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FINE AND WILF'S THEOREM FOR k-ABELIAN PERIODS *

JUHANI KARHUMÄKI

Department of Mathematics and Statistics, University of Turku, FI-20014 Turku, Finland
karhumak@utu.fi

SVETLANA PUZYNINA

Department of Mathematics and Statistics, University of Turku, FI-20014 Turku, Finland, and Sobolev Institute of Mathematics, Novosibirsk, Russia
svepuz@utu.fi, puzynina@math.nsc.ru

ALEKSI SAARELA

Department of Mathematics and Statistics, University of Turku, FI-20014 Turku, Finland
amsaar@utu.fi

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Two words u and v are *k-abelian equivalent* if they contain the same number of occurrences of each factor of length at most k. This leads to a hierarchy of equivalence relations on words which lie properly in between the equality and abelian equality. The goal of this paper is to analyze Fine and Wilf's periodicity theorem with respect to these equivalence relations. Fine and Wilf's theorem tells exactly how long a word with two periods p and q can be without having the greatest common divisor of p and q as a period. Recently, the same question has been studied for abelian periods. In this paper we show that for k-abelian periods the situation is similar to the abelian case: In general, there is no bound for the lengths of such words, but the values of the parameters p, q and k for which the length is bounded can be characterized. In the latter case we provide nontrivial upper and lower bounds for the maximal lengths of such words. In some cases (e.g., for $k = 2$) we found the maximal length precisely.

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1. Introduction

In the paper we deal with the following question: How far two "periodic" processes must coincide in order to guarantee a common "period"? In 1965, Fine and Wilf proved their famous periodicity theorem [7] characterizing this for words. It tells

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exactly how long a word with two periods p and q can be without having the greatest common divisor of p and q as a period. Many variations of the theorem have been considered. For example, there are several articles on generalizations for more than two periods, see, e.g., [4, 12]. Periods of partial words were studied in, e.g., [1, 2, 11]. Periodicity with respect to an involution was considered in [10]. Particularly interesting in the context of this article is the variation related to abelian equivalence. This was first considered by Constantinescu and Ilie in 2006 [6]. They found an upper bound for the lengths of such words in the case of relatively prime periods and stated that otherwise there are no upper bounds. Blanchet-Sadri, Tebbe and Veprauskas [3] gave an algorithm showing the optimality of the above bound, although they only provided the proof of the correctness of the algorithm in some cases.

In this paper the k-abelian versions of periodicity are studied. Two words are called k-abelian equivalent if they contain the same number of occurrences of each factor of length k, if their prefixes of length $k-1$ are the same and if their suffixes of length $k-1$ are the same. For k-abelian equivalence the problem is similar but more complicated than for abelian equivalence. Again, there does not always exist a bound: If $\gcd(p, q) > k$, then there are infinite words having k-abelian periods p and q but not $\gcd(p, q)$. In all other cases a finite upper bound for the length of such words is obtained. In the case $k = 2$ and in some other special cases we give an exact variant of Fine and Wilf's theorem. In the general case, however, the problem seems to be rather intricate. Nontrivial upper bounds in the general case and lower bounds in some special cases are established, but many questions about the behavior of the problem remain open.

2. Preliminaries

We study words over a non-unary alphabet Σ. For a general reference on combinatorics on words, see, e.g., [5].

The *length* of a word $w \in \Sigma^*$ is denoted by $|w|$ and the product of n copies of w by w^n. If $w = tuv$, then u is a *factor* of w. If $|t| = 0$, then u is a *prefix* of w, and if $|v| = 0$, then u is a *suffix* of w. The notation $u \leq w$ means that u is a prefix of w. The prefix and suffix of length $m \leq |w|$ are denoted by $\mathrm{pref}_m(w)$ and $\mathrm{suff}_m(w)$. If $w = a_1 \dots a_n$, where $a_1, \dots, a_n \in \Sigma$, then we use the notation $w[i] = a_i$ and $w[i, j] = a_i \dots a_j$. The number of occurrences of a factor u in w is denoted by $|w|_u$ and the *reversal* of w by $w^R = a_n \dots a_1$. Occasionally we will also consider right-infinite words. Then $tu^\omega = tuuu \dots$ means the word consisting of t followed by infinitely many copies of u.

Let $w = a_1 \dots a_n$, where $a_1, \dots, a_n \in \Sigma$. A positive integer p is a *period* of w if $a_{i+p} = a_i$ for every $i \in \{1, \dots, n-p\}$. Equivalently, p is a period if there is a word u of length p, a prefix u' of u and a number m such that $w = u^m u'$.

Now we state Fine and Wilf's periodicity theorem, which was proved in [7].

Theorem 1 (Fine and Wilf) *Let $p, q > \gcd(p, q) = d$. Let w have periods p and*

q. If $|w| \geq p + q - d$, then w has period d. There are words of length $p + q - d - 1$ that have periods p and q but not period d.

Two words u and v are *abelian equivalent* if $|u|_a = |v|_a$ for every letter a.

If there are abelian equivalent words $u_0, \dots, u_{n+1}$ of length p and a non-negative integer $r \leq p - 1$ such that

$$w = \mathrm{suff}_r(u_0)u_1 \dots u_n \mathrm{pref}_{|w|-np-r}(u_{n+1}),$$

then w has *abelian period* p. If $r = 0$, then w has *initial* abelian period p.

In [6] it is proved that a word of length $2pq - 1$ having relatively prime abelian periods p and q has also period 1. The authors also conjectured that this bound is optimal.

Theorem 2 (Constantinescu and Ilie) *Let $p, q > \gcd(p, q) = 1$. Let w have abelian periods p and q. If*

$$|w| \geq 2pq - 1,$$

then w is unary.

In [3] an algorithm constructing optimal words was described, and a proof of correctness was provided for some pairs (p, q).

Initial abelian periods were not considered in [3, 6], but from the proofs it is quite easy to see that the value $2pq - 1$ could be replaced with pq if the periods are assumed to be initial.

Let k be a positive integer. Two words u and v are *k-abelian equivalent* if the following conditions hold:

- $|u|_t = |v|_t$ for every word t of length k,
- $\mathrm{pref}_{k-1}(u) = \mathrm{pref}_{k-1}(v)$ and $\mathrm{suff}_{k-1}(u) = \mathrm{suff}_{k-1}(v)$ (or $u = v$, if $|u| < k - 1$ or $|v| < k - 1$).

