Weakly Unambiguous Morphisms with Respect to Sets of Patterns with Constants

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Abstract. A non-erasing morphism is weakly unambiguous with respect to a pattern if no other non-erasing morphism maps the pattern to the same image. If the size of the target alphabet is at least three, then the patterns for which there exists a length-increasing weakly unambiguous morphism can be characterized using the concept of loyal neighbors of variables. In this article this characterization is generalized for patterns with constants. Two different generalizations are given for sets of patterns.

1 Introduction

Many fundamental topics of combinatorics on words are defined in terms of morphisms. One example is equality sets and the Post Correspondence Problem: Given two morphisms f and g, does there exist a non-empty word w such that $f(w) = g(w)$. Another example is given by word equations: A solution of a word equation $u = v$ is a morphism h such that $h(u) = h(v)$. For more on these and several other topics related to morphisms, see [5]. Also the theory of codes is concerned with morphisms [1], as is the theory of pattern languages [7].

This central role of morphisms in combinatorics on words means that it is important to understand the behavior of morphisms. For example, this might lead to the study of fixed points of morphisms, see e.g [6] and [11], or to the concept of unambiguity of morphisms, which is the topic of this paper.

A morphism is said to be unambiguous with respect to a pattern (or a word) if no other morphism maps the pattern to the same image. Unambiguity of morphisms was introduced by Freydenberger, Reidenbach and Schneider [4]. Two questions that have been studied in many papers [4, 3, 10, 9] are:

- For which patterns does there exist an unambiguous morphism?
- For which patterns does there exist a non-erasing unambiguous morphism?

Unambiguity is closely related to pattern languages, see e.g [8].

Many variations of unambiguity of morphisms exist. For example, it is possible to study unambiguity in the free semigroup, that is, assume that all morphism

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are non-erasing. This leads to the definition of weakly unambiguous morphisms: A non-erasing morphism is said to be weakly unambiguous with respect to a pattern if no other non-erasing morphism maps the pattern to the same image. Trivially, every 1-uniform morphism is weakly unambiguous with respect to every pattern, so the interesting question in this case is the following:

– For which patterns does there exist a non-erasing length-increasing weakly unambiguous morphism?

This question was studied by Freydenberger, Nevisi and Reidenbach [2]. Questions on unambiguity of morphisms often lead to complicated technical considerations, but the results on weakly unambiguous morphisms are relatively elegant. If the target alphabet is unary, then the question is quite simple, although not trivial. If the size of the target alphabet is at least three, then a characterization can be obtained by using so called loyal neighbors of variables. The binary case is complicated and only partial results are known.

In many questions about morphism, there can be constants (or terminal symbols), i.e. letters which must be mapped to themselves. For example, constants are often used in the theory of pattern languages. However, unambiguity has mostly been studied from the point of view of constant-free patterns. In this article weak unambiguity is studied for patterns with constants. We concentrate on the case of target alphabets with at least three letters. If the definition of loyal neighbors of variables is extended for patterns with constants in the right way, then also the characterization from [2] can be extended quite straightforwardly.

As another generalization, weak unambiguity with respect to several patterns is studied in this paper. If the patterns are constant-free, then a characterization that is similar to the one in [2] can be found easily. However, if the two generalizations are studied at the same time, that is there are many patterns with constants, then the situation is more complicated. The same characterization works only if the size of the target alphabet is at least two more than the number of patterns.

There is also another way to generalize unambiguity for sets of patterns. Instead of considering every pattern separately, they can be treated, in a sense, as a single pattern. Weak unambiguity and loyal neighbors can then be defined for sets of patterns and an analogous characterization can be proved.

Although this paper concentrates on weakly unambiguous morphisms, and only on the case of ternary or larger alphabets, it seems likely that unambiguity with respect to patterns with constants and with respect to multiple patterns could be studied also more generally.

2 Patterns with Constants

Let Σ be an alphabet of *constants* and Ξ an alphabet of *variables*. A word $\alpha \in (\Xi \cup \Sigma)^+$ is called a *pattern*. If $\alpha \in \Xi^*$, then α is *constant-free*. If Γ is the set of those variables that appear in α, then α is a *Γ-pattern*.

