Metrics and Quasimetrics Induced by Point Pair Function

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Abstract
We study the point pair function in subdomains G of $\mathbb{R}^n$. We prove that, for every domain $G \subsetneq \mathbb{R}^n$, this function is a quasi-metric with the constant less than or equal to $\sqrt{5}/2$. Moreover, we show that it is a metric in the domain $G = \mathbb{R}^n \backslash \{0\}$ with $n \geq 1$. We also consider generalized versions of the point pair function, depending on a parameter $\alpha > 0$, and show that in some domains these generalizations are metrics if and only if $\alpha \leq 12$.

1 Introduction

During the past few decades, several authors have contributed to the study of various metrics important for the geometric function theory. In this field of research, *intrinsic metrics* are the most useful because they measure distances in a way that takes into account not only how close the points are to each other but also how the points are located with respect to the boundary of the domain. These metrics are often used to estimate the hyperbolic metric and, while they share some but not all of its properties,

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Springer

intrinsic metrics are much simpler than the hyperbolic metric and therefore more applicable.

Let G be a proper subdomain of the real n-dimensional Euclidean space $\mathbb{R}^n$. Denote by $|x - z|$ the Euclidean distance in $\mathbb{R}^n$ and by $d_G(x)$ the distance from a point $x \in G$ to the boundary ∂G, i.e. $d_G(x) := \inf\{|x - z| | z \in \partial G\}$. One of the most interesting intrinsic measures of distance in G is the *point pair function* $p_G : G \times G \to [0, 1)$ defined as

$$p_G(x, y) = \frac{|x - y|}{\sqrt{|x - y|^2 + 4d_G(x)d_G(y)}}, \quad x, y \in G. \tag{1.1}$$

This function was first introduced in Chen et al. (2015, p. 685), named in Hariri et al. (2017b, a) and further studied in Hariri et al. (2020), Rainio (2021), Rainio and Vuorinen (2022a, b, c). In Chen et al. (2015, Rmk 3.1, p. 689) it was noted that the function p_G is not a metric when the domain G coincides with the unit disk $\mathbb{B}^2$.

In order to be a metric, a function needs to fulfill certain three properties, the third of which is called the *triangle inequality*. The point pair function has all the other properties of a metric but it only fulfills a relaxed version (2.1) of the original triangle inequality, as explained in Sect. 2. We call such functions *quasi-metrics* and study what is the best constant c such that the generalized inequality (2.1) holds. Namely, it was proven in Rainio (2021, Lemma 3.1, p. 2877) that the point pair function is a quasi-metric on every domain $G \subsetneq \mathbb{R}^n$ with a constant less than or equal to $\sqrt{2}$, but this result is not sharp for any domain G.

For this reason, we continue here the investigations initiated in the paper (Rainio 2021). We give an answer to the question posed in Rainio (2021, Conj. 3.2, p. 2877) by proving in Theorem 4.14 that, for all domains $G \subsetneq \mathbb{R}^n$, the point pair function is a quasi-metric with a constant less than or equal to $\sqrt{5}/2$. For the domain $G = \mathbb{R}^n \backslash \{0\}$ with $n \geq 1$, we prove Theorem 4.6 which states that the point pair function p_G defines a metric. In Lemma 4.17, we explain for which domains the constant $\sqrt{5}/2$ is sharp.

We also investigate what happens when the constant 4 in (1.1) is replaced by another constant $\alpha > 0$ to define a generalized version p_G^α of the point pair function p_G as in (5.1). In particular, we prove that, for $\alpha \in (0, 12]$, this function p_G^α is a metric if G is the positive real axis $\mathbb{R}^+$ (Theorem 5.2), the punctured space $\mathbb{R}^n \backslash \{0\}$ with $n \geq 2$ (Theorem 5.11), or the upper half-space $\mathbb{H}^n$ with $n \geq 2$ (Theorem 5.13). Furthermore, we also show in Theorem 5.15 that the function p_G^α is not a metric for any values of $\alpha > 0$ in the unit ball $\mathbb{B}^n$.

The structure of this article is as follows. In Sect. 2, we give necessary definitions and notations. First, in Sect. 3, we study the point pair function in the 1-dimensional case and then consider the general n-dimensional case in Sect. 4. In Sect. 5, we inspect the generalized version p_G^α of the point pair function in several domains. At last, in Sect. 6, we state some open problems.

2 Preliminaries

In this section, we introduce some notation and recall a few necessary definitions related to metrics.

We will denote by $[x, y]$ the Euclidean line segment between two distinct points $x, y \in \mathbb{R}^n$. For every $x \in \mathbb{R}^n$ and $r > 0$, $B^n(x, r)$ is the x-centered open ball of radius r, and $S^{n-1}(x, r)$ is its boundary sphere. If $x = 0$ and $r = 1$ here, we simply write $\mathbb{B}^n$ instead of $B^n(0, 1)$. Let $\mathbb{H}^n$ denote the upper-half space $\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n | x_n > 0\}$. Furthermore, hyperbolic sine, cosine and tangent are denoted as sh, ch and th, respectively, and their inverse functions are arsh, arch, and arth.

For a non-empty set G, a *metric* on G is a function $d : G \times G \to [0, \infty)$ such that for all $x, y, z \in G$ the   following three properties hold:

(1) Positivity: $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$,
(2) Symmetry: $d(x, y) = d(y, x)$,
(3) Triangle inequality: $d(x, y) \leq d(x, z) + d(z, y)$.

If a function $d : G \times G \to [0, \infty)$ satisfies (1), (2) and the inequality

$$d(x, y) \leq c(d(x, z) + d(z, y)) \tag{2.1}$$

for all $x, y, z \in G$ with some constant $c \geq 1$ independent of the points x, y, z, then the function d is a *quasi-metric* (Paluszyński and Stempak 2009, p. 4307; Stempak 2015, p. 603; Xia 2009, Def. 2.1, p. 453). Note that this term "quasi-metric" has slightly different meanings in some works, see for instance (Bachar and Mâagli 2009; Castro-Company et al. 2014; Sarwar et al. 2014).

The point pair function defined in (1.1) is a metric for some domains $G \subsetneq \mathbb{R}^n$ and a quasi-metric for other domains (Rainio 2021, Lemma 3.1, p. 2877). Note that *the triangular ratio metric*

$$s_G(x, y) = \frac{|x - y|}{\inf_{z \in \partial G}(|x - z| + |z - y|)},$$

introduced by Hästö (2002), is a metric for all domains $G \subsetneq \mathbb{R}^n$ (Hästö 2002, Lemma 6.1, p. 53; Chen et al. 2015, p. 683) and, because of the equality $p_{\mathbb{H}^n}(x, y) = s_{\mathbb{H}^n}(x, y)$ (Hariri et al. 2020, p. 460), the point pair function is a metric on $\mathbb{H}^n$. However, the point pair function is not a metric for either the unit ball (Chen et al. 2015, Rmk 3.1, p. 689) or a two-dimensional sector with central angle $\theta \in (0, \pi)$ (Rainio 2021, p. 2877).

3 The Point Pair Function in the One-Dimensional Case

In this section, we prove that, for every 1-dimensional domain G, the point pair function p_G is either a metric or a quasi-metric with the sharp constant $\sqrt{5}/2$, depending on the number of the boundary points of G (Corollary 3.8).

To do this, we need to establish the following lemma which is also required for the proof of another important result, Theorem 3.6.

Lemma 3.1 *Let the function* $f : [-1, 0] \times [0, 1] \to \mathbb{R}$ *be defined as*

$$f(x, y) = \frac{y - x}{\sqrt{(y - x)^2 + 4(1 + x)(1 - y)}} \quad \text{if } x \neq y, \quad f(x, y) = 0 \quad \text{otherwise,}$$

and, the function $g : [0, 1] \times [0, 1] \to \mathbb{R}$ *be defined as*

$$g(x, y) = \frac{y - x}{2 - x - y} \quad \text{if } x \neq y, \quad g(x, y) = 0 \quad \text{otherwise.}$$

Then, for all $-1 \leq x \leq 0 \leq z \leq y \leq 1$, *the inequality*

$$f(x, z) + g(z, y) \geq \frac{2}{\sqrt{5}} f(x, y) \tag{3.2}$$

holds. Furthermore, the equality in (3.2) *takes place if and only if either* $x = y = z = 0$ *or* $x = -1/3$, $z = 0$ *and* $y = 1/3$.

Proof (I) First we will investigate the function

$$F(x, z, y) = f(x, z) + g(z, y) - \frac{2}{\sqrt{5}} f(x, y)$$

in the domain $D = \{(x, z, y) \in \mathbb{R}^3 : -1 < x < 0 < z < y < 1\}$.

