

A NEW INTRINSIC METRIC AND QUASIREGULAR MAPS

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ABSTRACT. We introduce a new intrinsic metric in subdomains of a metric space and give upper and lower bounds for it in terms of well-known metrics. We also prove distortion results for this metric under quasiregular maps.

1. INTRODUCTION

Distance functions specific to a domain $G \subset \mathbb{R}^n$, $n \geq 2$, or, as we call them, intrinsic metrics, are some of the key notions of geometric function theory and are currently studied by many authors. See for instance the recent monographs [8, 9, 10, 15] and papers [4, 6, 7, 11, 12, 13]. In [8] intrinsic metrics are used as a powerful tool to analyse the properties of quasidisks and [10] provides a survey of some recent progress in the field. A list of twelve metrics recurrent in geometric function theory is given in [15, pp. 42-48].

In the classical case $n = 2$ one can define the hyperbolic metric of a simply connected domain by use of a conformal mapping given by the Riemann mapping theorem and the hyperbolic metric of the unit disk [3]. This metric is conformally invariant and therefore a most useful tool. For dimensions $n \geq 3$, by Liouville's theorem, conformal mappings $f: D \rightarrow D'$ of domains $D, D' \subset \mathbb{R}^n$ are of the form $f = g|_D$ where g is a Möbius transformation [9, pp. 64-75], [10, pp.11-12], and therefore there is no counterpart of the Riemann mapping theorem, and the planar procedure is not applicable. This state of affairs led many researchers to look for generalized hyperbolic geometries and metrics which share at least some but not all properties of the hyperbolic metric [10, Ch. 5]. For instance, the quasihyperbolic and distance ratio metrics studied in [4, 8, 9, 10] do not enjoy the full conformal invariance property for any dimension $n \geq 2$, both are invariant under similarity transformations only.

Here we study a function recently used as a tool by O. Dovgoshey, P. Hariri, and M. Vuorinen [6] and show that this function satisfies the triangle inequality and, indeed, defines an intrinsic metric of a domain. Moreover, we compare it to the distance ratio metric and find two-sided bounds for it. Finally, we study the behavior of this metric under quasiconformal mappings.

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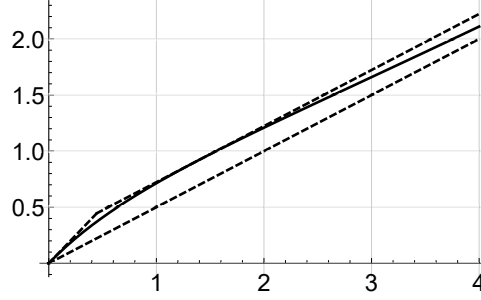


FIGURE 1. The graphs of the functions $y = \frac{t}{2}$, $y = \log(1 + 2 \sinh \frac{t}{2})$, and $y = \min\{t, \frac{t}{2} + \log \frac{5}{4}\}$ in Theorem 1.3.

For a proper nonempty open subset $D \subset \mathbb{R}^n$ and for all $x, y \in D$, the distance ratio metric j_D is defined as

$$j_D(x, y) = \log \left(1 + \frac{|x - y|}{\min\{d_D(x), d_D(y)\}} \right).$$

For a proof of the triangle inequality, see [8, Lemma 3.3.4], [1, 7.44]. If there is no danger of confusion, we write $d_D(x) = d(x) = d(x, \partial D) = \text{dist}(x, \partial D)$.

In this paper our goal is to prove that the expression (1.2) studied in [6] for $(X, \rho) = (D, j_D)$ is, in fact, a metric. We also prove several upper and lower bounds for this new metric.

Theorem 1.1. *Let (X, ρ) be a metric space and for $x, y \in X, c > 0$, let*

$$(1.2) \quad W(x, y) := \log \left(1 + 2c \sinh \frac{\rho(x, y)}{2} \right).$$

If $c \geq 1$, then W is a metric on X . Moreover, if $\rho = j_{\mathbb{B}^2}$ where \mathbb{B}^2 is the unit disk, then W is a metric on \mathbb{B}^2 if and only if $c \geq 1$.

Theorem 1.3. *Let G be a proper subdomain of \mathbb{R}^n . The following inequality holds for all $x, y \in G$*

$$\frac{j_G(x, y)}{2} \leq \log \left(1 + 2 \sinh \frac{j_G(x, y)}{2} \right) \leq \min \left\{ j_G(x, y), \frac{j_G(x, y)}{2} + \log \frac{5}{4} \right\}.$$

We conclude our paper by studying the behavior of the metric of Theorem 1.1 under quasiregular mappings defined on the unit disk and prove the following result, which is based on a recent sharp version of the Schwarz lemma for quasiregular mappings for $n = 2$ [17].

