Mutually Best Matches

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Abstract

We study iterated formation of mutually best matches (IMB) in college admissions problems. When IMB produces a non-wasteful matching, the matching has many good properties like Pareto optimality and stability. Moreover, in this case IMB selects the unique core allocation and truth-telling is a Nash equilibrium for students. If preferences satisfy a single peakedness condition, or have a single crossing property, then IMB is guaranteed to produce a non-wasteful matching. These properties guarantee also that the Deferred Acceptance algorithm (DA) and the Top Trading Cycles algorithm (TTC) produce the same matching as IMB. We compare these results with some well-known results about when DA is Pareto optimal, or when DA and TTC produce the same matching.

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1. Introduction

Gale and Shapley (1962) introduce the college admissions problem where no money is used to match agents in two disjoint sets with each other. These sets consist of students and colleges (or "schools") having strict preferences

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over the other set. In this paper we describe iterated formation of mutually best matches (IMB) algorithm as a way to solve the college admission problem.

By construction, IMB produces an individually rational matching. Furthermore, when IMB produces a non-wasteful matching[1], the matching has many good properties like Pareto optimality and stability. Moreover, the core is guaranteed to be a singleton, and truthful reporting of preferences is a Nash equilibrium for students in such case. We suggest that IMB is used as a preliminary test for a good allocation. If IMB produces a non-wasteful matching, then there cannot exist any other core allocations, and the selected allocation has all the nice properties one can hope for.

If preferences satisfy a single peakedness condition, or have a single crossing property with setwise increasing acceptability relations, then IMB always produces a non-wasteful matching.[2] It follows that in these cases DA produces the same Pareto optimal matching as IMB. Further, these properties guarantee that the most common algorithms – the student proposing DA, the school proposing DA, and the TTC – produce the same matching as IMB.

Ergin (2002) shows that a necessary and sufficient condition for DA to be Pareto optimal is that the preference structure of schools satisfies an acyclicity condition. Kesten (2006) introduces a slightly different acyclicity condition and shows that a necessary and sufficient condition for DA and TTC to produce the same matching is that the priority structure of the schools satisfies his acyclicity condition.

We don't need acyclicity conditions for our results. The reason is that our setup is different than that of Ergin or Kesten. Ergin and Kesten fix the priorities of schools, and ask when DA is Pareto optimal (Ergin) or when DA = TTC (Kesten), for *all* possible preferences of students. They find

[1] There does not exist a mutually acceptable student - school pair s, c such that s is unmatched and c has free capacity.

[2] See Chung (2000), Banerjee *et al.* (2006), Gabszewicz *et al.* (2012), and Milgrom and Shannon (1994) for versions of single peakedness and single crossing properties.

that the priority structure must be acyclic.

We assume that single peakedness or single crossing property hold for both sides of the market. Then if we look at a particular preference structure of the schools, there is only a subset of possible preferences for students such that single peakedness or single crossing assumptions remain valid.

A third example when IMB produces a non-wasteful matching is when one side of the market has identical preferences. In this case DA = TTC as well. Although this is a very restrictive assumption from theoretical viewpoint, in some real world applications it may hold approximately.

When IMB produces a wasteful matching, one may try to fix it by continuing the matching process with some other algorithm. We study a variant introduced by Morrill (2015) called *Always Clinch and Trade* (AC&T) in which one round of TTC is applied whenever IMB halts producing a wasteful matching. After that, IMB is applied again, *e.t.c.* When schools have capacity of one ("marriage market"), then AC&T = TTC. It follows that AC&T is not stable but it is Pareto optimal. Moreover, if some schools may accept more than just one student, AC&T no longer satisfies strategy proofness.

The paper is organized in the following way. In Section 2 we introduce the notation, axioms, and the used matching algorithms. Section 3 contains the main results. In Section 4 we study a way to "fix" IMB when it produces a wasteful matching. Section 5 concludes.

2. Preliminaries

Let us denote by S the nonempty finite set of students and by C the nonempty finite set of schools. A *matching* is a function $\mu : S \longrightarrow C \cup S$ such that $\mu(s) \notin C$ iff $\mu(s) = s$. We denote by $\mu^{-1}(c)$ the set of students that are matched with school c.

Student $s \in S$ has a strict preference order $\prec_s$ over acceptable schools: $c \prec_s c'$ means that student s strictly prefers school c' to school c. Notation $c \precsim_s c'$ means that $c \prec_s c'$ or $c = c'$. We may denote preferences by ordered lists like $\prec_s = c_1 c_2 \cdots c_k$ where c_1 is the best school for s and c_k

is the worst school that s finds acceptable. Student s a) strictly prefers being unmatched to being matched with an unacceptable school, b) strictly prefers any acceptable school to being unmatched.

A *preference profile* $(\prec_s)_s$ specifies a preference relation to each $s \in S$. Notation $(\prec'_s, \prec_{-s})$ means that student s has preferences $\prec'_s$ while the other students have the same preferences as in the profile $(\prec_s)_s$.

School $c \in C$ has a strict preferences $\prec_c$ over students. Notation $s \precsim_c s'$ means that $s \prec_c s'$ or $s = s'$. We may denote school preferences by ordered lists like $\prec_c = s_1 s_2 \cdots s_t$ where s_1 is the most preferred student, and s_t is the worst student that c finds acceptable.

Schools order subsets of students as well. We make the common but strong assumption that preferences are *responsive* (see Roth and Sotomayor 1992, p. 128). By this assumption we don't have to represent preferences over subsets of students explicitly.

A *preference profile* $(\prec_c)_c$ specifies a preference relation to each $c \in C$. Notation $(\prec'_c, \prec_{-c})$ means that school c has preferences $\prec'_c$ while the other schools have the same preferences as in the profile $(\prec_c)_c$. A school c has capacity $q_c > 0$ which tells the greatest number of students that a school c can accept. We denote by $q = (q_c)_c$ the vector of capacities.

A *mechanism* is a rule M that to each *matching problem* $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$ assigns a matching $M(P) : S \longrightarrow C \cup S$. We assume that every $s \in S$ has some acceptable school $c \in C$ and every school $c' \in C$ has some acceptable student $s' \in S$. In this paper mechanisms are sometimes called algorithms, since the mechanisms studied here are given in algorithmic form.

2.1. Properties of mechanisms and matchings

Given a problem $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$, we say that a pair s, c is *mutually acceptable*, if s is an acceptable student for school c and c is an acceptable school for student s. A mutually acceptable pair s, c is a *mutually best pair*, if c is the best school for the student s and s is among the best q_c students of school c.