We can replace the conditions with a single one and get an equivalent definition:

- $|u|_t = |v|_t$ for every word t of length at most k.

It is easy to see that k-abelian equivalence implies k'-abelian equivalence for every $k' < k$. In particular, it implies abelian equivalence, which is the same as 1-abelian equivalence. For more on k-abelian equivalence, see [8, 9].

We define k-abelian periodicity similarly to abelian periodicity: If there are k-abelian equivalent words $u_0, \dots, u_{n+1}$ of length p and a non-negative integer $r \leq p - 1$ such that

$$w = \mathrm{suff}_r(u_0)u_1 \dots u_n \mathrm{pref}_{|w|-np-r}(u_{n+1}),$$

then w has *k-abelian period* p with *offset* r. If $r = 0$, then w has *initial* k-abelian period p. If also $|w| \geq p$, then the k-abelian equivalence class of $\mathrm{pref}_p(w)$ is called the *k-abelian p-root* of w. Notice that if w has a k-abelian period p, then so has every factor of w and w^R. If w has initial k-abelian period p, then so has every prefix of w.

In this article we are mostly interested in initial k-abelian periods. Many of our results could be generalized for noninitial periods, but the bounds would be worse.

Example 3. *The initial abelian periods of* $w = babaaabaabb$ *are* $5, 7, 8, 9, 10, \ldots$. *In addition,* w *has abelian periods 3 and 6.*

If k is large compared to p, then k-abelian period p is also an ordinary period.

Lemma 4. *If* w *has a* k*-abelian period* $p \leq 2k - 1$*, then* w *has period* p.

Proof. Words of length $\leq 2k - 1$ are k-abelian equivalent iff they are equal. □

Let $k \geq 1$ and let $p, q \geq 2$ be such that neither of p and q divides the other. Let $d = \gcd(p, q)$. We define $L_k(p, q)$ to be the length of the longest word that has initial k-abelian periods p and q but does not have initial k-abelian period d. If there are arbitrarily long such words, then $L_k(p, q) = \infty$.

The following two questions can be asked:

- For which values of k, p and q is $L_k(p, q)$ finite?
- If $L_k(p, q)$ is finite, how large is it?

If w is a word of length pq/d that has initial k-abelian periods p and q but does not have initial k-abelian period d, then also the infinite word w^ω has initial k-abelian periods p and q but does not have initial k-abelian period d. So either $L_k(p, q) < pq/d$ or $L_k(p, q) = \infty$.

The first question is answered exactly in Section 3: $L_k(p, q)$ is finite if and only if $d \leq k$. The second question is answered exactly if $k = 2$. This is done in Section 4 by first proving upper bounds and then proving matching lower bounds. In Section 5, nontrivial upper bounds are proved for $L_k(p, q)$ in the case $d \leq k$.

The same questions can be asked also for non-initial k-abelian periods. Again, infinite words exist if and only if $d > k$, but the proof is omitted here.

The following lemma shows that the size of the alphabet is not important in our considerations (if there are at least two letters).

Lemma 5. *Let* w *be a word that has* k*-abelian periods* p *and* q*, and does not have* k*-abelian period* $d = \gcd(p, q)$*. Then there exists a binary word of length* $|w|$ *that has* k*-abelian periods* p *and* q*, and does not have* k*-abelian period* d*.*

Proof. First, let $d \leq k$, so k-abelian period d is also an ordinary period. For any two letters a and b, let h_{ab} be the morphism that maps a to b and every other letter to itself. We show that there are letters a, b such that $h_{ab}(w)$ has k-abelian periods p and q but does not have k-abelian period d.

Let $w = a_1 \ldots a_n$, where $a_1, \ldots, a_n \in \Sigma$. Let $A_j = \{a_i \mid i \equiv j \mod d\}$. Because w does not have period d, there is a number m and two letters a, c such that

$a, c \in A_m$. If b is a third letter and $B_j = \{h_{ab}(a_i) \mid i \equiv j \mod d\}$, then B_m contains at least b and c, so $h_{ab}(w)$ does not have period d. Clearly $h_{ab}(w)$ still has k-abelian periods p and q, so the claim is proved. This process can be repeated until a binary word is obtained.

If $d > k$, then the claim about the existence of a suitable h_{ab} does not necessarily hold. For example, $abcaacba$ does not have 2-abelian period 4, but for all two letters $a', b' \in \{a, b, c\}$, $h_{a'b'}(abcaacba)$ has 2-abelian period 4. However, the proof of Theorem 6 gives an infinite binary word that has k-abelian periods p and q but that does not have k-abelian period d. □

3. Existence of Bounds

In this section we characterize when $L_k(p, q)$ is finite: If $\gcd(p, q) > k$, then $L_k(p, q) = \infty$ by Theorem 6, otherwise $L_k(p, q) < pq/d$ by Theorem 9.

Theorem 6. *Let $p, q > \gcd(p, q) = d > k$. There is an infinite word that has initial k-abelian periods p and q but that does not have k-abelian period d.*

Proof. If $k = 1$, then $a^d bba^{d-2}(ba^{d-1})^\omega$ is such a word, and if $k > 1$, then $a^{2d-k-1}ba^{k-1}b(a^{d-1}b)^\omega$ is such a word. These words have initial k-abelian periods id for all $i > 1$ and hence initial k-abelian periods p and q. □

Assume that a word has k-abelian periods p, q. If $p, q \leq 2k - 1$, then they are ordinary periods, so Theorem 1 can be used. If $p \leq 2k - 1$ but $q > 2k - 1$, then we get the following result that is similar to Theorem 1 but slightly worse.

Theorem 7. *Let $p < 2k$ and $p \leq q$. Let w have k-abelian periods p and q. If*

$$|w| \geq 2p + 2q - 2k - 1 \qquad \textit{and} \qquad |w| \geq 2q - 1,$$

then w has period $\gcd(p, q)$.

