Abelian periods of factors of Sturmian words

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Abstract

We study the abelian period sets of Sturmian words, which are codings of irrational rotations on a one-dimensional torus. The main result states that the minimum abelian period of a factor of a Sturmian word of angle α with continued fraction expansion $[0; a_1, a_2, \ldots]$ is either tq_k with $1 \leq t \leq a_{k+1}$ (a multiple of a denominator q_k of a convergent of α) or $q_{k,\ell}$ (a denominator $q_{k,\ell}$ of a semiconvergent of α). This result generalizes a result of Fici et al. stating that the abelian period set of the Fibonacci word is the set of Fibonacci numbers. A characterization of the Fibonacci word in terms of its abelian period set is obtained as a corollary.

Keywords: Sturmian word, continued fraction, abelian equivalence, abelian period, singular word

1 Introduction

let $w = w_0 w_1 \cdots w_{|w|-1}$ to be a finite word of length $|w|$ composed of letters w_0, w_1, ..., $w_{|w|-1}$. The word w has period p if $w_i = w_{i+p}$ for all i with $0 \leq i \leq |w| - 1 - p$. For example, the word *abaab* has period 3. Periods of words have been extensively studied; see, e.g., [20, Ch. 8]. One famous result is the Theorem of Fine and Wilf which states that if a word w has two periods p and q and $|w| \geq p + q - \gcd(p, q)$, then w has period $\gcd(p, q)$ [12].

The majority of the research on periods has been about understanding the structure of periods of a single finite word. Much less attention has been paid to period sets. The period set of a (finite or infinite) given word is the set of minimum periods of all of its factors (subwords). For instance, the above word *abaab* has proper factors *a*, *b*, *ab*, *ba*, *aa*, *aba*, *baa*, *aab*, *abaa*, and *baab*, so its period set is $\{1, 2, 3\}$. It seems that the only papers written on period sets are the 2009 seminal paper [9] of J. Currie and K. Saari and the 2012 preprint [14] of D. Goč and J. Shallit. Currie and Saari study the period sets of infinite words. They show that the period set of the Thue-Morse word [2] is the set of positive integers, and they prove the following theorem on the period sets of Sturmian words, codings of irrational rotations on a one-dimensional torus.

Theorem 1.1. *[9, Cor. 3] The period set of a Sturmian word of slope α having continued fraction expansion $[0; a_1, a_2, \ldots]$ is $\{\ell q_k + q_{k-1} : k \geq 0, \ell = 1, \ldots, a_{k+1}\}$, where the sequence (q_k) is the sequence of denominators of convergents of α.*[1]

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[1]Sturmian words are binary, and the slope of a Sturmian word is the (irrational) frequency of the letter having frequency less than $\frac{1}{2}$.

When Theorem 1.1 is applied to the Fibonacci word whose slope has continued fraction expansion $[0; 2, \overline{1}]$ (by a bar, we indicate a repeating pattern), we obtain the following nice theorem.

Theorem 1.2. *[9, Cor. 4] The period set of the Fibonacci word is the set of Fibonacci numbers.*

In the 2016 paper [11] by the author and others, the period set of the Fibonacci word was studied with a generalized notion of a period called an abelian period, and an analogue of Theorem 1.2 was obtained in this generalized setting. The goal of this paper is to extend this result to all Sturmian words and obtain an analogue of Theorem 1.1 for abelian periods.

Sturmian words are central objects in combinatorics on words. Their study was initiated in the 1940 paper [22] by M. Morse and G. Hedlund. Sturmian words often exhibit extremal behavior among infinite words, and their properties have links to other areas of mathematics like number theory and discrete geometry. They admit many interesting combinatorial and dynamical generalizations. See [3, 13] and the references therein.

Two words u and v are called abelian equivalent if one is obtained from the other by permuting letters. If u_0, u_1, $\ldots$, u_{n-1} are abelian equivalent words of length m, then their concatenation $u_0 u_1 \cdots u_{n-1}$ is called an abelian power of period m and exponent n. For example, $aba \cdot baa \cdot aab$ is an abelian cube. This notion is a generalization of the concept of a power: a power is simply a repetition of the same word such as $aba \cdot aba \cdot aba$ (a cube). Recently it has been popular to generalize concepts and questions regarding ordinary powers and periods to this abelian setting. The foundational paper here is [27]; see [11] for additional references. For example, an appropriate generalization of the Theorem of Fine and Wilf was given in [8, 4, 31]. These papers naturally contain the definition of an abelian period, which we shall give next; cf. [30].

Let w be a finite word. Then w has abelian period m if w is a factor of an abelian power $u_0 u_1 \cdots u_{n-1}$ with $|u_0| = \ldots = |u_{n-1}| = m$. For example, the word *abaababa*, having minimum period 5, has abelian periods 2 and 3 because it is a factor of the abelian powers *babaababab* and *abaababaa* respectively. This indeed generalizes the concept of a period: a word has period p if and only if it is a factor of some power of a word of length p.

The abelian period set of an infinite word $\mathbf{w}$ is defined as the set of minimum abelian periods of its nonempty factors. As was done in [9] by Currie and Saari for the usual period set, we may now ask for a characterization of the abelian period set for a given word or class of words. For the Thue-Morse word, this is easy. The Thue-Morse word $\mathbf{t}$ is the fixed point of the substitution $0 \mapsto 01$, $1 \mapsto 10$ beginning with the letter 0, and it is clear that $\mathbf{t}$ is an infinite concatenation of the words 01 and 10. Thus every factor of $\mathbf{t}$ has abelian period 2. The minimum abelian period can equal 1, but this happens only for finitely many factors because 000 and 111 do not occur in $\mathbf{t}$. Hence the abelian period set of $\mathbf{t}$ is $\{1, 2\}$. This should be compared with [9, Thm. 2]: the period set of $\mathbf{t}$ is the set of positive integers.

Characterizing the abelian period sets of Sturmian words is significantly harder. The following result was proved in [11] for the Fibonacci word (which can be said to be the simplest Sturmian word). It should be compared with Theorem 1.2.

Theorem 1.3. *[11, Thm. 6.9], [11, Thm. 6.12] The abelian period set of the Fibonacci word is the set of Fibonacci numbers.*

What are then the abelian period sets of other Sturmian words? By simply replacing the word "period set" with "abelian period set" in the statement of Theorem 1.1 yields a false statement. Indeed, it was observed in [11, Remark 6.11] that, for example, the factor

$$00101 \cdot 0010010010100100100101001001 00 \cdot 10100,$$

of a Sturmian word of slope $[0; \overline{2, 1}]$ has minimum abelian period 6, which is not of the form $\ell q_k + q_{k-1}$ for this slope. This example showed that the proof of Theorem 1.3 in [11] is not generalizable to all Sturmian words. In this paper, we present new ideas that work for all Sturmian words and prove the following result, which is the main result of this paper.

Theorem 1.4. *If m is the minimum abelian period of a nonempty factor of a Sturmian word of slope α having continued fraction expansion $[0; a_1, a_2, \ldots]$, then either $m = tq_k$ for some $k \geq 0$ and some t such that $1 \leq t \leq a_{k+1}$ or $m = \ell q_k + q_{k-1}$ for some $k \geq 1$ and some ℓ such that $1 \leq \ell < a_{k+1}$, where the sequence (q_k) is the sequence of denominators of convergents of α.*

Theorem 1.4 essentially says that certain multiples of the numbers q_k must also be allowed as minimum abelian periods. Theorem 1.4 implies Theorem 1.3 (see the end of Section 5).

Notice that Theorem 1.4 does not characterize the abelian period sets completely. Indeed, we shall see at the end of Section 5 that the set of possible minimum abelian periods given by Theorem 1.4 can be unnecessarily large. The complete answer seems to depend on the slope α in a complicated way. To us Theorem 1.4 seems to be the best result obtainable without additional assumptions about the arithmetical nature of the slope α. Because of this, we leave the complete characterization open.

Theorem 1.4 allows an interesting characterization of the Fibonacci subshift, the shift orbit closure of the Fibonacci word, as the Sturmian subshift of slope α whose language $\mathcal{L}(\alpha)$ has the following property: the minimum abelian period of each $w \in \mathcal{L}(\alpha)$ is a denominator of a convergent of α. See Theorem 5.9 at the end of Section 5. This adds yet another property to the rather long list of extremal properties of the Fibonacci word [7, 10, 29].

Even though the problems considered in this paper have their background in combinatorics and formal languages, a large part of the proofs are completely number-theoretic. It was already observed in [11] (and independently in [28]) that abelian powers and their exponents in Sturmian words can be studied effectively using continued fractions; in fact it is almost impossible to do without them. We continue to use this powerful tool. We give combinatorial arguments to derive a certain inequality which must hold if a given number is the minimum abelian period of some factor. Then we proceed to study the inequality using continued fractions with little combinatorics involved. Some of the intermediate results presented could be of independent interest in the theory of continued fractions.

The paper is organized as follows. In Section 2, we give the necessary definitions and background information on continued fractions, Sturmian words, and abelian equivalence. Auxiliary results needed for the main proofs are then presented in Section 3. The central proof ideas and derivation of the main inequality are given in Section 4; the actual proofs of the main results are presented in Section 5. We conclude the paper by briefly considering the so-called minimum k-abelian periods of factors of Sturmian words in Section 6; this is a further generalization of the notion of a period.

2 Preliminaries

We shall use standard notions and notation from combinatorics on words. These are found in, e.g., [20], and we briefly repeat what we need here.

An *alphabet* is a finite nonempty set of *letters*. A *word* $a_0 a_1 \cdots a_{n-1}$ of length n over A is a finite sequence of letters of A. We refer to the empty word with the symbol ε. The length of a word w is denoted by $|w|$. In this paper, we only consider binary words, and we take them to be over the alphabet $\{0, 1\}$. By $|w|_0$ (resp. $|w|_1$), we refer to the number of letters 0 (resp. 1) in the word w. An *infinite word* $\mathbf{w}$ is a map from $\mathbb{N}$ to an alphabet A, and we write, as is usual, $\mathbf{w} = a_0 a_1 \cdots$ with $a_i \in A$ (we always index from 0). We refer to infinite words in boldface symbols. Many of the notions given here extend naturally to infinite words.

Given two words u and v, their product uv is formed by concatenating their letters. A word z is a *factor* of the word w if $w = uzv$ for some words u and v. If $u = \varepsilon$ (resp. $v = \varepsilon$), then z is a *prefix* (resp. *suffix*) of w. The word z is a *proper* prefix (resp. proper suffix) if $z \neq \varepsilon$ and $v \neq \varepsilon$ (resp. $u \neq \varepsilon$). With $u^{-1}w$ and wv^{-1} we respectively refer to the words zv and uz. By w^n, we mean the word $w \cdots w$ where w is repeated n times. Such a word is called an *nth power*, or a *repetition*. If $w = uzv$, then we say that z *occurs* in w in position $|u|$. In other words, the position $|u|$ defines an *occurrence* of z in w. When we say that a factor z occurs in w in *phase* n modulo q, we mean that z occurs in w in a position i such that $i \equiv n \pmod{q}$.

Let $w = a_0 a_1 \cdots a_{n-1}$ with $a_i \in A$. As mentioned in the introduction, the word w has *period* p if $a_i = a_{i+p}$ for all i with $0 \leq i \leq n-1-p$. The *reversal* $\widetilde{w}$ of w is defined to be the word $a_{n-1} \cdots a_1 a_0$. The word w is a *palindrome* if $\widetilde{w} = w$. If a word u has w as a prefix and a suffix and contains exactly two occurrences of w, then we say that u is a *complete first return to* w. A word u is a *complete first return to* w *in the same phase* if u has w as a prefix and as a suffix, $|u| \equiv 0 \pmod{|w|}$, u contains at least two occurrences of w, and if w occurs in u in position i such that $i \equiv 0 \pmod{|w|}$, then $i = 0$ or $i = |u| - |w|$. For example, the word 01001 is a complete first return to 01, but not a complete first return to 01 in the same phase. The word 01001101 is not a complete first return to 01, but it is a complete first return to 01 in the same phase.

An infinite word $\mathbf{x}$ is *recurrent* if each of its factors occur in it infinitely many times. Let $\mathbf{x} = x_0 x_1 \cdots$ and $\mathbf{y} = y_0 y_1 \cdots$ be two infinite words over an alphabet A. We endow $A^{\mathbb{N}}$, the set of infinite words over A, with the topology determined by the metric d defined by

$$d(\mathbf{x}, \mathbf{y}) = 2^{-k},$$

where k is the least integer such that $x_k \neq y_k$ if $x \neq y$, and $k = \infty$ otherwise. The *shift map* $T\colon A^{\mathbb{N}} \to A^{\mathbb{N}}$ is defined by setting the nth letter of $T\mathbf{x}$ to be the $(n+1)$th letter of $\mathbf{x}$. In other words, T maps (x_n) to (x_{n+1}). A *subshift* is a closed and T-invariant subset of $A^{\mathbb{N}}$.

Before defining abelian equivalence and the related concepts precisely, let us first recall some facts on continued fractions and define Sturmian words. For a more extensive introduction to continued fractions and Sturmian words, we refer the reader to [24, Ch. 4]. Good books on continued fractions are, e.g., [15, 19] whereas [20, 26] are good sources on Sturmian words.

