;
- (3) $f: \overline{G} \to \overline{G'}$ is a homeomorphism and $f|_{\partial G}$ is quasimobius;
- (4) f is quasimöbius in G.

Moreover, one can obtain the following invariance of φ -uniformity of domains in \mathbb{R}^n under quasimobius mappings. Recently, Hästö, Klén, Sahoo and Vuorinen [12] studied the geometric properties of φ -uniform domains in \mathbb{R}^n . They proved that φ -uniform domains are preserved under quasiconformal mappings of \mathbb{R}^n . We restate this result in a stronger form which is more practical to check as follows.

Theorem 1.5. Let $G \subsetneq \mathbb{R}^n$ be a φ -uniform domain and let $f : G \to G' \subsetneq \mathbb{R}^n$ be a θ -quasimöbius homeomorphism. Then G' is φ' -uniform with φ' depending only on φ , θ and n.

The rest of this paper is organized as follows. In Section 2, we recall some definitions and preliminary results. Section 3 is devoted to the proofs of our main results.

2. Preliminaries

2.1. Notation. We denote by \mathbb{R}^n the Euclidean *n*-space and by $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ the one point compactification of \mathbb{R}^n , *G* and *G'* are proper domains in \mathbb{R}^n .

2.2. Uniform domains. In 1979, uniform domains were introduced by Martio and Sarvas [15]. A domain $G \subsetneq \mathbb{R}^n$ is called *uniform* provided there exists a constant c with the property that each pair of points $x, y \in G$ can be joined by a rectifiable curve γ in G satisfying

(1) $\ell(\gamma) \leq c |x - y|$, and

(2) $\min\{\ell(\gamma[x,z]), \ell(\gamma[z,y])\} \le c d_G(z) \text{ for all } z \in \gamma,$

where $d_G(z) = \text{dist}(z, \partial G)$, $\ell(\gamma)$ denotes the arc length of γ , $\gamma[x, z]$ the part of γ between x and z.

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There is an important characterization of uniform domains in terms of an inequality for j-metric

$$j_G(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d_G(x), d_G(y)\}}\right)$$

and the quasi-hyperbolic metric

$$k_G(x,y) = \inf \int_{\gamma} \frac{|dx|}{d_G(x)},$$

where the infimum is taken over all rectifiable curves joining x and y in G.

Theorem 2.3. ([7, Theorem 1]) A domain $G \subset \mathbb{R}^n$ is uniform if and only if there exist constants c and d such that for all $x, y \in G$

$$k_G(x,y) \le cj_G(x,y) + d.$$

This form of the definition for uniform domains is due to Gehring and Osgood [7] and subsequently, it was shown by Vuorinen [29, 2.50(2)] that the additive constant can be chosen to be zero. This observation leads to the definition of φ -uniform domains introduced in [29]. Let $\varphi : [0, \infty) \to [0, \infty)$ be a homeomorphism. A domain $G \subsetneq \mathbb{R}^n$ is called φ -uniform if for all x, y in G

$$k_G(x,y) \le \varphi(r_G(x,y))$$
 where $r_G(x,y) = \frac{|x-y|}{\min\{d_G(x), d_G(y)\}}.$

In order to give a simple criterion for φ -uniform domains, consider domains G satisfying the following property [29, Examples 2.50 (1)]: there exists a constant $C \geq 1$ such that each pair of points $x, y \in G$ can be joined by a rectifiable path $\gamma \in G$ with $\ell(\gamma) \leq C |x - y|$ and $\min\{d_G(x), d_G(y)\} \leq C d(\gamma, \partial G)$. Then G is φ -uniform with $\varphi(t) = C^2 t$. In particular, every convex domain is φ -uniform with $\varphi(t) = t$. However, in general, convex domains need not be uniform.

2.4. Natural domains. Suppose that $\emptyset \neq A \subset G \subsetneq \mathbb{R}^n$. We write

$$r_G(A) = \sup\{r_G(x, y) : x \in A, y \in A\}.$$

Clearly,

$$\frac{d(A)}{2d(A,\partial G)} \le r_G(A) \le \frac{d(A)}{d(A,\partial G)}$$

where d(A) denotes the diameter of set A and $d(A, \partial G)$ is the distance from set A to the boundary ∂G .

