COMPLETE CHARACTERIZATION OF EXTREME QUANTUM OBSERVABLES IN INFINITE DIMENSIONS

JUHA-PEKKA PELLONPÄÄ

Abstract. We give a complete characterization for extreme quantum observables, i.e. for normalized positive operator valued measures (POVMs) which are extremals in the convex set of all POVMs. The characterization is valid both in discrete and continuous cases, and also in the case of an infinite dimensional Hilbert space. We show that sharp POVMs are pre-processing clean, i.e. they cannot be irreversibly connected to another POVMs via quantum channels.

PACS numbers: 03.65.Ta, 03.67.–a

1. Introduction

In the modern formalism of quantum mechanics, observables are described by normalized positive operator (valued) measures (POVMs) which have found numerous applications in various areas of quantum physics, ranging from quantum theory of open systems to detection, estimation and quantum information theories. POVMs generalize the traditional concept of an observable, a selfadjoint operator or a normalized projection (valued) measure (PVM), which turned out to be a too restrictive idealization to efficiently describe actual experimental settings such as fuzzy position and momentum measurements or (inefficient) photon counting and phase measurements in quantum optics [1, 2].

A fundamental problem is to characterize the most precise and informative observables. One crucial property of such optimal POVMs is the lack of noise; classical or quantum. The present article focuses on the determination of noise-free observables. Here we consider two types of noise: classical noise associated with the randomness due to fluctuations in the measuring procedure and quantum noise due to the possibility of irreversibly manipulating the state before a measurement (using a quantum channel).

POVMs form a convex set if the measurement outcome space Ω, the σ-algebra $\Sigma \subseteq 2^\Omega$, and the Hilbert space $\mathcal{H}$ are fixed. For any two POVMs P and P' from Σ to $\mathcal{L}(\mathcal{H})$ (bounded operators on $\mathcal{H}$), a convex combination $X \mapsto a\mathsf{P}(X) + (1-a)\mathsf{P}'(X)$, $0 < a < 1$, corresponds to a classical randomization or mixing between P and P'. Such mixing is a source of classical noise. A POVM $\mathsf{P} : \Sigma \to \mathcal{L}(\mathcal{H})$ is *extreme* (or pure) if, for any POVM $\mathsf{P}' : \Sigma \to \mathcal{L}(\mathcal{H})$ and

$a \in (0,1)$, the condition $a\mathsf{P}(X) + (1-a)\mathsf{P}'(X)$ for all $X \in \Sigma$ implies $\mathsf{P}' = \mathsf{P}$. Thus, extreme POVMs are free from the classical noise arising from this type of randomization [1].

For a finite dimensional system ($\dim \mathcal{H} < \infty$), a simple criterion for extremality can be given [3] and Chiribella *et al.* [4] showed that all extreme POVMs are concentrated on a finite number of outcomes. However, in the infinite case ($\dim \mathcal{H} = \infty$), there exist extreme (nonsharp) POVMs with continuous measurement outcome spaces [5, 6].

In this paper, we fully characterize all extreme POVMs using a diagonalization technique of Hytönen *et al.* [7]. This result is a generalization of the finite dimensional characterization [3]. We also introduce a simple polynomial method which can be used to show that a POVM is extreme. This method is very useful in many areas of quantum physics, e.g. in continuous variable quantum information.

Finally, we show that any PVM $\mathsf{P} : \Sigma \to \mathcal{L}(\mathcal{H})$ can be connected to any (P-continuous) POVM $\mathsf{P}' : \Sigma \to \mathcal{L}(\mathcal{H}')$ via a quantum channel Φ (i.e. $\Phi^*(\mathsf{P}(X)) = \mathsf{P}'(X)$ for all $X \in \Sigma$), or in other words, P' is obtained from P via pre-processing [8]. The pre-processing can change the POVM irreversibly, reducing the information from the measurement. Our result shows that PVMs are (pre-processing) *clean* or 'undisturbed' in the sense that they are not irreversibly connected to another POVMs. Thus they do not have any additional 'extrinsic' quantum noise [8].

2. Extreme POVMs

Let us briefly recall the mathematical description of quantum observables via *normalized positive operator measures* (POVMs) [1, 2]. Consider a quantum system with a separable (complex) Hilbert space[1] $\mathcal{H}$ and suppose that the measurement outcomes form a set Ω. We assume that Ω is arbitrary and Σ is any σ-algebra of subsets of Ω. Usually, in 'continuous' cases, Ω is a manifold and Σ is its Borel σ-algebra $\mathcal{B}(\Omega)$.

Let $\mathcal{L}(\mathcal{H})$ (resp. $\mathcal{T}(\mathcal{H})$) be the set of bounded (resp. trace-class) operators on $\mathcal{H}$. A POVM is a function P which associates to each $X \in \Sigma$ a positive operator $\mathsf{P}(X) \in \mathcal{L}(\mathcal{H})$. It is required that for every state (a density operator) $\varrho \in \mathcal{T}(\mathcal{H})$, $\varrho \geq 0$, $\mathrm{tr}\,[\varrho] = 1$, the mapping $X \mapsto \mathrm{tr}\,[\varrho\mathsf{P}(X)]$ is a probability measure. Especially, P satisfies the normalization condition $\mathsf{P}(\Omega) = I$. The number $\mathrm{tr}\,[\varrho\mathsf{P}(X)]$ is the probability of getting a measurement outcome x belonging to X, when

[1] We use the same symbol $\langle\,\cdot\,|\,\cdot\,\rangle$ (resp. I) for the inner product (resp. the identity operator) of *any* Hilbert space. We assume that an inner product is linear with respect to its second argument. We let $\|\cdot\|$ denote the norm associated with $\langle\,\cdot\,|\,\cdot\,\rangle$, and also the norm of a bounded operator.

the system is in the state ϱ and the measurement of P is performed. A POVM $\mathsf{P} : \Sigma \to \mathcal{L}(\mathcal{H})$ is a *normalized projection measure* (PVM), or a *sharp* POVM, if $\mathsf{P}(X)^2 = \mathsf{P}(X)$ for all $X \in \Sigma$. It is well known that PVMs are extreme (see example 1). Next we show that any POVM can be diagonalized (see a) of Theorem 1).

Fix an (arbitrary) orthonormal (ON) basis $\{e_n\}_{n=1}^{\dim \mathcal{H}}$ of $\mathcal{H}$, and let V be the (dense) linear subspace of $\mathcal{H}$ consisting of all finite linear combinations of the basis vectors e_n. Let $V^\times$ be the algebraic antidual of the vector space V. Recall that $V^\times$ can be identified with the linear space of formal series $c = \sum_{n=1}^{\dim \mathcal{H}} c_n e_n$ where the c_n's are arbitrary complex numbers. Denote the dual pairing $\langle \psi | c \rangle := \sum_{n=1}^{\dim \mathcal{H}} \langle \psi | e_n \rangle c_n$ and $\langle c | \psi \rangle := \overline{\langle \psi | c \rangle}$ for all $\psi \in V$ and $c \in V^\times$. Note that $V \subseteq \mathcal{H} \subseteq V^\times$ and the above notation for the dual pairing is consistent with the inner product $\langle \cdot | \cdot \rangle$ of $\mathcal{H}$.