A matching μ is *individually rational*, if $c = \mu(s)$ implies that s, c is a mutually acceptable pair.

Axiom 1 (Individual rationality)**.** The matching $M(P)$ is individually rational for all $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$.

A matching μ is *non-wasteful*[3], if there does not exists a mutually acceptable pair s, c such that s is unmatched and c has free capacity. That is, if $\mu(s) = s$ and $|\mu^{-1}(c)| < q_c$, then s, c is not a mutually acceptable pair.

Axiom 2 (Non-wastefulness)**.** The matching $M(P)$ is non-wasteful for all $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$.

Given a matching μ, a mutually acceptable pair s, c is a *blocking pair*, if a) $\mu(s) \neq c$, and b) $\mu(s) \prec_s c$ and $s' \prec_c s$ for some $s' \in \mu^{-1}(c)$, or $\mu(s) \prec_s c$ and c has free capacity. A matching μ is *stable*, if it is individually rational, and there does not exist a blocking pair.

A matching μ is in the *core*, if there does not exist a matching μ' and a nonempty coalition $A \subset S \cup C$ such that every agent $i \in A$ at least weakly prefers μ' to μ and some agent $j \in A$ strictly prefers μ' to μ. With strict and responsive preferences the core equals the set of stable matchings (Roth and Sotomayor 1992, 167).

Axiom 3 (Stability)**.** The matching $M(P)$ is stable for all $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$.

Given a mechanism M, we say that *truth-telling is a Nash equilibrium* at a problem $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$, if $M(P') \precsim_s M(P)$ for all s and for all $\prec'_s$, where $P' = \{S, C, (\prec'_s, \prec_{-s}), (\prec_c)_c, q\}$, and $(\prec_s)_s$ and $(\prec_c)_c$ are the true preference profiles. We say that *truth-telling is a dominant strategy* for student s, if reporting the true preferences $\prec_s$ is optimal for s at every problem P in which s is present with preferences $\prec_s$.

Axiom 4 (Strategy proofness)**.** Truth-telling is a dominant strategy for all students s.

[3]Balinski and Sönmez (1999) define *non-wastefulness* the following way: "matching μ is *non-wasteful* if $\mu(s) \prec_s c$ implies $|\mu^{-1}(c)| = q_c$ for all $s \in S$ and for all $c \in C$". Both definitions can be used in this paper, but our definition is slightly weaker.

A matching μ is *Pareto optimal*, if there does not exist an alternative matching μ' such that $\mu(s) \precsim_s \mu'(s)$ for all $s \in S$, and $\mu(s) \prec_s \mu'(s)$ for some $s \in S$. In this paper we study truth-telling and efficiency from the viewpoint of students only.

Axiom 5 (Pareto optimality)**.** The matching $M(P)$ is Pareto optimal for all $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$.

2.2. Matching mechanisms

If the choice is restricted to subsets $S' \subset S$ and $C' \subset C$, the preferences on S' and C' are the original orders restricted to these subsets. We may denote by $A_s(C')$ and $A_c(S')$ the schools and students that are acceptable in subsets C' and and S' for student s and school c, respectively. Note that $A_s(C') = A_s(C) \cap C'$ and $A_c(S') = A_c(S) \cap S'$.

Student $s \in S$ and school $c \in C$ are mutually acceptable if $s \in A_c(S)$ and $c \in A_s(C)$. Matching problems without mutually acceptable pairs are not interesting from the viewpoint of matching theory.

The best known mechanisms are the *deferred acceptance* algorithm (Gale and Shapley 1962) and the *top trading cycles* mechanism (Shapley and Scarf 1974, Abdulcadiroğlu and Sönmez 2003).

The deferred acceptance algorithm (hereafter DA), more precisely a *student proposing* version of it, is defined by the following steps.

1. Any student s names her best school. Any school c tentatively accept the best q_c of those students that named c, and permanently rejects the rest of the students that named c.
2. Any student s rejected in the previous step names her best school among the schools that have not yet rejected him. A student who has been tentatively accepted cannot name any school. Any school c compares the new applicants to the ones she already has, and tentatively accepts the best q_c of them.

DA ends when any student who is not tentatively accepted by some school does not have any acceptable schools left. The students tentatively accepted by schools are permanently matched with these schools. In school proposing DA, the students accept proposals from schools. Student proposing DA produces a student optimal stable matching while school proposing DA produces a school optimal stable matching.

The top trading cycle mechanism (hereafter TTC), is defined by the following steps.

1. Each student s names the best school c in the market that finds s acceptable. Each school c names the best student in the market. If there are no cycles, go to 3. If there are cycles ($s \to c \to \ldots \to s$), match each student in a cycle permanently with the school that the student named. Update the preferences and capacities of the agents. Permanently matched students and those schools whose capacity is full leave the market. Also those students (schools) who no longer have acceptable schools (students) leave the market (this step is repeated if necessary). Go to 2.
2. Let S' and C' denote the students and schools that are still in the market. If both S' and C' are nonempty, then go to 1, and apply the procedure to S' and C'. If S' or C' is empty, go to 3.
3. End.

In step 1. of TTC, we must require that an student s names the best school c that finds s acceptable. Without this specification TTC could match a student with a school that does not accept s. Also students and schools that do not have acceptable matches must be removed from the market, since otherwise a school might name a student who has no acceptable school and TTC could produce a wasteful matching.

Recall that s, c is a mutually best pair, if c is the best school for student s, and s is among the best q_c students for school c. By *iterated formation of mutually best matches* (IMB) we mean the following process.

1. Students (schools) report their preferences over schools (students) that are in the market. If there are no mutually best pairs, then go to 3. If there are mutually best pairs, the mechanism matches permanently

all such pairs. Update the preferences and capacities of the agents. Permanently matched students and those schools whose capacity is full leave the market. Also those students (schools) who no longer have acceptable schools (students) leave the market (this step is repeated if necessary). Go to 2.

2. Let S' and C' denote the students and schools that are still in the market. If both S' and C' are nonempty, then go to 1, and apply the procedure to S' and C'. If S' or C' is empty, go to 3.
3. End.

In the next example we demonstrate how the algorithm is applied in practice.

Example 1. Let $S = \{s_1, s_2, s_3\}$ and $C = \{c_1, c_2\}$. All student – school pairs s, c are mutually acceptable. Preferences are: $\prec_{s_1} = c_2 c_1$, $\prec_{s_2} = c_1 c_2$, $\prec_{s_3} = c_2 c_1$. Schools' preferences are $\prec_{c_1} = s_1 s_2 s_3$, $\prec_{c_2} = s_2 s_3 s_1$. School c_1 has capacity of two and school c_2 has capacity of one.

