INEQUALITIES FOR THE GENERALIZED TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

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ABSTRACT. The generalized trigonometric functions occur as an eigenfunction of the Dirichlet problem for the one-dimensional $p-$Laplacian. The generalized hyperbolic functions are defined similarly. Some classical inequalities for trigonometric and hyperbolic functions, such as Mitrinović-Adamović inequality, Lazarević's inequality, Huygens-type inequalities, Wilker-type inequalities, and Cuza-Huygens-type inequalities, are generalized to the case of generalized functions.

KEYWORDS. Generalized trigonometric functions, generalized hyperbolic functions, Mitrinović-Adamović inequality, Lazarević's inequality, Huygens-type inequalities, Wilker-type inequalities, and Cuza-Huygens-type inequalities

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1. INTRODUCTION

It is well known from basic calculus that

$$\arcsin(x) = \int_0^x \frac{1}{(1-t^2)^{1/2}}dt, \quad 0 \le x \le 1,$$

and

$$\frac{\pi}{2} = \arcsin(1) = \int_0^1 \frac{1}{(1-t^2)^{1/2}}dt.$$

We can define the function sin on $[0, \pi/2]$ as the inverse of arcsin and extend it on $(-\infty, \infty)$.

Let $1 < p < \infty$. We can generalize the above functions as follows:

$$\mathrm{arcsin}_p(x) \equiv \int_0^x \frac{1}{(1-t^p)^{1/p}}dt, \quad 0 \le x \le 1,$$

and

$$\frac{\pi_p}{2} = \mathrm{arcsin}_p(1) \equiv \int_0^1 \frac{1}{(1-t^p)^{1/p}}dt.$$

The inverse of arcsin_p on $[0, \pi_p/2]$ is called the *generalized sine function* and denoted by $\sin_p$. By standard extension procedures as the sine function we get a differentiable function on the whole of $(-\infty, \infty)$ which coincides with sin when $p = 2$. It is easy to see that the function $\sin_p$ is strictly increasing and concave on $[0, \pi_p/2]$. In the same way we can define the generalized cosine function, the generalized tangent function, and their inverses, and also the corresponding hyperbolic functions.

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The generalized sine function $\sin_p$ occurs as an eigenfunction of the Dirichlet problem for the one-dimensional $p-$Laplacian. There are several different definitions for these generalized trigonometric and hyperbolic functions [LE, L1, L2, LP1]. Recently, these functions have been studied very extensively (see [BV2, BE, EGL, LE, L1, L2, LP1, LP2]). In particular, the reader is referred to [L1, L2, LP1, LP2]. These generalized functions are similar to the classical functions in various aspects. Some of these functions can be expressed in terms of the Gaussian hypergeometric series (see [BV1, BV2]).

In this paper we will generalize some classical inequalities for trigonometric and hyperbolic functions, such as *Mitrinović-Adamović inequality* (Theorem 3.6), *Lazarević's inequality* (Theorem 3.8), *Huygens-type inequalities* (Theorem 3.13 and Theorem 3.16), *Wilker-type inequality* (Corollary 3.19), and *Cuza-Huygens-type inequalities* (Theorem 3.22 and Theorem 3.24) to the case of generalized functions. For the classical cases, these inequalities have been extended and sharpened extensively (see the very recent survey [AVZ]).

2. Definitions and formulas

In this section we define the generalized cosine function, the generalized tangent function, and their inverses, and also the corresponding hyperbolic functions.