Theorem 1.4. *Let $f: \mathbb{B}^2 \rightarrow \mathbb{B}^2$ be a non-constant K -quasiregular mapping, where $K \geq 1$. Denote by $\rho = \rho_{\mathbb{B}^2}$ the hyperbolic metric and let $W_\lambda(x, y) = \log \left(1 + 2\lambda \sinh \frac{\rho(x, y)}{2} \right)$, where $\lambda \geq 1$ and let $c(K)$ be the constant in Theorem 4.1. For all $x, y \in \mathbb{B}^2$*

$$(1.5) \quad W_\lambda(f(x), f(y)) \leq 2\lambda c(K) \max \left\{ W_\lambda(x, y)^{1/K}, W_\lambda(x, y) \right\}.$$

2. PRELIMINARIES

We recall the definition of the hyperbolic distance $\rho_{\mathbb{B}^n}(x, y)$ between two points $x, y \in \mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$ [2, Thm 7.2.1, p. 130]:

$$(2.1) \quad \tanh \frac{\rho_{\mathbb{B}^n}(x, y)}{2} = \frac{|x - y|}{\sqrt{|x - y|^2 + (1 - |x|^2)(1 - |y|^2)}}.$$

One of the main properties of the hyperbolic metric is its invariance under a Möbius self-mapping $T_a: \mathbb{B}^n \rightarrow \mathbb{B}^n$, with $T_a(a) = 0$, $|a| < 1$, of the unit ball \mathbb{B}^n . In other words, the mapping T_a is an isometry. By [2, p.35] we have for $x, y \in \mathbb{H}^n = \{z \in \mathbb{R}^n : z_n > 0\}$

$$(2.2) \quad \cosh \rho_{\mathbb{H}^n}(x, y) = 1 + \frac{|x - y|^2}{2x_n y_n}.$$

For $D \in \{\mathbb{B}^n, \mathbb{H}^n\}$ and all $x, y \in D$ we have by [10, Lemma 4.9]

$$(2.3) \quad j_D(x, y) \leq \rho_D(x, y) \leq 2j_D(x, y).$$

By means of the Riemann mapping theorem one can extend the definition of the hyperbolic metric to the case of simply connected plane domains [3, Thm 6.3, p. 26].

3. A NEW METRIC

Theorem 3.1. [6] *Let D be a nonempty open set in a metric space (X, ρ) and let $\partial D \neq \emptyset$. Then the function*

$$h_{D,c}(x, y) = \log \left(1 + c \frac{\rho(x, y)}{\sqrt{d_D(x)d_D(y)}} \right),$$

is a metric for every $c \geq 2$. The constant 2 is best possible here.

This metric is listed in [5] and it has found some applications in [14].

Proposition 3.2. [6, Prop. 2.7] *For $c, t > 0$, let*

$$F_c(t) = \log \left(1 + 2c \sinh \frac{t}{2} \right).$$

Then the double inequality

$$\frac{c}{2(1+c)}t < F_c(t) < ct$$

holds for $c \geq \frac{1}{2}$ and $t > 0$.

Lemma 3.3. [6, Lemma 4.4] *Let D be a proper subdomain of \mathbb{R}^n . Then for $c > 0$ and $x, y \in D$*

$$\log \left(1 + 2c \sinh \frac{j_D(x, y)}{2} \right) \leq h_{D,c}(x, y) \leq c j_D(x, y).$$

We will now prove that the expression on the left hand side of the inequality of Lemma 3.3 satisfies the triangle inequality and for that purpose we need the following refined form of Proposition 3.2 for $c \geq 1$. This refined result and some of the lower bounds that will be proved below for the function F_c in Proposition 3.2, also lead to improved constants in some of the results of [6].

Lemma 3.4. *The function $F_c(t)/t$ is decreasing from $(0, \infty)$ onto $(1/2, c)$ if and only if $c \geq 1$.*

Proof. Let $w(t) = 1 + 2c \sinh(t/2)$. Differentiation yields

$$\left(\frac{F_c(t)}{t}\right)' = \frac{1}{t^2}g(t), \quad g(t) := \left(\frac{ct \cosh(t/2)}{w(t)} - \log(w(t))\right),$$

$$g'(t) = \frac{ct}{2(w(t))^2} \left(\sinh\left(\frac{t}{2}\right) - 2c\right).$$

The equation $\sinh\left(\frac{t}{2}\right) = 2c$ has the unique solution

$$t_1 = 2 \log\left(2c + \sqrt{4c^2 + 1}\right) > 0.$$

We have $g'(t) < 0$ for $0 < t < t_1$ and $g'(t) > 0$ for $t > t_1$. Then g is strictly decreasing on $[0, t_1]$ and strictly increasing on $[t_1, \infty)$. Note that $g(t) < g(0) = 0$ for $0 < t \leq t_1$.