In the first round, the only mutually best pair is s_2, c_1. After matching these agents, we remove student s_2 from the market and reduce the quota of school 1 by one. After updating the preferences, we can move on to round 2. In the second round there is only one mutually best pair s_3, c_2. In the last round school 1 and student s_1 are a mutually best pair. No more mutually best pairs can be found and the algorithm ends in a non-wasteful matching.

Note that both IMB and DA produce a matching $\mu = \{(s_1, c_1), (s_2, c_1), (s_3, c_2)\}$ while the TTC produces a matching $\mu' = \{(s_1, c_2), (s_2, c_1), (s_3, c_1)\}$.

Let us modify this example so that IMB produces a wasteful matching. Suppose both schools have capacity of one. Now the IMB stops immediately as there are no mutually best matches. Yet, all schools are acceptable for all students. Furthermore, s_1 can manipulate her preferences and the wasteful matching by dropping c_2 from her preferences. With manipulated preferences $\prec'_{s_1} = c_1$ the resulting matching using IMB is $\mu'' = \{(s_1, c_1), (s_2, c_2)\}$. Matching μ'' is preferred by s_1 to not being matched. $\triangleleft$

Note that since the IMB matches mutually acceptable pairs iteratively, it satisfies Axiom 1 (individual rationality) by construction. As we saw from

Example 1, IMB does not satisfy Axiom 2 (non-wastefulness) in general. Moreover, a wasteful matching produced by IMB is manipulable and IMB does not satisfy Axiom 4 (strategy proofness) in general.

3. Results

The first result says that when IMB produces a non-wasteful matching for $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$, then the matching is stable and Pareto optimal, and truth-telling is a Nash equilibrium for students at P.

Proposition 1. *If IMB produces a non-wasteful matching μ for the problem $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$, then μ is stable and Pareto optimal. Hence DA produces μ as well. Moreover, truth-telling is a Nash equilibrium for students at P.*

Proof. Stability. If μ is not stable there exists a blocking pair s, c, since IMB is individually rational. Since μ is a non-wasteful matching, c cannot have free capacity, and $s' \prec_c s$ must hold for some s' that is matched with c. Under IMB, it is not possible that s is matched with some school c' before or at the same time s' is matched with c. This holds since s is above s' in the preferences of c, so c' would then be a better school for s than c, a contradiction. But then it follows that s will remain among the q_c best students of c even after s' is matched with c. The worst thing that can happen to s is that she will be eventually matched with c, a contradiction again. Hence μ is stable.

Pareto optimality. Let S_k be the set of students that are matched in round k of IMB. Those matched in round 1 are matched with their best schools. Those matched in round $k = 2$ are matched with the best schools still in the market, and so on. So it is impossible to find a Pareto improving reallocation among the students that are matched with some school. Hence if μ' is a Pareto improving matching, μ' matches all students in subsets S_k the same way as μ. Hence μ' can be different only if some s that was unmatched under μ is placed in some school under μ'. By the definition of

IMB, any school c that has free capacity under μ finds s unacceptable, or else c is unacceptable to s. Hence μ is Pareto optimal.

Let DA produce a matching μ''. Since μ'' Pareto dominates any other stable matching (Roth and Sotomayor 1992), $\mu'' = \mu$.

Truth-telling. Let S_k be the set of students that are matched in round k of IMB. If s is matched in round $k = 1$, then she is matched with her best school and therefore untruthful reporting of preferences cannot be beneficial. If $k = 2$, then untruthful reporting would benefit s only if she would be matched with some school c' that is better than $c = \mu(s)$, and that got its capacity filled already in round $k = 1$. But that is possible only if s was among the $q_{c'}$ best students for school c' in round $k = 1$. If this holds, then c' has still free capacity in round $k = 2$, a contradiction with the assumption that s is matched with a worse school c. Apply induction on k and conclude that no student that is matched with a school can benefit from misreporting preferences. Finally note that a similar argument holds also for students that are unmatched under μ. □

As we saw in Example 1, TTC can produce a different matching than IMB even when IMB produces a non-wasteful matching. The following result shows that TTC is equivalent to IMB when the capacities of the schools are restricted to one and IMB produces a non-wasteful matching.

Corollary 1. *If IMB produces a non-wasteful matching μ for the problem $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$, and each school has capacity of one, then IMB generates the same matching as DA and TTC.*

Proof. By construction IMB produces only short cycles $s \to c \to s$. Because each school has capacity of one, a pair s, c is mutually best, if and only if s is the best student for school c and c is the best school for student s. These kind of mutually best pairs are matched also by TTC. Hence every pair s, c matched by IMB will also be matched by TTC. Since TTC produces an individually rational matching, and by assumption IMB produces a non-wasteful matching, we are done. □

The next Theorem shows that when IMB produces a non-wasteful matching μ, the core is a singleton.

Theorem 1. *If IMB produces a non-wasteful matching μ for the problem $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$, then μ is the unique element of the core.*

Proof. Let IMB produce a non-wasteful matching μ. By Proposition 1, student proposing DA also produces matching μ. Suppose the core is not a singleton. Hence the school proposing DA produces a matching $\mu' \neq \mu$.

By the Rural Hospital Theorem (Roth 1986), the set of matched students is the same in both matchings μ and μ', and if a school c has free capacity in μ, then school c has exactly the same students in μ and μ'. Since μ' is a school optimal stable matching, at least one school c strictly prefers μ' to μ. Hence for this school c it must hold that there exists students s', s such that $\mu'(s') = c \neq \mu(s')$, $\mu(s) = c \neq \mu'(s)$, and s' is strictly better for c than s.

Now apply the school proposing DA algorithm and IMB on a given problem P. In the school proposing DA, school c sends an offer to the top students in its preference relation, up to the capacity constraint, and we may assume *w.l.o.g* that offers are sent to those students only that find c acceptable.

Since IMB produces a non-wasteful matching, there is at least one mutually best pair s, c at round 1 of IMB. By the definition of the school proposing DA and the definition of a mutually best pair, this school c is tentatively accepted by student s at round 1 in problem P. Note that since s, c is a mutually best pair, the student s cannot get a better offer on later rounds of the school proposing DA algorithm.

Denote by μ^1 all the mutually best pairs (s, c) matched by IMB in the first round, and note that these pairs (s, c) will be eventually matched by the school proposing DA as well. In the school proposing DA, if school c was rejected by a student s', school c may never again make a proposal to s'.

