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Starlike functions associated with the generalized Koebe function

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Abstract

In this paper we study some properties of functions f which are analytic and normalized (i.e. $f(0) = 0 = f'(0) - 1$) such that satisfy the following subordination relation

$$\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{z}{(1-pz)(1-qz)},$$

where $(p, q) \in [-1, 1] \times [-1, 1]$. These types of functions are starlike related to the generalized Koebe function. Some of the features are: radius of starlikeness of order $\gamma \in [0, 1)$, image of $f(\{z : |z| < r\})$ where $r \in (0, 1)$, radius of convexity, estimation of initial and logarithmic coefficients, and Fekete–Szegö problem.

Keywords Starlikeness · Convexity · Coefficient estimates · Fekete–Szegö problem · Logarithmic coefficients · Subordination · Koebe function

Mathematics Subject Classification 30C45

1 Introduction

Let $\mathcal{A}$ be the class of functions f analytic in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ normalized by the condition $f(0) = 0 = f'(0) - 1$. Each function f belonging to the

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class $\mathcal{A}$ has the following form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \Delta). \tag{1.1}$$

The subclass of $\mathcal{A}$ consisting of all univalent functions f in Δ will be denoted by $\mathcal{S}$. A function $f \in \mathcal{A}$ is subordinate to $g \in \mathcal{A}$, written as $f(z) \prec g(z)$ or $f \prec g$, if there exists an analytic function w, known as a Schwarz function, with $w(0) = 0$ and $|w(z)| \leq |z|$, such that $f(z) = g(w(z))$ for all $z \in \Delta$. Moreover, if $g \in \mathcal{S}$, then $f(z) \prec g(z) \Leftrightarrow f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$ (c.f. [25]).

For $\gamma < 1$, a function $f \in \mathcal{A}$ is called starlike of order γ if, and only if, $\operatorname{Re}\left\{zf'(z)/f(z)\right\} > \gamma$ in Δ. The class of such functions will be denoted by $\mathcal{S}^*(\gamma)$. A function $f \in \mathcal{A}$ is called convex of order γ if, and only if, $zf'(z) \in \mathcal{S}^*(\gamma)$. Indeed, f is convex of order γ if, and only if,

$$\operatorname{Re}\left\{1 + \frac{zf'(z)}{f(z)}\right\} > \gamma \quad (z \in \Delta).$$

We denote by $\mathcal{K}(\gamma)$ the class of convex functions of order γ. The classes $\mathcal{S}^*(\gamma)$ and $\mathcal{K}(\gamma)$ for $0 \leq \gamma < 1$ are subclasses of the univalent functions (e.g., see [4]) and the function

$$\mathbf{K}_\gamma(z) := \frac{z}{(1-z)^{2(1-\gamma)}} = z + \sum_{n=2}^{\infty} \tau_n(\gamma) z^n \quad (z \in \Delta, 0 \leq \gamma < 1),$$

where

$$\tau_n(\gamma) := \frac{\prod_{k=2}^{n}(k-2\gamma)}{(n-1)!} \quad (n \geq 2),$$

is the well-known extremal function for the class $\mathcal{S}^*(\gamma)$. Observe that $\mathbf{K}_0(z)$ is the famous standard Koebe function. In particular $\mathcal{S}^* \equiv \mathcal{S}^*(0)$ and $\mathcal{K} \equiv \mathcal{K}(0)$ are the classes of starlike and convex functions in Δ, respectively. It is well-known that $\mathcal{K} \subset \mathcal{S}^*$.

Another one of the generalizations of Koebe function was proposed by Gasper [6]. Namely, he proposed some extension of the Löwner theory and de Branges's inequality, in which the natural extension of Koebe function is

$$k_q(z) = \frac{z}{(1-z)(1-qz)} \quad (z \in \Delta),$$

where $-1 \leq q \leq 1$. We now recall from [26], a two-parameter family of functions as follows:

$$k_{p,q}(z) := \frac{z}{(1-pz)(1-qz)} = z + \sum_{n=2}^{\infty} \mathfrak{A}_n z^n \quad ((p,q) \in [-1,1] \times [-1,1]), \tag{1.2}$$

where

$$\mathfrak{A}_n = \begin{cases} \frac{p^n - q^n}{p-q} & p \neq q, \\ np^{n-1} & p = q, \end{cases} \tag{1.3}$$

or

$$\mathfrak{A}_n = p^{n-1} + p^{n-2}q + \cdots + pq^{n-2} + q^{n-1} = \sum_{i=0}^{n-1} p^{n-i-1} q^i.$$

We note that $k_{1,1} \equiv \mathbf{K}_0$ and $k_{1,q} \equiv k_q$, therefore we understand the function $k_{p,q}$ as it's generalization. We also notice that the function $k_{p,q}$ is strictly related to the generalized Chebyshev polynomials of the second kind and maps the unit disk Δ onto a domain symmetric with respect to real axis. Here, we recall that the generalized Chebyshev polynomials of the second kind $U_n(p,q;e^{i\theta})$ are defined by

$$\Psi_{p,q}(e^{i\theta};z) = \frac{1}{(1-pze^{i\theta})(1-qze^{-i\theta})} = \sum_{n=0}^{\infty} U_n(p,q;e^{i\theta}) z^n \quad (z \in \Delta), \tag{1.4}$$

where $0 \le \theta \le 2\pi$ and $-1 \le p, q \le 1$. From (1.4) we have

$$U_0(p,q;e^{i\theta}) = 1, \quad U_1(p,q;e^{i\theta}) = pe^{i\theta} + qe^{-i\theta}$$

and

$$U_n(p,q;e^{i\theta}) = \frac{p^{n+1}e^{i(n+1)\theta} - q^{n+1}e^{-i(n+1)\theta}}{pe^{i\theta} - qe^{-i\theta}} \quad (n = 2, 3, 4, \ldots).$$

For more details about another properties of the function $k_{p,q}$ one can refer to [8, §2].

It was proved that [26, Proposition 1] for $-1 \le p, q \le 1$ (p and q at the same time are not zero) the function $k_{p,q}$ is starlike of order $\gamma_1 \in [0, 1)$ in Δ where

$$\gamma_1 := \gamma_1(p,q) = \frac{1}{2}\left(\frac{1-|p|}{1+|p|} + \frac{1-|q|}{1+|q|}\right)$$

and is convex in the disk $|z| < r(p,q)$ where

$$r(p,q) = \frac{2}{x + \sqrt{x^2 - 4|p||q|}} \quad \text{with} \quad x = \frac{|p| + |q| + \sqrt{|p|^2 + |q|^2 + 34|p||q|}}{2}.$$

The above results are sharp if $pq > 0$. Also, the function $k_{p,q}$ is convex of order $\gamma_2 \in [0, 1)$ in Δ (see [8, Lemma 2.4]) where

$$\gamma_2 := \gamma_2(p,q) = \frac{2(1-|pq|)}{(1+|p|)(1+|q|)} - \frac{1+|pq|}{1-|pq|} \qquad (-1 \le p, q \le 1, |pq| \ne 1).$$

It is easy to check that each of the results cited above is true with a wider assumption $(p, q) \in [-1, 1] \times [-1, 1]$.

In [8] were given bounds of minimum and maximum of the real part of function $k_{p,q}$. We quote them in the following lemma.

