EXTENSION IN GENERALIZED ORLICZ SPACES

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ABSTRACT. We prove that a Φ -function can be extended from a domain Ω to all of \mathbb{R}^n while preserving crucial properties for harmonic analysis on the generalized Orlicz space L^{Φ} .

1. INTRODUCTION

Generalized Orlicz spaces, also known as Musielak–Orlicz spaces, and related differential equations have been studied with increasing intensity recently, see, e.g., the references [1, 2, 3, 4, 6, 7, 9, 14, 15, 16] published since 2018. This year, we published the monograph [10] in which we present a new framework for the basics of these spaces. In contrast to earlier studies, we emphasize properties which are invariant under equivalence of Φ -functions. This means, in particular, that we replace convexity by the assumption that $\frac{\varphi(t)}{t}$ be almost increasing. Within this framework we can more easily use techniques familiar from the L^p -context, see for instance the papers [12, 13] with applications to PDE.

In the pre-release version of the book [10], we had included a section on extension of the Φ -function. However, this was removed from the final, published version, as we could not at that point prove a result with which we were satisfied. In this article we remedy this short-coming.

In many places, extension offers an easy way to prove result in $\Omega \subset \mathbb{R}^n$ from results in \mathbb{R}^n . For example, in variable exponent spaces we may argue

$$\|Mf\|_{L^{p(\cdot)}(\Omega)} \leqslant \|Mf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|f\|_{L^{p(\cdot)}(\Omega)}$$

where f is a zero extension of $f : \Omega \to \mathbb{R}$ to \mathbb{R}^n , provided that $p : \Omega \to [1, \infty]$ can be extended as well (cf. [8, Proposition 4.1.17]) and the maximal operator is bounded in \mathbb{R}^n .

In generalized Orlicz spaces this is not as simple. In [10] we showed that many results of harmonic analysis hold in L^{φ} provided φ satisfies conditions (A0), (A1) and (A2) (see the next section for definitions). However, the extension requires a stronger condition than (A1), namely (A1)_Ω, which is known to be equivalent with (A1) only in quasiconvex domains (Lemma 3.3). Furthermore, we show in the main result of this paper, Theorem 3.5, that (A1)_Ω also is necessary. Thus we solve the extension problem for the assumptions of [10]. In some recent papers, e.g. [13], also stronger assumptions have been used to obtain higher regularity. Whether the extension can be chosen to preserve those conditions remains an open problem.

2. PRELIMINARIES

In this section we introduce the terminology and auxiliary results needed in this paper. The notation $f \leq g$ means that there exists a constant C > 0 such that $f \leq Cg$. The notation $f \approx g$ means that $f \leq g \leq f$.

Date: October 10, 2019.

²⁰¹⁰ Mathematics Subject Classification. 46E30, 26B25.

Key words and phrases. Generalized Orlicz space, Musielak-Orlicz space, Phi-function, extension.

Definiton 2.1. A function $g: (0,\infty) \to \mathbb{R}$ is *almost increasing* if there exists a constant $a \ge 1$ such that $g(s) \le ag(t)$ for all 0 < s < t. Almost decreasing is defined analogously.

Increasing and decreasing functions are included in the previous definition as the special case a = 1.

Definiton 2.2. Let $f: \Omega \times [0, \infty) \to \mathbb{R}$ and p, q > 0. We say that f satisfies

(Inc)_p if $\frac{f(x,t)}{t^p}$ is increasing; (aInc)_p if $\frac{f(x,t)}{t^p}$ is almost increasing; (Dec)_q if $\frac{f(x,t)}{t^q}$ is decreasing; (aDec)_q if $\frac{f(x,t)}{t^q}$ is almost decreasing;

all conditions should hold for almost every $x \in \Omega$ and the almost increasing/decreasing constant should be independent of x.

Suppose that φ satisfies $(aInc)_{p_1}$. Then it satisfies $(aInc)_{p_2}$ for $p_2 < p_1$ and it does not satisfy $(aDec)_q$ for $q < p_1$. Likewise, if φ satisfies $(aDec)_{q_1}$, then it satisfies $(aDec)_{q_2}$ for $q_2 > q_1$ and it does not satisfy (aInc)_p for $p > q_1$.

