J. Appl. Prob. **58,** 1–21 (2021)
*doi:*10.1017/jpr.2020.80

ERGODIC CONTROL OF DIFFUSIONS WITH RANDOM INTERVENTION TIMES

HARTO SAARINEN,* ** AND
JUKKA LEMPA,* *University of Turku*

Abstract

We study an ergodic singular control problem with constraint of a regular one-dimensional linear diffusion. The constraint allows the agent to control the diffusion only at the jump times of an independent Poisson process. Under relatively weak assumptions, we characterize the optimal solution as an impulse-type control policy, where it is optimal to exert the exact amount of control needed to push the process to a unique threshold. Moreover, we discuss the connection of the present problem to ergodic singular control problems, and illustrate the results with different well-known cost and diffusion structures.

Keywords: Bounded variation control; ergodic control; diffusion process; resolvent operator; Poisson process; singular stochastic control

2010 Mathematics Subject Classification: Primary 49N25
Secondary 49L20;60J60

1. Introduction

In many biological and economical control problems, the decision maker is faced with the situation where the information of the evolving system is not available all the time. Instead, the decision maker might observe the state of the system only at discrete times, for example daily or weekly. Thus, in the following we model the times when the controller receives the information of the evolving system as jump times of a Poisson process with a parameter λ. It is assumed that the decision maker can only exert control at these exogenously given times, in other words, they cannot act in the dark. Also, we restrict ourselves to controls of impulse type. Whenever control is applied, the decision maker has to pay a cost which is directly proportional to the size of the impulse. Otherwise, when there are no interventions, we assume that the system evolves according to one-dimensional linear diffusion X that is independent of the Poisson process. In the literature these types of restriction processes on the controllability of X are often referred to as *constraints* or *signals*, see [24, 26, 30, 31, 32].

In the classical case, the decision maker has continuous and complete information, and hence control is allowed whenever the decision maker wishes. The objective criterion to be minimized is often either a discounted cost or an ergodic cost (average cost per unit time). Both discounted cost and ergodic problems have been studied in the literature, but the ergodic

Received 13 September 2019; revision received 8 June 2020.
* Postal address: Department of Mathematics and Statistics, FI - 20014 Turun Yliopisto, Finland.
** Email address: hoasaa@utu.fi

problems have received less attention. This is because they are often mathematically more involved. However, from the point of view of many applications this is a bit surprising, as the discounting factor is often very hard or impossible to estimate. Also, outside of financial applications the discounting factor might not have a very clear interpretation.

The simplest case in the classical setting is where control is assumed to be costless. As a result, the optimal policy is often a local time of X at the corresponding boundaries; see [1, 29] for discounted problems and [3, 4, 19] for ergodic problems. One drawback of this model is that the optimal strategies are often singular with respect to Lebesgue measure, which makes them unappealing for applications. One way to make the model more realistic is to add a fixed transaction cost on the control. Then the optimal policy is often a sequential impulse control where the decision maker chooses a sequence of stopping times $\{\tau_1, \tau_2, \ldots\}$ to exert control, and corresponding impulse sizes $\{\zeta_1, \zeta_2, \ldots\}$; see [2, 5, 18]. In addition, it is possible that the flow of information is continuous but imperfect. This type of problem, often referred to as *filtering problems*, is also widely studied; see [8, 14, 33] for a textbook treatment and further references. In this case, the disturbance in the information flow is assumed to be such that the decision maker sees the underlying process all the time, but only observes a noisy version of it.

As in the model at hand, another possibility is to allow the decision maker to control only at certain discrete exogenously given times. These times can be, for example, multiples of integers, as in [23, 35], or given by a signal process. Often, as in our model, the times between arrivals of the signal process are assumed to be exponentially distributed; see [26, 36, 39]. In [36], this framework was used as a simple model for liquidity effects in a classical investment optimization problem. Reference [39] investigates both discounted cost and an ergodic cost criterion while tracking a Brownian motion under quadratic cost, and [26] generalizes the discounted problem to a more general payoff and underlying structure. Related studies in optimal stopping are [12, 24]. In [12], the authors consider a perpetual American call with underlying geometric Brownian motion, and in [24] the results are generalized to a larger class of underlying processes. Studies related to more general signal processes are found in [30, 31, 32]. In these, the signal process can be a general, not necessarily independent, renewal process, and the underlying process is a general Markov–Feller process. There are also multiple studies that are less directly related, where an underlying Poisson process brings a different friction to the model by either affecting the structure of the underlying diffusion [17, 20] or the payoff structure [6, 25, 27].

The main contribution of this paper is that we allow the underlying stochastic process X to follow a general one-dimensional diffusion process, and also allow a rather general cost function. This is a substantial generalization of [39], where the case of Brownian motion with quadratic cost is considered. We emphasize this in the illustrations in Section 5 by explicitly solving multiple examples with different underlying dynamics and cost functions. These generalizations have not, to the best of our knowledge, been studied before in the literature. Furthermore, we are able to connect the problem to a related problem in optimal ergodic singular control [3].

The rest of the paper is organized as follows. In Section 2 we define the control problem and prove auxiliary results. In Section 3, we first investigate the necessary conditions of optimality by forming the associated free boundary problem, followed by the verification. We connect the problem to a similar problem of singular control in Section 4, and then illustrate the results by explicitly solving a few examples in Section 5. Finally, Section 6 concludes our study.

2. The control problem

2.1. The underlying dynamics

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space which satisfies the usual conditions. We consider an uncontrolled process X defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ that evolves in $\mathbb{R}_+$ and is modelled as a solution to regular linear Itô diffusion

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \qquad X_0 = x,$$

where W_t is the Wiener process and the functions $\mu, \sigma : (0, \infty) \to \mathbb{R}$ are continuous and satisfy the condition $\int_{x-\varepsilon}^{x+\varepsilon} \frac{1+|\mu(y)|}{\sigma^2(y)}\mathrm{d}y < \infty$. These assumptions guarantee that the diffusion has a unique weak solution (see [22, Section 5.5]). Although we consider the case where the process evolves in $\mathbb{R}_+$, we remark that this is done only for notational convenience, and the results would remain the same, with obvious changes, if the state space were replaced with any interval of $\mathbb{R}$.

We define the second-order linear differential operator $\mathcal{A}$ that represents the infinitesimal generator of the diffusion X as

$$\mathcal{A} = \mu(x)\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}\sigma^2(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2},$$

and for a given $\lambda > 0$ we respectively denote the increasing and decreasing solutions to the differential equation $(\mathcal{A} - \lambda)f = 0$ by $\psi_\lambda > 0$ and $\varphi_\lambda > 0$.

The differential operator $\lambda - \mathcal{A}$ has an inverse operator called the resolvent R_λ defined by

$$(R_\lambda f)(x) = \mathbb{E}_x\left[\int_0^\tau \mathrm{e}^{-\lambda s} f(X_s)\mathrm{d}s\right]$$

for all $x \in \mathbb{R}_+$ and functions $f \in \mathcal{L}_1^\lambda$, where $\mathcal{L}_1^\lambda$ is the set of functions f on $\mathbb{R}_+$ which satisfy the integrability condition $\mathbb{E}_x[\int_0^\tau \mathrm{e}^{-\lambda s}|f(X_s)|\mathrm{d}s] < \infty$. Here, τ is the first exit time from $\mathbb{R}_+$, i.e. $\tau = \inf\{t \geq 0 \mid X_t \notin \mathbb{R}_+\}$. We also define the scale density of the diffusion by

$$S'(x) = \exp\left(-\int^x \frac{2\mu(z)}{\sigma^2(z)}\mathrm{d}z\right),$$

which is the (non-constant) solution to the differential equation $\mathcal{A}f = 0$, and the speed measure of the diffusion by

$$m'(x) = \frac{2}{\sigma^2(x)S'(x)}.$$

It is well known that the resolvent and the solutions ψ_λ and φ_λ are connected by the formula

$$\begin{aligned}(R_\lambda f)(x) = {} & B_\lambda^{-1}\psi_\lambda(x)\int_x^\infty \varphi_\lambda(z)f(z)m'(z)\mathrm{d}z \\ & + B_\lambda^{-1}\varphi_\lambda(x)\int_0^x \psi_\lambda(z)f(z)m'(z)\mathrm{d}z,\end{aligned} \tag{1}$$

where

$$B_\lambda = \frac{\psi_\lambda'(x)}{S'(x)}\varphi_\lambda(x) - \frac{\varphi_\lambda'(x)}{S'(x)}\psi_\lambda(x)$$

denotes the Wronskian determinant (see [10, p. 19]). We remark that the value of B_λ does not depend on the state variable x because an application of the harmonicity properties of ψ_λ and φ_λ yields

$$\frac{\mathrm{d}B_\lambda(x)}{\mathrm{d}x}=0.$$

In calculations, it is sometimes also useful to use the identity

$$\int_x^y \mu(z)m'(z)\mathrm{d}z=\frac{1}{S'(y)}-\frac{1}{S'(x)}. \tag{2}$$

2.2. The control problem

We consider a control problem where the goal is to minimize the average cost per unit time, so that the controller is only allowed to control the underlying process at exogenously given times. These times are given as the arrival times of an independent Poisson process, called the *signal process* or *constraint*, and thus the interarrival times are exponentially distributed.