Assume that the limit $L(c) := \lim_{t \rightarrow \infty} g(t)$ is finite. Then:

- a) if $L(c) \leq 0$, then $g(t) < L(c) \leq 0$ for $t > t_1$. In this case, $g(t) < 0$ for all $t > 0$. It follows that $F_c(t)/t$ is strictly decreasing.
- b) if $L(c) > 0$, then there exist a unique point $t_2 > t_1$ such that $g(t_2) = 0$. In this case $g(t) > 0$ for all $t > t_2$. It follows that $F_c(t)/t$ is strictly decreasing on $[0, t_2]$ and strictly increasing on $[t_2, \infty)$.

Now we compute $L(c)$. Setting $s = t/2$ we see that

$$L(c) = \lim_{s \rightarrow \infty} \left(\frac{2cs \cosh(s)}{w(s)} - \log(w(s))\right).$$

Now we use the change of variable $\frac{1}{\sinh(s)} = u$ when $s > 0$. Then $1 + 2c \sinh(s) = \frac{u+2c}{u}$ and $\cosh(s) = \sqrt{1 + \frac{1}{u^2}} = \frac{\sqrt{u^2+1}}{u}$. Moreover, $\sinh(s) = \frac{1}{u} > 0$ implies $s = \log\left(\frac{1}{u} + \sqrt{1 + \frac{1}{u^2}}\right)$.

Write $v = 2c \frac{\sqrt{u^2+1}}{u+2c}$. It follows that

$$(3.5) \quad L(c) = \lim_{u \rightarrow 0} \left(v \log\left(\frac{1}{u} + \sqrt{1 + \frac{1}{u^2}}\right) - \log\left(1 + \frac{2c}{u}\right)\right)$$

$$= \lim_{u \rightarrow 0} \left[v \log\left(1 + \sqrt{u^2 + 1}\right) - \log(u + 2c) + (1 - v) \log u\right],$$

where

$$\lim_{u \rightarrow 0} v \log\left(1 + \sqrt{u^2 + 1}\right) = \log 2, \quad \lim_{u \rightarrow 0} \log(u + 2c) = \log(2c).$$

The limit $\lim_{u \rightarrow 0} (1 - v) \log u$ has the indeterminate form $0 \cdot \infty$. But

$$(3.6) \quad \lim_{u \rightarrow 0} (1 - v) \log u = \lim_{u \rightarrow 0} \left(\frac{(u + 2c)^2 - 4c^2(u^2 + 1)}{(u + 2c)(u + 2c + 2c\sqrt{u^2 + 1})}\right) \log u$$

$$= \frac{1}{8c^2} \lim_{u \rightarrow 0} \left((1 - 4c^2)u^2 \log u + 4cu \log u\right).$$

Since $\lim_{u \rightarrow 0} u \log u = \lim_{u \rightarrow 0} u^2 \log u = 0$, (3.5) and (3.6) imply $L(c) = \log 2 - \log(2c)$.

We have $L(c) \leq 0$ if $c \geq 1$ and $L(c) > 0$ if $0 < c < 1$.

In conclusion, $F_c(t)/t$ is strictly decreasing on $(0, \infty)$ if and only if $c \geq 1$ and its limit values at 0 and ∞ follow easily. \square

3.7. Proof of Theorem 1.1. The proof for $c \geq 1$ follows readily from Lemma 3.4 and a general property of metrics [1, 7.42(1)]. The well-known fact that $j_G(x, y)$ is a metric is recorded e.g. in [1, 7.44].