Definiton 2.3. Let $\varphi \colon [0,\infty) \to [0,\infty]$ be increasing with $\varphi(0) = 0$, $\lim_{t\to 0^+} \varphi(t) = 0$ and $\lim_{t\to\infty}\varphi(t)=\infty$. We say that such φ is a (*weak*) Φ -function if it satisfies (aInc)₁ on $(0,\infty)$. The set of weak Φ -functions is denoted by Φ_w .

We mention the epithet "weak" only when special emphasis is needed. Note that when we speak about Φ -functions, we mean the weak Φ -functions of the previous definition, whereas many other authors use this term for convex Φ -functions, possibly with additional assumptions as well. If φ is convex and $\varphi(0) = 0$, then we obtain for 0 < s < t that

(2.4)
$$\varphi(s) = \varphi\left(\frac{s}{t}t + 0\right) \leqslant \frac{s}{t}\varphi(t) + \left(1 - \frac{s}{t}\right)\varphi(0) = \frac{s}{t}\varphi(t),$$

i.e. $(Inc)_1$ holds.

Definiton 2.5. A function $\varphi \colon \Omega \times [0, \infty) \to [0, \infty]$ is said to be a *(generalized weak)* Φ *function*, denoted $\varphi \in \Phi_w(\Omega)$, if $x \mapsto \varphi(y, |f(x)|)$ is measurable for every measurable f, $\varphi(y, \cdot)$ is a weak Φ -function for almost every $y \in \Omega$ and φ satisfies $(aInc)_1$.

Unless there is danger of confusion, we will drop the word "generalized". Note that if $x \mapsto \varphi(y,t)$ is measurable for every $t \ge 0$ and $t \mapsto \varphi(x,t)$ is left-continuous for almost every $x \in \Omega$, then $x \mapsto \varphi(y, |f(x)|)$ is measurable for every measurable f, [10, Theorem 2.5.4].

Two functions φ and ψ are *equivalent*, $\varphi \simeq \psi$, if there exists $L \ge 1$ such that $\varphi(x, \frac{t}{L}) \le 1$ $\psi(x,t) \leq \varphi(x,Lt)$ for all x and all $t \geq 0$. Short calculations show that \simeq is an equivalence relation. Note that if $\varphi \simeq \psi$, then $L^{\varphi}(\Omega) = L^{\psi}(\Omega)$, see Theorem 3.2.6 in [10].

Let us define a left-inverse of φ by

$$\varphi^{-1}(x,\tau) := \inf\{t \ge 0 : \varphi(x,t) \ge \tau\}.$$

Note that φ^{-1} is left-continuous when φ is increasing and $\varphi(0) = 0$. Moreover if $\varphi \in \Phi_w$, then φ satisfies (aInc)_p if and only if φ^{-1} satisfies (aDec)_{1/p}; and φ satisfies (aDec)_q if and only if φ^{-1} satisfies $(aInc)_{1/q}$. These and other properties can be found in [10, Chapter 2.3].

Definiton 2.6. Let $\varphi \in \Phi_w(\Omega)$. We define three conditions:

(A0) There exists $\beta \in (0,1]$ such that $\beta \leq \varphi^{-1}(x,1) \leq \frac{1}{\beta}$ for almost every $x \in \Omega$.

(A1) There exists $\beta \in (0, 1)$ such that

$$\beta \varphi^{-1}(x,t) \leqslant \varphi^{-1}(y,t)$$

for every $t \in [1, \frac{1}{|B|}]$, almost every $x, y \in B \cap \Omega$ and every ball B with $|B| \leq 1$.

(A2) For every s > 0 there exist $\beta \in (0, 1]$ and $h \in L^1(\Omega) \cap L^{\infty}(\Omega)$ such that

$$\beta \varphi^{-1}(x,t) \leqslant \varphi^{-1}(y,t)$$

for almost every $x, y \in \Omega$ and every $t \in [h(x) + h(y), s]$.

By [10, Lemma 4.2.7] (A2) is equivalent with the following condition: there exist $\varphi_{\infty} \in \Phi_{w}$, $h \in L^{1}(\Omega) \cap L^{\infty}(\Omega)$, s > 0 and $\beta \in (0, 1]$ such that

(2.7)
$$\varphi(x,\beta t) \leq \varphi_{\infty}(t) + h(x) \text{ and } \varphi_{\infty}(\beta t) \leq \varphi(x,t) + h(x)$$

for almost every $x \in \Omega$.