Proof. If $q \leq 2k - 1$, then the claim follows from Lemma 4 and Theorem 1, so let $q \geq 2k$. By Lemma 4, p is a period. Let q be k-abelian period with offset r. Because $|w| \geq 2q - 1$, there is an integer j such that

$$0 \leq r + jq \leq \frac{|w|}{2} \leq r + (j + 1)q \leq |w|.$$

Then there are words t, u, s such that $|t| = r + jq$, $|u| = q$ and $w = tus$. Because

$$|st| = |w| - q \geq 2p + q - 2k - 1 \geq 2p - 1,$$

one of t and s has length at least p. The other has length at least $\lceil |w|/2 \rceil - q \geq p - k$. It follows that w has a factor $v = t'us'$, where $|t's'| = p - 1$, s' is a prefix of s and of $\mathrm{pref}_{k-1}(u)$ and t' is a suffix of t and of $\mathrm{suff}_{k-1}(u)$. Then v has periods p and q. By Theorem 1, v has period $\gcd(p, q)$. Because w has period p and its factor of length p has period $\gcd(p, q)$, w has period $\gcd(p, q)$. □

Lemma 8. *If w has a k-abelian period p and some factor of w of length $2p-1$ has at most k factors of length k, then w has a period $d \leq k$ that divides p.*

Proof. If $p \leq k$, then we can set $d = p$ by Lemma 4, so let $p > k$. There are k-abelian equivalent words $u_0, \dots, u_n$ of length p such that $w = tu_1 \dots u_{n-1}s$, where t is a suffix of u_0 and s is a prefix of u_n. Every factor v of w of length $2p-1$ has a factor of the form $v' = t'u_m s'$, where t' is a suffix of every u_i, s' is a prefix of every u_i and $|t's'| = k-1$. Every factor of w of length k is a factor of v'. Because v can be selected so that it has at most k factors of length k, it follows that also w has at most k factors of length k. Thus w has a period $d_1 \leq k$. By Theorem 7, w has period $\gcd(d_1, p)$. □

Theorem 9. *Let w have initial k-abelian periods p and q, $d = \gcd(p, q) < p, q$ and $d \leq k$. If*

$$|w| \geq \operatorname{lcm}(p, q),$$

then w has period d.

Proof. If $p \leq k$ or $q \leq k$, then the claim follows from Theorem 7, so let $p, q > k$. Let $p = dp'$ and $q = dq'$ and let w' be the prefix of w of length $pq/d = p'q'd$. There is a word u of length p and a word v of length q such that w' is k-abelian equivalent with $u^{q'}$ and $v^{p'}$. Let s be the common prefix of u and v of length $k-1$. If $x \in \Sigma^k$, then

$$|w's|_x = |u^{q'}s|_x = q'|us|_x \qquad \text{and} \qquad |w's|_x = |v^{p'}s|_x = p'|vs|_x.$$

Thus $|w's|_x$ is divisible by both p' and q', so it is divisible by $p'q'$. In particular, it is either 0 or at least $p'q'$. Because

$$\sum_{x \in \Sigma^k} |w's|_x = |w's| - k + 1 = |w'| = p'q'd,$$

there can be at most d factors $x \in \Sigma^k$ such that $|w's|_x \geq p'q'$. This means that $w's$ can have at most d different factors of length k. By Lemma 8, w has a period $d_1 \leq k$ that divides p. By Theorem 7, w has period $\gcd(d_1, q)$. This divides d, so w has period d. □

4. Initial 2-Abelian Periods

In this section the exact value of $L_2(p, q)$ is determined. We start with upper bounds and then give matching lower bounds. First we state the following lemma, which is very useful also later in the general k-abelian case.

Lemma 10. *Let $p = dp'$, $q = dq'$, $\gcd(p', q') = 1$ and $p', q' \geq 2$. For every i satisfying $0 < |i| < \min\{p', q'\}$ there are numbers $m_i \in \{1, \dots, q'-1\}$ and $n_i \in \{1, \dots, p'-1\}$ such that*

$$n_i q - m_i p = id. \tag{1}$$

The notation of Lemma 10 is used in this section and in the later sections, that is, m_i and n_i are always numbers such that (1) holds. The equalities $n_1 q = m_1 p + d$ and $m_{-1} p = n_{-1} q + d$ are particularly important.

Upper Bounds. The following lemma gives an upper bound in the 2-abelian case.

Lemma 11. *Let $p, q \geq 2$ and $\gcd(p, q) = 1$. Let w have initial 2-abelian periods p and q. If*

$$|w| \geq \max\{n_1 q, m_{-1} p\},$$

then w is unary.

Proof. The word w has prefixes

$$u_1 \dots u_{m_1} a = v_1 \dots v_{n_1} \qquad \text{and} \qquad u_1 \dots u_{m_{-1}} = v_1 \dots v_{n_{-1}} a', \tag{2}$$

where the u_i's are 2-abelian equivalent words of length p, the v_i's are 2-abelian equivalent words of length q and a, a' are letters. Both a and a' are first letters of every u_i and v_i, so they are equal.

For any letter $b \neq a$, it follows from (2) that

$$m_1 |u_1|_b = n_1 |v_1|_b \qquad \text{and} \qquad m_{-1} |u_1|_b = n_{-1} |v_1|_b$$

and thus

$$m_1 n_{-1} |u_1|_b |v_1|_b = n_1 m_{-1} |u_1|_b |v_1|_b. \tag{3}$$

By (1), $m_1 p < n_1 q$ and $m_{-1} p > n_{-1} q$ and thus

$$m_1 n_{-1} < n_1 m_{-1}. \tag{4}$$

Both (3) and (4) can hold only if $|u_1|_b |v_1|_b = 0$. It follows that $w \in a^*$. □

Lower Bounds. The following lemma gives a lower bound in the 2-abelian case.

Lemma 12. *Let $q > p \geq 2$, $\gcd(p, q) = 1$, x, y be the smallest positive integers such that $xq - yp = \pm 1$. Then there exists a non-unary word w of length $(p - x)q$ in the case $xq - yp = +1$ (or $(q - y)p$ in the case $xq - yp = -1$) which has initial 2-abelian periods p and q.*

Remark that the pair (x, y) is either (n_1, m_1) or (n_{-1}, m_{-1}) (the one with smaller numbers), and the pair $(p - x, q - y)$ is the other one.

Proof. First we describe the construction (actually, the algorithm producing the word w), then give an example, and finally we prove that the algorithm works correctly, i.e. it indeed produces a word with initial 2-abelian periods p and q.