The *empty word* is denoted by ε, the length of a word w by $|w|$, and the number of occurrences of a letter a in w by $|w|_a$.

A *morphism* is a mapping $h : (\Xi \cup \Sigma)^* \to \Sigma^*$ such that $h(\alpha\beta) = h(\alpha)h(\beta)$ for all $\alpha, \beta \in (\Xi \cup \Sigma)^*$ and $h(a) = a$ for all $a \in \Sigma$. Thus all morphisms are assumed to be constant-preserving. A morphism h is *non-erasing* if $h(x) \neq \varepsilon$ for all $x \in \Xi$. A non-erasing morphism h is *Γ-increasing* if $|h(x)| \geq 2$ for some $x \in \Gamma$. Two morphisms h and g are *Γ-equivalent* if $h(x) = g(x)$ for all $x \in \Gamma$. This is denoted by $h \sim_\Gamma g$, and non-equivalence is denoted by $h \nsim_\Gamma g$

Let α be a Γ-pattern. A non-erasing morphism h is *weakly unambiguous with respect to* α if there is no non-erasing morphism $g \nsim_\Gamma h$ such that $h(\alpha) = g(\alpha)$. It is easy to see that if h is not Γ-increasing, then h is weakly unambiguous with respect to every Γ-pattern. Thus we study the following question: Given a Γ-pattern α, does there exist a Γ-increasing morphism that is weakly unambiguous with respect to α?

Example 1. Let $\Xi = \{x, y\}$ and $\Sigma = \{a, b\}$. Consider the pattern xay. The morphism defined by $x \mapsto a, y \mapsto ba$ is weakly unambiguous with respect to xay, because no other non-erasing morphism maps xay to $aaba$. The morphism defined by $x \mapsto a, y \mapsto ab$ is not weakly unambiguous with respect to xay, because also the morphism defined by $x \mapsto aa, y \mapsto b$ maps xay to $aaab$.

If the alphabet Σ is unary, say $\Sigma = \{a\}$, then the addition of constants in patterns is not very interesting. Let $\alpha \in (\Xi \cup \{a\})^+$ and let α' be the pattern obtained from α by removing every occurrence of a. For all morphisms h and g, $h(\alpha) = g(\alpha)$ if and only if $h(\alpha') = g(\alpha')$. Thus h is weakly unambiguous with respect to α if and only if it is weakly unambiguous with respect to the constant-free pattern α' and the result in [2] can be used directly.

If the alphabet Σ is binary, then only partial results are known on weak unambiguity of morphisms with respect to constant-free patterns. In this article we concentrate on the case where Σ has at least three letters, since this case is well-understood for constant-free patterns.

Let $\alpha = a_0 a_1 \dots a_n a_{n+1}$, where $a_0 = a_{n+1} = \varepsilon$ and $a_1, \dots, a_n \in \Xi \cup \Sigma$. The set of *left neighbors of x in α* is

$$L_\alpha(x) = \{a_i \mid 0 \leq i \leq n, a_{i+1} = x\},$$

and the set of *right neighbors of x in α* is

$$R_\alpha(x) = \{a_i \mid 1 \leq i \leq n+1, a_{i-1} = x\}.$$

Both $L_\alpha(x)$ and $R_\alpha(x)$ are subsets of $\Xi \cup \Sigma \cup \{\varepsilon\}$.

It was defined in [2] that if α is a constant-free pattern, then a variable x has *loyal neighbors in α* if at least one of the following two conditions is satisfied:

$$\varepsilon \notin L_\alpha(x) \text{ and } R_\alpha(y) = \{x\} \text{ for all } y \in L_\alpha(x),$$
$$\varepsilon \notin R_\alpha(x) \text{ and } L_\alpha(y) = \{x\} \text{ for all } y \in R_\alpha(x).$$

This definition must be generalized for patterns with constants. This is done by treating the constants in the same way as the beginning and end of the pattern (or in the same way as ε in $L_\alpha(x)$ and $R_\alpha(x)$). So, given a pattern α with constants, a variable x has *loyal neighbors in* α if at least one of the following two conditions is satisfied:

$$L_\alpha(x) \subseteq \Xi \text{ and } R_\alpha(y) = \{x\} \text{ for all } y \in L_\alpha(x), \tag{1}$$

$$R_\alpha(x) \subseteq \Xi \text{ and } L_\alpha(y) = \{x\} \text{ for all } y \in R_\alpha(x). \tag{2}$$

Theorem 6 justifies that this is the right definition.