Assumption 1. *The Poisson process N_t and the controlled process X_t are assumed to be independent, and the process N_t is $\{\mathcal{F}_t\}_{t\geq 0}$-adapted.*

More precisely, the set of admissible controls $\mathcal{Z}$ is given by those non-decreasing left-continuous processes $\zeta_{t\geq 0}$ that have the representation

$$\zeta_t=\int_{[0,t)}\eta_s\mathrm{d}N_s,$$

where N is the signal process and the integrand η is $\{\mathcal{F}_t\}_{t\geq 0}$-predictable. The controlled dynamics are then given by the Itô integral

$$X_t^\zeta=X_0+\int_0^{\tau_0^\zeta\wedge t}\mu(X_s^\zeta)\mathrm{d}s+\int_0^{\tau_0^\zeta\wedge t}\sigma(X_s^\zeta)\mathrm{d}W_s-\zeta_t, \qquad 0\leq t\leq \tau_0^\zeta,$$

where τ_0^ζ is the first exit time of X_t^ζ from $\mathbb{R}_+$, i.e. $\tau_0^\zeta=\inf\{t\geq 0 \mid X_t^\zeta\notin\mathbb{R}_+\}$.

Define the average cost per unit time or ergodic cost criterion as

$$J(x,\zeta):=\liminf_{T\to\infty}\frac{1}{T}\mathbb{E}_x\left[\int_0^T(\pi(X_s^\zeta)\mathrm{d}s+\gamma\mathrm{d}\zeta_s)\right],$$

where γ is a given positive constant and $\pi:\mathbb{R}_+\to\mathbb{R}$ is a function measuring the cost of continuing the process. Now define the value function

$$V(T,x)=\inf_{\zeta\in\mathcal{Z}}\mathbb{E}_x\left[\int_0^T(\pi(X_s)\mathrm{d}s+\gamma\mathrm{d}\zeta_s)\right] \tag{3}$$

and denote by β the minimum average cost. The objective of the control problem is to minimize $J(x,\zeta)$ over all the admissible controls $\zeta\in\mathcal{Z}$ and to find, if possible, the optimal control ζ^* such that $\beta=\inf_{\zeta\in\mathcal{Z}}J(x,\zeta)=J(x,\zeta^*)$.

We now define the auxiliary functions $\pi_\gamma:\mathbb{R}_+\to\mathbb{R}$,

$$\pi_\gamma(x)=\pi(x)+\gamma\lambda x,$$

and $\pi_\mu : \mathbb{R}_+ \to \mathbb{R}$,

$$\pi_\mu(x) = \pi(x) + \gamma\mu(x).$$

In order for our solution to be well behaved, we must pose the following assumptions:

Assumption 2. *We assume that*

(i) *the lower boundary* 0 *and upper boundary* ∞ *are natural;*

(ii) *the cost* π *is continuous, non-negative, and minimized at* 0*;*

(iii) *the function* π_μ *and* $\mathrm{id} : x \mapsto x$ *are in* $\mathcal{L}_1^\lambda$*;*

(iv) *there exists a unique state* $x^* \in \mathbb{R}_+$ *such that* π_μ *is decreasing on* $(0, x^*)$ *and increasing on* $[x^*, \infty)$*. Also,* $\lim_{x\uparrow\infty} \pi_\mu(x) > 0$.

The boundaries of the state space are assumed to be natural, which means that, in the absence of interventions, the process cannot become infinitely large or infinitely close to zero in finite time. In biological applications these boundary conditions guarantee that the population does not explode or become extinct in the absence of harvesting. We refer to [10, pp. 18–20] for a more thorough discussion of the boundary behaviour of one-dimensional diffusions. Also, it is worth mentioning that no second-order properties of π_μ are assumed.

In addition, the following limiting and integrability conditions on the scale density and speed measure must be satisfied. These conditions assure the existence of a stationary distribution of the underlying diffusion.

Assumption 3. *We assume that*

(i) $m(0, y) = \int_0^y m'(z)\mathrm{d}z < \infty$ *and* $\int_0^y \pi_\mu(z)m'(z)\mathrm{d}z < \infty$ *for all* $y \in \mathbb{R}_+$*;*

(ii) $\lim_{x\downarrow 0} S'(x) = \infty$.

Remark 1. The conditions of Assumption 3 alone guarantee that the lower boundary 0 should be either natural or entrance, and hence unattainable. However, in the proof of Lemma 1 we must exclude the possibility of entrance to assure that $L(x)$ (defined below) also attains negative values. If we wanted to include this possibility, we would also have to assume that $\lim_{x\to 0} \pi_\mu(x) = \infty$; see the proof of Lemma 1.

2.3. Auxiliary results

Define the auxiliary functions $L : \mathbb{R}_+ \to \mathbb{R}$ and $H : \mathbb{R}_+ \to \mathbb{R}$ as

$$L(x) = \lambda \int_x^\infty \pi_\mu(z)\varphi_\lambda(z)m'(z)\mathrm{d}z + \frac{\varphi'_\lambda(x)}{S'(x)}\pi_\mu(x),$$

$$H(0, x) = \int_0^x \pi_\mu(z)m'(z)\mathrm{d}z - \pi_\mu(x)m(0, x).$$

These functions will offer a convenient representation of the optimality equation in Section 3, and thus their properties play a key role when determining the optimal control policy.

Lemma 1. *Under Assumption 1, the functions* $L(x)$ *and* $H(0,x)$ *aer such that there exists a unique* $\tilde{x} < x^*$ *and a unique* $\hat{x} > x^*$ *such that*

(i) $L(x) \underset{>}{\overset{<}{=}} 0$ *when* $x \underset{>}{\overset{<}{=}} \tilde{x}$,

(ii) $H(0, x) \underset{>}{\overset{<}{=}} 0$ *when* $x \underset{<}{\overset{>}{=}} \hat{x}$.

Proof. The proof of the claim on L is similar to that of [26, Lemma 3.3]. However, to show that the results are in accordance with Remark 1, we need to adjust the argument on finding a point $x_1 < x^*$ such that $L(x_1) < 0$. Thus, assume for a while that $\lim_{x\to 0} \pi_\mu(x) = \infty$, and that $x^* > y > x$. Then

$$\begin{aligned} L(x) - L(y) &= \lambda \int_x^y \pi_\mu(z)\varphi_\lambda(z)m'(z)\mathrm{d}z \\ &\quad + \left[\pi_\mu(x)\frac{\varphi'_\lambda(x)}{S'(x)} - \pi_\mu(y)\frac{\varphi'_\lambda(y)}{S'(y)}\right] \\ &\le \frac{\varphi'_\lambda(y)}{S'(y)}(\pi_\mu(x) - \pi_\mu(y)), \end{aligned}$$

which shows that $\lim_{x\to 0} L(x) = -\infty$.