Form a sub-problem $P^1 = \{S^1, C^1, (\prec_s^1)_s, (\prec_c^1)_c, q^1\}$, where $S \setminus S^1$ consists of those students that were matched by IMB in the first round, $C \setminus C^1$

consists of those schools whose capacity was filled at this point, and preferences $\prec^1$ and capacities q^1 are updated for the remaining students and schools accordingly.

Start round 2 of IMB on the original problem P. There exists at least mutually best pair s, c. Now $s \in S^1$ and $c \in C^1$ and hence student s will tentatively accept c when the school proposing DA is applied on P^1. Note that student s cannot get an offer from a better school anymore when DA is applied, and therefore s will be permanently matched with c. But this holds also when the school proposing DA is applied to the original problem P, because in that case c will make proposal to s in the second round of DA.

Denote by μ^2 all the mutually best pairs (s, c) matched by IMB in the second round. We have shown that the pairs $(s, c) \in \mu^2$ will be eventually matched by the school proposing DA as well.

Continue recursively by forming a subproblem P^{k+1} from P^k in the same manner as P^1 was formed P, for $k \geq 1$. It follows that the mutually best pairs μ^{k+1} identified by IMB in the round k will be permanently matched by the school proposing DA.

Let n be the round when IMB halts. Then the non-wasteful matching μ generated by IMB satisfies $\mu = \mu^1 \cup \cdots \cup \mu^n$. By Proposition 1 the student proposing DA also produces μ. On the other hand, we have shown that $\mu^k \subset \mu'$ for all $k \leq n$, where μ' is the matching produced by the school proposing DA. Hence $\mu \subset \mu'$. The Rural Hospital Theorem implies $\mu = \mu'$ □

Note that Theorem 1 is a necessary but not a sufficient condition for a singleton core. This is shown in the following example slightly modified from Example 4 in Morrill (2015).

Example 2. $S = \{s_1, \ldots, s_5\}$, $C = \{c_1, \ldots, c_4\}$. All student-school pairs are mutually acceptable. Students' preferences are as follows, unlisted schools could be in any order after the top schools: $\prec_{s_1} = c_3c_2c_1$, $\prec_{s_2} = c_1$, $\prec_{s_3} = c_2c_1$, $\prec_{s_4} = c_4c_1c_3$, $\prec_{s_5} = c_3$. Schools' preferences are as follows, unlisted students could be in any order after the top students: $\prec_{c_1} = s_1s_4s_2s_3$, $\prec_{c_2} = s_2s_3s_1$,

$\prec_{c_3} = s_4 s_5$, $\prec_{c_4} = s_5 s_4$. School c_1 has capacity of two, the other schools have capacity of one.

There are no mutually best matches to eliminate and IMB produces a wasteful empty matching. Both DA algorithms produce the matching μ^*:

$$\mu^* = \{(s_1, c_1), (s_2, c_1), (s_3, c_2), (s_4, c_4), (s_5, c_3)\}.$$

This shows that the core can be singleton even when the IMB produces a wasteful matching. $\lhd$

For the remaining part of this section we concentrate on studying situations where restricted preference domains allow IMB to produce a non-wasteful matching. One such situation is when one side of the market has identical preferences over the other side of the market.

Proposition 2. *If $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$ is such that all students have identical preferences, then IMB produces the same non-wasteful matching μ for P as TTC and DA. This holds also if all schools have identical preferences but may have different capacities.*

Proof. Identical student preferences. Index the schools so that the common preference of students is $\prec_s = c_1 c_2 \cdots c_k$, where c_k is the last acceptable school. In the first round of IMB, school c_1 gets the top q_{c_1} students according to its preferences $\prec_{c_1}$ or fewer if c_1 has smaller number of acceptable students. After that school c_2 gets the top q_{c_2} students according to its preferences $\prec_{c_2}$ or fewer if c_2 has smaller number of acceptable students. Continue this way as long as there are acceptable students for some $c_t, t \leq k$.

Now apply TTC, and note that school c_1 gets first the same students she got under IMB. After that, school c_2 gets the same students she got under IMB. Continuing this way we observe that TTC produces the same matching μ as IMB. Hence μ is a non-wasteful individually rational matching. By Proposition 1 DA produces μ.

Identical school preferences. Index the students so that the common preferences of schools is $\prec_c = s_1 s_2 \cdots s_k$, where s_k is the last acceptable student. In the first round of TTC only s_1 is matched, and she is matched with her best school, say c^1. IMB also matches s_1 with c^1. By the same argument, during each round $t \leq k$ of TTC only student s_t is matched, and she is matched with the best school still in the market, say c^t. IMB also matches s_t with c^t. Therefore if TTC matches a student s with a school c, then also IMB matches s with c. The matching μ produced by TTC is Pareto optimal and individually rational. Hence μ is also a non-wasteful matching. By Proposition 1 both IMB and DA produces μ as well. □

Next we relax the identical preference structure with a *single peakedness* property. Single peakedness has sound economic rationale in school choice. Such a situation would arise when schools and students apply to their proximate counterparts first. Eeckhout (2000) defines conditions for a singleton core in a balanced marriage market in terms of single peaked preference structure. Our definition is more general since we allow for multiple agents to have the same most preferred choice in an unbalanced college admissions setup. To define single peakedness for problems $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$, we assume that sets S and C of students and schools are subsets of some finite dimensional real space $\mathbb{R}^p$ which is equipped with a norm $\|\cdot\|$.[4] The following holds for any student $s \in S$, and any schools c, c':

$$c \prec_s c' \iff \|s - c\| > \|s - c'\|, \tag{1}$$

and similarly for $c \in C$, and any students s, s':

$$s \prec_c s' \iff \|c - s\| > \|c - s'\|. \tag{2}$$

Single peaked preferences are based on the notion of distance between the ideal points s and c of the agents. Student s can have the same preferences as student s' if s' is sufficiently close to s, and similarly for schools. We continue to assume that all preferences are strict and hence there are no indifferences.

[4]For example, $\|\cdot\|$ could be the Euclidean norm $\|x\| = \sqrt{\sum_i x_i^2}$, but any norm on $\mathbb{R}^p$ will do.

We define the acceptability relations A_s of students to be *distance consistent*, if

$$c \in A_s(C) \text{ and } \|s - c'\| < \|s - c\| \Longrightarrow c' \in A_s(C). \tag{3}$$

The distance consistent acceptability relations A_c for school c is defined analogously.

Single peaked preferences guarantee that IMB produces a non-wasteful matching. This is shown in the next proposition.