2.1 Continued Fractions

Every irrational real number α has a unique infinite continued fraction expansion:

$$\alpha = [a_0; a_1, a_2, a_3, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}} \tag{1}$$

with $a_0 \in \mathbb{Z}$ and $a_t \in \mathbb{Z}_+$ for $t \geq 1$. The numbers a_i are called the *partial quotients* of α. The rational numbers $[a_0; a_1, a_2, a_3, \ldots, a_k]$, denoted by p_k/q_k, are called *convergents* of α. The convergents satisfy the following recurrences:

$$\begin{array}{llll} p_0 = a_0, & p_1 = a_1 a_0 + 1, & p_k = a_k p_{k-1} + p_{k-2}, & k \geq 2, \\ q_0 = 1, & q_1 = a_1, & q_k = a_k q_{k-1} + q_{k-2}, & k \geq 2. \end{array}$$

For convenience, we set $p_{-1} = 1$ and $q_{-1} = 0$. The *semiconvergents* (or intermediate fractions) $p_{k,\ell}/q_{k,\ell}$ of α are defined as the fractions

$$\frac{\ell p_{k-1} + p_{k-2}}{\ell q_{k-1} + q_{k-2}}$$

for $1 \leq \ell < a_k$ and $k \geq 2$ (if they exist). Notice that semiconvergents are not a subtype of convergents. We often do not refer to convergents or semiconvergents, but to their denominators q_k or $q_{k,\ell}$, so we let $\mathcal{Q}_\alpha$ denote the set of denominators of convergents of α and $\mathcal{Q}_\alpha^+$ denote the set of denominators of the convergents and semiconvergents of α. We emphasize that the above

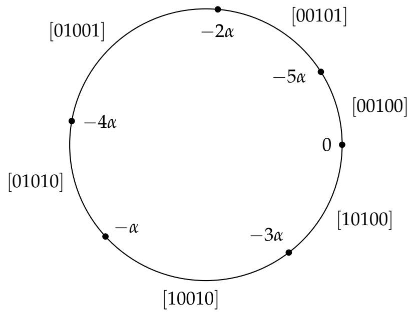

Figure 1: The points 0, $\{-\alpha\}$, $\{-2\alpha\}$, ..., $\{-5\alpha\}$ on the circle $\mathbb{T}$ when $\alpha = 1/\varphi^2$. The intervals of the factors of length 5 of the Fibonacci word are also included.

number q_{-1} defined for convenience does not belong to the sets $\mathcal{Q}_\alpha$ and $\mathcal{Q}_\alpha^+$. Throughout the paper, we make the convention that α always refers to some fixed irrational number in $(0,1)$ with continued fraction expansion (1), convergents q_k, and semiconvergents $q_{k,\ell}$.

Example 2.1. Let φ be the Golden ratio, that is, set $\varphi = (1+\sqrt{5})/2$. Then $\varphi = [1;\overline{1}] \approx 1.62$. The number $1/\varphi^2$, approximately 0.38, has continued fraction expansion $[0;2,\overline{1}]$. Its convergents, related to the Fibonacci numbers, are

$$\frac{0}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{5}, \frac{3}{8}, \frac{5}{13}, \ldots.$$

Notice that this number does not have semiconvergents.

For a real number x, we let $\{x\}$ to be its fractional part and $\|x\| = \min\{\{x\}, 1-\{x\}\}$. Here $\|x\|$ measures the distance of x to the nearest integer. It is often useful to reduce numbers of the form $n\alpha$ modulo 1 and imagine them lying on the circle $\mathbb{T}$ having circumference 1, which we identify with the unit interval $[0,1)$. See Figure 1 for a picture of the numbers $\{-\alpha\}$, $\{-2\alpha\}$, ..., $\{-5\alpha\}$ lying on $\mathbb{T}$ when $\alpha = 1/\varphi^2$. In fact, adding α to its multiple can be viewed as the rotation

$$R\colon \mathbb{T} \to \mathbb{T}, R(x) = \{x+\alpha\}$$

on $\mathbb{T}$.

The denominators of convergents of α satisfy the *best approximation property*:

$$\|q_k\alpha\| = \min_{0<n<q_{k+1}} \|n\alpha\|.$$

This means that the point $\{q_k\alpha\}$ is closer to the point 0 on $\mathbb{T}$ than the points $\{\alpha\}$, $\{2\alpha\}$, ..., $\{(q_{k+1}-1)\alpha\}$. Information on the quality of approximation of the numbers $\{q_{k,\ell}\alpha\}$ related to semiconvergents is given in [23, Prop. 2.2], but this information is not needed in this paper. For deeper understanding how the special points $\{q_k\alpha\}$ and $\{q_{k,\ell}\alpha\}$ lie on $\mathbb{T}$ see Figure 2 (ignore the negative signs for now; they are needed when we work with Sturmian words). The details on why the picture is correct are found in the proof of Lemma 3.3. It is important to understand how the next point closest to 0 is formed from the previously closest points $\{q_k\alpha\}$ and $\{q_{k-1}\alpha\}$. Notice that $q_{k+1,1} = q_k + q_{k-1}$. The point $\{q_{k+1,1}\alpha\}$ related to the denominator of the convergent or semiconvergent $q_{k+1,1}$ is formed by performing q_k rotations on the point $\{q_{k-1}\alpha\}$. This point $\{q_{k+1,1}\alpha\}$ is closer to 0 than $\{q_{k-1}\alpha\}$—as is evident from Figure 2—but it is not necessarily closer than $\{q_k\alpha\}$ if $a_{k+1} > 1$. In fact, we have

$$\|q_{k+1,1}\alpha\| = \|q_{k-1}\alpha\| - \|q_k\alpha\|.$$

Then successive q_k rotations are added forming the points $\{q_{k+1,2}\alpha\}, \ldots, \{q_{k+1,a_{k+1}-1}\alpha\}$ that are successively closer to 0 than $\{q_{k-1}\alpha\}$, but not closer than $\{q_k\alpha\}$. Finally the point $\{q_{k+1,a_{k+1}}\alpha\}$, i.e., the point $\{q_{k+1}\alpha\}$, is closer to 0 than $\{q_k\alpha\}$.

By the preceding description, we see that $\|q_{k,\ell}\alpha\| = \|q_{k,\ell-1}\alpha\| - \|q_{k-1}\alpha\|$. From this identity, it is not difficult to derive by induction that

$$\alpha = p_{k+1}\|q_k\alpha\| + \|q_{k+1}\alpha\| \tag{2}$$

for all $k \geq 1$ when $a_0 = 0$ (i.e., when $\alpha \in (0,1)$). Let then α_t for $t \geq 1$ denote the number with the continued fraction expansion $[a_t; a_{t+1}, a_{t+2}, \ldots]$. A short proof by induction shows that

$$\frac{\|q_k\alpha\|}{\|q_{k+1}\alpha\|} = \alpha_{k+2} \tag{3}$$

for all $k \geq 1$. The following identity is well-known (see, e.g., [15, Sect. 10.7]) for $k \geq 0$:

$$\alpha - \frac{p_k}{q_k} = \frac{(-1)^k}{q_k(\alpha_{k+1}q_k + q_{k-1})}.$$

This identity shows that

$$\|q_k\alpha\| = \frac{1}{\alpha_{k+1}q_k + q_{k-1}} \tag{4}$$

for $k \geq 0$.

We conclude by a simple lemma needed in Section 3 and in the proof of Lemma 5.3.

Lemma 2.2. *Let ℓ be a nonnegative integer. Then*

$$\frac{1}{\|q_{k-1}\alpha\| + \ell\|q_k\alpha\|} = \frac{\alpha_{k+1}q_k + q_{k-1}}{\alpha_{k+1} + \ell}$$

for all $k \geq 1$.

Proof. By (4), we have for all $k \geq 1$ that

$$\begin{aligned}
\|q_{k-1}\alpha\| &= (\alpha_k q_{k-1} + q_{k-2})^{-1} = (q_k + [0; a_{k+1}, \ldots]q_{k-1})^{-1} \\
&= (q_k + (a_{k+1} + \alpha_{k+2}^{-1})^{-1}q_{k-1})^{-1} \\
&= ((a_{k+1} + \alpha_{k+2}^{-1})^{-1}((a_{k+1} + \alpha_{k+2}^{-1})q_k + q_{k-1}))^{-1} \\
&= ((a_{k+1} + \alpha_{k+2}^{-1})^{-1}(\alpha_{k+1}q_k + q_{k-1}))^{-1}
\end{aligned}$$

(the computation indeed works with the convention $q_{-1} = 0$ when $k = 1$), so

$$\|q_{k-1}\alpha\| + \ell\|q_k\alpha\| = \frac{\ell + a_{k+1} + \alpha_{k+2}^{-1}}{\alpha_{k+1}q_k + q_{k-1}} = \frac{\ell + \alpha_{k+1}}{\alpha_{k+1}q_k + q_{k-1}}.$$

□

2.2 Sturmian Words

For the purposes of this paper, Sturmian words are best defined as codings of orbits of irrational rotations on the circle $\mathbb{T}$. For alternative definitions and proofs of the facts listed below, we refer the reader to [20, 26].

Let $\alpha \in (0,1)$ be an irrational real number, and divide $\mathbb{T}$ into two disjoint intervals I_0 and I_1 by the points 0 and $1-\alpha$. The map $R\colon \mathbb{T} \to \mathbb{T}$, $R(\rho) = \{\rho + \alpha\}$ defines an irrational rotation on $\mathbb{T}$. We shall code the orbit of a point ρ as follows. Let ν be the coding function

$$\nu(x) = \begin{cases} 0, & \text{if } x \in I_0, \\ 1, & \text{if } x \in I_1, \end{cases}$$

and let $\mathbf{s}_{\rho,\alpha}$ be the infinite binary word whose nth letter (indexing from 0) equals $\nu(R^n(\rho))$. We call this infinite word $\mathbf{s}_{\rho,\alpha}$ a *Sturmian word of slope α and intercept ρ*. Our definition leaves the behavior of ν on the endpoints of I_0 and I_1 ambiguous. To fix this, we dictate that there are exactly two options: either select $I_0 = [0, 1-\alpha)$ and $I_1 = [1-\alpha, 1)$ (case $0 \in I_0$) or set $I_0 = (0, 1-\alpha]$ and $I_1 = (1-\alpha, 1]$ (case $0 \notin I_0$). For a typical intercept ρ, the choice makes no difference, but a difference is seen if ρ is of the form $\{-n\alpha\}$ for some $n \geq 0$. We define the *Sturmian subshift of slope α*, denoted by Ω_α, to be the set of all Sturmian words of slope α and intercept ρ obtained in both cases $0 \in I_0$ and $0 \notin I_0$. We refer to the words of Ω_α as the Sturmian words of slope α. We remark that $\Omega_\alpha \cap \Omega_\beta \neq \emptyset$ if and only if $\alpha = \beta$.

Example 2.3. Let $\alpha = 1/\varphi^2$ where φ is the Golden ratio. Here $\alpha = [0; 2, \overline{1}] \approx 0.38$. The Sturmian word

$$\mathbf{s}_{\alpha,\alpha} = 0100101001001010010100100101001001010010100100101001010010010100 10\cdots$$

of slope α and intercept α is the Fibonacci word $\mathbf{f}$ mentioned in the introduction. The subshift Ω_α of slope α is called the *Fibonacci subshift*.

The decision if $0 \in I_0$ or $0 \notin I_0$ is often irrelevant because all words in Ω_α have the same *language* (set of factors) $\mathcal{L}(\alpha)$. It is irrelevant in this paper too with the exceptions of the proofs of two minor claims. However, both options are needed for proving equivalence with the alternative definitions of Sturmian words and are needed to make Ω_α a subshift.

Let w be a word $a_0 a_1 \cdots a_{n-1}$ of length n in $\mathcal{L}(\alpha)$, and set

$$[w] = I_{a_0} \cap R^{-1}(I_{a_1}) \cap \cdots \cap R^{-(n-1)}(I_{a_{n-1}}).$$

Then $[w]$ is the unique subinterval of $\mathbb{T}$ such that $\mathbf{s}_{\rho,\alpha}$ begins with w if and only if $\rho \in [w]$. The points 0, $\{-\alpha\}$, …, $\{-n\alpha\}$ partition the circle $\mathbb{T}$ into $n+1$ subintervals that are in one-to-one correspondence with the words of $\mathcal{L}(\alpha)$ of length n. See Figure 1 for the intervals of the factors of length 5 of the Fibonacci word. We let $I(x,y)$, $\{x\} < \{y\}$, stand for the interval $[\{x\},\{y\})$ if $0 \in I_0$ and for $(\{x\},\{y\}]$ if $0 \notin I_0$. We call the words of $\mathcal{L}(\alpha)$ the *factors of slope α*.

Moreover, Sturmian words are recurrent, and the language $\mathcal{L}(\alpha)$ is *closed under reversal*: for each word w in $\mathcal{L}(\alpha)$, its reversal $\widetilde{w}$ is also in $\mathcal{L}(\alpha)$. The only difference between Sturmian words of slope $[0; 1, a_2, a_3, \ldots]$ and Sturmian words of slope $[0; a_2 + 1, a_3, \ldots]$ is that the roles of the letters 0 and 1 are reversed. Thus we make the typical assumption that $a_1 \geq 2$ in (1). This means that $\alpha \in (0, \frac{1}{2})$.

2.3 Abelian Powers, Repetitions, and Periods

Many of the notions and results presented in this subsection and the following subsection are found in [11]. However, we use the notation of [24, Ch. 4.7].

Let w be a finite binary word over the alphabet $\{0,1\}$. The *Parikh vector* (or abelianization) $\mathcal{P}(w)$ of w is defined to be the vector $(|w|_0, |w|_1)$ counting the number of occurrences of the letters 0 and 1 in w. Two words u and v are *abelian equivalent* if $\mathcal{P}(u) = \mathcal{P}(v)$. If $\mathcal{P}$ and $\mathcal{Q}$ are two Parikh vectors and $\mathcal{P}$ is componentwise less than or equal to $\mathcal{Q}$ but is not equal to $\mathcal{Q}$, then we say that $\mathcal{P}$ is *contained* in $\mathcal{Q}$.

Using the above notions, we generalize the notion of a period to the abelian setting.

Definition 2.4. An *abelian decomposition* of a word w is a factorization $w = u_0 u_1 \cdots u_{n-1} u_n$ such that $n \geq 2$, the words $u_1, \ldots, u_{n-1}$ have a common Parikh vector $\mathcal{P}$ (i.e., they are abelian equivalent), and the Parikh vectors of u_0 and u_n are contained in $\mathcal{P}$. The words u_0 and u_n are respectively called the *head* and the *tail* of the decomposition. The common length m of the words $u_1, \ldots, u_{n-1}$ is called an *abelian period* of w. The *minimum abelian period* (i.e., the shortest) of w is denoted by μ_w.

If $n \geq 3$, then we say that w is an *abelian repetition* of period m and exponent $|w|/m$. If $n \geq 3$ and the head u_0 and the tail u_n are empty, then we say that w is an *abelian power* of period m and exponent $|w|/m$. If $n \leq 2$, then we say that w is a *degenerate abelian repetition* (of period m) or a *degenerate abelian power* (of period m) if the head and tail are empty.

For example, the word $abaababa$ has abelian decompositions $a \cdot ba \cdot ab \cdot ab \cdot a$ (of period 2 and exponent $8/2$) and $\varepsilon \cdot aba \cdot aba \cdot ba$ (of period 3 and exponent $8/3$).