The *generalized cosine function* $\cos_p$ is defined as

$$\cos_p(x) \equiv \frac{d}{dx}\sin_p(x).$$

It is clear from the definitions that

$$\cos_p(x) = (1-\sin_p(x)^p)^{1/p}, \qquad x \in [0,\pi_p/2],$$

and

$$|\sin_p(x)|^p + |\cos_p(x)|^p = 1, \qquad x \in \mathbb{R}. \tag{2.1}$$

It is easy to see that

$$\frac{d}{dx}\cos_p(x) = -\cos_p(x)^{2-p}\sin_p(x)^{p-1}, \qquad x \in [0,\pi_p/2].$$

The *generalized tangent function* is defined as in the classical case:

$$\tan_p(x) \equiv \frac{\sin_p(x)}{\cos_p(x)}, \qquad x \in \mathbb{R}\setminus\{k\pi_p + \frac{\pi_p}{2} : k \in \mathbb{Z}\}.$$

It follows from (2.1) that

$$\frac{d}{dx}\tan_p(x) = 1 + |\tan_p(x)|^p, \quad x \in (-\pi_p/2, \pi_p/2).$$

Similarly, the *generalized inverse hyperbolic sine function*

$$\mathrm{arcsinh}_p(x) \equiv \begin{cases} \int_0^x \frac{1}{(1+t^p)^{1/p}}dt, & x \in [0,\infty), \\ -\mathrm{arcsinh}_p(-x), & x \in (-\infty,0) \end{cases}$$

generalizes the classical inverse hyperbolic sine function. The inverse of arcsinh_p is called the *generalized hyperbolic sine function* and denoted by $\sinh_p$. The *generalized hyperbolic cosine function* is defined as

$$\cosh_p(x) \equiv \frac{d}{dx}\sinh_p(x).$$

The definitions show that

$$\cosh_p(x)^p - |\sinh_p(x)|^p = 1, \qquad x \in \mathbb{R},$$

and

$$\frac{d}{dx}\cosh_p(x) = \cosh_p(x)^{2-p}\sinh_p(x)^{p-1}, \qquad x \geq 0.$$

The *generalized hyperbolic tangent function* is defined as

$$\tanh_p(x) \equiv \frac{\sinh_p(x)}{\cosh_p(x)},$$

and hence we have

$$\frac{d}{dx}\tanh_p(x) = 1 - |\tanh_p(x)|^p.$$

It is clear that all these generalized functions coincide with the classical ones when $p = 2$.

3. Inequalities

The l'Hôpital Monotone Rule (LMR), Lemma 3.1, is the key tool in proofs of our generalizations.

3.1. **Lemma.** [AVV] (l'Hôpital Monotone Rule). Let $-\infty < a < b < \infty$, and let $f, g : [a, b] \to \mathbb{R}$ be continuous functions that are differentiable on (a, b), with $f(a) = g(a) = 0$ or $f(b) = g(b) = 0$. Assume that $g'(x) \neq 0$ for each $x \in (a, b)$. If f'/g' is increasing (decreasing) on (a, b), then so is f/g.

Some other applications of the l'Hôpital Monotone Rule (LMR) in special functions one is referred to the survey [AVZ].

3.2. **Lemma.** *For $p > 2$, the function $f(x) \equiv \tan_p(x)^{p-2} - \tanh_p(x)^{p-2}$ is strictly increasing in $(0, \pi_p/2)$.*

Proof. By differentiation, we have

$$f'(x) = (p-2)(\tan_p(x)^{p-3}(1 + \tan_p(x)^p) - \tanh_p(x)^{p-3}(1 - \tanh_p(x)^p)).$$

For $p \geq 3$,

$$f'(x) \geq (p-2)(\tan_p(x)^{p-3} - \tanh_p(x)^{p-3}) > 0,$$

since $\tan_p(x) > \tanh_p(x)$.