Lemma 1.1 *Let* $(p, q) \in [-1, 1] \times [-1, 1]$ *and* $|pq| \ne 1$. *The values of*

$$\max_{0 \le t \le 2\pi} \operatorname{Re}\left\{k_{p,q}(e^{it})\right\} \quad and \quad \min_{0 \le t \le 2\pi} \operatorname{Re}\left\{k_{p,q}(e^{it})\right\}$$

are the following

$$\min_{0 \le t \le 2\pi} \operatorname{Re}\left\{k_{p,q}(e^{it})\right\} = \begin{cases} \frac{-1}{1+p} & for \quad q = 0, \\ \frac{-1}{1+q} & for \quad p = 0, \\ \frac{-1}{(1+p)(1+q)} & for \quad pq < 0, \\ \frac{-1}{(1+p)^2} & for \quad q = p, \\ \frac{(1+pq)^2}{2(1-pq)[2\sqrt{pq(1-p^2)(1-q^2)}-(p+q)(1-pq)]} & for \quad p, q \in (0,1),\ p \ne q, \\ \frac{-1}{(1+p)(1+q)} & for \quad p, q \in (-1,0),\ p \ne q. \end{cases}$$

$$\max_{0 \le t \le 2\pi} \operatorname{Re}\left\{k_{p,q}(e^{it})\right\} = \begin{cases} \frac{1}{1-p} & for \quad q = 0, \\ \frac{1}{1-q} & for \quad p = 0, \\ \frac{1}{(1-p)(1-q)} & for \quad pq < 0, \\ \frac{1}{(1-p)^2} & for \quad q = p, \\ \frac{1}{(1-p)(1-q)} & for \quad p, q \in (0,1),\ p \ne q, \\ \frac{-(1+pq)^2}{2(1-pq)[(p+q)(1-pq)+2\sqrt{pq(1-p^2)(1-q^2)}]} & for \quad p, q \in (-1,0),\ p \ne q. \end{cases}$$

In 1992, Ma and Minda (see [19]) introduced the class $\mathcal{S}^*(\varphi)$ as follows

$$\mathcal{S}^*(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\},$$

where φ is analytic univalent function with $\operatorname{Re}\{\varphi(z)\} > 0$ $(z \in \Delta)$ and normalized by $\varphi(0) = 1$ and $\varphi'(0) > 0$. For special choices of φ, the class $\mathcal{S}^*(\varphi)$ becomes the well-known subclasses of the starlike functions. The class $\mathcal{S}^*((1 + Az)/(1 + Bz)) =:$

$\mathcal{S}^*[A, B]$ $(-1 \leq B < A \leq 1)$ was introduced by Janowski in [7]. If we let $\varphi(z) := (1+(1-2\gamma)z)/(1-z)$, then the class $\mathcal{S}^*(\varphi)$ $(0 \leq \gamma < 1)$ becomes the familiar class of the starlike functions of order γ. Letting $\varphi(z) := (1+(1-2\beta)z)/(1-z)$ $(\beta > 1)$ we have the class $\mathcal{M}(\beta)$ which was introduced and investigated by Uralegaddi et al. [35] as follows

$$\mathcal{M}(\beta) := \left\{ f \in \mathcal{A} : \mathrm{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} < \beta \right\} = \mathcal{S}^*\left(\frac{1+(1-2\beta)z}{1-z} \right).$$

Table 1 shows more details about some another subclasses of the starlike functions with different choices for φ.

We remark that all of the above special cases for φ are univalent in Δ. But in 2011, Dziok et al. [5] defined the class $\mathcal{S}_F^*$ related to the non-univalent function $\widetilde{p}(z)$ which includes of all functions $f \in \mathcal{A}$ so that satisfy the following subordination relation

$$\frac{zf'(z)}{f(z)} \prec \widetilde{p}(z),$$

where

$$\widetilde{p}(z) = \frac{1+t^2z^2}{1-tz-t^2z^2} \quad (t := (1-\sqrt{5})/2).$$

The function $\widetilde{p}(t)$ is related to the Fibonacci numbers and maps the open unit disc Δ onto a shell-like domain in the right-half plane.

Motivated by the above defined classes, we introduce a new subclass of the starlike functions associated with the generalized Koebe function $k_{p,q}$ which is defined in (1.2). We denote this subclass by $\mathcal{S}_k^*(p,q)$ which is defined as follows.

Definition 1.1 Let $f \in \mathcal{A}$ and $(p,q) \in [-1,1] \times [-1,1]$. Then the function $f \in \mathcal{S}_k^*(p,q)$ if and only if

$$\left(\frac{zf'(z)}{f(z)} - 1 \right) \prec k_{p,q}(z),$$

where $k_{p,q}$ is defined in (1.2).

With a simple calculation, we see that the function

$$\begin{aligned} f_{p,q}(z) &:= z \exp\left(\int_0^z \frac{k_{p,q}(t)}{t} \mathrm{d}t \right) = z \left(\frac{1-qz}{1-pz} \right)^{\frac{1}{p-q}} \\ &= z + z^2 + \frac{1}{2}(p+q+1)z^3 + \frac{1}{6}\left[2p^2 + p(2q+3) + 2q^2 + 3q + 1\right] z^4 + O(z^5), \end{aligned} \tag{1.5}$$

belongs to the class $\mathcal{S}_k^*(p,q)$. Since the function $f_{p,q}$ is not univalent in Δ (see Fig. 1), we conclude that the members of the class $\mathcal{S}_k^*(p,q)$ may not be univalent in the

Table 1 Some subclass of $\mathcal{S}^*$

Author/s	$\varphi(z)$	Year	Ref.
Ma and Minda	$1+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$	1992	[20]
Sokół and Stankiewicz	$\sqrt{1+z}$	1996	[34]
Sokół	$\frac{3}{3+(\alpha-3)z-\alpha z^2}$, $(-3<\alpha\le 1)$	2011	[33]
Kuroki and Owa	$1+\frac{\beta-\alpha}{\pi}i\log\left(\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}z}}{1-z}\right)$, $(0\le\alpha<1, \beta>1)$	2011	[18]
Mendiratta et al.	$\sqrt{2}-(\sqrt{2}-1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}}$	2014	[21]
Mendiratta et al.	e^z	2015	[22]
Raina and Sokół	$z+\sqrt{1+z^2}$	2015	[30]
Sharma et al.	$1+\frac{4z}{3}+\frac{2z^2}{3}$	2016	[32]
Kumar and Ravichandran	$1+(z/k)\frac{k+z}{k-z}$, $(k=1+\sqrt{2})$	2016	[17]
Kargar et al.	$1+\frac{1}{2i\sin\alpha}\log\left(\frac{1+ze^{i\alpha}}{1+ze^{-i\alpha}}\right)$, $(\pi/2\le\alpha<\pi)$	2017	[11]
Kargar et al.	$1+\frac{z}{1-\alpha z^2}$, $(0\le\alpha\le 1)$	2019	[10]
Cho et al.	$1+\sin z$	2019	[3]
Kumar et al.	$\exp(e^z-1)$	2019	[16]
Naraghi et al.	$\frac{1}{(1-z)^\alpha}$, $(0<\alpha\le 1)$	2021	[23]

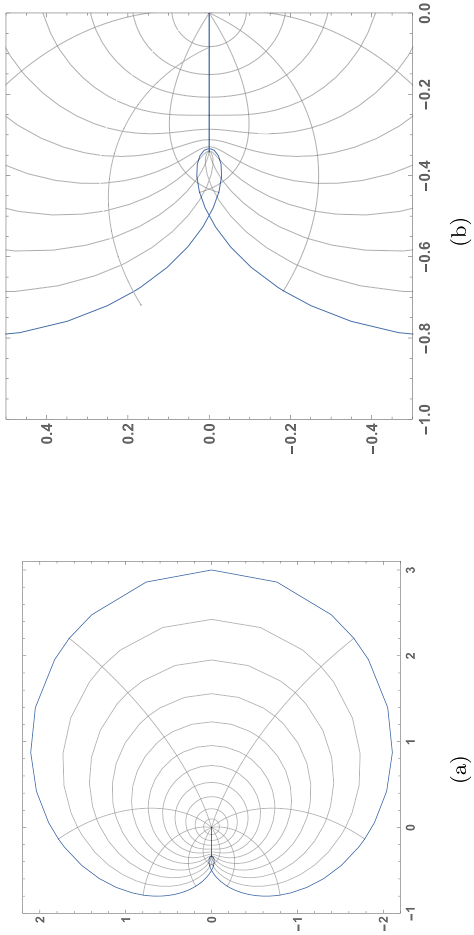


Fig. 1 **a** The image of Δ under the function $f_{-0.5,0.5}(z)$ **b** Zoom of (**a**)

whole disc Δ. Thus it will be interesting to find the radius of univalency of functions $f \in \mathcal{S}_k^*(p, q)$.

