# EXTENSION IN GENERALIZED ORLICZ SPACES 

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#### Abstract

We prove that a $\Phi$-function can be extended from a domain $\Omega$ to all of $\mathbb{R}^{n}$ while preserving crucial properties for harmonic analysis on the generalized Orlicz space $L^{\Phi}$.


## 1. Introduction

Generalized Orlicz spaces, also known as Musielak-Orlicz spaces, and related differential equations have been studied with increasing intensity recently, see, e.g., the references $[1,2,3,4,6,7,9,14,15,16]$ published since 2018. This year, we published the monograph [10] in which we present a new framework for the basics of these spaces. In contrast to earlier studies, we emphasize properties which are invariant under equivalence of $\Phi$-functions. This means, in particular, that we replace convexity by the assumption that $\frac{\varphi(t)}{t}$ be almost increasing. Within this framework we can more easily use techniques familiar from the $L^{p}$-context, see for instance the papers [12,13] with applications to PDE.

In the pre-release version of the book [10], we had included a section on extension of the $\Phi$-function. However, this was removed from the final, published version, as we could not at that point prove a result with which we were satisfied. In this article we remedy this short-coming.

In many places, extension offers an easy way to prove result in $\Omega \subset \mathbb{R}^{n}$ from results in $\mathbb{R}^{n}$. For example, in variable exponent spaces we may argue

$$
\|M f\|_{L^{p(\cdot)}(\Omega)} \leqslant\|M f\|_{L^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \leqslant\|f\|_{L^{p(\cdot)}\left(\mathbb{R}^{n}\right)}=\|f\|_{L^{p(\cdot)}(\Omega)}
$$

where $f$ is a zero extension of $f: \Omega \rightarrow \mathbb{R}$ to $\mathbb{R}^{n}$, provided that $p: \Omega \rightarrow[1, \infty]$ can be extended as well (cf. [8, Proposition 4.1.17]) and the maximal operator is bounded in $\mathbb{R}^{n}$.

In generalized Orlicz spaces this is not as simple. In [10] we showed that many results of harmonic analysis hold in $L^{\varphi}$ provided $\varphi$ satisfies conditions (A0), (A1) and (A2) (see the next section for definitions). However, the extension requires a stronger condition than (A1), namely $(\mathrm{A} 1)_{\Omega}$, which is known to be equivalent with (A1) only in quasiconvex domains (Lemma 3.3). Furthermore, we show in the main result of this paper, Theorem 3.5, that (A1) $)_{\Omega}$ also is necessary. Thus we solve the extension problem for the assumptions of [10]. In some recent papers, e.g. [13], also stronger assumptions have been used to obtain higher regularity. Whether the extension can be chosen to preserve those conditions remains an open problem.

## 2. Preliminaries

In this section we introduce the terminology and auxiliary results needed in this paper. The notation $f \lesssim g$ means that there exists a constant $C>0$ such that $f \leqslant C g$. The notation $f \approx g$ means that $f \lesssim g \lesssim f$.

Definiton 2.1. A function $g:(0, \infty) \rightarrow \mathbb{R}$ is almost increasing if there exists a constant $a \geqslant 1$ such that $g(s) \leqslant a g(t)$ for all $0<s<t$. Almost decreasing is defined analogously.

Increasing and decreasing functions are included in the previous definition as the special case $a=1$.
Definiton 2.2. Let $f: \Omega \times[0, \infty) \rightarrow \mathbb{R}$ and $p, q>0$. We say that $f$ satisfies
(Inc) $)_{p}$ if $\frac{f(x, t)}{t^{p}}$ is increasing;
(aInc) $)_{p}$ if $\frac{f(x, t)}{t^{p}}$ is almost increasing;
$(\mathrm{Dec})_{q}$ if $\frac{f(x, t)}{t^{q}}$ is decreasing;
$(\mathrm{aDec})_{q}$ if $\frac{f(x, t)}{t^{q}}$ is almost decreasing;
all conditions should hold for almost every $x \in \Omega$ and the almost increasing/decreasing constant should be independent of $x$.