By the identities $\sin_p(x)^p + \cos_p(x)^p = 1$ and $\cosh_p(x)^p - \sinh_p(x)^p = 1$,

$$\begin{aligned} f'(x) &= (p-2)\left(\frac{\sin_p(x)^{p-3}}{\cos_p(x)^{2p-3}} - \frac{\sinh_p(x)^{p-3}}{\cosh_p(x)^{2p-3}}\right) \\ &\geq (p-2)\sinh_p(x)^{p-3}\left(\frac{1}{\cos_p(x)^{2p-3}} - \frac{1}{\cosh_p(x)^{2p-3}}\right) > 0 \end{aligned}$$

for $p \in [2,3)$ since $\sin_p(x) < \sinh_p(x)$. This completes the proof. $\square$

3.3. **Lemma.** *For $p > 1$, the function $f(x) \equiv \cos_p(x)\cosh_p(x)$ is strictly decreasing from $(0, \pi_p/2)$ onto $(0,1)$. In particular, for all $p \in (1,\infty)$ and $x \in (0,\pi_p/2)$,*

$$\cos_p(x) < \frac{1}{\cosh_p(x)}.$$

Proof. After simple computations we get

$$f'(x) = \cos_p(x)\cosh_p(x)(\tanh_p(x)^{p-1} - \tan_p(x)^{p-1}) < 0,$$

which implies that f is strictly decreasing, and hence $\cos_p(x)\cosh_p(x) < 1$. $\square$

3.4. **Theorem.** *For $p \in [2,\infty)$ and $x \in (0,\pi_p/2)$,*

$$\frac{\sin_p(x)}{x} < \frac{x}{\sinh_p(x)}. \tag{3.5}$$

Proof. Let $f_1(x) \equiv \sin_p(x)\sinh_p(x)$, $f_2(x) \equiv x^2$ and $f_1(0) = f_2(0) = 0$. By simple computations, we have

$$\frac{f_1''(x)}{f_2''(x)} = \cos_p(x)\cosh_p(x) - \frac{1}{2}\sin_p(x)\sinh_p(x)(\tan_p(x)^{p-2} - \tanh_p(x)^{p-2})$$

which is strictly decreasing for any $p \geq 2$ by Lemma 3.2 and Lemma 3.3. Hence the monotonicity of $f_1(x)/f_2(x)$ follows from the l'Hôpital Monotone Rule, and this implies

$$\frac{\sin_p(x)\sinh_p(x)}{x^2} < 1.$$

$\square$

The next two theorems generalize the *Mitrinović-Adamović inequality* and *Lazarević's inequality* (see [M]). For the classical case of Theorem 3.8 also see [LWC].

3.6. **Theorem.** *For $p \in (1,\infty)$, the function*

$$f(x) \equiv \frac{\log(\sin_p(x)/x)}{\log\cos_p(x)}$$

is strictly decreasing from $(0,\pi_p/2)$ onto $(0, 1/(1+p))$. In particular, for all $p \in (1,\infty)$ and $x \in (0,\pi_p/2)$,

$$\cos_p(x)^\alpha < \frac{\sin_p(x)}{x} < 1 \tag{3.7}$$

with the best constant $\alpha = 1/(1+p)$.

Proof. Write $f_1(x) \equiv \log(\sin_p(x)/x)$ and $f_2(x) \equiv \log \cos_p(x)$. Then $f_1(0) = f_2(0) = 0$ and, by simple computations,

$$\frac{f_1'(x)}{f_2'(x)} = \frac{\tan_p(x) - x}{x \tan_p(x)^p} = \frac{f_{11}(x)}{f_{22}(x)},$$

with $f_{11}(x) \equiv \tan_p(x) - x$, $f_{22}(x) \equiv x \tan_p(x)^p$, and $f_{11}(0) = f_{22}(0) = 0$.

$$\frac{f_{11}'(x)}{f_{22}'(x)} = \frac{1}{1 + p\, g(x)}$$

with

$$g(x) \equiv \frac{x}{\sin_p(x)} \frac{1}{\cos_p(x)^{p-1}}$$

which is strictly increasing. By the l'Hôpital Monotone Rule we see that $f(x)$ is strictly decreasing. The limiting values follow from l'Hôpital's Rule easily. $\square$

3.8. **Theorem.** *For $p \in (1, \infty)$, the function*

$$f(x) \equiv \frac{\log(\sinh_p(x)/x)}{\log \cosh_p(x)}$$

is strictly increasing from $(0, \infty)$ onto $(1/(1+p), 1)$. In particular, for all $p \in (1, \infty)$ and $x \in (0, \infty)$,

$$\cosh_p(x)^{\alpha} < \frac{\sinh_p(x)}{x} < \cosh_p(x)^{\beta} \tag{3.9}$$

with the best constants $\alpha = 1/(1 + p)$ and $\beta = 1$.