We next show that for $c \in (0, 1)$ the function

$$W(x, y) = \log \left(1 + 2c \sinh \frac{j_{\mathbb{B}^2}(x, y)}{2} \right)$$

fails to satisfy the triangle inequality in the unit disk \mathbb{B}^2 . Write

$$E(x, y) = 1 + \frac{|x - y|}{\min\{1 - |x|, 1 - |y|\}}.$$

The inequality $W(x, z) > W(x, y) + W(y, z)$ is equivalent to

$$(3.8) \quad \frac{E(x, z) - 1}{\sqrt{E(x, z)}} > \frac{E(x, y) - 1}{\sqrt{E(x, y)}} + \frac{E(y, z) - 1}{\sqrt{E(y, z)}} + c \frac{E(x, y) - 1}{\sqrt{E(x, y)}} \frac{E(y, z) - 1}{\sqrt{E(y, z)}}.$$

Assume that $0 < y < z < 1$ and $x = -y$. In this case, (3.8) writes as

$$\begin{aligned} & \frac{z + y}{\sqrt{(1 + y)(1 - z)}} \\ & > \frac{2y}{\sqrt{(1 - y)(1 + y)}} + \frac{z - y}{\sqrt{(1 - y)(1 - z)}} + c \frac{2y(z - y)}{(1 - y)\sqrt{(1 + y)(1 - z)}}, \end{aligned}$$

which is equivalent to

$$H(y, z) := p(y)z + q(y)\sqrt{1 - z} - r(y) < 0,$$

where

$$\begin{aligned} p(y) &:= \frac{1}{\sqrt{1 - y}} + \frac{2cy}{(1 - y)\sqrt{1 + y}} - \frac{1}{\sqrt{1 + y}}, & q(y) &:= \frac{2y}{\sqrt{(1 - y)(1 + y)}} \\ r(y) &:= y \left(\frac{1}{\sqrt{1 - y}} + \frac{2cy}{(1 - y)\sqrt{1 + y}} + \frac{1}{\sqrt{1 + y}} \right). \end{aligned}$$

Note that

$$\lim_{z \nearrow 1} H(y, z) = p(y) - r(y) = \sqrt{1 - y} + \frac{2cy}{\sqrt{1 + y}} - \sqrt{1 + y}$$

and that $\lim_{y \nearrow 1} (p(y) - r(y)) = \sqrt{2}(c - 1) < 0$. Take $0 < a < 1$ such that $p(a) - r(a) < 0$. Since $\lim_{z \nearrow 1} H(a, z) = p(a) - r(a) < 0$, we may choose $a < b < 1$ such that $H(a, b) < 0$. The latter inequality implies $W(-a, b) > W(-a, a) + W(a, b)$. \square

Our next result refines, for $(2c-1)t > 1$, the upper bound in Proposition 3.2. Because Lemma 3.9 will not be used and its proof is straightforward and tedious, its proof is placed in an appendix at the end of the paper.

Lemma 3.9. *The inequality*

$$(3.10) \quad F_c(t) \leq \frac{1}{2} \frac{t^2 + (2c+1)t}{t+1},$$

holds for $c > 0$ and $t > 0$.

The next result refines the lower bound of Proposition 3.2 for $c \geq 1, t \geq 0$ and the upper bound of Lemma 3.9 for each $c > 0$ and large enough t .

Lemma 3.11. *For $t \geq 0$, the following inequalities hold*

$$(1) \quad \frac{t}{t+1} \log c + \frac{t}{2} \leq F_c(t), \quad c \geq 1,$$

$$(2) \quad F_c(t) \leq \log \left(1 + \frac{1}{4c} \right) + \frac{t}{2}, \quad c > 0.$$

Equality holds in (1) if and only if $t = 0$, respectively in (2) if and only if $c \geq \frac{1}{2}$ and $t = 2 \log(2c)$.

Proof. (1) For each fixed $t > 0$, we consider the expression

$$L_t(c) = 1 + c(e^{\frac{t}{2}} - e^{-\frac{t}{2}}) - c^{\frac{t}{t+1}} e^{\frac{t}{2}}$$

as a function of c .

The derivative is

$$L'_t(c) = e^{\frac{t}{2}} - e^{-\frac{t}{2}} - \frac{t}{t+1} c^{-\frac{1}{t+1}} e^{\frac{t}{2}},$$

and the equation $L'_t(c) = 0$ has the unique solution

$$(3.12) \quad c = c_0 := \left(\frac{t}{t+1} \frac{1}{1 - e^{-t}} \right)^{t+1}.$$

The inequality $e^t > t+1$ for $t > 0$, implies that $c_0 < 1$, and hence $L'_t(c) > 0$ holds for $c \geq 1$. Since $L_t(1) = 1 - e^{-\frac{t}{2}} > 0$, we have $L_t(c) > 0$ for $c \geq 1$.

Hence the following inequality holds for $c \geq 1$ and $t > 0$,

$$c^{\frac{t}{t+1}} e^{\frac{t}{2}} < 1 + c(e^{\frac{t}{2}} - e^{-\frac{t}{2}}).$$

Considering the logarithms of both sides, we have the assertion.