Let $\psi : [0, \infty) \to [0, \infty)$ be an increasing function. A domain $G \subsetneq \mathbb{R}^n$ is called ψ -natural if

$$k_G(A) \le \psi(r_G(A))$$

for every nonempty connected set $A \subset G$ with $r_G(A) < \infty$, where $k_G(A)$ denotes the quasihyperbolic diameter of set A.

We note that a φ -uniform domain is φ -natural, and every convex domain is ψ -natural with $\psi(t) = t$ (see, [20, Theorems 2.8 and 2.9]). In fact, the next result from [27] shows that the class of natural domains is fairly large. Note that the growth of the function $\psi_n(t)$ in Lemma 2.5 is $\approx t^n$.

Lemma 2.5. ([27, Corollary 2.18]) Every proper domain in \mathbb{R}^n is ψ_n -natural with ψ_n depending only on n.

It should be noted that Lemma 2.5 is only valid in the finite dimensional case. In an infinite dimensional Hilbert space, the broken tube construction in [26, 2.3] provides an example of a domain, which is not natural.

2.6. Gromov hyperbolic domains. A geodesic metric space X is called δ -hyperbolic, $\delta \geq 0$, if for all triples of geodesics [x, y], [y, z], [z, x] in X every point in [x, y] is within distance δ from $[y, z] \cup [z, x]$. The property is often expressed by saying that geodesic triangles in X are δ -thin. In general, we say that a space is *Gromov hyperbolic* if it is δ -hyperbolic for some δ .

We shall use the term *Gromov hyperbolic domain* (δ -hyperbolic) for those proper domains in \mathbb{R}^n that are Gromov hyperbolic in the quasi-hyperbolic metric.

Example 2.7. The real line is 0-hyperbolic. A classical example of a hyperbolic space is the Poincaré half space $x_n > 0$ in \mathbb{R}^n with the hyperbolic metric defined by the element of length $\frac{|dx|}{x_n}$. This space is δ -hyperbolic with $\delta = \log 3$ [3]. More generally, uniform domains in \mathbb{R}^n with the quasihyperbolic metric are hyperbolic [2, Theorem 1.11].

Some examples of nonhyperbolic domains are: (1) $G = \mathbb{R}^2 \setminus \{ne_1 : n \in \mathbb{Z}\}; (2) G = \{x \in \mathbb{R}^3 : 0 < x_3 < 1\} [24, 2.11].$

2.8. Quasimöbius mapping. Let X and Y be metric spaces. A quadruple in a space X is an ordered sequence Q = (a, b, c, d) of four distinct points in X. The cross ratio of Q is defined to be the number

$$\tau(Q) = |a, b, c, d| = \frac{|a - c|}{|a - d|} \cdot \frac{|b - d|}{|b - c|}.$$

Observe that the definition is extended in the well known manner to the case where one of the points is ∞ . For example,

$$|a, b, c, \infty| = \frac{|a - c|}{|b - c|}.$$

If $X_0 \subset \dot{X} = X \cup \{\infty\}$ and if $f : X_0 \to \dot{Y} = Y \cup \{\infty\}$ is an injective map, the image of a quadruple Q in X_0 is the quadruple fQ = (fa, fb, fc, fd).

Definition 2.9. Let $\theta : [0, \infty) \to [0, \infty)$ be a homeomorphism. An embedding $f : X_0 \to \dot{Y}$ is said to be θ -quasimöbius, or briefly θ -QM, if the inequality $\tau(f(Q)) \leq \theta(\tau(Q))$ holds for each quadruple in X_0 . In particular, if $\theta(t) = C \max\{t^{\lambda}, t^{1/\lambda}\}$, then we say that f is power quasimöbius.

