On location-domination of set of vertices in cycles and paths

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Abstract

The motivation to study location-domination comes from finding objects in sensor networks. In this paper, we consider location-domination of both single vertices and sets of vertices in cycles and paths. In many cases, optimal codes, i.e. codes with the smallest cardinalities, are found.

1 Introduction

Let $G = (V, E)$ be a simple connected and undirected graph with V as the set of vertices and E as the set of edges. Let u and v be vertices in V. If u and v are adjacent to each other, then the edge joining u and v is denoted by uv. The *distance* $d(u, v)$ is the number of edges in any shortest path between u and v. For the rest of the paper, assume that r is a positive integer. We say that u *r-covers* v if the distance $d(u, v)$ is at most r. The *ball of radius r centered at u* is defined as

$$B_r(u) = \{x \in V \mid d(u, x) \leq r\}.$$

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Furthermore, if X is a subset of V, then we define

$$B_r(X) = \bigcup_{x \in X} B_r(x).$$

A non-empty subset of V is called a *code*, and its elements are called *codewords*. Let $C \subseteq V$ be a code. An *I-set* (or an *identifying set*) of the subset X of V with respect to the code C is defined as

$$I_r(C; X) = I_r(X) = B_r(X) \cap C.$$

If $X = \{x_, x_2, \ldots, x_\ell\}$, then we denote in short $I_r(X) = I_r(x_1, x_2, \ldots, x_\ell)$.

Let X and Y be subsets of V. The *symmetric difference* of X and Y is defined as $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$. We say that the vertices u and v are *r-separated* by a code $C \subseteq V$ (or by a codeword of C) if the symmetric difference $I_r(u) \triangle I_r(v)$ is non-empty.

We say that $C \subseteq V$ is an r-locating-dominating code in G if for all $u, v \in V \setminus C$ we have $I_r(u) \neq \emptyset$ and $I_r(u) \neq I_r(v)$. In other words, C is an r-locating-dominating code in G if each non-codeword is r-covered by a codeword of C and each pair of non-codewords are r-separated by C. This definition is due to Slater [13] in the case $r = 1$ and due to Carson [2] when $r \geq 2$. Furthermore, Honkala *et al.* [10] introduced two generalizations of r-locating-dominating codes, which instead of single vertices consider sets of vertices. These definitions are as follows:

Definition 1.1. Let r and ℓ be positive integers. A code $C \subseteq V$ is $(r, \leq \ell)$-locating-dominating of type A — $(r, \leq \ell)$-LDA for short — in G if for all $X, Y \subseteq V$ such that $X \neq Y$, $|X| \leq \ell$ and $|Y| \leq \ell$ we have $X \cap C \neq Y \cap C$ or $I_r(X) \neq I_r(Y)$.

The second variant is similar to the previous definition. However, now we only consider subsets of $V \setminus C$.

Definition 1.2. Let r and ℓ be non-negative integers. A code $C \subseteq V$ is $(r, \leq \ell)$-locating-dominating of type B — $(r, \leq \ell)$-LDB for short — in G if for all $X, Y \subseteq V \setminus C$ such that $X \neq Y$, $|X| \leq \ell$ and $|Y| \leq \ell$ we have $I_r(X) \neq I_r(Y)$.

Notice that the definition of $(r, \leq \ell)$-locating-dominating codes of type A and type B both reduces to the one of r-locating-dominating codes when $\ell = 1$. It is also clear that an $(r, \leq \ell)$-locating-dominating code of type A is always an $(r, \leq \ell)$-locating-dominating code of type B.

The smallest cardinalities of an $(r, \leq \ell)$-locating-dominating code of type A and type B in a finite graph G are denoted by $M^{LDA}_{(r,\leq \ell)}(G)$ and $M^{LDB}_{(r,\leq \ell)}(G)$, respectively. Notice that there always exist an $(r, \leq \ell)$-locating-dominating code of type A and type B in G. An $(r, \leq \ell)$-locating-dominating code of type A or type B attaining the smallest cardinality is called *optimal*. The smallest cardinality of an r-locating-dominating code in G is denoted in short by $M_r^{LD}(G)$.

Locating-dominating codes are also known as locating-dominating *sets* in the literature. The locating-dominating codes have been studied in various papers such as [6, 7, 9, 10, 12, 15, 16]. For other papers on the subject, we refer to the Web site [11]. Moreover, location-domination in cycles and paths have been examined in [1, 3, 4, 5, 8, 13, 14].

Assume throughout the paper that n is an integer such that $n \geq 3$. A cycle $\mathcal{C}_n = (V_n, E_n)$ is a graph such that the set of vertices $V_n = \{v_i \mid i \in \mathbb{Z}_n\}$ and the set of edges

$$E_n = \{v_i v_{i+1} \mid i = 0, 1, \ldots, n-1\} \cup \{v_{n-1} v_0\}.$$

For the rest of the paper, we assume that the indices of $v_i \in V_n$ are calculated modulo n. Hence, the set of edges can be written as $E_n = \{v_i v_{i+1} \mid i \in \mathbb{Z}_n\}$. Similarly, we define a path $\mathcal{P}_n = (V_n, E'_n)$ as a graph with the set of vertices V_n as above and the set of edges $E'_n = E_n \setminus \{v_{n-1} v_0\}$. (Notice that the problems concerning location-domination in paths of length one or two are trivial.)

In what follows, in Section 2, we first recall some known results on r-locating-dominating codes in cycles and paths. Then we proceed with some improvements on these results. In Section 3, we consider $(r, \leq \ell)$-locating-dominating codes of type A in $\mathcal{C}_n$ and $\mathcal{P}_n$. Finally, in Section 4, we study $(r, \leq \ell)$-locating-dominating codes of type B in $\mathcal{C}_n$ and $\mathcal{P}_n$.