The following lemma is immediate.

Lemma 2.5. *Let u be a factor of a word w. Then $\mu_w \geq \mu_u$. On the other hand, if w has an abelian period m such that $m \leq |u|$, then m is also an abelian period of u.*

Definition 2.6. Let w be a finite or infinite word. Then the set

$$\{\mu_u : u \text{ is a nonempty factor of } w\}$$

is called the *abelian period set of w*.

2.4 Abelian Powers in Sturmian Words

The starting point of the study of abelian equivalence in Sturmian words is the following result stating that factors of length n of a Sturmian word belong to exactly two abelian equivalence classes and that these classes can be identified with the subintervals of $\mathbb{T}$ separated by the points 0 and $\{-n\alpha\}$. Let w be a factor of slope α. If w contains the minimum (resp. maximum) number of occurrences of the letter 1 among factors of length $|w|$, then we say that w is *light* (resp. *heavy*).

Proposition 2.7. *[11, Prop. 3.3], [28, Thm. 19] Each factor of length n in $\mathcal{L}(\alpha)$ is either light or heavy. A factor w in $\mathcal{L}(\alpha)$ is light if and only if $[w] \subseteq I(0, -|w|\alpha)$. Moreover, if $\{-n\alpha\} \geq 1-\alpha$, then all heavy factors of length n begin and end with 1, while if $\{-n\alpha\} \leq 1-\alpha$, then each light factor of length n begins and ends with 0.*

The following proposition is a direct consequence of Proposition 2.7, but it is best to state it for clarity. See [11, Lemma 4.2] for more precise information.

Proposition 2.8. *Let $\mathbf{s}_{\rho,\alpha} = a_0 a_1 \cdots$ be a Sturmian word of slope α and intercept ρ. Then its factor $a_n \cdots a_{n+m-1} \cdots a_{n+em-1}$ is an abelian power of period m and exponent e, $e \geq 2$, if and only if the e points $\{\rho + (n+im)\alpha\}$, $i = 0, \ldots, e-1$, are all either in the interval $I(0, -m\alpha)$ or in the interval $I(-m\alpha, 1)$.*

Remark 2.9. Consider factors of a Sturmian word of slope α and of length q_k for some $k \geq 0$. By the best approximation property, the point $\{-q_k\alpha\}$ is closest to the point 0 among the points $\{-\alpha\}, \{-2\alpha\}, \ldots, \{-q_k\alpha\}$. This means that the interval separated by the points $\{-q_k\alpha\}$ and 0 is the interval $[s]$ of a unique word s of length q_k. The word s is called the *singular factor* of length q_k. By Proposition 2.7, we see that the factors of length q_k that do not equal s are abelian equivalent.

The singular factors play a crucial role in deriving the main inequality in Section 4. Singular factors have been studied before in other contexts; see [21, 5]. The previous approaches have been combinatorial, but here we derive the needed results by number-theoretic means.

We need the following result on singular factors.

Lemma 2.10. *The singular factor s of length q_k has the following properties:*

(i) s begins and ends with the same letter;

(ii) s is a palindrome; and

(iii) the Parikh vectors of proper prefixes and suffixes of s are contained in the Parikh vectors of all factors of length q_k.

Proof. The property (i) is directly implied by Proposition 2.7. Namely if $\{-q_k\alpha\} \geq 1 - \alpha$, then s is heavy by Remark 2.9 and s begins and ends with 1 by Proposition 2.7. If $\{-q_k\alpha\} \leq 1 - \alpha$, then s is light and begins and ends with 0.

Since the language $\mathcal{L}(\alpha)$ is closed under reversal, we have $\widetilde{s} \in \mathcal{L}(\alpha)$ (recall that $\widetilde{s}$ is the reversal of s). By Remark 2.9, the singular factor s uniquely corresponds to its Parikh vector among factors of length q_k. Since a Parikh vector is invariant under reversal, it follows that $\widetilde{s} = s$. This establishes property (ii).

Let us then consider the final claim. If $|s| = 1$, then there is nothing to prove, so suppose that $|s| > 1$. Write $s = as'a$ for a letter a. It is sufficient to prove that the Parikh vector $\mathcal{P}(as')$ of as' is contained in the Parikh vectors of all factors of length q_k because $\mathcal{P}(s'a) = \mathcal{P}(as')$. Let w be a factor of length q_k such that $w \neq s$. Suppose that $a = 0$. The first paragraph of this proof shows that s is light. This means that w is heavy. Thus $|w|_1 > |s|_1 = |as'|_1$. In fact, since all factors of fixed length are either heavy or light by Proposition 2.7, it must be that $|w|_1 = |s|_1 + 1$. In other words, $|w|_0 = |s|_0 - 1 = |as'|_0$. Hence $\mathcal{P}(as')$ is contained in $\mathcal{P}(w)$. The case $a = 1$ is similar. □

Proposition 2.7 allows continued fractions and geometric arguments to be applied to the study of abelian powers in Sturmian words. Let $\mathcal{A}e(m)$ denote the maximum exponent of an abelian power of period m occurring in a Sturmian word of slope α. The number $\mathcal{A}e(m)$ is always finite and is easily computed using the following result.

Proposition 2.11. *[11, Thm. 4.7] We have $\mathcal{A}e(m) = \left\lfloor \frac{1}{\|m\alpha\|} \right\rfloor$.*

Proof Sketch. We sketch the proof here because similar arguments are needed in the proofs of Lemma 4.7 and Lemma 5.3. Say $\{-m\alpha\} < \frac{1}{2}$; the case $\{-m\alpha\} > \frac{1}{2}$ is similar. Consider two points ρ and $\{\rho + m\alpha\}$ on $\mathbb{T}$. Because the distance between these points is $\|m\alpha\|$, they cannot both belong to the interval $I(0, -m\alpha)$. If they lie on the interval $I(-m\alpha, 1)$ of length $1 - \|m\alpha\|$, then the word $\mathbf{s}_{\rho,\alpha}$ begins with an abelian square of period m by Proposition 2.8. To find the maximum exponent of an abelian power of period m that is a prefix of $\mathbf{s}_{\rho,\alpha}$, it thus suffices to see how many times $\|m\alpha\|$ divides $1 - \|m\alpha\|$. This proves the claim. □

Since $\|m\alpha\|$ can be made as small as desired, the preceding proposition shows that each Sturmian word contains abelian powers of arbitrarily high exponent. A similar result for a broader class of words is given in [27, Thm. 1.8].

3 Lemmas on Abelian Exponents

In this section, we prove several inequalities concerning the abelian exponents of factors of slope α needed mainly in Section 5. The results presented here are purely arithmetical in their nature and do not, as such, provide any significant insight for proving the main results. The reader might want to read Section 4 before studying this section in detail.

The first lemma relates an abelian exponent to a convergent of α.

Lemma 3.1. *If $\|m\alpha\| \geq \|q_{k-1}\alpha\| + \|q_k\alpha\|$ for some $k \geq 1$, then $\mathcal{A}e(m) < q_k$.*

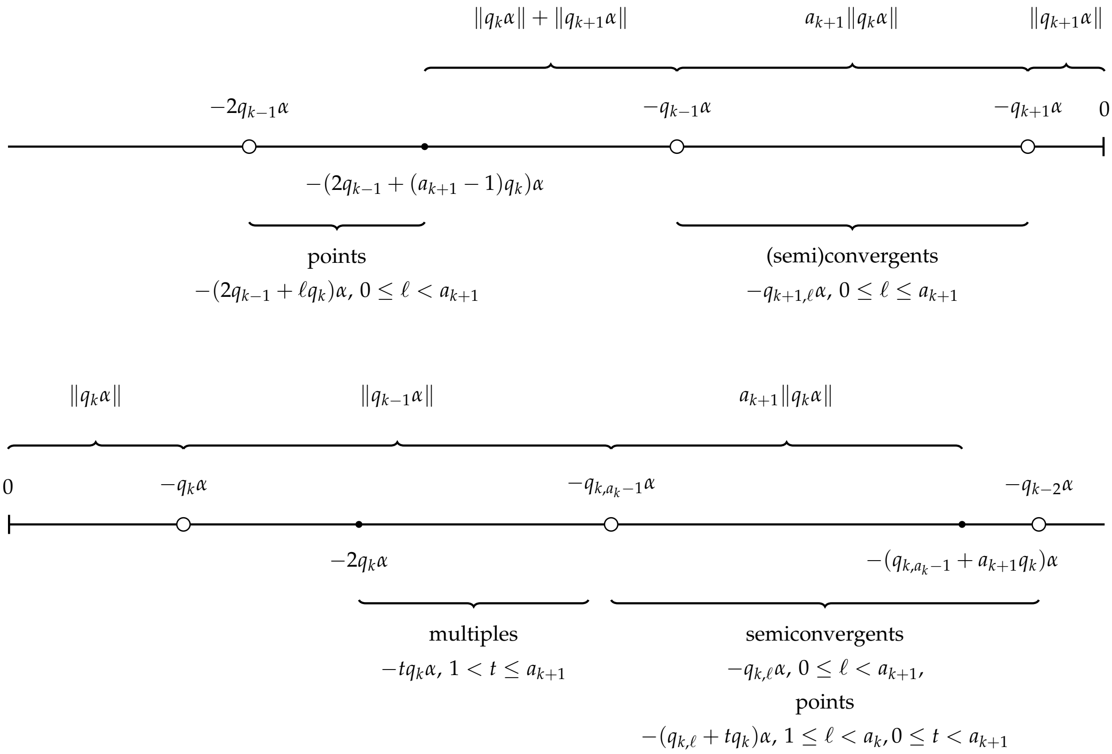

Figure 2: The points $\{-i\alpha\}$ with $i \le q_{k+1}$ that are closest to 0. The picture is in scale; $a_k = a_{k+1} = 2$ was used for drawing.

Proof. Suppose that $\|m\alpha\| \ge \|q_{k-1}\alpha\| + \|q_k\alpha\|$ for some $k \ge 1$. By Proposition 2.11, it suffices to establish that

$$\frac{1}{\|q_{k-1}\alpha\| + \|q_k\alpha\|} < q_k.$$

By Lemma 2.2, this inequality is equivalent to the inequality

$$\alpha_{k+1}q_k + q_{k-1} < \alpha_{k+1}q_k + q_k,$$

which is obviously true because $q_{k-1} < q_k$. $\square$

Lemma 3.1 is sharp in the sense that, for suitable partial quotients, it is possible that $\mathcal{A}e(m) = q_k - 1$ for some k. For instance, if $\alpha = [0; 2, \overline{1}]$ and $m = 4$, then $q_2 = 3 < m < q_3 = 5$, $\|m\alpha\| \approx 0.47 > 0.38 \approx \|q_1\alpha\| + \|q_2\alpha\|$, and $\mathcal{A}e(m) = 2 = q_2 - 1$.

In some cases, we need the following improvement of Lemma 3.1.

Lemma 3.2. *If $\|m\alpha\| \ge \|q_{k-1}\alpha\| + (a_{k+1} + 1)\|q_k\alpha\|$ for some $k \ge 2$, then $\mathcal{A}e(m) < q_k - 1$.*

Proof. Suppose that $\|m\alpha\| \ge \|q_{k-1}\alpha\| + (a_{k+1} + 1)\|q_k\alpha\|$ for some $k \ge 2$. Then, by Lemma 2.2, it suffices to show that

$$\frac{\alpha_{k+1}q_k + q_{k-1}}{\alpha_{k+1} + a_{k+1} + 1} < q_k - 1.$$

This inequality is equivalent to

$$q_{k-1} < (a_{k+1} + 1)q_k - (\alpha_{k+1} + a_{k+1} + 1). \tag{5}$$

Now $q_k \geq q_{k-1} + q_{k-2}$, so it is enough to show that

$$a_{k+1}q_{k-1} + (a_{k+1}+1)q_{k-2} > \alpha_{k+1} + a_{k+1} + 1. \tag{6}$$

Since $k \geq 2$, we have $q_{k-1} \geq q_1 \geq 2$ and $q_{k-2} \geq q_0 = 1$. Thus $a_{k+1}q_{k-1} + (a_{k+1}+1)q_{k-2} \geq 3a_{k+1} + 1$ and $3a_{k+1} + 1 > \alpha_{k+1} + a_{k+1} + 1$ if and only if $2a_{k+1} > \alpha_{k+1}$ (recall that $\alpha_{k+1} < a_{k+1} + 1$). Since $a_{k+1} \geq 1$, this final inequality is true. This means that (6) holds. □

In order to apply Lemma 3.2, we need the following lemma. Its proof essentially argues that Figure 2 is correctly drawn. This figure is important for the proofs in Section 5. It depicts the points of the form $\{-i\alpha\}$ with $i \leq q_{k+1}$ that are closest to 0. The presented arguments contain ingredients for proving the Three Distance Theorem; see [1] and its references, especially [32].

Lemma 3.3. *Let $k \geq 1$, and suppose that m is an integer such that $a_{k+1}q_k < m < q_{k+1}$. If $m \neq q_{k+1,a_{k+1}-1}$ and $m \neq (a_{k+1}-1)q_k + 2q_{k-1}$ when $a_k = 1$, then $\|m\alpha\| \geq \|q_{k-1}\alpha\| + (a_{k+1}+1)\|q_k\alpha\|$.*

Proof. Suppose that m does not equal $q_{k+1,a_{k+1}-1}$, and assume moreover that if $a_k = 1$, then $m \neq (a_{k+1}-1)q_k + 2q_{k-1}$. For the proof, we omit the negative signs and consider points of the form $\{i\alpha\}$ with i positive instead the points $\{-i\alpha\}$ that are the endpoints of the intervals of the factors of slope α.