Note how the conditions (A0)–(A2) are formulated in terms of the inverse function φ . This turns out to be very convenient in many cases, since the appropriate range of t for which the comparison can be done is easily expressed for the inverse function. We also use the inverse function for the extension. In [10, Proposition 2.5.14], we showed that f is the inverse of some Φ -function if and only if it satisfies the following conditions:

- (1) f is increasing;
- (2) f is left-continuous;
- (3) f satisfies (aDec)₁;
- (4) f(t) = 0 if and only if t = 0, and, $f(t) = \infty$ if and only if $t = \infty$;
- (5) $x \mapsto f(x, t)$ is measurable for all $t \ge 0$

3. EXTENSION

In the following version of (A1) we can use any size of t, but have to pay in terms of a smaller constant for large t.

Definiton 3.1. Let $\Omega \subset \mathbb{R}^n$. We say that $\varphi \in \Phi_w(\Omega)$ satisfies (A1)_{Ω}, if there exist a constant $\beta \in (0,1]$ such that $\beta^{|x-y|t^{1/n}+1}\varphi^{-1}(y,t) \leq \varphi^{-1}(x,t)$ for all $x, y \in \Omega$ and $t \ge 1$.

By Theorem 2.3.6 of [10] we have $\varphi \simeq \psi$ if and only if $\varphi^{-1} \approx \psi^{-1}$. Hence we obtain the following lemma.

Lemma 3.2. The condition $(A1)_{\Omega}$ is invariant under equivalence of weak Φ -functions.

A domain $\Omega \subset \mathbb{R}^n$ is *quasi-convex*, if there exists a constant $K \ge 1$ such that every pair $x, y \in \Omega$ can be connect by a rectifiable path $\gamma \subset \Omega$ with the length $\ell(\gamma) \le K|x-y|$.

Lemma 3.3. If $\Omega \subset \mathbb{R}^n$ is quasi-convex, then $\varphi \in \Phi_w(\Omega)$ satisfies (A1) if and only if it satisfies (A1)_{Ω}.

Proof. Assume first that $(A1)_{\Omega}$ holds. Let B be a ball with $|B| \leq 1$ and $x, y \in \Omega \cap B$. Since $|x - y| \leq \operatorname{diam}(B), |x - y|t^{\frac{1}{n}} \leq c(n)$ for $t \in [1, \frac{1}{|B|}]$. Hence (A1) holds with constant $\beta^{c(n)+1}$.

Assume then that (A1) holds. Let $x, y \in \Omega$, $t \ge 1$ and $\gamma \subset \Omega$ be a path connecting x and y of length at most K | x - y |. Let $x_0 := x$ and ω_n be the measure of the unit ball. Choose points $x_j \in \gamma$ such that $\ell(\gamma(x, x_j)) = \frac{j}{(\omega_n t)^{1/n}}$ for $j = 1, \ldots, k - 1$ when possible and finally set $x_k = y$. Then $|x_{j-1} - x_j| \le \frac{1}{(\omega_n t)^{1/n}}$ for all j. Let B_{j+1} be an open ball such that $x_j, x_{j+1} \in B_{j+1}$ and diam $(B_{j+1}) = 2|x_j - x_{j+1}|$, see Figure 1. Then $\frac{1}{|B|} = \frac{1}{\omega_n |x_j - x_{j+1}|^n} \ge t$. Thus t is in the allowed range for (A1) and so $\beta \varphi^{-1}(x_{j+1}, t) \le \varphi^{-1}(x_j, t)$. With this chain of inequalities, we obtain that $\beta^k \varphi^{-1}(y, t) \le \varphi^{-1}(x, t)$. On the other hand, at most

$$k = \frac{\ell(\gamma(x, x_{k-1}))}{\operatorname{diam}(B)} + 1 \leqslant \frac{K|x-y|}{\operatorname{diam}(B)} + 1 \leqslant \frac{K|x-y|}{2/(\omega_n t)^{1/n}} + 1 = c'Kt^{1/n}|x-y| + 1$$