We need the following notion. Let $m \geq l \geq 0$, $c, d \in \{a, b\}$. Define $K_2(m, l, c, d)$ to be the set of binary words satisfying the following conditions:

- words of length m

- containing l letters b (and hence $m-l$ letters a)
- b's in them are isolated (i.e., with no occurrence of factor bb)
- the first letter being $c \in \{a,b\}$, the last letter being $d \in \{a,b\}$

The following properties are easy to conclude:

(i) The set $K_2(m,l,a,a)$ is non-empty for $l < m/2$, and the set $K_2(m,l,a,b)$ is non-empty for $0 < l \leq m/2$.

(ii) $K_2(m,l,c,d)$ is a 2-abelian equivalence class of words (if it is non-empty). For $c = d = a$ the words in it contain l occurrences of ab, l occurrences of ba, $m-2l-1$ occurrences of aa and no occurrences of bb. For $c = a$, $d = b$ they contain l occurrences of ab, $l-1$ occurrences of ba, $m-2l$ occurrences of aa and no occurrences of bb.

(iii) If $u \in K_2(m,l,c,d)$, $u' \in K_2(m',l',c',d')$, and at least one of the letters d and c' is a, then $uu' \in K_2(m+m', l+l', c, d')$.

Now, our construction is done as follows:

(1) Find the smallest positive integers x, y satisfying $xq - yp = \pm 1$. In the case $xq - yp = -1$ we construct a word w with 2-abelian p-root $K_2(p,x,a,a)$ and q-root $K_2(q,y,a,b)$. Note that in this case $x < p/2$, $y \leq q/2$. Indeed, the pair (x,y) is chosen to be one of the pairs (m_1,n_1) and (m_{-1},n_{-1}), the one with the smallest numbers. Since $m_1 + m_{-1} = p$ and $n_1 + n_{-1} = q$ and $x/p < y/q$, we have the above inequalities. If $xq - yp = 1$, we take the p-root to be $K_2(p,x,a,b)$ and q-root to be $K_2(q,y,a,a)$. In this case $x \leq p/2$, $y < q/2$. To be definite, assume that $xq - yp = -1$, in the other case the construction is symmetric.

(2) Now we start building our word based on 2-abelian periods indicated in step (1). We mark the positions ip and jq for $i = 0, \dots, q-y$, $j = 0, \dots, p-x-1$, and denote these positions by t_m in increasing order, $m = 0, \dots, q-y+p-x-1$. Now we will fill in the factors $v_m = w[t_{m-1}, t_m - 1]$ one after another.

   (a) If in v_m we have that $t_{m-1} = (i-1)p$ and $t_m = ip$ for some i (including v_1), then put v_m equal to any word from the 2-abelian equivalence class $K_2(p,x,a,a)$.

   (b) If in v_m we have that $t_{m-1} = ip$ and $t_m = jq$ for some i and j, then fill it with any word from $K_2(jq-ip, jy-ix, a, b)$. Then the word $w[t_m - q, t_m - 1]$ is from the 2-abelian equivalence class $K_2(q,y,a,b)$.

   (c) If in v_m we have that $t_{m-1} = jq$ and $t_m = ip$ for some i and j, then fill it with any word from $K_2(ip-jq, ix-jy, a, a)$. Then the word $w[t_m - p, t_m - 1]$ is from the 2-abelian equivalence class $K_2(p,x,a,a)$.

Example 13. $p = 7$, $q = 10$. *We find* $x = 2$, $y = 3$, *so we take the 2-abelian equivalence class of the word aaababa as p-root and the 2-abelian equivalence class of aaaaababab as q-root, and the length of word is* $p(q-y) = 49$. *One of the words* *given by the construction is*

$$aaababa.aab^{\cdot}aaba.aaabab^{\cdot}a.aaababa.ab^{\cdot}aaaba.aabab^{\cdot}aa.aaababa.$$

Here the lower dots are placed at positions $7i$*, and the upper dots at positions* $10j$*. This word has initial 2-abelian periods* 7 *and* 10*. In the example each time we chose the lexicographically biggest word* v_i*, though we actually have some flexibility. E.g., one might take* $v_1 = abaaaba$*, so the word is not unique.*

To prove the correctness of the algorithm, we will prove that on each step (2b) and (2c) the corresponding 2-abelian equivalence classes are non-empty, so that one can indeed choose such a word. This would mean that on each step m we obtain a word $v_1 \dots v_m$ such that all its prefixes of lengths divisible by p and q are 2-abelian p- and q-periodic, respectively (in other words, we have periodicity in full periods up to length t_m), and the last incomplete period (either p- or q-period) starts with a.

Correctness of step (2b). At step (2b), we should add a word $v_m \in K_2(jq - ip, jy - ix, a, b) = K_2(t_m - t_{m-1}, l, a, b)$, where the length $t_m - t_{m-1} = jq - ip < p$ and the number l of b's is as large as required so that $w[t_m - q, t_m - 1]$ has the same number of b's as there are in the words of the q-root $K_2(q, y, a, b)$. In view of properties (i)–(iii), these conditions are sufficient to guarantee that $w[t_m - q, t_m - 1] \in K_2(q, y, a, b)$. The only thing we should care about is that we can indeed choose such a word, i.e., that the set $K_2(jq - ip, jy - ix, a, b)$ is non-empty. So, we should check the required number $l = jy - ix$ of b's: it should not be larger than $|v_m|/2$ and it should not be less than 1. I.e., we should prove that

$$1 \le jy - ix \le \frac{jq - ip}{2}. \tag{5}$$

We will use that

$$xq - yp = -1 \tag{6}$$

$$jq - ip \ge 2 \tag{7}$$

$$i < q - y \tag{8}$$

$$y \le q/2 \tag{9}$$

First we prove the left inequality from (5). The equality (6) implies that $y/q > x/p$, the inequality (7) implies that $j/p > i/q$. Multiplying these inequalities with positive values one gets that $(j/p) \cdot (y/q) > (i/q) \cdot (x/p)$. Therefore, $jy - ix > 0$, and since $jy - ix$ is integer, we get the required inequality.