By differentiation, we obtain

$$\frac{\partial f(x, y)}{\partial x} = -\frac{2(1 - y)(2 + x + y)}{(\sqrt{(y - x)^2 + 4(1 + x)(1 - y)})^3},$$

$$\frac{\partial f(x, y)}{\partial y} = \frac{2(1 + x)(2 - x - y)}{(\sqrt{(y - x)^2 + 4(1 + x)(1 - y)})^3},$$

$$\frac{\partial g(x, y)}{\partial x} = -\frac{2(1 - y)}{(2 - x - y)^2}, \quad \frac{\partial g(x, y)}{\partial y} = \frac{2(1 - x)}{(2 - x - y)^2}.$$

Denote

$$A = \sqrt{(z - x)^2 + 4(1 + x)(1 - z)}, \quad B = \sqrt{(y - x)^2 + 4(1 + x)(1 - y)}.$$

Then

$$\frac{\partial F(x, z, y)}{\partial x} = \frac{\partial f(x, z)}{\partial x} - \frac{2}{\sqrt{5}} \frac{\partial f(x, y)}{\partial x} = -\frac{2(1 - z)(2 + x + z)}{A^3}$$
$$+ \frac{2}{\sqrt{5}} \frac{2(1 - y)(2 + x + y)}{B^3},$$

$$\frac{\partial F(x, z, y)}{\partial y} = \frac{\partial g(z, y)}{\partial y} - \frac{2}{\sqrt{5}} \frac{\partial f(x, y)}{\partial y} = \frac{2(1 - z)}{(2 - z - y)^2} - \frac{2}{\sqrt{5}} \frac{2(1 + x)(2 - x - y)}{B^3},$$

$$\frac{\partial F(x, z, y)}{\partial z} = \frac{\partial f(x, z)}{\partial z} + \frac{\partial g(z, y)}{\partial z} = \frac{2(1 + x)(2 - x - z)}{A^3} - \frac{2(1 - y)}{(2 - z - y)^2}.$$

At every critical point of $F(x, y, z)$, we have $\nabla F(x, y, z) = 0$ and, therefore,

$$-\frac{2(1 - z)(2 + x + z)}{A^3} + \frac{2}{\sqrt{5}}\frac{2(1 - y)(2 + x + y)}{B^3} = 0,$$
$$\frac{2(1 - z)}{(2 - z - y)^2} - \frac{2}{\sqrt{5}}\frac{2(1 + x)(2 - x - y)}{B^3} = 0,$$
$$\frac{2(1 + x)(2 - x - z)}{A^3} - \frac{2(1 - y)}{(2 - z - y)^2} = 0. \tag{3.3}$$

From the two latter equalities above, we can deduce that

$$A^3 = \frac{(1 + x)(2 - x - z)(2 - z - y)^2}{1 - y},$$
$$B^3 = \frac{2}{\sqrt{5}}\frac{(1 + x)(2 - x - y)(2 - z - y)^2}{1 - z}.$$

By combining these expressions of A and B with the equality (3.3), we have

$$\frac{A^3}{B^3} = \frac{\sqrt{5}}{2}\frac{(2 - x - z)(1 - z)}{(2 - x - y)(1 - y)} = \frac{\sqrt{5}}{2}\frac{(1 - z)(2 + x + z)}{(1 - y)(2 + x + y)}$$

and, consequently, $x + y = x + z$. This implies that $y = z$ and we see that $F(x, z, y)$ has no extrema in the domain D.

(II) Now, let us investigate the case where (x, z, y) is a boundary point of the aforementioned domain D. If $x = -1$, $x = 0$, $z = y$ or $y = 1$, then evidently $F(x, y, z) \geq 0$. Thus, we only have to consider the case $z = 0$. Without loss of generality, we can assume that $x > -1$ and $y < 1$.

Since

$$f(x, 0) = \frac{-x}{2 + x}, \quad g(0, y) = \frac{y}{2 - y},$$

the inequality (3.2) in the case $z = 0$ is equivalent to the inequality

$$\frac{-x}{2 + x} + \frac{y}{2 - y} \geq \frac{2}{\sqrt{5}}\frac{y - x}{\sqrt{(y - x)^2 + 4(1 + x)(1 - y)}}, \quad -1 < x < 0 < y < 1. \tag{3.4}$$

By denoting $s = -x/(2 + x)$ and $t = y/(2 - y)$ for $0 \le s, t \le 1$, we have

$$x = \frac{-2s}{1+s}, \quad y = \frac{2t}{1+t}, \quad 1+x = \frac{1-s}{1+s}, \quad 1-y = \frac{1-t}{1+t},$$

$$y - x = \frac{2(s+t+2st)}{(1+s)(1+t)},$$

$$(y-x)^2 + 4(1+x)(1-y) = 4\frac{(s+t+2st)^2 + (1-s^2)(1-t^2)}{(1+s)^2(1+t)^2},$$

and (3.2) is equivalent to

$$s+t \ge \frac{2}{\sqrt{5}} \frac{s+t+2st}{\sqrt{(s+t+2st)^2 + (1-s^2)(1-t^2)}}, \quad 0 < s, t < 1.$$

This can also be written as

$$(s+t+2st)^2 + (1-s^2)(1-t^2) \ge \frac{4}{5}\left(\frac{s+t+2st}{s+t}\right)^2. \tag{3.5}$$

Now, let $u = s + t$ and $v = st$. Then $0 < 4v \le u^2 < 4$, and we can write (3.5) in the form

$$(u+2v)^2 + 1 - u^2 + 2v + v^2 \ge \frac{4}{5}\left(1 + \frac{2v}{u}\right)^2.$$

After simple transformations, we obtain

$$5v + 4u + 2 + \frac{1}{5v} \ge \frac{16}{5}\left(\frac{1}{u} + \frac{v}{u^2}\right).$$

Since $v/u^2 \le 1/4$, we only need to prove that

$$5v + 4u + \frac{6}{5} + \frac{1}{5v} \ge \frac{16}{5}\frac{1}{u} \quad \Leftrightarrow \quad 5uv + 4u^2 + \frac{6}{5}u + \frac{u}{5v} \ge \frac{16}{5}.$$

Because $u \ge 2\sqrt{v}$, it is sufficient to establish that

$$10v^{3/2} + 16v + \frac{12}{5}v^{1/2} + \frac{2}{5v^{1/2}} \ge \frac{16}{5}.$$

By denoting $v^{1/2} = \zeta$, we can write the last inequality in the form

$$h(\zeta) \equiv 10\zeta^4 + 16\zeta^3 + \frac{12}{5}\zeta^2 - \frac{16}{5}\zeta + \frac{2}{5} \ge 0.$$

It is easy to see that $h''(\zeta) \geq 0$ for $\zeta \geq 0$, therefore h is convex for positive ζ. Consequently,

$$h(\zeta) = 10(\zeta - 1/5)^2 \left((\zeta - 1/5)^2 + \frac{12}{5}(\zeta - 1/5) + \frac{36}{25} \right).$$

Since $h(1/5) = h'(1/5) = 0$, the convexity of h implies $h(\zeta) \geq 0$ for $\zeta \geq 0$. Thus, the inequality (3.4) is proven. Furthermore, from the arguments above it follows that the equality in (3.4) holds if and only if $\zeta = 1/5$, and in this case $v = 1/25$, $u = 2\sqrt{v} = 2/5$, $s = t = 1/5$ or, equivalently, $x = -1/3$ and $y = 1/3$. □

Theorem 3.6 *The point pair function p_G is a quasi-metric on the domain $G = (-1, 1) \subset \mathbb{R}$ with a sharp constant $\sqrt{5}/2$.*

Proof We need to show that the function

$$p_G(x, y) = \frac{|x - y|}{\sqrt{|x - y|^2 + 4(1 - |x|)(1 - |y|)}} \tag{3.7}$$

satisfies the inequality (2.1) for all points x, y, $z \in (-1, 1)$ with the constant $c = \sqrt{5}/2$. If x, y and z are all either non-negative or non-positive, then $p_G(x, y) \leq p_G(x, z) + p_G(z, y)$ trivially. In the opposite case, either one of the points is negative and other two points are non-negative, or we have one positive and two non-positive points. Because of symmetry, we can just consider the first possibility. If z is negative, then the inequality $p_G(x, y) \leq p_G(x, z) + p_G(z, y)$ holds for all x, $y \in [0, 1)$. Consequently, we can assume that $x < 0 \leq z \leq y$. In this case, our inequality can be simplified to

$$\frac{y - x}{\sqrt{(y - x)^2 + 4(1 + x)(1 - y)}} \leq \frac{\sqrt{5}}{2} \left(\frac{x - z}{\sqrt{(x - z)^2 + 4(1 + x)(1 - z)}} + \frac{z - y}{2 - z - y} \right).$$

The inequality above follows from Lemma 3.1. Since, in the case $x = -1/3$, $z = 0$ and $y = 1/3$, the equality holds we see that the constant $\sqrt{5}/2$ is the best possible. □

We note that, for any 1-dimensional domain $G \subsetneq \mathbb{R}$, the boundary ∂G consists of either one or two points. Using this fact, we formulate:

Corollary 3.8 *If $G \subsetneq \mathbb{R}$ is a 1-dimensional domain, then the point pair function p_G is a metric if $\operatorname{card}(\partial G) = 1$ and a quasi-metric if $\operatorname{card}(\partial G) = 2$. Moreover, in the second case the best possible constant c in the inequality $p_G(x, y) \leq c(p_G(x, z) + p_G(z, y))$, x, y, $z \in G$, equals $\sqrt{5}/2$.*

Proof First, we note that the point pair function is invariant under translation and stretching by a nonzero factor.

If $\operatorname{card}(\partial G) = 1$, then for some x_0 we have $\partial G = \{x_0\}$ and with the help of the function $f : x \mapsto a(x - x_0)$, $a \neq 0$, we can map the domain G onto the positive real axis $\mathbb{R}^+$. The function f preserves the point pair function, i.e. $p_{\mathbb{R}^+}(f(x), f(y)) = p_G(x, y)$ for all x, $y \in G$. Therefore, from the very beginning we can assume that

$G = \mathbb{R}^+$. Since $p_{\mathbb{R}^+}(x, y) = |x - y|/(x + y)$ coincides with the triangular ratio metric $s_{\mathbb{R}^+}(x, y)$ for all $x, y \in \mathbb{R}^+$, we conclude that in this case the point pair function is a metric.