2.10. **Remark.** ([22]) We remark that the inverse map f^{-1} of a θ -QM is θ' -QM with $\theta'(t) = \theta^{-1}(t^{-1})^{-1}$ for t > 0. If $f : A \to \dot{Y}$ is θ_1 -QM and $g : f(A) \to \dot{Z}$ is θ_2 -QM, then the composition $g \circ f$ is θ -QM with $\theta(t) = \theta_2(\theta_1(t))$. If f is θ -QM with $\theta(t) = t$, then we say that f is a Möbius map. In particular, the inversion u defined by $u(x) = \frac{x}{|x|^2}$ is Möbius in an inner product space.

2.11. Roughly bilipschitz mappings and quasiconformal mappings.

A homeomorphism $f : (G, m_G) \to (G', m_{G'})$ is said to be an *M*-roughly *C*-bilipschitz in the *m* metric, if $M \ge 1, C \ge 0$, and

$$\frac{m_G(x,y) - C}{M} \le m_{G'}(f(x), f(y)) \le Mm_G(x,y) + C$$

for all $x, y \in G$. A homeomorphism $f : G \to G'$ is said to be an (M, C)-roughly Apollonian bilipschitz, if it is M-roughly C-bilipschitz in the Apollonian metric. This means that f is a homeomorphism such that

$$\frac{\alpha_G(x,y) - C}{M} \le \alpha_{G'}(f(x), f(y)) \le M\alpha_G(x,y) + C$$

for all $x, y \in G$. Similarly, we say that a homeomorphism f is C-coarsely M-quasihyperbolic, abbreviated (M, C)-CQH if it is M-roughly C-bilipschitz in the quasihyperbolic metric. This means that f is a homeomorphism such that

$$\frac{k_G(x,y) - C}{M} \le k_{G'}(f(x), f(y)) \le Mk_G(x,y) + C$$

for all $x, y \in G$.

The basic theory of quasiconformal mappings in \mathbb{R}^n , $n \geq 2$ is given in Väisälä's book [18]. There are plenty of mutually equivalent definitions for quasiconformality in \mathbb{R}^n . In this paper we adopt the following simplified version of the metric definition. Let $n \geq 2$, let G and G' be domains in \mathbb{R}^n , and let $f: G \to G'$ be a homeomorphism. For $x \in G$, The *linear dilatation* of f at $x \in G$ is defined by

$$H_f(x) := \limsup_{r \to 0} \frac{\sup\{|f(x) - f(y)| : |x - y| = r\}}{\inf\{|f(x) - f(z)| : |x - z| = r\}}.$$

For $1 \leq K < \infty$, we say that $f : G \to G'$ is *K*-quasiconformal if $H_f(x) \leq K$ for all $x \in G$, and that f is quasiconformal if it is *K*-quasiconformal for some K.

For a K-quasiconformal mapping we have the following property.

Lemma 2.12. ([7, Theorem 3]) For $n \ge 2$, $K \ge 1$, there exist constants c and μ depending only on n and K with the following property. If $G, G' \subset \mathbb{R}^n$ and $f : G \to G'$ is a K-quasiconformal mapping, then for all $x, y \in G$,

$$k_{G'}(f(x), f(y)) \le c \max\{k_G(x, y), (k_G(x, y))^{\mu}\}.$$

The next result deals with the case when the mapping is defined in \mathbb{R}^n .

Lemma 2.13. ([12, Lemma 2.3]) For $n \ge 2$, $K \ge 1$, there exist constants c and μ depending only on n and K with the following property. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a K-quasiconformal mapping, $G, G' \subset \mathbb{R}^n$ are domains, and fG = G', then for all $x, y \in G$,

$$j_{G'}(f(x), f(y)) \le c \max\{j_G(x, y), (j_G(x, y))^{\mu}\}.$$

2.14. **Remark.** Let G_1 and G_2 be proper domains of \mathbb{R}^n . We know from Lemma 2.5 that G_i (i = 1, 2) is ψ_i -natural with ψ_i depending only on n. Suppose that $f: G_1 \to G_2$ is a K-quasiconformal mapping of \mathbb{R}^n which maps G_1 onto G_2 , then we see from Lemmas 2.12 and 2.13 that G_2 is ψ_2 -natural with $\psi_2 = \psi_2(\psi_1, n, K)$.