Now we will prove right inequality from (5). In the proof we will make use of the following obvious claim:

Claim 1. Let c, d, e, f be positive numbers. The following two inequalities are equivalent:

$$\frac{c}{d} > \frac{c+e}{d+f}, \qquad \frac{e}{f} < \frac{c+e}{d+f}.$$

Now, using (6), one gets that (8) is equivalent to

$$\begin{aligned} i(xq-yp) &> y-q, \\ ixq+q &> ipy+y, \\ \frac{ix+1}{ip+1} &> \frac{jy}{jq} = \frac{ix+1+(jy-ix-1)}{ip+1+(jq-ip-1)}. \end{aligned}$$

Applying Claim 1 for $c = ix+1$, $d = ip+1$, $e = jy-ix-1$, $f = jq-ip-1$, we get that

$$\begin{aligned} \frac{jy-ix-1}{jq-ip-1} &< \frac{y}{q} \leq^{(9)} \frac{1}{2}, \\ jy-ix &< \frac{jq-ip+1}{2}, \end{aligned}$$

and since all numbers in these inequality are integer, we finally get that

$$jy-ix \leq \frac{jq-ip}{2}.$$

Correctness of step (2c). At step (2c), we should add a word $v_m \in K_2(ip-jq, ix-jy, a, a) = K_2(t_m - t_{m-1}, l, a, a)$, where the length $t_m - t_{m-1} = ip - jq < p$ and the number l of b's is as large as required so that $w[t_m - p, t_m - 1]$ has the same number of b's as there are in words of $K_2(p, x, a, a)$. In view of properties (i)–(iii), these conditions imply that $w[t_m - p, t_m - 1] \in K_2(p, x, a, a)$. To prove the correctness, we should show that the set $K_2(ip-jq, ix-jy, a, a)$ is non-empty, i.e., the number $ix - jy$ of b's satisfies

$$0 \leq ix - jy < \frac{ip-jq}{2}. \tag{10}$$

We will use that

$$xq - yp = -1 \tag{11}$$
$$ip - jq \geq 1 \tag{12}$$
$$j < p - x \tag{13}$$
$$x < p/2 \tag{14}$$

First we prove the right inequality from (10). The equality (11) implies that $y/q > x/p$, or, equivalently,

$$\frac{jy}{jq} > \frac{ix}{ip} = \frac{jy+(ix-jy)}{jq+(ip-jq)}.$$

Applying Claim 1 for $c = jy$, $d = jq$, $e = ix - jy$, $f = ip - jq$, we get that

$$\begin{aligned} \frac{ix-jy}{ip-jq} &< \frac{ix}{ip} <^{(14)} \frac{1}{2}, \\ ix-jy &< \frac{ip-jq}{2} \end{aligned}$$

Now we will prove the left inequality from (10). In the proof we will use another obvious auxiliary claim:

Claim 2. Let c, d, e, f, t be positive numbers such that $(c+z)/(d+t) = e/f$ and $c/d < e/f$. Then $z > 0$.

Direct computations show that

$$\frac{(jy-1)+(ix-jy+1)}{jq+(ip-jq)} = \frac{ix}{ip} = \frac{x}{p} = \frac{qx}{qp} = \frac{py-1}{qp}$$

Now we apply Claim 2 for $c = jy-1$, $d = jq$, $e = py-1$, $f = qp$, $t = ip-jq$, $z = ix-jy+1$. Indeed, we check the conditions:

$$\frac{c}{d} = \frac{jy-1}{jq} = \frac{1}{q}\left(y - \frac{1}{j}\right) <^{(13)} \frac{1}{q}\left(y - \frac{1}{p}\right) = \frac{py-1}{pq} = \frac{e}{f},$$

and due to (12) we have $t > 0$. So, we get that $z = ix - jy + 1 > 0$. Taking into account that all numbers in this inequality are integer, we have that this is equivalent to $ix - jy \geq 0$.

So, we built a word w of length $(q-y)p$ having initial 2-abelian p-period and initial 2-abelian q-period till length $(q-y)p - q + 1$ (within full periods). It remains to check that $\mathrm{suff}_{q-1}(w)$ can be extended till a word of the 2-abelian equivalence class $K_2(q, y, a, b)$ of the q-period. It is easy to see that it can be extended in this way by adding letter b. □

By a similar construction we find optimal words for the abelian case. We construct such words satisfying additional condition, which we use later for k-abelian case:

Lemma 14. *Let $q > p \geq 2$, $\gcd(p,q) = 1$. Then there exists a non-unary word w of length $pq-1$ which has initial abelian periods p and q, and moreover $w[ip] = w[ip+p]$ and $w[jq] = w[jq+q]$ for all integers i, j for which the indices are defined.*

Proof. We briefly describe the construction, since everything here is similar to the proof of Lemma 12.

We need the following notation. Let $m \geq l \geq 0$, $c \in \{a, b\}$. Define

$$K(m,l,c) = \{v \in \{a,b\}^m |\ |v|_b = l, v[m] = c\}.$$

Clearly, all words from $K(m,l,c)$ are abelian equivalent and end in the same letter.

Now, our construction is done as follows:

(1) Find the smallest integers x, y satisfying $xq - yp = \pm 1$. In the case $xq - yp = -1$ (resp., $xq - yp = 1$) we construct a word w which is a prefix of a word in $K(p, x, a)^*$ (resp., $K(p, x, b)^*$) and also a prefix of a word in $K(q, y, b)^*$ (resp., $K_2(q, y, a)^*$). To be definite, assume that $xq - yp = -1$, in the other case the construction is symmetric.

(2) Now we start building our word based on abelian periods indicated in step (1). Denote the positions ip and jq for $i = 0, \ldots, q-1$, $j = 0, \ldots, p-1$, by t_m in increasing order, $m = 0, \ldots, q+p-2$, $t_{q+p-1} = pq-1$. Now we will fill in the factors $v_m = w[t_{m-1}, t_m - 1]$ as follows:
   (a) If in v_m we have that $t_{m-1} = (i-1)p$ and $t_m = ip$ for some i (including v_1), then put $v_m = a^{p-x-1}b^x a$ (or any other word from $K(p,x,a)$).
   (b) If in v_m we have that $t_{m-1} = ip$ and $t_m = jq$ for some i and j, then put $v_m = a^{jq-ip-jy+ix}b^{jy-ix}$ (or any other word from $K(jq-ip, jy-ix, b)$). Then $w[t_m - q, t_m - 1] \in K(q, y, b)$.
   (c) If in v_m we have that $t_{m-1} = jq$ and $t_m = ip$ for some i and j, then put $v_m = a^{ip-jq-ix+jy-1}b^{ix-jy}a$ (or any other word from $K(ip-jq, ix-jy, a)$. Then $w[t_m - p, t_m - 1] \in K(p, x, a)$.
   (d) For $m = p+q-1$, put $v_m = a^{p-x-1}b^x$.