Assume first that the point $\{m\alpha\}$ is on the same side of the point 0 as the point $\{q_{k+1}\alpha\}$. By this we mean that if $\{q_{k+1}\alpha\} < \frac{1}{2}$, then also $\{m\alpha\} < \frac{1}{2}$ and if $\{q_{k+1}\alpha\} > \frac{1}{2}$, then $\{m\alpha\} > \frac{1}{2}$. By the best approximation property, the point $\{m\alpha\}$ cannot be closer to 0 than $\{q_{k+1}\alpha\}$. Let D_1 be the distance of $\{m\alpha\}$ to 0 through the point $\{q_{k+1}\alpha\}$. Points $\{i\alpha\}$ with $i < q_{k+1}$ between the points $\{q_{k-1}\alpha\}$ and $\{q_{k+1}\alpha\}$ are exactly the points $\{q_{k+1,\ell}\alpha\}$ for $1 \leq \ell < a_{k+1}$ because of the best approximation property and the fact that the distance between $\{q_{k+1,\ell}\alpha\}$ and $\{q_{k+1,\ell+1}\alpha\}$ is $\|q_k\alpha\|$. Since $a_{k+1}q_k < m$ and $m \neq q_{k+1,a_{k+1}-1}$, we conclude that $D_1 > \|q_{k-1}\alpha\|$. Let us consider next points between $\{q_{k-1}\alpha\}$ and $\{2q_{k-1}\alpha\}$. The points $\{(2q_{k-1} + \ell q_k)\alpha\}$, $1 \leq \ell \leq a_{k+1}$, lie between $\{q_{k-1}\alpha\}$ and $\{2q_{k-1}\alpha\}$. As the distance between two consecutive such points is $\|q_k\alpha\|$, the points between $\{q_{k-1}\alpha\}$ and $\{2q_{k-1}\alpha\}$ of the form $\{i\alpha\}$ with $i < q_{k+1}$ are among these points $\{(2q_{k-1} + \ell q_k)\alpha\}$. Say $m = 2q_{k-1} + \ell q_k$ for some ℓ such that $1 \leq \ell \leq a_{k+1}$. Then the assumption $a_{k+1}q_k < m < q_{k+1}$ implies that $\ell = a_{k+1} - 1$ and $q_k < 2q_{k-1}$. The inequality $q_k < 2q_{k-1}$ implies that $a_k = 1$. This case is however excluded by our assumptions. Thus $\{m\alpha\}$ does not lie between $\{q_{k-1}\alpha\}$ and $\{2q_{k-1}\alpha\}$. Consider then the point $\{i\alpha\}$ with $i < q_{k+1}$ that is closest to the point $\{2q_{k-1}\alpha\}$. By the best approximation property, the distance from $\{i\alpha\}$ to $\{2q_{k-1}\alpha\}$ cannot be less than or equal to $\|q_{k+1}\alpha\|$. Therefore it must be at least $\|q_k\alpha\|$. Therefore (see Figure 2)

$$\begin{aligned}
D_1 &\geq 2\|q_{k-1}\alpha\| + \|q_k\alpha\| \\
&= \|q_{k-1}\alpha\| + a_{k+1}\|q_k\alpha\| + \|q_{k+1}\alpha\| + \|q_k\alpha\| \\
&> \|q_{k-1}\alpha\| + (a_{k+1}+1)\|q_k\alpha\|.
\end{aligned}$$

Assume then that the point $\{m\alpha\}$ is on the same side as $\{q_k\alpha\}$. Again $\{m\alpha\}$ cannot be closer to 0 than the point $\{q_k\alpha\}$ due to the best approximation property. Let D_2 be the distance of $\{m\alpha\}$ to 0 through the point $\{q_k\alpha\}$. If $\{i\alpha\}$ with $i < q_{k+1}$ is a point between $\{q_k\alpha\}$ and $\{q_{k,a_k-1}\alpha\}$, then i is a multiple of q_k. This means that $\{m\alpha\}$ is not between $\{q_k\alpha\}$ and $\{q_{k,a_k-1}\alpha\}$. The points closest to $\{q_{k,a_k-1}\alpha\}$ that are not between $\{q_k\alpha\}$ and $\{q_{k,a_k-1}\alpha\}$ are the points $\{(q_{k,a_k-1} + \ell q_k)\alpha\}$ with $1 \leq \ell \leq a_k$. If $q_{k,a_k-1} + \ell q_k > a_{k+1}q_k$, then $\ell = a_{k+1}$. Thus $D_2 \geq \|(q_{k,a_k-1} + a_{k+1}q_k)\alpha\|$. The claim follows since

$$\|(q_{k,a_k-1} + a_{k+1}q_k)\alpha\| = a_{k+1}\|q_k\alpha\| + \|q_{k,a_k-1}\alpha\| = a_{k+1}\|q_k\alpha\| + \|q_{k-1}\alpha\| + \|q_k\alpha\|$$

and $\|m\alpha\| \geq \min\{D_1, D_2\}$. □

The next result contains a lower bound for an abelian exponent.

Lemma 3.4. *If $\|m\alpha\| \leq \|q_k\alpha\|$, then $\mathcal{A}e(m) \geq q_{k+1}$. If $\|m\alpha\| \geq \|q_k\alpha\|$, then $\mathcal{A}e(m) < q_{k+1} + q_k$.*

Proof. The claim follows directly from (4): $1/\|q_k\alpha\| = \alpha_{k+1}q_k + q_{k-1} = q_{k+1} + [0; a_{k+2}, \ldots]q_k$ and $0 < [0; a_{k+2}, \ldots] < 1$. □

4 Idea of the Proof and Derivation of the Main Inequality

The idea of the proof of Theorem 1.3 that is given in [11] is, roughly speaking, to show that near the beginning of an occurrence of a factor w with abelian period m in $\mathbf{f}$, the Fibonacci word, there begins an abelian power of period F_k, where F_k is the largest Fibonacci number such that $F_k \leq m$, and large exponent that contains w completely. This shows by Lemma 2.5 that F_k is an abelian period of w. Thus the minimum abelian period of w must be a Fibonacci number. Recall that all Sturmian words with a common slope have the same language. Therefore it often suffices to study the factors of a single Sturmian word of slope α.

As is mentioned in the introduction, an explicit counterexample showed that the above proof idea, as such, does not generalize to other Sturmian words. In fact, we shall show in Proposition 5.7 that such a counterexample exists in all cases except in the case of the slope $1/\varphi^2$. This means that the proof of [11] for Theorem 1.3 is specific to the Fibonacci subshift. While this specific proof could be modified to work more generally, this line of reasoning seems to be unworkable. Thus new ideas are necessary.

Let us now consider Sturmian words of slope α. Instead of looking for abelian powers of period q_k with large exponent that can cover some factor of slope α, the idea is to see what it means if the period q_k is avoided. We shall soon see that abelian powers of period q_k cover almost all of a Sturmian word of slope α. This means that a factor w avoiding the period q_k must be rather long. This in turn means that the abelian exponent $\mathcal{A}e(m)$, related to the minimum abelian period m of w, must be large whenever m is not too large compared to q_k. Since $\mathcal{A}e(m) = \lfloor 1/\|m\alpha\| \rfloor$ by Proposition 2.11, it must be that $\|m\alpha\|$ is small. The analysis of Section 5 indicates that $\|m\alpha\|$ has to be so small that m relates to a rather good rational approximation of α. Precise analysis of the quality of the approximation leads to the statement of Theorem 1.4.

We let $\mathcal{M}_\alpha$ denote the set $\{tq_k : k \geq 0 \text{ and } 1 \leq t \leq a_{k+1}\}$. With the new notation from Section 2, we now rephrase the main result, Theorem 1.4, as follows.

Theorem 4.1. *If m is the minimum abelian period of a nonempty factor of slope α, then $m \in \mathcal{Q}_\alpha^+ \cup \mathcal{M}_\alpha$.*

Let us consider the minimum abelian period m of a nonempty word w in $\mathcal{L}(\alpha)$ (we shall use the notation introduced here throughout this section). In view of Theorem 4.1, we suppose that $m \notin \mathcal{M}_\alpha$. Let k be the largest integer such that $q_k < m$, and let t to be the largest integer such that $tq_k < m$ with $1 \leq t \leq a_{k+1}$. Notice that our assumptions imply that $k \geq 1$ because $q_0 = 1$. By taking the exponent and head and tail length to be maximal, we see that $|w| \leq (\mathcal{A}e(m) + 2)m - 2$. The main task of this section is to derive the following lower bound for the length of w:

$$(q_{k+1} + 2t - 1)q_k - q_{k+1} \leq |w| \tag{7}$$

This establishes the key inequality

$$(q_{k+1} + 2t - 1)q_k - q_{k+1} \leq (\mathcal{A}e(m) + 2)m - 2. \tag{8}$$

In other words, our aim is to establish the following proposition.

Proposition 4.2. *Consider a factor w of slope α with minimum abelian period m. Let k be the largest integer such that $q_k \leq m$, and let t to be the largest integer such that $tq_k \leq m$ with $1 \leq t \leq a_{k+1}$. If $m \notin \mathcal{M}_\alpha$, then both (7) and (8) hold.*

As mentioned above, the main point of this paper is to show that left side of (8) is so large that it also forces $\mathcal{A}e(m)$ to be relatively large, that is, it forces $\|m\alpha\|$ to be small, so small that m has to correspond to a good rational approximation of α.

Example 4.3. Recall that the slope α of the Fibonacci word equals $1/\varphi^2$, where φ is the Golden ratio. Now $\alpha = [0; 2, \overline{1}] \approx 0.38$. The inequality (8) predicts that a factor of the Fibonacci word having minimum abelian period 9 must have length at least $(13 + 2 \times 1 - 1) \times 8 - 13 = 99$. On the other hand, we have $\|9\alpha\| \approx 0.44$, so $\mathcal{A}e(9) = \lfloor 1/\|9\alpha\| \rfloor = 2$ by Proposition 2.11. Thus the upper bound of (8) is $(2+2) \times 9 - 2 = 34$. The conclusion is that there is no factor with minimum abelian period 9 in the Fibonacci word.

Claim 4.4. *The word w contains at least q_k occurrences of the singular factor s of length q_k.*

Proof. By Remark 2.9, all factors of length q_k belong to the same abelian equivalence class except the singular factor s of length q_k. Thus whenever we factorize a factor of slope α of length nq_k as a product $u_1 \cdots u_n$ with $|u_1| = \ldots = |u_n| = q_k$ and none of the words u_i equal s, then $u_1 \cdots u_n$ is an abelian power of period q_k and exponent n. The word w cannot be a factor of such an abelian power $u_1 \cdots u_n$. This means that w contains the singular factor s of length q_k in all phases modulo q_k. Otherwise there is a phase which does not contain s or it contains s only partially (a suffix of s as a prefix or a prefix of s as a suffix). As the Parikh vectors of the proper prefixes and suffixes of s are contained in the Parikh vectors of any factor of length q_k by Lemma 2.10 (iii), it follows that it is possible to cover w with an abelian repetition of period q_k. This is contrary to our assumptions. Consequently, the word w contains at least q_k occurrences of s. $\square$

The next result is crucial in obtaining a lower bound for $|w|$.

Lemma 4.5. *The return times of the singular factor of length q_k are q_{k+1} and $q_{k+2,1}$.*

Proof. The interval $[s]$ of the singular factor s is $I(0, -q_k\alpha)$ or $I(-q_k\alpha, 1)$ by Remark 2.9. Let $x \in [s]$. Then the word $\mathbf{s}_{x,\alpha}$ begins with s. The word s occurs in $\mathbf{s}_{x,\alpha}$ at position n, $n > 0$, if $\{x + n\alpha\} \in [s]$. The return time of the prefix s in $\mathbf{s}_{x,\alpha}$ is determined by the least such n. The length of the interval $[s]$ is $\|q_k\alpha\|$, so it must be that the distance between x and $\{x + n\alpha\}$ is less than $\|q_k\alpha\|$, that is, $\|n\alpha\| < \|q_k\alpha\|$. By the best approximation property, we thus conclude that $n \geq q_{k+1}$. Let y be a point such that $y \in [s]$ and $\|y\| = \|q_{k+1}\alpha\|$. If $x \in I(-q_k\alpha, y) \subseteq [s]$, then $\{x + q_{k+1}\alpha\} \in [s]$ and $n = q_{k+1}$. Suppose then that $\|x\| < \|y\|$. Now $\{x + q_{k+1}\alpha\} \notin [s]$, so $n > q_{k+1}$. On the other hand, $\{x + q_{k+2,1}\alpha\} \in [s]$ because $\|q_{k+2,1}\alpha\| = \|q_k\alpha\| - \|q_{k+1}\alpha\|$ and the distance between x and $\{-q_k\alpha\}$ is at least $\|q_k\alpha\| - \|q_{k+1}\alpha\|$. Therefore $n \leq q_{k+2,1}$. If $n < q_{k+2,1}$, then both $\{x + n\alpha\}$ and $\{x + q_{k+2,1}\alpha\}$ lie on $[s]$. Then we have $q_{k+2,1} - n \geq q_{k+1}$ by the best approximation property. Therefore $q_{k+2,1} \geq q_{k+1} + n > 2q_{k+1} > q_{k+1} + q_k = q_{k+2,1}$; a contradiction. The conclusion is that $n = q_{k+2,1}$. We are left with the case $x = y$. Recall that the interval $[s]$ is half-open. If $0 \in [s]$, then $\{x + q_{k+1}\alpha\} = 0 \in [s]$, and the claim is clear. Otherwise $\{-q_k\alpha\} \in [s]$, $\{x + q_{k+2,1}\alpha\} = \{-q_k\alpha\} \in [s]$, and $\{x + q_{k+1}\alpha\} \notin [s]$, so the preceding arguments show that $n = q_{k+2,1}$. $\square$

In fact, every factor of a Sturmian word has exactly two returns (the return times are distinct), and Sturmian words can be characterized as the recurrent infinite words whose each factor has exactly two returns [33, 16]. Lemma 4.5 states that the maximum return time of the singular factor of length q_k is $q_{k+2,1}$. This is in fact the longest return time among all factors of length q_k. This is important in determining the so-called recurrence quotients of Sturmian words [22, 6].

Before proceeding any further, we need the following two technical lemmas on complete first returns to s in the same phase. The first lemma essentially says that a Sturmian word $\mathbf{s}$ of slope α is "covered" by abelian powers of period q_k. By this we mean that whenever we factorize $\mathbf{s}$ as blocks of length q_k, the singular factor s is seen only rarely. Intuitively, it is "hard" to avoid

$$\mathbf{f} = 01001010010010100101\mathbf{00100}1010010010100101\mathbf{00100}10100101001001010010010100\cdots$$

Figure 3: The Fibonacci word $\mathbf{f}$ factorized as a product of blocks of length 5 in phases 0 and 1 modulo 5. The singular factor 00100 of length 5 is seen only rarely in each phase.

having abelian period q_k for factors whose length does not significantly differ from $\mathcal{A}e(q_k)q_k$. See Figure 3 for a picture of two such factorizations of the Fibonacci word, one for phase 0 modulo 5 and another for phase 1 modulo 5. Notice that $\mathcal{A}e(q_k) \geq q_{k+1}$; see Lemma 3.4.