Moreover, we see from Lemma 2.12 and [19, Theorem 4.14] that a quasiconformal mapping is CQH, which we state as follows.

Lemma 2.15. ([7, Theorem 3] and [19, Theorem 4.14]) For $n \geq 2$, $K \geq 1$, there exist constants $M \geq 1$, C > 0 such that if $G, G' \subset \mathbb{R}^n$ and $f: G \to G'$ is K-quasiconformal, then f is (M, C)-CQH.

2.16. **Remark.** Let G be a proper domain of \mathbb{R}^n . We consider the Apollonian metric α_G , Seittenranta's metric δ_G which is defined as ([16])

$$\delta_G(x,y) = \log(1 + \sup_{a,b \in \partial G} |a, x, b, y|),$$

the metric $h_{G,c}$ $(c \ge 2)$ which is defined as ([4])

$$h_{G,c}(x,y) = \log(1 + c \frac{|x-y|}{\sqrt{d_G(x)d_G(y)}})$$

and j_G metric. We see from [16, Theorems 3.4 and 3.11] and [4, Lemma 4.4] that the inequalities

$$(2.17) j_G \le \delta_G \le 2j_G$$

(2.18)
$$\alpha_G \le \delta_G \le \log(e^{\alpha_G} + 2) \le \alpha_G + \log 3$$

and

$$\frac{c}{2(1+c)}j_G \le h_{G,c} \le cj_G$$

hold for every proper domain G of \mathbb{R}^n . Hence, the identity map id: $(G, m_1) \to (G, m_2)$ is roughly bilipschitz, where $m_1, m_2 \in \{\alpha_G, j_G, \delta_G, h_{G,c}\}$. 2.19. Quasi-isotropic. The concept of quasi-isotropy which is a kind of local comparison property was introduced by Hästö in [8], and was the focus of [9]. Let $G \subsetneq \mathbb{R}^n$. We recall that a metric space (G, d) is *L*-quasi-isotropic $(L \ge 1)$ [8] if

$$\limsup_{r \to 0} \frac{\sup\{d(x, z) : |x - z| = r\}}{\inf\{d(x, y) : |x - y| = r\}} \le L$$

for every $x \in G$, where |x - z| means the Euclidean distance of x and z. In this paper, we say a domain G is L-quasi-isotropic if (G, α_G) is L-quasi-isotropic.

3. The proofs of main results

3.1. **Basic lemmas.** In this section, we shall give the proofs of our main results. We first introduce some basic inequalities which are important to our proofs.

Lemma 3.2. Let $G \subsetneq \mathbb{R}^n$ be a domain.

(1) ([16, Theorems 3.4 and 3.11]) For all $x, y \in G$,

$$j_G(x,y) \le \alpha_G(x,y) + \log 3; \ \frac{1}{2}\alpha_G(x,y) \le j_G(x,y) \le k_G(x,y);$$

(2) ([21, Theorem 3.9]) If $|x - y| \le d_G(x)/2$ or $k_G(x, y) \le 1$ with $x, y \in G$, then

$$\frac{|x-y|}{2d_G(x)} \le k_G(x,y) \le \frac{2|x-y|}{d_G(x)}$$

(3) If G is a c-uniform domain, then we have

 $k_G(x, y) \le c_1 j_G(x, y) \le c_1(\alpha_G(x, y) + \log 3),$

where $c_1 = c_1(c)$. Moreover, for the uniform domain $G = \mathbb{R}^n \setminus \{0\}$ we note that there does not exist any constant $c_2 \ge 0$ such that $k_G(x, y) \le c_2 \alpha_G(x, y)$ holds for all $x, y \in G$.

P. Hästö [8] investigated domains G for which $\alpha_G(x, y) \ge K j_G(x, y)$. He found a sufficient condition on the domain under which this holds, in particular this sufficient condition fails for $\mathbb{R}^n \setminus \{0\}$ and requires that the boundary of the domain is "thick".