To prove the correctness of the algorithm, it is enough to verify that on step (2b) it holds

$$1 \leq jy - ix \leq jq - ip.$$

and on step (2c) it holds

$$0 \leq ix - jy \leq ip - jq - 1.$$

The verifications of the inequalities are very similar to the verifications from Lemma 12. The only difference is that the inequalities to be proved are weaker, but they should hold for bigger indices, i.e., we should consider $j < p$ and $i < q$ instead of (8) and (13) (since we build a longer word of length $pq-1$).

So, we built a word w of length $pq-1$ having initial abelian p-period till length $pq-p$ and $w[ip] = a$ for all $i < q$ and initial abelian q-period till length $pq-q$ and $w[jq] = b$ for all $j < p$. To conclude the proof, we notice that $\mathrm{suff}_{q-1}(w)$ (resp., $\mathrm{suff}_{p-1}(w)$) can be extended with a letter b (resp., a) till a word in $K(q,y,b)$ (resp., $K(p,x,a)$). □

Remark 15. *We note that the constructions from Lemmas 12 and 14 are very similar. Actually, one can be obtained from the other applying a morphism $a \mapsto a$, $b \mapsto ab$, adding a letter in the end and changing numbers p and q accordingly.*

Lemma 16. *Let $q > p > \gcd(p,q) = d = k$. Then there exists a non-unary word w of length $pq/d - 1$ which has initial k-abelian periods p and q and no k-abelian period d.*

Proof. The word w is constructed from the word w' given by construction from Lemma 14 for p/d and q/d: $w = \varphi(w')a^{k-1}$, where the morphism φ is given by $\varphi(a) = a^k, \varphi(b) = a^{k-1}b$. □

Optimal Values. Combining the previous results gives two exact theorems.

Theorem 17. *Let* $p, q > \gcd(p, q) = k$. *Then*

$$L_k(p, q) = \frac{pq}{k} - 1.$$

Proof. Follows from Theorem 9 and Lemmas 14 and 16. □

Now we get a version of Fine and Wilf's theorem for initial 2-abelian periods.

Theorem 18. *Let* $p, q > \gcd(p, q) = d$. *Then*

$$L_2(p, q) = \begin{cases} \max\{m_1 p, n_{-1} q\} & \text{if } d = 1, \\ pq/2 - 1 & \text{if } d = 2, \\ \infty & \text{if } d \geq 3. \end{cases}$$

Proof. The case $d = 2$ follows from Theorem 17 and the case $d \geq 3$ from Theorem 6. Let $d = 1$. It follows from Lemma 11 that

$$L_2(p, q) < \max\{n_1 q, m_{-1} p\} = \max\{m_1 p, n_{-1} q\} + 1.$$

In Lemma 12, if $xq - yp = 1$, then $(x, y) = (n_1, m_1)$ and

$$L_2(p, q) \geq (p - x)q = n_{-1} q = \max\{m_1 p, n_{-1} q\},$$

and if $xq - yp = -1$, then $(x, y) = (n_{-1}, m_{-1})$ and

$$L_2(p, q) \geq (q - y)p = m_1 p = \max\{m_1 p, n_{-1} q\}.$$ □

Remark 19. *We note that in the case of 2-abelian Fine and Wilf's theorem optimal words are typically not unique, contrary to the case of normal words.*

The size of $\max\{m_1 p, n_{-1} q\}$ depends a lot on the particular values of p and q. The extreme cases are $p = 2$, which gives $n_{-1} q = q$, and $q = p + 1$, which gives $n_{-1} q = pq - q$. In general we get the following corollary.

Corollary 20. *Let* $q > p > \gcd(p, q) = 1$. *Then*

$$\frac{pq}{2} + \frac{p}{2} - 1 \leq L_2(p, q) \leq pq - q.$$

Proof. By Theorem 18, $L_2(p, q) = \max\{m_1 p, n_{-1} q\}$. If it were $m_1 = m_{-1}$, then it would be $n_1 q = m_1 p + 1 = m_{-1} p + 1 = n_{-1} q + 2$. This is impossible, because $q > 2$. Thus $|m_1 - m_{-1}|p \geq p$. From this and $m_1 p + m_{-1} p = 2pq$ it follows that $2 \max\{m_1, m_{-1}\} p \geq 2pq + p$. Then

$$\max\{m_1 p, n_{-1} q\} = \max\{m_1 p, m_{-1} p - 1\} \geq \max\{m_1, m_{-1}\} p - 1 \geq pq + \frac{p}{2} - 1.$$

The upper bound follows from

$$\max\{m_1 p, n_{-1} q\} = \max\{n_1 q - 1, n_{-1} q\} \leq \max\{n_1, n_{-1}\} q \leq (p - 1)q.$$ □

5. General Upper Bounds

In this section $L_k(p,q)$ is studied for $k \geq 3$. We are not able to give the exact value in all cases, but we will prove an upper bound that is optimal for an infinite family of pairs (p,q). We start with an example.

Example 21. *Let $k \geq 2$, $p \geq 2k-1$ and $q = p+1$. The word*

$$(a^{p-k+1}ba^{k-2})^{q-2k+2}a^{p-k+1}$$

of length $(q-2k+2)p + p - k + 1 = pq - 2kq + 3q + k - 2$ has initial k-abelian periods p and q but does not have period $\gcd(p,q) = 1$.

Recall the notation of Lemma 10: $m_i \in \{1, \dots, q'-1\}$ and $n_i \in \{1, \dots, p'-1\}$ are numbers such that $n_i q - m_i p = id$. This is used in the following lemmas and theorems. The proofs of Lemmas 22 and 23 are in some sense more complicated and technical versions of the proof of Lemma 11.