(2) By the arithmetic-geometric mean inequality, $2 \leq x + \frac{1}{x}$ holds for all $x > 0$, hence

$$2 - 2ce^{-\frac{t}{2}} \leq \frac{1}{2c} e^{\frac{t}{2}}.$$

Adding $2ce^{\frac{t}{2}}$ to the both sides of this inequality, dividing by 2 and taking the logarithm, we obtain (2).

Equality holds in (2) if and only if $2ce^{-\frac{t}{2}} = 1$, i.e. $t = 2 \log(2c)$. \square

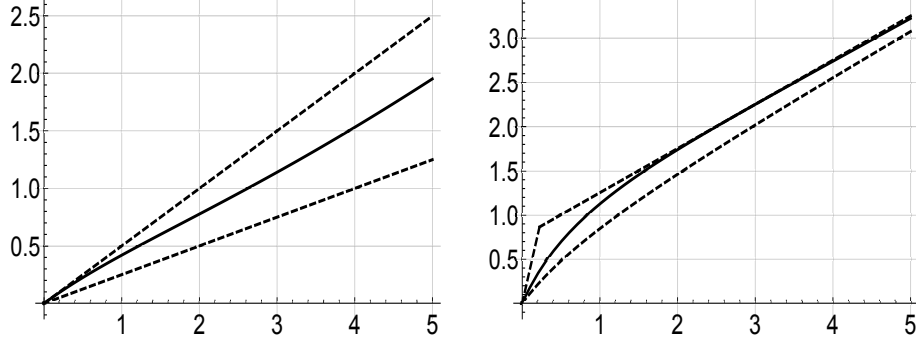


FIGURE 2. The graphs of the functions $y = l(c, t)$, $y = F_c(t)$, and $y = u(c, t)$ in Lemma 3.13.

Left: $c = 0.5$, right: $c = 2$.

Lemma 3.13. *Let*

$$(3.14) \quad l(c, t) = \begin{cases} \frac{t}{t+1} \log c + \frac{t}{2} & \text{if } c \geq 1, t \geq 0 \\ \frac{ct}{2} & \text{if } 0 < c \leq 1, t \geq 0 \end{cases}$$

and

$$(3.15) \quad u(c, t) = \min \left\{ \log \left(c + \frac{1}{4c} \right) + \frac{t}{2}, \log(1 + ct) + c(e^t - 1) \right\}.$$

Then, for $t > 0$ and $c > 0$, the following inequalities hold

$$(3.16) \quad l(c, t) < F_c(t) \leq u(c, t).$$

The upper bound is attained, for $t > 0$ and $c > 0$, if and only if $c > \frac{1}{2}$ and $t = 2 \log(2c)$.

Proof. We first prove the lower bounds. By Lemma 3.11 it is enough to prove the case $0 < c \leq 1$. For each fixed $t > 0$, let

$$f_t(c) = 1 + c(e^{\frac{t}{2}} - e^{-\frac{t}{2}}) - e^{\frac{ct}{2}}.$$

Then $f_t(0) = 0$ and

$$f'_t(c) = e^{\frac{t}{2}} - e^{-\frac{t}{2}} - \frac{t}{2} e^{\frac{ct}{2}}$$

is decreasing on the real axis. Because $f'_t(0) = e^{\frac{t}{2}} - e^{-\frac{t}{2}} - \frac{t}{2} > 0$, the sign of $f'_t(1) = e^{\frac{t}{2}} \left(1 - \frac{t}{2} - e^{-t} \right)$ depends on t . The equation $1 - \frac{t}{2} - e^{-t} = 0$ has a unique positive solution $t_0 \in (1, 2)$ and we have $f'_t(1) > 0$ for $0 < t < t_0$ and $f'_t(1) < 0$ for $t > t_0$.

If $0 < t \leq t_0$, it follows that f_t is increasing on $[0, 1]$, therefore $f_t(c) > 0$ for $0 < c \leq 1$.

If $t > t_0$, there is a unique positive critical point $c = c_0$ as a solution of

$$f'_t(c) = 0.$$

The function f_t attains the maximum at $c = c_0$. Since $f_t(0) = 0$ and $f_t(1) = 1 - e^{-\frac{t}{2}} > 0$, we have $f_t(c) > 0$ for $0 < c \leq 1$. Therefore

$$e^{\frac{ct}{2}} < 1 + 2c \sinh \frac{t}{2}$$

holds for $0 < c \leq 1$ which yields the desired inequality.