Let $\mathsf{P} : \Sigma \to \mathcal{L}(\mathcal{H})$ be a POVM, and let $\mu : \Sigma \to [0, \infty)$ be *any* measure such that P is absolutely continuous with respect to μ (see [7], Remark 3.8). For example, μ can be chosen to be a probability measure $\mu : \Sigma \to [0, 1]$,

$$\mu(X) := \sum_{n=1}^{\dim \mathcal{H}} \lambda_n \langle e_n | \mathsf{P}(X) e_n \rangle, \qquad X \in \Sigma,$$

where $\lambda_n > 0$ for all n and $\sum_{n=1}^{\dim \mathcal{H}} \lambda_n = 1$. Note that, in the discrete case, μ is a (weighted) counting measure and all integrals below reduce to sums.

We say that $f : \Omega \to \mathbb{C}$ is μ-measurable if it is measurable with respect to the Lebesgue extension of Σ with respect to μ. Moreover, a mapping $c : \Omega \to V^\times$, $x \mapsto \sum_{n=1}^{\dim \mathcal{H}} c_n(x) e_n$ is (weak*-)μ-measurable if its components $x \mapsto c_n(x)$ are μ-measurable; for measurability of (vector valued) maps, see [7], Section 4. In what follows, *we write briefly (m.) for (μ-measurable).* Also we write a.a. for μ-almost all.

Let $\int_\Omega^\oplus \mathcal{H}_{n(x)} \mathrm{d}\mu(x)$ be the direct integral of Hilbert spaces $\mathcal{H}_{n(x)}$ where $\mathcal{H}_l$ is an l-dimensional Hilbert space spanned by vectors $e_1, e_2, \ldots, e_l$, $\mathcal{H}_0 := \{0\}$ and $\mathcal{H}_\infty := \mathcal{H}$ (if $\dim \mathcal{H} = \infty$). A bounded decomposable operator $D = \int_\Omega^\oplus D(x) \mathrm{d}\mu(x)$ on $\int_\Omega^\oplus \mathcal{H}_{n(x)} \mathrm{d}\mu(x)$ consists of a (m.) field of bounded operators $D(x)$ on $\mathcal{H}_{n(x)}$, and D operates as $(D\psi)(x) = D(x)\psi(x)$ for all $\psi \in \int_\Omega^\oplus \mathcal{H}_{n(x)} \mathrm{d}\mu(x)$ and a.a. $x \in \Omega$. The norm $\|D\|$ of D is the essential supremum of $\{\|D(x)\| \,|\, x \in \Omega\}$. We consider any essentially bounded (m.) function $f : \Omega \to \mathbb{C}$ (e.g. the characteristic function χ_X of $X \in \Sigma$) as a multiplicative (diagonalizable) bounded operator on $\int_\Omega^\oplus \mathcal{H}_{n(x)} \mathrm{d}\mu(x)$. Obviously, f is decomposable. As usual, $\sum_{k=1}^{0} (\cdots) := 0$.

Theorem 1. Let $\mathsf{P} : \Sigma \to \mathcal{L}(\mathcal{H})$ be a POVM.

a) There are (m.) mappings $\Omega \ni x \mapsto n(x) \in \{0, 1, \ldots, \dim \mathcal{H}\}$ and $d_k : \Omega \to V^\times$ such that, for all $x \in \Omega$, $d_k(x) \neq 0$ for all $k = 1, \ldots, n(x)$, and

$$\langle \varphi | \mathsf{P}(X) \psi \rangle = \int_X \sum_{k=1}^{n(x)} \langle \varphi | d_k(x) \rangle \langle d_k(x) | \psi \rangle \mathrm{d}\mu(x), \qquad \varphi,\, \psi \in V,\ X \in \Sigma.$$

b) There are (m.) maps $\Omega \ni x \mapsto \psi_m(x) \in \mathcal{H}_{n(x)} \subseteq \mathcal{H} \subseteq V^\times$ such that, for all $X \in \Sigma$,

$$\begin{aligned} \mathsf{P}(X) &= \sum_{n,m=1}^{\dim \mathcal{H}} \int_X \langle \psi_n(x) | \psi_m(x) \rangle \mathrm{d}\mu(x) | e_n \rangle \langle e_m | \\ &= \Big(\sum_m | \chi_X \psi_m \rangle \langle e_m | \Big)^* \Big(\sum_m | \chi_X \psi_m \rangle \langle e_m | \Big) \end{aligned}$$

(weakly) and the set of linear combinations of vectors $\chi_X \psi_m$ is dense in $\int_\Omega^\oplus \mathcal{H}_{n(x)} \mathrm{d}\mu(x)$ *(the minimal Kolmogorov decomposition of* P*).*

c) $\mathsf{P}(X) = J^* \overline{\mathsf{P}}(X) J$, $X \in \Sigma$, where $J := \sum_{m=1}^{\dim \mathcal{H}} | \psi_m \rangle \langle e_m |$ is an isometry, i.e. $J^* J = I$, and $X \mapsto \overline{\mathsf{P}}(X) := \chi_X$ is the (canonical) PVM on $\mathcal{H}_{\overline{\mathsf{P}}} := \int_\Omega^\oplus \mathcal{H}_{n(x)} \mathrm{d}\mu(x)$ *(the minimal Naimark dilation of* P*).*

d) P is a PVM if and only if $\{\psi_m\}_{m=1}^{\dim \mathcal{H}}$ is an ON basis of $\mathcal{H}_{\overline{\mathsf{P}}}$. Then $\mathcal{H}_{\overline{\mathsf{P}}}$ can be identified with $\mathcal{H}$, i.e. $J J^* = I$ and J is a unitary operator. [9]

Proof. a) and b) follow from Theorems 4.5 and 5.1 of [7] by defining $\psi_m(x) := \sum_{k=1}^{n(x)} \langle d_k(x) | e_m \rangle e_k$, and c) follows from b) and Theorem 3.6 of [7]. Finally, d) follows from Corollary 5.2 of [7]. □

Remark 1. Note that $\int_\Omega^\oplus \mathcal{H}_{n(x)} \mathrm{d}\mu(x)$ is a closed subspace of $L^2(\Omega, \mu, \mathcal{H}) \cong L^2(\Omega, \mu, \mathbb{C}) \otimes \mathcal{H}$, the space of ($\mu$-square integrable) 'wave functions' $\psi : \Omega \to \mathcal{H}$, such that $\psi(x) \in \mathcal{H}_{n(x)}$ for a.a. $x \in \Omega$. Usually, when Ω is a manifold and μ is a 'volume' form, $\int_\Omega^\oplus \mathcal{H}_{n(x)} \mathrm{d}\mu(x)$ can be seen as a space of square integrable vector fields (e.g. spinors), that is, sections of a vector bundle over Ω with fibers $\mathcal{H}_{n(x)}$. The map J can be viewed as a transformation from the 'matrix mechanics' to the 'wave mechanics' generated by the POVM P.

Remark 2. For all $x \in \Omega$, define (possibly unbounded) operators $\mathsf{A}(x) : V \to \mathcal{H}_{n(x)}$,

$$\mathsf{A}(x) := \sum_{k=1}^{n(x)} | e_k \rangle \langle d_k(x) | = \sum_{m=1}^{\dim \mathcal{H}} | \psi_m(x) \rangle \langle e_m |.$$

By a) of Theorem 1, one can write

$$\langle \varphi | \mathsf{P}(X) \psi \rangle = \int_X \langle \varphi | \mathsf{A}(x)^* \mathsf{A}(x) \psi \rangle \mathrm{d}\mu(x) \tag{2.1}$$

for all φ, $\psi \in V$ and $X \in \Sigma$. We use a brief notation

$$\mathsf{P}(X) = \int_X \mathsf{A}(x)^*\mathsf{A}(x)\mathrm{d}\mu(x)$$

for (2.1), which we interpret as a sesquilinear form $V \times V \to \mathbb{C}$. *We use this interpretation for equations containing operators* $\mathsf{A}(x)$ *or vectors* $d_k(x)$.