2 On r-locating-dominating codes in $\mathcal{C}_n$ and $\mathcal{P}_n$

We first present useful characterizations of the r-locating-dominating codes in cycles and paths. For this, we need the concept of C-consecutive vertices introduced in [1]. Let i and j be positive integers. We say that (v_i, v_j) is a pair of *C-consecutive vertices* in $\mathcal{C}_n$ if $v_i, v_j \in V_n \setminus C$ and $v_k \in C$ either for all $k = i+1, i+2, \ldots, j-1$ or for all $k = j+1, j+2, \ldots, i-1$. In the case of paths, we again say that (v_i, v_j) is a pair of C-consecutive vertices if $v_i, v_j \in V_n \setminus C$ and all the vertices between v_i and v_j belong to C.

Now we are ready to present the following characterization, which is introduced in [1, Remark 3], for r-locating-dominating codes in paths.

Lemma 2.1 ([1]). *A code $C \subseteq V_n$ is r-locating-dominating in $\mathcal{P}_n$ if and only if (i) each vertex $u \in V_n \setminus C$ is r-covered by a codeword of C and (ii) for each pair (u, v) of C-consecutive vertices in $\mathcal{P}_n$ the vertices u and v are r-separated by a codeword of C.*

Similar result for r-locating-dominating codes in cycles have been shown in [5].

Lemma 2.2 ([5])**.** *A code $C \subseteq V_n$ is r-locating-dominating in $\mathcal{C}_n$ if and only if*

(i) each vertex $u \in V_n \setminus C$ is r-covered by a codeword of C,

(ii) each pair (u, v) of C-consecutive vertices in $\mathcal{C}_n$ is r-separated by C and

(iii) there exists at most one vertex $u \in V_n \setminus C$ such that $I_r(u) = C$.

The smallest cardinalities of 1-locating-dominating codes in $\mathcal{P}_n$ and $\mathcal{C}_n$ have been solved by Slater in [13] and [14]. In particular, he showed that $M_1^{LD}(\mathcal{C}_n) = M_1(\mathcal{P}_n) = \lceil 2n/5 \rceil$. For general r, we have the following lower bounds for the smallest cardinalities in $\mathcal{C}_n$ and $\mathcal{P}_n$ by Bertrand *et al.* [1]:

$$M_r^{LD}(\mathcal{C}_n) \geq \left\lceil \frac{n}{3} \right\rceil \tag{1}$$

and

$$M_r^{LD}(\mathcal{P}_n) \geq \left\lceil \frac{n+1}{3} \right\rceil . \tag{2}$$

In [8], it is shown that the lower bound (2) is always attained when $r = 2$, i.e. $M_2^{LD}(\mathcal{P}_n) = \lceil (n+1)/3 \rceil$ for all n. Moreover, in [4], the exact values of $M_3^{LD}(\mathcal{P}_n)$ and $M_4^{LD}(\mathcal{P}_n)$ are determined. In particular, it is proved that also in these cases the lower bound can be attained when n is large enough. By [4], the exact values of $M_r^{LD}(\mathcal{P}_n)$ are known when $3 \leq n \leq 7r + 3$. Furthermore, the following theorem (in [4]) shows that the lower bound (2) can always be attained when $r \geq 5$ and n is large enough. This result settles a conjecture stated in [1, Conjecture 1].

Theorem 2.3 ([4])**.** *If $r \geq 5$ and $n \geq 3r+2+3(2r+1)((r-3)(2r+1)+r)$, then*

$$M_r^{LD}(\mathcal{P}_n) = \left\lceil \frac{n+1}{3} \right\rceil .$$

In [5], the following analogous result has been shown in the case of cycles.

Theorem 2.4 ([5])**.** *Let $r \geq 5$ and $n \geq 12r+5+2r((r-3)(6r+3)+3r+3)$.*

(i) If $n \not\equiv 3 \pmod 6$, then $M_r^{LD}(\mathcal{C}_n) = \lceil n/3 \rceil$.

(ii) If $n \equiv 3 \pmod 6$, then $n/3 \leq M_r^{LD}(\mathcal{C}_n) \leq n/3 + 1$.

The exact values of $M_2^{LD}(\mathcal{C}_n)$ are determined in [3]. In particular, it is shown that for $n > 6$ if $n \equiv 3 \pmod 6$, then $M_2^{LD}(\mathcal{C}_n) = n/3 + 1$, else $M_2^{LD}(\mathcal{C}_n) = \lceil n/3 \rceil$. In [5], the exact values of $M_3^{LD}(\mathcal{C}_n)$ and $M_4^{LD}(\mathcal{C}_n)$ are solved. Moreover, as in the case when $r = 2$, it is proved that if $n \equiv 3 \pmod 6$, then we have $M_3^{LD}(\mathcal{C}_n) = n/3 + 1$ and $M_4^{LD}(\mathcal{C}_n) = n/3 + 1$. Hence, it is conjectured that in the case (ii) of Theorem 2.4 we actually have $M_r^{LD}(\mathcal{C}_n) = n/3 + 1$ for any r.

Theorem 2.4 can be improved when r is odd in the sense that the results in the cases (i) and (ii) hold when $n = \Omega(r^2)$ instead of the previous bound $n = \Omega(r^3)$. The proof of the following theorem (omitted here) can be found in Appendix.

Theorem 2.5. *Let $n \geq 6r + 1 + (r-1)(3r+3)$ and r be an odd integer such that $r \geq 5$.*

(i) If $n \not\equiv 3 \pmod 6$, then $M_r^{LD}(\mathcal{C}_n) = \lceil n/3 \rceil$.

(ii) If $n \equiv 3 \pmod 6$, then $n/3 \leq M_r^{LD}(\mathcal{C}_n) \leq n/3 + 1$.

3 On $(r, \leq \ell)$-LDA codes in $\mathcal{C}_n$ and $\mathcal{P}_n$

In this section, we are going to consider $(r, \leq \ell)$-locating-dominating codes of type A in cycles and paths when $\ell \geq 2$. If $\ell \geq 3$, then an $(r, \leq \ell)$-LDA code in any cycle or path trivially contains all the vertices of the considered graph. Indeed, if C is an $(r, \leq \ell)$-LDA code in $\mathcal{C}_n$ ($n \geq 3$) and v_i is a vertex such that $v_i \in V_n \setminus C$, then $\{v_{i-1}, v_i, v_{i+1}\} \cap C = \{v_{i-1}, v_{i+1}\} \cap C$ and $I_r(v_{i-1}, v_i, v_{i+1}) = I_r(v_{i-1}, v_{i+1})$ (a contradiction). The reasoning in the case of paths is analogous.