We now prove the upper bound. By Lemma 3.11 it is enough to prove that

$$\log\left(1 + 2c \sinh \frac{t}{2}\right) < \log(1 + ct) + c(e^t - 1)$$

holds for $c > 0$ and $t > 0$. For each fixed $t > 0$, let

$$g_t(c) = (1 + ct)e^{c(e^t - 1)} - 1 - c(e^{\frac{t}{2}} - e^{-\frac{t}{2}}).$$

Then, $g_t(c)$ is increasing on $(0, \infty)$, because

$$g'_t(c) = e^{c(e^t - 1)}((1 + ct)(e^t - 1) + t) - (e^{\frac{t}{2}} - e^{-\frac{t}{2}})$$

is clearly increasing and

$$g'_t(0) = e^t - 1 + t - (e^{\frac{t}{2}} - e^{-\frac{t}{2}}) = (e^{\frac{t}{2}} - e^{-\frac{t}{2}})(e^{\frac{t}{2}} - 1) + t > 0.$$

Therefore, $g_t(c) > 0$ holds as $g_t(0) = 0$. Hence, we have

$$1 + c(e^{\frac{t}{2}} - e^{-\frac{t}{2}}) < (1 + ct)e^{c(e^t - 1)},$$

which yields the assertion.

Moreover, for $t > 0$, $c > 0$, the equation $\log\left(1 + 2c \sinh \frac{t}{2}\right) = u(c, t)$ leads to

$$\log\left(1 + 2c \sinh \frac{t}{2}\right) = \log\left(c + \frac{1}{4c}\right) + \frac{t}{2} < \log(1 + ct) + c(e^t - 1),$$

which holds if and only if $c > \frac{1}{2}$ and $t = 2 \log(2c)$. \square

3.17. Proof of Theorem 1.3. The upper bound follows from Lemmas 3.3, 3.11 and the lower bound from Lemma 3.11. \square

The above results readily give the following theorem.

Theorem 3.18. *For points $x, y \in G$, and a number $c \geq 1$, we have*

$$L j_G(x, y) \leq W(x, y) \leq U j_G(x, y)$$

where W is the metric

$$W(x, y) = \log\left(1 + 2c \sinh \frac{j_G(x, y)}{2}\right)$$

and

$$L = \frac{1}{2} + \frac{\log(c)}{1 + j_G(x, y)}, \quad U = \frac{j_G(x, y) + (2c + 1)}{2(1 + j_G(x, y))}.$$

Proposition 3.19. *Let (G, ρ_G) and (D, ρ_D) be two metric spaces and let $\omega_{G,c} = \log\left(1 + 2c \sinh \frac{\rho_G}{2}\right)$ and $\omega_{D,c} = \log\left(1 + 2c \sinh \frac{\rho_D}{2}\right)$, where $c \geq 1$. If $f: (G, \rho_G) \rightarrow (D, \rho_D)$ is an L -Lipschitz function, then $f: (G, \omega_{G,c}) \rightarrow (D, \omega_{D,c})$ is L' -Lipschitz, where $L' = L$ if $L \geq 1$ and $L' = cL$ if $L > 0$. Conversely, if $f: (G, \omega_{G,c}) \rightarrow (D, \omega_{D,c})$ is an L' -Lipschitz function with $L' > 0$, then $f: (G, \rho_G) \rightarrow (D, \rho_D)$ is an $2cL'$ -Lipschitz function.*

Proof. Denote $F_c(t) = \log(1 + 2c \sinh(\frac{t}{2}))$ as in Lemma 3.4. By Theorem 1.1, $\omega_{G,c} = F_c \circ \rho_G$ and $\omega_{D,c} = F_c \circ \rho_D$ are metrics on G and D , respectively.

Fix distinct points $x, y \in G$.

Assume that $f: (G, \rho_G) \rightarrow (D, \rho_D)$ is L -Lipschitz. We have to prove that

$$(3.20) \quad F_c(\rho_D(f(x), f(y))) \leq L' F_c(\rho_G(x, y)),$$

where $L' = L$ if $L \geq 1$ and $L' = cL$ whenever $L > 0$.

Since $f: (G, \rho_G) \rightarrow (D, \rho_D)$ is L -Lipschitz and F_c is increasing,

$$(3.21) \quad F_c(\rho_D(f(x), f(y))) \leq F_c(L\rho_G(x, y)).$$

Assume first that $L \geq 1$. By Lemma 3.4, $\frac{F_c(t)}{t}$ is decreasing on $(0, \infty)$, therefore $F_c(Lt) \leq LF_c(t)$ for all $t > 0$, as $L \geq 1$. Then $F_c(L\rho_G(x, y)) \leq LF_c(\rho_G(x, y))$. The latter inequality and (3.21) imply (3.20) with $L' = L$.