Lemma 22. *Let $p = dp'$, $q = dq'$, $\gcd(p',q') = 1$ and $2 \leq p' < q'$. Let $k-1 = dk'$ and $1 \leq k' < p'/2$. Let w have initial k-abelian periods p and q. Let $u = \mathrm{pref}_p(w)$, $v = \mathrm{pref}_q(w)$ and $s = \mathrm{pref}_d(w)$. If there are indices*

$$i \in \{-1, 1\}, \qquad j \in \{-k', k'\}, \qquad l \in \{-2k'+1, \dots, -1\} \cup \{1, \dots, 2k'-1\}$$

such that i, j, l do not all have the same sign and

$$m_i p, n_i q \leq |w| - k + 1 + d, \tag{15}$$

$$m_j p, n_j q \leq |w|, \tag{16}$$

$$m_l p, n_l q \leq |w| + k - 1 - d, \tag{17}$$

then

$$\mathrm{pref}_{k-1}(u) = \mathrm{pref}_{k-1}(v) = s^{k'} \qquad \text{and} \qquad \mathrm{suff}_{k-1}(u) = \mathrm{suff}_{k-1}(v) = s^{k'}.$$

Proof. Let

$$w = u_1 \dots u_m \mathrm{pref}_{|w|-mp}(u_{m+1}) = v_1 \dots v_n \mathrm{pref}_{|w|-nq}(v_{n+1}),$$

where $u_1, \dots, u_{m+1}$ are k-abelian equivalent to u and $v_1, \dots, v_{n+1}$ are k-abelian equivalent to v. Thus

$$\begin{aligned}
&\mathrm{pref}_{k-1}(u) = \mathrm{pref}_{k-1}(u_{m'}) = \mathrm{pref}_{k-1}(w) = \mathrm{pref}_{k-1}(v) = \mathrm{pref}_{k-1}(v_{n'}),\\
&\mathrm{suff}_{k-1}(u) = \mathrm{suff}_{k-1}(u_{m'}),\\
&\mathrm{suff}_{k-1}(v) = \mathrm{suff}_{k-1}(v_{n'})
\end{aligned}$$

for all m', n'.

For the rest of the proof, let $i = 1$ (the case $i = -1$ is similar). By (1) and (15),

$$\mathrm{suff}_{k-1}(u_{m_i})\mathrm{pref}_{k-1}(u_{m_i+1})s' = t'\mathrm{suff}_{k-1}(v_{n_i})\mathrm{pref}_{k-1}(v_{n_i+1}), \tag{18}$$

where $|s'| = |t'| = d$. On the left-hand side, $\mathrm{pref}_{k-1}(u_{m_i+1}) = \mathrm{pref}_{k-1}(w)$ occurs starting at position $k-1$. On the right-hand side, $\mathrm{suff}_d(v_{n_i})\mathrm{pref}_{k-1}(v_{n_i+1}) =$

$\mathrm{suff}_d(v_{n_i})\mathrm{pref}_{k-1}(w)$ occurs starting at position $k-1$. It follows that $\mathrm{pref}_{k-1}(w)$ is a prefix of $\mathrm{suff}_d(v_{n_i})\mathrm{pref}_{k-1}(w)$. Thus $\mathrm{pref}_{k-1}(u) = \mathrm{pref}_{k-1}(v) = \mathrm{pref}_{k-1}(w) = s^{k'}$.

If $j = -k'$, then by (1) and (16),

$$t''\mathrm{suff}_{k-1}(u_{m_j})\mathrm{pref}_{k-1}(u_{m_j+1}) = \mathrm{suff}_{k-1}(v_{n_j})\mathrm{pref}_{k-1}(v_{n_j+1})s'',$$

where $|s''| = |t''| = k'd = k-1$. The factor of length $k-1$ starting at position $k-1$ is $\mathrm{suff}_{k-1}(u_{m_j})$ on the left-hand side and $\mathrm{pref}_{k-1}(v_{n_j+1})$ on the right-hand side. It follows that $\mathrm{suff}_{k-1}(u) = \mathrm{suff}_{k-1}(u_{m_j}) = \mathrm{pref}_{k-1}(v_{n_j+1}) = \mathrm{pref}_{k-1}(v) = s^{k'}$ and then also $\mathrm{suff}_{k-1}(v) = \mathrm{suff}_{k-1}(v_{n_i}) = s^{k'}$ by (18). (In this case the index l was not needed.)

If $j = k'$, then by (1) and (16),

$$\mathrm{suff}_{k-1}(u_{m_j})\mathrm{pref}_{k-1}(u_{m_j+1})s'' = t''\mathrm{suff}_{k-1}(v_{n_j})\mathrm{pref}_{k-1}(v_{n_j+1}),$$

where $|s''| = |t''| = k'd = k-1$. The factor of length $k-1$ starting at position $k-1$ is $\mathrm{pref}_{k-1}(u_{m_j+1})$ on the left-hand side and $\mathrm{suff}_{k-1}(v_{n_j})$ on the right-hand side. It follows that $\mathrm{suff}_{k-1}(v) = \mathrm{suff}_{k-1}(v_{n_j}) = \mathrm{pref}_{k-1}(u_{m_j+1}) = \mathrm{pref}_{k-1}(u) = s^{k'}$ and then $\mathrm{suff}_{k-1-d}(u) = \mathrm{suff}_{k-1-d}(u_{m_i}) = s^{k'-1}$ by (18). Now $l < 0$, so by (1) and (17),

$$t'''\mathrm{pref}_d(\mathrm{suff}_{k-1}(u_{m_l})) \leq \mathrm{suff}_{k-1}(v_{m_l})\mathrm{pref}_{k-1}(v_{m_l+1}) = s^{2k'},$$

where $|t'''| = -ld$. It follows that $\mathrm{pref}_d(\mathrm{suff}_{k-1}(u)) = \mathrm{pref}_d(\mathrm{suff}_{k-1}(u_{m_l})) = s$ and then $\mathrm{suff}_{k-1}(u) = \mathrm{pref}_d(\mathrm{suff}_{k-1}(u))\mathrm{suff}_{k-1-d}(u) = s^{k'}$. □

Lemma 23. *Let $p = dp'$, $q = dq'$, $\gcd(p', q') = 1$ and $2 \leq p' < q'$. Let $k-1 = dk'$ and $1 \leq k' < p'/2$. Let w have initial k-abelian periods p and q. Let $u = \mathrm{pref}_p(w)$, $v = \mathrm{pref}_q(w)$ and $s = \mathrm{pref}_d(w)$. Let*

$$\mathrm{pref}_{k-1}(u) = \mathrm{pref}_{k-1}(v) = s^{k'} \qquad \text{and} \qquad \mathrm{suff}_{k-1}(u) = \mathrm{suff}_{k-1}(v) = s^{k'}.$$

If there are indices

$$i, j \in \{-2k', \ldots, -1\} \cup \{1, \ldots, 2k'\}$$

such that $m_i n_j \neq m_j n_i$ and

$$m_i p, n_i q, m_j p, n_j q \leq |w| + k - 1, \tag{19}$$

then w has period d.