Suppose that $\varphi$ satisfies (aInc) $)_{p_{1}}$. Then it satisfies (aInc) $p_{p_{2}}$ for $p_{2}<p_{1}$ and it does not satisfy $(\mathrm{aDec})_{q}$ for $q<p_{1}$. Likewise, if $\varphi$ satisfies $(\mathrm{aDec})_{q_{1}}$, then it satisfies $(\mathrm{aDec})_{q_{2}}$ for $q_{2}>q_{1}$ and it does not satisfy (aInc) ${ }_{p}$ for $p>q_{1}$.
Definiton 2.3. Let $\varphi:[0, \infty) \rightarrow[0, \infty]$ be increasing with $\varphi(0)=0, \lim _{t \rightarrow 0^{+}} \varphi(t)=0$ and $\lim _{t \rightarrow \infty} \varphi(t)=\infty$. We say that such $\varphi$ is a (weak) $\Phi$-function if it satisfies (aInc) $)_{1}$ on $(0, \infty)$. The set of weak $\Phi$-functions is denoted by $\Phi_{\mathrm{w}}$.

We mention the epithet "weak" only when special emphasis is needed. Note that when we speak about $\Phi$-functions, we mean the weak $\Phi$-functions of the previous definition, whereas many other authors use this term for convex $\Phi$-functions, possibly with additional assumptions as well. If $\varphi$ is convex and $\varphi(0)=0$, then we obtain for $0<s<t$ that

$$
\begin{equation*}
\varphi(s)=\varphi\left(\frac{s}{t} t+0\right) \leqslant \frac{s}{t} \varphi(t)+\left(1-\frac{s}{t}\right) \varphi(0)=\frac{s}{t} \varphi(t) \tag{2.4}
\end{equation*}
$$

i.e. $(\mathrm{Inc})_{1}$ holds.

Definiton 2.5. A function $\varphi: \Omega \times[0, \infty) \rightarrow[0, \infty]$ is said to be a (generalized weak) $\Phi$ function, denoted $\varphi \in \Phi_{\mathrm{w}}(\Omega)$, if $x \mapsto \varphi(y,|f(x)|)$ is measurable for every measurable $f$, $\varphi(y, \cdot)$ is a weak $\Phi$-function for almost every $y \in \Omega$ and $\varphi$ satisfies (aInc) $)_{1}$.

Unless there is danger of confusion, we will drop the word "generalized". Note that if $x \mapsto \varphi(y, t)$ is measurable for every $t \geqslant 0$ and $t \mapsto \varphi(x, t)$ is left-continuous for almost every $x \in \Omega$, then $x \mapsto \varphi(y,|f(x)|)$ is measurable for every measurable $f$, [10, Theorem 2.5.4].

Two functions $\varphi$ and $\psi$ are equivalent, $\varphi \simeq \psi$, if there exists $L \geqslant 1$ such that $\varphi\left(x, \frac{t}{L}\right) \leqslant$ $\psi(x, t) \leqslant \varphi(x, L t)$ for all $x$ and all $t \geqslant 0$. Short calculations show that $\simeq$ is an equivalence relation. Note that if $\varphi \simeq \psi$, then $L^{\varphi}(\Omega)=L^{\psi}(\Omega)$, see Theorem 3.2.6 in [10].
Let us define a left-inverse of $\varphi$ by

$$
\varphi^{-1}(x, \tau):=\inf \{t \geqslant 0: \varphi(x, t) \geqslant \tau\}
$$

Note that $\varphi^{-1}$ is left-continuous when $\varphi$ is increasing and $\varphi(0)=0$. Moreover if $\varphi \in \Phi_{\mathrm{w}}$, then $\varphi$ satisfies $(\mathrm{aInc})_{p}$ if and only if $\varphi^{-1}$ satisfies $(\mathrm{aDec})_{1 / p}$; and $\varphi$ satisfies $(\mathrm{aDec})_{q}$ if and only if $\varphi^{-1}$ satisfies (aInc) $)_{1 / q}$. These and other properties can be found in [10, Chapter 2.3].
Definiton 2.6. Let $\varphi \in \Phi_{\mathrm{w}}(\Omega)$. We define three conditions:
(A0) There exists $\beta \in(0,1]$ such that $\beta \leqslant \varphi^{-1}(x, 1) \leqslant \frac{1}{\beta}$ for almost every $x \in \Omega$.
(A1) There exists $\beta \in(0,1)$ such that