If card$(\partial G) = 2$, then we have $G = (x_0, x_1)$ for some $x_0, x_1 \in \mathbb{R}$. Now, we apply the function $f : x \mapsto (2x - (x_0 + x_1))/(x_1 - x_0)$ which maps G onto the interval $(-1, 1)$. We see that, as above, f preserves the point pair function, therefore we can assume that $G = (-1, 1)$, and the result follows from Theorem 3.6. □

4 The Point Pair Function in the *n*-Dimensional Case

In this section, we investigate the quasi-metric property of the point pair function by considering its behaviour in *n*-dimensional domains, $n \geq 1$. Our main results are Theorems 4.6 and 4.14. First, we will establish Lemma 4.1, which has quite complicated and technical inequalities but is necessary for the proof of Theorem 4.6. Note that Mocanu (2021) established results similar to those described in Lemmas 4.1 and 5.8.

Lemma 4.1 *Let* $A = \operatorname{sh}(x + y)$, $a \leq \operatorname{sh}(u + v)$, $B = \operatorname{sh} x$, $C = \operatorname{sh} y$, $b = \operatorname{sh} u$, $c = \operatorname{sh} v$, *and* $x, y, u, v \geq 0$. *Then*

$$\operatorname{arsh}\sqrt{A^2 + a^2} \leq \operatorname{arsh}\sqrt{B^2 + b^2} + \operatorname{arsh}\sqrt{C^2 + c^2} \tag{4.2}$$

or, what is equivalent,

$$\sqrt{A^2 + a^2} \leq \sqrt{B^2 + b^2}\sqrt{1 + C^2 + c^2} + \sqrt{C^2 + c^2}\sqrt{1 + B^2 + b^2}.$$

Moreover,

$$\sqrt{\frac{A^2 + a^2}{1 + A^2 + a^2}} \leq \sqrt{\frac{B^2 + b^2}{1 + B^2 + b^2}} + \sqrt{\frac{C^2 + c^2}{1 + C^2 + c^2}}. \tag{4.3}$$

Proof It is sufficient to prove that

$$\sqrt{\operatorname{sh}^2(x + y) + \operatorname{sh}^2(u + v)} \leq \sqrt{\operatorname{sh}^2 x + \operatorname{sh}^2 u}\sqrt{1 + \operatorname{sh}^2 y + \operatorname{sh}^2 v}$$
$$+ \sqrt{\operatorname{sh}^2 y + \operatorname{sh}^2 v}\sqrt{1 + \operatorname{sh}^2 x + \operatorname{sh}^2 u}. \tag{4.4}$$

By squaring both sides of (4.4), we have

$$\operatorname{sh}^2(x + y) + \operatorname{sh}^2(u + v)$$
$$\leq (\operatorname{sh}^2 x + \operatorname{sh}^2 u)(1 + \operatorname{sh}^2 y + \operatorname{sh}^2 v) + (\operatorname{sh}^2 y + \operatorname{sh}^2 v)(1 + \operatorname{sh}^2 x + \operatorname{sh}^2 u)$$
$$+ 2\sqrt{\operatorname{sh}^2 x + \operatorname{sh}^2 u}\sqrt{1 + \operatorname{sh}^2 y + \operatorname{sh}^2 v}\sqrt{\operatorname{sh}^2 y + \operatorname{sh}^2 v}\sqrt{1 + \operatorname{sh}^2 x + \operatorname{sh}^2 u}$$

or

$$(\text{sh}\,x\,\text{ch}\,y + \text{sh}\,y\,\text{ch}\,x)^2 + (\text{sh}\,u\,\text{ch}\,v + \text{sh}\,v\,\text{ch}\,u)^2$$
$$\leq \text{sh}^2x(\text{ch}^2y + \text{sh}^2v) + \text{sh}^2u(\text{ch}^2v + \text{sh}^2y)$$
$$+\,\text{sh}^2y(\text{ch}^2x + \text{sh}^2u) + \text{sh}^2v(\text{ch}^2u + \text{sh}^2x)$$
$$+2\sqrt{\text{sh}^2x + \text{sh}^2u}\,\sqrt{1 + \text{sh}^2y + \text{sh}^2v}\,\sqrt{\text{sh}^2y + \text{sh}^2v}\,\sqrt{1 + \text{sh}^2x + \text{sh}^2u}.$$

After simple transformations, we obtain

$$2\text{sh}\,x\,\text{ch}\,y\,\text{sh}\,y\,\text{ch}\,x + 2\text{sh}\,u\,\text{ch}\,v\,\text{sh}\,v\,\text{ch}\,u$$
$$\leq \text{sh}^2x\,\text{sh}^2v + \text{sh}^2u\,\text{sh}^2y + \text{sh}^2y\,\text{sh}^2u + \text{sh}^2v + \text{sh}^2x$$
$$+2\sqrt{\text{sh}^2x + \text{sh}^2u}\,\sqrt{1 + \text{sh}^2y + \text{sh}^2v}\,\sqrt{\text{sh}^2y + \text{sh}^2v}\,\sqrt{1 + \text{sh}^2x + \text{sh}^2u}.$$

To establish the inequality above, it is sufficient to prove that

$$\text{sh}\,x\,\text{ch}\,y\,\text{sh}\,y\,\text{ch}\,x + \text{sh}\,u\,\text{ch}\,v\,\text{sh}\,v\,\text{ch}\,u$$
$$\leq \sqrt{\text{sh}^2x + \text{sh}^2u}\,\sqrt{1 + \text{sh}^2y + \text{sh}^2v}\,\sqrt{\text{sh}^2y + \text{sh}^2v}\,\sqrt{1 + \text{sh}^2x + \text{sh}^2u}.$$

Squaring this inequality, we have

$$\text{sh}^2x\,\text{ch}^2y\,\text{sh}^2y\,\text{ch}^2x + \text{sh}^2u\,\text{ch}^2v\,\text{sh}^2v\,\text{ch}^2u$$
$$+\,2\,\text{sh}\,x\,\text{ch}\,y\,\text{sh}\,y\,\text{ch}\,x\,\text{sh}\,u\,\text{ch}\,v\,\text{sh}\,v\,\text{ch}\,u$$
$$\leq (\text{sh}^2x + \text{sh}^2u)(1 + \text{sh}^2y + \text{sh}^2v)(\text{sh}^2y + \text{sh}^2v)(1 + \text{sh}^2x + \text{sh}^2u)$$

or

$$\text{sh}^2x\,\text{ch}^2y\,\text{sh}^2y\,\text{ch}^2x + \text{sh}^2u\,\text{ch}^2v\,\text{sh}^2v\,\text{ch}^2u$$
$$+2\,\text{sh}\,x\,\text{ch}\,y\,\text{sh}\,y\,\text{ch}\,x\,\text{sh}\,u\,\text{ch}\,v\,\text{sh}\,v\,\text{ch}\,u$$
$$\leq [\text{sh}^2x(1 + \text{sh}^2y + \text{sh}^2v) + \text{sh}^2u(1 + \text{sh}^2y + \text{sh}^2v)]$$
$$\times[\text{sh}^2y(1 + \text{sh}^2x + \text{sh}^2u) + \text{sh}^2v(1 + \text{sh}^2x + \text{sh}^2u)]$$
$$= [\text{sh}^2x\,\text{ch}^2y + \text{sh}^2x\,\text{sh}^2v + \text{sh}^2u\,\text{ch}^2v + \text{sh}^2u\,\text{sh}^2y]$$
$$\times[\text{sh}^2y\,\text{ch}^2x + \text{sh}^2y\,\text{sh}^2u + \text{sh}^2v\,\text{ch}^2u + \text{sh}^2v\,\text{sh}^2x]. \qquad (4.5)$$

By the inequality of arithmetic and geometric means, we have

$$2\text{sh}\,x\,\text{ch}\,y\,\text{sh}\,y\,\text{ch}\,x\,\text{sh}\,u\text{ch}\,v\,\text{sh}\,v\,\text{ch}\,u$$
$$\leq \text{sh}^2x\,\text{ch}^2y\,\text{sh}^2v\,\text{ch}^2u + \text{sh}^2u\,\text{ch}^2v\,\text{sh}^2y\,\text{ch}^2x,$$

therefore,

$$\mathrm{sh}^2 x\, \mathrm{ch}^2 y\, \mathrm{sh}^2 y\, \mathrm{ch}^2 x + \mathrm{sh}^2 u\, \mathrm{ch}^2 v\, \mathrm{sh}^2 v\, \mathrm{ch}^2 u$$
$$+ 2\mathrm{sh}\, x\, \mathrm{ch}\, y\, \mathrm{sh}\, y\, \mathrm{ch}\, x\, \mathrm{sh}\, u\, \mathrm{ch}\, v\, \mathrm{sh}\, v\, \mathrm{ch}\, u$$
$$\leq [\mathrm{sh}^2 x \mathrm{ch}^2 y + \mathrm{sh}^2 u \mathrm{ch}^2 v][\mathrm{sh}^2 y \mathrm{ch}^2 x + \mathrm{sh}^2 v \mathrm{ch}^2 u].$$

This inequality implies (4.5), therefore, (4.2) is proved. The inequality (4.3) can be obtained by applying the function th to both sides of (4.2). □

Theorem 4.6 *For $n \geq 1$, the point pair function is a metric on $G = \mathbb{R}^n \backslash \{0\}$.*

Proof Because the point pair function trivially satisfies the properties (1) and (2) of a metric, we only need to prove the triangle inequality. Therefore, we will show that $p_G(x, y) \leq p_G(x, z) + p_G(z, y)$ for $x, y, z \in G = \mathbb{R}^n \backslash \{0\}$. Note that, for all points x, y in this domain,

$$p_G(x, y) = \frac{|x - y|}{\sqrt{|x - y|^2 + 4|x||y|}}.$$

(1) First we consider the case $n = 2$. Then we can identify points of $\mathbb{R}^2$ with complex numbers.