Proof. Let

$$w = u_1 \ldots u_m \mathrm{pref}_{|w|-mp}(u_{m+1}) = v_1 \ldots v_n \mathrm{pref}_{|w|-nq}(v_{n+1}),$$

where $u_1, \ldots, u_{m+1}$ are k-abelian equivalent to u and $v_1, \ldots, v_{n+1}$ are k-abelian equivalent to v.

The infinite word s^ω has at most d factors of length d. Let x be a word of length d that is not a factor of s^ω.

If $s' = \mathrm{pref}_{d-1}(s)$, then by (1) and (19),

$$u_1 \dots u_{m_i} s^i s' = v_1 \dots v_{n_i} s' \qquad \text{or} \qquad u_1 \dots u_{m_i} s' = v_1 \dots v_{n_i} s^i s'.$$

The factor x can only appear in these words inside the factors $u_1, \dots, u_{m+1}$ and $v_1, \dots, v_{n+1}$, because these begin and end with $s^{k'}$ and x is not a factor of s^ω. Thus

$$m_i|u|_x = |u_1 \dots u_{m_i} s^i s'|_x = |v_1 \dots v_{n_i} s'|_x = n_i|v|_x \qquad \text{or}$$
$$m_i|u|_x = |u_1 \dots u_{m_i} s'|_x = |v_1 \dots v_{n_i} s^i s'|_x = n_i|v|_x.$$

In both cases $m_i|u|_x = n_i|v|_x$. Similarly $n_j|v|_x = m_j|u|_x$. Multiplying these two equations gives $m_i n_j |u|_x |v|_x = m_j n_i |u|_x |v|_x$. Because $m_i n_j \neq m_j n_i$, it must be $|u|_x = 0$ or $|v|_x = 0$, so $|w|_x = 0$.

We conclude that the factors of w of length d are the factors of s^ω of length d, so w has period d. □

Theorem 24. *Let $p = dp'$, $q = dq'$, $\gcd(p', q') = 1$ and $2 \leq p' < q'$. Let $k-1 = dk'$ and $1 \leq k' \leq p'/4$. Let w have initial k-abelian periods p and q. If*

$$|w| \geq \frac{pq}{d} - \frac{2(k-1)q}{d} + q + k - 1$$

then w has period d.

Proof. If $n_i q - m_i p = id$, then $(p' - n_i)q - (q' - m_i)p = -id$. It follows that for every i, either $m_i p, n_i q \leq pq/(2d)$ or $m_{-i}p, n_{-i}q \leq pq/(2d)$. Because $p \geq 4(k-1)$, $|w| \geq pq/(2d) + q + k - 1$. Then

$$|w| \geq \max\{m_i p, n_i q\} + \max\{k - d - 1, 0\}$$

for $i = 1$ or $i = -1$ and for $i = k'$ or $i = -k'$. Thus the indices i, j in Lemma 22 exist.

If the above indices i and j have a different sign, then l exists (for example, $l = i$ will do). If i and j have the same sign, then there are $2k' - 1$ candidates for l. All of these have the same sign, so for these l, the numbers n_l are different. If we select l so that n_l is as small as possible, then $n_l \leq p' - 2k' + 1$. Now

$$m_l p, n_l q \leq n_l q + |l| d \leq (p' - 2k' + 1)q + (2k' - 1)d,$$

so in order to (17) be satisfied, it is sufficient that

$$|w| \geq (p' - 2k' + 1)q + (2k' - 1)d - k + 1 + d.$$

This is the bound of the theorem, so l exists and Lemma 22 can be used.

We need to prove the existence of the indices i and j in Lemma 23; the other assumptions are satisfied by Lemma 22. By the argument that was used for the existence of the index l above, there exists $i \in \{1, \dots, 2k'\}$ such that $m_i p, n_i q \leq |w| + k - 1$ and $j \in \{-1, \dots, -2k'\}$ such that $m_j p, n_j q \leq |w| + k - 1$. Because $m_i p < n_i q$ and $n_j q < m_j p$, it follows that $m_i n_j < n_i m_j$ and Lemma 23 can be used to complete the proof. □

If $d = 1$, then Theorem 24 gives that $L_k(p, q) \leq pq - 2kq + 3q + k - 2$. By Example 21, there is an equality if $q = p+1$. The next example shows that for some p and q the exact value is much smaller.

Example 25. *Let $k \geq 2$, $r \geq 2$, $p = rk+1$ and $q = rk+k+1$. Then $\gcd(p, q) = 1$. The word $w = ((a^{k-1}b)^r a)^{r+2} a^{k-2}$ has initial k-abelian periods p and q. Because $n_{-1} = r$, $m_{-1} = r+1$, $m_{k-1} = r+2$ and $n_{k-1} = r+1$, it follows from Lemmas 22 and 23 and the above word w that*

$$L_k(p, q) = (r+2)p + k - 2 = \frac{pq}{k} + q - \frac{q}{k} - 1.$$

6. Conclusion

We conclude with a summary of the results related to initial k-abelian periods. Let $d = \gcd(p, q) < p < q$ and $d \leq k$.

- By Theorem 17, if $d = k$, then
$$L_k(p, q) = \frac{pq}{d} - 1.$$
- By Theorem 18, if $d = 1$, then
$$L_2(p, q) = \max\{m_1 p, n_{-1} q\}.$$
- By Theorem 24, if $2 \leq k \leq p/4 + 1$ and $k - 1$ is divisible by d, then
$$L_k(p, q) \leq \frac{pq}{d} - \frac{2(k-1)q}{d} + q + k - 2.$$
This is optimal if $q = p + 1$.
- By Theorem 7, if $k \geq (p+1)/2$, then
$$L_k(p, q) \leq \max\{2p + 2q - 2k - 2, 2q - 2\}.$$
- By Lemma 4 and Theorem 1, if $k \geq (q+1)/2$, then
$$L_k(p, q) = p + q - d - 1.$$

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