$$
\beta \varphi^{-1}(x, t) \leqslant \varphi^{-1}(y, t)
$$

for every $t \in\left[1, \frac{1}{|B|}\right]$, almost every $x, y \in B \cap \Omega$ and every ball $B$ with $|B| \leqslant 1$.
(A2) For every $s>0$ there exist $\beta \in(0,1]$ and $h \in L^{1}(\Omega) \cap L^{\infty}(\Omega)$ such that

$$
\beta \varphi^{-1}(x, t) \leqslant \varphi^{-1}(y, t)
$$

for almost every $x, y \in \Omega$ and every $t \in[h(x)+h(y), s]$.
By [10, Lemma 4.2.7] (A2) is equivalent with the following condition: there exist $\varphi_{\infty} \in$ $\Phi_{\mathrm{w}}, h \in L^{1}(\Omega) \cap L^{\infty}(\Omega), s>0$ and $\beta \in(0,1]$ such that

$$
\begin{equation*}
\varphi(x, \beta t) \leqslant \varphi_{\infty}(t)+h(x) \quad \text { and } \quad \varphi_{\infty}(\beta t) \leqslant \varphi(x, t)+h(x) \tag{2.7}
\end{equation*}
$$

for almost every $x \in \Omega$.
Note how the conditions (A0)-(A2) are formulated in terms of the inverse function $\varphi$. This turns out to be very convenient in many cases, since the appropriate range of $t$ for which the comparison can be done is easily expressed for the inverse function. We also use the inverse function for the extension. In [10, Proposition 2.5.14], we showed that $f$ is the inverse of some $\Phi$-function if and only if it satisfies the following conditions:
(1) $f$ is increasing;
(2) $f$ is left-continuous;
(3) $f$ satisfies $(\mathrm{aDec})_{1}$;
(4) $f(t)=0$ if and only if $t=0$, and, $f(t)=\infty$ if and only if $t=\infty$;
(5) $x \mapsto f(x, t)$ is measurable for all $t \geqslant 0$

## 3. Extension

In the following version of (A1) we can use any size of $t$, but have to pay in terms of a smaller constant for large $t$.

Definiton 3.1. Let $\Omega \subset \mathbb{R}^{n}$. We say that $\varphi \in \Phi_{\mathrm{w}}(\Omega)$ satisfies (A1) $)_{\Omega}$, if there exist a constant $\beta \in(0,1]$ such that $\beta^{|x-y| t^{1 / n}+1} \varphi^{-1}(y, t) \leqslant \varphi^{-1}(x, t)$ for all $x, y \in \Omega$ and $t \geqslant 1$.

By Theorem 2.3.6 of [10] we have $\varphi \simeq \psi$ if and only if $\varphi^{-1} \approx \psi^{-1}$. Hence we obtain the following lemma.
Lemma 3.2. The condition $(\mathrm{A} 1)_{\Omega}$ is invariant under equivalence of weak $\Phi$-functions.
A domain $\Omega \subset \mathbb{R}^{n}$ is quasi-convex, if there exists a constant $K \geqslant 1$ such that every pair $x, y \in \Omega$ can be connect by a rectifiable path $\gamma \subset \Omega$ with the length $\ell(\gamma) \leqslant K|x-y|$.
Lemma 3.3. If $\Omega \subset \mathbb{R}^{n}$ is quasi-convex, then $\varphi \in \Phi_{\mathrm{w}}(\Omega)$ satisfies (A1) if and only if it satisfies (A1) ${ }_{\Omega}$.

Proof. Assume first that (A1) $)_{\Omega}$ holds. Let $B$ be a ball with $|B| \leqslant 1$ and $x, y \in \Omega \cap B$. Since $|x-y| \leqslant \operatorname{diam}(B),|x-y| t^{\frac{1}{n}} \leqslant c(n)$ for $t \in\left[1, \frac{1}{|B|}\right]$. Hence (A1) holds with constant $\beta^{c(n)+1}$.