Proposition 3. *If $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$ is such that preferences satisfy the single peakedness property, and the acceptability relations are distance consistent, then IMB produces a non-wasteful matching μ for P.*

Proof. We assume that there is at least one mutually acceptable pair s, c, since otherwise the empty matching is trivially non-wasteful. There exists a pair s', c' that minimizes the norm $\|s - c\|$, since there are finitely many students and schools in the market. We will show that s', c' is a mutually best pair.

We show first that s', c' is a mutually acceptable pair. Assume first that s' does not accept c'. Since s' has some acceptable school c'' in the market, and acceptability relations are distance consistent, we must have $\|s' - c''\| < \|s' - c'\|$. This contradicts the choice of s', c'. In the same manner we can show that c' finds s' acceptable, because any norm is a symmetric function: $\|s - c\| = \|c - s\|$. Therefore s', c' is a mutually acceptable pair.

Now c' must be the best school in the market for s', since otherwise $\|s' - c''\| < \|s' - c'\|$ would hold because acceptability relations are distance consistent. Similarly, s' must be the best student for c', and therefore s', c' is a mutually best pair.

If there are any mutually acceptable pairs left, then IMB can be continued and the step above can be repeated. If there are no mutually acceptable pairs we are done. □

Whenever IMB produces a non-wasteful matching μ, by Proposition 1 DA produces matching μ. Our next result shows that single peaked preferences guarantee that TTC produces μ as well regardless of school capacities.

Proposition 4. *If $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$ is such that preferences satisfy the single peakedness property, and the acceptability relations are distance consistent, then IMB generates the same matching μ as DA and TTC. Furthermore, matching μ is stable, Pareto optimal, the unique element of the core, and truth-telling is a Nash equilibrium for students at P.*

Proof. IMB = DA, stability, Pareto-optimality, and the truth-telling property follow from Propositions 3 and 1. Uniqueness of the core follows from Theorem 1. By definition, IMB produces only short cycles $s \to c \to s$. For TTC = IMB to hold, we need to show that there are no cycles of the form $s \to c \to s' \to c' \to s$, where $s \neq s'$. For such cycles the following inequalities have to hold for $s, s' \in S$ and $c, c' \in C$:

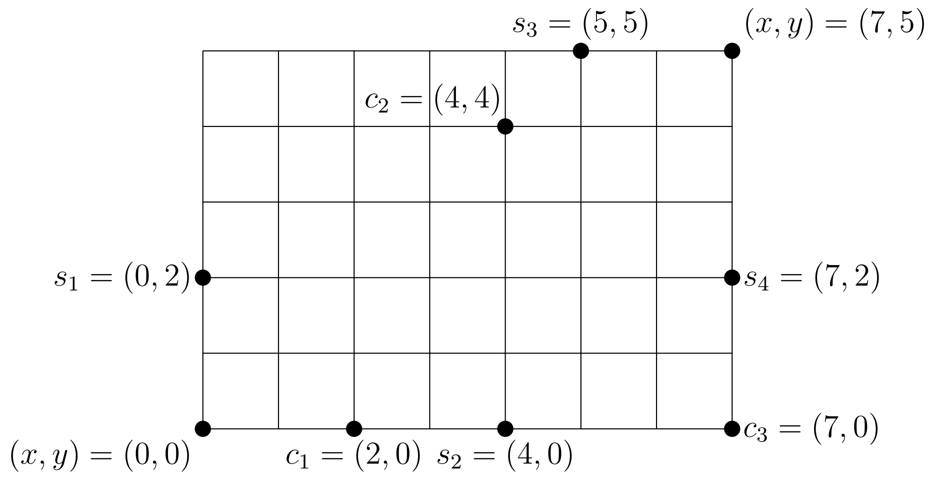
$$\begin{aligned} s &: \|s - c'\| > \|s - c\|, \\ c &: \|c - s\| > \|c - s'\|, \\ s' &: \|s' - c\| > \|s' - c'\|, \\ c' &: \|c' - s'\| > \|c' - s\|. \end{aligned}$$

It follows that $\|s - c\| > \|c - s'\| > \|s' - c'\| > \|c' - s\| > \|s - c\|$ has to hold by symmetry of $\|\cdot\|$, a contradiction. Same reasoning can be applied to all cycles $s \to c \to s' \to \cdots \to c' \to s$ since the inequalities $\|s - c\| > \|c - s'\| > \cdots > \|c' - s\| > \|s - c\|$ cannot hold. $\square$

Example 3. Let $S = \{s_1, s_2, s_3, s_4\}$ and $C = \{c_1, c_2, c_3\}$, all schools are acceptable to all students and all students are acceptable to all schools. Let the capacities of the schools be 1 except for school c_2 let $q_{c_2} = 2$. Let the used norm be the taxicab norm so that for student s_1 and school c_2 the following holds $\|s_1 - c_2\| = |s_1^x - c_2^x| + |s_1^y - c_2^y| = |0 - 4| + |2 - 4| = 6$. The distances can be easily calculated from Figure 1 where each agent is represented by coordinates on $x - y$ plane.

Now the students preferences are as follows: $\prec_{s_1} = c_1c_2c_3$, $\prec_{s_2} = c_1c_3c_2$, $\prec_{s_3} = c_2c_3c_1$, $\prec_{s_4} = c_3c_2c_1$. Similarly the schools' preferences are: $\prec_{c_1} = s_2s_1s_4s_3$, $\prec_{c_2} = s_3s_2s_4s_1$, $\prec_{c_3} = s_4s_2s_3s_1$. Clearly the preferences are strict and satisfy single peakedness.

Figure 1: Taxicab norm single peaked preferences.



Applying IMB, the first mutually best matches are (s_2, c_1), (s_3, c_2), and (s_4, c_3). After that, the only mutually best match is (s_1, c_2). IMB produces the matching $\mu = \{(s_1, c_2), (s_2, c_1), (s_3, c_2), (s_4, c_3)\}$. DA and TTC produce μ as suggested by Proposition 4. $\triangleleft$

If students' and schools' preferences satisfy the "single crossing" property, then again IMB produces a non-wasteful matching for the problem. Clark (2006) defines the single crossing property in a balanced marriage market, and shows that the core is a singleton. We define the single crossing property in the framework of possibly unbalanced college admissions problems.