Proof. Write $f_1(x) \equiv \log(\sinh_p(x)/x)$ and $f_2(x) \equiv \log \cosh_p(x)$. Then $f_1(0) = f_2(0) = 0$ and, by simple computations,

$$\frac{f_1'(x)}{f_2'(x)} = \frac{x - \tanh_p(x)}{x \tanh_p(x)^p} = \frac{f_{11}(x)}{f_{22}(x)},$$

with $f_{11}(x) \equiv x - \tanh_p(x)$, $f_{22}(x) \equiv x \tanh_p(x)^p$, and $f_{11}(0) = f_{22}(0) = 0$.

$$\frac{f_{11}'(x)}{f_{22}'(x)} = \frac{1}{1 + p\, g(x)}$$

with

$$g(x) \equiv \frac{x}{\sinh_p(x)} \frac{1}{\cosh_p(x)^{p-1}}$$

which is strictly decreasing. By the l'Hôpital Monotone Rule we see that $f(x)$ is strictly increasing. The limiting values follow from l'Hôpital's Rule easily. $\square$

3.10. **Corollary.** *For all $p \in [2, \infty)$ and $x \in (0, \pi_p/2)$,*

$$\left(\frac{x}{\sinh_p(x)}\right)^{1+p} < \frac{1}{\cosh_p(x)} < \frac{\tanh_p(x)}{x} < \frac{\sin_p(x)}{x} < \frac{x}{\sinh_p(x)}. \tag{3.11}$$

Proof. The first inequality of (3.11) follows by the left side of (3.9). The second inequality follows by $\sinh_p(x)/x > 1$, while the third by $\sin_p(x) > \tanh_p(x)$. The last inequality is the inequality (3.5) $\square$

3.12. **Conjecture.** *For $p \in [2, \infty)$, the function*

$$f(x) \equiv \frac{\log(x/\sin_p(x))}{\log(\sinh_p(x)/x)}$$

is strictly increasing in $(0, \pi_p/2)$.

Next two theorems show the *Huygens-type inequalities* for the generalized trigonometric and hyperbolic functions.

3.13. **Theorem.** *Let $p > 1$. Then the following inequalities hold*

$$(p+1)\frac{\sin_p(x)}{x} + \frac{1}{\cos_p(x)} > p+2 \quad for \quad x \in (0, \pi_p/2), \tag{3.14}$$

and

$$(p+1)\frac{\sinh_p(x)}{x} + \frac{1}{\cosh_p(x)} > p+2 \quad for \quad x > 0. \tag{3.15}$$

Proof. The well-known weighted arithmetic-geometric inequality states that

$$ta + (1-t)b > a^t b^{1-t},$$

for $a, b > 0$, $a \neq b$, and $0 < t < 1$. Putting $t = (p+1)/(p+2)$, $a = \sin_p(x)/x$, and $b = 1/\cos_p(x)$, and combining the left side of (3.7), we have

$$(p+1)\frac{\sin_p(x)}{x} + \frac{1}{\cos_p(x)} > (p+2)\left(\frac{\sin_p(x)}{x}\right)^{(p+1)/(p+2)}\left(\frac{1}{\cos_p(x)}\right)^{1/(p+2)} > p+2.$$