Assume then that (A1) holds. Let $x, y \in \Omega, t \geqslant 1$ and $\gamma \subset \Omega$ be a path connecting $x$ and $y$ of length at most $K|x-y|$. Let $x_{0}:=x$ and $\omega_{n}$ be the measure of the unit ball. Choose points $x_{j} \in \gamma$ such that $\ell\left(\gamma\left(x, x_{j}\right)\right)=\frac{j}{\left(\omega_{n} t\right)^{1 / n}}$ for $j=1, \ldots, k-1$ when possible and finally set $x_{k}=y$. Then $\left|x_{j-1}-x_{j}\right| \leqslant \frac{1}{\left(\omega_{n} t\right)^{1 / n}}$ for all $j$. Let $B_{j+1}$ be an open ball such that $x_{j}, x_{j+1} \in B_{j+1}$ and $\operatorname{diam}\left(B_{j+1}\right)=2\left|x_{j}-x_{j+1}\right|$, see Figure 1. Then $\frac{1}{|B|}=\frac{1}{\omega_{n}\left|x_{j}-x_{j+1}\right|^{n}} \geqslant t$. Thus $t$ is in the allowed range for (A1) and so $\beta \varphi^{-1}\left(x_{j+1}, t\right) \leqslant \varphi^{-1}\left(x_{j}, t\right)$. With this chain of inequalities, we obtain that $\beta^{k} \varphi^{-1}(y, t) \leqslant \varphi^{-1}(x, t)$. On the other hand, at most

$$
k=\frac{\ell\left(\gamma\left(x, x_{k-1}\right)\right)}{\operatorname{diam}(B)}+1 \leqslant \frac{K|x-y|}{\operatorname{diam}(B)}+1 \leqslant \frac{K|x-y|}{2 /\left(\omega_{n} t\right)^{1 / n}}+1=c^{\prime} K t^{1 / n}|x-y|+1
$$



Figure 1. Points $x_{j}$ and balls $B_{j}$ in the proof of Lemma 3.3.
points $x_{j}$ are needed, so that $\beta^{c^{\prime} K t^{1 / n}|x-y|+1} \varphi^{-1}(y, t) \leqslant \varphi^{-1}(x, t)$ for all $x, y \in \Omega$ and $t \geqslant$ 1.

Question 3.4. Does there exist $\Omega \subset \mathbb{R}^{n}$ and $\varphi \in \Phi_{\mathrm{w}}(\Omega)$ such that (A1) holds but (A1) $)_{\Omega}$ does not?

We say that $\psi \in \Phi_{\mathrm{w}}\left(\mathbb{R}^{n}\right)$ is an extension of $\varphi \in \Phi_{\mathrm{w}}(\Omega)$ if $\left.\psi\right|_{\Omega} \simeq \varphi$. Since we consider properties which hold up to equivalence of $\Phi$-functions this is a natural definition. However, if one wants identity in $\Omega$ this is easily achieved by choosing $\psi_{2}:=\varphi \chi_{\Omega}+\psi \chi_{\mathbb{R}^{n} \backslash \Omega}$, which is equivalent to $\psi$ and hence has the same properties.

The next theorem was proved in [11, Proposition 5.2] but the proof was incorrect: the function $f$ constructed was not increasing, its measurability was unclear, and $\left.\psi\right|_{\Omega} \simeq \varphi$ was shown only for $t \geqslant 1$.

Theorem 3.5. Suppose that $\Omega \subset \mathbb{R}^{n}$ and $\varphi \in \Phi_{\mathrm{w}}(\Omega)$. Then there exists an extension $\psi \in \Phi_{\mathrm{w}}\left(\mathbb{R}^{n}\right)$ of $\varphi$ which satisfies (A0), (A1) and (A2), if and only if $\varphi$ satisfies (A0), (A1) $)_{\Omega}$ and ( A 2 ).
If $\varphi$ satisfies $(\mathrm{aInc})_{p}$ and/or $(\mathrm{aDec})_{q}$, then the extension can be taken to satisfy it/them, as well.