Lemma 4.6. *Let s be the singular factor of length q_k for some $k \geq 0$, and let w in $\mathcal{L}(\alpha)$ be a complete first return to s in the same phase. Then the word $s^{-1}ws^{-1}$ is an abelian power of period q_k having exponent $\mathcal{A}e(q_k) - 1$ or $\mathcal{A}e(q_k)$.*

Proof. Suppose that w is a prefix of a Sturmian word $\mathbf{s}_{x,\alpha}$. As w begins with s, we have $x \in [s]$. By Remark 2.9, the interval $[s]$ has endpoints 0 and $\{-q_k\alpha\}$. The occurrences of s in $\mathbf{s}_{x,\alpha}$ in the same phase as the prefix s correspond to points of the form $\{x + nq_k\alpha\}$ that are interior points of the interval $[s]$.[2] Consider the smallest such positive n (such a number exists because the sequence $(nq_k\alpha)_n$ is dense in $\mathbb{T}$ by the well-known Kronecker Approximation Theorem; see [15, Ch. XXIII]). The points $\{x + q_k\alpha\}, \ldots, \{x + (n-1)q_k\alpha\}$ lie on the interval $I(-q_k\alpha, 1)$ (if $\{-q_k\alpha\} < 1/2$) or on the interval $I(0, -q_k\alpha)$ (if $\{-q_k\alpha\} > 1/2$). Thus Proposition 2.8 implies that the word $s^{-1}ws^{-1}$ is an abelian power of period q_k and exponent $n-1$. Moreover, the exponent is found by adding one to the times the length $\|q_k\alpha\|$ fits into the interval $I(x + q_k\alpha, 1)$ (if $\{-q_k\alpha\} < 1/2$) or into the interval $I(x + q_k\alpha, -q_k\alpha)$ (if $\{-q_k\alpha\} > 1/2$). Suppose that $\{-q_k\alpha\} < 1/2$. If x is arbitrarily close to 0, then $\{x + q_k\alpha\}$ is arbitrarily close to $\{-q\alpha\}$, so in this case $n - 1 = 1 + \lfloor (1 - \|q_k\alpha\|)/\|q_k\alpha\| \rfloor$, that is, we have $n - 1 = \mathcal{A}e(q_k)$. If x is arbitrarily close to $\{-q_k\alpha\}$, then analogously we have $n - 1 = \mathcal{A}e(q_k) - 1$. The case $\{-q_k\alpha\} > 1/2$ is similar. □

Lemma 4.7. *Let s be the singular factor of length q_k for some $k \geq 0$. Let w in $\mathcal{L}(\alpha)$ be a complete first return to s in the same phase, and write $s^{-1}ws^{-1} = u_0u_1\cdots u_{n-1}$ with $|u_0| = \ldots = |u_{n-1}| = q_k$. Then the words $u_0, u_1, \ldots, u_{\lambda-1}$ end with the same letter as s and the words $u_{n-\lambda}, u_{n-\lambda+1}, \ldots, u_{n-1}$ begin with the same letter as s when*

$$\lambda = \begin{cases} q_{k+1} - p_{k+1} - 1, & \text{if } k \text{ is odd,} \\ p_{k+1} - 1, & \text{if } k \text{ is even.} \end{cases}$$

Moreover, the singular factor s ends and begins with the same letter.

Proof. If $k = 0$, then the claim is true, so suppose that $k \geq 1$. Let w be a prefix of a Sturmian word $\mathbf{s}_{x,\alpha}$. Then $x \in [s]$, and we can assume without loss of generality that x is an interior point of $[s]$. We consider first the latter claim concerning the first letters of the words u_j. Assume that $\{-q_k\alpha\} > 1/2$ so that s begins with the letter 1. If the points $\{x + iq_k\alpha\}, \{x + (i+1)q_k\alpha\}, \ldots, \{x + (n-1)q_k\alpha\}$ lie on the interval $[1]$ of length α, then the words $u_i, \ldots, u_{n-1}$ begin with the letter 1. Notice that the distance between two consecutive points is $\|q_k\alpha\|$ and that $\{x + nq_k\alpha\}$ lies on $[1]$. The worst case scenario is that the point $\{x + nq_k\alpha\}$ is very close to the point $\{-q_k\alpha\}$, and then it must be that the $n - i$ consecutive distances $\|q_k\alpha\|$ must fit into $\alpha - \|q_k\alpha\|$. Suppose next that $\{-q_k\alpha\} < 1/2$. In this case, the word s begins with the letter 0. Similarly we need to see if the points $\{x + iq_k\alpha\}, \ldots, \{x + (n-1)q_k\alpha\}$ are placed on the interval $[0]$ of length $1 - \alpha$. This time we need to check how many times $\|q_k\alpha\|$ fits into $1 - \alpha - \|q_k\alpha\|$. Thus i is maximal when

[2] Since $\mathbf{s}_{x,\alpha}$ is recurrent and α is irrational, we may assume that none of these points coincide with the endpoints of $[s]$.

$n-1-i+1$ equals $\lfloor \alpha/\|q_k\alpha\| \rfloor - 1$ (if $\{-q_k\alpha\} > 1/2$) or $\lfloor (1-\alpha)/\|q_k\alpha\| \rfloor - 1$ (if $\{-q_k\alpha\} < 1/2$). Consider the former case $\{-q_k\alpha\} > 1/2$. By applying (2), we obtain that

$$\frac{\alpha}{\|q_k\alpha\|} = \frac{p_{k+1}\|q_k\alpha\| + \|q_{k+1}\alpha\|}{\|q_k\alpha\|} \geq p_{k+1}.$$

Suppose then that $\{-q_k\alpha\} < 1/2$. In this case, we derive using (2), (3), and (4) that

$$\begin{aligned}
\frac{1-\alpha}{\|q_k\alpha\|} &= \frac{1}{\|q_k\alpha\|} - p_{k+1} - \frac{\|q_{k+1}\alpha\|}{\|q_k\alpha\|} \\
&= \alpha_{k+1}q_k + q_{k-1} - p_{k+1} - \frac{\|q_{k+1}\alpha\|}{\|q_k\alpha\|} \\
&= q_{k+1} - p_{k+1} + [0; a_{k+2}, \ldots]q_k - \frac{\|q_{k+1}\alpha\|}{\|q_k\alpha\|} \\
&= q_{k+1} - p_{k+1} + \frac{1}{\alpha_{k+2}}q_k - \frac{1}{\alpha_{k+2}} \\
&= q_{k+1} - p_{k+1} + \frac{q_k - 1}{\alpha_{k+2}} \\
&\geq q_{k+1} - p_{k+1}.
\end{aligned}$$

Together the two preceding inequalities establish the latter claim on first letters of the words u_j.

Consider then the former claim about the last letters of the words u_j. Suppose first that $\{-q_k\alpha\} > 1/2$. The final letter of s is determined by the point $\{x + (q_k - 1)\alpha\}$. Since $[s] = I(0, -q_k\alpha)$, we have $\{x + q_k\alpha\} \in I(0, q_k\alpha)$, and hence $1 - \alpha < \{x + (q_k - 1)\alpha\} < 1$ because $0 < \{q_k\alpha\} < \alpha$. This means that s ends with the letter 1. As long as the points $\{x + 2q_k\alpha\}, \ldots, \{x + (i+1)q_k\alpha\}$ lie between the points 0 and α, the words $u_0, \ldots, u_{i-1}$ end with the letter 1. Again, it is clearly sufficient to compute $\lfloor \alpha/\|q_k\alpha\| \rfloor - 1$. The final case $\{-q_k\alpha\} < 1/2$ is analogous. □

Remark 4.8. In the proof of Lemma 4.7, we derived lower bounds for both $\alpha/\|q_k\alpha\|$ and $(1-\alpha)/\|q_k\alpha\|$, the lower bound for the former being p_{k+1}. Since $1 - \alpha > \alpha$, we derive a common lower bound p_{k+1} for both quantities, that is, $\lambda \geq p_{k+1} - 1$ for all $k \geq 1$. It is straightforward to see that $p_k \geq a_k$ for all $k \geq 1$, so we conclude that for all $k \geq 1$ the $a_{k+1} - 1$ consecutive factors of length q_k preceding (resp. following) each occurrence of the singular factor s of length q_k begin (resp. end) with the same letter as s. This is the consequence of Lemma 4.7 we need in this paper.

We may now continue to derive the inequality (8). The factor w contains at least q_k occurrences of s (Claim 4.4) and, by Lemma 4.5, the minimum return time of s is q_{k+1}. Thus if w contains at least $q_k + 2$ occurrences of s then, $|w| \geq (q_k + 1)q_{k+1} + q_k$. In this case (7) holds as it is straightforward to compute that $(q_k + 1)q_{k+1} + q_k - ((q_{k+1} + 2t - 1)q_k - q_{k+1}) > 0$ (recall from the paragraph following Theorem 4.1 that $t \leq a_{k+1}$). We may thus assume that w contains at most $q_k + 1$ occurrences of s. Suppose then that $t = 1$. If w contains q_k occurrences of s, then we have $|w| \geq (q_k - 1)q_{k+1} + q_k$, that is, (7) holds. If w contains $q_k + 1$ occurrences of s then, by the same logic, we see that $|w| \geq q_k q_{k+1} + q_k$. Then (7) is true as is easy to verify. Therefore we may assume that $t > 1$.

Suppose first that w contains exactly $q_k + 1$ occurrences of s. By Lemma 4.5, the occurrences of s do not overlap, so we may write $w = u_0 s u_1 s u_2 \cdots s u_{q_k+1}$ for some words $u_0, u_1, \ldots, u_{q_k+1}$. Since both return times of s, q_{k+1} and $q_{k+2,1}$, equal q_{k-1} modulo q_k, it follows that the occurrences of s in w are all in different phases modulo q_k except the first and final one that are in the same phase (it is easy to see by induction that $\gcd(q_{k-1}, q_k) = 1$). By Lemma 4.6, the word $(u_0 s)^{-1} w (s u_{q_k+1})^{-1}$ is an abelian power of period q_k and exponent E, where E equals $\mathcal{A}e(q_k) - 1$ or $\mathcal{A}e(q_k)$. Write $(u_0 s)^{-1} w (s u_{q_k+1})^{-1} = \beta_0 \cdots \beta_{E-1}$ with $|\beta_0| = \ldots = |\beta_{E-1}| = q_k$. Consider the words $\beta_0 \cdots \beta_{E-1}$, $\beta_1 \cdots \beta_{E-1}$, ..., $\beta_{t-1} \cdots \beta_{E-1}$. These words can be viewed as (possibly degenerate) abelian repetitions with period tq_k and empty head.

Claim 4.9. *The abelian repetitions $\beta_0 \cdots \beta_{E-1}$, $\beta_1 \cdots \beta_{E-1}$, ..., $\beta_{t-1} \cdots \beta_{E-1}$ of period tq_k can be extended to have head and tail of maximum length $tq_k - 1$.*

Proof. Consider the word $\beta_n \cdots \beta_{E-1}$ with $0 \leq n \leq t-1$. Suppose that the singular factor s begins with letter x, and let y to be the letter such that $y \neq x$. The word s also ends with the letter x by Lemma 4.7. Let i equal the number of letters x in the nonsingular factors of length q_k and set $j = q_k - i$. The word $\beta_0 \cdots \beta_{E-1}$ is preceded by the word s, and $|s|_x = i+1$ and $|s|_y = j-1$. Further, the word $s\beta_0 \cdots \beta_{E-1}$ is preceded by an abelian power of period q_k and exponent at least $\mathcal{A}e(q_k) - 1$. This power might extend beyond the starting position of w. For our purposes, it is irrelevant how w is extended to the left; all that matters is that by recurrence the left extension of w exists in $\mathcal{L}(\alpha)$. We conclude that the first s of w is preceded by an abelian power $\gamma_0 \cdots \gamma_{t-(n+2)}$ of period q_k with $|\gamma_0| = \ldots = |\gamma_{t-(n+2)}| = q_k$. By Remark 4.8, the words γ_0, ..., $\gamma_{t-(n+2)}$ all begin with the letter x. Thus

$$|x^{-1}\gamma_0 \cdots \gamma_{t-(n+2)} s \beta_0 \cdots \beta_{n-1}|_x = ((t-(n+1))i - 1) + (i+1) + ni = ti$$

and

$$|x^{-1}\gamma_0 \cdots \gamma_{t-(n+2)} s \beta_0 \cdots \beta_{n-1}|_y = tj - 1.$$

Therefore the Parikh vector of the factor $x^{-1}\gamma_0 \cdots \gamma_{t-(n+2)} s \beta_0 \cdots \beta_{n-1}$ is contained in the Parikh vector of $\beta_n \cdots \beta_{n+t-1}$. Thus the abelian repetition $\beta_n \cdots \beta_{E-1}$ can be extended to have a head of maximal length $tq_k - 1$.

Let r be the largest integer such that $rt \leq E - n$. Then the abelian repetition $\beta_n \cdots \beta_{E-1}$ has tail $\beta_{n+rt} \cdots \beta_{E-1}$. This tail is followed by the word s, which is in turn followed by an abelian power $\delta_0 \cdots \delta_{(r+1)t-E+n-2}$ of period q_k with $|\delta_0| = \ldots = |\delta_{(r+1)t-E+n-2}| = q_k$. By Remark 4.8, the words δ_0, ..., $\delta_{(r+1)t-E+n-2}$ end with the letter x. Similarly to above, if we remove the final letter of the word $\beta_{n+rt} \cdots \beta_{E-1} s \delta_0 \cdots \delta_{(r+1)t-E+n-2}$ it will cancel the additional letter x in s, and we see that the tail of $\beta_n \cdots \beta_{E-1}$ can be extended to maximum length $tq_k - 1$. □

Let then n be an integer such that $0 \leq n \leq t-1$, and let λ_n to be the abelian repetition $\beta_n \cdots \beta_{E-1}$ extended to have head and tail of length $tq_k - 1$. We define the *left overhang* of λ_n, denoted by $L(\lambda_n)$, to be its prefix that comes before the occurrence of s that coincides with the first s in w if it exists; otherwise we set $L(\lambda_n) = \varepsilon$. Using the notation of the proof of Claim 4.9, we thus set $L(\lambda_n) = x^{-1}\gamma_0 \cdots \gamma_{t-(n+2)}$ when $n < t-1$. Similarly, we define the word $R(\lambda_n)$, the *right overhang* of λ_n, as the suffix of λ_n that comes after the final s in w if it exists, that is, $R(\lambda_n) = \delta_0 \cdots \delta_{(r+1)t-E+n-2} x^{-1}$, $n \neq E - (r+1)t + 2$, in the notation of the proof of Claim 4.9. Here r is the largest integer such that $rt \leq E - n$. In particular, we have

$$|L(\lambda_n)| = (t-(n+1))q_k - 1$$

and

$$|R(\lambda_n)| = ((r+1)t - E + n - 1)q_k - 1$$

when $L(\lambda_n)$ and $R(\lambda_n)$ are nonempty.