Assume then that $\ell = 2$. The following lemma gives a useful characterization of $(r, \leq \ell)$-LDA codes in cycles.

Lemma 3.1. *A code $C \subseteq V_n$ is $(r, \leq 2)$-locating-dominating of type A in $\mathcal{C}_n$ if and only if the following conditions are satisfied:*

(i) if $v_i \in V_n \setminus C$, then $v_{i-r} \in C$ and $v_{i+r} \in C$.

(ii) if sets $X, Y \subseteq V_n$ of size at most two are such that $X \cap C = Y \cap C$ and $I_r(X) = I_r(Y) = C$, then $X = Y$.

Proof. Let first C be an $(r, \leq 2)$-locating-dominating code of type A. Now the condition (ii) immediately follows. Assume then that $v_i \in V_n \setminus C$. Since $\{v_{i-1}, v_i\} \cap C = \{v_{i-1}\} \cap C$, the symmetric difference of $I_r(v_{i-1}, v_i)$ and $I_r(v_{i-1})$ is non-empty. Therefore, the vertex v_{i+r} belongs to C. Analogous reasoning implies that also $v_{i-r} \in C$. Thus, the condition (ii) is also satisfied.

Assume now that C is a code satisfying the conditions (i) and (ii). By the definition, C is an $(r, \leq 2)$-LDA code in $\mathcal{C}_n$ if each set $X \subseteq V_n$ of size at most two is uniquely determined by the sets $X \cap C$ and $I_r(X)$. Let then $X \subseteq V_n$ be a set of size at most two. Clearly, if $|X \cap C| = 2$, then X is uniquely determined ($X = X \cap C$).

Assume then that $X \cap C = \emptyset$. If $I_r(X) = C$, then X is uniquely determined by the condition (ii). Assume then that $I_r(X) \neq C$. Now, by the condition (i), there exists a vertex $v_i \in I_r(X)$ such that $v_{i-1} \notin B_r(X)$. Furthermore, by (i), it can be concluded that $v_{i+r} \in X$. Similarly, there exists $v_j \in I_r(X)$ such that $v_{j+1} \notin B_r(X)$. This implies that $v_{j-r} \in X$. (It is possible that $v_{i+r} = v_{j-r}$.) Thus, the set X can be uniquely determined (using the available information). The case when $|X \cap C| = 1$ is similar to the previous one. In conclusion, C is an $(r, \leq 2)$-LDA code in C_n. □

A characterization similar to the previous lemma can also be presented in the case of paths. The proof is analogous to the one of the previous lemma.

Lemma 3.2. *A code $C \subseteq V_n$ is $(r, \leq 2)$-locating-dominating of type A in $\mathcal{P}_n$ if and only if the following conditions are satisfied:*

(i) $\{v_0, v_1, \dots, v_{r-1}\}$ and $\{v_{n-r}, v_{n-r+1}, \dots, v_{n-1}\}$ are subsets of C.

(ii) if $v_i \in V_n \setminus C$, then $v_{i-r} \in C$ and $v_{i+r} \in C$.

For future considerations, we say that a code $T \subseteq V$ is a *transversal* of a graph $G = (V, E)$ if for each edge $e = uv \in E$ the vertex u or the vertex v belongs to T. A transversal is also sometimes called a *vertex cover* [17, p. 102] or an *edge-covering set* [18] of G.

Let then t be a positive integer. Define graphs $\mathcal{C}'_{(n,t)} = (V_n, F_n)$ and $\mathcal{P}'_{(n,t)} = (V_n, F'_n)$, where $F_n = \{v_i v_{i+t} \mid i \in \mathbb{Z}_n\}$ and $F'_n = \{v_i v_{i+t} \mid 0 \leq i \leq n - t - 1\}$. Now we are ready to present the following lower bound on $(r, \leq 2)$-LDA codes in cycles.

Theorem 3.3. *For all integers $n \geq 3$ and $r \geq 1$, we have*

$$M^{LDA}_{(r,\leq 2)}(\mathcal{C}_n) \geq \gcd(r, n) \left\lceil \frac{n}{2\gcd(r, n)} \right\rceil .$$

Proof. Let C be an $(r, \leq 2)$-locating-dominating code of type A in $\mathcal{C}_n$. By Lemma 3.1 (i), C is a transversal of $\mathcal{C}'_{(n,r)}$. The graph $\mathcal{C}'_{(n,r)}$ consists of $\gcd(r, n)$ disjoint cycles on $n/\gcd(r, n)$ vertices, where $\gcd(r, n)$ stands for the greatest common divisor of r and n. For each cycle of length $n/\gcd(r, n)$, the minimum cardinality of a transversal is clearly $\lceil n/(2\gcd(r, n))\rceil$. Therefore, the claim immediately follows. □

The previous lower bound can be attained when $n \geq 6r+3$ as is shown in the following theorem.

Theorem 3.4. *For all integers $n \geq 6r+3$ and $r \geq 1$, we have*

$$M^{LDA}_{(r,\leq 2)}(\mathcal{C}_n) = \gcd(r,n)\left\lceil \frac{n}{2\gcd(r,n)} \right\rceil.$$

Proof. Let $d = \gcd(r,n)$ and $n' = n/d$. The graph $\mathcal{C}'_{(n,r)}$ consists of d disjoint cycles on n' vertices. For all $i \in \mathbb{Z}_d$ define

$$T_i = \{v_{i+jr} \mid 0 \leq j \leq n'-1,\ j \text{ is even}\}.$$

Furthermore, define

$$T = \bigcup_{i=0}^{d-1} T_i.$$

By the construction, T is a transversal of $\mathcal{C}'_{(n,r)}$ and the number of vertices in T is equal to $\gcd(r,n)\lceil n/(2\gcd(r,n))\rceil$. Therefore, T satisfies the condition (i) of Lemma 3.1.