Because of the homogeneity of $p_G(x, y)$, we can assume that $z = 1$, so that the triangle inequality becomes

$$\frac{|x - y|}{\sqrt{|x - y|^2 + 4|x||y|}} \leq \frac{|x - 1|}{\sqrt{|x - 1|^2 + 4|x|}} + \frac{|1 - y|}{\sqrt{|1 - y|^2 + 4|y|}}. \tag{4.7}$$

Let $x = Re^{i2\phi}$, $y = re^{i2\psi}$, $R, r > 0$, $\phi, \psi \in \mathbb{R}$. We can assume that $R \geq r$.

First we will show that if either $0 < r \leq R \leq 1$ or $1 \leq r \leq R$, then (4.7) holds. Let us fix some u and v such that $x = u^2$, $y = v^2$. Then (4.7) is equivalent to the inequality

$$\frac{|u/v - v/u|}{\sqrt{|u/v - v/u|^2 + 4}} \leq \frac{|u - 1/u|}{\sqrt{|u - 1/u|^2 + 4}} + \frac{|v - 1/v|}{\sqrt{|v - 1/v|^2 + 4}}. \tag{4.8}$$

If $0 < r \leq R \leq 1$, then $|u|, |v| \leq 1$ and

$$|u/v - v/u| \leq |u/v - uv| + |uv - v/u|$$
$$= |u||v - 1/v| + |v||u - 1/u| \leq |u - 1/u| + |v - 1/v|. \tag{4.9}$$

Therefore, if we put

$$t = \mathrm{arsh}\, \frac{|u/v - v/u|}{2}, \quad q = \mathrm{arsh}\, \frac{|u - 1/u|}{2}, \quad s = \mathrm{arsh}\, \frac{|v - 1/v|}{2},$$

then, by (4.9), we have $\operatorname{sh} t \leq \operatorname{sh} q + \operatorname{sh} s$. But this immediately implies $t \leq q + s$ and $\operatorname{th} t \leq \operatorname{th} q + \operatorname{th} s$ which is equivalent to (4.7).

Since the inequality (4.8) does not change after replacing u and v with u^{-1} and v^{-1}, we see that for the case $1 \leq r \leq R$ the inequality (4.9) is also valid.

Thus, we only need to consider the case $r \leq 1 \leq R$. We have

$$|x - y| = \sqrt{R^2 + r^2 - 2Rr\cos[2(\phi - \psi)]}, \quad |x - 1| = \sqrt{R^2 + 1 - 2R\cos 2\phi},$$

$$|1 - y| = \sqrt{r^2 + 1 - 2r\cos 2\psi},$$

consequently, the inequality (4.7) can be written in the form

$$\sqrt{\frac{R^2 + r^2 - 2Rr\cos[2(\phi - \psi)]}{R^2 + r^2 + 2Rr(2 - \cos[2(\phi - \psi)])}}$$

$$\leq \sqrt{\frac{R^2 + 1 - 2R\cos 2\phi}{R^2 + 1 + 2R(2 - \cos 2\phi)}} + \sqrt{\frac{r^2 + 1 - 2Rr\cos 2\psi}{r^2 + 1 + 2r(2 - \cos 2\psi)}}.$$

This can be simplified to

$$\sqrt{\frac{(R - r)^2 + 4Rr\sin^2(\phi - \psi)}{(R + r)^2 + 4Rr\sin^2(\phi - \psi)}}$$

$$\leq \sqrt{\frac{(R - 1)^2 + 4R\sin^2\phi}{(R + 1)^2 + 4R\sin^2\phi}} + \sqrt{\frac{(1 - r)^2 + 4r\sin^2\psi}{(r + 1)^2 + 4r\sin^2(\psi)}}. \tag{4.10}$$

If we denote

$$A = \frac{|R - r|}{2\sqrt{Rr}}, \quad B = \frac{|R - 1|}{2\sqrt{R}}, \quad C = \frac{|1 - r|}{2\sqrt{r}}, \tag{4.11}$$

then the inequality (4.10) takes the form

$$\sqrt{\frac{A^2 + \sin^2(\phi - \psi)}{1 + A^2 + \sin^2(\phi - \psi)}} \leq \sqrt{\frac{B^2 + \sin^2\phi}{1 + B^2 + \sin^2\phi}} + \sqrt{\frac{C^2 + \sin^2\psi}{1 + C^2 + \sin^2\psi}}. \tag{4.12}$$

Let $a = |\sin(\phi - \psi)|$, $b = |\sin\phi|$, $c = |\sin\psi|$, $u = \operatorname{arsh} b$ and $v = \operatorname{arsh} c$. Then $a \leq \operatorname{sh}(u + v)$, since

$$a = |\sin\phi\cos\psi - \sin\psi\cos\phi| \leq |\sin\phi\cos\psi| + |\sin\psi\cos\phi|$$

$$= b\sqrt{1 - c^2} + c\sqrt{1 - b^2}$$

$$\leq b\sqrt{1 + c^2} + c\sqrt{1 + b^2}.$$

If A, B and C are as in (4.11), then we have $A = \text{sh}\,(\text{arsh}\,B + \text{arsh}\,C)$. By applying the function th to the inequality (4.2) of Lemma 4.1 and combining this with the inequality $\text{th}\,(u + v) \leq \text{th}\,u + \text{th}\,v$ for $u, v \geq 0$, we obtain (4.12).

(2) Now we consider the case $n \neq 2$. If $n = 1$, then the statement of the theorem immediately follows from the case (1). Therefore, we will assume that $n \geq 3$. Consider the subspace E of $\mathbb{R}^n$ containing the points 0, x, y and z. Since the Euclidean distance and the function p_G are invariant under orthogonal transformations of $\mathbb{R}^n$ and for $n \geq 2$ the triangle inequality is valid, we can assume that E coincides with $\mathbb{R}^3$.

Without loss of generality we can put $|z| = 1$. Now consider the vectors Ox, Oy, and Oz from the origin to the points x, y, and z, respectively. Let 2α be the angle between Ox and Oz, 2β be the angle between Oz and Oy, and 2γ be the angle between Ox and Oy; $\alpha, \beta, \gamma \in [0, \pi/2]$. Then, by the law of cosines,

$$|x - y| = \sqrt{R^2 + r^2 - 2Rr\cos 2\gamma}, \quad |x - z| = \sqrt{R^2 + 1 - 2R\cos 2\alpha},$$
$$|z - y| = \sqrt{r^2 + 1 - 2r\cos 2\beta},$$

where $R = |x|$ and $r = |y|$. Applying the same arguments as above in the case $n = 2$, we see that we only need to prove the inequality

$$\sqrt{\frac{A^2 + \sin^2 \gamma}{1 + A^2 + \sin^2 \gamma}} \leq \sqrt{\frac{B^2 + \sin^2 \alpha}{1 + B^2 + \sin^2 \alpha}} + \sqrt{\frac{C^2 + \sin^2 \beta}{1 + C^2 + \sin^2 \beta}}, \quad (4.13)$$

where A, B and C is defined by (4.11).

Denote $a = \sin \gamma$, $b = \sin \alpha$, $c = \sin \beta$. Consider the three vectors Ox, Oy and Oz. While they are not necessarily coplanar, the three pairs of vectors Ox, Oy, and Oy, Oz and Oz, Ox are coplanar and form planar angles 2α, 2β and 2γ. Since each plane angle of a triangular angle is less than the sum of its other two plane angles, we obtain $2\gamma \leq 2\alpha + 2\beta$, therefore, $\gamma \leq \alpha + \beta$.

Now, we will show that this implies the inequality $\sin \gamma \leq \sin \alpha + \sin \beta$. Actually, if $\alpha + \beta \leq \frac{\pi}{2}$, then $\sin \gamma \leq \sin(\alpha + \beta) \leq \sin \alpha + \sin \beta$. If $\alpha + \beta > \frac{\pi}{2}$, then $\beta > \frac{\pi}{2} - \alpha$ and $\sin \alpha + \sin \beta \geq \sin \alpha + \sin(\frac{\pi}{2} - \alpha) = \sin \alpha + \cos \alpha = \sqrt{2}\sin(\alpha + \frac{\pi}{4}) \geq 1 \geq \sin \gamma$, since $\frac{\pi}{4} \leq \alpha + \frac{\pi}{4} \leq \frac{3\pi}{4}$.

Thus, we have $a \leq b + c$ and this implies that $a \leq b\sqrt{1 + c^2} + c\sqrt{1 + b^2}$, and we can continue the proof just as in the Case (1) to show that (4.13) is valid. $\qquad \square$

Theorem 4.14 *On every domain $G \subsetneq \mathbb{R}^n$, $n \geq 1$, the point pair function p_G is a quasi-metric with a constant less than or equal to $\sqrt{5}/2$.*

Proof To prove that the point pair function p_G is a quasi-metric, we only need to find such a constant $c \geq 1$ that

$$p_G(x, y) \leq c(p_G(x, z) + p_G(z, y))$$

for all points x, y, $z \in G$. Let

$$c(x, y, z; G) = \frac{p_G(x, y)}{p_G(x, z) + p_G(z, y)},$$

where x, y, and z are distinct points from G. Define

$$c^* = \sup_{x,y,z,G} c(x, y, z; G), \qquad (4.15)$$

where the supremum is taken over all domains $G \subsetneq \mathbb{R}^n$ and triples of distinct points. We will call such domains and triples admissible. We need to prove that $c^* = \sqrt{5}/2$.