Using the concept of subordination and univalency of the function $k_{p,q}(z)$ and also by suitable choices for p and q, we describe some geometric properties of functions f belonging to the class $\mathcal{S}_k^*(p, q)$.

Remark 1.1 Let $k_{p,q}$ be given by (1.2). Then we have:

(1) Suppose that $p = q = 0$. If $f \in \mathcal{S}_k^*(0, 0)$, i.e. f satisfies the following subordination relation

$$\left(\frac{zf'(z)}{f(z)} - 1\right) \prec z, \tag{1.6}$$

then

$$0 < \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < 2 \quad (z \in \Delta).$$

This means that if f satisfies the above subordination relation (1.6), then it belongs to the class $\mathcal{S}(0, 2)$, where the class $\mathcal{S}(\gamma, \beta)$ $(0 \leq \gamma < 1, \beta > 1)$ was recently introduced by Kuroki and Owa [18].

(2) The case $p = q = 1$ in the equation (1.2) leads to the famous standard Koebe function. It is well-known that this function maps the unit disk onto the complex plane without the slit $(-\infty, -1/4]$ along the real axis. So if $f \in \mathcal{S}_k^*(1, 1)$, then it is starlike respect to $3/4$.

(3) Putting $p = q = -1$ in the equation (1.2) we have the famous function $z/(1+z)^2$ that maps the unit disk onto the complex plane without the slit $[1/4, \infty)$ along the real axis. Consequently if $f \in \mathcal{S}_k^*(-1, -1)$, then is starlike respect to $5/4$.

(4) If we set $p = -q$ in the equation (1.2), then we have the function $F_q(z) = \frac{z}{1-q^2z^2}$. The function $F_q(z)$ was studied in [27,28]. The function $F_q(z)$ is a starlike univalent when $q^2 < 1$. Also $F_q(\Delta) = D(q)$, where

$$D(q) := \left\{x + iy \in \mathbb{C} : \ \left(x^2 + y^2\right)^2 - \frac{x^2}{(1-q^2)^2} - \frac{y^2}{(1+q^2)^2} < 0\right\}$$

and

$$D(1) := \{x + iy \in \mathbb{C} : \ (\forall t \in (-\infty, -i/2] \cup [i/2, \infty))\, [x + iy \neq it]\}\,.$$

It should be noted that the curve

$$\left(x^2 + y^2\right)^2 - \frac{x^2}{(1-q^2)^2} - \frac{y^2}{(1+q^2)^2} = 0 \qquad (x, y) \neq (0, 0),$$

is the Booth lemniscate of elliptic type (see [27]). In the case $|q| = 1$, the function $F_q(z)$ becomes the function $G(z) := z/(1 - z^2)$ and thus $G(\Delta) = D(1)$. With a simple calculation if $f \in \mathcal{S}_k^*(-q, q)$, then

$$\frac{q^2}{q^2-1} < \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < \frac{2-q^2}{1-q^2} \quad (z \in \Delta).$$

The function class that satisfy the last two-sided inequality was introduced by Kargar et al. [10], and studied in [1,12,13].

(5) If we take $p = 0$ and $q \neq 0$ in (1.2), then we get

$$k_{p,q} \equiv k_q(z) := \frac{z}{1-qz} \quad (z \in \Delta).$$

Thus by Lemma 1.1 we have

$$\frac{-1}{1+q} < \operatorname{Re}\left\{k_q(z)\right\} < \frac{1}{1-q} \quad (z \in \Delta).$$

Furthermore, if $f \in \mathcal{A}$ belongs to the class $\mathcal{S}_k^*(0, q) \equiv \mathcal{S}_k^*(q)$, then

$$\frac{q}{1+q} < \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < \frac{2-q}{1-q} \quad (z \in \Delta).$$

(6) Let $pq < 0$. If the function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}_k^*(p, q)$, then

$$1 - \frac{1}{(1+p)(1+q)} < \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < 1 + \frac{1}{(1-p)(1-q)} \quad (z \in \Delta).$$

(7) Let $p = q$. If the function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}_k^*(p, q)$, then

$$1 - \frac{1}{(1+p)^2} < \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < 1 + \frac{1}{(1-p)^2} \quad (z \in \Delta).$$

(8) Assume that $p \neq q$ and $0 < p, q < 1$. If the function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}_k^*(p, q)$, then

$$1 + \frac{(1+pq)^2}{2(1-pq)[2\sqrt{pq(1-p^2)(1-q^2)} - (p+q)(1-pq)]} < \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < 1 + \frac{1}{(1-p)(1-q)}.$$

(9) Let $p \neq q$ and $-1 < p, q < 0$. If the function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}_k^*(p, q)$, then

$$1 - \frac{1}{(1+p)(1+q)} < \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}$$

$$< 1 - \frac{(1+pq)^2}{2(1-pq)[(p+q)(1-pq) + 2\sqrt{pq(1-p^2)(1-q^2)}]}.$$

The structure of the paper is as follows. In Sect. 2 we obtain some radius problems for the class $\mathcal{S}_k^*(p,q)$. In Sect. 3 we estimate the initial coefficients and logarithmic coefficients of the function f of the form (1.1) belonging to the class $\mathcal{S}_k^*(p,q)$.

2 The radius of starlikeness and convexity

The first result of this section is contained in the following theorem.

Theorem 2.1 *Let $(p,q) \in [-1,1] \times [-1,1]$ and $\gamma \in [0,1)$. If $f \in \mathcal{S}_k^*(p,q)$, then f is starlike of order γ in the disc $|z| < r_s(p,q,\gamma)$ where*

$$r_s(p,q,\gamma) = \begin{cases} \frac{1-\gamma}{1+(1-\gamma)|q|} & if \quad p = 0, \\ \frac{1-\gamma}{1+(1-\gamma)|p|} & if \quad q = 0, \\ \frac{1+(1-\gamma)(|p|+|q|)-\sqrt{1+2(1-\gamma)(|p|+|q|)+(1-\gamma)^2(|p|-|q|)^2}}{2(1-\gamma)||pq|} & if \quad pq \neq 0. \end{cases} \tag{2.1}$$

The result is sharp.

Proof Let $f \in \mathcal{S}_k^*(p,q)$ and $(p,q) \in [-1,1] \times [-1,1]$. Then from the definition of the class we have

$$\left(\frac{zf'(z)}{f(z)} - 1\right) \prec k_{p,q}(z),$$

where $k_{p,q}(z)$ is defined by (1.2). Therefore by the subordination principle there exists a Schwarz function $\omega : \Delta \to \Delta$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that

$$\frac{zf'(z)}{f(z)} - 1 = \frac{\omega(z)}{(1-p\omega(z))(1-q\omega(z))} \quad (z \in \Delta) \tag{2.2}$$

and consequently:

$$\begin{aligned} \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} &= \mathrm{Re}\left\{1 + \frac{\omega(z)}{(1-p\omega(z))(1-q\omega(z))}\right\} \\ &= 1 + \mathrm{Re}\left\{\frac{\omega(z)}{(1-p\omega(z))(1-q\omega(z))}\right\}. \end{aligned}$$