Let us then show that there does not exist a set $X \subseteq V_n$ such that $|X| \leq 2$ and $I_r(X) = T$. If X is such a set, then there exist vertices $v_i, v_{i+r} \in V_n$ such that $v_i, v_{i+r} \notin B_r(X)$ (since $n \geq 2(2r+1)+2r+1$). This leads to a contradiction since at least one of the vertices v_i and v_{i+r} belongs to T. Thus, T is an $(r,\leq 2)$-locating-dominating code of type A in $\mathcal{C}_n$. □

Consider then $(r,\leq 2)$-locating-dominating codes of type A in paths. If $3 \leq n \leq 2r$, then by Lemma 3.2 (i) we obtain that $M^{LDA}_{(r,\leq 2)}(\mathcal{P}_n) = n$. The following theorem solves the problem in the remaining cases.

Theorem 3.5. *Let $n = qr+p$, where $q \geq 2$ and $0 \leq p \leq r-1$. Then we have*

$$M^{LDA}_{(r,\leq 2)}(\mathcal{P}_n) = p\left\lfloor \frac{q+1}{2} \right\rfloor + (r-p)\left\lfloor \frac{q}{2} \right\rfloor + r.$$

Proof. Let C be an $(r,\leq 2)$-locating-dominating code of type A in $\mathcal{P}_n$. The graph $\mathcal{P}'_{(n,r)}$ consists of p and $r-p$ disjoint paths on $q+1$ and q vertices, respectively. Since C satisfies the conditions (i) and (ii) of Lemma 3.2, the number of codewords in the previous paths of length $q+1$ and q is at least $\lfloor (q+1)/2 \rfloor + 1$ and $\lfloor q/2 \rfloor + 1$, respectively. Therefore, we have

$$|C| \geq p\left(\left\lfloor \frac{q+1}{2} \right\rfloor + 1\right) + (r-p)\left(\left\lfloor \frac{q}{2} \right\rfloor + 1\right).$$

For the construction attaining the previous lower bound, we define

$$T_i = \{v_{i+jr} \mid 0 \leq i+jr \leq n-1,\ j \text{ is even}\},$$

where i is an integer such that $0 \leq i \leq r-1$. Furthermore, define

$$T = \bigcup_{i=0}^{r-1} T_i \cup \{v_{n-r}, v_{n-r+1}, \ldots, v_{n-1}\}.$$

By Lemma 3.2, T is an $(r, \leq 2)$-LDA code in $\mathcal{P}_n$. Furthermore, it is straightforward to verify that the number of codewords of T is equal to the previous lower bound. Thus, the claim follows. $\square$

4 On $(r, \leq \ell)$-LDB codes in $\mathcal{C}_n$ and $\mathcal{P}_n$

In this section, we are going to consider $(r, \leq \ell)$-locating-dominating codes of type B in cycles and paths when $\ell \geq 2$. With small n (compared to r), the smallest cardinalities of $(r, \leq \ell)$-LDB codes in cycles and paths are easy to determine. Indeed, if $3 \leq n \leq 2r+1$, then we clearly have $M^{LDB}_{(r,\leq \ell)}(\mathcal{C}_n) = n-1$ for all r. Furthermore, if $3 \leq n \leq r+1$, then we immediately obtain $M^{LDB}_{(r,\leq \ell)}(\mathcal{P}_n) = n-1$ for all r. Let us then consider more closely the case with $\ell = 2$.

The following lemma gives a useful characterization of $(r, \leq 2)$-LDB codes in cycles.

Lemma 4.1. *A code $C \subseteq V_n$ is $(r, \leq 2)$-locating-dominating of type B in $\mathcal{C}_n$ if and only if the following conditions are satisfied:*

(i) if (u, v) is a pair of C-consecutive vertices in $\mathcal{C}_n$, then the sets $I_r(u) \setminus I_r(v)$ and $I_r(v) \setminus I_r(u)$ are both non-empty.

(ii) if sets $X, Y \subseteq V_n \setminus C$ of size at most two are such that $I_r(X) = I_r(Y) = C$, then $X = Y$.

Proof. Let first C be an $(r, \leq 2)$-locating-dominating code of type B in $\mathcal{C}_n$. Clearly, the condition (ii) is now satisfied. Let then (u, v) be a pair of C-consecutive vertices. Since $I_r(v) \neq I_r(u, v)$, we have $I_r(u) \setminus I_r(v) \neq \emptyset$. Similarly, we have $I_r(v) \setminus I_r(u) \neq \emptyset$. Hence, the condition (i) is also satisfied.

Assume then that $C \subseteq V_n$ satisfies the conditions (i) and (ii). By the definition, C is an $(r, \leq 2)$-LDB code in $\mathcal{C}_n$ if each set $X \subseteq V_n \setminus C$ of size at most two is uniquely determined by the set $I_r(X)$. Let then $X \subseteq V_n \setminus C$ be a set of size at most two. If $I_r(X) = C$, then X is uniquely determined by the condition (ii).

Assume then that $I_r(X) \neq C$. Let now u be the leftmost vertex of $I_r(X)$. It is straightforward to determine that there exists a unique pair (v, w) of C-consecutive vertices such that $u \in I_r(v) \setminus I_r(w)$. Hence, by the condition (i), v belongs to the set X. Similarly, for the rightmost vertex u' of $I_r(X)$, there exists a unique pair (v', w') of C-consecutive vertices such that $u' \in I_r(w') \setminus I_r(v')$. Therefore, by (i), w' belongs to the set X. Thus, the set X can be uniquely determined (using only the I-set $I_r(X)$). In conclusion, C is an $(r, \leq 2)$-LDB code in C_n. □

A characterization similar to the previous lemma can also be presented in the case of paths.

Lemma 4.2. *A code $C \subseteq V_n$ is $(r, \leq 2)$-locating-dominating of type B in $\mathcal{P}_n$ if and only if the following conditions are satisfied:*

(i) sets $\{v_0, v_1, \ldots, v_r\}$ and $\{v_{n-r-1}, v_{n-r}, \ldots, v_{n-1}\}$ both contain at least r codewords of C.

(ii) if (u, v) is a pair of C-consecutive vertices in $\mathcal{P}_n$, then the sets $I_r(u) \setminus I_r(v)$ and $I_r(v) \setminus I_r(u)$ are both non-empty.