To prove the second part, assume first that $y > x > x^*$. Since the function π_μ is increasing on (x^*, ∞), we see that

$$\begin{aligned} H(0, y) - H(0, x) &= \int_x^y \pi_\mu(z)m'(z)\mathrm{d}z - \pi_\mu(y)m(0, y) + \pi_\mu(x)m(0, x) \\ &< \pi_\mu(y)(m(x, y) - m(0, y)) + \pi_\mu(x)m(0, x) \\ &= m(0, x)(\pi_\mu(x) - \pi_\mu(y)) \\ &< 0, \end{aligned}$$

proving that H is decreasing on (x^*, ∞). It also follows from

$$H(0, y) - H(0, x) < m(0, x)(\pi_\mu(x) - \pi_\mu(y))$$

that $\lim_{y\to\infty} H(0, y) < 0$. Next, assume that $x^* > y > x$. Because π_μ is decreasing on $(0, x^*)$, we find similarly that

$$\begin{aligned} H(0, y) - H(0, x) &= \int_x^y \pi_\mu(z)m'(z)\mathrm{d}z - \pi_\mu(y)m(0, y) + \pi_\mu(x)m(0, x) \\ &> \pi_\mu(y)(m(x, y) - m(0, y)) + \pi_\mu(x)m(0, x) \\ &= m(0, x)(\pi_\mu(x) - \pi_\mu(y)), \\ &> 0, \end{aligned}$$

implying that H is increasing on $(0, x^*)$. Furthermore, H is positive when $x < x^*$. Hence, by continuity, H has a unique root, which we denote by $\hat{x}$. □

Proposition 1. *There exists a unique solution* $\bar{x} \in (\tilde{x}, \hat{x})$ *to the equation*

$$S'(x)m(0, x)L(x) = -\varphi'_\lambda(x)H(0, x).$$

Proof. Define the function

$$P(x) = S'(x)m(0, x)L(x) + \varphi'_\lambda(x)H(0, x).$$

Assuming that $x_1 > \hat{x} > x^*$, we get, by Lemma 1, that

$$P(x_1) = S'(x_1)m(0, x_1)L(x_1) + \varphi'_\lambda(x_1)H(0, x_1) \geq 0.$$

Similarly, when $x_2 < \tilde{x} < x^*$ we have that

$$P(x_1) = S'(x_2)m(0, x_2)L(x_2) + \varphi'_\lambda(x_2)H(0, x_2) \leq 0. \tag{4}$$

By continuity, the function $P(x)$ must have at least one root. We denote one of these roots by z.

To prove that the root z is unique, we first notice that the naturality of the upper boundary implies that [10, p. 19]

$$\lim_{x\to\infty} \frac{\varphi'_\lambda(x)}{S'(x)} = 0.$$

Hence,

$$-\frac{1}{\lambda}\frac{\varphi'_\lambda(y^*)}{S'(y^*)} = \int_{y^*}^{\infty} \varphi_\lambda(z)m'(z)\mathrm{d}z. \tag{5}$$

Thus, we see that the equation $P(x) = 0$ is equivalent to

$$\frac{\int_x^\infty \pi_\mu(y)\varphi_\lambda(y)m'(y)\mathrm{d}y}{\int_x^\infty \varphi_\lambda(y)m'(y)\mathrm{d}y} = \frac{\int_0^x \pi_\mu(y)m'(y)\mathrm{d}y}{\int_0^x m'(y)\mathrm{d}y}.$$

Now, differentiating the left-hand side yields

$$\frac{\varphi_\lambda(x)m'(x)L(x)}{I(x)^2},$$

where $I(x) = \int_x^\infty \varphi_\lambda(y)m'(y)\mathrm{d}y$. Differentiating the right-hand side and evaluating it at z, we get, using the equation $P(z) = 0$,

$$\frac{\pi_\mu(z)m'(z)}{m(0, z)} - \frac{\int_0^z \pi_\mu(y)m'(y)\mathrm{d}y}{m(0, z)}\frac{m'(z)}{m(0, z)} = \frac{-m'(z)L(z)}{I(z)m(0, z)}.$$

Because $L(y) > 0$ in the region $(\tilde{x}, \hat{x})$, and all the other terms are positive everywhere, by comparing the derivatives we find that

$$\frac{-m'(z)L(z)}{I(z)m(0, z)} < \frac{\varphi_\lambda(z)m'(z)L(z)}{I(z)^2}.$$

Therefore, by continuity, the intersection between the curves

$$I(x)^{-1}\int_x^\infty \pi_\mu(y)\varphi_\lambda(y)m'(y)\mathrm{d}y$$

and

$$m(0, x)^{-1}\int_0^x \pi_\mu(y)m'(y)\mathrm{d}y$$

is unique. This unique point is denoted by $\bar{x}$. □

In the next lemma we make some further computations that are needed for the sufficient conditions of the control problem. Define the functions $J:\mathbb{R}_+\to\mathbb{R}$ and $I:\mathbb{R}_+\to\mathbb{R}$ as

$$J(x)=\frac{\gamma-(R_\lambda\pi_\gamma)'(x)}{\varphi'_\lambda(x)},\qquad I(x)=\frac{\int_0^x\pi_\mu(x)m'(t)\mathrm{d}t}{m(0,x)}.$$

Lemma 2. *Under Assumption 2,*

(i) $J'(x)\overset{>}{\underset{<}{=}}0$ *when* $x\overset{>}{\underset{<}{=}}\tilde{x}$,

(ii) $I'(x)\overset{>}{\underset{<}{=}}0$ *when* $x\overset{>}{\underset{<}{=}}\hat{x}$.

Here, $\tilde{x}$ and $\hat{x}$ are as in Lemma 1.

Proof. The first claim follows from the formula

$$J'(x)=\frac{2S'(x)}{\sigma^2(x)\varphi'_\lambda(x)^2}L(x),$$

which can be derived using representation (1) and straightforward differentiation (see [26, Lemma 3] for details). The claim on I follows similarly, as differentiation yields

$$I'(x)=-\frac{m'(x)}{m^2(0,x)}H(x).$$

□

3. The solution

3.1. Necessary conditions

Denote the candidate solution for (3) as $F(T,x)$. We use the heuristic that $F(T,x)$ can be separated for large T as

$$F(T,x)\sim\beta T+W(x).\tag{6}$$

In mathematical finance literature, the constant β usually denotes the minimum average cost per unit time and $W(x)$ is the potential cost function (see [15, 37]). The fact that the leading term βT is independent of x is, of course, dependent on the ergodic properties of the underlying process. We also note that this heuristic can be used as a separation of variables to solve a partial differential equation of parabolic type related to the expectation in (3) via the Feynman–Kac formula [16].

We shall proceed as in [26]. We assume that the optimal control policy exists and is given by the following. When the process is below some threshold y^* (called the *waiting region*) we let the process run, but if the process is above the threshold value y^* (called the *action region*) and the Poisson process jumps we exert the exact amount of control to push the process back to the boundary y^* and start it anew.

In the waiting region $[0,y^*]$ we expect that the candidate solution satisfies Bellman's principle,

$$F(T,x)=\mathbb{E}\left[\int_0^U\pi(X_s)\mathrm{d}s+F(T-U,X_U)\right],$$

where U is an exponentially distributed random variable with mean $1/\lambda$.

Using the heuristic (6) and noticing the connection between the random times U and the resolvent, we get, by independence and the strong Markov property, that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^U \pi(X_s)\mathrm{d}s + F(T-U, X_U)\right] \\
&= \lim_{r\to 0}(R_r\pi)(x) + \mathbb{E}_x[W(X_U)] - \frac{\beta}{\lambda} + \beta T - \mathbb{E}_x\left[\int_U^\infty \pi(X_s)\mathrm{d}s\right] \\
&= \lim_{r\to 0}(R_r\pi)(x) + \lambda(R_\lambda W)(x) - \frac{\beta}{\lambda} + \beta T - \mathbb{E}_x\left[\lim_{r\to 0}(R_r\pi)(X_U)\right] \\
&= \lim_{r\to 0}(R_r\pi)(x) + \lambda(R_\lambda W)(x) - \frac{\beta}{\lambda} + \beta T - \lambda \lim_{r\to 0}(R_\lambda R_r\pi)(x).
\end{aligned}$$

Hence, we arrive at the equation

$$W(x) - \lim_{r\to 0}(R_r\pi)(x) = \lambda R_\lambda(W(x) - \lim_{r\to 0}(R_r\pi)(x)) - \frac{\beta}{\lambda}.$$

We next choose $f(x) = W(x) - \lim_{r\to 0}(R_r\pi)(x)$ in [24, Lemma 2.1], and notice that the lemma remains unchanged even if we add a constant β/λ. Thus, we expect, by our heuristic arguments, that the pair (W, β) satisfies the differential equation

$$\mathcal{A}W(x) + \pi(x) = \beta.$$

This type of equation often arises in ergodic control problems and there is lots of literature on sufficient conditions for the existence of a solution [7, 15, 34]. We remark here that usually these conditions rely heavily on the solution of the corresponding discounted infinite-horizon control problem, and thus apply the so-called vanishing discount method. However, in our case we can proceed by explicit calculations.