Example 2. Let $\Xi = \{x, y, z, t\}$, $\Sigma = \{a\}$, and $\alpha = xayzyt$. The variable y has loyal neighbors in α because $R_\alpha(y) = \{z, t\}$ and $L_\alpha(z) = L_\alpha(t) = \{y\}$. The other variables do not have loyal neighbors in α:

- x does not, because $\varepsilon \in L_\alpha(x)$ and $a \in R_\alpha(x)$.
- z does not, because $L_\alpha(z) = \{y\}$ but $R_\alpha(y) \neq \{x\}$, and $R_\alpha(z) = \{y\}$ but $L_\alpha(y) \neq \{x\}$.
- t does not, because $L_\alpha(t) = \{y\}$ but $R_\alpha(y) \neq \{t\}$, and $\varepsilon \in R_\alpha(t)$.

Next we will characterize, in the case $\#\Sigma \geq 3$, those Γ-patterns with respect to which there exists a Γ-increasing weakly unambiguous morphism. There are many similarities between the proofs here and the proofs in [2]. The proofs are self-contained, so we do not need to refer to any previous results.

Lemma 3. *Let $u_1, \ldots, u_n, v_1, \ldots, v_n \in \Sigma^*$. If $u_1 \ldots u_n$ is a factor of $v_1 \ldots v_n$, then either $u_i = v_i$ for all i or u_i is a proper factor of v_i for some i.*

Proof. Let $v_1 \ldots v_n = u_0 u_1 \ldots u_n u_{n+1}$ and consider the numbers

$$k_i = |v_1 \ldots v_i| - |u_0 \ldots u_i|$$

for $i \in \{0, \ldots, n\}$.

If $k_i = 0$ for all i, then $u_i = v_i$ for all i.

If $k_i < 0$ for some i, then let j be the largest index such that $k_j < 0$. Because $k_n \geq 0$, it must be $j < n$, and $k_{j+1} \geq 0$. This means that u_{j+1} is a proper factor of v_{j+1}.

If $k_i > 0$ for some i, then let j be the smallest index such that $k_j > 0$. Because $k_0 \leq 0$, it must be $j > 0$, and $k_{j-1} \leq 0$. This means that u_j is a proper factor of v_j. □

Lemma 4. *Let α be a Γ-pattern and h a non-erasing morphism. If there is $x \in \Gamma$ such that $|h(x)| > 1$ and x has loyal neighbors in α, then h is not weakly unambiguous with respect to α.*

Proof. Assume that (1) is satisfied for x (the case where (2) is satisfied is symmetric). It must be $x \notin L_\alpha(x)$, because it is not possible that $R_\alpha(x) = \{x\}$. Let $h(x) = au$ where $a \in \Sigma$ and $u \in \Sigma^+$. If g is the morphism defined by $g(x) = u$, $g(y) = h(y)a$ for all $y \in L_\alpha(x)$ and $g(z) = h(z)$ for all $z \in \Xi \smallsetminus L_\alpha(x) \smallsetminus \{x\}$, then $h(\alpha) = g(\alpha)$. □

Lemma 5. *Let α be a Γ-pattern and x a variable that does not have loyal neighbors in α. Let $a, b, c \in \Sigma$ be different letters such that $L_\alpha(x) \cap \Sigma \neq \{a\}$ and $R_\alpha(x) \cap \Sigma \neq \{b\}$. The morphism h defined by $h(x) = ab$ and $h(y) = c$ for all $y \in \Xi \setminus \{x\}$ is weakly unambiguous with respect to α.*

Proof. We assume that $g \not\sim_\Gamma h$ is a Γ-increasing morphism such that $h(\alpha) = g(\alpha)$ and derive a contradiction. Let $\alpha = a_1 \dots a_n$, where $a_1, \dots, a_n \in \Xi \cup \Sigma$. Lemma 3 is used with $g(a_1), \dots, g(a_n)$ as $u_1, \dots, u_n$ and $h(a_1), \dots, h(a_n)$ as $v_1, \dots, v_n$. Because $g \not\sim_\Gamma h$, it follows from Lemma 3 that there is an i such that $g(a_i)$ is a proper factor of $h(a_i)$. In particular, $|h(a_i)| > |g(a_i)| \geq 1$, so $a_i = x$. Thus $g(x)$ is a proper factor of $h(x) = ab$. By symmetry, it can be assumed that $g(x) = a$. Then $g(y)$ cannot contain a's for any variable $y \in \Gamma \setminus \{x\}$, because otherwise $g(\alpha)$ would contain more a's than $h(\alpha)$.