Proof. Let C be an $(r, \leq 2)$-locating-dominating code of type B in $\mathcal{P}_n$. Let us first show that $\{v_0, v_1, \ldots, v_r\}$ always contains at least r codewords of C. If this is not the case, then there exist two vertices $v_i \in V_n \setminus C$ and $v_j \in V_n \setminus C$ such that $0 \leq i < j \leq r$. This leads to a contradiction since now $I_r(v_i) = I_r(v_i, v_j)$. Analogous arguments can also be applied to $\{v_{n-r-1}, v_{n-r}, \ldots, v_{n-1}\}$. Hence, the condition (i) is satisfied. The proof of the condition (ii) is similar to the one of Lemma 4.1.

In order to show that C is an $(r, \leq 2)$-LDB code in $\mathcal{P}_n$ if the conditions (i) and (ii) are satisfied, we again refer to the proof of Lemma 4.1. □

The characterization of Lemma 4.1 gives rise to the following lower bound.

Theorem 4.3. *For all integers $n \geq 3$ and $r \geq 1$, we have*

$$M^{LDB}_{(r,\leq 2)}(\mathcal{C}_n) \geq \left\lceil \frac{n}{2} \right\rceil .$$

Proof. Let C be an $(r, \leq 2)$-locating-dominating code of type B in $\mathcal{C}_n$. By Lemma 4.1 (i), each pair of C-consecutive vertices has to be r-separated by at least two codewords. On the other hand, each codeword of C can clearly r-separate at most two pairs of C-consecutive vertices. Therefore, we have $2|C| \geq 2(n - |C|)$. Thus, the claim immediately follows. □

The following theorem shows that the lower bound can be attained when $n \geq 4r + 5$.

Theorem 4.4. *For all integers $n \geq 4r + 5$ and $r \geq 1$, we have*

$$M^{LDB}_{(r,\leq 2)}(\mathcal{C}_n) = \left\lceil \frac{n}{2} \right\rceil .$$

Proof. For the construction, define first

$$C = \{v_i \mid i \in \mathbb{Z}_n,\, i \text{ is even}\}.$$

For any pair (u, v) of C-consecutive vertices, we have $|B_r(u) \setminus B_r(v)| \geq 2$ and $|B_r(v) \setminus B_r(u)| \geq 2$. Therefore, we obtain that $I_r(u) \setminus I_r(v) \neq \emptyset$ and $I_r(v) \setminus I_r(u) \neq \emptyset$. Hence, the condition (i) of Lemma 4.1 is satisfied. Since $n \geq 2(2r+1)+3$, there does not exist $X \subseteq V_n \setminus C$ such that $|X| \leq 2$ and $I_r(X) = C$. Therefore, the condition (ii) of Lemma 4.1 is satisfied. Thus, the claim follows. □

The following theorem provides a lower bound for the size of $(r, \leq 2)$-LDB codes in paths.

Theorem 4.5. *For all integers $n \geq 3$ and $r \geq 1$, we have*

$$M^{LDB}_{(r,\leq 2)}(\mathcal{P}_n) \geq \left\lceil \frac{n+r-1}{2} \right\rceil .$$

Proof. Let C be an $(r, \leq 2)$-locating-dominating code of type B in $\mathcal{P}_n$. By Lemma 4.2 (ii), each pair of C-consecutive vertices has to be r-separated by at least two codewords. On the other hand, each codeword of C can clearly r-separate at most two pairs of C-consecutive vertices. Moreover, each codeword belonging to $\{v_0, v_1, \ldots, v_r\}$ or $\{v_{n-r-1}, v_{n-r}, \ldots, v_{n-1}\}$ can r-separate at most one pair of C-consecutive vertices. Therefore, by Lemma 4.2 (i), we have

$$2(|C| - 2r) + 2r \geq 2(n - |C| - 1).$$

Thus, the claim immediately follows. □

Let us then consider constructions for $(r, \leq 2)$-LDB codes in paths. First assume that $r = 1$. By the previous theorem, the smallest cardinality of a $(1, \leq 2)$-locating-dominating code of type B in $\mathcal{P}_n$ is at least $\lceil n/2 \rceil$. The following results, which show that the lower bound can always be attained, are straightforward to verify using Lemma 4.2:

- If $n \geq 3$ and n is odd, then

  $$\{v_i \mid 0 \leq i \leq n-1,\, i \text{ is even}\}$$

  is a $(1, \leq 2)$-LDB code in $\mathcal{P}_n$ with $\lceil n/2 \rceil$ codewords.

- If $n = 4k$, where k is an integer such that $k \geq 1$, then

  $$C = \bigcup_{i=0}^{k-1} \{v_{4i+1}, v_{4i+2}\}$$

  is a $(1, \leq 2)$-LDB code in $\mathcal{P}_n$ with $\lceil n/2 \rceil$ codewords.

- If $n = 4k+2$, where k is an integer such that $k \geq 1$, then $C \cup \{v_{4k+1}\}$ is a $(1, \leq 2)$-LDB code in $\mathcal{P}_n$ with $\lceil n/2 \rceil$ codewords.

In conclusion, we have $M^{LDB}_{(1,\leq 2)}(\mathcal{P}_n) = \lceil n/2 \rceil$ for any n.

In general, for each r, we have an infinite family of n such that $M^{LDB}_{(r,\leq 2)}(\mathcal{P}_n) = \lceil (n+r-1)/2 \rceil$. Indeed, by Lemma 4.2,

$$\bigcup_{i=0}^{k} \{v_{2i(r+1)+1}, v_{2i(r+1)+2}, \ldots, v_{2i(r+1)+r+1}\}$$

is an $(r, \leq 2)$-locating-dominating code of type B in $\mathcal{P}_n$, where $k \geq 1$ is an integer and $n = (2k+1)(r+1)+2$. Moreover, the size $(k+1)(r+1)$ of the code attains the lower bound of Theorem 4.5.

Let $\ell \geq 3$. Consider then $(r, \leq \ell)$-locating-dominating codes of type B in cycles and paths. In comparison to the $(r, \leq \ell)$-locating-dominating codes of type A, these codes are not trivial. The following theorem provides optimal $(r, \leq \ell)$-LDB codes in cycles.