Next, we shall determine the equation for the pair (W, β) in the action region $[y^*, \infty]$. The Poisson process jumps in infinitesimal time with probability $\lambda\mathrm{d}t$, and in that case the agent has to pay a cost $\gamma(x-y^*) + F(T, y^*)$. On the other hand, the Poisson process does not jump with probability $1-\lambda\mathrm{d}t$, and in this case the agent has to pay $\pi(x)\mathrm{d}t + \mathbb{E}_x[F(T-\mathrm{d}t, X_{\mathrm{d}t})]$. Thus, the candidate function F should satisfy the condition

$$F(T, x) = \lambda\mathrm{d}t(\gamma(x-y^*) + F(T, y^*)) + (1-\lambda\mathrm{d}t)\,(\pi(x)\mathrm{d}t + \mathbb{E}_x[F(T-\mathrm{d}t, X_{\mathrm{d}t})]).$$

Now, again using the heuristic (6) and that, intuitively, $\mathrm{d}t^2 = 0$, we find that

$$W(x) = \lambda\mathrm{d}t(\gamma(x-y^*) + W(y^*)) + \pi(x)\mathrm{d}t - \beta\mathrm{d}t + (1-\lambda\mathrm{d}t)\mathbb{E}_x[W(X_{\mathrm{d}t})].$$

By formally using Dynkin's formula on the last term and simplifying we get

$$0 = \lambda\mathrm{d}t(\gamma(x-y^*) + W(y^*)) + \pi(x)\mathrm{d}t - \beta\mathrm{d}t + (\mathcal{A}-\lambda)W(x)\mathrm{d}t.$$

We conclude that in the action region, the pair (W, β) should satisfy the differential equation

$$(\mathcal{A}-\lambda)W(x) = -(\pi(x) + \lambda(\gamma(x-y^*) + W(y^*)) - \beta).$$

Now, we first observe that

$$\mathcal{A}W(x) + \pi(x) - \beta = \begin{cases} 0, & x < y^*, \\ \lambda(W(x) - \gamma x - (W(y^*) - \gamma y^*)), & x \geq y^*, \end{cases}$$

which implies that $W(x)$ satisfies the C^1-condition $W'(y^*)=\gamma$. We have thus arrived at the following free boundary problem: find a function $W(x)$ and constants y^* and β such that

$$W\in C^2,$$

$$W'(y^*)=\gamma,$$

$$(\mathcal{A}-\lambda)W(x)+\pi(x)+\lambda(\gamma(x-y^*)+W(y^*))=\beta, \qquad x\in[y^*,\infty), \tag{7}$$

$$\mathcal{A}W(x)+\pi(x)=\beta, \qquad x\in(0,y^*). \tag{8}$$

Remark 2. Another common approach to heuristically form the Hamilton–Jacobi–Bellman (HJB) equation of the problem is to use the value function $J_r(x)$ of the corresponding discounted problem (see [26, p. 4]) and the vanishing discount limits $rJ_r(\bar{x})\to\beta$ and $W_r(x)=J_r(x)-J_r(\bar{x})\to W(x)$, where $\bar{x}$ is a fixed point in $\mathbb{R}_+$ (see [34, p. 284] and [15, p. 427]). This argument yields the HJB equation

$$\mathcal{A}W(x)+\pi(x)-\lambda(W'(x)-\gamma)\mathbf{1}_{\{x\in S\}}=\beta,$$

where $S=[y^*,\infty)$ is the control region.

To solve the free boundary problem, we consider (8) first. In this case we write the differential operator $\mathcal{A}$ as [38, p. 285]

$$\mathcal{A}=\frac{\mathrm{d}}{\mathrm{d}m(x)}\frac{\mathrm{d}}{\mathrm{d}S(x)},$$

which allows us to find that

$$\frac{\mathrm{d}W'(x)}{\mathrm{d}S'(x)}=(\beta-\pi(x))m'(x). \tag{9}$$

Therefore, integrating over the interval $(0,y^*)$ gives

$$\frac{W'(y^*)}{S'(y^*)}=\beta\int_0^{y^*}m'(z)\mathrm{d}z-\int_0^{y^*}\pi(z)m'(z)\mathrm{d}z.$$

Hence, by Assumption 3 and the C^1-condition $W'(y^*)=\gamma$, we get

$$\beta=\left[\int_0^{y^*}m'(z)\mathrm{d}z\right]^{-1}\left[\int_0^{y^*}\pi(z)m'(z)\mathrm{d}z+\frac{\gamma}{S'(y^*)}\right].$$

Finally, using (2), we arrive at

$$\beta=\left[\int_0^{y^*}m'(z)\mathrm{d}z\right]^{-1}\left[\int_0^{y^*}\pi_\mu(z)m'(z)\mathrm{d}z\right]. \tag{10}$$

Next, we consider (7). We immediately find that a particular solution is

$$W(x)=(R_\lambda\pi_\gamma)(x)-\frac{\beta}{\lambda}-\gamma y^*+W(y^*).$$

Hence, we conjecture analogously to [26, p. 113] that the solution to (7) is

$$W(x)=(R_\lambda\pi_\gamma)(x)-\frac{\beta}{\lambda}-\gamma y^*+W(y^*)+C\varphi_\lambda(x). \tag{11}$$

To find the constants C and β, we first use the continuity of W at the boundary y^*, which allows us to substitute $x = y^*$ into (11). This yields

$$0 = (R_\lambda \pi_\gamma)(y^*) - \frac{\beta}{\lambda} - \gamma y^* + C\varphi_\lambda(y^*). \tag{12}$$

Then, by applying the condition $W'(y^*) = \gamma$ to (11), we find that

$$C = \frac{\gamma - (R_\lambda \pi_\gamma)'(y^*)}{\varphi'_\lambda(y^*)}.$$

Combining this with (12) gives

$$\beta = \lambda(R_\lambda \pi_\gamma)(y^*) - \lambda\gamma y^* + \frac{\gamma - (R_\lambda \pi_\gamma)'(y^*)}{\varphi'_\lambda(y^*)}\lambda\varphi_\lambda(y^*).$$

To rewrite this expression, we first notice that a straightforward differentiation gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\varphi'_\lambda(x)}{S'(x)} - \frac{\varphi_\lambda(x)}{S'(x)}\right) = -m'(x)\varphi_\lambda(x)(\mu(x) - \lambda x).$$

Thus, by the fundamental theorem of calculus and the naturality of the upper boundary, we get

$$y^*\frac{\varphi'_\lambda(y^*)}{S'(y^*)} - \frac{\varphi_\lambda(y^*)}{S'(y^*)} = \int_{y^*}^{\infty} m'(z)\varphi_\lambda(z)(\mu(z) - \lambda z)\mathrm{d}z. \tag{13}$$

Next, using (1), we find that

$$(R_\lambda \pi_\gamma)(y^*)\varphi'_\lambda(y^*) - (R_\lambda \pi_\gamma)'(y^*)\varphi_\lambda(y^*) = -S'(y^*)\int_{y^*}^{\infty} \varphi_\lambda(z)\pi_\gamma(z)m'(z)\mathrm{d}z.$$

Combining these observations with (5), the constant β reads

$$\beta = \left[\int_{y^*}^{\infty} \varphi_\lambda(z)m'(z)\mathrm{d}z\right]^{-1}\left[\int_{y^*}^{\infty} \varphi_\lambda(z)\pi_\gamma(z)m'(z)\mathrm{d}z + \gamma\int_{y^*}^{\infty} m'(z)\varphi_\lambda(z)(\mu(z) - \lambda z)\mathrm{d}z\right].$$

Finally, by recalling the definition of $\pi_\mu(x)$, we have

$$\beta = \left[\int_{y^*}^{\infty} \varphi_\lambda(z)m'(z)\mathrm{d}z\right]^{-1}\left[\int_{y^*}^{\infty} \varphi_\lambda(z)\pi_\mu(z)m'(z)\mathrm{d}z\right]. \tag{14}$$

Now, by equating the representations (10) and (14) of β, we find the optimality condition

$$\int_0^{y^*} \pi_\mu(z)m'(z)\mathrm{d}z\int_{y^*}^{\infty} \varphi_\lambda(z)m'(z)\mathrm{d}z = \int_0^{y^*} m'(z)\mathrm{d}z\int_{y^*}^{\infty} \varphi_\lambda(z)\pi_\mu(z)m'(z)\mathrm{d}z,$$

which can be re-expressed, using the functions $L(x)$ and $H(0, x)$, as

$$m(0, y^*)L(y^*) = -\varphi'_\lambda(y^*)H(0, y^*). \tag{15}$$

We proved in Proposition 1 that there exists a unique solution $\bar{x}$ to (15), and thus we will assume in the following that $y^* = \bar{x}$.