Proof. Suppose first that there exists an extension $\psi \in \Phi_{\mathrm{w}}\left(\mathbb{R}^{n}\right)$ which satisfies (A0), (A1) and (A2). Since $\left.\varphi \simeq \psi\right|_{\Omega}$ we find that $\varphi$ satisfies (A0) and (A2). Let $x, y \in \Omega$. Since $\psi$ satisfies (A1) and $\mathbb{R}^{n}$ is quasi-convex, Lemma 3.3 implies that $\psi$ satisfies $(\mathrm{A} 1)_{\mathbb{R}^{n}}$, so that $\beta^{t^{1 / n}|x-y|+1} \psi^{-1}(y, t) \leqslant \psi^{-1}(x, t)$, Since $\left.\psi\right|_{\Omega} \simeq \varphi$, this implies (A1) $)_{\Omega}$ of $\varphi$ by Lemma 3.2. So we can move on to the converse implication.

Let $\varphi \in \Phi_{\mathrm{w}}(\Omega)$ satisfy (A0), (A1) $)_{\Omega}$, (A2) and (aInc) ${ }_{p}$ for some $p \geqslant 1$. Then $\varphi^{-1}$ satisfies $(\mathrm{aDec})_{1 / p}$. Let $\beta_{0}$ be a constant form (A0) of $\varphi$ and $\beta$ be from (A1) $)_{\Omega}$ of $\varphi$. Next we use (A2).

By [10, Lemma 4.2.7], there exists $\varphi_{\infty} \in \Phi_{\mathrm{w}}$ and $h \in L^{1}(\Omega) \cap L^{\infty}(\Omega)$ such that

$$
\varphi\left(x, \beta_{2} t\right) \leqslant \varphi_{\infty}(t)+h(x) \quad \text { and } \quad \varphi_{\infty}(t) \leqslant \varphi\left(x, \beta_{2} t\right)+h(x)
$$

for almost every $x \in \Omega$ when $\varphi_{\infty}(t) \leqslant \beta_{0}$ and $\varphi(x, t) \leqslant \beta_{0}$, respectively. Note that $\varphi_{\infty}$ is equivalent to $\liminf _{x \rightarrow \infty} \varphi(x, \cdot)$; hence satisfies (A0) and (aDec) $)_{1 / p}$ if $\varphi$ does.

Denote $\hat{\Omega}:=\Omega \cap \mathbb{Q}^{n}$ and define $f: \mathbb{R}^{n} \times[0, \infty] \rightarrow[0, \infty]$ by

$$
f(x, t):= \begin{cases}\beta_{0}^{2} \varphi^{-1}(x, t) \chi_{\Omega}(x)+\beta_{0}^{2} \varphi_{\infty}^{-1}(t) \chi_{\mathbb{R}^{n} \backslash \Omega}(x) & \text { if } t \in[0,1] \\ \min \left\{\left(\varphi_{\Omega}^{-}\right)^{-1}(t), \inf _{y \in \hat{\Omega}} \beta^{-|x-y| t^{1 / n}} \varphi^{-1}(y, t)\right\} & \text { if } t \in(1, \infty) \\ \infty & \text { if } t=\infty\end{cases}
$$

The following properties follow directly from the corresponding properties of $\varphi^{-1}$ and the definition of $f$ :
(1) $f(x, t)=0$ iff $t=0$, and $f(x, t)=\infty$ iff $t=\infty$.
(2) $f$ satisfies $(\mathrm{aDec})_{1 / p}$ in $[0,1]$.
(3) $f$ satisfies (A0).

For properties (A0)-(A2) we think of $f$ as the inverse in the conditions, i.e. $\varphi^{-1}$ is replaced by $f$, not $f^{-1}$, in the inequalities. We next prove the following properties of $f$ :
(4) $f$ is increasing.
(5) $f$ is left-continuous and measurable.
(6) $\left.f\right|_{\Omega} \approx \varphi^{-1}$.
(7) $f$ satisfies (A1).
(8) $f$ satisfies $(\mathrm{aInc})_{1 / q}$ provided that $\varphi$ satisfies $(\mathrm{aDec})_{q}$.