After application of the Schwarz lemma we have

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq 1 - \frac{|\omega(z)|}{|(1-p\omega(z))(1-q\omega(z))|} = 1 - \frac{|\omega(z)|}{|1-p\omega(z)| \cdot |1-q\omega(z)|}$$

$$\geq 1 - \frac{|\omega(z)|}{(1 - |p||\omega(z)|)(1 - |q||\omega(z)|)} \geq 1 - \frac{|z|}{(1 - |p||z|)(1 - |q||z|)}$$
$$= 1 - \frac{r}{(1 - |p|r)(1 - |q|r)}$$

where $r = |z| < 1$. Consider now the function $h(r) := 1 - \frac{r}{(1-|p|r)(1-|q|r)}$ $(r \in [0, 1])$. Its derivative has a form

$$h'(r) = -\frac{(1 - |p|r)(1 - |q|r) + r[|p|(1 - |q|r) + |q|(1 - |p|r)]}{(1 - |p|r)^2(1 - |q|r)^2},$$

so under assumptions of theorem we have $h'(r) < 0$ for $r \in [0, 1]$. From this we find that $h(r)$ is a strictly decreasing function on the interval $[0, 1]$ and it decreases from $h(0) = 1$ to the value $h(1) = 1 - \frac{1}{(1-|p|)(1-|q|)} < 0$. Therefore we conclude that there is only one root of the equation $h(r) = \gamma$ in $(0, 1)$. We can write this equation in the following equivalent form:

$$(1 - \gamma)|pq|r^2 - [1 + (1 - \gamma)(|p| + |q|)]r + 1 - \gamma = 0. \tag{2.3}$$

Denote the polynomial in (2.3) by $Q(r)$. In the case when p or q are zero, the equation $Q(r) = 0$ is linear equation so it has one solution $r = \frac{1-\gamma}{1+(1-\gamma)|q|}$ or $r = \frac{1-\gamma}{1+(1-\gamma)|p|}$ respectively. It is easy to see that in this both cases solutions are in the interval $(0, 1)$.

Assume now that $pq \neq 0$. Then Q is a quadratic polynomial with determinant of the form

$$\Lambda = 1 + 2(1 - \gamma)(|p| + |q|) + (1 - \gamma)^2(|p| - |q|)^2$$

and we can see that this determinant is positive for all $p, q;\ pq \neq 0$. In consequence, there are two roots of Q:

$$r_1 = \frac{1 + (1 - \gamma)(|p| + |q|) - \sqrt{1 + 2(1 - \gamma)(|p| + |q|) + (1 - \gamma)^2(|p| - |q|)^2}}{2(1 - \gamma)||pq|}$$

and

$$r_2 = \frac{1 + (1 - \gamma)(|p| + |q|) + \sqrt{1 + 2(1 - \gamma)(|p| + |q|) + (1 - \gamma)^2(|p| - |q|)^2}}{2(1 - \gamma)||pq|}$$

with $r_1 < r_2$. Observe that $Q(0) = 1 - \gamma > 0$. From this it follows that the roots r_1, r_2 both are positive numbers. Let us recall that the equation $h(r) = \gamma$ has strictly one solution in $(0, 1)$ so the equation $Q(r) = 0$ has. From this it follows that this solution is r_1. Therefore f is starlike of order γ in the disc $|z| < r < r_s(p, q, \gamma)$ where $r_s(p, q, \gamma)$ is given by (2.1).

For the sharpness consider the function $f_{p,q}$ given by (1.5). It is easy to see that

$$\frac{zf'_{p,q}(z)}{f_{p,q}(z)} = 1 + \frac{z}{(1 - pz)(1 - qz)} \quad (z \in \Delta).$$

With the same argument as above we get the result. Here the proof ends. □

Putting $\gamma = 0$ in the previous theorem we obtain the following result:

Corollary 2.1 *Let* $(p, q) \in [-1, 1] \times [-1, 1]$ *and* $\gamma \in [0, 1)$. *If* $f \in \mathcal{S}_k^*(p, q)$, *then* f *is starlike univalent in the disc* $|z| < r_s(p, q)$ *where*

$$r_s(p, q) = \begin{cases} \frac{1}{1+|q|} & if \quad p = 0, \\ \frac{1}{1+|p|} & if \quad q = 0, \\ \frac{1+(|p|+|q|)-\sqrt{1+2(|p|+|q|)+(|p|-|q|)^2}}{2|pq|} & if \quad pq \neq 0. \end{cases}$$

The result is sharp.

Theorem 2.2 *Let the number* $r \in (0, 1]$ *be given and* $(p, q) \in [-1, 1] \times [-1, 1]$. *If*

$$|q| < \frac{r|p| + r - 1}{r^2|p| - r}, \tag{2.4}$$

then each function $f \in \mathcal{S}_k^*(p, q)$ *maps a disc* $|z| < r$ *onto a starlike domain. The result is sharp.*

Proof Let $(p, q) \in [-1, 1] \times [-1, 1]$ satisfy (2.4) for given $r \in (0, 1]$. After repeating the same reasoning as in the proof of Theorem 2.1 we have that for $f \in \mathcal{S}_k^*(p, q)$ the following condition holds

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq 1 - \frac{|z|}{(1 - |p||z|)(1 - |q||z|)} \quad (z \in \Delta).$$

Moreover for $|z| < r$ we obtain

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq 1 - \frac{r}{(1 - |p|r)(1 - |q|r)} =: l(p, q).$$

It is easy to observe that under our assumptions, the function $l(p, q)$ has positive values. In conclusion we obtain the thesis. The function $f_{p,q}$ shows that the result is sharp concluding the proof. □

Now we shall find the range of parameters p, q that satisfy the assumptions of Theorem 2.2. For given $r \in (0, 1]$, let $D(r)$ by the set of solutions of the inequality (2.4). Observe that due to the form of this inequality, $D(r)$ must be symmetrical about both axes. Let us find its part lying in the first quadrant of the coordinate system. If $p \geq 0$ and $q \geq 0$ then (2.4) reduces to the condition

$$q < \frac{rp + r - 1}{r^2 p - r}.$$

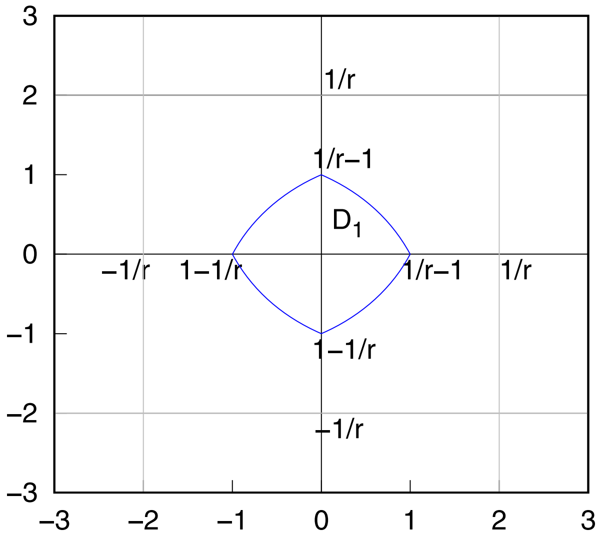


Fig. 2 The set $D(r)$

Note that, for the homography $q(p) = q = \frac{rp+r-1}{r^2p-r}$ its vertical asymptote and horizontal asymptote are given by the equations $p = \frac{1}{r}$ and $q = \frac{1}{r}$, respectively. Moreover, zero of this homography is the point $p = \frac{1}{r} - 1 < \frac{1}{r}$ and $0 < q(0) = \frac{1}{r} - 1 < \frac{1}{r}$. The suitable set of the (p, q) is bounded by one of the branches of hyperbola and by p-axis and q-axis (domain D_1, Fig. 2).