Remark 3. As often in ergodic optimal control problems, the potential value function $W(x)$ satisfies second-order differentiability across the boundary $\lim_{x\downarrow y^*} W''(x) = \lim_{x\uparrow y^*} W''(x)$. This can be verified as follows. When $x > y^*$, the differentiation and the harmonicity properties $(R_\lambda(\mathcal{A}-\lambda)\pi_\gamma)(x) + \pi_\gamma(x) = 0$ and $(\mathcal{A}-\lambda)\varphi_\lambda(x) = 0$ yield

$$\begin{aligned}\lim_{x\downarrow y^*} W''(x) &= (R_\lambda\pi_\gamma)''(y^*) + C\varphi_\lambda''(y^*)\\ &= \frac{2}{\sigma^2(y^*)}\bigg[\lambda(R_\lambda\pi_\gamma)(y^*) - \pi_\gamma(y^*) - \mu(y^*)(R_\lambda\pi_\gamma)'(y^*)\\ &\qquad + \frac{\gamma-(R_\lambda\pi_\gamma)'(y^*)}{\varphi_\lambda'(y^*)}(\lambda\varphi_\lambda(y^*) - \mu(y^*)\varphi_\lambda'(y^*))\bigg],\end{aligned}$$

which after cancellation and applying (13) and (1) equals

$$\frac{2}{\sigma^2(y^*)}\bigg[-\frac{\lambda S'(y^*)}{\varphi_\lambda'(y^*)}\int_{y^*}^{\infty}\varphi_\lambda(z)m'(z)(\pi_\gamma(z)-\gamma\lambda z+\mu(z)\gamma)\mathrm{d}z - \pi_\mu(y^*)\bigg].$$

Therefore, by using (5) and (14) we find that

$$\lim_{x\downarrow y^*} W''(x) = \frac{2}{\sigma^2(y^*)}[\beta - \pi_\mu(y^*)].$$

On the other hand, when $x < y^*$ we notice that

$$\frac{\mathrm{d}}{\mathrm{d}x}\bigg[\frac{W'(x)-\gamma}{S'(x)}\bigg] = (\mathcal{A}W'(x) - \gamma\pi(x))m'(x) = (\beta - \pi_\mu(x))m'(x).$$

Hence, by differentiating the left-hand side and plugging in $x = y^*$, we find by the first-order condition $W'(y^*) = \gamma$ that

$$\lim_{x\uparrow y^*} W''(x) = \frac{2}{\sigma^2(y^*)}[\beta - \pi_\mu(y^*)].$$

3.2. Sufficient conditions

We begin with an initial remark. When $x > y^*$, we get, by differentiating (11) and using Lemma 2, that

$$W'(x) - \gamma = \varphi'_\lambda(x)\bigg[\frac{(R_\lambda\pi_\gamma)'(x)-\gamma}{\varphi'_\lambda(x)} - \frac{(R_\lambda\pi_\gamma)'(y^*)-\gamma}{\varphi'_\lambda(y^*)}\bigg] > 0.$$

In the opposite case, when $x < y^*$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\bigg[\frac{W'(x)-\gamma}{S'(x)}\bigg] = (\beta - \pi_\mu(x))m'(x).$$

Thus, by integrating over the interval $(0, x)$, and using (10) and Lemma 2, we find that

$$\frac{W'(x)-\gamma}{S'(x)} = m(0, x)\bigg[\frac{\int_0^{y^*}\pi_\mu(t)m'(t)\mathrm{d}t}{m(0, y^*)} - \frac{\int_0^{x}\pi_\mu(t)m'(t)\mathrm{d}t}{m(0, x)}\bigg] < 0.$$

These observations imply that, under the standing assumptions, the function $W(x) - \gamma x$ has a global minimum at y^*, which shows that $W(x)$ satisfies the variational equality

$$\mathcal{A}W(x) + \pi(x) + \lambda\bigg[\inf_{y\le x}\{(W(y)-\gamma y) - (W(x)-\gamma x)\}\bigg] = \beta. \tag{16}$$

Proposition 2. (Verification.) *Under Assumptions 1, 2, and 3, the optimal policy is as follows. If the controlled process X^ζ is above the threshold y^* at a jump time of N, i.e. $X^\zeta_{T_-} > y^*$, the decision maker should take the controlled process X^ζ to y^*. Further, the threshold y^* is uniquely determined by (15), and the constant β characterized by (10) and (14) gives the minimum average cost,*

$$\beta = \inf_\zeta \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}_x\bigg[\int_0^T (\pi(X_s)\mathrm{d}s + \gamma \mathrm{d}\zeta_s)\bigg].$$

Proof. Define the function

$$\begin{aligned}\Phi(x) &:= \inf_{y\le x}\{(W(y)-\gamma y)-(W(x)-\gamma x)\}\\ &= \{W(y^*)-W(x)+\gamma(x-y^*)\}\mathbf{1}_{[y^*,\infty)}(x),\end{aligned}$$

and a family of almost surely finite stopping times $\tau(\rho)_{\rho>0}$ as $\tau(\rho) := \tau_0^\zeta \wedge \rho \wedge \tau_\rho^\zeta$, where $\tau_\rho^\zeta = \inf\{t \ge 0 : X_t^\zeta \notin (1/\rho, \rho)\}$. By applying the Doléans–Dade–Meyer change of variables formula to the process $W(X_t)$, we obtain

$$\begin{aligned}W(X_{t\wedge\tau(\rho)}) - W(x) &= \int_0^{t\wedge\tau(\rho)} \mathcal{A}W\big(X_s^\zeta\big)\mathrm{d}s + \int_0^{t\wedge\tau(\rho)} \sigma\big(X_s^\zeta\big)W'\big(X_s^\zeta\big)\mathrm{d}B_s\\ &\quad + \sum_{0\le s\le t\wedge\tau(\rho)}\big[W\big(X_s^\zeta\big)-W\big(X_{s-}^\zeta\big)\big].\end{aligned}$$

Because the control ζ jumps only if the Poisson process N jumps, we have that $W(X_s^\zeta) - W(X_{s-}^\zeta) + \gamma(\Delta\zeta_s) \ge \Phi(X_{s-}^\zeta)$. By combining these two observations with (16), we get

$$W(X_{t\wedge\tau(\rho)}) \ge W(x) + \beta(t\wedge\tau(\rho)) - \int_0^{t\wedge\tau(\rho)}\big[\pi\big(X_s^\zeta\big)\mathrm{d}s + \gamma\mathrm{d}\zeta_s\big] + Z_{t\wedge\tau(\rho)} + M_{t\wedge\tau(\rho)}, \tag{17}$$

where

$$M_t := \int_0^t \sigma\big(X_s^\zeta\big)W'\big(X_s^\zeta\big)\mathrm{d}B_s, \qquad Z_t := \int_0^t \Phi\big(X_s^\zeta\big)\mathrm{d}\tilde{N}_s.$$

Here, $\tilde{N}_t = (N_t - \lambda t)_{t\ge 0}$ is the compensated Poisson process. It follows from the calculation above that $Z_{t\wedge\tau(\rho)} + M_{t\wedge\tau(\rho)}$ is a submartingale and thus $\mathbb{E}_x[Z_{t\wedge\tau(\rho)} + M_{t\wedge\tau(\rho)}] \ge 0$. Taking expectation on both sides, dividing by $t\wedge\tau(\rho)$, and letting t, $\rho \to \infty$, we find that

$$\liminf_{T\to\infty}\frac{1}{T}\mathbb{E}_x\bigg[W\big(X_T^\zeta\big) + \int_0^T \big(\pi\big(X_s^\zeta\big)\mathrm{d}s + \gamma\mathrm{d}\zeta_s\big)\bigg] \ge \beta.$$

Thus, if $\liminf_{T\to\infty}\frac{1}{T}\mathbb{E}_x[W(X_T^\zeta)] = 0$, it follows that $J(x, \zeta) \ge \beta$; we postpone the proof of this limiting property to the following lemma.