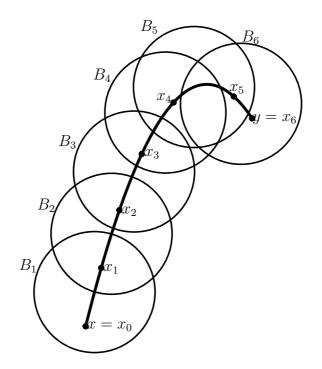


FIGURE 1. Points x_i and balls B_i in the proof of Lemma 3.3.

points x_j are needed, so that $\beta^{c'Kt^{1/n}|x-y|+1}\varphi^{-1}(y,t) \leq \varphi^{-1}(x,t)$ for all $x, y \in \Omega$ and $t \geq 1$.

Question 3.4. Does there exist $\Omega \subset \mathbb{R}^n$ and $\varphi \in \Phi_w(\Omega)$ such that (A1) holds but (A1)_{Ω} does not?

We say that $\psi \in \Phi_w(\mathbb{R}^n)$ is an *extension* of $\varphi \in \Phi_w(\Omega)$ if $\psi|_{\Omega} \simeq \varphi$. Since we consider properties which hold up to equivalence of Φ -functions this is a natural definition. However, if one wants identity in Ω this is easily achieved by choosing $\psi_2 := \varphi \chi_{\Omega} + \psi \chi_{\mathbb{R}^n \setminus \Omega}$, which is equivalent to ψ and hence has the same properties.

The next theorem was proved in [11, Proposition 5.2] but the proof was incorrect: the function f constructed was not increasing, its measurability was unclear, and $\psi|_{\Omega} \simeq \varphi$ was shown only for $t \ge 1$.

Theorem 3.5. Suppose that $\Omega \subset \mathbb{R}^n$ and $\varphi \in \Phi_w(\Omega)$. Then there exists an extension $\psi \in \Phi_w(\mathbb{R}^n)$ of φ which satisfies (A0), (A1) and (A2), if and only if φ satisfies (A0), (A1)_{Ω} and (A2).

If φ satisfies $(aInc)_p$ and/or $(aDec)_q$, then the extension can be taken to satisfy it/them, as well.

Proof. Suppose first that there exists an extension $\psi \in \Phi_w(\mathbb{R}^n)$ which satisfies (A0), (A1) and (A2). Since $\varphi \simeq \psi|_{\Omega}$ we find that φ satisfies (A0) and (A2). Let $x, y \in \Omega$. Since ψ satisfies (A1) and \mathbb{R}^n is quasi-convex, Lemma 3.3 implies that ψ satisfies (A1)_{\mathbb{R}^n}, so that $\beta^{t^{1/n}|x-y|+1}\psi^{-1}(y,t) \leq \psi^{-1}(x,t)$, Since $\psi|_{\Omega} \simeq \varphi$, this implies (A1)_{Ω} of φ by Lemma 3.2. So we can move on to the converse implication.

Let $\varphi \in \Phi_{w}(\Omega)$ satisfy (A0), (A1) $_{\Omega}$, (A2) and (aInc) $_{p}$ for some $p \ge 1$. Then φ^{-1} satisfies $(aDec)_{1/p}$. Let β_{0} be a constant form (A0) of φ and β be from (A1) $_{\Omega}$ of φ . Next we use (A2).

By [10, Lemma 4.2.7], there exists $\varphi_{\infty} \in \Phi_{w}$ and $h \in L^{1}(\Omega) \cap L^{\infty}(\Omega)$ such that

$$\varphi(x,\beta_2 t) \leqslant \varphi_{\infty}(t) + h(x) \text{ and } \varphi_{\infty}(t) \leqslant \varphi(x,\beta_2 t) + h(x)$$

for almost every $x \in \Omega$ when $\varphi_{\infty}(t) \leq \beta_0$ and $\varphi(x,t) \leq \beta_0$, respectively. Note that φ_{∞} is equivalent to $\liminf_{x\to\infty} \varphi(x,\cdot)$; hence satisfies (A0) and $(a\text{Dec})_{1/p}$ if φ does.