By Lemma 3.4, $\frac{t}{2} \leq F_c(t) \leq ct$ for all $t \geq 0$, if $c \geq 1$.

For all $L > 0$, (3.21) implies

$$F_c(\rho_D(f(x), f(y))) \leq F_c(L\rho_G(x, y)) \leq cL\rho_G(x, y).$$

Now assume that (3.20) holds. Then

$$\begin{aligned} \rho_D(f(x), f(y)) &\leq 2F_c(\rho_D(f(x), f(y))) \leq L' F_c(\rho_G(x, y)) \\ &\leq 2cL' \rho_G(x, y). \end{aligned}$$

□

4. METRICS AND QUASIREGULAR MAPS

If $D \in \{\mathbb{B}^n, \mathbb{H}^n\}$ and ρ_D is the hyperbolic metric on D , then the metric defined on D by $W_c(x, y) = \log\left(1 + 2c \sinh\left(\frac{\rho_D(x, y)}{2}\right)\right)$, where $c \geq 1$, is invariant under Möbius self maps of D , due to the Möbius invariance of the hyperbolic metric.

We also recall some notation about special functions and the fundamental distortion result of quasiregular maps, a variant of the Schwarz lemma for these maps, see [10]. For $r \in (0, 1)$ and $K > 0$, we define the distortion function

$$\varphi_K(r) = \mu^{-1}(\mu(r)/K),$$

where $\mu(r)$ is the modulus of the planar Grötzsch ring, a decreasing homeomorphism $\mu: (0, 1) \rightarrow (0, \infty)$, see [1, pp. 92-94], [10, pp.120-125].

Theorem 4.1. *Let G_1 and G_2 be simply-connected domains in \mathbb{R}^2 and let $f: G_1 \rightarrow G_2 = f(G_1)$ be a K -quasiregular mapping. Then for all $x, y \in G_1$*

$$\rho_{G_2}(f(x), f(y)) \leq c(K) \max\{\rho_{G_1}(x, y), \rho_{G_1}(x, y)^{1/K}\}$$

where $c(K)$ is as in [10, Thm 16.39, p. 313], [17, Theorem 3.6].

Remark 4.2. By [10, Thm 16.39, p. 313],

$$K \leq c(K) \leq \log(2(1 + \sqrt{1 - 1/e^2}))(K - 1) + K$$

and, in particular, $c(K) \rightarrow 1$, when $K \rightarrow 1$.

4.3. Proof of Theorem 1.4. By Theorem 4.1 [10, Thm 16.39],

$$(4.4) \quad \rho(f(x), f(y)) \leq c(K) \max \left\{ \rho(x, y)^{1/K}, \rho(x, y) \right\}.$$

According to Lemma 3.4,

$$(4.5) \quad \frac{t}{2} < \log \left(1 + 2\lambda \sinh \frac{t}{2} \right) < \lambda t \text{ for every } t \in (0, \infty).$$

Then for all $x, y \in \mathbb{B}^2$

$$\begin{aligned} W_\lambda(f(x), f(y)) &\leq \lambda \rho(f(x), f(y)) \leq \lambda c(K) \max \left\{ \rho(x, y)^{1/K}, \rho(x, y) \right\} \\ &\leq \lambda c(K) \max \left\{ 2^{1/K} W_\lambda(x, y)^{1/K}, 2W_\lambda(x, y) \right\} \end{aligned}$$

and (1.5) follows. \square

5. APPENDIX

We give here the proof of Lemma 3.9. We shall apply the inequality

$$(5.1) \quad 1 - e^{-t} = \frac{e^t - 1}{e^t} > \frac{t}{t+1}$$

which easily follows from $e^t - (t+1) > 0, t > 0$.

5.2. Proof of Lemma 3.9. The right hand side of (3.10) can be written as $\frac{1}{2}t + \frac{ct}{t+1}$. Taking the exponential function of both sides of (3.10), we need to check that for each fixed $t > 0$ the following inequality holds,

$$(5.3) \quad E_t(c) := e^{\frac{t}{2}} \cdot e^{\frac{ct}{t+1}} - 1 - c(e^{\frac{t}{2}} - e^{-\frac{t}{2}}) > 0, \quad (c > 0).$$

First, we remark that $E_t(0) = e^{\frac{t}{2}} - 1 > 0$ holds for $t > 0$.