Lemma 3.3. Let $G, G' \subseteq \mathbb{R}^n$ be domains and let a homeomorphism $f: G \to G'$ be an (M, C)-roughly Apollonian bilipschitz mapping. Then $f: G \to G'$ is an (M', C')-CQH with (M', C') depending on M, C and n only.

Proof. We may assume that there are constants $M \ge 1$ and $C \ge 0$ such that $f : G \to G'$ is (M, C)-roughly Apollonian bilipschitz. Thanks to [21, Lemma 2.3] and by symmetry, we only need to estimate $k_{G'}(f(x), f(y))$ for all $x, y \in G$ with $k(x, y) \le \frac{1}{8}$, because (G, k_G) and $(G', k_{G'})$ are geodesic metric spaces and evidently *c*-quasi-convex with c = 1.

Towards this end, first by Lemma 3.2, we have

$$r_G(x,y) = \frac{|x-y|}{\min\{d_G(x), d_G(y)\}} \le 2k_G(x,y) \le \frac{1}{4},$$

and so the segment $A = [x, y] \subset G$. Moreover, we have

$$r_G(A) \le \frac{\operatorname{diam}(A)}{\operatorname{dist}(A, \partial G)} \le \frac{r_G(x, y)}{1 - r_G(x, y)} \le 1/2$$

Then for all $a, b \in A$, we get

$$r_G(a,b) \le r_G(A) \le 1/2$$

which, together with Lemma 3.2, implies that

$$\alpha_G(a,b) \le 2j_G(a,b) = 2\log(1 + r_G(a,b)) < 2\log 2.$$

Furthermore, on one hand, since $f: G \to G'$ is (M, C)-roughly Apollonian bilipschitz, we have

$$\alpha_{G'}(f(a), f(b)) \le M\alpha_G(a, b) + C < 2M\log 2 + C,$$

On the other hand, again by using Lemma 3.2, we obtain

$$r_{G'}(f(a), f(b)) = e^{j_{G'}(f(a), f(b))} - 1 \le e^{2M \log 2 + C + \log 3} - 1 =: L,$$

so $r_{G'}(f(A)) \leq L$. Hence we see from Lemma 2.5 that there is an increasing function $\psi_n : [0, \infty) \to [0, \infty)$ such that

$$k_{G'}(f(x), f(y)) \le k_{G'}(f(A)) \le \psi_n(r_{G'}(f(A)) \le \psi_n(L).$$

The proof is complete.

Lemma 3.4. Suppose that $f: G \to G'$ is a θ -quasimobius homeomorphism between two proper domains of \mathbb{R}^n , then f is an (M, C)-roughly Apollonian bilipschitz mapping with M, C depending only on θ .

Proof. We first observe from [22, Theorem 3.19] that f has a quasimobius extension $\overline{f}: \overline{G} \to \overline{G'}$. To show that f is roughly Apollonian bilipschitz, we only need to prove that f is power quasimobius, that is, there exist constants $C \ge 1$ and $\lambda \ge 1$ depending only on θ such that f is θ_1 -QM with $\theta_1(t) = C \max\{t^{\lambda}, t^{1/\lambda}\}$. Indeed, this can be seen as follows. For any $x, y \in G$ and $a, b \in \partial G$, we note that

$$|a, y, x, b| = |b, y, x, a|^{-1}$$
 and $\alpha_G(x, y) = \log \sup_{a, b \in \partial G} |a, y, x, b|.$

Without loss of generality we may assume that $|a, y, x, b| \ge 1$. Since f is θ_1 -QM with $\theta_1(t) = C \max\{t^{\lambda}, t^{1/\lambda}\}$, we have

 $\log |f(a), f(y), f(x), f(b)| \le \lambda \log |a, y, x, b| + \log C \le \lambda \alpha_G(x, y) + \log C$, so by the arbitrariness of $a, b \in \partial G$, we get

$$\alpha_{G'}(f(x), f(y)) \le \lambda \alpha_G(x, y) + \log C.$$

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Since the inverse map of power quasimöbius is also power quasimöbius, by symmetry, the assertion follows.