The following theorem characterizes extreme POVMs, that is, the extreme points of a convex set consisting of all POVMs from Σ to $\mathcal{L}(\mathcal{H})$. Retaining the notations of Theorem 1 we have:

Theorem 2. A POVM $\mathsf{P} : \Sigma \to \mathcal{L}(\mathcal{H})$ is extreme if and only if, for any bounded decomposable operator $D = \int_\Omega^\oplus D(x)\mathrm{d}\mu(x)$ on $\mathcal{H}_{\overline{\mathsf{P}}}$ the condition $\int_\Omega \langle\psi_n(x)|D(x)\psi_m(x)\rangle\mathrm{d}\mu(x) = 0$ for all n, m implies that $D = 0$.

Proof. Suppose that there exists a nonzero bounded $D = \int_\Omega^\oplus D(x)\mathrm{d}\mu(x)$ such that

$$\int_\Omega \langle\psi_n(x)|D(x)\psi_m(x)\rangle\mathrm{d}\mu(x) = 0$$

for all n, m. Redefining D as $i(D - D^*)$ (if $D^* \neq D$) and then scaling D by $1/\|D\|$, one may assume that $D^* = D$, $\|D\| \leq 1$ and, thus, $D_\pm := I \pm D \geq 0$ and $D_+ \neq D_-$. Since vectors $\chi_X\psi_m$ span $\mathcal{H}_{\overline{\mathsf{P}}}$ there exists a set $X' \in \Sigma$ and n', m' such that $\int_{X'}\langle\psi_{n'}(x)|D_+(x)\psi_{m'}(x)\rangle\mathrm{d}\mu(x) \neq \int_{X'}\langle\psi_{n'}(x)|D_-(x)\psi_{m'}(x)\rangle\mathrm{d}\mu(x)$ implying that POVMs

$$\mathsf{P}_\pm(X) := \sum_{n,m}\int_X \langle\psi_n(x)|D_\pm(x)\psi_m(x)\rangle\mathrm{d}\mu(x)|e_n\,\rangle\langle\,e_m|$$

are distinct and $\mathsf{P} = (\mathsf{P}_+ + \mathsf{P}_-)/2$ so that P is not extreme.

Suppose then that P is not extreme, that is, of the form $\mathsf{P} = (\mathsf{P}_+ + \mathsf{P}_-)/2$ for some POVMs $\mathsf{P}_\pm : \Sigma \to \mathcal{L}(\mathcal{H})$ such that $\mathsf{P}_+ \neq \mathsf{P}_-$. Now $\mathsf{P}_\pm(X) \leq 2\mathsf{P}(X)$, $X \in \Sigma$, so that, POVMs $\mathsf{P}_\pm$ are absolutely continuous with respect to μ and, by Theorem 1,

$$\mathsf{P}_\pm(X) = \sum_{n,m=1}^{\dim\mathcal{H}}\int_X \langle\psi_n^\pm(x)|\psi_m^\pm(x)\rangle\mathrm{d}\mu(x)|e_n\,\rangle\langle\,e_m| = \int_X \mathsf{A}(x)_\pm^*\mathsf{A}(x)_\pm\mathrm{d}\mu(x)$$

where $\mathsf{A}(x)_\pm := \sum_m |\psi_m^\pm(x)\,\rangle\langle\,e_m|$. In addition, $\langle\varphi|\mathsf{P}_\pm(X)\varphi\rangle \leq 2\langle\varphi|\mathsf{P}(X)\varphi\rangle$ (for all $\varphi \in V$) implies that $\|\mathsf{A}(x)_\pm\varphi\| \leq \sqrt{2}\|\mathsf{A}(x)\varphi\|$ (for all $\varphi \in V$) holds for a.a. $x \in \Omega$. Hence, for a.a. $x \in \Omega$, one can define bounded (well-defined) operators $C_\pm(x)$ on $\mathcal{H}_{n(x)}$ as follows: (a) define $C_\pm(x)\big(\mathsf{A}(x)\varphi\big) := \mathsf{A}(x)_\pm\varphi$, (b) extend $C_\pm(x)$ to the closure of $\mathsf{A}(x)V$, and (c) extend $C_\pm(x)$ to the whole fiber $\mathcal{H}_{n(x)}$ by setting $C_\pm(x)$ to zero on the orthogonal complement of the closure of $\mathsf{A}(x)V$. Define then (linear) operators $C_\pm$ by $(C_\pm\psi)(x) := C_\pm(x)\psi(x)$ where ψ is a linear combination of vectors $\chi_X\psi_m$. Since $\|C_\pm(x)\| \leq \sqrt{2}$, $C_\pm(\chi_X\psi_m) = \chi_X\psi_m^\pm$ and vectors $\chi_X\psi_m$

span $\mathcal{H}_{\overline{\mathsf{P}}}$, one can extend $C_\pm$ to the whole space $\mathcal{H}_{\overline{\mathsf{P}}}$ and $C_\pm = \int_\Omega^\oplus C_\pm(x)\mathrm{d}\mu(x)$ is bounded. Define then $D_\pm(x) := C_\pm(x)^* C_\pm(x)$ to get

$$\mathsf{P}_\pm(X) = \sum_{n,m=1}^{\dim\mathcal{H}} \int_X \langle\psi_n(x)|D_\pm(x)\psi_m(x)\rangle \mathrm{d}\mu(x) |e_n\,\rangle\langle\, e_m|$$

since $C_\pm(x)\psi_m(x) = \psi_m^\pm(x)$. From the assumption $\mathsf{P}_+ \neq \mathsf{P}_-$ one gets

$$D := \int_\Omega^\oplus [D(x)_+ - D(x)_-]\mathrm{d}\mu(x) \neq 0.$$

But, for all n, m, $\int_\Omega \langle\psi_n(x)|D(x)\psi_m(x)\rangle\mathrm{d}\mu(x) = \langle e_n|[\mathsf{P}_+(\Omega) - \mathsf{P}_-(\Omega)]e_m\rangle = \delta_{nm} - \delta_{nm} = 0.$ $\square$

The condition of Theorem 2 can also be written in the form

$$J^* D J = \int_\Omega \mathsf{A}(x)^* D(x) \mathsf{A}(x)\mathrm{d}\mu(x) = 0,$$

or (formally) in the form

$$\int_\Omega \sum_{k,l=1}^{n(x)} \langle e_k|D(x)e_l\rangle |d_k(x)\,\rangle\langle\, d_l(x)|\mathrm{d}\mu(x) = 0. \tag{2.2}$$

Hence, Theorem 2 is a ('continuous' and infinite) generalization of [3]. Formally, one could say that P is extreme if and only if

> 'the (overcomplete) system of generalized coherent states $d_k(x)$ generates a linearly independent set of operators $|d_k(x)\,\rangle\langle\, d_l(x)|$ in the sense that (2.2) implies that $D = 0$.'