Next, we prove that $J(x, \zeta^*) \le \beta$. We proceed as above and note that (17) holds as equality when $\zeta = \zeta^*$. Hence, the local martingale term $M_T + Z_T$ is now uniformly bounded from below by $-W(x) - \beta T$, and is therefore a supermartingale. Thus, we have

$$\mathbb{E}_x\bigg[\int_0^T \big(\pi\big(X_s^\zeta\big)\mathrm{d}s + \gamma\mathrm{d}\zeta_s^*\big)\bigg] \le \beta T + W(x) - \mathbb{E}_x[W(X_T)] \le \beta T + W(x).$$

Finally, dividing by T and letting $T \to \infty$, we get $J(x, \zeta^*) \le \beta$, which completes the proof. □

As is usual in ergodic control problems, we noticed in the proof that the verification theorem holds under the assumption that $\liminf_{T\to\infty} \frac{1}{T}\mathbb{E}_x[W(X_T^\zeta)] = 0$. Thus, in the following lemma we give the sufficient condition on $\pi(x)$ under which the limit equals zero.

Lemma 3. *The limit*

$$\liminf_{T\to\infty} \frac{1}{T}\mathbb{E}_x\big[W\big(X_T^\zeta\big)\big] = 0 \tag{18}$$

holds if $\pi(x) \geq C(x^\alpha - 1)$, *where* α *and* C *are positive constants.*

Proof. Let $x > y^*$. Then $W(x)$ reads

$$\begin{aligned} W(x) &= (R_\lambda \pi_\gamma)(x) - \frac{\beta}{\lambda} - \gamma y^* + W(y^*) + C\varphi_\lambda(x), \\ W(x) &= (R_\lambda \pi_\mu)(x) - \frac{\beta}{\lambda} + \gamma(x - y^*) + W(y^*) + C\varphi_\lambda(x). \end{aligned}$$

Because $\varphi_\lambda(x)$ is bounded in this region, we only need to deal with the resolvent term. By the Markov property and the substitution $k = s + T$, we find that

$$\begin{aligned} \mathbb{E}_x[(R_\lambda \pi_\gamma)(X_T)] &= \int_0^\infty \mathrm{e}^{-\lambda s}\mathbb{E}_x\big[\mathbb{E}_{X_T}\big[\pi_\gamma(X_s)\big]\big]\mathrm{d}s \\ &= \int_0^\infty \mathrm{e}^{-\lambda s}\mathbb{E}_x\big[\mathbb{E}_x\big[\pi_\gamma(X_{s+T}) \mid \mathcal{F}_T\big]\big]\mathrm{d}s \\ &= \int_0^\infty \mathrm{e}^{-\lambda s}\mathbb{E}_x\big[\pi_\gamma(X_{s+T})\big]\mathrm{d}s \\ &= \mathrm{e}^{\lambda T}\int_T^\infty \mathrm{e}^{-\lambda k}\mathbb{E}_x\big[\pi_\gamma(X_k)\big]\mathrm{d}k. \end{aligned}$$

By l'Hôpital's rule and the assumption that id and π are elements of $\mathcal{L}_1^\lambda$, we find that

$$\liminf_{T\to\infty} \frac{\mathrm{e}^{\lambda T}}{T}\int_T^\infty \mathrm{e}^{-\lambda k}\mathbb{E}_x\big[\pi_\gamma(X_k)\big]\mathrm{d}k = \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}_x\big[\pi_\gamma(X_T)\big].$$

On the other hand, if

$$\liminf_{T\to\infty} \frac{1}{T}\mathbb{E}_x[X_T] > 0,$$

there exists T_1 such that

$$\mathbb{E}_x[X_s] > \varepsilon\frac{s}{(\alpha+1)^{1/\alpha}}$$

for all $s > T_1$. Together with the assumption $\pi(x) \geq C(x^\alpha - 1)$, this leads to contradiction as

$$\begin{aligned}\infty &> \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}_x\Big[\int_0^T (\pi(X_s)\mathrm{d}s + \gamma\,\mathrm{d}\zeta_s)\Big]\\ &\geq \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}_x\Big[\int_0^T \pi(X_s)\mathrm{d}s\Big]\\ &\geq -C + C\liminf_{T\to\infty} \frac{1}{T}\mathbb{E}_x\Big[\int_0^T X_s^\alpha\,\mathrm{d}s\Big]\\ &\geq -C + C\liminf_{T\to\infty} \frac{1}{T}\mathbb{E}_x\Big[\frac{\varepsilon^\alpha}{\alpha+1}\int_0^T s^\alpha\,\mathrm{d}s\Big]\\ &= -C + C\varepsilon^\alpha \liminf_{T\to\infty} T^\alpha = \infty.\end{aligned}$$

Similarly, we must have

$$\liminf_{T\to\infty} \frac{1}{T}\mathbb{E}_x[\pi(X_T)] = 0,$$

and thus conclude that the limit (18) must vanish.

In the opposite case $x < y^*$, we find by integrating in (9) that

$$\frac{W'(y)}{S'(y)} = \frac{\gamma}{S'(y^*)} + \int_y^{y^*} m'(z)(\pi(z) - \beta)\mathrm{d}z.$$

Multiplying by $S'(x)$ and integrating over the interval (x, y^*), we have

$$W(x) = W(y^*) - \frac{\gamma}{S'(y^*)}\int_x^{y^*} S'(z)\mathrm{d}z - \int_x^{y^*}\int_y^{y^*} m'(z)(\pi(z) - \beta)\mathrm{d}z S'(y)\mathrm{d}y.$$

Since the second term is negative and $\pi(x)$ is positive everywhere, this has the upper bound

$$W(x) \leq W(y^*) + \beta\int_x^{y^*}\int_y^{y^*} m'(z)\mathrm{d}z S'(y)\mathrm{d}y.$$

As the last integral is positive, we can by Assumption 3 expand the region of the inner integral to get

$$W(x) \leq W(y^*) + \beta\int_x^{y^*}\int_0^{y^*} m'(z)\mathrm{d}z S'(y)\mathrm{d}y.$$

Thus,

$$W(x) \leq W(y^*) + \beta m(0, y^*)(S(y^*) - S(x)).$$

Consequently, the upper bound is of the form $W(x) \leq C_0 S(x) + C_1$, where C_0 and C_1 are constants. Since $S(X_t)$ is a non-negative local martingale [9, p. 88] and hence a supermartingale, we have

$$\begin{aligned}\mathbb{E}_x[W(X_T)] \leq C_0\mathbb{E}_x[S(X_T)] + C_1 \leq C_0\mathbb{E}_x[S(X_0)] + C_1\\ = C_0 S(x) + C_1.\end{aligned}$$

Hence, also in this case the limit (18) must vanish. □

Remark 4. Another approach to see that the limit vanishes is to get a suitable upper bound for $W(x)$. Indeed, if $W(x) \leq A_0 + A_1\pi(x)$ for some constant A_0 and A_1, then the result also holds by [39, Lemma 3.1].

4. Ergodic singular control problem: Connecting the problems

The singular control problem, where the agent is allowed to control the process X_t without any constraints, is studied in the case of Brownian motion in [21] and in the case of a more general one-dimensional diffusion in [3, 19]. In this corresponding singular problem the optimal policy is a local time reflecting barrier policy. The threshold y_s^* characterizing the optimal policy is the unique solution to the optimality condition [3, p. 17]

$$H(0, y_s^*) = 0. \tag{19}$$

Heuristically, one would expect that in the limit $\lambda \to \infty$, this optimal boundary y_s^* coincides with the optimal boundary y^*. This is because, in the limit, the decision maker has more frequent opportunities to exercise control. This is shown in the next proposition after an auxiliary lemma.

Lemma 4. *Let $\varphi_\lambda(x)$ be the decreasing solution to the differential equation $(\mathcal{A} - \lambda)f = 0$, and assume that $x < z$; then*

$$\frac{\varphi_\lambda(z)}{\varphi_\lambda(x)} \xrightarrow{\lambda \to \infty} 0.$$

Proof. Taking the limit $\lambda \to \infty$ in [10, p. 18]

$$\mathbb{E}_x[\mathrm{e}^{-\lambda \tau_z}] = \frac{\varphi_\lambda(z)}{\varphi_\lambda(x)},$$

where $\tau_z = \inf\{t \geq 0 \mid X_t = z\}$ is the first hitting time to z, yields the result by monotone convergence. □

We now have the following result.

Proposition 3. *Define a function $G : \mathbb{R}_+ \to \mathbb{R}$ as*

$$\hat{G}(x) = L(x) + \frac{\varphi_\lambda'(x)}{S'(x)m(0, x)} H(0, x).$$

Let y^* and y_s^* be the unique solutions to $\hat{G}(x) = 0$ and $H(0, x) = 0$, respectively. Then $\hat{G}(y_s^*) \to 0$ as $\lambda \to \infty$.