Theorem 4.6. *Let n and ℓ be integers such that $n \geq 2r+2$ and $\ell \geq 3$.*

- *If $n \not\equiv r+1 \pmod{2r+2}$, then $M^{LDB}_{(r,\leq \ell)}(\mathcal{C}_n) = \lceil rn/(r+1) \rceil$.*
- *If $n \equiv r+1 \pmod{2r+2}$, then $M^{LDB}_{(r,\leq \ell)}(\mathcal{C}_n) = \lceil rn/(r+1) \rceil + 1$.*

Proof. Let C be an $(r, \leq \ell)$-locating-dominating code of type B in $\mathcal{C}_n$. Let then (v_i, v_j) and (v_j, v_k) $(v_i \neq v_k)$ be pairs of C-consecutive vertices. (If such pairs of C-consecutive vertices do not exist, then the number of non-codewords is at most two and the lower bound immediately follows.) The total number of codewords in $\{v_{i+1}, v_{i+2}, \ldots, v_{j-1}\}$ and $\{v_{j+1}, v_{j+2}, \ldots, v_{k-1}\}$ is at least $2r$ since $I_r(v_i, v_j, v_k) \neq I_r(v_i, v_k)$. The number of such triples of non-codewords is equal to $n - |C|$. On the other hand, each codeword can associate with at most two of such triples. Therefore, we have

$$2|C| \geq 2r(n - |C|).$$

Hence, we have $|C| \geq \lceil rn/(r+1) \rceil$. Moreover, using the previous notations, we obtain that if n is divisible by $r+1$ and $|C| = rn/(r+1)$, then the sum of the number of codewords in $\{v_{i+1}, v_{i+2}, \ldots, v_{j-1}\}$ and $\{v_{j+1}, v_{j+2}, \ldots, v_{k-1}\}$ is equal to $2r$.

Let $n = 2q(r+1)+p$, where q and p are integers such that $q \geq 1$ and $0 \leq p \leq 2r+1$. The lower bound obtained above can now be written as follows:

$$M^{LDB}_{(r,\leq \ell)}(\mathcal{C}_n) \geq \begin{cases} 2qr+p & \text{if } 0 \leq p \leq r \\ 2qr+p-1 & \text{otherwise.} \end{cases}$$

Assume that $0 \leq p \leq r$. Let then $D_1 \subseteq V_n$ be a code such that

$$V_n \setminus D_1 = \bigcup_{i=0}^{q-1} \{v_{2(r+1)i}, v_{2(r+1)i+r+2}\}.$$

The number of codewords in D_1 is equal to $2qr + p$. For each vertex $v \in V_n \setminus D_1$ there exists a codeword u such that v is r-covered by u and u is not r-covered by any other non-codeword. Indeed, $v_{2(r+1)i+1}$ and $v_{2(r+1)i+r+1}$ are such codewords for $v_{2(r+1)i}$ and $v_{2(r+1)i+r+2}$, respectively. Therefore, D_1 is an $(r, \leq \ell)$-LDB code in $\mathcal{C}_n$.

Assume then that $r + 2 \leq p \leq 2r + 1$. Let $D_2 \subseteq V_n$ be a code such that

$$V_n \setminus D_2 = (V_n \setminus D_1) \cup \{v_{2q(r+1)}\}.$$

Similarly as above, it can be shown that D_2 is an $(r, \leq \ell)$-LDB code in $\mathcal{C}_n$ with $2qr + p - 1$ codewords.

Finally, assume that $p = r + 1$. Let us then show that in this case the lower bound can be increased by one. Let C be an $(r, \leq \ell)$-LDB code in $\mathcal{C}_n$ (attaining the lower bound) with $(2q + 1)r$ codewords. Notice that the number of non-codewords is now equal to $2q + 1$. Without loss of generality, we may assume that $v_0 \in V_n \setminus C$. Let then (v_i, v_0), (v_0, v_j) and (v_j, v_k) be pairs of C-consecutive vertices. By the considerations in the first paragraph of the proof, the total number of codewords in $\{v_1, v_2, \ldots, v_{j-1}\}$ and $\{v_{j+1}, v_{j+2}, \ldots, v_{k-1}\}$ is equal to $2r$. Denote the number of codewords in $\{v_1, v_2, \ldots, v_{j-1}\}$ by s. Then the number of codewords in $\{v_{j+1}, v_{j+2}, \ldots, v_{k-1}\}$ is equal to $2r - s$. Therefore, by continuing in the same way through all the triples, we obtain that the number of codewords in $\{v_{i+1}, v_{i+2}, \ldots, v_{-1}\}$ is equal to s. On the other hand, since $|\{v_1, v_2, \ldots, v_{j-1}\}| = s$, we have $|\{v_{i+1}, v_{i+2}, \ldots, v_{-1}\}| = 2r - s$. Hence, we have $s = r$. This leads to a contradiction since now $I_r(v_i, v_0, v_j) = I_r(v_i, v_j)$. Thus, we have $M^{LDB}_{(r,\leq \ell)}(\mathcal{C}_n) \geq \lceil rn/(r+1) \rceil + 1$. On the other hand, it can be verified (as in the previous cases) that a code D_3 satisfying $V_n \setminus D_3 = V_n \setminus D_1$ is an $(r, \leq \ell)$-LDB code in $\mathcal{C}_n$ with $2qr + p = \lceil rn/(r + 1) \rceil + 1$ codewords. Hence, the claim follows. □

The following theorem provides optimal $(r, \leq \ell)$-LDB codes in paths when $\ell \geq 3$.