Example 1. Similarly, as in the case of position and momentum, $\mathcal{H}$ can be identified with the 'P-reprensentation' space $\mathcal{H}_{\mathsf{P}} := \mathrm{ran}\, J = \mathcal{P}\mathcal{H}_{\overline{\mathsf{P}}} = \mathcal{P}L^2(\Omega, \mu, \mathcal{H})$ where

$$\mathcal{P} := JJ^* = \sum_{m=1}^{\dim\mathcal{H}} |\psi_m\,\rangle\langle\,\psi_m|$$

is a projection. Theorem 2 can be restated in the following form: P is extreme if and only if $\mathcal{P}D\mathcal{P} = 0$ implies $D = \int_\Omega^\oplus D(x)\mathrm{d}\mu(x) = 0$. If P is a PVM then $\mathcal{P} = I$ and hence $\mathcal{P}D\mathcal{P} = D$ so that any PVM is extreme.

From Theorem 2 one gets the following necessary conditions for P to be extreme. Let P be an extreme POVM:

- For any bounded (m.) function $\Omega \ni x \mapsto f(x) \in \mathbb{C}$, the condition $\int_\Omega f(x)\mathrm{d}\mathsf{P}(x) = 0$ implies $f(x) = 0$ (for a.a. $x \in \Omega$). (Put $D = f$ in Theorem 2.)

- For any *fixed* $X \in \Sigma$, the condition $\int_X \langle \psi_n(x) | D(x) \psi_m(x) \rangle \mathrm{d}\mu(x) = 0$ for all n, m implies that $D(x) = 0$ for a.a. $x \in X$. (Replace D with $\chi_X D$ in Theorem 2.)
- For any disjoint sets $X_1, \ldots, X_p \in \Sigma$ such that $\mathsf{P}(X_i) \neq 0$ for all $i = 1, \ldots, p$ the condition $\sum_{i=1}^p c_i \mathsf{P}(X_i) = 0$ implies $c_1 = \cdots = c_p = 0$, i.e. effects $\mathsf{P}(X_i)$ are linearly independent. (Now $D = \sum_i c_i \chi_{X_i}$.)

Next we introduce a simple concrete polynomial method which gives a sufficient criterion for extremality of (continuous) quantum observables.

2.1. **The polynomial method.** Collecting the fibers of the same dimension together, one can write

$$\int_\Omega^\oplus \mathcal{H}_{n(x)} \mathrm{d}\mu(x) = L^2(\Omega_1, \mu_1, \mathcal{H}_1) \oplus L^2(\Omega_2, \mu_2, \mathcal{H}_2) \oplus \cdots \oplus L^2(\Omega_N, \mu_N, \mathcal{H}_N),$$

where $N := \dim \mathcal{H}$, $\Omega_l := \{x \in \Omega \,|\, n(x) = l\}$ and μ_l is the restriction of μ to Ω_l, i.e. $\mu_l(X) := \mu(X)$ for all $X \subseteq \Omega_l$ such that $X \in \Sigma$. We have the following necessary extremality condition for a POVM P: If P is extreme then

$$\int_{\Omega_l} \langle \psi_n(x) | D(x) \psi_m(x) \rangle \mathrm{d}\mu(x) = 0$$

for all n, m implies that $\int_{\Omega_l}^\oplus D(x) \mathrm{d}\mu(x) = 0$. Hence, it is not very restrictive to consider the case $\mathcal{H}_{\overline{\mathsf{P}}} = L^2(\Omega, \mu, \mathcal{H}_l)$ where $l \in \{1, \ldots, N\}$.

Assume then that $\mathcal{H}_{\overline{\mathsf{P}}} = L^2(\Omega, \mu, \mathcal{H}_l)$. Usually in physically relevant 'continuous' cases $\Omega \subseteq \mathbb{R}^p$ and, by choosing suitable coordinates, Ω is of the form $\mathcal{I} := \mathcal{I}_1 \times \cdots \times \mathcal{I}_p$ where $\mathcal{I}_i \subseteq \mathbb{R}$ is an interval. (Without restricting generality, we may even assume that any $\mathcal{I}_i$ is either $[-1, 1]$, $[0, \infty)$ or $\mathbb{R}$.) Moreover, in practice, $\mathrm{d}\mu(x) = w_1(x^1) \cdots w_p(x^p) \mathrm{d}x^1 \cdots \mathrm{d}x^p$ (where $x = (x^1, \ldots, x^p)$ and any 'weight function' $y \mapsto w_i(y) > 0$ is integrable) and an ON basis of $L^2(\mathcal{I}, \mu, \mathcal{H}_l)$ is

$$\{f^1_{k_1} \otimes \cdots \otimes f^p_{k_p} \otimes e_n\} \qquad \text{(here } n = 1, \ldots, l)$$

where $\{f^i_{k_i}\}_{k_i}$ is an ON polynomial basis of $L^2(\mathcal{I}_i, w_i(x^i) \mathrm{d}x^i, \mathbb{C})$.

Remark 3. In many cases, $f^i_{k_i}(y)$ is a classical ON polynomial: when $\mathcal{I}_i = [-1, 1]$ (and $w_i(y) = (1 - y)^\nu (1 + y)^\mu$), $f^i_{k_i}(y)$ is a Jacobi polynomial $P_n^{(\nu,\mu)}(y)$, when $\mathcal{I}_i = [0, \infty)$ (and $w_i(y) = y^\nu e^{-y}$), $f^i_{k_i}(y)$ is an associated Laguerre polynomial $L_n^\nu(y)$, and when $\mathcal{I}_i = \mathbb{R}$ (and $w_i(y) = e^{-y^2}$), $f^i_{k_i}(y)$ is a Hermite polynomial $H_n(y)$. If $\mathcal{I}_i = [0, 2\pi)$ then a suitable basis could be a trigonometric polynomial basis $\{e^{in\theta}\}_{n \in \mathbb{Z}}$. (Note that here we implicitly assume that the above polynomials are normalized.)

Next we give a simple version of the polynomial method in the form of a proposition: Assume that $n(x) \equiv 1$, i.e. $l = 1$ and $\mathcal{H}_{\overline{\mathsf{P}}} \cong L^2(\mathcal{I}, \mu, \mathbb{C})$ with the above assumptions. Hence $\psi_m(x) \in \mathbb{C}$. From Theorem 2 we get:

Proposition 1. P *is extreme if the linear span of* $\big\{\overline{\psi_n(x)}\psi_m(x)\big\}_{n,m}$ *is dense in* $L^2(\mathcal{I}, \mu, \mathbb{C})$. *Especially, if any* $\psi_m(x)$ *is a polynomial and* $\mathrm{lin}\big\{\overline{\psi_n(x)}\psi_m(x)\big\}_{n,m}$ *contains all polynomials then* P *is extreme.*

Proof. P is extreme if and only if, for any essentially bounded (m.) $\lambda : \mathcal{I} \to \mathbb{C}$, the condition $\int_\Omega \lambda(x)\overline{\psi_n(x)}\psi_m(x)\mathrm{d}\mu(x) = 0$ for all n, m implies $\lambda = 0$. Assume that $\mathrm{lin}\{\overline{\psi_n(x)}\psi_m(x)\}_{n,m}$ is dense in $L^2(\mathcal{I}, \mu, \mathbb{C})$. Since $\lambda \in L^2(\mathcal{I}, \mu, \mathbb{C})$ it follows that $\lambda = 0$. Moreover, since the space of all polynomials is dense in $L^2(\mathcal{I}, \mu, \mathbb{C})$ we get the last claim of the proposition. $\square$