Proof. Because $H(0, y_s^*) = 0$ and the upper boundary ∞ is natural, we have

$$\frac{\hat{G}(y_s^*)}{\varphi_\lambda(y_s^*)} = \frac{L(y_s^*)}{\varphi_\lambda(y_s^*)} = \int_{y_s^*}^{\infty} \frac{\varphi_\lambda(z)}{\varphi_\lambda(y_s^*)} (\pi_\mu(z) - \pi_\mu(y_s^*)) m'(z) \mathrm{d}z.$$

Thus, taking the limit $\lambda \to \infty$ yields the result by Lemma 4. □

It is also reasonable to expect that when λ increases, it is more likely that the decision maker postpones the exercise of control as they have more information about the underlying process available. Therefore, we expect that the optimal threshold y^* is increasing as a function of λ. The next proposition shows that this is indeed the case.

Proposition 4. *Assume that $\mu(x) > 0$. Then the unique root y_λ^* of the function*

$$G_\lambda(x) = \frac{L(x)}{\varphi_\lambda'(x)} + \frac{H(0, x)}{S'(x)m(0, x)}$$

is increasing in λ.

Proof. Let $\hat{\lambda} > \lambda$. From the proof of Lemma 4, we find that, for every $x < z$,

$$\frac{\varphi_\lambda(z)}{\varphi_\lambda(x)} \geq \frac{\varphi_{\hat{\lambda}}(z)}{\varphi_{\hat{\lambda}}(x)},$$

which is equivalent to

$$\frac{\varphi_\lambda(z)}{\varphi_{\hat{\lambda}}(z)} \geq \frac{\varphi_\lambda(x)}{\varphi_{\hat{\lambda}}(x)}. \tag{20}$$

Because $\hat{\lambda} > \lambda$, there exists $r > 0$ such that $\hat{\lambda} = \lambda + r$. Thus, utilizing the fact that $(\mathcal{A} - \lambda)\varphi_{\lambda+r} = (\mathcal{A} - (\lambda + r))\varphi_{\lambda+r} + r\varphi_{\lambda+r} = r\varphi_{\lambda+r}$ with [2, Corollary 3.2], we have

$$\varphi_\lambda(x)\varphi_{\hat{\lambda}}{}'(x) - \varphi_{\hat{\lambda}}(x)\varphi_\lambda'(x) = -rS'(x)\int_x^\infty \varphi_\lambda(y)\varphi_{\lambda+r}(y)m'(y)\mathrm{d}y \leq 0.$$

Reorganizing the above, we get

$$\frac{\varphi_\lambda(x)}{\varphi_{\hat{\lambda}}(x)} \geq \frac{\varphi_\lambda'(x)}{\varphi_{\hat{\lambda}}{}'(x)}. \tag{21}$$

Combining (20) and (21), we deduce that

$$\frac{\varphi_\lambda(z)}{\varphi_\lambda'(x)} \leq \frac{\varphi_{\hat{\lambda}}(z)}{\varphi_{\hat{\lambda}}{}'(x)}.$$

Hence, the function $G_\lambda(x)$ satisfies

$$G_\lambda(x) = \int_x^\infty \frac{\varphi_\lambda(z)}{\varphi_\lambda'(x)}\pi_\mu(z)m'(z)\mathrm{d}z + \frac{\int_0^x \pi_\mu(z)m'(z)\mathrm{d}z}{S'(x)m(0, x)} \leq G_{\hat{\lambda}}(x).$$

This implies that $y_\lambda^* \leq y_{\hat{\lambda}}^*$ as, by (4), $G_\lambda(x)$ is positive in the interval $(0, y_\lambda^*)$ and has a unique root. □

Remark 5. The assumption that $\mu(x) > 0$ is somewhat restricting, and is there to guarantee that $\pi_\mu(x) > 0$. It would be enough that

$$\frac{L(y_\lambda^*)}{\varphi_\lambda'(y_\lambda^*)} \leq \frac{L(y_\lambda^*)}{\varphi_{\hat{\lambda}}{}'(y_\lambda^*)}.$$

It is often hard to show this exactly; however, in applications it can be verified numerically.

Remark 6. Denote by β_s the average cost per unit time of the singular problem. Then $\beta \xrightarrow{\lambda\to\infty} \beta_s$ [28, p. 12].

5. Illustrations

5.1. Verhulst–Pearl diffusion

We consider a standard Verhulst–Pearl diffusion

$$\mathrm{d}X_t = \mu X_t(1 - \beta X_t)\mathrm{d}t + \sigma X_t \mathrm{d}W_t, \qquad X_0 = x \in \mathbb{R}_r,$$

TABLE 1: The values for the optimal threshold y^* for some values of the intensity λ.

λ	y^*
5	0.317
10	0.496
50	0.656
100	0.684
1000	0.726

where $\mu > 0$, $\sigma > 0$, and $\beta > 0$. This diffusion is often used as a model for stochasticly fluctuating populations [4, 13]. The scale density and speed measure in this case are

$$S'(x) = x^{-\frac{2\mu}{\sigma^2}} \mathrm{e}^{\frac{2\mu\beta}{\sigma^2} x}, \qquad m'(x) = \frac{2}{\sigma^2} x^{\frac{2\mu}{\sigma^2} - 2} \mathrm{e}^{-\frac{2\mu\gamma}{\sigma^2} x}.$$

We assume that the cost $\pi(x) = x^2$ and $\gamma = 1$. Hence, $\pi_\mu(x) = x^2 - x\mu(1 - \beta x)$. In this setting, we note that if $\mu > \sigma^2/2$ then

$$m(0, x) = \frac{2}{\sigma^2} \left(\frac{\sigma^2}{2\mu\beta} \right)^{\frac{2\mu}{\sigma^2} - 1} \left(\left[\Gamma\left(\frac{2\mu}{\sigma^2} - 1 \right) - \Gamma\left(\frac{2\mu}{\sigma^2} - 1, \frac{2\mu\beta x}{\sigma^2} \right) \right] \right).$$

The minimal excessive functions read [11, pp. 201–203]

$$\varphi_\lambda(x) = x^{\alpha_1} U\left(\alpha_1, 1 + \alpha_1 - \alpha_2, \frac{2\mu\beta x}{\sigma^2} \right),$$
$$\psi_\lambda(x) = x^{\alpha_1} M\left(\alpha_1, 1 + \alpha_1 - \alpha_2, \frac{2\mu\beta x}{\sigma^2} \right),$$

where U and M are Kummer's confluent hypergeometric functions of the second and first kind respectively, and

$$\alpha_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2} \right)^2 + \frac{2\lambda}{\sigma^2}}, \qquad \alpha_2 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2} \right)^2 + \frac{2\lambda}{\sigma^2}}.$$

We see that our assumptions are satisfied, and thus the result applies. Unfortunately, (15) for the optimal threshold y^*, and the formula for the minimum average cost β (14), are complicated and therefore left unstated. However, we can illustrate the results numerically. In Table 1 the optimal threshold y^* is calculated with the values $\mu = 1$, $\sigma = 1$, and $\beta = 0.01$ for a few different values of λ. We see from the table that as λ increases the threshold y^* gets closer to the corresponding threshold $y_s^* \approx 0.743$ of the singular control problem (19).

5.2. The standard Ornstein–Uhlenbeck process

As remarked in the introduction, the results also hold for $\mathbb{R}$ with straightforward changes. Indeed, we only have to adjust the assumptions slightly, by changing the lower boundary from 0 to $-\infty$ in Assumptions 2 and 3, and change all the formulas accordingly. With this change we can study a larger class of processes.

TABLE 2: The value of the optimal threshold y^* for a controlled Ornstein–Uhlenbeck process for $\beta = 0.1$ and a few choices of λ.