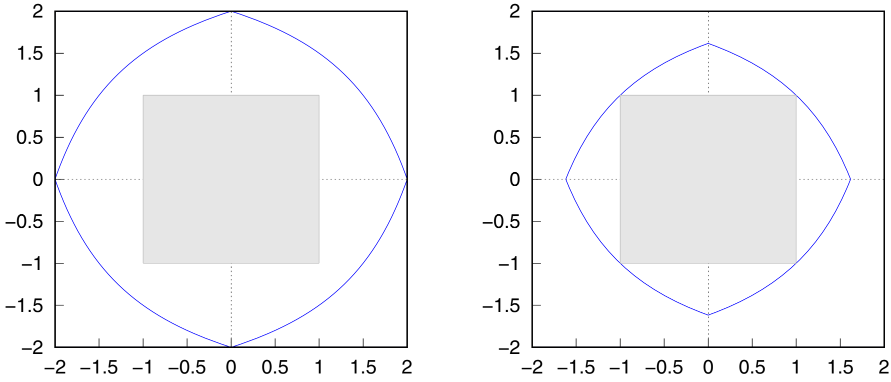
Therefore taking into account the symmetry of the set $D(r)$ we conclude that it has a form as in Fig. 2.

Remark 2.1 Note that regardless of the value of $r \in (0, 1)$, the asymptotes of the hyperbolas, whose fragments are components of the boundary of $D(r)$, do not have any common points with the square $[-1, 1] \times [-1, 1]$. Moreover, it is easy to observe that the domain $D(r)$ is growing if $r \longrightarrow 0$ and it is decreasing if $r \longrightarrow 1$. For this reason, as the value of r changes, the location of the set $D(r)$ relative to the square $[-1, 1] \times [-1, 1]$ also changes. Note that the point $(1, 1)$ is situated on the hyperbola given by the equation $q = \frac{rp+r-1}{r^2p-r}$ if and only if $r = \frac{3-\sqrt{5}}{2}$. For such r, also the other three vertices of the square are located on the boundary of the set $D(r)$. Hence, in this case, as well as for all $0 < r < \frac{3-\sqrt{5}}{2}$, the whole square is covered by $D(r)$.

Below we present various examples of the range of the parameters p, q that satisfy the assumptions of Theorem 2.2, i.e. the sets $D(r) \cap [-1, 1] \times [-1, 1]$ for selected values of r (Figs. 3, 4, 5).

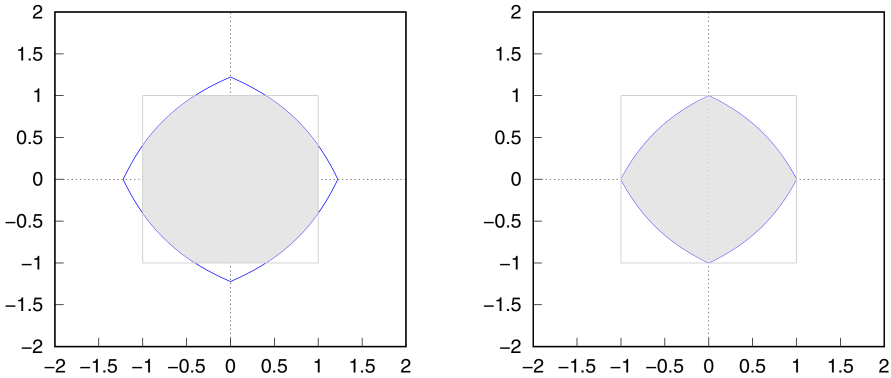
In view of *Remark* 2.1 we have the following result:

Corollary 2.2 *Let* $0 < r \leq \frac{3-\sqrt{5}}{2} = 0.381966\ldots$ *be given. Then for each function* $f \in \mathcal{S}_k^*(p, q)$ *the set* $f(|z| < r)$ *is a starlike domain.*



(a) (b)

Fig. 3 **a** The range of the parameters p, q for $r = \frac{1}{3}$, **b** The range of the parameters p, q for $r = \frac{3-\sqrt{5}}{2}$



(a) (b)

Fig. 4 **a** The range of the parameters p, q for $r = 0.45$, **b** The range of the parameters p, q for $r = \frac{1}{2}$

Theorem 2.3 *Let a function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}_k^*(p,q)$. Then f is convex univalent in the disk $|z| < \delta$ where δ is the smallest positive root of equation*

$$1 - \frac{r}{(1-|p|r)(1-|q|r)} - \left(\frac{2|p||q|r + 1 + |p| + |q|}{(1-|p|r)(1-|q|r) - r} + \frac{|p|}{1-|p|r} + \frac{|q|}{1-|q|r}\right)\frac{r}{1-r^2} = 0.$$

Proof Since $f \in \mathcal{S}_k^*(p,q)$, it follows that there exists a Schwarz function w such that (2.2) holds true. A logarithmic differentiation of (2.2) gives

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{w(z)}{(1-pw(z))(1-qw(z))}$$

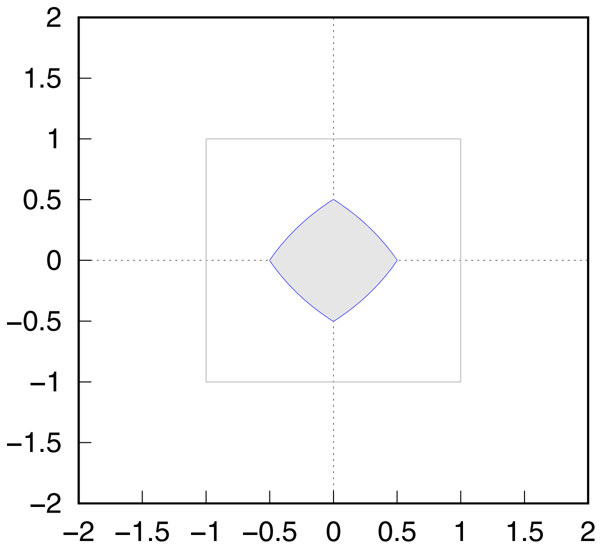


Fig. 5 The range of the parameters p, q for $r = \frac{2}{3}$

$$+\left(\frac{2pqw(z)+1-p-q}{(1-pw(z))(1-qw(z))+w(z)}+\frac{p}{1-pw(z)}+\frac{q}{1-qw(z)}\right)zw'(z). \quad (2.5)$$

The Schwarz-Pick lemma (see [24]) states that for a Schwarz function w the following sharp estimate holds

$$|w'(z)| \leq \frac{1-|w(z)|^2}{1-|z|^2} \quad (z \in \Delta).$$

Also if w is a Schwarz function then $|w(z)| \leq |z|$ (cf. [4]). According to what came above and using definition of convexity, it follows from (2.5) that

$$\begin{aligned}
\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} &\geq 1-\frac{|w(z)|}{(1-|p||w(z)|)(1-|q||w(z)|)} \\
&-\left(\frac{2|p||q||w(z)|+1+|p|+|q|}{(1-|p||w(z)|)(1-|q||w(z)|)-|w(z)|}+\frac{|p|}{1-|p||w(z)|}+\frac{|q|}{1-|q||w(z)|}\right)|zw'(z)| \\
&\geq 1-\frac{r}{(1-|p|r)(1-|q|r)} \\
&-\left(\frac{2|p||q|r+1+|p|+|q|}{(1-|p|r)(1-|q|r)-r}+\frac{|p|}{1-|p|r}+\frac{|q|}{1-|q|r}\right)\frac{r}{1-r^2} =: F(p,q,r). \quad (2.6)
\end{aligned}$$

It is a simple exercise that $F(p,q,r) > 0$ if and only if $0 < r \leq \delta$ where δ is the smallest positive root of

$$1-\frac{r}{(1-|p|r)(1-|q|r)}-\left(\frac{2|p||q|r+1+|p|+|q|}{(1-|p|r)(1-|q|r)-r}+\frac{|p|}{1-|p|r}+\frac{|q|}{1-|q|r}\right)\frac{r}{1-r^2}=0.$$

This is the end of proof. □

Remark 2.2 Let $F(p, q, r)$ be defined as (2.6). It is easy to check that $F(1/2, 1/2, r) = 0$ has two real roots as follows

$$r_1 \approx -1.35474 \quad \text{and} \quad r_2 \approx 0.177348.$$

Therefore if $f \in \mathcal{S}_k^*(1/2, 1/2)$, then f is convex univalent in the disk $|z| < r_2$. Also if $f \in \mathcal{S}_k^*(0, 0)$, then f is convex univalent in the disk $|z| < r_3$ where $r_3 \approx 0.55496$, because $F(0, 0, r)$ has three real roots

$$r_3 \approx 0.55496, \quad r_4 \approx -0.80194 \quad \text{and} \quad r_5 \approx 2.2470.$$

3 On coefficients of $f \in \mathcal{S}_k^*(p, q)$

Following, we shall estimate the initial coefficients and Fekete–Szegö problem for the function f of the form (1.1) belonging to the class $\mathcal{S}_k^*(p, q)$. The following lemmas will be useful.