Let us assume that the single crossing property holds at a problem $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$. Then the students $S = \{s_1, \dots, s_n\}$ and schools $C = \{c_1, \dots, c_m\}$ can be thought as being real numbers and indexed so that $s_i < s_{i+1}$ and $c_k < c_{k+1}$. Further, the following holds for $s_i, s_j \in S$, $c_k, c_p \in C$:

$$c_k \prec_{s_i} c_p \text{ and } c_k < c_p \Longrightarrow c_k \prec_{s_j} c_p, \text{ if } s_i < s_j \tag{4}$$

$$s_i \prec_{c_k} s_j \text{ and } s_i < s_j \Longrightarrow s_i \prec_{c_p} s_j, \text{ if } c_k < c_p. \tag{5}$$

If all students would accept the same schools and all schools would accept all students, then single crossing alone would be sufficient for IMB to generate a non-wasteful matching. If these assumptions do not hold, we have to impose some restrictions on the acceptability relations as well.

Acceptability relation $A_s(C)$ for student s is an *interval*, if $c, c' \in A_s(C)$, $c \le c'$, implies $c'' \in A_s(C)$, for all schools c'' such that $c \le c'' \le c'$. Interval acceptability relation for a school c is defined in the same manner.

Given a nonempty subset A of S, we denote by $\max A$ and $\min A$ the greatest and least elements of A, respectively. Notation $\max B$ and $\min B$ is defined analogously for nonempty subsets B of C.

Acceptability relations A_s and A_c are *setwise increasing intervals*, if they are intervals, and

$$s < s' \Longrightarrow \min A_s(C) \le \min A_{s'}(C), \text{ and } \max A_s(C) \le \max A_{s'}(C) \quad (6)$$

$$c < c' \Longrightarrow \min A_c(S) \le \min A_{c'}(S), \text{ and } \max A_c(S) \le \max A_{c'}(S). \quad (7)$$

Constant relations $A_s(C) = A_{s'}(C)$, $A_c(S) = A_{c'}(S)$ for all s, s' and c, c' are special cases of setwise increasing relations.

The following lemma is a version of some well-known results from the literature of strategic complements (see *e.g.* Milgrom and Shannon 1994), but we give a proof here for the sake of completeness. For any $s \in S$ let $c(s)$ denote the best school for s. That is, $c(s)$ is the best school in the set $A_s(C)$. For any $c \in C$ let $s(c)$ denote the best student for c.

Lemma 1. *Suppose that the single crossing property holds at a problem* $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$ *and that the acceptability relations of schools and students are setwise increasing intervals.*

Then $s < s'$ *implies* $c(s) \le c(s')$ *and* $c < c'$ *implies* $s(c) \le s(c')$.

Proof. It suffices to prove the lemma for students only because the proof for schools is identical.

Take any $s, s' \in S$ such that $s < s'$. We want to show $c(s) \le c(s')$. If this does not hold then $c(s') < c(s)$. Since acceptability relations are setwise increasing intervals, we have that $c(s'), c(s) \in A_s(C) \cap A_{s'}(C)$ because $s < s'$. But then single crossing property implies that student s' strictly prefers $c(s)$ to $c(s')$, since $s < s'$, $c(s') < c(s)$, and $c(s') \prec_s c(s)$. A contradiction with the definition of $c(s')$. Hence $c(s) \le c(s')$. □

When acceptability relations $A_c(S)$ and $A_s(C)$ on S and C are setwise increasing intervals, then their restrictions $A_c(S')$ and $A_s(C')$ to subsets S', C' are also setwise increasing intervals of these subsets. The next lemma states that when acceptability relations are setwise increasing intervals, the single crossing property is a "hereditary property" in the sense that it holds for subsets of S and C.

Lemma 2. *Suppose that the single crossing property holds at a problem $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$, all acceptability relations are setwise increasing intervals, and $S' \subset S$ and $C' \subset C$ are nonempty subsets. Then the student preferences restricted to C' and school preferences restricted to S' satisfy the single crossing property, and nonempty acceptability relations are setwise increasing intervals of S' and C'. In particular, Lemma 1 holds for preferences restricted to C' and S'.*

Proof. It suffices to give a proof for students only. Let $S' \subset S$ and $C' \subset C$ be nonempty subsets. Preferences restricted to subsets C' and S' satisfy the single crossing property since they have this property on C and S.

Take any $s, s' \in S$ such that $s < s'$. Since $A_s(C)$ is an interval of C, also $A_s(C') = C' \cap A_s(C)$ is an interval of C'. If both $A_s(C')$ and $A_{s'}(C')$ are nonempty, then it follows immediately that these relations are setwise increasing.

The proof of Lemma 1 holds if S and C are replaced by S' and C', so the function $c(s)$ is increasing on S'. □

We have the following.

Proposition 5. *If $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$ is such that preferences satisfy the single crossing property, and the acceptability relations are setwise increasing intervals, then IMB produces a non-wasteful matching μ for P.*

Proof. We assume that there is at least one mutually acceptable pair s, c, since the empty matching is trivially non-wasteful. Form a sequence $s^1, s^2, \ldots$ of students and a sequence $c^1, c^2, \ldots$ of schools by the following rule. Let $s^1 = s_n$, and $c^1 = c(s^1)$, and given s^t, let $c^t = c(s^t)$ and $s^{t+1} = s(c^t)$. In words, we start from the highest indexed student s_n, and choose the best school c^1 for him. Then we choose the best student s^2 for school c^1, and after that the best school c^2 for s^2, and so on.

Note that $s^2 \leq s^1$, and therefore $c^2 \leq c^1$ by Lemma 1. It follows by Lemma 1 that sequences $\{s^t\}$ and $\{c^t\}$ are decreasing. By finiteness of S and C, these sequence have limits s^* and c^*. The limits satisfy $c^* = c(s^*)$ and $s^* = s(c^*)$, so s^*, c^* is a mutually best pair.

Match all mutually best pairs, remove matched students from S, and remove those schools from C whose capacity is full. Let S' (C') be the set of students (schools) who are still in the market. If either S' or C' is empty, then the process ends. If both S' and C' are nonempty, update the preferences and capacities of these agents and continue with IMB.

IMB can be applied as long as there are mutually acceptable pairs left, and as shown above, in such a case there is at least one mutually best pair. This completes the proof. □

In fact, when single crossing property holds, the IMB produces the same matching as TTC and DA.

Proposition 6. *If $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$ is such that preferences satisfy the single crossing property, and the acceptability relations are setwise increasing intervals, then IMB, DA, and TTC produce the same matching μ for P. Furthermore, matching μ is stable, Pareto optimal, the unique element of the core, and truth-telling is a Nash equilibrium for students at P.*

Proof. By Proposition 5, IMB produces a non-wasteful matching μ for P. We only have to prove that TTC produces μ, since the other results follow immediately from Proposition 1 and Theorem 1.