Example 2. Let $\mathsf{Q}(X) = \chi_X$, $X \in \mathcal{B}(\mathbb{R})$, be the PVM of the canonical position operator $(Q\psi)(x) = x\psi(x)$ of a particle moving on $\mathbb{R}$. Using (normalized) Hermite functions $h_n(x) = c_n H_n(x) e^{-x^2/2}$, $c_n > 0$, we can write (weakly)

$$\mathsf{Q}(X) = \sum_{m,n=0}^{\infty} \int_X h_n(x) h_m(x) \mathrm{d}x |h_n\rangle\langle h_m|.$$

Let $P_k := I - |h_k\rangle\langle h_k|$ be a projection, $\mathcal{H} = P_k L^2(\mathbb{R}, \mathrm{d}x, \mathbb{C})$, $\mathrm{d}\mu(x) = e^{-x^2}\mathrm{d}x$, and $\mathsf{Q}_k(X) = P_k \mathsf{Q}(X) P_k$ a POVM with vectors $\psi_n(x) = c_n H_n(x)$ where now $n \in \{0, 1, \ldots\} \setminus \{k\}$. If, say, $k = 2$ then Q_2 is extreme by the polynomial method since $\{H_n(x)H_m(x)\}_{n\neq 2, m\neq 2}$ contains at least one polynomial of each degree: $H_0(x)H_m(x)$ is a polynomial of degree $m \neq 2$ and $H_1(x)H_1(x)$ is of degree 2.

Similarly, using the polynomial method, one easily sees that the POVM associated to the measurement of the quantum optical Q-function [5] and the canonical phase observable [6] are extreme. Next we consider the case of the Q-function and note that one needs to slightly modify Proposition 1.

Example 3. Let

$$\mathsf{E}(Z) := \frac{1}{\pi} \int_Z |z\rangle\langle z| \mathrm{d}^2 z = \sum_{n,m=0}^{\infty} \int_Z \frac{r^{n+m} e^{i(n-m)\theta}}{\sqrt{n!m!}} e^{-r^2} \mathrm{d}r^2 \frac{\mathrm{d}\theta}{2\pi} |n\rangle\langle m|, \qquad Z \in \mathcal{B}(\mathbb{C}),$$

be the POVM associated to the measurement of the Q-function $z \mapsto \langle z|\varrho|z\rangle$ of a state ϱ, where $z = re^{i\theta} \in \mathbb{C}$, $r = |z|$, and

$$|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle$$

is a coherent state (in the Hilbert space spanned by the number states $|n\rangle$, $n = 0, 1, \ldots$).

For all $s = 0, 1, \ldots$, define a measure $\mu_s(X) := \int_X t^{s/2} e^{-t} \mathrm{d}t$, $X \in \mathcal{B}([0,\infty))$, and note that polynomials are dense in $L^2\big([0,\infty), \mu_s, \mathbb{C}\big)$ since the (normalized) associated Laguerre polynomials $L_n^{s/2}(t)$, $n = 0, 1, \ldots,$, constitute an ON basis of $L^2\big([0,\infty), \mu_s, \mathbb{C}\big)$.

Denoting $t = r^2$ we get

$$r^{n+m} e^{i(n-m)\theta} = t^m e^{ik\theta} t^{k/2} = t^n e^{-il\theta} t^{l/2}$$

where $k := n - m$ and $l := m - n$. From Theorem 2 one sees that E is extreme if

$$\begin{aligned} \int_0^\infty \left[\int_0^{2\pi} \lambda(\theta, t) e^{ik\theta} \mathrm{d}\theta \right] t^m \mathrm{d}\mu_k(t) &= 0, \qquad k, m \in \{0, 1, \ldots\}, \\ \int_0^\infty \left[\int_0^{2\pi} \lambda(\theta, t) e^{-il\theta} \mathrm{d}\theta \right] t^n \mathrm{d}\mu_l(t) &= 0, \qquad l, n \in \{0, 1, \ldots\}, \end{aligned}$$

implies $\lambda = 0$ (where λ is an essentially bounded (m.) complex function). Since polynomials are dense in $L^2\big([0,\infty), \mu_s, \mathbb{C}\big)$ and both m and n run from 0 to ∞ it follows that

$$\int_0^{2\pi} \lambda(\theta, t) e^{ik\theta} \mathrm{d}\theta = 0$$

for all $k \in \mathbb{Z}$, i.e. $\lambda = 0$ and E is pure.

3. Clean POVMs

In this section, we show that any PVM P can be connected to any (P-continuous) POVM P' via a channel Φ.

Let P and P' be POVMs with the same outcome space Ω (and Σ) but *acting possibly different separable Hilbert spaces* $\mathcal{H}$ and $\mathcal{H}'$. Recall that a channel $\Phi : \mathcal{T}(\mathcal{H}') \to \mathcal{T}(\mathcal{H})$ is a completely positive trace-preserving linear map between state spaces associated to physical systems with Hilbert spaces $\mathcal{H}'$ and $\mathcal{H}$. Its dual map $\Phi^* : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}')$ (i.e. $\mathrm{tr}\,[\varrho \Phi^*(B)] = \mathrm{tr}\,[\Phi(\varrho) B]$ for all $\varrho \in \mathcal{T}(\mathcal{H}')$ and $B \in \mathcal{L}(\mathcal{H})$) is identity-preserving (unital): $\Phi^*(I) = I$.

Physically, the condition $\Phi^* \mathsf{P} = \mathsf{P}'$ (i.e. $\mathrm{tr}\,[\varrho \mathsf{P}'(X))] = \mathrm{tr}\,[\Phi(\varrho) \mathsf{P}(X)]$ for all $\varrho \in \mathcal{T}(\mathcal{H}')$ and $X \in \Sigma$) between POVMs means that to get the measurement outcome statistics of P' in the state ϱ one can equally measure P in the state $\Phi(\varrho)$. A POVM P is (pre-processing) *clean* if, for any (P-continuous) POVM P' and a channel $\tilde{\Phi}$ such that $\mathsf{P} = \tilde{\Phi}^* \mathsf{P}'$ there exists a channel Φ such that $\mathsf{P}' = \Phi^* \mathsf{P}$. Hence, a clean POVM cannot be obtained by (irreversibly) manipulating the state before the measurement and then measuring some other POVM.

Let $\{e_n'\}$ be an ON basis of $\mathcal{H}'$. Similarly, as is the case of P (see Theorem 1), we let $\mu'(X)$, $n'(x)$, $\psi_n'(x)$, etc. denote the corresponding maps related to P'. Suppose that there exists a channel Φ such that $\Phi^*(\mathsf{P}(X)) = \mathsf{P}'(X)$ for all $X \in \Sigma$. Then, if $\mathsf{P}(X) = 0$ one has $\mathsf{P}'(X) = 0$ so that (by the Radon-Nikodým theorem) $\mathrm{d}\mu'(x) = w(x)\mathrm{d}\mu(x)$ where $x \mapsto w(x) \geq 0$ is μ-integrable; we say that P' is P-*continuous.* If P' is P-continuous, one can absorb $\sqrt{w(x)}$ into functions $\psi'_n(x)$ and redefine $\psi'_n(x)$ to be $\sqrt{w(x)}\psi'_n(x)$. Hence, without restricting generality, we may (and will) assume that $\mu' = \mu$.