To this end, by auxiliary translations we may assume that $0 \in \partial G$ and that either $\overline{f}(0) = 0$ or $\overline{f}(0) = \infty$. Let u be the inversion $u(x) = \frac{x}{|x|^2}$. We note from Remark 2.10 that u is Möbius. If $\overline{f}(0) = 0$, we define $g: u(G) \to u(G')$ by $g(x) = u \circ f \circ u(x)$. If $\overline{f}(0) = \infty$, we define $g: u(G) \to G'$ by $g(x) = f \circ u(x)$. In both cases, we have that g is θ -QM. Since $g(x) \to \infty$ as $x \to \infty$, g is θ -QS, see [22, Theorem 3.10]. Moreover, by [17, Corollary 3,12] we have that g is θ_1 -QS with $\theta_1(t) = C \max\{t^{\lambda}, t^{1/\lambda}\}$, where $C \ge 1$ and $\lambda \ge 1$ depend only on θ . Hence, we get that f is θ_1 -QM with $\theta_1(t) = C \max\{t^{\lambda}, t^{1/\lambda}\}$ as desired. Hence the proof of this Lemma is complete.

3.5. The proof of Theorem 1.2. $(1) \Rightarrow (2)$: This implication follows from [8, Proposition 6.6].

 $(2) \Rightarrow (3)$: It follows from the assumption, [8, Corollary 5.4] and [8, Proposition 6.6] that there is $M \ge 1$ such that f is M-bilipschitz with respect to the quasihyperbolic metrics. Let $x_0 \in G$ and $r \in (0, \frac{d_G(x_0)}{2M})$. For all $x, y \in \mathbb{S}^{n-1}(x_0, r)$, we have $k_G(x, x_0) \le \frac{1}{M}$ by means of [21, Theorem 3.9], and so $k_{G'}(f(x), f(x_0)) \le 1$. Again by [21, Theorem 3.9], we obtain that

$$\frac{|f(x) - f(x_0)|}{2d_{G'}(f(x_0))} \le k_{G'}(f(x), f(x_0)) \le \frac{2|f(x) - f(x_0)|}{d_{G'}(f(x_0))}.$$

Hence we have

$$\limsup_{r \to 0+} \frac{|f(x) - f(x_0)|}{|f(y) - f(x_0)|} \leq \limsup_{r \to 0+} \frac{4k_{G'}(f(x), f(x_0))}{k_{G'}(f(y), f(x_0))} \leq \limsup_{r \to 0+} \frac{4M^2k_G(x, x_0)}{k_G(y, x_0)} \leq 16M^2,$$

as desired.

 $(3) \Rightarrow (1)$: This implication follows from Theorem 1.1.

 $(3) \Rightarrow (4)$: Assume that f is quasiconformal in G, then Theorem 1.1 yields that f(G) is A-uniform. Hence, we get from the fact "an A-uniform domain is uniform" and [22, Theorem 5.6] that f is quasimöbius, as desired.

 $(4) \Rightarrow (3)$: This implication follows from [22, Theorem 5.2].

3.6. The proof of Theorem 1.3. Let $f: G \to G'$ be roughly Apollonian bilipschitz. Then by Lemma 3.3 we may assume that f is (M, C)roughly Apollonian bilipschitz and (M, C)-CQH for some constants $M \ge 1$ and $C \ge 0$. Hence, we see from Lemma 3.2 that

(3.7)
$$j_G(x,y) \le 2M j_{G'}(f(x), f(y)) + C + \log 3.$$

We first prove part (1), that is, if G is uniform, then G' is uniform. Suppose that G is c-uniform for some constant $c \ge 1$. According to Theorem 2.3, there exist positive constants c_1 and c_2 such that

$$k_G(x,y) \le c_1 j_G(x,y) + c_2$$

for all $x, y \in G$. One computes from these facts that

$$\begin{aligned} k_{G'}(f(x), f(y)) &\leq M k_G(x, y) + C \\ &\leq M [c_1 j_G(x, y) + c_2] + C \\ &\leq c_1 M [M j_{G'}(f(x), f(y)) + C + \log 3] + c_2 M + C. \end{aligned}$$

Again by Theorem 2.3, we immediately see that G' = f(G) is uniform. Hence, part (1) holds.