When single crossing property holds, it is impossible that TTC generates a cycle that contains at least two students and schools. To see this, assume that there is cycle $s \to c \to s' \to c' \to s$. If $s < s'$, then $c \leq c'$ by single crossing property, and $c = c'$ would imply that $s = s'$. Hence $c < c'$, and therefore $s' \leq s$ a contradiction. In the same manner all longer cycles are impossible under TTC.

Therefore under TTC only cycles $s \to c \to s$ will be formed. In each round of IMB, there is at least one cycle $s \to c \to s$ and s will be matched with c. Every cycle $s \to c \to s$ that is formed during the first round of IMB, will be formed also under TTC, so at least these matchings are the same. In the first round of IMB, there could also be matchings s, c such that c is the best school for s, and s is among the best q_c students for c although not the best one. Under each round of TTC, c is either matched with the best student for c, or c gets no students at all during this round. This implies that s will eventually be matched with c also under TTC. Hence all students that are matched in the first round of IMB will eventually be matched with the same schools under TTC as well.

Continuing this way we can conclude that any match s, c that is formed during round k of IMB will eventually be formed under TTC as well. Since IMB produces a non-wasteful matching μ, TTC must produce μ as well. □

In the next example a matching problem is presented that satisfies the single crossing property. However, the acyclicity conditions of Ergin (2002) and Kesten (2006) are violated. We show that DA produces a Pareto optimal matching although Ergin's condition is violated and that DA = TTC holds although Kesten's condition is violated. A similar situation for single peaked preferences was shown in Example 3.

As explained in the Introduction, this is due to the fact that our setup is different than theirs. They seek a condition for a *fixed* priority structure such that DA is Pareto optimal or TTC = DA, no matter what the students' preferences are. In our setup, students' and schools' preferences cannot vary totally independently in the class of problems with the single crossing property.

Example 4. Let $S = \{s_1, s_2, s_3, s_4\}$ and $C = \{c_1, c_2, c_3\}$, all schools are acceptable to all students and all students are acceptable to all schools. Each school has capacity of one. Let the students' preferences be as follows: $\prec_{s_1} = c_1c_2c_3$, $\prec_{s_2} = c_1c_3c_2$, $\prec_{s_3} = c_3c_1c_2$, $\prec_{s_4} = c_3c_2c_1$. Let schools' preferences be as follows: $\prec_{c_1}= s_1s_2s_3s_4$, $\prec_{c_2}= s_3s_4s_1s_2$, $\prec_{c_3}= s_4s_3s_2s_1$. Now all agents have different preferences.

Order students in the order given by their indices: $s_1 < s_2 < s_3 < s_4$. Order schools in the order given by their indices: $c_1 < c_2 < c_3$. Student s_3 prefers c_1 to c_2, and so do also students s_2 and s_1. On the other hand student s_2 prefers c_3 to c_2, and so do also students s_3 and s_4.

School c_2 orders s_3 and s_4 higher than s_1 and s_2, and so does school c_3 who is on top in the linear order of schools. On the other hand, c_2 ranks s_3 higher than s_4 and s_1 higher than s_2, and so does c_1 who is the least school in the linear order of schools.

In this way one can check that single crossing property holds for preferences.

Applying IMB, the first mutually best matches are (s_1, c_1) and (s_4, c_3). After that, the only mutually best match is (s_3, c_2), and student s_2 is left unmatched. Hence the matching is $\mu = \{(s_1, c_1), (s_3, c_2), (s_4, c_3)\}$ and s_2 is left unmatched. By Proposition 6 DA and TTC also produce this Pareto optimal matching.

Both Ergin's and Kesten's acyclicity conditions are violated, since $s_3 \prec_{c_1} s_2 \prec_{c_1} s_1$ and $s_1 \prec_{c_3} s_2 \prec_{c_3} s_3$, and the Scarcity condition of both author's is satisfied since all schools have capacity of one (for details, see Ergin 2002 and Kesten 2006). $\lhd$

Single crossing works nicely for IMB when the student and school sets are totally ordered, or "one dimensional". If student and school sets have more complicated lattice structure then IMB may fail as shown in the next example.

Example 5. Consider the subset $\{(0,0), (0,1), (1,0), (1,1)\}$ of $\mathbb{R}^2$. When equipped with the usual order of $\mathbb{R}^2$ this set becomes a lattice. Let $S = \{s_{00}, s_{01}, s_{10}, s_{11}\}$ and $C = \{c_{00}, c_{01}, c_{10}, c_{11}\}$ and order the students and schools by their indices. So $s_{00} \leq s_{01}, s_{10} \leq s_{11}$, and s_{01} and s_{10} are incomparable, and analogously for schools. All schools have capacity of one.

Students' preferences: $\prec_{s_{00}}= c_{00}c_{10}c_{01}c_{11}$, $\prec_{s_{10}}= c_{11}c_{01}c_{10}c_{00}$, $\prec_{s_{01}}= c_{11}c_{10}c_{01}c_{00}$, $\prec_{s_{11}}= c_{11}c_{01}c_{10}c_{00}$. All schools are acceptable to all students.

Schools' preferences: $\prec_{c_{00}} = s_{00}s_{01}s_{10}c_{11}$, $\prec_{c_{10}} = s_{11}s_{10}s_{01}s_{00}$, $\prec_{c_{01}} = s_{11}s_{01}s_{10}s_{00}$, $\prec_{c_{11}} = s_{11}s_{10}s_{01}c_{00}$. All students are acceptable to all schools.

Note that preferences satisfy the single crossing property. They satisfy also a condition called "quasi supermodularity". This condition says for student s_{00} that c_{00} must be better than c_{10} *because* c_{01} is better than c_{11} (for details, see Milgrom and Shannon 1994).

IMB matches first s_{00} with c_{00} and s_{11} with c_{11}. But then IMB halts: s_{10} prefers c_{01} to c_{10} but s_{10}, c_{01} is not a mutually best pair, and s_{01} prefers c_{10} to c_{01} but , s_{01}, c_{10} is not a mutually best pair. $\lhd$

4. Fixing IMB?

IMB may halt before it has generated a non-wasteful matching. Sometimes it may fail to match any pairs of students and schools. In this section we look at a possibility to modify IMB in such a way that it always produces a non-wasteful matching.

The IMB modification that is of interest coincides with *Always Clinch and Trade* (AC& T) first introduced by Morrill (2015). The definition is simple: if IMB halts and the matching is wasteful, then apply one round of TTC to the sets of students S' and C' still in the market. There must be at least one cycle. Remove the matched students from S'. Remove the schools whose capacity is full from C'. Update the preferences and capacities of the remaining students S'' and schools C'', and try to apply the usual IMB again. And so on. It is clear that the outcome will be a non-wasteful matching.