Claim (4): Since $\varphi^{-1}, \varphi_{\infty}$ and $\left(\varphi_{\Omega}^{-}\right)^{-1}$ are increasing, the claim follows in each subinterval. To show that $f(x, 1) \leqslant f(x, t)$ for $1<t$ we note that $\left(\varphi_{\Omega}^{-}\right)^{-1}(t) \geqslant\left(\varphi_{\Omega}^{-}\right)^{-1}(1) \geqslant \beta_{0}$ and $\beta^{-|x-y| t^{1 / n}} \varphi^{-1}(y, t) \geqslant \varphi^{-1}(y, 1) \geqslant \beta_{0}$. Thus $f(x, t) \geqslant \beta_{0}$. On the other hand, $f(x, 1) \leqslant$ $\beta_{0}^{2}\left(\varphi_{\Omega}^{-}\right)^{-1}(1) \leqslant \frac{\beta_{0}^{2}}{\beta_{0}}=\beta_{0}$.

Claim (5): Since $\varphi^{-1}, \varphi_{\infty}^{-1}$ and $\left(\varphi_{\Omega}^{-}\right)^{-1}$ are left-continuous and the minimum of leftcontinuous functions is left-continuous, the claim follows by the definition of $f$. By Lemma 2.5.12 of [10], $y \mapsto \varphi^{-1}(y, t)$ is measurable. The infimum of measurable functions over countable sets is measurable. Thus $x \mapsto f(x, t)$ is measurable for every $t$.

Claim (6): We show that $f \approx \varphi^{-1}$ in $\Omega$. For $t \in[0,1]$ this holds by the definition of $f$. If $t>1$, then by $(\mathrm{A} 1)_{\Omega}$ for $x, y \in \Omega$ we have

$$
\beta^{-|x-y| t^{1 / n}} \varphi^{-1}(y, t) \geqslant \beta^{-|x-y| t^{1 / n}} \beta^{|x-y| t^{1 / n}+1} \varphi^{-1}(x, t)=\beta \varphi^{-1}(x, t) .
$$

Since $\left(\varphi_{\Omega}^{-}\right)^{-1}(t) \geqslant \varphi^{-1}(x, t)$, this yields that $f(x, t) \geqslant \beta \varphi^{-1}(x, t)$. On the other hand by $(\mathrm{A} 1)_{\Omega}$ we have for $y \in \hat{\Omega}$ that

$$
f(x, t) \leqslant \beta^{-|x-y| t^{1 / n}} \varphi^{-1}(y, t) \leqslant \beta^{-|x-y| t^{1 / n}} \beta^{-|x-y| t^{1 / n}-1} \varphi^{-1}(x, t) \rightarrow \frac{1}{\beta} \varphi^{-1}(x, t)
$$

as $y \rightarrow x$ and hence $f(x, t) \leqslant \frac{1}{\beta} \varphi^{-1}(x, t)$.
Claim (7): Let us then prove (A1). The required inequality for $t=1$ follows from (A0), so we take $t \in\left(1, \frac{1}{|B|}\right]$ and $x, y \in B$. Note that $|x-y| t^{1 / n} \leqslant c(n)$. Let $z_{0} \in \hat{\Omega}$ be such that $\inf _{z \in \hat{\Omega}} \beta^{-|x-z| t^{1 / n}} \varphi^{-1}(z, t) \geqslant \frac{1}{2} \beta^{-\left|x-z_{0}\right| t^{1 / n}} \varphi^{-1}\left(z_{0}, t\right)$. Then by the definition of infimum, the
triangle inequality, and the choice of $z_{0}$ we obtain that

$$
\begin{aligned}
\inf _{z \in \hat{\Omega}} \beta^{-|x-z| t^{1 / n}} \varphi^{-1}(z, t) & \leqslant \beta^{-\left|y-z_{0}\right| t^{1 / n}} \varphi^{-1}\left(z_{0}, t\right) \leqslant \beta^{-\left(|y-x|+\left|x-z_{0}\right|\right) t^{1 / n}} \varphi^{-1}\left(z_{0}, t\right) \\
& \leqslant 2 \beta^{-c(n)} \inf _{z \in \hat{\Omega}} \beta^{-|x-z| t^{1 / n}} \varphi^{-1}(z, t)
\end{aligned}
$$

The part $\left(\varphi_{\Omega}^{-}\right)^{-1}$ satisfies (A1) since it is independent of $x$. A short calculation shows that the minimum of two functions that satisfy (A1) also satisfies it.