Denote $\hat{\Omega} := \Omega \cap \mathbb{Q}^n$ and define $f : \mathbb{R}^n \times [0, \infty] \to [0, \infty]$ by

$$f(x,t) := \begin{cases} \beta_0^2 \varphi^{-1}(x,t) \chi_\Omega(x) + \beta_0^2 \varphi_\infty^{-1}(t) \chi_{\mathbb{R}^n \backslash \Omega}(x) & \text{if } t \in [0,1];\\ \min\left\{ (\varphi_\Omega^-)^{-1}(t), \inf_{y \in \hat{\Omega}} \beta^{-|x-y|t^{1/n}} \varphi^{-1}(y,t) \right\} & \text{if } t \in (1,\infty);\\ \infty & \text{if } t = \infty. \end{cases}$$

The following properties follow directly from the corresponding properties of φ^{-1} and the definition of f:

- (1) f(x,t) = 0 iff t = 0, and $f(x,t) = \infty$ iff $t = \infty$.
- (2) f satisfies $(aDec)_{1/p}$ in [0, 1].
- (3) f satisfies (A0).

For properties (A0)–(A2) we think of f as the inverse in the conditions, i.e. φ^{-1} is replaced by f, not f^{-1} , in the inequalities. We next prove the following properties of f:

- (4) f is increasing.
- (5) f is left-continuous and measurable.
- (6) $f|_{\Omega} \approx \varphi^{-1}$.
- (7) f satisfies (A1).
- (8) f satisfies $(aInc)_{1/q}$ provided that φ satisfies $(aDec)_q$.

Claim (4): Since φ^{-1} , φ_{∞} and $(\varphi_{\Omega}^{-})^{-1}$ are increasing, the claim follows in each subinterval. To show that $f(x,1) \leq f(x,t)$ for 1 < t we note that $(\varphi_{\Omega}^{-})^{-1}(t) \geq (\varphi_{\Omega}^{-})^{-1}(1) \geq \beta_{0}$ and $\beta^{-|x-y|t^{1/n}}\varphi^{-1}(y,t) \geq \varphi^{-1}(y,1) \geq \beta_{0}$. Thus $f(x,t) \geq \beta_{0}$. On the other hand, $f(x,1) \leq \beta_{0}^{2}(\varphi_{\Omega}^{-})^{-1}(1) \leq \frac{\beta_{0}^{2}}{\beta_{0}} = \beta_{0}$.

<u>Claim (5)</u>: Since φ^{-1} , φ_{∞}^{-1} and $(\varphi_{\Omega}^{-})^{-1}$ are left-continuous and the minimum of leftcontinuous functions is left-continuous, the claim follows by the definition of f. By Lemma 2.5.12 of [10], $y \mapsto \varphi^{-1}(y, t)$ is measurable. The infimum of measurable functions over countable sets is measurable. Thus $x \mapsto f(x, t)$ is measurable for every t.

Claim (6): We show that $f \approx \varphi^{-1}$ in Ω . For $t \in [0, 1]$ this holds by the definition of f. If t > 1, then by $(A1)_{\Omega}$ for $x, y \in \Omega$ we have

$$\beta^{-|x-y|t^{1/n}}\varphi^{-1}(y,t) \ge \beta^{-|x-y|t^{1/n}}\beta^{|x-y|t^{1/n}+1}\varphi^{-1}(x,t) = \beta\varphi^{-1}(x,t).$$

Since $(\varphi_{\Omega}^{-})^{-1}(t) \ge \varphi^{-1}(x,t)$, this yields that $f(x,t) \ge \beta \varphi^{-1}(x,t)$. On the other hand by (A1)_Ω we have for $y \in \hat{\Omega}$ that

$$f(x,t) \leqslant \beta^{-|x-y|t^{1/n}} \varphi^{-1}(y,t) \leqslant \beta^{-|x-y|t^{1/n}} \beta^{-|x-y|t^{1/n}-1} \varphi^{-1}(x,t) \to \frac{1}{\beta} \varphi^{-1}(x,t)$$

as $y \to x$ and hence $f(x,t) \leq \frac{1}{\beta} \varphi^{-1}(x,t)$.