Lemma 3.1 (Nehari [24, p. 172]) *Let w be a Schwarz function of the form*

$$w(z) = \sum_{n=1}^{\infty} c_n z^n \quad (z \in \Delta). \tag{3.1}$$

Then

$$|c_1| \le 1 \quad \textit{and} \quad |c_n| \le 1 - |c_1|^2 \quad (n = 2, 3, \ldots).$$

Lemma 3.2 (Prokhorov and Szynal [29]) *If w is a Schwarz function of the form* (3.1)*, then for any complex numbers ρ and τ the following sharp estimate holds:*

$$|c_3 + \rho c_1 c_2 + \tau c_1^3| \le H(\rho, \tau),$$

where

$$H(\rho,\tau)=\begin{cases}1 & \textit{for } (\rho,\tau)\in\Omega_1\cup\Omega_2\\ |\rho| & \textit{for } (\rho,\tau)\in\bigcup_{k=3}^{7}\Omega_k;\\ \frac{2}{3}(|\rho|+1)\left(\frac{|\rho|+1}{3(|\rho|+1+\tau)}\right)^{\frac{1}{2}} & \textit{for } (\rho,\tau)\in\Omega_8\cup\Omega_9\\ \frac{\tau}{3}\left(\frac{\rho^2-4}{\rho^2-4\tau}\right)^{\frac{1}{2}} & \textit{for } (\rho,\tau)\in\Omega_{10}\cup\Omega_{11}\setminus\{\pm 2,1\}\\ \frac{2}{3}(|\rho|-1)\left(\frac{|\rho|-1}{3(|\rho|-1-\tau)}\right)^{\frac{1}{2}} & \textit{for} (\rho,\tau)\in\Omega_{12}.\end{cases} \tag{3.2}$$

The extremal functions, up to rotations, are of the form

$$w(z)=z^3,\quad w(z)=z,\quad w(z)=w_0(z)=\frac{[(1-\lambda)\epsilon_2+\lambda\epsilon_1]z-\epsilon_1\epsilon_2 z}{1-[(1-\lambda)\epsilon_1+\lambda\epsilon_2]z},$$
$$w(z)=w_1(z)=\frac{z(t_1-z)}{1-t_1 z},\quad w(z)=w_2(z)=\frac{z(t_2+z)}{1+t_2 z},$$
$$|\epsilon_1|=|\epsilon_2|=1,\quad \epsilon_1=t_0-e^{\frac{-i\theta_0}{2}}(a\mp b),\quad \epsilon_2=-e^{\frac{-i\theta_0}{2}}(ia\pm b),$$
$$a=t_0\cos\frac{\theta_0}{2},\quad b=\sqrt{1-t_0^2\sin^2\frac{\theta_0}{2}},\quad \lambda=\frac{b\pm a}{2b},$$
$$t_0=\left(\frac{2\tau(\rho^2+2)-3\rho^2}{3(\tau-1)(\rho^2-4\tau)}\right)^{\frac{1}{2}},\quad t_1=\left(\frac{|\rho|+1}{3(|\rho|+1+\tau)}\right)^{\frac{1}{2}},$$
$$t_2=\left(\frac{|\rho|-1}{3(|\rho|-1-\tau)}\right)^{\frac{1}{2}},\quad \cos\frac{\theta_0}{2}=\frac{\rho}{2}\left[\frac{\tau(\rho^2+8)-2(\rho^2+2)}{2\tau(\rho^2+2)-3\rho^2}\right].$$

The sets Ω_i, $i=1,2,\ldots,12$ *are defined as follows:*

$$\Omega_1=\left\{(\rho,\tau):|\rho|\le\frac{1}{2},|\tau|\le 1\right\},$$
$$\Omega_2=\left\{(\rho,\tau):\frac{1}{2}\le|\rho|\le 2,\frac{4}{27}(|\rho|+1)^3-(|\rho|+1)\le\tau\le 1\right\},$$
$$\Omega_3=\left\{(\rho,\tau):|\rho|\le\frac{1}{2},\tau\le -1\right\},$$
$$\Omega_4=\left\{(\rho,\tau):|\rho|\ge\frac{1}{2},\tau\le -\frac{2}{3}(|\rho|+1)\right\},$$
$$\Omega_5=\{(\rho,\tau):|\rho|\le 2,\tau\ge 1\},$$
$$\Omega_6=\left\{(\rho,\tau):2\le|\rho|\le 4,\tau\ge\frac{1}{12}(\rho^2+8)\right\},$$

$$\Omega_7 = \left\{(\rho,\tau) : |\rho| \geq 4, \tau \geq \frac{2}{3}(|\rho|-1)\right\},$$
$$\Omega_8 = \left\{(\rho,\tau) : \frac{1}{2} \leq |\rho| \leq 2, -\frac{2}{3}(|\rho|+1) \leq \tau \leq \frac{4}{27}(|\rho|+1)^3 - (|\rho|+1)\right\},$$
$$\Omega_9 = \left\{(\rho,\tau) : |\rho| \geq 2, -\frac{2}{3}(|\rho|+1) \leq \tau \leq \frac{2|\rho|(|\rho|+1)}{\rho^2+2|\rho|+4}\right\},$$
$$\Omega_{10} = \left\{(\rho,\tau) : 2 \leq |\rho| \leq 4, \frac{2|\rho|(|\rho|+1)}{\rho^2+2|\rho|+4} \leq \tau \leq \frac{1}{12}(\rho^2+8)\right\},$$
$$\Omega_{11} = \left\{(\rho,\tau) : |\rho| \geq 4, \frac{2|\rho|(|\rho|+1)}{\rho^2+2|\rho|+4} \leq \tau \leq \frac{2|\rho|(|\rho|-1)}{\rho^2-2|\rho|+4}\right\},$$
$$\Omega_{12} = \left\{(\rho,\tau) : |\rho| \geq 4, \frac{2|\rho|(|\rho|-1)}{\rho^2-2|\rho|+4} \leq \tau \leq \frac{2}{3}(|\rho|-1)\right\}.$$

Lemma 3.3 (Keogh and Merkes [15]) *Let w be a Schwarz function of the form* (3.1). *Then for any complex number μ we have*

$$|c_2 - \mu c_1^2| \leq \max\{1, |\mu|\}.$$

The result is sharp for the functions $w(z) = z^2$ or $w(z) = z$.

Theorem 3.1 *Let f of the form* (1.1) *belong to the class $\mathcal{S}_k^*(p,q)$ where $(p,q) \in [-1,1] \times [-1,1]$. Then the following inequalities for the coefficients of f hold*

$$|a_2| \leq 1, \tag{3.3}$$
$$|a_3| \leq \begin{cases} \frac{1}{2}(1+|p|+|q|), & p \neq q; \\ \frac{1}{2}(1+2|p|), & p = q \end{cases} \tag{3.4}$$

and

$$|a_4| \leq \frac{1}{3}H(\rho,\tau) \tag{3.5}$$

with

$$\rho = 2(p+q) + \frac{3}{2}, \quad \tau = p^2 + pq + q^2 + \frac{3}{2}(p+q) + \frac{1}{2},$$

where $H(\rho,\tau)$ is of the form as in Lemma 3.2. Further

$$|a_3 - \mu a_2^2| \leq \frac{1}{2}\max\{1, |2\mu - (p+q+1)|\}.$$

All inequalities are sharp.