Similarly, the inequality (3.15) follows from the left side of (3.9). $\square$

3.16. **Theorem.** *For $p > 1$, the following inequalities hold*

$$\frac{p\sin_p(x)}{x} + \frac{\tan_p(x)}{x} > 1+p, \quad 0 < x < \frac{\pi_p}{2}, \tag{3.17}$$

and

$$\frac{p\sinh_p(x)}{x} + \frac{\tanh_p(x)}{x} > 1+p, \quad x > 0. \tag{3.18}$$

Proof. Let $f(x) \equiv p\sin_p(x) + \tan_p(x) - (1+p)x$. After some elementary computations, we get

$$f'(x) = p\cos_p(x) + \tan_p(x)^p - p$$

and

$$f''(x) = p\tan_p(x)^{p-1}(1 - \cos_p(x) + \tan_p(x)^p) > 0,$$

which implies that $f'(x) > 0$ and f is strictly increasing. Hence we have $f(x) > 0$, and the inequality (3.17) follows.

Similarly, put $g(x) \equiv p\sinh_p(x) + \tanh_p(x) - (1+p)x$. We have

$$g'(x) = p\cosh_p(x) - \tanh_p(x)^p - p$$

and

$$g''(x) = p\tanh_p(x)^{p-1}(\tanh_p(x)^p + \cosh_p(x) - 1) > 0,$$

from which we get $g'(x) > 0$, implying $g(x) > 0$. This finishes the proof. $\square$

3.19. **Corollary.** *For $p > 1$ and $x > 0$,*

$$\left(\frac{\sinh_p(x)}{x}\right)^p + \frac{\tanh_p(x)}{x} > 2. \tag{3.20}$$

Proof. The well-known Bernoulli inequality states that, for $a > 1$ and $t > 0$,

$$(1+t)^a > 1 + at. \tag{3.21}$$

Setting $t = \sinh_p(x)/x - 1$ and $a = p$ in (3.21), and then combining the inequality (3.18), we have

$$\left(\frac{\sinh_p(x)}{x}\right)^p > 1 + p\left(\frac{\sinh_p(x)}{x} - 1\right) > 2 - \frac{\tanh_p(x)}{x},$$

which implies (3.20). $\square$

The inequality (3.20) is the so-called *Wilker's inequality*. The following Theorem 3.22 and 3.24 present the famous *Cusa-Huygens-type inequalities* for the generalized trigonometric and hyperbolic functions, respectively.

3.22. **Theorem.** *For $p \in (1,2]$, the following inequalities*

$$\frac{\sin_p(x)}{x} < \frac{\cos_p(x) + p}{1+p} \leq \frac{\cos_p(x) + 2}{3} \tag{3.23}$$

hold for all $x \in (0, \pi_p/2]$.

Proof. Let $f(x) \equiv x\cos_p(x) + px - (1+p)\sin_p(x)$. By differentiation, we have

$$f'(x) = -\cos_p(x)(x\tan_p(x)^{p-1} + p) + p \equiv -g(x) + p,$$

and

$$g'(x) = \cos_p(x)\tan_p(x)^{p-2}\left((p-1)(x - \tan_p(x)) + (p-2)x\tan_p(x)^p\right) < 0,$$

which implies $g(x) < g(0) = p$ and $f'(x) > 0$. Hence $f(x)$ is strictly increasing and $f(x) > f(0) = 0$ which implies the inequality (3.23) .

The second inequality in (3.23) is clear since $\cos_p(x) \leq 1$. $\square$

3.24. **Theorem.** *For all $x > 0$,*

$$\frac{\sinh_p(x)}{x} < \frac{\cosh_p(x) + p}{1+p}, \quad if \quad p \in (1,2], \tag{3.25}$$

and

$$\frac{\sinh_p(x)}{x} < \frac{\cosh_p(x) + 2}{3}, \quad if \quad p \in [2,\infty). \tag{3.26}$$

Proof. Let $f(x) \equiv x\cosh_p(x) + px - (1+p)\sinh_p(x)$. By differentiation, we have

$$f'(x) = \cosh_p(x)(x\tanh_p(x)^{p-1} - p) + p$$

and

$$f''(x) = \cosh_p(x)\tanh_p(x)^{p-2}\left((p-1)(x - \tanh_p(x)) + (2-p)x\tanh_p(x)^p\right) > 0,$$

which implies $f'(x) > 0$. Hence $f(x)$ is strictly increasing, and $f(x) > f(0) = 0$ which implies the inequality (3.25) .