Let us fix a domain $G \subsetneq \mathbb{R}^n$ and two distinct points x, $y \in G$. Since $G \neq \mathbb{R}^n$, the boundary $\partial G \neq \emptyset$, therefore, there exist points u, $v \in \partial G$ such that $d_G(x) = |x - u|$ and $d_G(y) = |y - v|$. In the general case, the points u and v might not be unique because there can be several boundary points on the spheres $S^{n-1}(x, d_G(x))$ and $S^{n-1}(y, d_G(y))$.

We note that the value p_G decreases as G grows, i.e. if $G \subset G_1$, then $p_{G_1}(\tilde{x}, \tilde{y}) \leq p_G(\tilde{x}, \tilde{y})$ for all $\tilde{x}, \tilde{y} \in G$.

Consider the following two cases.

(1) If $u = v$, then we can set $G_1 = \mathbb{R}^n \setminus \{u\}$. It is clear that $G \subset G_1$ and $p_G(x, y) = p_{G_1}(x, y)$. Taking into account the invariance of p_G under the shifts of $\mathbb{R}^n$, we can assume that $u = 0$. Then, by Theorem 4.6, we have $p_{G_1}(x, y) \leq p_{G_1}(x, z) + p_{G_1}(z, y)$ for all $z \in G_1$. From the monotonicity of p_G with respect to G, we obtain

$$p_G(x, y) = p_{G_1}(x, y) \leq p_{G_1}(x, z) + p_{G_1}(z, y) \leq p_G(x, z) + p_G(z, y).$$

Therefore, $c(x, y, z; G) \leq 1$.

(2) Let now $u \neq v$. We put $G_1 = G_1^{u,v} = \mathbb{R}^n \setminus \{u, v\}$. Then $G \subset G_1$. Moreover, $p_G(x, y) = p_{G_1}(x, y)$ and $p_{G_1}(x, z) + p_{G_1}(z, y) \leq p_G(x, z) + p_G(z, y)$. Consequently, $c(x, y, z; G) \leq c(x, y, z; G_1)$ and the supremum in (4.15) is attained on domains of the type $G_1^{u,v}$.

Denote $a = |x - z|$, $b = |z - y|$, $c = |x - y|$, $\rho = |x - u|$, $r = |y - v|$. By the triangle inequality, we have

$$c \leq a + b, \quad \rho = |x - u| \leq |x - v| \leq c + r \leq a + b + r,$$
$$r = |y - v| \leq |y - u| \leq c + \rho \leq a + b + \rho.$$

Now consider the segment Δ on $\mathbb{R}^1$ with endpoints $\tilde{u} := -a - \rho$, $\tilde{v} := b + r$. Let $\tilde{z} = 0$, $\tilde{x} = -a$, $\tilde{y} = b$. Then $\tilde{u} < \tilde{x} < \tilde{z} < \tilde{y} < \tilde{v}$ and

$$|\tilde{x} - \tilde{u}| = \rho = |x - u|, \quad |\tilde{y} - \tilde{v}| = r = |y - v|, \quad |\tilde{x} - \tilde{y}| = a + b \geq c = |x - y|,$$
$$|\tilde{x} - \tilde{z}| = a = |x - z|, \quad |\tilde{z} - \tilde{y}| = b = |z - y|.$$

With the help of the triangle inequality, we also have

$$|\widetilde{x} - \widetilde{v}| = a + b + r \geq |x - u| = |\widetilde{x} - \widetilde{u}|,$$
$$|\widetilde{y} - \widetilde{u}| = a + b + \rho \geq |y - v| = |\widetilde{y} - \widetilde{v}|.$$

Therefore, $d_\Delta(\widetilde{x}) = \rho$, $d_\Delta(\widetilde{y}) = r$.

At last,

$$|\widetilde{z} - \widetilde{u}| = a + \rho \geq |z - u|, \quad |\widetilde{z} - \widetilde{v}| = b + r \geq |z - v|,$$

and this implies $d_\Delta(\widetilde{z}) \geq d_{G_1}(z)$. Using the obtained inequalities and the fact that the function $t \mapsto t/\sqrt{t^2 + \gamma^2}$ is increasing on $\mathbb{R}_+$ when γ is a real nonzero constant, we have

$$p_\Delta(\widetilde{x}, \widetilde{y}) = \frac{|\widetilde{x} - \widetilde{y}|}{\sqrt{|\widetilde{x} - \widetilde{y}|^2 + 4\rho r}} \geq \frac{|x - y|}{\sqrt{|x - y|^2 + 4\rho r}} = p_{G_1}(x, y),$$

$$p_\Delta(\widetilde{x}, \widetilde{z}) = \frac{|\widetilde{x} - \widetilde{z}|}{\sqrt{|\widetilde{x} - \widetilde{z}|^2 + 4\rho d_\Delta(\widetilde{z})}} = \frac{|x - z|}{\sqrt{|x - z|^2 + 4\rho d_\Delta(\widetilde{z})}}$$

$$\leq \frac{|x - z|}{\sqrt{|x - z|^2 + 4\rho d_{G_1}(z)}} = p_{G_1}(x, z)$$

and, similarly, $p_\Delta(\widetilde{z}, \widetilde{y}) \leq p_{G_1}(z, y)$. From this, we deduce that $c(x, y, z; G_1) \leq c(\widetilde{x}, \widetilde{y}, \widetilde{z}; \Delta)$.

Since the point pair function is invariant under shifts and stretchings, we can assume that $\Delta = [-1, 1]$. But, by Theorem 3.6, the point pair function fulfills the inequality

$$p_\Delta(\widetilde{x}, \widetilde{y}) \leq \frac{\sqrt{5}}{2} (p_\Delta(\widetilde{x}, \widetilde{z}) + p_\Delta(\widetilde{z}, \widetilde{y})) \tag{4.16}$$

for all points $\widetilde{x}, \widetilde{y}, \widetilde{z} \in (-1, 1)$ with the constant $\sqrt{5}/2$. Therefore, we have $c^* \leq \sqrt{5}/2$.

To prove that $c^* = \sqrt{5}/2$, consider $\Delta = [-1, 1] \subset \mathbb{R}^1$ as a part of $\mathbb{R}^n$. Let $\widetilde{u} = -1$ and $\widetilde{v} = 1$ be the endpoints of Δ. Consider the domain $G_1 = G_1^{\widetilde{u}, \widetilde{v}}$. For all $\widetilde{x}, \widetilde{y} \in \Delta$ we have $p_{G_1}(\widetilde{x}, \widetilde{y}) = p_\Delta(\widetilde{x}, \widetilde{y})$. Since in (4.16) the constant $\sqrt{5}/2$ is sharp if we take $\widetilde{x}, \widetilde{y}$ and $\widetilde{z}$ from $(-1, 1)$, we obtain that it is sharp for G_1 and, therefore, for the class of proper subdomains on $\mathbb{R}^n$. The theorem is proved. $\qquad\square$

Now, we will investigate the sharpness of the constant $\sqrt{5}/2$, if a proper subdomain G of $\mathbb{R}^n$ is fixed.

Lemma 4.17 *If a domain $G \subsetneq \mathbb{R}^n$, $n \geq 1$, contains some ball $B^n(z_0, r)$ and there are two points $u, v \in \partial G$ such that the segment $[u, v]$ is a diameter of $B^n(z_0, r)$, then $c = \sqrt{5}/2$ is the best possible constant for which the inequality*

$$p_G(x, y) \leq c(p_G(x, z) + p_G(z, y)), \quad x, y, z \in G,$$

Fig. 1 A domain G and points $x, y, z \in G$ for which the equality (4.18) holds
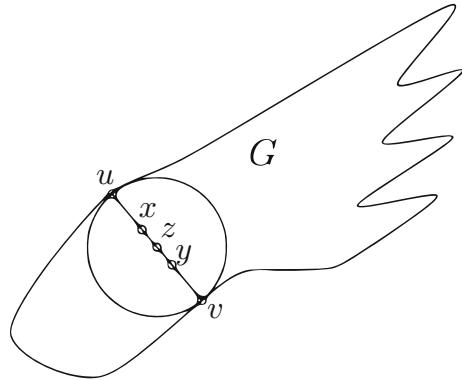
is valid.

Proof By Theorem 4.14, the point pair function is a quasi-metric with the constant $\sqrt{5}/2$. The sharpness of this constant follows from the fact that the equality

$$p_G(x, y) = (\sqrt{5}/2)(p_G(x, z) + p_G(z, y)) \tag{4.18}$$

holds for the points $x = z_0 + (u - z_0)/3$, $z = z_0$ and $y = z_0 + (v - z_0)/3$ (see Fig. 1). $\square$

It follows from Lemma 4.17 that the point pair function p_G is a quasi-metric with the best possible constant $\sqrt{5}/2$ if the domain G is, for instance, a ball, a hypercube, a hyperrectangle, a multipunctured real space of any dimension $n \geq 1$, or a two-dimensional, regular and convex polygon with an even number of vertices.