Since $tq_k < m$ and m is the minimum abelian period of w, none of the words λ_0, ..., λ_{t-1} can completely cover w. Let i be the largest integer such that $iq_k \leq |u_0|$. If $i \geq t-1$, then $|w| \geq (t-1)q_k + q_k q_{k+1} + q_k$, and it is elementary to verify that (7) holds. We hence assume that $i < t-1$. Now the left overhangs of the words λ_0, ..., $\lambda_{t-(i+1)-1}$ cover u_0, so it must be that w extends beyond their right overhangs. At least one of these right overhangs must have length at least $(t-i-2)q_k - 1$. Namely if $|R(\lambda_0)|$ is as small as possible, then $R(\lambda_0) = \varepsilon$ and $|R(\lambda_{t-(i+1)-1})| = (t-i-2)q_k - 1$. It follows that

$$\begin{aligned}|w| &\geq |u_0| + q_k q_{k+1} + q_k + (t-i-2)q_k \\ &\geq iq_k + (q_{k+1}+1)q_k + (t-i-2)q_k \\ &= (q_{k+1}+t-1)q_k,\end{aligned}$$

and it is again straightforward to verify that (7) holds.

Suppose finally that w contains exactly q_k occurrences of s. Factor w again according to the occurrences of s: $w = u_0 s u_1 \cdots s u_{q_k}$. Similarly as before, the factor $(u_0 s)^{-1} w$ is now a prefix of an abelian power $\beta_0 \cdots \beta_{E-1}$. The arguments of Claim 4.9 can now be repeated to see that the abelian power $\beta_0 \cdots \beta_{E-1}$ can be extended to an abelian repetition with period tq_k and head of length $tq_k - 1$. Since m is the minimum abelian period of w, this head cannot cover u_0 completely, so $|u_0| \geq (t-1)q_k$. Consider next the reversal $\widetilde{w}$ of w. Since the word s is a palindrome by Lemma 2.10 (ii), we have $\widetilde{w} = \widetilde{u}_{q_k} s \cdots \widetilde{u}_1 s \widetilde{u}_0$. Since the language $\mathcal{L}(\alpha)$ is closed under reversal, we have $\widetilde{w} \in \mathcal{L}(\alpha)$. The minimum abelian period is invariant under reversal so, by repeating the preceding arguments, we see that $|u_{q_k}| \geq (t-1)q_k$. A short computation shows that (7) holds also in this final case. This ends the proof of Proposition 4.2.

5 Proofs of the Main Results

In this section, we prove the main results, Theorems 4.1 and 5.9. Throughout this section, we continue to use the notation of Section 4. We consider an abelian period m of a factor w of slope α, and we assume that $m \notin \mathcal{Q}_\alpha^+ \cup \mathcal{M}_\alpha$. We let k to be the largest integer such that $q_k < m$ and t to be the largest integer such that $tq_k < m$ with $1 \leq t \leq a_{k+1}$. Recall that $k \geq 1$. The assumption $m \notin \mathcal{Q}_\alpha^+ \cup \mathcal{M}_\alpha$ implies that $\|m\alpha\| > \|q_{k-1}\alpha\| + \|q_k\alpha\|$. This is most easily seen from Figure 2. Consider first the lower part of the figure. Point with distance at most $\|q_{k-1}\alpha\| + \|q_k\alpha\|$ is either in $\mathcal{M}_\alpha$ or equals $\{-q_{k,a_k-1}\alpha\}$. The latter option is ruled out because $m > q_k > q_{k,a_k-1}$. In the upper part, all points with distance at most $\|q_{k-1}\alpha\|$ are in $\mathcal{Q}_\alpha^+$. Since $q_k < m < q_{k+1}$, the best approximation property shows that the distance from $\{-m\alpha\}$ to $\{-q_{k-1}\alpha\}$ is greater than $\|q_k\alpha\|$. The claim follows.

The next lemma is used repeatedly in the following proofs. It will be used to show Theorem 4.1 to be true in the fairly typical case $\mathcal{Ae}(m) < q_k - 1$; the remaining case $\mathcal{Ae}(m) = q_k - 1$ is the difficult one.

Lemma 5.1. *Consider a factor w of slope α with abelian period m and exponent E. Let k be the largest integer such that $q_k \leq m$. If $m \notin \mathcal{M}_\alpha$ and $E < q_k - 1$, then m is not the minimum abelian period of w.*

Proof. If $k = 0$, then there is nothing to prove as E is always positive. Assume that $k \geq 1$, $m \notin \mathcal{M}_\alpha$, and $E < q_k - 1$. Clearly $|w| \leq (E+2)m - 2$. Suppose for a contradiction that m is the minimum abelian period of w. Then Proposition 4.2 gives

$$(q_{k+1} + 2t - 1)q_k - q_{k+1} \leq (E+2)m - 2. \tag{9}$$

Let us assume first that $t < a_{k+1}$. Using the upper bound $m < (t+1)q_k$, we obtain from (9) that

$$(q_{k+1} + 2t - 1)q_k - q_{k+1} \leq q_k((t+1)q_k - 1) - 2.$$

Using the equality $q_{k+1} = a_{k+1}q_k + q_{k-1}$, we obtain by rearrangement the equivalent inequality

$$((a_{k+1} - (t+1))q_k + q_{k-1} + 2t - a_{k+1})q_k \leq q_{k-1} - 2. \tag{10}$$

The right side of (10) is at less than q_k, so it must be that the coefficient of q_k on the left is at most 0. If $k \geq 2$, then $q_k, q_{k-1} \geq q_1 \geq 2$, and we obtain

$$\begin{aligned}(a_{k+1} - (t+1))q_k + q_{k-1} + 2t - a_{k+1} &\geq 2(a_{k+1} - (t+1)) + q_{k-1} + 2t - a_{k+1} \\ &= a_{k+1} + q_{k-1} - 2 \\ &\geq a_{k+1},\end{aligned}$$

which is impossible as $a_{k+1} \geq 1$. If $k = 1$, then the right side of (10) is negative. Analogous calculation now gives $(a_2 - (t+1))q_1 + q_0 + 2t - a_2 \geq a_2 + q_0 - 2 = a_2 - 1 \geq 0$, which is again contradictory.

Suppose then that $t = a_{k+1}$. Now the upper bound $m < q_{k+1}$ and (9) give

$$(q_{k+1} + 2a_{k+1} - 1)q_k - q_{k+1} \leq q_k(q_{k+1} - 1) - 2.$$

Rearranging like above then gives

$$2a_{k+1}q_k \leq q_{k+1} = a_{k+1}q_k + q_{k-1},$$

which is again impossible since $q_k > q_{k-1}$. □

Next we do not make restrictions on $\mathcal{A}e(m)$ and prove Theorem 4.1 in almost every case.

Lemma 5.2. *If $k \geq 4$, then m is not the minimum abelian period of w.*

Proof. Suppose for a contradiction that m is the minimum abelian period of w. Since $\|m\alpha\| > \|q_{k-1}\alpha\| + \|q_k\alpha\|$, we have $\mathcal{A}e(m) < q_k$ by Lemma 3.1. Thus we obtain from Proposition 4.2 that

$$(q_{k+1} + 2t - 1)q_k - q_{k+1} \leq (q_k + 1)m - 2.$$

Consider first the case $t < a_{k+1}$. We obtain from the previous inequality that

$$(q_{k+1} + 2t - 1)q_k - q_{k+1} \leq (q_k + 1)((t+1)q_k - 1) - 2.$$

By rearranging and writing $q_{k+1} = a_{k+1}q_k + q_{k-1}$, we obtain

$$((a_{k+1} - (t+1))q_k + q_{k-1} + t - 1)q_k \leq q_{k+1} - 3. \tag{11}$$

In order for this inequality to hold, it is necessary that the coefficient of q_k on the left side is at most a_{k+1}. Since $k > 1$, we have $q_k \geq q_2 \geq 3$ and $q_{k-1} \geq q_1 \geq 2$. It follows that

$$a_{k+1} \geq 3(a_{k+1} - t - 1) + t + 1,$$

which yields $a_{k+1} \leq t + 1$. This is true only if $t = a_{k+1} - 1$. It is also clear from (11) that we must have $q_{k-1} \leq 2$ if $t = a_{k+1} - 1$. This means that $k = 2$ and $a_1 = 2$. However, now the left side of (11) is a_3q_2 and the right side is $a_3q_2 + q_1 - 3$. This is impossible as $q_1 = 2$.

Suppose then that $t = a_{k+1}$. Let us first assume that $m \leq (a_{k+1} - 1)q_k + 2q_{k-1}$ to obtain from (8) that

$$(q_{k+1} + 2a_{k+1} - 1)q_k - q_{k+1} \leq (q_k + 1)((a_{k+1} - 1)q_k + 2q_{k-1}) - 2.$$

This inequality is equivalent to

$$(q_k - q_{k-1} + a_{k+1})q_k \leq q_{k+1} + 2q_{k-1} - 2.$$

Substituting $q_{k+1} = a_{k+1}q_k + q_{k-1}$ gives the equivalent inequality

$$(q_k - q_{k-1})q_k \leq 3q_{k-1} - 2. \tag{12}$$

Since $k \geq 4$, we have $q_k - q_{k-1} \geq q_4 - q_3 \geq q_2 \geq 3$. This together with (12) gives $3q_k < 3q_{k-1}$, which is obviously false. We may thus assume that $m > (a_{k+1} - 1)q_k + 2q_{k-1}$. Then Lemma 3.3 implies that $\|m\alpha\| \geq \|q_{k-1}\alpha\| + (a_{k+1} + 1)\|q_k\alpha\|$. This means that we can improve the bound $\mathcal{A}e(m) < q_k$ to $\mathcal{A}e(m) < q_k - 1$ by Lemma 3.2. The desired contradiction follows now from Lemma 5.1. □

By Lemma 5.2, we are left with the cases $k = 1$, $k = 2$, and $k = 3$. However in the last two cases, the arguments of the proof of Lemma 5.2 apply under suitable conditions. Let us analyze the situation. The only place where we really need the assumption $k \geq 4$ is when a contradiction is derived from (12). Here we needed $k \geq 4$ to establish that $q_k - q_{k-1} \geq 3$. Let us see when $q_k - q_{k-1} \leq 2$. Now $q_k - q_{k-1} = (a_k - 1)q_{k-1} + q_{k-2}$, so if $q_k - q_{k-1} \leq 2$, then it must be that $a_k = 1$ whenever $q_{k-1} \geq 2$. Moreover, by Lemma 3.2, we do not need to know that $q_k - q_{k-1} \geq 3$ in order to improve the bound to $\mathcal{A}e(m) < q_k - 1$ if we know that $m \neq (a_{k+1} - 1)q_k + 2q_{k-1}$. We are thus left to prove Theorem 4.1 in the following cases:

- $k = 1$;
- $m = (a_{k+1} - 1)q_k + 2q_{k-1}$, $a_k = 1$ when $k = 2$ or $k = 3$.

It must be emphasized that the proof of Lemma 5.2 does not work in these final cases: the inequality (8) is not enough. Indeed, if $\alpha = [0; 2, \overline{1}]$ and $m = (a_3 - 1)q_2 + 2q_1 = 4$, then $\mathcal{A}e(m) = 2$ and the left side of (8) is 13, but the right side is 14. A more interesting example is perhaps $\alpha = [0; 2, 3, 1, 6, \overline{1}]$. When $k = 3$, we have $m = (a_4 - 1)q_3 + 2q_2 = (6 - 1) \times 9 + 2 \times 7 = 59$ and $\mathcal{A}e(m) = 8 = q_3 - 1$. The left side of (8) is $(61 + 2 \times 6 - 1) \times 9 - 61 = 587$, and the right side is 588.

The following general lemma handles the above cases $k = 2, 3$ and a subcase of $k = 1$.

Lemma 5.3. *If either*

(i) $k = 1$, $a_1 \geq 3$, $a_2 > 1$, *and* $m = (a_2 - 1)q_1 + 2q_0$ *or*

(ii) $k \geq 2$, $a_k = 1$, *and* $m = (a_{k+1} - 1)q_k + 2q_{k-1}$,

then m is not the minimum abelian period of w.

Proof. Let $m = (a_{k+1} - 1)q_k + 2q_{k-1}$ for $k \geq 1$, and assume for a contradiction that there exists a factor w of slope α with minimum abelian period m. Notice that we have $(a_{k+1} - 1)q_k < m < a_{k+1}q_k$ in the case (i), and $a_{k+1}q_k < m < q_{k+1}$ in the case (ii). Write $w = u_0 s u_1 \cdots s u_\lambda$ according to the λ occurrences of the singular factor s of length q_k in w. Recall that $\lambda \geq q_k$ by Claim 4.4. Factorize $w = \beta_0\beta_1 \cdots \beta_E\beta_{E+1}$, with $|\beta_1| = \ldots = |\beta_E| = m$, according to the minimum abelian period m of w. We may assume that $E = q_k - 1$ for otherwise the claim is clear by Lemma 5.1. Suppose first that $\lambda > q_k$. Then

$$|w| - |u_0| \geq q_k q_{k+1} + q_k$$

according to Lemma 4.5. Assume then that $\lambda = q_k$. When the inequality (7) was derived, we showed that $|u_0|, |u_\lambda| \geq (a_{k+1} - 1)q_k$. In particular, we have $|u_\lambda| \geq (a_{k+1} - 1)q_k$, and thus

$$|w| - |u_0| \geq (q_k - 1)q_{k+1} + a_{k+1}q_k. \tag{13}$$

Thus no matter the value of λ, the inequality (13) holds.