Next, we will show that $E'_t(c) > 0$ holds for $c > 0$. It is clear that the derivative

$$(5.4) \quad E'_t(c) = e^{\frac{t}{2}} \left(\frac{t}{t+1} e^{\frac{t}{t+1}c} - 1 + e^{-t} \right)$$

is an increasing function with respect to c , and satisfies $\lim_{c \rightarrow \infty} E'_t(c) = \infty$. From (5.1), we have

$$E'_t(0) = e^{\frac{t}{2}} \left(\frac{t}{t+1} - (1 - e^{-t}) \right) < 0.$$

Hence, $E'_t(c) = 0$ has the unique positive root c_0 , and $E_t(c)$ attains the minimum at $c = c_0$. Moreover, the root c_0 is given by the formula

$$c_0 = \frac{t+1}{t} \log \left(\frac{t+1}{t} (1 - e^{-t}) \right).$$

Then,

$$E_t(c_0) = \frac{t+1}{t} \left(1 - \log \left(\frac{t+1}{t} (1 - e^{-t}) \right) \right) (e^{\frac{t}{2}} - e^{-\frac{t}{2}}) - 1.$$

Therefore, we need to show that

$$(5.5) \quad C(t) := \left(1 - \log\left(\frac{t+1}{t}(1 - e^{-t})\right)\right)(e^{\frac{t}{2}} - e^{-\frac{t}{2}}) - \frac{t}{t+1} > 0 \quad (t > 0).$$

By using (5.1), we have

$$\begin{aligned} C(t) &\geq \left(1 - \log\left(\frac{t+1}{t}(1 - e^{-t})\right)\right)(e^{\frac{t}{2}} - e^{-\frac{t}{2}}) - \frac{e^t - 1}{e^t} \\ &= \left(1 - \log\left(\frac{t+1}{t}(1 - e^{-t})\right) - e^{-\frac{t}{2}}\right)(e^{\frac{t}{2}} - e^{-\frac{t}{2}}). \end{aligned}$$

Since $e^{\frac{t}{2}} - e^{-\frac{t}{2}} > 0$, we have to check that

$$\begin{aligned} \tilde{C}(t) &:= 1 - \log\left(\frac{t+1}{t}(1 - e^{-t})\right) - e^{-\frac{t}{2}} \\ &= 1 - \log(t+1) + \log t - \log(e^t - 1) + t - e^{-\frac{t}{2}} > 0. \end{aligned}$$

Thus $\tilde{C}(0) = 1 - \log 1 - e^0 = 0$ (note that $\lim_{t \rightarrow 0} \frac{1 - e^{-t}}{t} = 1$), and

$$\begin{aligned} \tilde{C}'(t) &= \frac{1}{t} - \frac{1}{t+1} - \frac{e^t}{e^t - 1} + 1 + \frac{1}{2}e^{-\frac{t}{2}} \\ &= \frac{2(e^t - 1) - 2t(t+1) + t(t+1)(e^{\frac{t}{2}} - e^{-\frac{t}{2}})}{2t(t+1)(e^t - 1)}. \end{aligned}$$

Applying $2t(t+1)(e^t - 1) > 0$, we need to check that

$$\hat{C}(t) := 2(e^t - 1) - 2t(t+1) + t(t+1)(e^{\frac{t}{2}} - e^{-\frac{t}{2}}) > 0.$$

Then, $\hat{C}(0) = 0$ and

$$\hat{C}'(t) = 2e^t - 2(2t+1) + (2t+1)(e^{\frac{t}{2}} - e^{-\frac{t}{2}}) + \frac{t(t+1)}{2}(e^{\frac{t}{2}} + e^{-\frac{t}{2}}).$$

Similarly, $\hat{C}'(0) = 0$ and

$$\begin{aligned} \hat{C}''(t) &= 2(e^t - 1) + (e^{\frac{t}{2}} + e^{-\frac{t}{2}} - 2) + 2t(e^{\frac{t}{2}} + e^{-\frac{t}{2}}) \\ &\quad + \frac{t(t+1) + 8}{4}(e^{\frac{t}{2}} - e^{-\frac{t}{2}}). \end{aligned}$$

Therefore, $\hat{C}''(t) > 0$, since the coefficient of each term is positive for $t > 0$. Then, $\hat{C}'(t) > 0$, since $\hat{C}'(t)$ is increasing function with $\hat{C}'(0) = 0$. Similarly, $\hat{C}(t) > 0$.

Hence $\tilde{C}(t) > \lim_{s \searrow 0} \tilde{C}(s) = 0$ for all $t > 0$.

In conclusion, we have $C(t) > 0$ for each $t > 0$. Finally, we have $E_t(c) > 0$ ($c > 0$) for each $t > 0$, and we have the assertion. \square

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