5 The Generalized Point Pair Function

In this section, we will consider the generalized version of the point pair function. Namely, note that, by replacing the constant 4 with some $\alpha > 0$, we obtain the function

$$p_G^\alpha(x, y) = \frac{|x - y|}{\sqrt{|x - y|^2 + \alpha d_G(x) d_G(y)}}. \tag{5.1}$$

Let us first consider the case where the domain G is the positive real axis.

Theorem 5.2 *For a constant $\alpha > 0$, the function*

$$p_{\mathbb{R}_+}^\alpha(x, y) = \frac{|x - y|}{\sqrt{(x - y)^2 + \alpha x y}}, \quad x, y > 0,$$

is a metric if and only if $\alpha \leq 12$.

Proof To prove that for every fixed $0 < \alpha \le 12$, the function $p_{\mathbb{R}+}^{\alpha}(x, y)$ is a metric on the positive real axis, it is sufficient to establish the triangle inequality $p_{\mathbb{R}+}^{\alpha}(x, y) \le p_{\mathbb{R}+}^{\alpha}(x, z) + p_{\mathbb{R}+}^{\alpha}(z, y)$. Fix first two points x, $y > 0$. By symmetry, we can assume that $x \le y$. Next, we fix z such that the sum $p_{\mathbb{R}+}^{\alpha}(x, z) + p_{\mathbb{R}+}^{\alpha}(z, y)$ is at minimum. Without loss of generality, we can assume that $0 < x \le z \le y$ because, for all $z \in (0, x)$,

$$p_{\mathbb{R}+}^{\alpha}(x, 2x - z) + p_{\mathbb{R}+}^{\alpha}(2x - z, y) < p_{\mathbb{R}+}^{\alpha}(x, z) + p_{\mathbb{R}+}^{\alpha}(z, y),$$

and, if $y < z$, then the triangle inequality $p_{\mathbb{R}+}^{\alpha}(x, y) \le p_{\mathbb{R}+}^{\alpha}(x, z) + p_{\mathbb{R}+}^{\alpha}(z, y)$ holds trivially. Because the function $p_{\mathbb{R}+}^{\alpha}$ is invariant under any stretching by any factor $r > 0$, we can assume that $z = 1$.

Our aim is to prove $p_{\mathbb{R}+}^{\alpha}(x, y) \le p_{\mathbb{R}+}^{\alpha}(x, 1) + p_{\mathbb{R}+}^{\alpha}(1, y)$ or, equivalently,

$$F(x, z) := \frac{1 - x}{\sqrt{(1 - x)^2 + \alpha x}} + \frac{y - 1}{\sqrt{(y - 1)^2 + \alpha y}} - \frac{y - x}{\sqrt{(x - y)^2 + \alpha xy}} \ge 0, \tag{5.3}$$

for $0 < x \le 1 \le y$. By denoting $u = 1/\sqrt{x}$, $v = \sqrt{y}$, we can write (5.3) as

$$G(u, v) := \frac{u - u^{-1}}{\sqrt{(u - u^{-1})^2 + \alpha}} + \frac{v - v^{-1}}{\sqrt{(v - v^{-1})^2 + \alpha}} - \frac{uv - (uv)^{-1}}{\sqrt{(uv - (uv)^{-1})^2 + \alpha}} \ge 0, \tag{5.4}$$

$u, v \ge 1$. Now, we will find the critical points of $G(u, v)$. We have

$$\frac{\partial G(u, v)}{\partial u} = \alpha \frac{1 + u^{-2}}{\sqrt{((u - u^{-1})^2 + \alpha)^3}} - \alpha v \frac{1 + (uv)^{-2}}{\sqrt{((uv - (uv)^{-1})^2 + \alpha)^3}},$$

$$\frac{\partial G(u, v)}{\partial v} = \alpha \frac{1 + v^{-2}}{\sqrt{((v - v^{-1})^2 + \alpha)^3}} - \alpha u \frac{1 + (uv)^{-2}}{\sqrt{((uv - (uv)^{-1})^2 + \alpha)^3}}.$$

If

$$\frac{\partial G(u, v)}{\partial u} = \frac{\partial G(u, v)}{\partial v} = 0,$$

then it is easy to show that

$$\frac{u + u^{-1}}{\sqrt{((u - u^{-1})^2 + \alpha)^3}} = \frac{v + v^{-1}}{\sqrt{((v - v^{-1})^2 + \alpha)^3}},$$

consequently,

$$\frac{u + u^{-1}}{\sqrt{((u + u^{-1})^2 + \alpha - 4)^3}} = \frac{v + v^{-1}}{\sqrt{((v + v^{-1})^2 + \alpha - 4)^3}}. \tag{5.5}$$

The function $t \mapsto t(t^2 + \alpha - 4)^{-3/2}$ is monotonic on $[2, \infty)$. Since $u + u^{-1} \geq 2$ and $v + v^{-1} \geq 2$ for $u, v \geq 1$, from (5.5) we deduce that $u + u^{-1} = v + v^{-1}$. From the monotonicity of the function $u + u^{-1}$ on $[1, \infty)$, it follows that $u = v$. Consequently, all the critical points of $G(u, v)$ are on the line $u = v$.

If $u \to u_0$ or $v \to v_0$ and either u_0 or v_0 equals 1, then $G(u, v)$ tends to a non-negative value. Similarly, this condition holds if u_0 or v_0 equals $+\infty$. Therefore, to prove that the inequality (5.4) holds we only need to show that $G(u, u) \geq 0$, $u \geq 1$ or, equivalently,

$$2 \frac{u - u^{-1}}{\sqrt{(u - u^{-1})^2 + \alpha}} - \frac{u^2 - u^{-2}}{\sqrt{(u^2 - u^{-2})^2 + \alpha}} \geq 0.$$

This inequality can be written as

$$2\sqrt{(u^2 - u^{-2})^2 + \alpha} \geq (u + u^{-1})\sqrt{(u - u^{-1})^2 + \alpha}$$

or

$$4(u^4 + u^{-4} + \alpha - 2) \geq (u^2 + u^{-2} + 2)(u^2 + u^{-2} + \alpha - 2). \tag{5.6}$$

By denoting $t = u^2 + u^{-2}$, we will have $t \geq 2$, and the inequality (5.6) takes the form

$$4(t^2 + \alpha - 4) \geq (t + 2)(t + \alpha - 2)$$

or, equivalently,

$$3t^2 - \alpha t + 2\alpha - 12 = (t - 2)(3t - (\alpha - 6)) \geq 0, \quad t \geq 2. \tag{5.7}$$

The inequality (5.7) is valid for $\alpha \geq 12$ and $t \geq 2$ because for such α and t we have

$$3t - (\alpha - 6) \geq 6 - (\alpha - 6) = 12 - \alpha \geq 0.$$

Consequently, the inequality (5.3) holds and, for $0 < \alpha \leq 12$, the function $p_{\mathbb{R}_+}^\alpha$ is a metric. It also follows that the constant 12 here is sharp because, for $\alpha > 12$, we have

$$3t^2 - \alpha t + 2\alpha - 12 < 0, \quad 2 < t < (\alpha - 6)/3,$$

and, therefore, the inequality (5.7) is not valid at every point of $[2, +\infty)$. $\square$

In Theorem 5.11, we prove a result about the function p_G^α similar to Theorem 5.2 but for the case where $G = \mathbb{R}^n \setminus \{0\}$. See Fig. 2 for the disks of the function p_G^α in $\mathbb{R}^2 \setminus \{0\}$. However, in order to prove Theorem 5.11, we first need to consider the following lemma.

Lemma 5.8 *If $A, B, C > 0$ and $a, b, c \geq 0$ are chosen so that the inequalities*

$$\frac{A}{\sqrt{1 + A^2}} \leq \frac{B}{\sqrt{1 + B^2}} + \frac{C}{\sqrt{1 + C^2}} \tag{5.9}$$

and $a \le b + c$ hold, then

$$\sqrt{\frac{A^2 + a^2}{1 + A^2 + a^2}} \le \sqrt{\frac{B^2 + b^2}{1 + B^2 + b^2}} + \sqrt{\frac{C^2 + c^2}{1 + C^2 + c^2}}.$$

Proof Since the functions $t \mapsto t/\sqrt{1+t^2}$ and $t \mapsto \sqrt{t/(1+t)}$ are increasing on $[0, \infty)$, we can assume that the equality takes place in (5.9).