Let $|\alpha|_x = k$ and

$$\alpha = w_0 x w_1 x \dots w_{k-1} x w_k.$$

If $j = |w_0 \dots w_{i-1}|_a + i$ for some $i \in \{1, \dots, k\}$, then the jth a in $h(\alpha) = g(\alpha)$ is followed by b. Thus $g(y)$ begins with b for all $y \in R_\alpha(x)$. This means that $\varepsilon, d \notin R_\alpha(x)$ for all $d \in \Sigma \setminus \{b\}$. By the definition of b, $R_\alpha(x) \subseteq \Xi$.

The number of b's in $h(\alpha)$ is

$$|\alpha|_b + |\alpha|_x$$

and in $g(\alpha)$ it is at least

$$|\alpha|_b + \sum_{y \in R_\alpha(x)} |\alpha|_y.$$

These numbers should be the same, but because x does not have loyal neighbors in α,

$$|\alpha|_x < \sum_{y \in R_\alpha(x)} |\alpha|_y.$$

This is a contradiction. □

Theorem 6. *Let $\#\Sigma \geq 3$ and let α be a Γ-pattern. There is a Γ-increasing morphism h that is weakly unambiguous with respect to α if and only if at least one variable does not have loyal neighbors in α.*

Proof. Assume first that all variables have loyal neighbors in α and h is a Γ-increasing morphism. Then some variable x satisfies the conditions of Lemma 4, so h is not weakly unambiguous with respect to α.

Assume then that a variable x does not have loyal neighbors in α. Because $\#\Sigma \geq 3$, the three letters of Lemma 5 exist, and there is a Γ-increasing morphism that is weakly unambiguous with respect to α. □

Theory of word equations was mentioned in the introduction as one area where morphisms are important. Theorem 6 can be formulated in terms of word equations, although this is probably just a curiosity.

Corollary 7. *Let $\alpha \in (\Xi \cup \Sigma)^+$. There is a $\beta \in \Sigma^+$ such that $|\beta| > |\alpha|$ and the word equation $\alpha = \beta$ has a unique non-erasing solution if and only if at least one variable does not have loyal neighbors in α.*

3 Many Patterns

Weak unambiguity can be generalized for sets of patterns in two ways. The first way is to study the existence of morphisms that are weakly unambiguous with respect to multiple patterns. The next theorem proves a result about constant-free patterns.

Theorem 8. *Let $\#\Sigma \geq 3$ and let α_i be a constant-free Γ_i-pattern for each $i \in \{1, \ldots, n\}$. Let $\Gamma = \bigcap_{i=1}^{n} \Gamma_i$. There is a Γ-increasing morphism h that is weakly unambiguous with respect to every α_i if and only if at least one variable does not have loyal neighbors in any α_i.*

Proof. Assume first that for every variable x there is an index i_x such that x has loyal neighbors in α_{i_x}. Assume also that h is a Γ-increasing morphism. There is a variable x such that $|h(x)| > 1$. By Lemma 4, h is not weakly unambiguous with respect to α_{i_x}.

Assume then that a variable x does not have loyal neighbors in any α_i. Because the patterns are constant-free, any three letters a, b, c satisfy the conditions of Lemma 5, and there is a Γ-increasing morphism that is weakly unambiguous with respect to every α_i. □

To generalize Theorem 8 for patterns with constants, a larger alphabet Σ is needed.