Theorem 4.7. *Let n and ℓ be integers such that $n \geq r + 2$ and $\ell \geq 3$. Then we have*

$$M^{LDB}_{(r,\leq \ell)}(\mathcal{P}_n) = \left\lceil \frac{r(n-1)+1}{r+1} \right\rceil.$$

Proof. Let C be an $(r, \leq \ell)$-locating-dominating code of type B in $\mathcal{P}_n$. For the lower bound, first denote

$$V_n \setminus C = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\},$$

where $0 \leq i_1 < i_2 < \cdots < i_k \leq n-1$ and $k = n - |C|$. Further, denote

$$\begin{aligned}
A(v_{i_1}) &= \{v_0, v_1, \ldots, v_{i_1-1}\} \cup \{v_{i_1+1}, v_{i_1+2}, \ldots, v_{i_2-1}\}, \\
A(v_{i_k}) &= \{v_{i_{k-1}+1}, v_{i_{k-1}+2}, \ldots, v_{i_k-1}\} \cup \{v_{i_k+1}, v_{i_k+2}, \ldots, v_{n-1}\} \text{ and} \\
A(v_{i_j}) &= \{v_{i_{j-1}+1}, v_{i_{j-1}+2}, \ldots, v_{i_j-1}\} \cup \{v_{i_j+1}, v_{i_j+2}, \ldots, v_{i_{j+1}-1}\},
\end{aligned}$$

where $2 \leq j \leq k-1$. Finally, denote $a_1 = |\{v_0, v_1, \ldots, v_{i_1-1}\}|$ and $a_k = |\{v_{i_k+1}, v_{i_k+2}, \ldots, v_{n-1}\}|$. Since the sets $I_r(v_{i_1}, v_{i_2})$ and $I_r(v_{i_2})$ are not equal, we obtain that $|A(v_{i_1})| \geq r$ and if $a_1 = 0$, then $|A(v_{i_1})| \geq r+1$. Analogous arguments also hold for $A(v_{i_k})$. Now, using similar reasoning as in the proof of Theorem 4.6, we have $|A(v_{i_j})| \geq 2r$ when $2 \leq j \leq k-1$. On the other hand, each codeword belongs to at most two sets $A(v_{i_j})$ $(1 \leq j \leq k)$ except the ones belonging to $\{v_0, v_1, \ldots, v_{i_1-1}\}$ and $\{v_{i_k+1}, v_{i_k+2}, \ldots, v_{n-1}\}$, which are only contained in one such set. Therefore, the following inequality is obtained:

$$2(|C| - a_1 - a_k) + a_1 + a_k \geq 2r + f(a_i) + f(a_k) + 2r(n - |C| - 2),$$

where $f(0) = 1$ and $f(a) = 0$ for any positive integer a. Hence, we have

$$M^{LDB}_{(r,\leq \ell)}(\mathcal{P}_n) \geq \left\lceil \frac{r(n-1)+1}{r+1} \right\rceil .$$

Let $n = (2q+1)(r+1) + p$, where q and p are integers such that $q \geq 0$ and $0 \leq p \leq 2r+1$. The lower bound obtained above can now be written as follows:

$$M^{LDB}_{(r,\leq \ell)}(\mathcal{P}_n) \geq \begin{cases} (2q+1)r + p & \text{if } 0 \leq p \leq 1 \\ (2q+1)r + p - 1 & \text{if } 2 \leq p \leq r+2 \\ (2q+1)r + p - 2 & \text{if } r+3 \leq p \leq 2r+1. \end{cases}$$

Assume first that $0 \leq p \leq 1$. Let then $D_1 \subseteq V_n$ be a code such that

$$V_n \setminus D_1 = \{v_1\} \cup \bigcup_{i=0}^{q-1} \{v_{2(r+1)i+r+1}, v_{2(r+1)i+2r+3}\}.$$

The number of codewords in D_1 is equal to $(2q+1)r + p$. Similarly to the proof of Theorem 4.6, it can be shown that D_1 is an $(r, \leq \ell)$-LDB code in $\mathcal{P}_n$ attaining the lower bound. Assume then that $2 \leq p \leq r+2$. Let $D_2 \subseteq V_n$ be a code such that $V_n \setminus D_2 = (V_n \setminus D_1) \cup \{v_{(2q+1)(r+1)}\}$. Similarly as before, it can be shown that D_2 is an $(r, \leq \ell)$-LDB code in $\mathcal{P}_n$ with $(2q+1)r+p-1$ codewords. Finally, assume that $r+3 \leq p \leq 2r+1$. Then it can be shown that a code $D_3 \subseteq V_n$ satisfying $V_n \setminus D_3 = (V_n \setminus D_2) \cup \{v_{(2q+1)(r+1)+r+2}\}$ is an $(r, \leq \ell)$-LDB code in $\mathcal{P}_n$ with $(2q+1)r + p - 2$ codewords. $\square$

Appendix

In what follows, we present the proof of Theorem 2.5. Let $r \geq 5$ be an odd integer and s be a non-negative integer. Define

$$K(s) = \bigcup_{i=0}^{(r-1)/2} \{v_{s+2i}, v_{s+2r+2i}\}.$$

Using Lemma 2.2, it is straightforward to verify that $K(0)$ is an r-locating-dominating code in $\mathcal{C}_{3r}$, $\mathcal{C}_{3r+1}$, $\mathcal{C}_{3r+2}$ and $\mathcal{C}_{3r+3}$ with $r+1$ codewords. It can also be shown that $K'(s) = K(s) \cup \{v_{s+3r+2}\}$ is an r-locating-dominating code in $\mathcal{C}_{3r+5}$ with $r+2$ codewords when $s = 0$. Define then

$$L(s) = K(s) \cup \bigcup_{i=0}^{(r-5)/2} \{v_{s+3r+2+2i}, v_{s+5r+2+2i}\} \cup \{v_{s+4r}, v_{s+5r-1}\}$$

and

$$L'(s) = K(s) \cup \bigcup_{i=0}^{(r-3)/2} \{v_{s+3r+2+2i}, v_{s+5r+2+2i}\} \cup \{v_{s+5r+1}\}.$$

Again, by Lemma 2.2, $L(0)$ and $L'(0)$ are r-locating-dominating codes in $\mathcal{C}_{6r}$ and $\mathcal{C}_{6r+1}$ with $2r$ and $2r+1$ codewords, respectively.

In what follows, we are going to present constructions of r-locating-dominating codes in cycles $\mathcal{C}_n$ depending on the length n modulo 6. In particular, we show that $M_r^{LD}(\mathcal{C}_n) \leq n/3 + 1$ if $n \equiv 3 \pmod 6$ and $M_r^{LD}(\mathcal{C}_n) \leq \lceil n/3 \rceil$ otherwise.