Claim (8): Assume that $\varphi$ satisfies (aDec) ${ }_{q}$. Then $\varphi^{-1}$ and $\varphi_{\infty}^{-1}$ satisfy (aInc) $)_{1 / q}$, which imply the condition for $f$ when $t \in[0,1]$. For $1<t<s$, we use the condition for $\varphi^{-1}$ as well as $\beta^{-|x-y| t^{1 / n}} \leqslant \beta^{-|x-y| s^{1 / n}}$ to conclude that

$$
\begin{aligned}
\frac{f(x, t)}{t^{1 / q}} & =\min \left\{\frac{\left(\varphi_{\Omega}^{-}\right)^{-1}(t)}{t^{1 / q}}, \inf _{y \in \hat{\Omega}} \beta^{-|x-y| t^{1 / n}} \frac{\varphi^{-1}(y, t)}{t^{1 / q}}\right\} \\
& \lesssim \min \left\{\frac{\left(\varphi_{\Omega}^{-}\right)^{-1}(s)}{s^{1 / q}}, \inf _{y \in \hat{\Omega}} \beta^{-|x-y| t^{1 / n}} \frac{\varphi^{-1}(y, s)}{s^{1 / q}}\right\} \leqslant \frac{f(x, s)}{s^{1 / q}} .
\end{aligned}
$$

The case $1=t<s$ follows as $\varepsilon \rightarrow 0^{+}$since $f$ is increasing:

$$
\frac{f(x, 1)}{1^{1 / q}} \leqslant(1+\varepsilon)^{1 / q} \frac{f(x, 1+\varepsilon)}{(1+\varepsilon)^{1 / q}} \lesssim(1+\varepsilon)^{1 / q} \frac{f(x, s)}{s^{1 / q}} .
$$

The function $f$ does not satisfy $(\mathrm{aDec})_{1}$, so it is not the inverse of any $\Phi$-function. We therefore make a regularization to ensure this growth condition. Recall that $\varphi$ satisfies (aInc) ${ }_{p}$ for some $p \geqslant 1$. We define $g(x, 0):=0, g(x, \infty):=\infty$ and

$$
g(x, t):=t^{1 / p} \inf _{0<s \leqslant t} \frac{f(x, s)}{s^{1 / p}} .
$$

Since $f$ satisfies $(\mathrm{aDec})_{1 / p}$ on $[0,1]$, we have

$$
\begin{equation*}
t^{1 / p} \inf _{0<s \leqslant t} \frac{f(x, s)}{s^{1 / p}} \approx f(x, t) \tag{3.6}
\end{equation*}
$$

for $t \in[0,1]$, so that $g \approx f$ in the same range of $t$. From the corresponding properties of $f$ we conclude that $g$ is left-continuous and satisfies (A0).

Let us prove the following properties for $g$ :
(i) $g$ is measurable.
(ii) $g$ satisfies $(\mathrm{Dec})_{1 / p}$.
(iii) $\left.g\right|_{\Omega} \approx \varphi^{-1}$.
(iv) $g$ satisfies (A1).
(v) $g(x, t)=0$ iff $t=0$, and $g(x, t)=\infty$ iff $t=\infty$.
(vi) $g$ is increasing; and $g$ satisfies (aInc) $)_{1 / q}$ provided that $\varphi$ satisfies (aDec) ${ }_{q}$.