Theorem 9. *Let $\#\Sigma \geq n+2$ and let α_i be a Γ_i-pattern for each $i \in \{1, \ldots, n\}$. Let $\Gamma = \bigcap_{i=1}^{n} \Gamma_i$. There is a Γ-increasing morphism h that is weakly unambiguous with respect to every α_i if and only if at least one variable does not have loyal neighbors in any α_i.*

Proof. Assume first that for every variable x there is an index i_x such that x has loyal neighbors in α_{i_x}. Assume also that h is a Γ-increasing morphism. There is a variable x such that $|h(x)| > 1$. By Lemma 4, h is not weakly unambiguous with respect to α_{i_x}.

Assume then that a variable x does not have loyal neighbors in any α_i. There can be at most n letters a such that $L_{\alpha_i}(x) \cap \Sigma = \{a\}$ for some i, so there is a letter a such that $L_{\alpha_i}(x) \cap \Sigma \neq \{a\}$ for all i. There can be at most n letters b such that $R_{\alpha_i}(x) \cap \Sigma = \{b\}$ for some i, so there is a letter $b \neq a$ such that $R_{\alpha_i}(x) \cap \Sigma \neq \{b\}$ for all i. By Lemma 5, there is a Γ-increasing morphism that is weakly unambiguous with respect to every α_i. □

The next example shows that the assumption $\#\Sigma \geq n+2$ in Theorem 9 is necessary. Finding a characterization for smaller alphabets remains an open question. It is of course possible that this question is very complicated, like in the binary case for patterns with constants.

Example 10. Let $\Xi = \{x, y_1, y_2, z_1, z_2, t_1, t_2\}$ and $\Sigma = \{a_1, \ldots, a_n, b\}$. Let $a_0 = a_n$ and

$$\alpha_i = y_1 y_2 a_i x z_1 z_2 x a_{i+1} t_1 t_2$$

for $i \in \{0, \dots, n-1\}$. The variable x does not have loyal neighbors in any α_i, but there does not exist a Ξ-increasing morphism that would be weakly unambiguous with respect to every α_i. This can be seen as follows. If h would be a Ξ-increasing morphism that is weakly unambiguous with respect to α_0, then $|h(x)| > 1$ by Lemma 4, because all variables except x have loyal neighbors in α_0. If $h(x)$ starts with a_i, say $h(x) = a_i u$, and g is the morphism defined by $g(x) = u$, $g(y_2) = h(y_2)a_i$, $g(z_2) = h(z_2)a_i$ and $g(s) = h(s)$ for other variables s, then $h(\alpha_i) = g(\alpha_i)$. Similarly, if $h(x)$ ends with a_{i+1}, then h is not weakly unambiguous with respect to α_i. The only possibility is that $h(x) = bub$. But if g is the morphism defined by $g(x) = b$, $g(z_1) = ubh(z_1)$, $g(z_2) = h(z_2)bu$ and $g(s) = h(s)$ for other variables s, then $h(\alpha) = g(\alpha)$.

4 Sets of Patterns

The second way to generalize weak unambiguity for sets of patterns is to use the following definitions.

If A is a set of patterns and Γ is the set of those variables that appear in some $\alpha \in A$, then A is a *Γ-set of patterns*.

Let A be a Γ-set of patterns. A non-erasing morphism h is *weakly unambiguous with respect to A* if there is no non-erasing morphism $g \not\sim_\Gamma h$ such that $h(\alpha) = g(\alpha)$ for every $\alpha \in A$.

The set of *left neighbors of x in A* is

$$L_A(x) = \bigcup_{\alpha \in A} L_\alpha(x)$$

and the set of *right neighbors of x in A* is

$$R_A(x) = \bigcup_{\alpha \in A} R_\alpha(x)$$

A variable x has *loyal neighbors in A* if at least one of the following two conditions is satisfied:

$$L_A(x) \subseteq \Xi \text{ and } R_A(y) = \{x\} \text{ for all } y \in L_A(x), \tag{3}$$

$$R_A(x) \subseteq \Xi \text{ and } L_A(y) = \{x\} \text{ for all } y \in R_A(x). \tag{4}$$

Lemmas 11 and 12 and Theorem 13 are simple modifications of Lemmas 4 and 5 and Theorem 6.