Consider the function

$$F(x, y) := \sqrt{\frac{B^2 + x^2}{1 + B^2 + x^2}} + \sqrt{\frac{C^2 + y^2}{1 + C^2 + y^2}} - \sqrt{\frac{A^2 + (x + y)^2}{1 + A^2 + (x + y)^2}}.$$

We need to prove that $F(x, y) \ge 0$, $x, y \ge 0$. We have $F(0, 0) = 0$. Assume that for some $x, y \ge 0$, not equal to zero at the same time, $F(x, y) < 0$. Consider now the function $g(t) = F(tx, ty)$, $t \in [0, 1]$. It is continuous and $g(0) = 0$, $g(1) < 0$. We will show that for small positive t, the inequality $g(t) > 0$ holds. Actually,

$$g'(t)/t = \frac{x^2}{\sqrt{(B^2 + t^2 x^2)(1 + B^2 + t^2 x^2)^3}} + \frac{y^2}{\sqrt{(C^2 + t^2 y^2)(1 + C^2 + t^2 y^2)^3}}$$

$$- \frac{(x + y)^2}{\sqrt{(A^2 + t^2(x + y)^2)(1 + A^2 + t^2(x + y)^2)^3}}$$

$$\rightarrow \frac{x^2}{B\sqrt{(1 + B^2)^3}} + \frac{y^2}{C\sqrt{(1 + C^2)^3}} - \frac{(x + y)^2}{A\sqrt{(1 + A^2)^3}}, \quad \text{as} \quad t \rightarrow 0.$$

Now, we will show that

$$\frac{x^2}{B\sqrt{(1 + B^2)^3}} + \frac{y^2}{C\sqrt{(1 + C^2)^3}} - \frac{(x + y)^2}{A\sqrt{(1 + A^2)^3}} > 0. \tag{5.10}$$

Denote

$$a_1 = \frac{A}{\sqrt{1 + A^2}}, \quad b_1 = \frac{B}{\sqrt{1 + B^2}}, \quad c_1 = \frac{C}{\sqrt{1 + C^2}}.$$

Then $a_1 = b_1 + c_1 < 1$ and

$$A = \frac{a_1}{\sqrt{1 - a_1^2}}, \quad B = \frac{b_1}{\sqrt{1 - b_1^2}}, \quad C = \frac{c_1}{\sqrt{1 - c_1^2}}.$$

$$\sqrt{1 + A^2} = \frac{1}{\sqrt{1 - a_1^2}}, \quad \sqrt{1 + B^2} = \frac{1}{\sqrt{1 - b_1^2}}, \quad \sqrt{1 + C^2} = \frac{1}{\sqrt{1 - c_1^2}}.$$

In this notation, the inequality (5.10) can be written in the form

$$\frac{x^2(1 - b_1^2)^2}{b_1} + \frac{y^2(1 - c_1^2)^2}{c_1} - \frac{(x + y)^2(1 - a_1^2)^2}{a_1} > 0.$$

It is easy to prove that

$$\frac{(x + y)^2}{a_1} = \frac{(x + y)^2}{b_1 + c_1} \leq \frac{x^2}{b_1} + \frac{y^2}{c_1},$$

and it is therefore sufficient to show that

$$(1 - a_1^2)^2 < (1 - b_1^2)^2, \quad (1 - a_1^2)^2 < (1 - c_1^2)^2,$$

which follows from the fact that $a_1 > b_1$ and $a_1 > c_1$.

Since for small positive t, we have $g(t) > 0$, and $g(1) < 0$, we conclude that there exists $t_0 \in (0, 1)$ and $\varepsilon > 0$ such that $g(t_0) = 0$ and $g(t) < 0$ on $(t_0, t_0 + \varepsilon)$. Therefore, $g'(t_0) \leq 0$. Denote $x_0 = t_0 x$, $y_0 = t_0 y$. Then

$$\sqrt{\frac{A^2 + (x_0 + y_0)^2}{1 + A^2 + (x_0 + y_0)^2}} = \sqrt{\frac{B^2 + x_0^2}{1 + B^2 + x_0^2}} + \sqrt{\frac{C^2 + y_0^2}{1 + C^2 + y_0^2}}.$$

By denoting

$$A_1 = \sqrt{A^2 + (x_0 + y_0)^2}, \quad B_1 = \sqrt{B^2 + x_0^2}, \quad C_1 = \sqrt{C^2 + y_0^2},$$

we have

$$\frac{A_1}{\sqrt{1 + A_1^2}} = \frac{B_1}{\sqrt{1 + B_1^2}} + \frac{C_1}{\sqrt{1 + C_1^2}}.$$

$$g'(t_0)/t_0 = \frac{x^2}{B_1\sqrt{(1 + B_1^2)^3}} + \frac{y^2}{C_1\sqrt{(1 + C_1^2)^3}} - \frac{(x + y)^2}{A_1\sqrt{(1 + A_1^2)^3}}.$$

Reasoning as above but by replacing A, B and C with A_1, B_1 and C_1, we show that $g'(t_0) > 0$. The contradiction proves the theorem. □

Theorem 5.11 *For a constant $\alpha > 0$, the function*

$$p_{\mathbb{R}^n \setminus \{0\}}^\alpha (x, y) = \frac{|x - y|}{\sqrt{|x - y|^2 + \alpha |x| \, |y|}}, \quad x, y \in \mathbb{R}^n \setminus \{0\},$$

$$n \geq 2,$$

is a metric if and only if $\alpha \leq 12$.

Fig. 2 Disks
$\{x \in G | p_G^\alpha(x, 0.5) < r\}$ with
center 0.5 and radii
$r = 0.5, 0.6, 0.7, 0.75$ for the
metric p_G^α, $\alpha = 3.5$, in the
domain $G = \mathbb{R}^2 \backslash \{0\}$
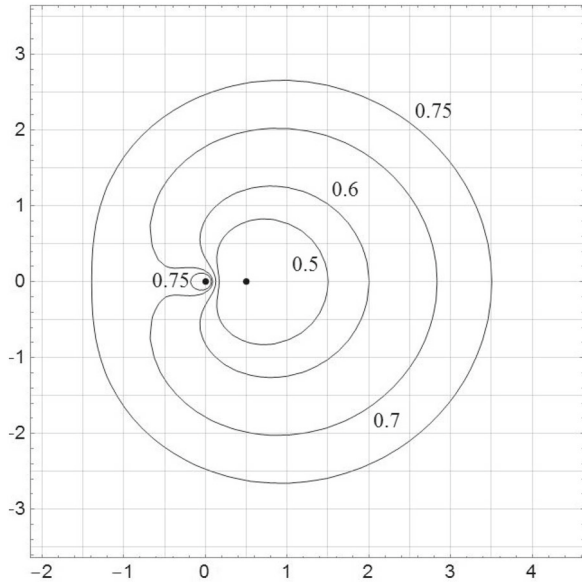
Proof We will only outline the proof because it is similar to that of Theorem 4.6 but, instead of (4.11), we use the following values:

$$A = \frac{|R - r|}{\sqrt{\alpha R r}}, \quad B = \frac{|R - 1|}{\sqrt{\alpha R}}, \quad C = \frac{|1 - r|}{\sqrt{\alpha r}}, \tag{5.12}$$

By Theorem 5.2, such values satisfy (5.9). We also note that the parameters a, b and c, considered in both the first and second parts of the proof of Theorem 4.6, satisfy the inequality $0 \le a \le b + c$. Therefore, we can apply Lemma 5.8 instead of Lemma 4.1, to prove (4.12) and (4.13) and establish the triangle inequality. □

Now, we will study a generalization of the point pair function in the upper half-space $\mathbb{H}^n$.

Theorem 5.13 *For a constant $\alpha > 0$, the function*

$$p_{\mathbb{H}^n}^\alpha(x, y) = \frac{|x - y|}{\sqrt{|x - y|^2 + \alpha x_n y_n}},$$

$$x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{H}^n, n \ge 2, \tag{5.14}$$

is a metric if and only if $\alpha \le 12$.

Proof It is evident that this function fulfills the first two properties of a metric if $\alpha > 0$ and, therefore, we only need to investigate the fulfillment of the triangle inequality. If we consider some points $x = (0, 0, \ldots, 0, x_n)$ and $y = (0, 0, \ldots, 0, y_n)$ with x_n, $y_n > 0$, then $p_{\mathbb{H}^n}^\alpha(x, y) = p_{\mathbb{R}^+}^\alpha(x_n, y_n)$ and, from Theorem 5.2, it follows that the function $p_{\mathbb{H}^n}^\alpha$ is not a metric for $\alpha > 12$, since $p_{\mathbb{R}^+}^\alpha$ is not a metric for such α.

Now, we will consider the case $\alpha \in (0, 12]$ and prove that, in this case, $p_{\mathbb{H}^n}^\alpha$ is a metric. Let us fix some distinct points $x, y \in \mathbb{H}^n$. Consider a two-dimensional plane Π in $\mathbb{R}^n$, which is orthogonal to the hyperplane $\{x_n = 0\}$ and contains the points x and y. If x and y do not lie on the same line, orthogonal to $\{x_n = 0\}$, then Π is defined in a unique way; in the opposite case, we fix any of the possible planes. Now, we take any point $z \in \mathbb{H}^n$ and find its orthogonal projection $\widetilde{z}$ to Π. Since $|\widetilde{z}-x| \leq |z-x|, |\widetilde{z}-y| \leq |z-y|$ and $\widetilde{z}_n = z_n$, we obtain $p_{\mathbb{H}^n}^\alpha(x, z) + p_{\mathbb{H}^n}^\alpha(z, y) \geq p_{\mathbb{H}^n}^\alpha(x, \widetilde{z}) + p_{\mathbb{H}^n}^\alpha(\widetilde{z}, y)$. Therefore, we only need to prove that $p_{\mathbb{H}^n}^\alpha(x, \widetilde{z}) + p_{\mathbb{H}^n}^\alpha(\widetilde{z}, y) \geq p_{\mathbb{H}^n}^\alpha(x, y)$ for every three points x, y and $\widetilde{z}$ lying in a two-dimensional plane Π.

Thus, without loss of generality we can assume that $n = 2$ and the points x, y and $z \in \mathbb{H}^2$ are complex numbers.

Consider the two following cases.