Claim (7): Let us then prove (A1). The required inequality for t = 1 follows from (A0), so we take $t \in (1, \frac{1}{|B|}]$ and $x, y \in B$. Note that $|x - y|t^{1/n} \leq c(n)$. Let $z_0 \in \hat{\Omega}$ be such that $\inf_{z \in \hat{\Omega}} \beta^{-|x-z|t^{1/n}} \varphi^{-1}(z, t) \ge \frac{1}{2} \beta^{-|x-z_0|t^{1/n}} \varphi^{-1}(z_0, t)$. Then by the definition of infimum, the

triangle inequality, and the choice of z_0 we obtain that

$$\inf_{z\in\hat{\Omega}}\beta^{-|x-z|t^{1/n}}\varphi^{-1}(z,t) \leqslant \beta^{-|y-z_0|t^{1/n}}\varphi^{-1}(z_0,t) \leqslant \beta^{-(|y-x|+|x-z_0|)t^{1/n}}\varphi^{-1}(z_0,t) \\
\leqslant 2\beta^{-c(n)}\inf_{z\in\hat{\Omega}}\beta^{-|x-z|t^{1/n}}\varphi^{-1}(z,t).$$

The part $(\varphi_{\Omega}^{-})^{-1}$ satisfies (A1) since it is independent of x. A short calculation shows that the minimum of two functions that satisfy (A1) also satisfies it.

<u>Claim (8)</u>: Assume that φ satisfies (aDec)_q. Then φ^{-1} and φ^{-1}_{∞} satisfy (aInc)_{1/q}, which imply the condition for f when $t \in [0, 1]$. For 1 < t < s, we use the condition for φ^{-1} as well as $\beta^{-|x-y|t^{1/n}} \leq \beta^{-|x-y|s^{1/n}}$ to conclude that

$$\frac{f(x,t)}{t^{1/q}} = \min\left\{\frac{(\varphi_{\Omega}^{-})^{-1}(t)}{t^{1/q}}, \inf_{y\in\hat{\Omega}}\beta^{-|x-y|t^{1/n}}\frac{\varphi^{-1}(y,t)}{t^{1/q}}\right\}$$
$$\lesssim \min\left\{\frac{(\varphi_{\Omega}^{-})^{-1}(s)}{s^{1/q}}, \inf_{y\in\hat{\Omega}}\beta^{-|x-y|t^{1/n}}\frac{\varphi^{-1}(y,s)}{s^{1/q}}\right\} \leqslant \frac{f(x,s)}{s^{1/q}}.$$

The case 1 = t < s follows as $\varepsilon \to 0^+$ since f is increasing:

$$\frac{f(x,1)}{1^{1/q}} \leqslant (1+\varepsilon)^{1/q} \frac{f(x,1+\varepsilon)}{(1+\varepsilon)^{1/q}} \lesssim (1+\varepsilon)^{1/q} \frac{f(x,s)}{s^{1/q}}.$$

The function f does not satisfy $(aDec)_1$, so it is not the inverse of any Φ -function. We therefore make a regularization to ensure this growth condition. Recall that φ satisfies $(aInc)_p$ for some $p \ge 1$. We define g(x, 0) := 0, $g(x, \infty) := \infty$ and

$$g(x,t) := t^{1/p} \inf_{0 < s \leq t} \frac{f(x,s)}{s^{1/p}}.$$

Since f satisfies $(aDec)_{1/p}$ on [0, 1], we have

(3.6)
$$t^{1/p} \inf_{0 < s \le t} \frac{f(x,s)}{s^{1/p}} \approx f(x,t)$$

for $t \in [0, 1]$, so that $g \approx f$ in the same range of t. From the corresponding properties of f we conclude that g is left-continuous and satisfies (A0).

Let us prove the following properties for g:

- (i) g is measurable.
- (ii) g satisfies $(Dec)_{1/p}$.
- (iii) $g|_{\Omega} \approx \varphi^{-1}$.
- (iv) g satisfies (A1).
- (v) g(x,t) = 0 iff t = 0, and $g(x,t) = \infty$ iff $t = \infty$.

(vi) g is increasing; and g satisfies $(aInc)_{1/q}$ provided that φ satisfies $(aDec)_q$.

Claim (i): Since f is left-continuous, we obtain that

$$g(x,t) = t^{1/p} \inf_{s \in (0,t] \cap Q} \frac{f(x,s)}{s^{1/p}}.$$

Since $x \mapsto f(x,t)$ is measurable, this implies that $x \mapsto g(x,t)$ is measurable as the infimum of countable many measurable functions.