Let p and q be non-negative integers. Let then $m = m' + p \cdot 6r + q(3r+3)$, where $m' = 3r$, $m' = 3r+1$, $m' = 3r+2$ or $m' = 3r+3$. Define

$$C_1 = \bigcup_{i=0}^{p-1} L(i \cdot 6r) \cup \bigcup_{i=0}^{q} K(p \cdot 6r + i(3r+3)).$$

The code C_1 is r-locating-dominating in $\mathcal{C}_m$ since the codes $K(0)$ and $L(0)$ are r-locating-dominating in $\mathcal{C}_{m'}$ and $\mathcal{C}_{6r}$, respectively. (Indeed, since $K(s)$ is also a part of $L(s)$, the I-sets of all vertices are analogous in both cases.)

Notice that the greatest common divisor of $6r$ and $3r+3$ is equal to 6. Therefore, if n is an integer such that $n \geq m' + (r-1)(3r+3)$ and $n \equiv m' \pmod 6$, then there exist such p and q that $n = m' + p \cdot 6r + q(3r+3)$. Thus, if n is an integer such that $n \geq m' + (r-1)(3r+3)$ and $n \equiv 0, 3, 4$ or $5 \pmod 6$, then by the previous construction $M_r^{LD}(\mathcal{C}_n) \leq n/3 + 1$ if $n \equiv 3 \pmod 6$ and $M_r^{LD}(\mathcal{C}_n) \leq \lceil n/3 \rceil$ otherwise.

Let then $m = 3r + 5 + p \cdot 6r + q(3r + 3)$. Define then

$$C_2 = \bigcup_{i=0}^{p-1} L(i \cdot 6r) \cup \bigcup_{i=0}^{q-1} K(p \cdot 6r + i(3r + 3)) \cup K'(p \cdot 6r + q(3r + 3)).$$

Again, using similar arguments as above, it can be shown that C_2 is an r-locating-dominating code in $\mathcal{C}_m$ with $\lceil m/3 \rceil$ vertices. Thus, if n is an integer such that $n \geq 3r + 5 + (r - 1)(3r + 3)$ and $n \equiv 2 \pmod 6$, then by the previous construction $M_r^{LD}(\mathcal{C}_n) \leq \lceil n/3 \rceil$.

Finally, let then $m = 6r + 1 + p \cdot 6r + q(3r + 3)$. Define

$$C_3 = \bigcup_{i=0}^{p-1} L(i \cdot 6r) \cup \bigcup_{i=0}^{q-1} K(p \cdot 6r + i(3r + 3)) \cup L'(p \cdot 6r + q(3r + 3)).$$

Again, using similar arguments as above, it can be shown that C_3 is an r-locating-dominating code in $\mathcal{C}_m$ with $\lceil m/3 \rceil$ vertices. Thus, if n is an integer such that $n \geq 6r + 1 + (r - 1)(3r + 3)$ and $n \equiv 1 \pmod 6$, then by the previous construction $M_r^{LD}(\mathcal{C}_n) \leq \lceil n/3 \rceil$.

Combining the previous results with the lower bound (1), we immediately obtain Theorem 2.5.

References

[1] N. Bertrand, I. Charon, O. Hudry, and A. Lobstein. Identifying and locating-dominating codes on chains and cycles. *European J. Combin.*, 25(7):969–987, 2004.

[2] D. I. Carson. On generalized location-domination. In *Graph theory, combinatorics, and algorithms, Vol. 1, 2 (Kalamazoo, MI, 1992)*, Wiley-Intersci. Publ., pages 161–179. Wiley, New York, 1995.

[3] C. Chen, C. Lu, and Z. Miao. Identifying codes and locating-dominating sets on paths and cycles. *arXiv*, 2009.

[4] G. Exoo, V. Junnila, and T. Laihonen. Locating-dominating codes in paths. 2009.

[5] G. Exoo, V. Junnila, and T. Laihonen. Locating-dominating codes in cycles. 2010.

[6] S. Gravier, R. Klasing, and J. Moncel. Hardness results and approximation algorithms for identifying codes and locating-dominating codes in graphs. *Algorithmic Oper. Res.*, 3(1):43–50, 2008.

[7] I. Honkala. An optimal locating-dominating set in the infinite triangular grid. *Discrete Math.*, 306(21):2670–2681, 2006.

[8] I. Honkala. On r-locating-dominating sets in paths. *European J. Combin.*, 30(4):1022–1025, 2009.

[9] I. Honkala and T. Laihonen. On locating-dominating sets in infinite grids. *European J. Combin.*, 27(2):218–227, 2006.

[10] I. Honkala, T. Laihonen, and S. Ranto. On locating-dominating codes in binary Hamming spaces. *Discrete Math. Theor. Comput. Sci.*, 6(2):265–281, 2004.

[11] A. Lobstein. Identifying and locating-dominating codes in graphs, a bibliography. Published electronically at `http://perso.enst.fr/~lobstein/debutBIBidetlocdom.pdf`.

[12] P. J. Slater. Domination and location in graphs. Research report 93, National University of Singapore, 1983.

[13] P. J. Slater. Domination and location in acyclic graphs. *Networks*, 17(1):55–64, 1987.

[14] P. J. Slater. Dominating and reference sets in a graph. *J. Math. Phys. Sci.*, 22:445–455, 1988.

[15] P. J. Slater. Locating dominating sets and locating-dominating sets. In *Graph Theory, Combinatories and Applications: Proceedings of the Seventh Quadrennial International Conference on the Theory and Applications of Graphs*, volume 2, pages 1073–1079. Wiley, 1995.

[16] P. J. Slater. Fault-tolerant locating-dominating sets. *Discrete Math.*, 249(1–3):179–189, 2002.

[17] D. B. West. *Introduction to graph theory.* Prentice Hall Inc., Upper Saddle River, NJ, 1996.

[18] M. Xu, K. Thulasiraman, and X.-D. Hu. Identifying codes of cycles with odd orders. *European J. Combin.*, 29(7):1717–1720, 2008.