(1) If $\mathrm{Re}\, x = \mathrm{Re}\, y$, then we denote by L the line, orthogonal to the real axis and containing x and y, and replace z with its orthogonal projection z' to L. Reasoning as above, we have $p_{\mathbb{H}^2}^\alpha(x, z) + p_{\mathbb{H}^2}^\alpha(z, y) \geq p_{\mathbb{H}^2}^\alpha(x, z') + p_{\mathbb{H}^2}^\alpha(z', y)$, and, therefore, we can reduce the problem to the one-dimensional case. From Theorem 5.2, it follows that $p_{\mathbb{H}^2}^\alpha(x, z') + p_{\mathbb{H}^2}^\alpha(z', y) = p_{\mathbb{R}^+}^\alpha(\mathrm{Im}\, x, \mathrm{Im}\, z') + p_{\mathbb{R}^+}^\alpha(\mathrm{Im}\, z', \mathrm{Im}\, y) \geq p_{\mathbb{R}^+}^\alpha(\mathrm{Im}\, x, \mathrm{Im}\, y) = p_{\mathbb{H}^2}^\alpha(x, y)$ and the triangle inequality is valid.

(2) If $\mathrm{Re}\, x \neq \mathrm{Re}\, y$, then we consider the circle C containing the points x and y and orthogonal to the real axis. Let C^+ be the upper half of C. There exists a Möbius transformation T that maps C^+ onto the positive part of the imaginary axis. Consider the points $x_1 = T(x)$, $y_1 = T(y)$ and $z_1 = T(z)$. For every $u, v \in \mathbb{H}^2$, $u \neq v$, we have $p_{\mathbb{H}^2}^\alpha(u, v) = (1 + \alpha t)^{-1/2}$ where $t = (\mathrm{Im}\, u \, \mathrm{Im}\, v)/(|u - v|^2)$. According to the well-known property of Möbius automorphisms of $\mathbb{H}^2$,

$$\frac{\mathrm{Im}\, u \, \mathrm{Im}\, v}{|u - v|^2} = \frac{\mathrm{Im}\, T(u) \, \mathrm{Im}\, T(v)}{|T(u) - T(v)|^2},$$

and we can conclude that T preserves the value $p_{\mathbb{H}^2}^\alpha(u, v)$, i.e. $p_{\mathbb{H}^2}^\alpha(T(u), T(v)) = p_{\mathbb{H}^2}^\alpha(u, v)$. Making use of this fact, we can replace x, y and z with x_1, y_1 and z_1; but for such points $\mathrm{Re}\, x_1 = \mathrm{Re}\, y_1$ and the triangle inequality, therefore, follows from Case 1). $\qquad\square$

It is interesting to study whether the point pair function defined as in (3.7) becomes a metric, if we replace the constant 4 with a smaller positive constant. The answer is negative, as proven below.

Theorem 5.15 *The function*

$$p_{\mathbb{B}^n}^\alpha(x, y) = \frac{|x - y|}{\sqrt{|x - y|^2 + \alpha(1 - |x|)(1 - |y|)}}, \quad x, y \in \mathbb{B}^n, \quad n \geq 1,$$

is not a metric for any constant $\alpha > 0$.

Proof Fix points $x = ke_1$, $y = -ke_1$, and $z = 0$, where $0 < k < 1$ and e_1 is the first unit vector. Then we have

$$p_{\mathbb{B}^n}^\alpha(x, y) > p_{\mathbb{B}^n}^\alpha(x, z) + p_{\mathbb{B}^n}^\alpha(z, y) \Leftrightarrow \frac{p_{\mathbb{B}^n}^\alpha(x, y)}{p_{\mathbb{B}^n}^\alpha(x, z) + p_{\mathbb{B}^n}^\alpha(z, y)}$$

$$= \sqrt{\frac{k^2 + \alpha(1 - k)}{4k^2 + \alpha(1 - k)^2}} > 1$$

$$\Leftrightarrow k^2 + \alpha(1 - k) > 4k^2 + \alpha(1 - k)^2$$

$$\Leftrightarrow -3k + \alpha(1 - k) > 0 \Leftrightarrow k < \alpha/(3 + \alpha).$$

Consequently, if we put $x = ke_1$, $y = -ke_1$, $k = \alpha/(4 + \alpha)$ and $z = 0$, then the triangle inequality does not hold. $\qquad\square$

6 Open Questions

On the base of numerical tests, we propose the following conjectures.

Conjecture 6.1 *For a constant $\alpha > 0$, the function*

$$p_{\mathbb{R}^n \setminus \overline{\mathbb{B}}^n}^\alpha(x, y) = \frac{|x - y|}{\sqrt{|x - y|^2 + \alpha(|x| - 1)(|y| - 1)}}, \quad x, y \in \mathbb{R}^n \setminus \overline{\mathbb{B}}^n, \ n \geq 2, \quad (6.2)$$

is a metric if and only if $\alpha \leq 12$.

Remark 6.3 Since the function $x \mapsto x/|x|^2$ maps $\mathbb{R}^n \setminus \overline{\mathbb{B}}^n$, $n \geq 2$, onto the domain $G = \mathbb{B}^n \setminus \{0\}$ and vice versa, it follows that if Conjecture 6.1 holds, then the quasimetric

$$\psi_G^\alpha(x, y) = p_{\mathbb{R}^n \setminus \overline{\mathbb{B}}^n}^\alpha\left(\frac{x}{|x|^2}, \frac{y}{|y|^2}\right)$$

$$= \frac{|xy|}{\sqrt{|x - y|^2 + \alpha|x|\,|y|(1 - |x|)(1 - |y|)}}, \quad x, y \in G, \quad (6.4)$$

is a metric on G if and only if $\alpha \in (0, 12]$. See Fig. 3 for the 'disks' of this function ψ_G^α.

Conjecture 6.5 *The point pair function p_G is a metric on the domain $G = \mathbb{R}^3 \setminus Z$, where Z is the z-axis.*

Remark 6.6 From the proof of Theorem 5.15 it follows that, for $\alpha > 0$, the generalized version $p_{\mathbb{B}^n}^\alpha$ of the point pair function can only be a quasi-metric in the unit ball with a constant $c(\alpha)$ that has the following lower bound:

$$c(\alpha) \geq \sup_{0 < k < 1} \sqrt{\frac{k^2 + \alpha(1 - k)}{4k^2 + \alpha(1 - k)^2}}. \quad (6.7)$$

Fig. 3 Disks
$\{x \in G \mid \psi_G^\alpha(x, 0.5) < r\}$ with
center 0.5 and radii
$r = 0.3, 0.5, 0.7, 0.8, 0.865, 0.9$
for the function ψ_G^α, $\alpha = 4$, and
$G = \mathbb{B}^2 \setminus \{0\}$, in the unit disk
$\mathbb{B}^2$
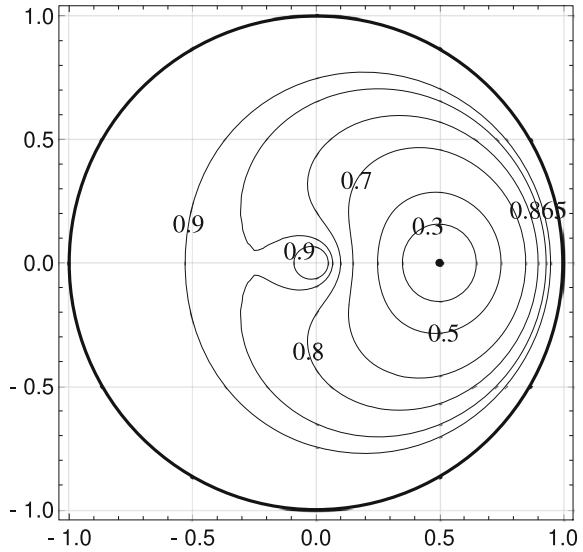
By differentiation, we have

$$\frac{\partial}{\partial k}\left(\frac{k^2 + \alpha(1-k)}{4k^2 + \alpha(1-k)^2}\right) = \frac{\alpha((\alpha+2)k^2 - 2(\alpha+3)k + \alpha)}{(4k^2 + \alpha(1-k)^2)^2} = 0$$

$$\Leftrightarrow (\alpha+2)k^2 - 2(\alpha+3)k + \alpha = 0$$

$$\Leftrightarrow k = \frac{\alpha + 3 \pm \sqrt{4\alpha + 9}}{\alpha + 2}.$$

It can be shown that the square-root expression on the right hand side of the inequality
(6.7) obtains its maximum with respect to $k \in (0, 1)$ at the point

$$k = \frac{\alpha + 3 - \sqrt{4\alpha + 9}}{\alpha + 2} \in (0, 1).$$

Consequently, the inequality (6.7) can be simplified to $c(\alpha) \geq c_*(\alpha)$ where

$$c_*(\alpha) = \sqrt{\frac{(\alpha + 3 - \sqrt{4\alpha + 9})^2 + \alpha(\alpha + 2)(\sqrt{4\alpha + 9} - 1)}{4(\alpha + 3 - \sqrt{4\alpha + 9})^2 + \alpha(1 - \sqrt{4\alpha + 9})^2}}.$$

Conjecture 6.8 *For $\alpha > 0$, the function*

$$p_{\mathbb{B}^n}^\alpha(x, y) = \frac{|x - y|}{\sqrt{|x - y|^2 + \alpha(1 - |x|)(1 - |y|)}}, \quad x, y \in \mathbb{B}^n, \, n \geq 1,$$

is a quasi-metric with the sharp constant $c_*(\alpha)$ or, equivalently, $c_*(\alpha)$ defined as above is the best of constants $c(\alpha)$ depending only on the value of α such that the inequality

$$p_{\mathbb{B}^n}^\alpha(x, y) \le c(\alpha)(p_{\mathbb{B}^n}^\alpha(x, z) + p_{\mathbb{B}^n}^\alpha(z, y))$$

holds for all $x, y, z \in \mathbb{B}^n$ and $n \ge 1$.

Declarations

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