For the inequality (3.26), let $h(x) \equiv x\cosh_p(x) + 2x - 3\sinh_p(x)$. By differentiation, we get

$$h'(x) = \cosh_p(x)\left(x\tanh_p(x)^{p-1} - 2\right) + 2$$

and

$$\begin{aligned} h''(x) &= \cosh_p(x)\tanh_p(x)^{p-2}\left(x\tanh_p(x)^p - \tanh_p(x) + (p-1)x(1 - \tanh_p(x)^p)\right) \\ &\geq \cosh_p(x)\tanh_p(x)^{p-2}\left(x\tanh_p(x)^p - \tanh_p(x) + x(1 - \tanh_p(x)^p)\right) \\ &= \cosh_p(x)\tanh_p(x)^{p-2}\left(x - \tanh_p(x)\right) > 0, \end{aligned}$$

which implies $h'(x) > h'(0) = 0$, and hence $h(x)$ is strictly increasing and $h(x) > h(0) = 0$. This implies the inequality (3.26). $\square$

3.27. **Theorem.** *For $p \in [2, \infty)$ and $x \in (0, \pi_p/2)$,*

$$\frac{\sinh_p(x)}{x} < \frac{3}{2 + \cos_p(x)}. \tag{3.28}$$

Proof. Let

$$f(x) \equiv 3x - 2\sinh_p(x) - \sinh_p(x)\cos_p(x).$$

Simple computations give

$$\begin{aligned} f'(x) &= 3 - 2\cosh_p(x) - \cosh_p(x)\cos_p(x) + \sinh_p(x)\sin_p(x)^{p-1}\cos_p(x)^{2-p} \\ &\geq 3 - 2\cosh_p(x) - \cosh_p(x)\cos_p(x) + \sinh_p(x)\sin_p(x)^{p-1} \\ &\equiv g(x) \end{aligned}$$

and

$$\begin{aligned} g'(x) &= -2\cosh_p(x)\tanh_p(x)^{p-1} - \sinh_p(x)\cos_p(x)\tanh_p(x)^{p-2} \\ &\quad + \cosh_p(x)\sin_p(x)^{p-1}\cos_p(x)^{2-p} + \cosh_p(x)\sin_p(x)^{p-1} \\ &\quad + (p-1)\sinh_p(x)\cos_p(x)\sin_p(x)^{p-2} \\ &\geq 2\cosh_p(x)(\sin_p(x)^{p-1} - \tanh_p(x)^{p-1}) \\ &\quad + \sinh_p(x)\cos_p(x)(\sin_p(x)^{p-2} - \tanh_p(x)^{p-2}) \\ &\geq 0, \end{aligned}$$

where the last inequality follows from $\sin_p(x) > \tanh_p(x)$. Now it is easy to see that $f(x) > f(0) = 0$ which implies the inequality (3.28). $\square$

3.29. **Conjecture.** *For* $p \in (2, \infty)$ *and* $x \in (0, \pi_p/2)$,

$$\frac{\sinh_p(x)}{x} < \frac{p+1}{p+\cos_p(x)}. \tag{3.30}$$

3.31. **Theorem.** *For* $p \in [2, \infty)$ *and* $x \in (0, \pi_p/2]$,

$$\frac{\sin_p(x)}{x} > \frac{p-1+\cos_p(x)}{p} \geq \frac{1+\cos_p(x)}{2}.$$

Proof. The second inequality is clear. For the first inequality, put $f(x) \equiv p\sin_p(x) - x\cos_p(x) - (p-1)x$. After some elementary computations, we get