Claim (i): Since $f$ is left-continuous, we obtain that

$$
g(x, t)=t^{1 / p} \inf _{s \in(0, t] \cap Q} \frac{f(x, s)}{s^{1 / p}} .
$$

Since $x \mapsto f(x, t)$ is measurable, this implies that $x \mapsto g(x, t)$ is measurable as the infimum of countable many measurable functions.
 definition of $g$ we obtain

$$
\frac{g(x, t)}{t^{1 / p}}=\inf _{0<s \leqslant t} \frac{f(x, s)}{s^{1 / p}} \geqslant \inf _{0<s \leqslant \tau} \frac{f(x, s)}{s^{1 / p}}=\frac{g(x, \tau)}{\tau^{1 / p}} .
$$

 same way as in (3.6).
 and $x, y \in \Omega$. Since $f$ satisfies (aDec $)_{1 / p}$ in $(0,1]$, we have

$$
\begin{equation*}
g(y, t) \approx t^{1 / p} \inf _{1 \leqslant s \leqslant t} \frac{f(y, s)}{s^{1 / p}} . \tag{3.7}
\end{equation*}
$$

Thus by (A1) of $f$ we obtain

$$
g(x, t) \approx t^{1 / p} \inf _{1 \leqslant s \leqslant t} \frac{f(x, s)}{s^{1 / p}} \lesssim t^{1 / p} \inf _{1 \leqslant s \leqslant t} \frac{f(y, s)}{s^{1 / p}} \approx g(y, t) .
$$

Claim (v): Using the inequality $f(x, 1) \leqslant f(x, s) \leqslant f(x, t)$ and (A0), we obtain by (3.7) for $1<t$, that

$$
0<\beta_{0} \leqslant g(x, t) \leqslant t^{1 / p} f(x, t)<\infty .
$$

By (3.6) we have $f \approx g$ for $s \in[0,1]$. Thus by the corresponding property of $f$ we have $g(x, t)=0$ if and only if $t=0$, and $g(x, t)=\infty$ if and only if $t=\infty$.

Claim (vi): Assume that $f$ satisfies (aInc) $)_{1 / q}$ with constant $L \geqslant 1$. Let $0<t<s, \varepsilon>0$ and choose $\theta \in(0,1]$ such that $g(x, s) \geqslant \theta^{-1 / p} f(x, \theta s)-\varepsilon$. Then by the definition of $g$, $(\mathrm{aDec})_{1 / q}$ of $f$ and the choice of $\theta$, we obtain that

$$
\frac{g(x, t)}{t^{1 / q}} \leqslant t^{1 / p-1 / q} \frac{f(x, \theta t)}{(\theta t)^{1 / p}}=\theta^{1 / q-1 / p} \frac{f(x, \theta t)}{(\theta t)^{1 / q}} \leqslant L \theta^{1 / q-1 / p} \frac{f(x, \theta s)}{(\theta s)^{1 / q}} \leqslant L \frac{g(x, s)+\varepsilon}{s^{1 / q}}
$$

Thus letting $\varepsilon \rightarrow 0^{+}$, we find that $g$ satisfies (aInc) $)_{1 / q}$. If we take $L=1$ and $1 / q=0$, this implies that $g$ is increasing (since $f$ is increasing).

We set $\psi:=g^{-1}$. Using (v)-(ii), increasing from (vi) and Proposition 2.5.14 in [10] we obtain that $\psi \in \Phi_{\mathrm{w}}\left(\mathbb{R}^{n}\right)$ and $\psi^{-1}=\left(g^{-1}\right)^{-1}=g$. Since $\varphi^{-1} \approx g=\psi^{-1}$, it follows that $\left.\psi\right|_{\Omega} \simeq \varphi$ [10, Theorem 2.3.6]. Now properties (A0) and (A1) for $\psi$ follow. Moreover, if $\varphi$ satisfies $(\mathrm{aDec})_{q}$, then by (vi) $\psi$ satisfies it as well.

When $x \in \mathbb{R}^{n} \backslash \Omega$ and $t \in[0,1]$, we have $g(x, t) \approx f(x, t)=\beta_{0}^{2} \varphi_{\infty}^{-1}(t)$. It follows that $\psi(x, t) \simeq \varphi_{\infty}(t)$ for sufficiently small values of $t$ and $x$ as before. Therefore, $\psi$ satisfies the condition of (2.7), and so $\psi$ satisfies (A2).

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