Next we want to show that $|\beta_0| > |u_0|$. Assume on the contrary that $|\beta_0| \le |u_0|$. Then

$$\begin{aligned}
|\beta_{E+1}| &= |w| - |\beta_0\beta_1 \cdots \beta_E| \\
&\ge |w| - |u_0| - |\beta_1 \cdots \beta_E| \\
&\ge (q_k - 1)q_{k+1} + a_{k+1}q_k - Em \\
&= (q_k - 1)q_{k+1} + q_k - (q_k - 1)((a_{k+1} - 1)q_k + 2q_{k-1}) + (a_{k+1} - 1)q_k \\
&= (q_{k+1} + 1)q_k - q_{k+1} - (a_{k+1} - 1)q_k q_k - 2q_{k-1}q_k + (a_{k+1} - 1)q_k + 2q_{k-1} + (a_{k+1} - 1)q_k \\
&= (q_{k+1} + 1 - (a_{k+1} - 1)q_k - 2q_{k-1})q_k - q_{k+1} + (a_{k+1} - 1)q_k + 2q_{k-1} + (a_{k+1} - 1)q_k \\
&= (q_k - q_{k-1} + 1)q_k - q_k + q_{k-1} + (a_{k+1} - 1)q_k.
\end{aligned}$$

Let $Q = (q_k - q_{k-1} + 1)q_k - q_k + q_{k-1} + (a_{k+1} - 1)q_k$. If (i) holds, then

$$Q = (q_1 - 2)q_1 + q_0 + (a_2 - 1)q_1 > 2q_0 + (a_2 - 1)q_1 = m.$$

If (ii) holds, then

$$\begin{aligned}
Q &= (q_{k-2} + 1)q_k - q_{k-2} + (a_{k+1} - 1)q_k \\
&= (q_{k-2} + 1)(q_{k-1} + q_{k-2}) - q_{k-2} + (a_{k+1} - 1)q_k \\
&= (q_{k-2} + 1)q_{k-1} + q_{k-2}^2 + (a_{k+1} - 1)q_k \\
&> 2q_{k-1} + (a_{k+1} - 1)q_k \\
&= m.
\end{aligned}$$

Thus $|\beta_{E+1}| \ge Q > m$. This is a contradiction, so we conclude that $|\beta_0| > |u_0|$.

We let J denote the longer of the two intervals separated by the points 0 and $\{-m\alpha\}$. Let $x \in [u_0^{-1}w]$. In particular, we have $x \in [s]$. Now $[s] = I(0, -q_k\alpha)$ or $[s] = I(-q_k\alpha, 1)$. Let $L = \|q_{k-1}\alpha\| + \|q_k\alpha\| + \|q_{k+1}\alpha\|$. The distance from $\{-m\alpha\}$ to 0 through the point $\{q_{k-1}\alpha\}$ equals L; see Figure 2. Our aim is to show that $L < \frac{1}{2}$. This establishes that the point x is not on the same side of 0 as $\{-m\alpha\}$ and that $\|m\alpha\| = L$.

Suppose first that $k \ge 2$. We have $\|q_{t+1}\alpha\| < \frac{1}{2}\|q_t\alpha\|$ for all $t \ge 0$, so

$$L < \frac{1}{2}(\|q_{k-2}\alpha\| + \|q_{k-1}\alpha\| + \|q_k\alpha\|) \le \frac{1}{2}(\|q_{k-2}\alpha\| + \|q_{k-2}\alpha\|) = \|q_{k-2}\alpha\| \le \alpha < \frac{1}{2}.$$

Assume then that $k = 1$. Since $a_2 > 1$, we have $m \ne 2$. The distance from $\{-m\alpha\}$ to $\{-\alpha\}$ is $\|q_1\alpha\| + \|q_2\alpha\|$, and the distance from $\{-m\alpha\}$ to $\{-2\alpha\}$ is $(a_2 - 1)\|q_1\alpha\|$. Since $a_1 \ge 3$, we see that $1 - L = (a_2 - 1)\|q_1\alpha\| + (a_1 - 2)\|\alpha\| + \|q_1\alpha\| = (a_1 - 2)\|\alpha\| + a_2\|q_1\alpha\|$. Clearly $1 - L > L$, so $L < \frac{1}{2}$.

Since the point x is not on the same side of 0 as $\{-m\alpha\}$, it follows that $x \in J$. Set $y = \{x + (|\beta_0| - |u_0|)\alpha\}$. Then $x \ne y$. The abelian power $\beta_1 \cdots \beta_E$ beginning at position $|\beta_0|$ of w is not degenerate: $E = q_k - 1 \ge 2$ when $k = 1$ and $E = q_k - 1 \ge q_2 - 1 \ge 2$ when $k \ge 2$. Therefore, by Proposition 2.8, the point y must also lie on J. Let D_1 be the distance of y to 0 through the point $\{-m\alpha\}$ and D_2 be the distance of y to 0 to the other direction. Since y lies on J, it follows that $D_1 \ge \|m\alpha\| \ge \|q_{k-1}\alpha\|$.

Our next aim is to find a lower bound to the distance D between x and y. Notice that since $|\beta_0| < m < q_{k+1}$, it must be that $|\beta_0| - |u_0| < q_{k+1}$. It thus follows from the best approximation property that $D > \|q_{k+1}\alpha\|$. If $D = \|q_k\alpha\|$, then it must be that $|\beta_0| - |u_0| = q_k$. This is however not the case because then y would be on the same side of 0 as the point $\{-m\alpha\}$. Therefore $D > \|q_k\alpha\|$. Since $y \in J$, it follows that $y \notin [s]$. In particular, we have $D_2 \ge D$. Since $D > \|q_k\alpha\|$, it follows from the best approximation property that $D \ge \|q_{k-1}\alpha\|$. The conclusion is that $\|y\| \ge \min\{D_1, D_2\} \ge \|q_{k-1}\alpha\|$.

By Proposition 2.8, the exponent E of the abelian power $\beta_1 \cdots \beta_E$ is the integer part of

$$\frac{1 - \|m\alpha\| - \|y\|}{\|m\alpha\|} + 1.$$

By the above, we obtain that

$$\frac{1 - \|m\alpha\| - \|y\|}{\|m\alpha\|} + 1 \leq \frac{1 - \|q_{k-1}\alpha\|}{\|m\alpha\|}.$$

Let $\mathcal{S}$ denote the right side of this inequality. We shall argue that $\mathcal{S} < q_k - 1$. This shows that $E < q_k - 1$ contradicting the maximality of E and ending the proof.

Recall that $\|m\alpha\| = \|q_{k-1}\alpha\| + \|q_k\alpha\| + \|q_{k+1}\alpha\|$. In particular, we have $\|m\alpha\| > \|q_{k-1}\alpha\| + \|q_k\alpha\|$. Hence

$$\mathcal{S} < \frac{1 - \|q_{k-1}\alpha\|}{\|q_{k-1}\alpha\| + \|q_k\alpha\|}.$$

By Lemma 2.2, we have

$$\frac{1}{\|q_{k-1}\alpha\| + \|q_k\alpha\|} = \frac{\alpha_{k+1}q_k + q_{k-1}}{\alpha_{k+1} + 1} \quad \text{and} \quad q_k - \frac{\alpha_{k+1}q_k + q_{k-1}}{\alpha_{k+1} + 1} = \frac{q_k - q_{k-1}}{\alpha_{k+1} + 1},$$

so

$$\begin{aligned}
&q_k - \frac{1}{\alpha_{k+1} + 1}\Big(q_k - q_{k-1} + \|q_{k-1}\alpha\|(\alpha_{k+1}q_k + q_{k-1})\Big) \\
&= q_k - \frac{q_k - q_{k-1}}{\alpha_{k+1} + 1} - \frac{\|q_{k-1}\alpha\|(\alpha_{k+1}q_k + q_{k-1})}{\alpha_{k+1} + 1} \\
&= \frac{\alpha_{k+1}q_k + q_{k-1}}{\alpha_{k+1} + 1} - \frac{\|q_{k-1}\alpha\|(\alpha_{k+1}q_k + q_{k-1})}{\alpha_{k+1} + 1} \\
&= \frac{1 - \|q_{k-1}\alpha\|}{\|q_{k-1}\alpha\| + \|q_k\alpha\|}.
\end{aligned}$$

In other words, we have

$$\mathcal{S} < q_k - \frac{1}{\alpha_{k+1} + 1}\Big(q_k - q_{k-1} + \|q_{k-1}\alpha\|(\alpha_{k+1}q_k + q_{k-1})\Big).$$

Now $\|q_{k-1}\| = (\alpha_k q_{k-1} + q_{k-2})^{-1}$ by (4), so

$$\mathcal{S} < q_k - \frac{1}{\alpha_{k+1} + 1}\left(q_k - q_{k-1} + \frac{\alpha_{k+1}q_k + q_{k-1}}{\alpha_k q_{k-1} + q_{k-2}}\right). \tag{14}$$

Clearly,

$$\frac{\alpha_{k+1}q_k + q_{k-1}}{\alpha_k q_{k-1} + q_{k-2}} = \frac{\alpha_{k+1}q_k + q_{k-1}}{q_k + \alpha_{k+1}^{-1}q_{k-1}} = \alpha_{k+1},$$

so it follows from (14) that $\mathcal{S} < q_k - 1$ because $q_k - q_{k-1} \geq 1$. $\square$

By Lemmas 5.2 and 5.3, Theorem 4.1 is true when $k \geq 2$, and we are left with the case $k = 1$. Let us first make some general observations.

Suppose that $k = 1$ and m is the minimum abelian period of w. If $t = a_2$, then $a_2 q_1 < m < q_2 = a_2 q_1 + 1$, and such m cannot exist. Thus we may assume that $a_2 > 1$ and $t < a_2$. Now $m \neq tq_1 + 1 = q_{2,t}$, so m must equal one of the numbers $tq_1 + 2, \ldots, tq_1 + q_1 - 1$. In particular, it must be that $q_1 = a_1 \geq 3$. The computation at the beginning of the proof of Lemma 5.2 leading to the inequality (11) shows that

$$((a_2 - (t+1))q_1 + t)q_1 \leq q_2 - 3.$$

The coefficient $(a_2 - (t+1))q_1 + t$ of q_1 on the left is at most a_2 and at least $3a_2 - 2t - 3$ because $q_1 \geq 3$. Hence $2a_2 \leq 2t + 3$. Since $t < a_2$, the only possibility is that $t = a_2 - 1$. By Lemma 5.3, we may further assume that $m > (a_2 - 1)q_1 + 2$. Notice that this implies that $a_1 \geq 4$. The rest of the case $k = 1$ is handled by the next lemma.

Lemma 5.4. *If $a_1 \geq 4$ and $(a_2 - 1)q_1 + 2 < m < a_2 q_1$, then m is not the minimum abelian period of w.*

Proof. By Lemma 3.3, we have $\|m\alpha\| \geq \|\alpha\| + (a_2 + 1)\|q_1\alpha\|$. In order to conclude that $\mathcal{Ae}(m) < q_1 - 1$, it is enough, by the beginning of the proof of Lemma 3.2, to verify the inequality (5) for $k = 1$, that is, we need to show that

$$q_0 + \alpha_2 + a_2 + 1 < (a_2 + 1)q_1.$$

Now $a_1 \geq 4$, so $(a_2 + 1)q_1 \geq 4a_2 + 4 > 3a_2 + \alpha_2 + 3 > q_0 + \alpha_2 + a_2 + 1$. Thus Lemma 5.1 implies the claim. □

We have established all the cases, and we are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let m be the minimum abelian period of a factor w of slope α. If $m < q_1$, then $m = tq_0$ with $1 \leq t < a_1$, that is, $m \in \mathcal{Q}_\alpha \cup \mathcal{M}_\alpha$. Suppose that $m \geq q_1$. Then there exists a positive integer k such that $q_k \leq m < q_{k+1}$. If $m \notin \mathcal{Q}_\alpha^+ \cup \mathcal{M}_\alpha$, then Lemmas 5.2, 5.3, and 5.4 (together with the discussions preceding them), imply that m cannot be the minimum abelian period of w. The conclusion is that $m \in \mathcal{Q}_\alpha^+ \cup \mathcal{M}_\alpha$. □

Notice that Theorem 4.1 directly implies Theorem 1.3 because the slope of the Fibonacci word has continued fraction expansion $[0; 2, \overline{1}]$.

Let us then see if Theorem 4.1 completely characterizes the minimum abelian periods of factors of slope α. We begin with the following proposition.

Proposition 5.5. *If $m \in \mathcal{Q}_\alpha$, then there exists a factor of slope α having minimum abelian period m.*

Proof. Let $q_k \in \mathcal{Q}_\alpha$ for some $k \geq 0$. Clearly the factor 0 has minimum abelian period q_0, so we may assume that $k \geq 1$. We suppose that $0 \in I_0$ if k is even and $0 \notin I_0$ otherwise. Consider the prefix w of $\mathbf{s}_{0,\alpha}$ of length $\mathcal{Ae}(q_k)q_k$. The factor w has abelian period q_k. Observe that $|w| \geq q_{k+1}q_k$ because $\mathcal{Ae}(q_k) \geq q_{k+1}$ by Lemma 3.4. Let m be the minimum abelian period of w, and suppose for a contradiction that $m < q_k$. Then we have $\|m\alpha\| \geq \|q_{k-1}\alpha\|$ by the best approximation property. It follows by Lemma 3.4 that $\mathcal{Ae}(m) < q_k + q_{k-1}$. Therefore

$$q_{k+1}q_k \leq |w| \leq (\mathcal{Ae}(m) + 2)m - 2 \leq (q_k + q_{k-1} + 1)(q_k - 1) - 2.$$

We thus obtain that

$$((a_{k+1} - 1)q_k - 1)q_k \leq -(q_k + q_{k-1} + 3).$$

The coefficient $(a_{k+1} - 1)q_k - 1$ of q_k on the left is at least -1, which shows that the inequality is impossible. The conclusion is that $m = q_k$. □

Theorem 4.1 and Proposition 5.5 together imply the following proposition. This was already implicitly present in [11] as a corollary of [11, Thm. 6.9] and [11, Thm. 6.12].