$$f'(x) = (p-1)\cos_p(x) + x\cos_p(x)\tan_p(x)^{p-1} - (p-1),$$

and

$$f''(x) = \cos_p(x)\tan_p(x)^{p-2}g(x),$$

where $g(x) = (p-2)x\tan_p(x)^p - (p-2)\tan_p(x) + (p-1)x$. We have to prove $g(x) > 0$ which follows from

$$g'(x) = p(p-2)x\tan_p(x)^{p-1}(1+\tan_p(x)^p) + 1 > 0.$$

□

3.32. **Lemma.** *For* $p > 1$,
(1) The functions $f_1(x) \equiv \sin_p(x)/x$ *is strictly decreasing from* $(0, \pi_p/2)$ *onto* $(2/\pi_p, 1)$. *In particular, for* $x \in (0,1)$,

$$\frac{x}{\operatorname{arcsin}_p(x)} < \frac{\sin_p(x)}{x} < \frac{2x/\pi_p}{\operatorname{arcsin}_p(2x/\pi_p)}. \tag{3.33}$$

(2) The function $f_2(x) \equiv \tan_p(x)/x$ *is strictly increasing from* $(0, \pi_p/2)$ *onto* $(1, \infty)$. *In particular, for* $x \in (0,k)$,

$$\frac{x}{\operatorname{arctan}_p(x)} < \frac{\tan_p(x)}{x} < \frac{ax}{\operatorname{arctan}_p(ax)}, \tag{3.34}$$

where $0 < k < \pi_p/2$ *and* $a = \tan_p(k)/k$.
(3) The function $f_3(x) \equiv \sinh_p(x)/x$ *is strictly increasing from* $(0, \infty)$ *onto* $(1, \infty)$. *In particular, for* $x \in (0,k)$,

$$\frac{x}{\operatorname{arcsinh}_p(x)} < \frac{\sinh_p(x)}{x} < \frac{bx}{\operatorname{arcsinh}_p(bx)}, \tag{3.35}$$

where $k > 0$ *and* $b = \sinh_p(k)/k$.
(4) The function $f_4(x) \equiv \tanh_p(x)/x$ *is strictly decreasing from* $(0, \infty)$ *onto* $(0,1)$. *In particular, for* $x \in (0,k)$,

$$\frac{x}{\operatorname{arctanh}_p(x)} < \frac{\tanh_p(x)}{x} < \frac{cx}{\operatorname{arctanh}_p(cx)}, \tag{3.36}$$

where $k > 0$ *and* $c = \tanh_p(k)/k$.

Proof. Since the proofs of part (1) to part (4) are similar to each other, we only prove the part (2) here. Since $\tan_p'(x) = 1 + \tan_p(x)^p$ is strictly increasing, the monotone form of l'Hôpital's Rule gives that the function f_2 is strictly increasing. Hence we have

$$1 < \frac{\tan_p(x)}{x} < \frac{\tan_p(k)}{k} = a,$$

and this is equivalent to

$$\operatorname{arctan}_p(x) < x < \operatorname{arctan}_p(ax).$$

By the monotonicity of f_2,

$$\frac{x}{\operatorname{arctan}_p(x)} = \frac{\tan_p(\operatorname{arctan}_p(x))}{\operatorname{arctan}_p(x)} < \frac{\tan_p(x)}{x} < \frac{\tan_p(\operatorname{arctan}_p(ax))}{\operatorname{arctan}_p(ax)} = \frac{ax}{\operatorname{arctan}_p(ax)}$$

$\square$

3.37. **Theorem.** *Let $p > 1$ and $x > 0$. Then*
(1) $f_1(t) \equiv \cos_p(x/t)^t$ is strictly increasing and logarithmic concave in $(2x/\pi_p, \infty)$;
(2) $f_2(t) \equiv \sin_p(x/t)^t$ is strictly decreasing and logarithmic concave in $(2x/\pi_p, \infty)$;
(3) $f_3(t) \equiv \sinh_p(x/t)^t$ is strictly decreasing and logarithmic concave in $(0, \infty)$;
(4) $f_4(t) \equiv \cosh_p(x/t)^t$ is strictly decreasing and logarithmic convex in $(0, \infty)$.