Proof Let $f \in \mathcal{S}_k^*(p,q)$. Then by Definition 1.1 and subordination principle, there exists a Schwarz function w of the form (3.1) with $|w(z)| < 1$ such that

$$\frac{zf'(z)}{f(z)} - 1 = k_{p,q}(w(z)) \quad (z \in \Delta),$$

where $k_{p,q}$ is defined in (1.2). Furthermore, since f has the form (1.1), it is easy to see that

$$\begin{aligned}\frac{zf'(z)}{f(z)} - 1 = a_2 z + \left(2a_3 - a_2^2\right) z^2 + \left(3a_4 - 3a_2a_3 + a_2^3\right) z^3 \\ + \left(4a_5 - 2a_3^2 - 4a_2a_4 + a_2^2\left(4a_3 - a_2^2\right)\right) z^4 + \cdots,\end{aligned} \tag{3.6}$$

Also, using (1.2) and (3.1), we get

$$\begin{aligned}k_{p,q}(w(z)) = c_1 z + \left(c_2 + \mathfrak{A}_2 c_1^2\right) z^2 + \left(c_3 + 2c_1c_2\mathfrak{A}_2 + \mathfrak{A}_3 c_1^3\right) z^3 \\ + \left(c_4 + \mathfrak{A}_2\left[2c_1c_3 + c_2^2\right] + 3c_1^2c_2\mathfrak{A}_3\right) z^4 + \cdots,\end{aligned} \tag{3.7}$$

where

$$\mathfrak{A}_2 = \begin{cases} p+q, & p \neq q; \\ 2p, & p = q \end{cases} \tag{3.8}$$

and

$$\mathfrak{A}_3 = \begin{cases} p^2 + pq + q^2, & p \neq q; \\ 3p^2, & p = q, \end{cases} \tag{3.9}$$

are defined in (1.2). Comparing (3.6) and (3.7), gives us

$$a_2 = c_1 \tag{3.10}$$

$$a_3 = \frac{1}{2}\left(c_2 + (\mathfrak{A}_2 + 1)c_1^2\right) \tag{3.11}$$

$$a_4 = \frac{1}{3}\left[\left(\mathfrak{A}_3 + \frac{3}{2}\mathfrak{A}_2 + \frac{1}{2}\right)c_1^3 + \left(2\mathfrak{A}_2 + \frac{3}{2}\right)c_1c_2 + c_3\right] \tag{3.12}$$

The inequality $|a_2| \leq 1$ follows directly from Lemma 3.1 and (3.10) with sharpness for the function $f_{p,q}$ given by (1.5). From Lemma 3.1 we have

$$\left|c_2 + (\mathfrak{A}_2 + 1)c_1^2\right| \leq |c_2| + (|\mathfrak{A}_2| + 1)|c_1|^2 \leq 1 - |c_1|^2 + |\mathfrak{A}_2||c_1|^2 + |c_1|^2 \leq 1 + |\mathfrak{A}_2|.$$

Therefore using (3.11) and the last estimate give that $|a_3| \leq \frac{1}{2}\left(1 + |\mathfrak{A}_2|\right)$. Now by (3.8) and since $(p,q) \in [-1,1] \times [-1,1]$, we get the desired inequality (3.4).

Now we shall find the estimation of the fourth coefficient. For this we first use (3.8) and (3.9) in (3.12) to obtain

$$a_4 = \frac{1}{3}\left[c_3 + \left(2(p+q) + \frac{3}{2}\right)c_1c_2 + \left(p^2 + pq + q^2 + \frac{3}{2}(p+q) + \frac{1}{2}\right)c_1^3\right].$$

By setting

$$\rho = 2(p+q) + \frac{3}{2}, \quad \tau = p^2 + pq + q^2 + \frac{3}{2}(p+q) + \frac{1}{2}$$

and by applying Lemma 3.2, we can write

$$|a_4| \le \frac{1}{3}H(\rho, \tau),$$

where the function H is defined in (3.2). Thus the result is established.

Now let μ be a complex number. From (3.10) and (3.11) we get

$$a_3 - \mu a_2^2 = \frac{1}{2}\left[c_2 - (2\mu - \mathfrak{A}_2 - 1)\,c_1^2\right].$$

Therefore using Lemma 3.3 we obtain

$$\begin{aligned}|a_3 - \mu a_2^2| &= \frac{1}{2}\left|c_2 - (2\mu - \mathfrak{A}_2 - 1)\,c_1^2\right| \le \frac{1}{2}\max\{1, |2\mu - \mathfrak{A}_2 - 1|\}\\ &= \frac{1}{2}\max\{1, |2\mu - (p+q+1)|\}.\end{aligned}$$

It easy to see that equalities in (3.3)–(3.4) occur for the function $f_{p,q}$ defined by (1.5). Sharpness the third inequality (3.5) also follows as an application of Lemma 3.2. This completes the proof. □

At the end of this paper we discuss the logarithmic coefficients $\gamma_n := \gamma_n(f)$ of the functions f belonging to the class $\mathcal{S}_k^*(p, q)$. We recall that the logarithmic coefficients γ_n of $f \in \mathcal{S}$ are defined with the following series expansion:

$$\log\left\{\frac{f(z)}{z}\right\} = \sum_{n=1}^{\infty} 2\gamma_n(f)z^n \quad (z \in \Delta). \tag{3.13}$$

The logarithmic coefficients have an important role in Geometric Function Theory. We remark that Kayumov [14] by use of these coefficients and under an additional condition solved the Brennan conjecture for conformal mappings or before de Branges by use of this concept, was able to prove the famous Bieberbach's conjecture [2]. We recall that the logarithmic coefficients γ_n of every function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$ satisfy the inequalities

$$|\gamma_1| \le 1, \quad |\gamma_2| \le \frac{1}{2}(1 + 2e^{-2}) \approx 0.635$$

and

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{\pi^2}{6}.$$

The sharp estimate of $|\gamma_n|$ when $n \geq 3$ and $f \in \mathcal{S}$ is still open.

In the sequel, we derive some inequalities involving the logarithmic coefficients in the class $\mathcal{S}_k^*(p, q)$.

Theorem 3.2 *If a function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}_k^*(p, q)$ and γ_n is the logarithmic coefficient of f, then the following sharp inequality holds*

$$|\gamma_1| \leq \frac{1}{2}, \quad |\gamma_2| \leq \begin{cases} \frac{1}{4}(|p| + |q|), \ p \neq q \& |p+q| \geq 1; \\ \\ \frac{1}{2}|p|, \qquad\qquad p = q \& |p| \geq 1/2 \end{cases}$$

and

$$|\gamma_3| \leq \frac{1}{6} H(\rho, \tau)$$

with

$$\rho = 2\mathfrak{A}_2, \quad \tau = \mathfrak{A}_3,$$

where $H(\rho, \tau)$ is of the form as in Lemma 3.2, and $\mathfrak{A}_2$ and $\mathfrak{A}_3$ are defined in (3.8) *and* (3.9)*, respectively. All inequalities are sharp.*