Claim (ii): Let us next show that g satisfies $(\text{Dec})_{1/p}$. For that let $0 < t < \tau$. By the definition of g we obtain

$$\frac{g(x,t)}{t^{1/p}} = \inf_{0 < s \leqslant t} \frac{f(x,s)}{s^{1/p}} \ge \inf_{0 < s \leqslant \tau} \frac{f(x,s)}{s^{1/p}} = \frac{g(x,\tau)}{\tau^{1/p}}.$$

<u>Claim (iii)</u>: Since $f \approx \varphi^{-1}$ in Ω and φ^{-1} satisfies $(aDec)_{1/p}$, we obtain $g|_{\Omega} \approx \varphi^{-1}$ in the same way as in (3.6).

<u>Claim (iv)</u>: Then we show that g satisfies (A1). Let B be a ball with $|B| \leq 1$, $t \in [1, \frac{1}{|B|}]$ and $x, y \in \Omega$. Since f satisfies (aDec)_{1/p} in (0, 1], we have

(3.7)
$$g(y,t) \approx t^{1/p} \inf_{1 \le s \le t} \frac{f(y,s)}{s^{1/p}}$$

Thus by (A1) of f we obtain

$$g(x,t) \approx t^{1/p} \inf_{1 \le s \le t} \frac{f(x,s)}{s^{1/p}} \lesssim t^{1/p} \inf_{1 \le s \le t} \frac{f(y,s)}{s^{1/p}} \approx g(y,t).$$

Claim (v): Using the inequality $f(x, 1) \leq f(x, s) \leq f(x, t)$ and (A0), we obtain by (3.7) for 1 < t, that

$$0 < \beta_0 \leqslant g(x,t) \leqslant t^{1/p} f(x,t) < \infty.$$

By (3.6) we have $f \approx g$ for $s \in [0, 1]$. Thus by the corresponding property of f we have g(x, t) = 0 if and only if t = 0, and $g(x, t) = \infty$ if and only if $t = \infty$.

<u>Claim (vi)</u>: Assume that f satisfies $(aInc)_{1/q}$ with constant $L \ge 1$. Let 0 < t < s, $\varepsilon > 0$ and choose $\theta \in (0, 1]$ such that $g(x, s) \ge \theta^{-1/p} f(x, \theta s) - \varepsilon$. Then by the definition of g, $(aDec)_{1/q}$ of f and the choice of θ , we obtain that

$$\frac{g(x,t)}{t^{1/q}} \leqslant t^{1/p-1/q} \frac{f(x,\theta t)}{(\theta t)^{1/p}} = \theta^{1/q-1/p} \frac{f(x,\theta t)}{(\theta t)^{1/q}} \leqslant L \theta^{1/q-1/p} \frac{f(x,\theta s)}{(\theta s)^{1/q}} \leqslant L \frac{g(x,s) + \varepsilon}{s^{1/q}}$$

Thus letting $\varepsilon \to 0^+$, we find that g satisfies $(aInc)_{1/q}$. If we take L = 1 and 1/q = 0, this implies that g is increasing (since f is increasing).

We set $\psi := g^{-1}$. Using (v)–(ii), increasing from (vi) and Proposition 2.5.14 in [10] we obtain that $\psi \in \Phi_w(\mathbb{R}^n)$ and $\psi^{-1} = (g^{-1})^{-1} = g$. Since $\varphi^{-1} \approx g = \psi^{-1}$, it follows that $\psi|_{\Omega} \simeq \varphi$ [10, Theorem 2.3.6]. Now properties (A0) and (A1) for ψ follow. Moreover, if φ satisfies (aDec)_q, then by (vi) ψ satisfies it as well.

When $x \in \mathbb{R}^n \setminus \Omega$ and $t \in [0, 1]$, we have $g(x, t) \approx f(x, t) = \beta_0^2 \varphi_{\infty}^{-1}(t)$. It follows that $\psi(x, t) \simeq \varphi_{\infty}(t)$ for sufficiently small values of t and x as before. Therefore, ψ satisfies the condition of (2.7), and so ψ satisfies (A2).

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