Lemma 1. There exists a channel Φ such that $\Phi^*(\mathsf{P}(X)) = \mathsf{P}'(X)$ for all $X \in \Sigma$ if and only if P' is P-continuous, there exist vectors v_n^s in a (separable) Hilbert space $\mathcal{M}$ such that $\sum_{s=1}^{\dim \mathcal{H}} \langle v_n^s | v_m^s \rangle = \delta_{nm}$, $1 \leq n, m < \dim \mathcal{H}' + 1$, and there exists a decomposable isometry $W = \int_\Omega W(x)\mathrm{d}\mu(x)$ from $\mathcal{H}'_{\mathsf{P}'}$ to $\mathcal{M} \otimes \mathcal{H}_{\mathsf{P}}$ such that $W(x)\psi'_n(x) = \sum_{s=1}^{\dim \mathcal{H}} v_n^s \otimes \psi_s(x)$.

Proof. Any channel $\Phi : \mathcal{T}(\mathcal{H}') \to \mathcal{T}(\mathcal{H})$ has a Kraus decomposition $\Phi(\varrho) = \sum_{k=1}^N A_k \varrho A_k^*$, $\varrho \in \mathcal{T}(\mathcal{H}')$, or, for the dual map $\Phi^* : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}')$,

$$\Phi^*(B) = \sum_{k=1}^N A_k^* B A_k$$

(ultraweakly) where $B : \mathcal{H} \to \mathcal{H}$ and $A_k : \mathcal{H}' \to \mathcal{H}$ are bounded operators and $N \leq \dim \mathcal{H} \dim \mathcal{H}'$ [10]. Let $\mathcal{M}$ be any Hilbert space with an ON basis $\{f_k\}_{k=1}^N$. Defining $v_n^s := \sum_k \langle e_s | A_k e'_n \rangle f_k \in \mathcal{M}$ one sees that $\sum_s \langle v_n^s | v_m^s \rangle = \delta_{nm}$ and $\|v_n^s\|^2 \leq \sum_s \|v_n^s\|^2 = 1$. Moreover, for all $B \in \mathcal{L}(\mathcal{H})$,

$$\Phi^*(B) = Y^*(B \otimes I)Y$$

(the Stinespring form) where $Y := \sum_{n,s} |e_s \otimes v_n^s\rangle\langle e'_n|$ is an isometry, i.e. $Y^*Y = I = \sum_n |e'_n\rangle\langle e'_n|$. Conversely, if there exist a Hilbert space $\mathcal{M}$ (with an ON basis $\{f_k\}_{k=1}^{\dim \mathcal{M}}$) and vectors $v_n^s \in \mathcal{M}$ such that $\sum_s \langle v_n^s | v_m^s \rangle = \delta_{nm}$, one can define bounded operators $A_k := \sum_{s,n} \langle f_k | v_n^s \rangle |e_s\rangle\langle e'_n|$, $\sum_k A_k^* A_k = \sum_n |e'_n\rangle\langle e'_n|$, an isometry $Y := \sum_{n,s} |e_s \otimes v_n^s\rangle\langle e'_n|$, and a channel $\Phi(\varrho) := \sum_{k=1}^{\dim \mathcal{M}} A_k \varrho A_k^*$ with the dual map $\Phi^*(B) = \sum_k A_k^* B A_k = Y^*(B \otimes I)Y$. To conclude, any channel Φ is characterized by (nonunique) vectors v_n^s as above and we have

$$\langle e'_n | \Phi^*(B) e'_m \rangle = \sum_{s,t=1}^{\dim \mathcal{H}} \langle v_n^s | v_m^t \rangle \langle e_s | B e_t \rangle.$$

Hence, there exists a channel Φ such that $\Phi^* \mathsf{P} = \mathsf{P}'$ if and only if P' is P-continuous and there exist vectors v_n^s in a (separable) Hilbert space $\mathcal{M}$ such that $\sum_{s=1}^{\dim \mathcal{H}} \langle v_n^s | v_m^s \rangle = \delta_{nm}$ and, for all

$X \in \Sigma$,

$$
\begin{aligned}
\int_X \langle \psi'_n(x) | \psi'_m(x) \rangle \mathrm{d}\mu(x) &= \langle \psi'_n | \chi_X \psi'_m \rangle = \sum_{s,t=1}^{\dim \mathcal{H}} \langle v_n^s | v_m^t \rangle \langle \psi_s | \chi_X \psi_t \rangle \\
&= \sum_{s,t=1}^{\dim \mathcal{H}} \int_X \langle v_n^s \otimes \psi_s(x) | v_m^t \otimes \psi_t(x) \rangle \mathrm{d}\mu(x). \qquad (3.1)
\end{aligned}
$$

If there exists a decomposable isometry[2] W such that $W(x)\psi'_n(x) = \sum_s v_n^s \otimes \psi_s(x)$ then (3.1) clearly follows. Conversely, if (3.1) holds then $\langle \chi_X \psi'_n | \chi_{\tilde{X}} \psi'_m \rangle = \sum_{s,t} \langle v_n^s \otimes \chi_X \psi_s | v_m^t \otimes \chi_{\tilde{X}} \psi_t \rangle$ and, since vectors $\chi_X \psi'_n$ span $\mathcal{H}'_{\overline{\mathsf{P}}'}$ by Theorem 1, there exists an isometry $W : \mathcal{H}'_{\overline{\mathsf{P}}'} \to \mathcal{M} \otimes \mathcal{H}_{\overline{\mathsf{P}}}$ such that $W(\chi_X \psi'_n) = \sum_s v_n^s \otimes (\chi_X \psi_s)$. But now W commutes with the linear combinations of functions χ_X so that it commutes with the von Neumann algebra $L^\infty(\Omega, \mu, \mathbb{C})$, the space of essentially bounded (m.) functions $\Omega \to \mathbb{C}$. (As usual, we consider any element of $L^\infty(\Omega, \mu, \mathbb{C})$ as a diagonalizable bounded operator.) Since a bounded operator is decomposable if and only if it commutes with every diagonalizable operator [11], it follows that W decomposable. □

Theorem 3. a) If P is sharp and P' *any* P-continuous POVM then there exists a channel such that $\Phi^*(\mathsf{P}(X)) = \mathsf{P}'(X)$ for all $X \in \Sigma$.
b) If there exists a channel Φ such that $\Phi^*(\mathsf{P}(X)) = \mathsf{P}'(X)$ for all $X \in \Sigma$ then $\Phi^*(B) = \overline{\Phi}^*(JBJ^*)$ for all $B \in \mathcal{L}(\mathcal{H})$ where $\overline{\Phi}^*$ is the dual of a channel $\overline{\Phi}$ which satisfies the relation $\overline{\Phi}^*(\overline{\mathsf{P}}(X)) = \mathsf{P}'(X)$ for all $X \in \Sigma$.