Proof Let the function $f \in \mathcal{A}$ belong to the class $\mathcal{S}_k^*(p, q)$. Then by definition we have

$$\frac{zf'(z)}{f(z)} - 1 = z \left(\log \left\{ \frac{f(z)}{z} \right\} \right)' \prec k_{p,q}(z),$$

where $k_{p,q}$ is given by (1.2). Now, by (1.2) and (3.13), the last subordination relation implies that

$$2 \sum_{n=1}^{\infty} n \gamma_n z^n \prec z + \sum_{n=2}^{\infty} \mathfrak{A}_n z^n,$$

where $\mathfrak{A}_n$ are defined in (1.3). By definition of subordination and (3.1) the last relation implies that

$$2 \sum_{n=1}^{\infty} n \gamma_n z^n = w(z) + \sum_{n=2}^{\infty} \mathfrak{A}_n w^n(z)$$

$$= c_1 z + \left(c_2 + \mathfrak{A}_2 c_1^2\right) z^2 + \left(c_3 + 2c_1c_2\mathfrak{A}_2 + \mathfrak{A}_3 c_1^3\right) z^3 + \cdots. \quad (3.14)$$

It follows from (3.14) that

$$2\gamma_1 = c_1, \quad 4\gamma_2 = c_2 + \mathfrak{A}_2 c_1^2$$

and

$$6\gamma_3 = c_3 + 2c_1c_2\mathfrak{A}_2 + \mathfrak{A}_3 c_1^3.$$

By Lemma 3.1, we obtain $2|\gamma_1| = |c_1| \leq 1$ or $|\gamma_1| \leq 1/2$. Thus the first inequality holds true. To obtain the second inequality by using Lemma 3.1 and (3.8) we get

$$\begin{aligned} 4|\gamma_2| &= |c_2 + \mathfrak{A}_2 c_1^2| \leq |c_2| + |\mathfrak{A}_2||c_1|^2 \leq 1 - |c_1|^2 + |\mathfrak{A}_2||c_1|^2 \\ &= (|\mathfrak{A}_2| - 1)|c_1|^2 + 1 \leq (|\mathfrak{A}_2| - 1) + 1 = |\mathfrak{A}_2| \end{aligned}$$

or

$$|\gamma_2| \leq \frac{1}{4}|\mathfrak{A}_2|.$$

This proves the second inequality. To estimate the third inequality it is enough to set $\rho = 2\mathfrak{A}_2$ and $\tau = \mathfrak{A}_3$ in Lemma 3.2.

For the sharpness we consider the function $f_{p,q}$ defined by (1.5). A simple check gives that

$$\begin{aligned} \sum_{n=1}^{\infty} 2\gamma_n(f_{p,q})z^n = \log\left\{\frac{f_{p,q}(z)}{z}\right\} &= \frac{1}{p-q}\log\frac{1-qz}{1-pz} \\ &= z + \frac{1}{2}(p+q)z^2 + \frac{1}{3}(p^2+pq+q^2)z^3 \\ &\quad + \frac{1}{4}(p^3+p^2q+pq^2+q^3)z^4 + \cdots. \end{aligned}$$

Comparison of the corresponding coefficients and an application of Lemma 3.2 show the result is sharp, therefore the proof is completed. □

For the next result we need the following theorem. By using this theorem we give the sharp inequality for sums involving logarithmic coefficients.

Theorem 3.3 *Let the function $f \in \mathcal{A}$ belong to the class $\mathcal{S}_k^*(p,q)$ and $k_{p,q}(z)$ be defined by* (1.2). *Then*

$$\log\left\{\frac{f(z)}{z}\right\} \prec \int_0^z \frac{k_{p,q}(t)}{t}\mathrm{d}t.$$

Moreover

$$K_{p,q}(z) := \int_0^z \frac{k_{p,q}(t)}{t} \mathrm{d}t \quad (z \in \Delta), \tag{3.15}$$

is a convex univalent function.

Proof The proof is similar to the proof of [9, Theorem 2.1], and thus we omit the details. □

By (1.2), it is easy to see that $K_{p,q}(z)$ has the following series expansion:

$$K_{p,q}(z) = \sum_{n=1}^{\infty} \frac{\mathfrak{A}_n}{n} z^n \quad (\mathfrak{A}_1 = 1), \tag{3.16}$$

where $\mathfrak{A}_n$ are defined in (1.3).

Theorem 3.4 *Let the $f \in \mathcal{A}$ belong to the class $\mathcal{S}_k^*(p, q)$. Then the logarithmic coefficients of f satisfy the following sharp inequality*

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \begin{cases} \frac{1}{4|p-q|^2} \sum_{n=1}^{\infty} \frac{1}{n^2} |p^n - q^n|^2, & p \neq q, p \neq 0, q \neq 0; \\ \frac{1}{4|p|^2} Li_2(|p|^2), & p \neq 0 = q; \\ \frac{1}{4|q|^2} Li_2(|q|^2), & q \neq 0 = p; \\ \frac{1}{4(1-|p|^2)}, & p = q \neq \pm 1, \end{cases}$$

where Li_2 is the well-known dilogarithm function.

Proof If a function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}_k^*(p, q)$, then by the previous Theorem 3.3 we have

$$\log \left\{ \frac{f(z)}{z} \right\} \prec \int_0^z \frac{k_{p,q}(t)}{t} \mathrm{d}t. \tag{3.17}$$

Replacing (3.13) and (3.16) into (3.17) we get

$$\sum_{n=1}^{\infty} 2\gamma_n z^n \prec \sum_{n=1}^{\infty} \frac{\mathfrak{A}_n}{n} z^n \quad (\mathfrak{A}_1 = 1). \tag{3.18}$$

Applying Rogosinski's theorem [31], we can obtain

$$4 \sum_{n=1}^{\infty} |\gamma_n|^2 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} |\mathfrak{A}_n|^2$$

$$= \begin{cases} \frac{1}{|p-q|^2} \sum_{n=1}^{\infty} \frac{1}{n^2} |p^n - q^n|^2, & p \neq q; \\ \sum_{n=1}^{\infty} |p|^{2(n-1)}, & p = q. \end{cases}$$

We consider the following cases:
Case 1. Let $q = 0 \neq p$. Then we have

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{1}{4|p|^2} \sum_{n=1}^{\infty} \frac{1}{n^2} |p|^{2n} = \frac{1}{4|p|^2} Li_2(|p|^2) \quad (|p|^2 \leq 1),$$

where Li_2 denotes the dilogarithm function.
Case 2. Let $p = 0 \neq q$. In this case we have

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{1}{4|q|^2} \sum_{n=1}^{\infty} \frac{1}{n^2} |q|^{2n} = \frac{1}{4|q|^2} Li_2(|q|^2) \quad (|q|^2 \leq 1).$$

Case 3. Let $p = q \neq \pm 1$. Thus we get

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{1}{4} \sum_{n=1}^{\infty} |p|^{2(n-1)} = \frac{1}{4(1 - |p|^2)}.$$

For the sharpness, it is enough to consider the function

$$\widetilde{K}_{p,q}(z) := z \exp(K_{p,q}(z)),$$

where $K_{p,q}$ is defined by (3.15). It is easily seen that $\widetilde{K}_{p,q}(z) \in \mathcal{S}_k^*(p, q)$ and

$$\gamma_n(\widetilde{K}_{p,q}(z)) = \frac{\mathfrak{A}_n}{2n}.$$

Thus the proof is complete. □

Remark 3.1 Since $K_{p,q}(z)$ is convex univalent in Δ, it follows from (3.18) and Rogosinski's theorem that $2|\gamma_n| \leq 1$ or $|\gamma_n| \leq 1/2$. This means that in Theorem 3.2 in the third inequality we have

$$|\gamma_3| \leq \frac{1}{6} H(\rho, \tau) \leq \frac{1}{2}.$$

Therefore $H(\rho, \tau) \leq 3/2$ and consequently $(\rho, \tau) \notin \Omega_1 \cup \Omega_2$ where ρ and τ are as Theorem 3.2.

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Declarations

Conflict of interest The authors have no conflicts of interest to declare that are relevant to the content of this article.

Author contributions Both authors contributed to the study's conception and design equally. They also read and approved the final manuscript.

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