Proof. For part (1), simple computations give

$$\frac{d}{dt}\log f_1(t) = \log\cos_p(s) + s\tan_p(s)^{p-1} \equiv g_1(s), \quad s = \frac{x}{t},$$

and

$$g_1'(s) = (p-1)s\tan_p(s)^{p-2}(1 + \tan_p(s)^p) > 0,$$

which implies that f_1 is logarithmic concave. For the monotonicity of f_1, we write $h(s) \equiv h_1(s)/h_2(s)$, $h_1(s) \equiv -\log\cos_p(s)$ and $h_2(s) \equiv s\tan_p(s)^{p-1}$ with $h_1(0) = h_2(0) = 0$, and

$$\frac{h_1'(s)}{h_2'(s)} = \frac{1}{1 + (p-1)l(s)}, \quad l(s) = \frac{s(1 + \tan_p(s)^p)}{\tan_p(s)} = \frac{l_1(s)}{l_2(s)}.$$

By differentiation, we have

$$\frac{l_1'(s)}{l_2'(s)} = 1 + ps\tan_p(s)^{p-1}$$

which is strictly increasing. Hence $h(s)$ is strictly decreasing by the l'Hôpital Monotone Rule, and $h(s) < h(0) = 1/p$ which is equivalent to $s\tan_p(s)^{p-1} > -p\log\cos_p(s)$. Now it is easy to see that $g_1(s) > (1-p)\log\cos_p(s) > 0$ which implies that f_1 is strictly increasing.

For part (2), it is easy to see that $-t\log(1/\sin_p(x/t))$ is strictly decreasing in t. Simple computations give

$$\frac{d}{dt}t\log\sin_p(x/t) = -\left(\log\frac{1}{\sin_p(s)} + \frac{s}{\tan_p(s)}\right), \quad s = \frac{x}{t},$$

which is strictly increasing in s and hence strictly decreasing in t.

For part (3), by differentiations we have

$$\frac{d}{dt}\log f_3(t) = \log \sinh_p(s) - \frac{s}{\tanh_p(s)} \equiv g_3(s), \quad s = \frac{x}{t},$$

and

$$g_3'(s) = \frac{s(1-\tanh_p(s)^p)}{\tanh_p(s)^2} > 0,$$

which implies that $g_3(s)$ is strictly increasing in s and hence decreasing in t. It follows that $\log f_3(t)$ is concave. Since $g_3(s) \le g_3(\infty) = 0$, f_3 is strictly decreasing.

For part (4), by differentiations we have

$$\frac{d}{dt}\log f_4(t) = \log \cosh_p(s) - s\tanh_p(s)^{p-1} \equiv g_4(s), \quad s = \frac{x}{t},$$

and

$$g_4'(s) = -(p-1)s\tanh_p(s)^{p-2}(1-\tanh_p(s)^p) < 0,$$

which implies that $g(s)$ is strictly decreasing in s and hence increasing in t. It follows that $\log f_4(t)$ is convex. Since $g_4(s) < g_4(0) = 0$, f_4 is strictly decreasing. □

3.38. **Open problem.** Recently, S. Takeuchi [T] has introduced functions depending on two parameters p and q that reduce to the functions studied in the present paper when $p = q$. In [BV1] the authors have continued the study of this family of generalized functions, and have suggested that many properties of classical functions also have a counterpart in this more general setting. It would be natural to generalize the properties of classical trigonometric and hyperbolic functions cited in the survey [AVZ] to the (p, q)-functions of Takeuchi.

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