Lemma 11. *Let A be a Γ-set of patterns and h a non-erasing morphism. If there is $x \in \Gamma$ such that $|h(x)| > 1$ and x has loyal neighbors in A, then h is not weakly unambiguous with respect to A.*

Proof. Assume that (3) is satisfied for x (the case where (4) is satisfied is symmetric). It must be $x \notin L_A(x)$, because it is not possible that $R_A(x) = \{x\}$. Let $h(x) = au$ where $a \in \Sigma$ and $u \in \Sigma^+$. If g is the morphism defined by $g(x) = u$, $g(y) = h(y)a$ for all $y \in L_A(x)$ and $g(z) = h(z)$ for all $z \in \Xi \setminus L_A(x) \setminus \{x\}$, then $h(\alpha) = g(\alpha)$ for every $\alpha \in A$. $\square$

Lemma 12. *Let A be a Γ-set of patterns and x a variable that does not have loyal neighbors in A. Let $a, b, c \in \Sigma$ be different letters such that $L_A(x) \cap \Sigma \neq \{a\}$ and $R_A(x) \cap \Sigma \neq \{b\}$. The morphism h defined by $h(x) = ab$ and $h(y) = c$ for all $y \in \Xi \setminus \{x\}$ is weakly unambiguous with respect to A.*

Proof. We assume that $g \not\sim_\Gamma h$ is a Γ-increasing morphism such that $h(\alpha) = g(\alpha)$ for all $\alpha \in A$ and derive a contradiction. There is a $\Gamma_1 \subseteq \Gamma$ and a Γ_1-pattern $\alpha_1 \in A$ such that $g \not\sim_{\Gamma_1} h$. Let $\alpha_1 = a_1 \dots a_n$, where $a_1, \dots, a_n \in \Xi \cup \Sigma$. Lemma 3 is used with $g(a_1), \dots, g(a_n)$ as $u_1, \dots, u_n$ and $h(a_1), \dots, h(a_n)$ as $v_1, \dots, v_n$. Because $g \not\sim_{\Gamma_1} h$, it follows from Lemma 3 that there is an i such that $g(a_i)$ is a proper factor of $h(a_i)$. In particular, $|h(a_i)| > |g(a_i)| \geq 1$, so $a_i = x$. Thus $g(x)$ is a proper factor of $h(x) = ab$. By symmetry, it can be assumed that $g(x) = a$. Then $g(y)$ cannot contain a's for any variable $y \in \Gamma \setminus \{x\}$, because otherwise $g(\alpha)$ would contain more a's than $h(\alpha)$ for some $\alpha \in A$.

Consider any $\alpha \in A$. Let $|\alpha|_x = k$ and

$$\alpha = w_0 x w_1 x \dots w_{k-1} x w_k.$$

If $j = |w_0 \dots w_{i-1}|_a + i$ for some $i \in \{1, \dots, k\}$, then the jth a in $h(\alpha) = g(\alpha)$ is followed by b. Thus $g(y)$ begins with b for all $y \in R_\alpha(x)$. This means that $\varepsilon, d \notin R_\alpha(x)$ for all $d \in \Sigma \setminus \{b\}$. By the definition of b, $R_A(x) \subseteq \Xi$.

The combined number of b's in all words $h(\alpha)$ is

$$\sum_{\alpha \in A} (|\alpha|_b + |\alpha|_x)$$

and in all words $g(\alpha)$ it is at least

$$\sum_{\alpha \in A} (|\alpha|_b + \sum_{y \in R_A(x)} |\alpha|_y).$$

These numbers should be the same, but because x does not have loyal neighbors in A,

$$\sum_{\alpha \in A} |\alpha|_x < \sum_{\alpha \in A} \sum_{y \in R_\alpha(x)} |\alpha|_y.$$

This is a contradiction. □

Theorem 13. *Let $\#\Sigma \geq 3$ and let A be a Γ-set of patterns. There is a Γ-increasing morphism h that is weakly unambiguous with respect to A if and only if at least one variable does not have loyal neighbors in A.*

Proof. Assume first that all variables have loyal neighbors in A and h is a Γ-increasing morphism. Then some variable x satisfies the conditions of Lemma 11, so h is not weakly unambiguous with respect to A.

Assume then that a variable x does not have loyal neighbors in A. Because $\#\Sigma \geq 3$, the three letters of Lemma 12 exist, and there is a Γ-increasing morphism that is weakly unambiguous with respect to A. □

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