Proof. a) Let P be a PVM and P' a P-continuous POVM. Let Ω' be the subset of Ω such that Ω' consists of all points $x \in \Omega$ for which $n(x) \neq 0$. Note that if $n(x) = 0$ then $n'(x) = 0$ (for a.a. $x \in \Omega$) by the P-continuity of P'. Let $\mu|_{\Omega'}$ be the restriction of μ to Ω', and let $\{\eta_t\}_{t=1}^M$ be an ON basis of $L^2(\Omega', \mu|_{\Omega'}, \mathbb{C})$ which is separable since $\mathcal{H}_{\mathsf{P}} = \mathcal{H}_{\overline{\mathsf{P}}}$ is separable by Theorem 1. Extend an ON set $\{\eta_t e_1\}_{t=1}^M$ of $\mathcal{H}_{\mathsf{P}}$ to an ON basis $\{\psi_s\}_{s=1}^{\dim \mathcal{H}}$ of $\mathcal{H}_{\mathsf{P}}$ (note that this forces $M \leq \dim \mathcal{H}$). Since functions $x \mapsto \langle e_k | \psi'_n(x) \rangle$ belong to $L^2(\Omega', \mu|_{\Omega'}, \mathbb{C})$ they can be represented as L^2-convergent series with respect to the basis $\{\eta_t\}_{t=1}^M$ and, hence,

$$
\langle e_k | \psi'_n(x) \rangle = \sum_{t=1}^{M} \tilde{c}_{kn}^t \eta_t(x) = \sum_{s=1}^{\dim \mathcal{H}} c_{kn}^s \langle e_1 | \psi_s(x) \rangle
$$

[2]Note that a decomposable operator $W = \int_\Omega W(x) \mathrm{d}\mu(x)$ is an isometry if and only if operators $W(x)$ are isometries for a.a. $x \in \Omega$.

where $\tilde{c}^t_{kn}$, $c^s_{kn} \in \mathbb{C}$ and, e.g. $\sum_{s,k} \overline{c^s_{kn}} c^s_{km} = \delta_{nm}$. Define vectors $v^s_n := \sum_{k=1}^\infty c^s_{kn} f_k \in \mathcal{M}$ where $\mathcal{M}$ is a Hilbert space with an ON basis $\{f_k\}_{k=1}^\infty$. Now $\sum_s \langle v^s_n | v^s_m \rangle = \sum_{s,k} \overline{c^s_{kn}} c^s_{km} = \delta_{nm}$ and

$$\langle \psi'_n | \chi_X \psi'_m \rangle = \int_X \sum_{k=1}^{n'(x)} \langle \psi'_n(x) | e_k \rangle \langle e_k | \psi'_m(x) \rangle \mathrm{d}\mu(x) = \sum_{s,t} \langle v^s_n | v^t_m \rangle \langle \psi_s \chi_X | \psi_t \rangle$$

so that there exists a channel Φ such that $\Phi^* \mathsf{P} = \mathsf{P}'$ by (3.1).

b) Assume that $\Phi^*\mathsf{P} = \mathsf{P}'$ and let v^s_n's be the vectors associated to Φ. It is easy to check that $\overline{\Phi}^*(\overline{B}) := \sum_{n,m=1}^{\dim \mathcal{H}'} \sum_{s,t=1}^{\dim \mathcal{H}} \langle v^s_n | v^t_m \rangle \langle \psi_s | \overline{B} \psi_t \rangle | e'_n \rangle \langle e'_m |$ (where $\overline{B}$ is a bounded operator on $\mathcal{H}_{\overline{P}}$) is the dual of a channel $\overline{\Phi}$ for which $\overline{\Phi}^*(\overline{\mathsf{P}}(X)) = \mathsf{P}'(X)$, $X \in \Sigma$, and $\Phi^*(B) = \overline{\Phi}^*(JBJ^*)$, $B \in \mathcal{L}(\mathcal{H})$. □

To close this section, we exhibit an example which demonstrates that it could be possible to replace a PVM P with an extreme POVM in a) of Theorem 3. The equation $\Phi^*\mathsf{P} = \mathsf{P}'$ could then hold in some approximate sense.

Example 4. Consider the canonical phase observable $\mathsf{P} : \mathcal{B}\big([0, 2\pi)\big) \to \mathcal{L}(\mathcal{H})$,

$$X \mapsto \mathsf{P}(X) := \sum_{n,m=0}^\infty \int_X e^{i(n-m)\theta} \frac{\mathrm{d}\theta}{2\pi} |n\rangle\langle m| = \int_X |\theta\rangle\langle\theta| \frac{\mathrm{d}\theta}{2\pi}$$

where $|\theta\rangle := \sum_{n=0}^\infty e^{in\theta} |n\rangle \in V^\times$ is the Susskind-Glogower phase state. Now $x = \theta$, $\mathrm{d}\mu(\theta) = \mathrm{d}\theta/(2\pi)$, $e_n = |n-1\rangle$, $n(\theta) \equiv 1$, $d_1(\theta) = |\theta\rangle$, and (since $\mathbb{C}|0\rangle \cong \mathbb{C}$) $\psi_n(\theta) = e^{-i(n-1)\theta}|0\rangle \cong e^{-i(n-1)\theta}$ where $n \in \{1, 2, \ldots\}$ (see Theorem 1). Using Proposition 1 one sees that the canonical phase observable is extreme.

Let $\mathsf{P}' : \mathcal{B}\big([0, 2\pi)\big) \to \mathcal{L}(\mathcal{H}')$ be any P-continuous POVM. Hence, P' is of the form

$$\mathsf{P}'(X) = \sum_{n,m=1}^{\dim \mathcal{H}'} \int_X \langle \psi'_n(\theta) | \psi'_m(\theta) \rangle \frac{\mathrm{d}\theta}{2\pi} | e'_n \rangle\langle e'_m |, \qquad X \in \mathcal{B}\big([0, 2\pi)\big).$$

Since each $\psi'_n \in L^2\big([0, 2\pi), \mu, \mathcal{H}'\big)$ it has the L^2-convergent Fourier series $\psi'_n(\theta) = \sum_{s=-\infty}^\infty \tilde{v}^s_n e^{-is\theta}$ where $\tilde{v}^s_n \in \mathcal{H}'$ and $\sum_{s=-\infty}^\infty \langle \tilde{v}^s_n | \tilde{v}^s_m \rangle = \langle \psi'_n | \psi'_m \rangle = \delta_{nm}$.

Let $\mathcal{X}$ be any finite subset of $\mathcal{B}\big([0, 2\pi)\big)$, and let $N < \dim \mathcal{H}' + 1$ and $\epsilon > 0$. One can pick an $M > 0$ such that

$$\left| \int_X \langle \psi'_n(\theta) | \psi'_m(\theta) \rangle \frac{\mathrm{d}\theta}{2\pi} - \int_X \langle \psi^M_n(\theta) | \psi^M_m(\theta) \rangle \frac{\mathrm{d}\theta}{2\pi} \right| < \epsilon$$

for all $X \in \mathcal{X}$ and $n, m \le N$ where $\psi^M_n(\theta) := \sum_{s=-M}^M \tilde{v}^s_n e^{-is\theta}$. (Note that ψ^M_n is a projection of ψ'_n.) Define a positive operator (valued) measure $\mathsf{P}'_M : \mathcal{B}\big([0, 2\pi)\big) \to \mathcal{L}(\mathcal{H}')$,

$$X \mapsto \mathsf{P}'_M(X) = \sum_{n,m=1}^{\dim \mathcal{H}'} \int_X \langle \psi^M_n(\theta) | \psi^M_m(\theta) \rangle \frac{\mathrm{d}\theta}{2\pi} | e'_n \rangle\langle e'_m | \le \sum_{n=1}^{\dim \mathcal{H}'} | e'_n \rangle\langle e'_n | = I$$

which is not necessarily normalized. However, one can consider P'_M as an approximation of P' when M is large. Since

$$e^{-iM\theta}\psi_n^M(\theta) = \sum_{s'=-M}^{M} \tilde{v}_n^{s'} e^{-i(s'+M)\theta} = \sum_{s=1}^{2M+1} v_n^s \psi_s(\theta),$$