Next, we prove part (2). Assume that G is φ -uniform, to prove G' is φ' -uniform, we only need to find a homeomorphism $\varphi' : [0, \infty) \to [0, \infty)$ such that

$$k_{G'}(f(x), f(y)) \le \varphi'(r_{G'}(f(x), f(y)))$$

for all $x, y \in G$. To this end, we divide the proof into two cases. Case A. $|f(x) - f(y)| \leq \frac{1}{2} \min\{d_{G'}(f(x)), d_{G'}(f(y))\}.$

Then by Lemma 3.2 we have

$$k_{G'}(f(x), f(y)) \le 2 \frac{|f(x) - f(y)|}{d_{G'}(f(x))} \le 2r_{G'}(f(x), f(y)),$$

which give the desired φ' with $\varphi'(t) = 2t$.

Case B.
$$|f(x) - f(y)| > \frac{1}{2} \min\{d_{G'}(f(x)), d_{G'}(f(y))\}$$
. Then

$$j_{G'}(f(x), f(y)) = \log(1 + \frac{|f(x) - f(y)|}{\min\{d_{G'}(f(x)), d_{G'}(f(y))\}}) > \log\frac{3}{2}.$$

Let $\varphi_1(t) = \varphi(e^t - 1)$. Then we see from (3.7) that

$$k_{G'}(f(x), f(y)) \leq Mk_G(x, y) + C \leq M\varphi_1(j_G(x, y)) + C \\ \leq M\varphi_1(2Mj_{G'}(f(x), f(y)) + C + \log 3))) + C \\ \leq M\varphi_1\left((2M + \frac{C + \log 3}{\log \frac{3}{2}})j_{G'}(f(x), f(y))\right) \\ + \frac{C}{\log \frac{3}{2}}j_{G'}(f(x), f(y)).$$

By letting $\varphi'(t) = M\varphi_1\left((2M + \frac{C + \log 3}{\log \frac{3}{2}})\log(1+t)\right) + \frac{C}{\log \frac{3}{2}}\log(1+t)$, we complete the proof in this case.

Combining Case A and Case B, we complete the proof of part (2).

Finally, we come to prove part (3). It follows from Lemma 3.3 and the fact that a Gromov hyperbolic domain under a CQH homeomorphism is still Gromov hyperbolic, see [2] (or [25, Theorem 3.18]). \Box

3.8. **Remark.** Let G be a proper domain of \mathbb{R}^n . We consider the Apollonian metric α_G , Seittenranta's metric δ_G , the metric $h_{G,c}$ ($c \ge 2$) and j_G metric. We see from Remark 2.16 and the proof of Lemma 3.3 that if we replace the Apollonian metric by $m_G \in \{j_G, \delta_G, h_{G,c}\}$, then we have

the following holds: If $f : (G, m_G) \to (G', m_{G'})$ is an *M*-roughly *C*bilipschitz mapping, then $f : G \to G'$ is an (M', C')-*CQH* with (M', C')depending on *M*, *C* and *n* only. Hence, we easily see that Theorem 1.3 is also true if we replace the Apollonian metric by $m_G \in \{j_G, \delta_G, h_{G,c}\}$.

3.9. The proof of Theorem 1.4. The equivalence of $(1) \Rightarrow (2) \Rightarrow (4) \Rightarrow (1)$ are from Theorem 1.3, [22, Theorem 5.6] and Lemma 3.4. It remains to show $(3) \Leftrightarrow (4)$.

The implication $(4) \Rightarrow (3)$ follows from [22, Theorem 3.19]; (3) $\Rightarrow (4)$: this implication follows from [23, Theorem 3.15].

3.10. The proof of Theorem 1.5. The proof follows from Theorem 1.3 and Lemma 3.4.

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