Proposition 5.6. *The abelian period set of the Fibonacci word is the set of Fibonacci numbers.*

We shall see next that it is not necessary that a denominator of a semiconvergent or a proper multiple of a denominator of a convergent is the minimum abelian period of some factor of slope α. We begin with the following observation.

Proposition 5.7. *Let $k \geq 1$, and suppose that $a_{k+1} > 1$. Then there exists a factor of slope α whose minimum abelian period equals $q_{k+1,1}$ or $2q_k$.*

Proof. A suitable factor was essentially constructed in Section 4. Let us repeat the construction. Let u be a factor of slope α that is a complete first return to s, the singular factor of length q_k, in the same phase. Such a factor exists by recurrence and Lemma 4.5. Indeed, the return times of s, q_{k+1} and $q_{k+2,1}$, equal q_{k-1} modulo q_k, so u has exactly $q_k + 1$ occurrences of s, all in different phases modulo q_k except the first and final occurrence. Let x be the final letter of s, and set $w = ux^{-1}$. It follows from what precedes that w cannot have abelian period q_k. Moreover, $|w| \geq (q_{k+1} + 1)q_k - 1$. Let m be the minimal abelian period of w. Suppose first that $m < q_k$. By the best approximation property, we have $\|m\alpha\| \geq \|q_{k-1}\alpha\|$, so $\mathcal{A}e(m) \leq \mathcal{A}e(q_{k-1})$. Now $\mathcal{A}e(q_{k-1}) < q_k + q_{k-1}$ by Lemma 3.4, so $|w| \leq (q_k + q_{k-1} + 1)(q_k - 1) - 2$. Therefore

$$(q_{k+1} + 1)q_k - 1 \leq (q_k + q_{k-1} + 1)(q_k - 1) - 2,$$

which is equivalent to

$$((a_{k+1} - 1)q_k)q_k \leq -(q_k + q_{k-1} + 2).$$

This is a contradiction as the left side is clearly nonnegative. Thus we conclude that $m > q_k$. Write $s^{-1}us^{-1} = \beta_0 \cdots \beta_{E-1}$ with $|\beta_1| = \ldots = |\beta_E| = q_k$. Observe that the words $\beta_0, \ldots, \beta_{E-1}$ are abelian equivalent and that $\beta_0 \cdots \beta_{E-1}$ is a prefix of $s^{-1}w$. Let $\gamma_i = \beta_{2i}\beta_{2i+1}$ for $i = 0, \ldots, r$, where $r = \frac{1}{2}(E-2)$ if E is even and $r = \frac{1}{2}(E-3)$ if E is odd. The words $\gamma_0, \ldots, \gamma_r$ are abelian equivalent and have length $2q_k$. We may write $w = s\gamma_0 \cdots \gamma_r v$, where $v = sx^{-1}$ if E is even and $v = \beta_{E-1}sx^{-1}$ otherwise. The Parikh vector of s is contained in the Parikh vector of γ_0 by Lemma 2.10 (iii). Similarly Lemma 2.10 (iii) shows that the Parikh vector of sx^{-1} is contained in the Parikh vector of β_0. Therefore the Parikh vector of v is contained in the Parikh vector of γ_0. Thus the word w is an abelian repetition of period $2q_k$ with head s and tail v. The conclusion is that $m \leq 2q_k$. Since $a_{k+1} > 1$, we have $q_{k+1} > 2q_k$. Hence Theorem 4.1 implies that $m \in \{q_{k+2,1}, 2q_k\}$. ☐

Let us then see through examples that we cannot improve on Proposition 5.7. Let $\alpha = [0; 2, 1, 2, 3, \overline{1}]$ (≈ 0.3711). Then the sequence of denominators of convergents is 2, 3, 8, 27, ... and the sequence of denominators of semiconvergents is 5, 11, 19, ... Set $m = 2q_2 = 6$. It can be verified with the help of a computer that no factor of slope α has minimum abelian period m. Since $\mathcal{A}e(m) = 6$, it is enough to compute the minimum abelian periods of factors up to length $(\mathcal{A}e(m) + 2)m - 2 = 34$. In fact, the minimum abelian periods of factors up to length 34 belong to the set $\{1, 2, 3, 5, 8\}$. Thus a proper multiple of a denominator of a convergent is not necessarily the minimum abelian period of some factor. Notice that the minimum abelian period 6 is not ruled out by (8): the left side of (8) equals 25. There are thus other, unknown reasons why 6 is not a minimum abelian period.

It is possible to have a minimum abelian period of the form tq_k with $t > 2$. Let $\alpha = [0; 2, 6, \overline{1}]$ (≈ 0.4649). Then the sequence of denominators of convergents is 2, 13, 15, ... and the sequence of denominators of semiconvergents is 3, 5, 7, 9, 11, ... The following factor of slope α of length 32 has minimal abelian period $4q_1$:

$$010100 \cdot 10101010 \cdot 10100101 \cdot 01010101 \cdot 01.$$

For the slope $[0; 2, 5, \overline{1}]$, no factor with minimum abelian period 8 exists.

Let finally $\alpha = [0; 2, 3, 2, \overline{1}]$ (≈ 0.4355). The sequence of denominators of convergents is 2, 7, 16, 23, ... and the sequence of denominators of semiconvergents is 3, 5, 9, ... It can be verified that there is no factor of slope α with minimum abelian period 9 (it is enough to study factors up to length 124). Therefore a denominator of a semiconvergent is not necessarily the minimum abelian period of some factor. The possible abelian periods of factors up to length 124 are in the set $\{1, 2, 3, 4, 5, 7, 14, 16\}$. The period 14 is included as predicted by Proposition 5.7.

It seems to us that the problem of characterizing the possible abelian periods of factors of slope α is significantly harder than proving Theorem 4.1. The above examples indicate that the answer depends heavily on the arithmetic nature of the slope. We leave this problem open. Based on computer experiments, we have the following conjecture.

Conjecture 5.8. *Let $\alpha = [0; \overline{2}]$. The abelian period set of a Sturmian word of slope α is $\mathcal{Q}_\alpha^+ \cup \mathcal{M}_\alpha$.*

By Proposition 5.7, there exists a factor with minimum abelian period that is not a denominator of a convergent whenever $a_k > 1$ for some $k \geq 2$. This gives the following interesting characterization of the Fibonacci subshift/the Golden ratio in terms of abelian periods.

Theorem 5.9. *Let α be an irrational in $(0, \frac{1}{2})$. Then $\alpha = 1/\varphi^2$, where φ is the Golden ratio, if and only if the minimum abelian period of every factor of slope α is in $\mathcal{Q}_\alpha$.*

Proof. Say $\alpha = 1/\varphi^2$. Then $\alpha = [0; 2, \overline{1}]$ and, by Theorem 4.1, the abelian periods of factors of slope α are in $\mathcal{Q}_\alpha$. Suppose then that the minimum abelian period of every factor of slope α is in $\mathcal{Q}_\alpha$. Proposition 5.7 shows that there is necessarily a factor with minimum abelian period outside the set $\mathcal{Q}_\alpha$ if $a_k > 1$ for some $k \geq 2$. Hence $a_k = 1$ for all $k \geq 2$. If $a_1 > 2$, then the factor 01 has abelian period 2 and $2 \notin \mathcal{Q}_\alpha$. Thus $a_1 = 2$. In other words, $\alpha = 1/\varphi^2$. $\square$

6 Remarks on k-abelian Periods

In this section, we briefly discuss what changes if abelian equivalence is replaced by the more general k-abelian equivalence.

Let k be a positive integer. Two words u and v are *k-abelian equivalent* if $|u|_w = |v|_w$ for each word w of length at most k. Here $|u|_w$ stands for the number of occurrences of w as a factor of u. When $k = 1$, the k-abelian equivalence relation is simply the abelian equivalence relation. If u and v are k-abelian equivalent, then we write $u \sim_k v$. Notice that if $u \sim_k v$, then $u \sim_{k-1} v$ for all $k > 1$. For example, the words 0101100 and 0011010 are 2-abelian equivalent, but they are not 3-abelian equivalent. If $u = 0110$ and $v = 1101$, then $|u|_w = |v|_w$ for each word of length 2, but $u \not\sim_2 v$ because u and v are not abelian equivalent. For words of length at least $k - 1$, we have $u \sim_k v$ if and only if u and v share a common prefix and a common suffix of length $k - 1$ and $|u|_w = |v|_w$ for each word w of length k [18, Lemma 2.4]. For words of length at most $2k - 1$, the relation $\sim_k$ is in fact the equality relation $=$ [18, Lemma 2.4]. If the words $u_0, \ldots, u_{e-1}$ are k-abelian equivalent, then their concatenation $u_0 \cdots u_{e-1}$ is a *k-abelian power* of period $|u_0|$ and exponent e.

The k-abelian equivalence is first introduced in the 1980 paper of J. Karhumäki [17] in relation to the Post Correspondence Problem. The 2013 paper [18] by J. Karhumäki, A. Saarela, and L. Zamboni contains the first deeper study of k-abelian equivalence and, most importantly, the first research on k-abelian equivalence in relation to Sturmian words. One of their result is a characterization of Sturmian words as the aperiodic binary words whose factors of length n belong to exactly $2k$ k-abelian equivalence classes if $n \geq 2k$ and to exactly $n + 1$ classes if $n \leq 2k - 1$ [18, Thm. 4.1]. Another nice result is a general theorem from which it follows that Sturmian words contain k-abelian powers of arbitrarily high exponent [18, Thm. 5.4]. The results of Karhumäki et al. are made more precise in the paper [25] by the author and M. Whiteland where an approach based on continued fractions is developed to study k-abelian powers in Sturmian words. This approach yields results similar to those of [11]. For example, the following analogue of Proposition 2.11 concerning the maximum exponent $\mathcal{A}e_{k,\alpha}(m)$ of a k-abelian power of period m occurring in a Sturmian word of slope α is obtained. Here $\min L(2k-2)$ (resp. $\max L(2k-2)$) is the length of the shortest (resp. longest) interval among the intervals of factors of length $2k-2$.

Proposition 6.1. *[25, Lemma 3.10] Let m be a positive integer and suppose that $\|m\alpha\| < \min L(2k-2)$. Then*

$$\left| \left\lfloor \frac{\max L(2k-2)}{\|m\alpha\|} \right\rfloor - \mathcal{A}e_{k,\alpha}(m) \right| \leq 1.$$

Let us next discuss the generalization of an abelian period to this setting of k-abelian equivalence. The following definition is compatible with Definition 2.4 when $k = 1$.

Definition 6.2. A word w has *k-abelian period m* if w is a factor of a k-abelian power of period m.

Example 6.3. Let $w = 0100110$. The minimum abelian period of w is 2 because of the (only possible) factorization $w = 0 \cdot 10 \cdot 01 \cdot 10$. Since $10 \not\sim_2 01$, we see that the 2-abelian period must be greater than 2. The only candidate factorization of w for 2-abelian period 3 is $w = 01 \cdot 001 \cdot 10$, but this is not good because no word of length 3 beginning with 10 can be 2-abelian equivalent to 001 (there must be a common prefix of length 1). Keeping in mind the requirement for a common prefix and a common suffix of length 1, we see that the relevant factorizations for period 4 are $01 \cdot 0011 \cdot 0$ and $010 \cdot 0110$. The prefix 01 of the first factorization cannot be completed to a word of length 4 that is 2-abelian equivalent to 0011 since such a completion must begin with 0 and then we are missing the factor 11. By a similar analysis for the second factorization, we see that the minimum 2-abelian period of w is at least 5. In fact, it is easily verified that it is 6.

The abelian period can also be generalized in another way based directly on Definition 2.4.

Definition 6.4. Let A be an alphabet and k a positive integer. Suppose that u_0, u_1, $\ldots$, u_t is an enumeration of the nonempty words over the alphabet A of length at most k in some fixed order. The *generalized Parikh vector* $\mathcal{P}_k(w)$ of a word w is the vector $(|w|_{u_0}, |w|_{u_1}, \ldots, |w|_{u_t})$. If u and v are words, then we say that $\mathcal{P}_k(u)$ is *contained* in $\mathcal{P}_k(v)$ if $\mathcal{P}_k(u)$ is componentwise less than or equal to $\mathcal{P}_k(v)$.

Definition 6.5. A word w over A has *k-abelian period m in the second sense* if it is possible to write $w = u_0 u_1 \cdots u_{n-1} u_n$ such that $n \geq 2$, $u_1 \sim_k \cdots \sim_k u_{n-1}$, and $\mathcal{P}_k(u_0)$ and $\mathcal{P}_k(u_n)$ are contained in $\mathcal{P}_k(u_1)$.

This latter definition is different from the first one. For example, the minimum 2-abelian period (in the second sense) of the word $w = 0100110$ of Example 6.3 is 4 due to the factorization $w = 010 \cdot 0110$. It is difficult to argue which of the two generalizations is more natural. The author's opinion is that the former one is the right definition.

Theorem 4.1 does not have immediate consequences on minimal k-abelian periods of factors of Sturmian words. Indeed, as was seen in Example 6.3, the minimum k-abelian period of a word might be larger than its minimum abelian period. The prefix 010010100 of the Fibonacci word has minimum abelian period 2 and minimum 2-abelian period 5 (in the first sense). No method presented in this paper is directly applicable to this more general setting, and hence we leave this problem open. It seems difficult to make any plausible conjecture in light of computer experiments. It would be natural to guess that direct analogues of Theorem 4.1 and Theorem 5.9 hold also in the k-abelian setting. Nonetheless, this is not true—at least not for all $k > 1$. It can be verified that the minimum 2-abelian and 3-abelian periods of each factor of the Fibonacci word of length at most 200 are Fibonacci numbers. The same seems to hold for k-abelian periods when $k = 4, \ldots, 6$, but for $k = 7$, the situation is different. The factor 010010010100100010100101 of the Fibonacci word has minimum 7-abelian period 16. Curiously, if the definition in the second sense is used, then the minimum 2-abelian periods of the factors of the Fibonacci word of length at most 200 are exactly 1, 2, 3, 4, 5, 8, 13, 21. The factor 0010100 indeed has minimum 2-abelian period 4 in the second sense. The corresponding minimum 3-abelian periods are 1, 2, 3, 5, 6, 7, 8, 10, 13, 21.

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