where $v_n^s := \tilde{v}_n^{s-M-1}$, following the proof of Lemma 1, one can show that there exists a completely positive linear mapping $\Phi_M : \mathcal{T}(\mathcal{H}') \to \mathcal{T}(\mathcal{H})$ such that $\Phi_M^*(\mathsf{P}(X)) = \mathsf{P}'_M(X)$, $X \in \mathcal{B}\big([0,2\pi)\big)$, where the (possibly nonunital) dual mapping $\Phi_M^* : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}')$ is of the form

$$\Phi_M^*(B) = Y_M^*(B\otimes I)Y_M, \qquad B \in \mathcal{L}(\mathcal{H}),$$

and $Y_M := \sum_{n=1}^{\dim\mathcal{H}'}\sum_{s=1}^{2M+1} |e_s \otimes v_n^s\,\rangle\langle\, e'_n|$ for which

$$Y_M^* Y_M = \sum_{n,m=1}^{\dim\mathcal{H}'} \langle\psi_n^M|\psi_m^M\rangle |e'_n\,\rangle\langle\, e'_m| \le I.$$

Hence, to obtain approximately a measurement outcome statistics of $\mathsf{P}' : \mathcal{B}\big([0,2\pi)\big) \to \mathcal{L}(\mathcal{H}')$ in a state $\varrho \in \mathcal{T}(\mathcal{H}')$, that is, a probability measure $X \mapsto \mathrm{tr}\,[\varrho\mathsf{P}'(X)]$, one can pick a large M and measure P in a state $\varrho_M := \Phi_M(\varrho)/\mathrm{tr}\,[\Phi_M(\varrho)]$ to get

$$\mathrm{tr}\,[\varrho\mathsf{P}'(X)] \approx \mathrm{tr}\,[\varrho\mathsf{P}'_M(X)] = \mathrm{tr}\,[\varrho_M\mathsf{P}(X)]\,\mathrm{tr}\,[\Phi_M(\varrho)]$$

for a *finite number* of sets X which can constitute, e.g., an arbitrarily dense discretization of the interval $[0,2\pi)$.

4. Discussion

In conclusion, we have shown that the traditional observables, PVMs, have a special role among quantum observables, namely, they are (extreme and) clean and free from any additional extrinsic quantum noise. However, the above example suggests that this result could also be approximately true for all extreme observables and, hence, the most accurate quantum observables should be described by extreme POVMs.

The physically significant POVMs usually satisfy certain properties of covariance with respect to a symmetry group of the theory [1]. For example, quantum optical phase observables are described by POVMs which transform covariantly with respect to phase shifts generated by the number operator. However, covariant phase POVMs are never sharp. Theorem 2 and Proposition 1 provide powerful tools (i) for constructing extreme POVMs describing 'canonical' (covariant) observables of physical quantities (phase, time, angle, etc.) and (ii) for studying whether (or not) a POVM associated to an actual measurement scheme (e.g. homodyne detection in quantum optics) is extreme.

Acknowledgments. The author thanks Paul Busch, Pekka Lahti, Jussi Schultz, and Kari Ylinen for useful discussions and comments on the manuscript.

References

[1] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982); *Statistical Structure of Quantum Theory* (Lecture Notes in Physics 67, Springer-Verlag, Berlin, 2001).

[2] E. B. Davies, *Quantum Theory of Open Systems* (Academic Press, London, 1976); C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976); G. Ludwig, *Foundations of Quantum Mechanics. Volume I.* (Springer-Verlag, Berlin, 1983); P. Busch, M. Grabowski and P. Lahti, *Operational Quantum Physics* (Springer-Verlag, Berlin, 1997); E. Prugovečki, *Quantum Geometry – A Framework for Quantum General Relativity* (Kluwer Academic Publishers, Dordrecht, 1992).

[3] See Theorem 2.4 of K. R. Parthasarathy, Inf. Dim. Anal. **2**, 557 (1999) or Theorem 2 of G. M. D'Ariano, P. L. Presti and P. Perinotti, J. Phys. A: Math. Gen. **38**, 5979 (2005).

[4] G. Chiribella, G. M. D'Ariano and D.-M. Schlingemann, Phys. Rev. Lett. **98**, 190403 (2007); J. Math. Phys. **51**, 022111 (2010); e-print arXiv:quant-ph/0703110.

[5] A. S. Holevo, Rep. Math. Phys **22**, 385 (1985).

[6] T. Heinosaari and J.-P. Pellonpää, Phys. Rev. A **80**, 040101(R) (2009).

[7] T. Hytönen, J.-P. Pellonpää and K. Ylinen, J. Math. Anal. Appl. **336**, 1287 (2007).

[8] F. Buscemi, G. M. D'Ariano, M. Keyl, P. Perinotti and R. Werner, J. Math. Phys. **46**, 082109 (2005).

[9] Note that d) can be *formally* expressed in the Dirac's formalism as follows: 'the resolution of the identity' $\int_\Omega \sum_{k=1}^{n(x)} |d_k(x)\rangle\langle d_k(x)|\mathrm{d}\mu(x) = I$ defines a PVM iff $\langle d_k(x)|d_l(y)\rangle = \delta_{kl}\delta_x(y)$ where δ_{kl} is the Kronecker's delta and δ_x is the 'Dirac's delta' concentrated on x. Then *formally* $\mathsf{A}(x)\mathsf{A}(y)^* = \delta_x(y)\sum_{k=1}^{n(x)} |e_k\,\rangle\langle\, e_k| = \sum_m |\psi_m(x)\,\rangle\langle\,\psi_m(y)|$ and

$$\begin{aligned}\langle\varphi|\psi\rangle &= \iint \langle\varphi(x)|\mathsf{A}(x)\mathsf{A}(y)^*\psi(y)\rangle \mathrm{d}\mu(y)\mathrm{d}\mu(x) \\ &= \sum_m \int \langle\varphi(x)|\psi_m(x)\rangle\mathrm{d}\mu(x) \int \langle\psi_m(y)|\psi(y)\rangle\mathrm{d}\mu(y)\end{aligned}$$

which implies that $\{\psi_m\}$ is an ON basis; moreover, $\mathsf{P}(X)^2 = \int_X\int_X \mathsf{A}(x)^*\mathsf{A}(x)\mathsf{A}(y)^*\mathsf{A}(y)\mathrm{d}\mu(x)\mathrm{d}\mu(y) = \int_X \left[\int_X \delta_x(y)\mathsf{A}(x)^*\mathsf{A}(y)\mathrm{d}\mu(y)\right]\mathrm{d}\mu(x) = \mathsf{P}(X)$.

If $\Omega = \mathbb{R}$ and P is the spectral measure associated to a selfadjoint operator P then '$Pd_k(x) = xd_k(x)$' so that x is an 'eigenvalue' of P with 'multiplicity' $n(x)$ [7], Section 8.

[10] K. Kraus, *States, Effects, and Operations* (Springer-Verlag, Berlin, 1983); M. Choi, Linear Algebra Appl. **10**, 285 (1975).

[11] See Theorem 1 in page 187 of J. Dixmier, *Von Neumann Algebras* (North-Holland, Amsterdam, 1981).

E-mail address: `juhpello@utu.fi`

Turku Centre for Quantum Physics, Department of Physics and Astronomy, University of Turku, FI-20014 Turku, Finland