The next result states that in marriage markets AC&T is actually TTC.

Proposition 7. *If each school has capacity of one, then AC&T = TTC.*

Proof. Note that under TTC, it is irrelevant in which order formed cycles are removed from the market. So we can apply TTC in such a way that all possible short cycles $s \to c \to s$ are first removed from the market. When there are no short cycles to be found, remove all long cycle $s' \to c' \to \ldots \to s'$. Iterate and produce a matching μ.

Since the quota of every school is one, IMB produces only such mutually best pairs in which a student points to her best school and a school points to its favorite student. This is equivalent to removing short cycles $s \to c \to s$ from the market under TTC. When no short cycles can be found, one round of TTC is applied by the definition of AC&T. This removes all long cycles $s' \to c' \to \ldots \to s'$. By iterating this process we end up with the same matching μ that was produced by TTC. $\square$

It follows from Proposition 7 that AC&T is not stable. However AC&T is Pareto optimal.

Proposition 8. *AC&T is Pareto optimal.*

Proof. Let AC&T generate a matching μ for problem P. Then μ is a non-wasteful matching. Suppose that μ is not Pareto optimal, and that μ' Pareto dominates it.

When AC&T is applied to P, let k be the first round such that some student gets a better match in μ' than in μ. Now $k = 1$ is impossible since all students matched in the first round of AC&T get their best match in P. Hence the students matched in the first round are matched with the same schools in μ and in μ'.

Let S', C' be the sets of agents left in the beginning of the second round, and update capacities and preferences. Let P^1 be the problem corresponding to this situation. Then it must still hold that AC&T applied to P^1 generates a matching μ^1 that is Pareto dominated by a matching μ'^1. Again, it holds that students matched in the first round are matched the same way in μ^1 and μ'^1. Hence $k = 2$ is impossible.

The proof is completed by applying induction on k. $\square$

A drawback of AC&T as compared to ordinary TTC is that it is not strategy proof. AC&T may also match a larger number of students with schools than TTC. This is show in the next example.

Example 2 (Continued). There are no mutually best matches to eliminate. By the definition of AC&T we must apply one round of TTC to S and

C. The only cycle is $s_4 \to c_4 \to s_5 \to c_3 \to s_4$. So the first matches are (s_4, c_4), (s_5, c_3) and these agents are removed. In the second round (s_2, c_1) is a mutually best match. In the third round (s_3, c_2) is a mutually best match. In the fourth round (s_1, c_1) is a mutually best match. So the matching μ^* generated by AC&T is

$$\mu^* = \{(s_1, c_1), (s_2, c_1), (s_3, c_2), (s_4, c_4), (s_5, c_3)\}.$$

Suppose s_1 reports that c_2 is her best school. Applying AC&T, we get first a cycle $s_1 \to c_2 \to s_2 \to c_1 \to s_1$. Hence s_1 will be matched with c_2, and therefore AC&T is not strategy proof.

Note that the ordinary TTC with true preferences generates the same matching μ' as AC&T with the false reporting by s_1:

$$\mu' = \{(s_1, c_2), (s_2, c_1), (s_3, c_1), (s_4, c_4), (s_5, c_3)\}.$$

Now change the preferences of student s_3 so that she accepts only c_2. That will not change the matching μ^* produced by AC&T when all students report true preferences. But TTC will now produce the matching

$$\mu = \{(s_1, c_2), (s_2, c_1), (s_4, c_4), (s_5, c_3)\},$$

so in this case AC&T matches more students with schools than TTC. $\triangleleft$

We saw from the last example that AC&T might produce a different matching compared to TTC when capacities of the schools are larger than one. However, we can guarantee that TTC = AC&T in all college admissions problems when we transform it to a *related marriage market* (see Roth and Sotomayor 1992, p 131). That is, whenever a school c has a capacity larger than one we create $|q_c|$ copies of the school, each copy maintaining the preferences of the original c with capacity one, and replace c by the string $c^1, \ldots c^{|q_c|}$ on preferences of the students. We assume that student s strictly prefers lower number indexed copies of school c to higher indexed copies. Thus for student s with preferences $\prec_s$: cc' we form new cloned preferences as $\prec_s$: $c^1 \ldots c^i c'^1 \ldots c'^j$, where $|q_c| = i$, $|q_{c'}| = j$, and $i, j \geq 1$. We can now present our final result.

Corollary 2. *For every college admissions problem* $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$ *transformed to a related marriage market, we have AC&T=TTC.*

Proof. As a related marriage market only requires us to make assumptions about preferences of the students and now for all schools we have a quota of one, it immediately follows from Proposition 7 that AC&T = TTC. □

There are of course many ways to solve the deadlock when IMB terminates producing a wasteful matching. Indeed, Morrill (2015) introduces more complex algorithms similar to AC&T which guarantee e.g. strategy proofness.

In the first version of this paper we defined IMB in such a way that the preferences $\prec_s$ ($\prec_c$) are defined over those schools (students) only that accept $s \in S$ ($c \in C$). While IMB produces a non-wasteful matching more often than with the specification used in this paper, the matching produced may be wasteful as was shown in Example 6. All the results that do not involve comparison between TTC and IMB hold with minor modifications in the proofs, but for example Corollary 1 and Proposition 7 fail when the alternative definition of IMB is applied. The reason is that under TTC, any school c names the best student s still in the market, but it is not necessary that s finds c acceptable. We are grateful to an anonymous referee for pointing out this fact.

In this section we tried to explore a reasonable way of maintaining the nice properties of IMB when it produces a wasteful matching. However, this seems a hard task to achieve without a tentative proposal construction a la deferred acceptance and demonstrates how carefully pairing has to be done when no mutually best pairs can be found. On a positive note AC&T, a simple fix on the IMB, can produce a larger matching than the TTC.

5. Conclusions

In this paper we have introduced an algorithm based on iterative formation of mutually best matches (IMB). When IMB produces a non-wasteful matching, this matching is the unique element of the core, and so IMB produces the same matching as the student proposing and school proposing deferred acceptance algorithms. Further, this matching satisfies strategy proofness. We suggest that IMB could be used as a first trial to find a "good" matching. If IMB produces a wasteful matching, then either it could be amended in some way like in Morrill (2015), or one could use his or her favorite algorithm.

With some tractable restrictions on preferences, such as single peakedness or single crossing, IMB produces a non-wasteful matching, and the top trading cycles algorithm produces the same matching as IMB. These kinds of assumptions about preferences are well-known and widely used in social choice literature.

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