λ	y^*
1	0.182
5	0.301
10	0.353
100	0.469
300	0.496

Consider dynamics that are characterized by the stochastic differential equation

$$\mathrm{d}X_t = -\beta X_t \mathrm{d}t + \mathrm{d}W_t, \qquad X_0 = x,$$

where $\beta > 0$. This diffusion is often used to model continuous-time systems that have mean-reverting behaviour. To illustrate the results we choose the running cost $\pi(x) = |x|$, and consequently $\pi_\mu(x) = |x| - \gamma\beta x$. The scale density and the density of speed measure in this case are

$$S'(x) = \exp\big(\beta x^2\big), \qquad m'(x) = 2\exp\big(-\beta x^2\big),$$

and the minimal excessive functions read [10, pp. 141]

$$\varphi_\lambda(x) = \mathrm{e}^{\frac{\beta x^2}{2}} D_{-\lambda/\beta}\Big(x\sqrt{2\gamma}\Big), \qquad \psi_\lambda(x) = \mathrm{e}^{\frac{\beta x^2}{2}} D_{-\lambda/\beta}\Big(-x\sqrt{2\gamma}\Big),$$

where $D_\nu(x)$ is a parabolic cylinder function. Equation (15) for the optimal threshold again takes a rather complicated form and thus the results are only illustrated numerically in Table 2. In the singular control case (19) gives $y_s^* \approx 0.535$. Thus, as expected, the threshold value y^* gets closer to y_s^* when λ increases.

6. Conclusions

We have considered ergodic singular control problems with the constraint of a regular one-dimensional diffusion. Relying on basic results from the classical theory of linear diffusions, we characterized the state after which the decision maker should apply an impulse control to the process. Our results are in agreement with the findings of [3], where the corresponding unconstrained singular control problem is studied. Indeed, no second-order or symmetry properties of the cost are needed. In addition, we proved that as the decision maker gets more frequent chances to exercise control, the value of the problem converges to that of the singular problem.

There are a few directions in which the constrained problems could be studied further. To the best of our knowledge, the finite-horizon problem with constraint remains open, even for the case of Brownian motion. Thus, it would be interesting if a similar analysis to [21] could be extended to also cover this case. In this case, we would expect similar connections between the finite-horizon time and the present problem as for those without any constraints [21, p. 241].

Moreover, the related two-sided problem, where the decision maker could control both downwards and upwards, but only at jump times of a Poisson process, could be studied. Unfortunately, these extension are outside the scope of the present study, and are therefore left for future research.

Acknowledgement

Two anonymous referees are acknowledged for helpful comments. We would like to gratefully acknowledge the emmy network foundation under the aegis of the Fondation de Luxembourg for its continued support.

References

[1] ALVAREZ, L. H. R. (1999). A class of solvable singular stochastic control problems. *Stoch. Stoch. Reports* **67**, 83–122.

[2] ALVAREZ, L. H. R. (2004). A class of solvable impulse control problems. *Appl. Math. Optimization* **49**, 265–295.

[3] ALVAREZ, L. H. R. (2018). A class of solvable stationary singular stochastic control problems. Preprint, arXiv:1803.03464.

[4] ALVAREZ, L. H. R. AND HENING, A. (2018). Optimal sustainable harvesting of populations in random environments. Preprint, arXiv:1807.02464.

[5] ALVAREZ, L. H. R. AND LEMPA, J. (2008). On the optimal stochastic impulse control of linear diffusions. *SIAM J. Control Optimization* **47**, 703–732.

[6] ALVAREZ, L. H. R. AND STENBACKA, R. (2001). Adoption of uncertain multi-stage technology projects: A real options approach. *J. Math. Econ.* **35**, 71–97.

[7] ARAPOSTATHIS, A., BORKAR, V. S. AND GHOSH, M. K. (2012). *Ergodic Control of Diffusion Processes*. Cambridge University Press.

[8] BAIN, A. AND CRISAN, D. (2009). *Fundamentals of Stochastic Filtering*. Springer, New York.

[9] BASS, R. (1998). *Diffusions and Elliptic Operators*. Springer, New York.

[10] BORODIN, A. N. AND SALMINEN, P. (2015). *Handbook of Brownian Motion – Facts and Formulae*, 2nd edn. Birkhäuser, Basel.

[11] DAYANIK, S. AND KARATZAS, I. (2003). On the optimal stopping problem for one-dimensional diffusions. *Stoch. Process. Appl.* **107**, 173–212.

[12] DUPUIS, P. AND WANG, H. (2002). Optimal stopping with random intervention times. *Adv. Appl. Prob.* **34**, 141–157.

[13] EVANS, S. N., HENING, A. AND SCHREIBER, S. J. (2015). Protected polymorphisms and evolutionary stability of patch-selection strategies in stochastic environments. *J. Math. Biol.* **71**, 325–359.

[14] FLEMING, W. H. (1968). Optimal control of partially observable diffusions. *SIAM J. Control Optimization* **6**, 194–214.

[15] FLEMING, W. H. (1999). Controlled Markov processes and mathematical finance. In *Nonlinear Analysis, Differential Equations and Control*, eds F. H. Clarke, R. J. Stern and G. Sabidussi. Kluwer, Dordrecht, 407–446.

[16] FLEMING, W. H. AND MCENEAEY, W. (1995). Risk sensitive control on an infinite time horizon. *SIAM J. Control Optimization* **33**, 1881–1915.

[17] GUO, X. AND ZHANG, Q. (2004). Closed-form solutions for perpetual American put options with regime switching. *SIAM J. Appl. Math.* **64**, 2034–2049.

[18] HARRISON, J. M., SELLKE, T. M. AND TAYLOR, A. J. (1983). Impulse control of Brownian motion. *Math. Operat. Res.* **8**, 454–466.

[19] JACK, A. AND ZERVOS, M. (2006). A singular control problem with an expected and pathwise ergodic performance criterion. *J. Appl. Math. Stoch. Anal.* **2006**, 82538.

[20] JIANG, Z. AND PISTORIUS, M. R. (2008). On perpetual American put valuation and first-passage in a regime-switching model with jumps. *Finance Stoch.* **12**, 331–355.

[21] KARATZAS, I. (1983). A class of singular stochastic control problems. *Adv. Appl. Prob.* **15**, 225–254.

[22] KARATZAS, I. AND SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*. Springer, New York.

[23] KUSHNER, H. J. AND DUPUIS, P. G. (1992). *Numerical Methods for Stochastic Control Problems in Continuous Time*. Springer, New York.

[24] LEMPA, J. (2012). Optimal stopping with information constraint. *Appl. Math. Optimization* **66**, 147–173.

[25] LEMPA, J. (2012). Optimal stopping with random exercise lag. *Math. Operat. Res.* **75**, 273–286.

[26] LEMPA, J. (2014). Bounded variation control of Itô diffusion with exogenously restricted intervention times. *Adv. Appl. Prob.* **46**, 102–120.

[27] LEMPA, J. (2017). A class of solvable multiple entry problems with forced exits. *Appl. Math. Optimization* **76**, 1–27.

[28] LEMPA, J. AND SAARINEN, H. (2019). On asymptotic relations between singular and constrained control problems of one-dimensional diffusions. Preprint, arXiv:1909.06115.

[29] MATOMÄKI, P. (2012). On solvability of a two-sided singular control problem. *Math. Operat. Res.* **76**, 239–271.

[30] MENALDI, J. L. AND ROBIN, M. (2016). On some optimal stopping problems with constraint. *SIAM J. Control Optimization* **54**, 2650–2671.
[31] MENALDI, J. L. AND ROBIN, M. (2017). On some impulse control problems with constraint. *SIAM J. Control Optimization* **55**, 3204–3225.
[32] MENALDI, J. L. AND ROBIN, M. (2018). On some ergodic impulse control problems with constraint. *SIAM J. Control Optimization* **56**, 2690–2711.
[33] PICARD, J. (1986). Nonlinear filtering of one-dimensional diffusions in the case of a high signal-to-noise ratio. *SIAM J. Appl. Math.* **46**, 1098–1125.
[34] ROBIN, M. (1983). Long-term average cost control problems for continuous time Markov processes: A survey. *Acta Appl. Math.* **1**, 281–299.
[35] ROGERS, L. C. G. (2001). The relaxed investor and parameter uncertainty. *Finance Stoch.* **5**, 131–154.
[36] ROGERS, L. C. G. AND ZANE, O. (2002). A simple model of liquidity effects. In *Advances in Finance and Stochastics*. Springer, Berlin, pp. 161–176.
[37] SETHI, S. P., ZHANG, H. AND ZHANG, Q. (2005). *Average-Cost Control of Stochastic Manufacturing Systems*. Springer, New York.
[38] REVUZ, D AND YOR, M (1991). *Continuous Martingales and Brownian Motion*. Springer, Berlin.
[39] WANG, H. (2001). Some control problems with random intervention times. *Adv. Appl. Prob.* **33**, 404–422.